A General Upper Bound for FSO Channel Capacity with Input-Dependent Gaussian Noise and the Corresponding Optimal Input Distribution

Kobra Akbari Dadamahalleh
Department of Electrical Engineering
Sadjad Institute for Higher Education
Mashhad, Iran
kobra_akbari20@yahoo.com

Ghosheh Abed Hodtani
Department of Electrical Engineering
Ferdowsi University of Mashhad
Mashhad, Iran
ghodtani@gmail.com

***Abstract*–In this paper we derive a new upper bound for the input-output mutual information of FSO channels with input-dependent Gaussian noise by using a simple mathematical inequality. Then, by maximizing the obtained upper bound over all the discrete input distributions with equally spaced mass points, and approximating, we reach to a third order equation for the optimum input distribution. Our equation (*i*) determines the optimum input distribution directly in contrary to the Farid-Hranilovic (FH) work where it is found numerically, and also, (*ii*) gives a previously derived second order equation for the optimum input distribution of FSO channels with input-independent Gaussian noise in a special case. Using numerical illustrations, we compare our input distribution with previous works.**

I. INTRODUCTION

Free space optical (FSO) channels, due to their high-speed, high security and economic links, have been studied extensively in recent years. A majority of wireless optical channels use intensity modulation with direct detection (IM/DD). In these channels all the transmitted signals are non-negative because data are transmitted by modulating the instantaneous intensity of a laser or a LED. In addition, an average optical power, i.e., average amplitude constraint is imposed on the transmitted signal, due to eye safety standards and physical limitations. A peak amplitude constraint is also applied due to safety.

In FSO channels, it is assumed that the corrupting noise is additive white Gaussian distributed that models both thermal and shot noise and is independent of the signal. This assumption is not always reasonable, specially when the light beam is weak. However, particularly at high power, the noise depends on the signal itself, because the photon emission in the laser diode is naturally random.

Previous Work

In [1], a channel model for input-dependent Gaussian noise was considered,

$$\mathrm{Y} = \mathrm{X} + \sqrt{\mathrm{X}}\, Z_1 + Z_0, \tag{1}$$

where, $\mathrm{X} \geq 0$ denoted the channel input, Y denoted the channel output, $Z_0 \sim \mathrm{N}(0,\sigma^2)$ was a zero-mean, variance-σ^2 Gaussian random variable describing the independent noise, and $Z_1 \sim \mathrm{N}(0,1)$ was a zero-mean, unit-variance Gaussian random variable describing the dependent noise, $Z_0 \perp\!\!\!\perp Z_1$. Without loss of generality input was assumed to be scaled such that Z_1 could be normalized to be of unit-variance. The less involved assumption of additive white Gaussian noise was kept, but the variance of the noise was made dependent on the current input signal to better reflect the physical properties of optical communication.

The conditional probability density function (PDF) of this input-dependent Gaussian noise channel was given by,

$$f_{y|x}(y\,|\,x) = \frac{1}{\sqrt{2\pi(\sigma^2+x)}}\, e^{-\frac{(y-x)^2}{2(\sigma^2+x)}}, \tag{2}$$

the constraints are the same as input-independent Gaussian noise, which are,

$$0 \leq \mathrm{X} \leq A, \qquad \mathrm{E}\{\mathrm{X}\} \leq P,$$

where, A is the peak amplitude constraint and P is the average power constraint. A parameter ρ was defined as *peak to average power ratio*,

$$\rho \triangleq \frac{A}{P}$$

In [1] an input distribution was chosen that maximized the source entropy, h(X), under the given constraints, and under the constraint that E{logY} was constant. The input distribution was given by [1, Eqn. (81)]. In order to upper bound the capacity, two concepts was considered in [1]. Firstly, a technique of using duality-based upper bounds on mutual information, and secondly the notion of input distributions that escape to infinity that allowed them to compute asymptotic expectations over the unknown capacity-achieving input distribution.

Our Work

By using a simple mathematical inequality, we find a new upper bound for the input-output mutual information of FSO channels with input-dependent Gaussian noise. We consider a family of discrete input distribution with equally spaced mass points. Then, by maximizing the obtained upper bound over all these input distributions and approximating, we reach to a third order equation for the optimum input distribution. The advantages of our work are, (i) our optimum input distributions are determined directly through a third order equation, in contrary to [3], [4] where the optimum input distributions are found numerically, (ii) gives the second order equation obtained in [2] for the optimum input distribution in FSO channels with input-independent Gaussain noise in a special case, (iii) in comparison with [1], where the upper bound is computed through complicated computations, our upper bound is computed easily through a simple mathematical inequality and has a simple analytic expression. Using numerical illustrations, we compare our results with previous works.

Paper Organization

This paper has 5 sections. In Section II, an upper bound for the mutual information of FSO channel with input-dependent Gaussian noise is obtained by using a simple mathematical inequality, then, by maximizing the obtained upper bound over all discrete input distributions with equally spaced mass points, we obtain the optimum input distribution. In section III, we approximate the obtained optimum input distribution with a third order equation. In section IV, numerical results and comparisons are shown in table I, II, and III. The paper concludes in section V.

II. MAIN RESULTS

In this section, first we determine a new upper bound for the input-output mutual information of FSO channel with input-dependent Gaussian noise by using a simple mathematical inequality, then, by maximizing the obtained upper bound over all discrete input distributions with equally spaced mass points, we find the corresponding optimum input distribution.

$$\mathrm{I(X;Y)} = H(\mathrm{Y}) - H(\mathrm{Y} \mid \mathrm{X}) =$$

$$-\int_{-\infty}^{\infty} f_y(\mathrm{y}) \log_2 f_y(\mathrm{y})\, \mathrm{dy} - \frac{1}{2}\mathrm{E}[\log_2 2\pi e\,(\sigma^2 + x)], \tag{3}$$

Using $f_{y|x}(y \mid x)$ given in (2), $f_y(y)$ is obtained,

$$f_y(\mathrm{y}) = \int_{-\infty}^{\infty} f_{y|x}(y \mid x)\, f_x(x)\, \mathrm{d}x =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\sigma^2+x)}} e^{-\frac{(y-x)^2}{2(\sigma^2+x)}} . \sum_{i=0}^{m} a_i \delta\,(x - x_i)\, \mathrm{d}x =$$

$$\sum_{i=0}^{m} \frac{a_i}{\sqrt{2\pi(\sigma^2+x_i)}} e^{-\frac{(y-x_i)^2}{2(\sigma^2+x_i)}} \tag{4}$$

Then,

$$\mathrm{I(X;Y)} =$$

$$-\int_{-\infty}^{\infty} \sum_{i=0}^{m} \frac{a_i}{\sqrt{2\pi(\sigma^2+x_i)}} e^{-\frac{(y-x_i)^2}{2(\sigma^2+x_i)}} .$$

$$\log_2 \sum_{k=0}^{m} \frac{a_k}{\sqrt{2\pi(\sigma^2+x_k)}} e^{-\frac{(y-x_k)^2}{2(\sigma^2+x_k)}} \mathrm{d}y$$

$$-\frac{1}{2}\mathrm{E}[\log_2 2\pi e(\sigma^2 + x)]. \tag{5}$$

In (5), the term $\mathrm{U}_k = a_k\, e^{-\frac{(y-x_k)^2}{2(\sigma^2+x_k)}}$ is positive and less than one and the term $\mathrm{V}_k = \frac{1}{\sqrt{2\pi(\sigma^2+x_k)}}$ is positive. Therefore, we have,

$$\sum_{k=0}^{m} \mathrm{U}_k \mathrm{V}_k \geq \prod_{k=0}^{m} \mathrm{U}_k \,.\, \sum_{k=0}^{m} \mathrm{V}_k \rightarrow$$

$$\log_2 \sum_{k=0}^{m} \mathrm{U}_k \mathrm{V}_k \geq \log_2\Big(\prod_{k=0}^{m} \mathrm{U}_k \,.\, \sum_{k=0}^{m} \mathrm{V}_k\Big) \rightarrow$$

$$-\log_2 \sum_{k=0}^{m} \mathrm{U}_k \mathrm{V}_k \leq -\log_2\Big(\prod_{k=0}^{m} \mathrm{U}_k \,.\, \sum_{k=0}^{m} \mathrm{V}_k\Big),$$

using the inequality above, it can be written,

$$\mathrm{I(X;Y)} \leq$$

$$-\int_{-\infty}^{\infty} \sum_{i=0}^{m} \frac{a_i}{\sqrt{2\pi(\sigma^2+x_i)}} e^{-\frac{(y-x_i)^2}{2(\sigma^2+x_i)}} .$$

$$\log_2\Big(\prod_{k=0}^{m} a_k\; e^{-\frac{(y-x_k)^2}{2(\sigma^2+x_k)}} . \sum_{k=0}^{m} \frac{1}{\sqrt{2\pi(\sigma^2+x_k)}}\Big)\, \mathrm{d}y$$

$$-\frac{1}{2}\,\mathrm{E}[\log_2 2\pi e(\sigma^2+x)] =$$

$$-\int_{-\infty}^{\infty}\sum_{i=0}^{m}\frac{a_i}{\sqrt{2\pi(\sigma^2+x_i)}}\,e^{-\frac{(y-x_i)^2}{2(\sigma^2+x_i)}}.$$

$$\Big(\underbrace{\log_2\sum_{k=0}^{m}\frac{1}{\sqrt{2\pi(\sigma^2+x_k)}}}_{\mathrm{a}}+\underbrace{\log_2\prod_{k=0}^{m}a_k\,e^{-\frac{(y-x_k)^2}{2(\sigma^2+x_k)}}}_{\mathrm{b}}\Big)$$

$$\underbrace{-\frac{1}{2}\,\mathrm{E}[\log_2 2\pi e\,(\sigma^2+\,x)]}_{\mathrm{c}}, \qquad (6)$$

where,

$$\mathrm{a} = -\frac{1}{2}\log_2 2\pi + \log_2\sum_{k=0}^{m}(\sigma^2+x_k)^{-\frac{1}{2}}$$

$$\mathrm{b} = \sum_{k=0}^{m}\Big(\log_2 a_k - \frac{(y-x_k)^2}{2(\sigma^2+x_k)}.\log_2 e\Big)$$

$$\mathrm{c} = \int_{-\infty}^{\infty}\log_2 2\pi e\,(\sigma^2+\,x)\,.\sum_{i=0}^{m}a_i\,\delta(x-x_i)\,\mathrm{d}x$$

$$= \sum_{i=0}^{m}a_i\log_2 2\pi e(\sigma^2+\,x_i),$$

Considering equal mass points pacing,

$$x_i = i\ell,$$

where, x_i is the position of the ith mass point, the following upper bound is obtained,

$$\mathrm{I(X;Y)} \le$$

$$-\frac{\log_2 e}{2} - \log_2\sum_{k=0}^{m}(\sigma^2+k\ell)^{-\frac{1}{2}} - \sum_{k=0}^{m}\log_2 a_k$$

$$+\frac{\log_2 e}{2}\,.\,[(\sigma^2+i\ell+(i\ell)^2)\sum_{k=0}^{m}\frac{1}{\sigma^2+k\ell}$$

$$-2i\ell^2\sum_{k=0}^{m}\frac{k}{\sigma^2+k\ell}+\ell^2\sum_{k=0}^{m}\frac{k^2}{\sigma^2+k\ell}]$$

$$-\frac{1}{2}\sum_{i=0}^{m}a_i\log_2(i\ell+\sigma^2). \qquad (7)$$

So we found a new upper bound for input-output mutual information of FSO channels with input-dependent Gaussian noise in (7) by using a simple mathematical inequality.

Determining the Optimal Input Distribution

Applying the method of Lagrange multipliers, we maximize the obtained upper bound over all discrete input distribution with equally spaced mass points,

$$\mathrm{J} = -\frac{\log_2 e}{2} - \log_2\sum_{k=0}^{m}(\sigma^2+k\ell)^{-\frac{1}{2}} - \sum_{k=0}^{m}\log_2 a_k$$

$$+\frac{\log_2 e}{2}.\,[(\sigma^2+i\ell+(i\ell)^2)\sum_{k=0}^{m}\frac{1}{\sigma^2+k\ell}$$

$$-2i\ell^2\sum_{k=0}^{m}\frac{k}{\sigma^2+k\ell}+\ell^2\sum_{k=0}^{m}\frac{k^2}{\sigma^2+k\ell}\,]$$

$$-\frac{1}{2}\sum_{i=0}^{m}a_i\log_2(i\ell+\sigma^2)+\lambda_1\Big(\sum_{i=0}^{m}a_i-1\Big)$$

$$+\lambda_2\Big(\sum_{i=0}^{m}\ell i a_i - P\Big).$$

$$\frac{\partial \mathrm{J}}{\partial a_i} = 0 \rightarrow a_i = \frac{\log_2 e}{-\frac{1}{2}\log_2(i\ell+\sigma^2)+\lambda_1+\lambda_2\ell i}, \qquad (8)$$

considering constraints,

$$\sum_{i=0}^{m}a_i = 1,\ \sum_{i=0}^{m}\ell i a_i = P,$$

the following optimal input distribution is obtained,

$$a_i = \frac{1}{\mathrm{D}_1+\mathrm{D}_2}, \qquad (9)$$

where,

$$\mathrm{D}_1 = -\frac{1}{2}ln(i\ell+\sigma^2)+(m+1)$$

$$+\frac{1}{2}\sum_{k=0}^{m}a_k ln(k\ell+\sigma^2)$$

$$\mathrm{D}_2 = \frac{i\ell-P}{P^2-\sum_{k=0}^{m}(k\ell)^2 a_k}\Big[(m+1)P-\frac{\ell m(m+1)}{2}$$

$$+\frac{1}{2}\sum_{k=0}^{m}a_k(P-k\ell)ln(k\ell+\sigma^2)\Big].$$

As we see in (9), the optimal input distributions depend on channel parameters. In the next part we approximate a_is which are derived from equation (9).

III. APPROXIMATION OF a_is

In this section, we approximate a_is which are derived from the previous part,

$$-\frac{a_i}{2}\sum_{k=0}^{m}(k\ell)^2 a_k.\sum_{k=0}^{m}a_k\,ln(k\ell+\sigma^2)$$

$$+\sum_{k=0}^{m}[a_i a_k\,[\frac{P^2}{2}ln(k\ell+\sigma^2)+\frac{(k\ell)^2}{2}ln(i\ell+\sigma^2)$$

$$-(\mathrm{m}+1)(k\ell)^2+\frac{(i\ell-P)(P-k\ell)ln(k\ell+\sigma^2)}{2}]+a_k\,(k\ell)^2]$$

$$+a_i\,[-\frac{P^2 ln(i\ell+\sigma^2)}{2}+P^2(m+1)+(i\ell-P)$$

$$.((m+1)P-\frac{\ell m(m+1)}{2})]-P^2 = 0, \qquad (10)$$

multiplying each term with $\sum_{k=0}^{m}\frac{6k^2}{m(m+1)(2m+1)}=1$ and using the approximation $\sum u_i.\sum v_i \approx \sum u_i v_i$, we obtain,

$$\sum_{k=0}^{m}(k\ell)^2[-\frac{a_i a_k^2}{2}ln(k\ell+\sigma^2)+a_i a_k(\frac{ln(i\ell+\sigma^2)}{2}$$

$$+\frac{3(i\ell P+k\ell P-ik\ell^2)ln(k\ell+\sigma^2)}{m(m+1)(2m+1)\ell^2}-m-1)+a_k$$

$$+a_i(\frac{-3ln(i\ell+\sigma^2)}{m(m+1)(2m+1)}\frac{P^2}{\ell^2}+\frac{6i}{m(2m+1)}\frac{P}{\ell}-\frac{3i}{2m+1}+\frac{3}{2m+1}\frac{P}{\ell})$$

$$-\frac{6P^2}{m(m+1)(2m+1)\ell^2}]=0, \quad (11)$$

or

$$-\frac{a_i a_k^2}{2}ln(k\ell+\sigma^2)+a_i a_k(\frac{ln(i\ell+\sigma^2)}{2}$$

$$+\frac{3(i\ell P+k\ell P-ik\ell^2)ln(k\ell+\sigma^2)}{m(m+1)(2m+1)\ell^2}-m-1)$$

$$+a_i(\frac{-3ln(i\ell+\sigma^2)}{m(m+1)(2m+1)}\frac{P^2}{\ell^2}+\frac{6i}{m(2m+1)}\frac{P}{\ell}-\frac{3i}{2m+1}+\frac{3}{2m+1}\frac{P}{\ell})$$

$$+a_k-\frac{6P^2}{m(m+1)(2m+1)\ell^2}=0, \quad (12)$$

equation (12) is also true for $i = k$,

$$a_i^3(-\frac{ln(i\ell+\sigma^2)}{2})+$$

$$a_i^2[(\frac{1}{2}+\frac{6i}{m(m+1)(2m+1)}\frac{P}{\ell}-\frac{3i^2}{m(m+1)(2m+1)}).$$

$$ln(i\ell+\sigma^2)-m-1]+a_i(\frac{-3ln(i\ell+\sigma^2)}{m(m+1)(2m+1)}\frac{P^2}{\ell^2}$$

$$+\frac{6i}{m(2m+1)}\frac{P}{\ell}-\frac{3i}{2m+1}+\frac{3}{2m+1}\frac{P}{\ell}+1)$$

$$-\frac{6P^2}{m(m+1)(2m+1)\ell^2}=0, \quad (13)$$

we found a third order equation for the optimal input distribution in (13). This equation depends on channel parameters.

IV. NUMERICAL RESULTS

Our approximated a_is and comparisons with previous works are shown in table I, II and III for different numbers of mass points and $\rho = 2, 4$.

Comparisons with Previous Works

(*a*) As we see, in all the tables, amplitudes of our approximated mass points located at zero are equal to the amplitudes of the corresponding mass points in [2]. This is because for $i = 0$ and $\sigma = 1$, our third order equation in (13) gives the second order equation in [2, Eqn. (28)]. At i=0 or $x_i = i\ell = 0$, $f_Y(y)$ in (4) takes a normal distribution with zero mean and variance σ^2 and loses its input-dependency nature.

(*b*) For $m = 1$, our mass point amplitudes in both $\rho = 2$ and $\rho = 4$ are equal to the mass point amplitudes in [2] and when $\rho = 4$ there is a great difference between our mass point located at zero and the corresponding mass point in [3], [4].

(*c*) As shown in table II and III, our mass point distributions are non-uniform at $\rho = 2$, in contrary to [2], [3], [4] where the mass point amplitudes have uniform distribution when $\rho = 2$. It means that for $m > 1$, uniform signaling for FSO channels with input-dependent Gaussian noise is not sufficient. Therefore, the use of non-uniform signaling is essential for FSO channels with input-dependent Gaussian noise even at $\rho = 2$. Also at $\rho = 2$, the mass points located at zero do not have the most weight, in contrary to [2], [3], [4] where the most weight is assigned to the mass point located at zero.

(*d*) Notice that just like [2], in table II and III, at $\rho = 4$ values of a_2 and a_3 are complex. It means that there are no mass points at these locations at $\rho = 4$.

(*e*) Notice that, because of approximation, sum of a_is does not yield to one. But in comparison with [2], sum of our a_is is closer to one at $\rho = 4$.

V. CONCLUSION

In this paper, we determined a new upper bound for the input-output mutual information of FSO channels with input-dependent Gaussian noise by using a simple mathematical inequality. Then, by maximizing the obtained upper bound over all the discrete input distributions with equally spaced mass points and approximating, we reached to a third order equation for the optimum input distributions. Our equation (*i*) determines the optimum input distribution directly in contrary to the FH work where the input distribution is found numerically, (*ii*) gives a previously derived second order equation for the optimum input distribution of FSO channels with input-independent Gaussian noise in a special case.

TABLE I

COMPARISON OF OUR APPROXIMATED a_is FOR $\sigma = 1$ WITH THE APPROXIMATED a_is IN [2] AND THE CAPACITY APPROACHING DISTRIBUTION IN [3], [4] FOR 2 MASS POINTS.

m=1	Our approximated a_i $\rho = 2$	Approximated a_i [2] $\rho = 2$	Capacity approaching distribution [3], [4] $\rho = 2$	Our approximated a_i $\rho = 4$	Approximated a_i [2] $\rho = 4$	Capacity approaching distribution [3], [4] $\rho = 4$
a_0	0.5	0.5	0.5	0.5702	0.5702	0.75
a_1	0.5	0.5	0.5	0.25	0.25	0.25

TABLE II

COMPARISON OF OUR APPROXIMATED a_is FOR $\sigma = 1$ WITH THE APPROXIMATED a_is IN [2] AND THE CAPACITY APPROACHING DISTRIBUTION IN [3], [4] FOR 3 MASS POINTS.

m=2	Our approximated a_i $\rho = 2$	Approximated a_i [2] $\rho = 2$	Capacity approaching distribution [3], [4] $\rho = 2$	Our approximated a_i $\rho = 4$	Approximated a_i [2] $\rho = 4$	Capacity approaching distribution [3], [4] $\rho = 4$
a_0	0.3333	0.33	0.33	0.3907	0.3907	0.6162
a_1	0.3599	0.33	0.33	0.3094	0.2721	0.2676
a_2	0.2106	0.33	0.33			0.1162

TABLE III

COMPARISON OF OUR APPROXIMATED a_is FOR $\sigma = 1$ WITH THE APPROXIMATED a_is IN [2] AND THE CAPACITY APPROACHING DISTRIBUTION IN [3], [4] FOR 4 MASS POINTS.

m=3	Our approximated a_i $\rho = 2$	Approximated a_i [2] $\rho = 2$	Capacity approaching distribution [3], [4] $\rho = 2$	Our approximated a_i $\rho = 4$	Approximated a_i [2] $\rho = 4$	Capacity approaching distribution [3], [4] $\rho = 4$
a_0	0.25	0.25	0.25	0.2965	0.2965	0.5285
a_1	0.2615	0.25	0.25	0.2540	0.2338	0.2680
a_2	0.2696	0.25	0.25	0.1846	0.1607	0.1366
a_3	0.1667	0.25	0.25			0.069

REFRENCES

[1] S. M. Moser, "Capacity results of an optical intensity channel with input-dependent Gaussian noise," *Information Theory*, *Laboratory Department of Electrical Engineering*, *National Chiao Tung University*, pp. 1–34, May 2011.

[2] A. Rezazadeh and G. A. Hodtani, "A general upper bound for the capacity of free space optical channel and the corresponding optimal input distribution," *Australian Communications Theory*, *Workshop*, pp. 34–39, 2012.

[3] A. Farid and S. Hranilovic, "Capacity of optical intensity channels with peak and average power constraints," *IEEE ICC*, *IEEE Communications Society Subject Matter Experts.*

[4] A. Farid and S. Hranilovic, "Channel capacity and non-uniform signaling for free-space optical intensity channels," *Selected Areas in Communications*, *IEEE Journal on*, vol. 27, no. 9, pp. 1553–1563, December 2009.