Lattice Based Codes for Insertion and Deletion Channels

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***Abstract*—Insertion/Deletion codes for the Levenshtein distance are constructed by truncation of lattices for the L^1 metric. These lattices are obtained from Construction A applied to binary codes and $\mathbb{Z}_4-$codes. Finally, Gilbert and Hamming type of bounds are derived.**

Keywords: Insertion/ Deletion codes, lattice, Lee metric, Construction A, weight enumerator, ν-series

I. Introduction

Coding for the insertion/deletion channel remains a major challenge for coding theorists. Part of the reason for this is that the use of standard block algebraic coding techniques (parity-checks, cosets, syndromes) is precluded due to the specificity of the channel which produces output vectors of variable lengths. A variation of this channel is the so-called segmented insertion/deletion channel where at most a fixed number of $r-1$ errors can occur within segments of given size [12], [11]. By looking at the input-output runlengths of symbols, the channel becomes a standard memoryless channel for which algebraic coding techniques can be used. Specifically, we construct lattice-based codes, which, in principle, can be decoded when obtained via Construction A from Lee metric codes with known decoding algorithms [6].

The proposed code constructions are analoguous to the so-called $(d,k)-$codes in magnetic recording where each codeword contains runs of zeros of length at least d and at most k while each run of one has unit length [7], [10]. Given d,k and assuming a constant number of runs of zeros, label the runs by integers modulo m and consider block codes over the ring of integers modulo m—the smallest possible m depends on d and k.

Our approach differs from the one in [7], [10] in two ways. First, we relax the unit length runlength of the ones. Second, we consider lattices rather than codes over the integers modulo m to allow a wider choice of parameters. Indeed our codes are obtained as sets of vectors in a lattice with certain metric properties. A code determines a lattice by Construction A but not conversely. We extend some results of [1], [15] on generalized theta series, called there $\nu-$series, to enumerate effectively these special sets of vectors in the lattice. In particular, if the lattice is obtained via Construction A from a code, the generalized $\nu-$series allows to enumerate these sets from the weight enumerators of the code.

The paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we state the main results on ν-series for Construction A lattices and provide some numerical results. In Section 4, we derive the analogue of the Gilbert and Hamming code size bounds for the L^1 metric space. In Section 5, we provide asymptotic bounds. In Section 6, we provide a few concluding remarks.

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II. Background and Statement of the Problem

Consider a binary sequence of length N, that starts with a zero and ends with a one, and that contains n' runs of zeros and n' runs of ones. (Similar considerations occur for different choices of starting/ending symbol).

Example: The sequence 0011100011 contains $n'=2$ runs of each symbol for a length of $N=10$.

Caveat: In the whole paper, we assume that n' is the same for all the vectors in any given code so that we can work in constant length.

There is a natural correspondence between such a sequence and a sequence defined over the natural integers. Let x_i and y_i denote the ith run length of zeros and ones, respectively. Then we can form a sequence of length $n=2n'$ over the integers defined by

$$(x_1, y_1, \ldots, x_i, y_i, \ldots, x_{n'}, y_{n'}).$$

Example: The binary sequence 0011100011 corresponds to $(2,3,3,2)$.

This approach is a natural generalization of [10] which considers the case where the y_i's are all equal to one.

Note that the integer sequence so constructed satisfies the constraint

$$N = \sum_{i=1}^{n'} (x_i + y_i).$$

Denote by ϕ the above correspondence from $\mathbb{F}_2^N$ to $\mathbb{Z}^n$. The **Levenshtein distance** between two binary vectors is the least number of insertions/deletions to go from one to the other. Alternatively it is the complement to the length of the length of the largest common subsequence. The L^1 **distance** between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$ is given by the expression

$$|\mathbf{x}-\mathbf{y}| = \sum_{i=1}^{n} |x_i - y_i|.$$

The following observation is trivial but crucial.

Proposition 2.1: The map ϕ is an isometry between the Levenshtein distance and the L^1 distance.

Proof: Let

$$\mathbf{z} = (x_1, y_1, \cdots, x_n, y_n)$$

denote the sequence of runs. Any insertion /deletion of j zeros (resp. ones) into run number i will result into a change of x_i (resp. y_i) into $x_i \pm j$ (resp. $y_i \pm j$) yielding a sequence $\mathbf{z}'$ at L^1 distance j away from $\mathbf{z}$.

■

The problem we consider is to characterize $A(n, d, N, r)$, the largest number of length n vectors of nonnegative integers at L^1 distance at least d apart and with coordinates summing up to N. Vectors are restricted to have runs of length at least r so that deletions of at most $r - 1$ bits do not destroy the runs' pattern at the receiving end.

Any set of length n vectors with integral entries $\geq r$, at L^1 distance at least d apart, and coordinates summing up to N, we refer to as an $(n, d, N, r)-$set.

III. Enumeration for construction A lattices

By a **code** C of $\mathbb{Z}_m^n$, we shall mean a $\mathbb{Z}_m-$ submodule of $\mathbb{Z}_m^n$. Define the **complete weight enumerator** (cwe) of C as

$$cwe_C(x_1, x_2, \ldots, x_m) = \sum_{c \in C} \prod_{i=0}^{m-1} x_i^{n_i(C)},$$

where $n_i(c)$ is the number of entries equal to i in the vector c. For $m = 2$, we let $W_C(x, y) = cwe_C(x, y)$, the **weight enumerator**. By a **lattice** of $\mathbb{R}^n$, we shall mean a discrete additive subgroup of $\mathbb{R}^n$. A lattice L is said to be obtained by **Construction A** from a code C of $\mathbb{Z}_m^n$ if it is the inverse image of C in $\mathbb{Z}^n$ by reduction modulo m componentwise. This will be denoted by $L = A(C)$. An important parameter of a lattice is its minimum distance (norm) which is given by the following proposition. Recall that the **Lee weight** of a symbol $x \in \mathbb{Z}_m = \{0, 1, \cdots, m-1\}$, is $\min(x, m - x)$. The weight is extended componentwise to vectors and the Lee distance of two vectors is the Lee weight of their difference. The **Lee distance** of a linear code $C \leq \mathbb{Z}_m^n$ is then the minimum nonzero weight of one its element.

Proposition 3.1 ([14]): Let L be a lattice constructed from a code C with reduction modulo m. Then its minimum distance

$$d = \min(d', m),$$

where d' is the minimum Lee distance of C.

Denote by $\nu_L(r; q)$ the shifted $\nu-$ series of the lattice L

$$\nu_L(r; q) = \sum \{q^{|\mathbf{x}|} : \mathbf{x} \in L, \& \min_i x_i \geq r\}.$$

This definition extends trivially to any discrete subset L of $\mathbb{R}^n$. The motivation for this generating function, a slight generalization of the $\nu-$series of [1], [13], is as follows.

Caveat: If the $q-$series $f = \sum_i f_i q^i$, we denote by $[q^i]f(q)$ the coefficient f_i.

Proposition 3.2: Keep the above notation. If L is a lattice of $\mathbb{R}^n$, of minimum L^1 distance d then the set of vectors of L with coordinate entries bounded below by r and L^1 norm N form a $(n, d, N, r)-$set of size $[q^N]\nu_L(r; q) \leq A(n, d, N, r)$.

Theorem 3.3: If $L = A(C)$ and $m = 2$, then

$$\nu_L(r; q) = W_C(\frac{q^a}{1-q^2}, \frac{q^b}{1-q^2}),$$

where a (resp. b) is the first even (resp. odd) integer $\geq r$. If $L = A(C)$ and $m = 4$, then

$$\nu_L(r; q) = cwe_C(\frac{q^a}{1-q^4}, \frac{q^b}{1-q^4}, \frac{q^c}{1-q^4}, \frac{q^d}{1-q^4}),$$

where a, b, c, d are the first integers $\geq r$, congruent to $0, 1, 2, 3$ modulo 4 respectively.

Proof: By the same argument as in [1], [15], writing $A(C)$ as a disjoint union of cosets of $m\mathbb{Z}^n$, we have

$$\nu_L(r; q) = W_C(\nu_{2\mathbb{Z}}(r; q), \nu_{2\mathbb{Z}+1}(r; q))$$

for $m = 2$, and

$$\nu_L(r; q) = cwe_C(\nu_{4\mathbb{Z}}(r; q), \nu_{4\mathbb{Z}+1}(r; q), \nu_{4\mathbb{Z}+2}(r; q), \nu_{4\mathbb{Z}+3}(r; q))$$

respectively, for $m = 4$. The result follows by summing appropriate geometric series. ■

In Table I and Table II we list, for some values of N and r, the size $[q^N]\nu_L(r; q)$ of $(n, d, N, r)-$set of the well-known lattices D_4, E_8, BW_{16} and Λ_{24} which are constructed from the extended Hamming code H_8, the Klemm code K_8, the code $RM(1, 4) + 2RM(2, 4)$ and the lifted Golay code QR_{24} respectively, where $K_8 = R_8 + 2P_8$ with R_8 being length-8 repetition code and $P_8 = R_8^{\perp}$ its dual and $RM(k, m)$ is the Reed-Muller code of order k.

Some cwe's for these codes can be found in [2], [3]; others were computed using Magma [4]. The cwe of K_n is easily seen to be

$$\frac{1}{2}[(x_0 + x_2)^n + (x_0 - x_2)^n + (x_1 + x_3)^n + (x_1 - x_3)^n].$$

Recall that Proposition 3.1 can only give a lower bound on the minimum distance of a $(n, d, N, r)-$set. To select good codes among the above mentioned lattices, we need exact minimum distances of the $(n, d, N, r)-$sets which characterize the capacity of error correction. These numerical results show, for instance, that for $r = 2$ and $N = 64$, among the three lattices E_8, BW_{16} and Λ_{24}, BW_{16} contains the largest code while Λ_{24} contains the largest one for $r = 1$ and $N = 64$.

IV. Bounds on $A(n, d, N, r)$

We will use the enumerative results of the previous section. First we recall a well-known identity of formal power series.

TABLE I

SIZE $[q^N]\nu_L(r;q)$ OF $(n,d,N,r)-$ SET WITH $L = D_4, E_8, d \geq 2, 4$ RESPECTIVELY AND $r = 1, 2$

N	$[q^N]\nu_{D_4}(1;q)$
8	1
10	8
12	50
14	232
16	835
18	2480
20	6372
22	14640
24	30789
26	60280
28	111254
30	195416
32	329095
34	534496
36	841160
N	$[q^N]\nu_{D_4}(2;q)$
16	1
18	8
20	50
22	232
24	835
26	2480
28	6372
30	14640
32	30789
34	60280
36	111254
38	195416
40	329095
42	534496
44	841160

N	$[q^N]\nu_{E_8}(1;q)$
8	1
12	36
16	331
20	1752
24	6765
28	21164
32	56823
36	135728
40	295545
44	596980
48	1133187
52	2041480
56	3517605
60	5832828
64	9354095
N	$[q^N]\nu_{E_8}(2;q)$
16	1
20	36
24	331
28	1752
32	6765
36	21164
40	56823
44	135728
48	295545
52	596980
56	1133187
60	2041480
64	3517605
68	5832828
72	9354095

TABLE II

SIZE $[q^N]\nu_L(r;q)$ OF $(n,d,N,r)-$ SET WITH $L = BW_{16}, \Lambda_{24}, d \geq 4$ AND $r = 1, 2$

N	$[q^N]\nu_{BW_{16}}(1;q)$
16	1
20	16
24	306
28	3984
32	39235
36	310176
40	2016996
44	11005344
48	51463749
52	210557360
56	767796630
60	2535136560
64	7680579975
68	21588192576
72	56814408136
N	$[q^N]\nu_{BW_{16}}(2;q)$
32	1
36	16
40	306
44	3984
48	39235
52	310176
56	2016996
60	11005344
64	51463749
68	210557360
72	767796630
76	2535136560
80	7680579975
84	21588192576
88	56814408136

N	$[q^N]\nu_{\Lambda_{24}}(1;q)$
24	1
28	24
32	300
36	2600
40	23415
44	299760
48	4144211
52	48058824
56	448956690
60	3450990152
64	22448210613
68	126639274800
72	632120648146
76	2837407970784
80	11605964888130
N	$[q^N]\nu_{\Lambda_{24}}(2;q)$
48	1
52	24
56	300
60	2600
64	23415
68	299760
72	4144211
76	48058824
80	448956690
84	3450990152
88	22448210613
92	126639274800
96	632120648146
100	2837407970784
104	11605964888130

Lemma 4.1: For any integer $n \geq 1$, we have

$$\frac{1}{(1-q)^n} = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} q^i.$$

Proof: Differentiate the geometric series

$$\frac{1}{(1-q)} = \sum_{i=0}^{\infty} q^i$$

with respect to q and use induction on n. ■

Using generating functions, we compute the volume $V(n,e)$ of the L^1 ball of radius e in $\mathbb{Z}^n$.

Lemma 4.2: For any integers $n \geq e \geq 1$, we have

$$V(n,e) = [q^e]\frac{(1+q)^n}{(1-q)^{n+1}} = \sum_{i=0}^{\min(n,e)} 2^i \binom{n}{i}\binom{e}{i}.$$

Proof:

$$\begin{aligned} V(n,e) &= \textstyle\sum_{i=0}^{e}[q^i]\nu_{\mathbb{Z}^n}(-\infty,q) = \sum_{i=0}^{e}[q^i](\frac{1+q}{1-q})^n \\ &= [q^e]\tfrac{(1+q)^n}{(1-q)^{n+1}}. \end{aligned}$$

The second expression is from [9]. It can be derived from the above generating series by expanding

$$(1+\frac{2q}{1-q})^{n+1} = \sum_{i=0}^{n}\binom{n}{i}2^i\frac{q^i}{(1-q)^{i+1}}$$

by Lemma 4.1. ■

By the same techniques, we can compute the volume of the ambient space $A(n,1,N,r)$.

Lemma 4.3: For any integer $N > nr$ and $r > e \geq 1$, we have

$$A(n,1,N,r) = \binom{N-nr+n-1}{n-1}.$$

Proof:

$$\begin{aligned} A(n,1,N,r) &= [q^N]\nu_{\mathbb{Z}^n}(r,q) = [q^N](q^r\tfrac{1}{1-q})^n \\ &= [q^{N-nr}]\tfrac{1}{(1-q)^n}. \end{aligned}$$

The result follows from Lemma 4.1. ■

We are now in a position to formulate the analogues of the Gilbert and Hamming bound in the present context.

Theorem 4.4: For any integers $N > nr, n \geq d$, and $r > e = \lfloor (d-1)/2 \rfloor \geq 1$, we have

$$\frac{\binom{N-nr+n-1}{n-1}}{V(n,d-1)} \leq A(n,d,N,r) \leq \frac{\binom{N-nr+n-1}{n-1}}{V(n,e)}.$$

Proof: Combine Lemma 4.2 and Lemma 4.3 with the standard arguments. ■

The lower and upper bound of $A(n,d,N,r)$ in Theorem 4.4 for lattices E_8 and BW_{16} are given in Table III and Table IV, where we use $I(n,d,N,r)$ and $S(n,e,N,r)$ to denote $\lceil \frac{\binom{N-nr+n-1}{n-1}}{V(n,d-1)} \rceil$ and $\lfloor \frac{\binom{N-nr+n-1}{n-1}}{V(n,e)} \rfloor$, respectively. The numerical results show that $[q^N]\nu_L(r;q)$ (a lower bound to $A(n,d,N,r)$ by Proposition 3.2), lies between $I(n,d,N,r)$ and $S(n,e,N,r)$ for many parameter values. Exceptions are, for instance, for BW_{16} with $r = 2$, and $N = 48,\ldots,96$. Whether these code constructions yield sizes between $I(n,d,N,r)$ and $S(n,e,N,r)$ for large N is an open issue.

Since all codewords have constant L^1 distance, it is natural to use the Johnson bound in the Lee metric.

Theorem 4.5: If $d > N(1-1/2n)$, then we have

$$A(n,d,N,r) \leq \frac{d}{d-N(1-1/2n)}.$$

Proof: Reduce all vectors modulo $Q = 2N$. Use Lemma 13.62 of [5] with $\overline{D} = Q/4 = N/2$, and $x = 1/n$. ∎

TABLE III
BOUNDS ON $A(n,d,N,r)$ WITH $L = E_8$ AND $r = 2,3,4$

N	$I(8,4,N,2)$	$[q^N]\nu_{E_8}(2;q)$	$S(8,1,N,2)$
24	8	331	378
28	61	1752	2964
32	295	6765	14421
36	1067	21164	52237
40	3157	56823	154680
44	8073	135728	395560
48	18465	295545	904761
52	38685	596980	1895536
56	75500	1133187	3699499
60	138986	2041480	6810300
64	243611	3517605	11936925
68	409544	5832828	20067614
72	664191	9354095	32545333
76	1043996	14567520	51155776
80	1596508	22105457	78228865
N	$I(8,4,N,3)$	$[q^N]\nu_{E_8}(3;q)$	$S(8,1,N,3)$
32	8	331	378
36	61	1752	2964
40	295	6765	14421
44	1067	21164	52237
48	3157	56823	154680
52	8073	135728	395560
56	18465	295545	904761
60	38685	596980	1895536
64	75500	1133187	3699499
68	138986	2041480	6810300
72	243611	3517605	11936925
76	409544	5832828	20067614
80	664191	9354095	32545333
84	1043996	14567520	51155776
88	1596508	22105457	78228865
N	$I(8,4,N,4)$	$[q^N]\nu_{E_8}(4;q)$	$S(8,1,N,4)$
40	8	331	378
44	61	1752	2964
48	295	6765	14421
52	1067	21164	52237
56	3157	56823	154680
60	8073	135728	395560
64	18465	295545	904761
68	38685	596980	1895536
72	75500	1133187	3699499
76	138986	2041480	6810300
80	243611	3517605	11936925
84	409544	5832828	20067614
88	664191	9354095	32545333
92	1043996	14567520	51155776
96	1596508	22105457	78228865

TABLE IV
BOUNDS ON $A(n,d,N,r)$ WITH $L = BW_{16}$ AND $r = 2,3,4$

N	$I(16,4,N,2)$	$[q^N]\nu_{BW_{16}}(2;q)$	$S(16,1,N,2)$
36	1	16	117
40	82	306	14858
44	2890	3984	526783
48	49949	39235	9107278
52	539795	310176	98422520
56	4178302	2016996	761843656
60	25184088	11005344	4591898687
64	124915457	51463749	22776251653
68	529944363	210557360	96626522164
72	1977679995	767796630	360596985630
76	6630474804	2535136560	1208956572561
80	20297778673	7680579975	3700961644542
84	57467324395	21588192576	10478208814512
88	152025004051	56814408136	27719225738485
92	378928483749	141077361984	69091293536850
96	896068510238	332674600329	163383158366718
N	$I(16,4,N,3)$	$[q^N]\nu_{BW_{16}}(3;q)$	$S(16,1,N,3)$
52	1	16	117
56	82	306	14858
60	2890	3984	526783
64	49949	39235	9107278
68	539795	310176	98422520
72	4178302	2016996	761843656
76	25184088	11005344	4591898687
80	124915457	51463749	22776251653
84	529944363	210557360	96626522164
88	1977679995	767796630	360596985630
92	6630474804	2535136560	1208956572561
96	20297778673	7680579975	3700961644542
100	57467324395	21588192576	10478208814512
104	152025004051	56814408136	27719225738485
108	378928483749	141077361984	69091293536850
112	896068510238	332674600329	163383158366718
N	$I(16,4,N,4)$	$[q^N]\nu_{BW_{16}}(4;q)$	$S(16,1,N,4)$
68	1	16	117
72	82	306	14858
76	2890	3984	526783
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84	539795	310176	98422520
88	4178302	2016996	761843656
92	25184088	11005344	4591898687
96	124915457	51463749	22776251653
100	529944363	210557360	96626522164
104	1977679995	767796630	360596985630
108	6630474804	2535136560	1208956572561
112	20297778673	7680579975	3700961644542
116	57467324395	21588192576	10478208814512
120	152025004051	56814408136	27719225738485
124	378928483749	141077361984	69091293536850
128	896068510238	332674600329	163383158366718

V. ASYMPTOTIC BOUNDS ON $A(n,d,N,r)$

We assume that r is fixed, that $N \to \infty$, and that $n \sim \eta N/r$, $d \sim \delta N$ for some constants η, δ with $\eta \in (0,1)$, and $\delta \geq 0$. Because each codeword has weight N, the triangle inequality in the L^1 metric shows that $\delta \in (0,2)$. Denote by R the asymptotic exponent of $A(n,d,N,r)$, that is

$$R = \limsup \frac{1}{N} \log A(n,d,N,r).$$

The asymptotic form of Theorem 4.5 shows that $\delta \in (0,1)$ whenever $R \neq 0$.

Let

$$L(x) = x\log_2 x + \log_2(x+\sqrt{x^2+1}) - x\log_2(\sqrt{x^2+1}-1).$$

It was proved in [8] that when $x \to \infty$ and $e \sim \epsilon n$, then

$$\lim \frac{1}{n} \log_2 V(n,e) = L(\epsilon).$$

For convenience, let $H(q) = -q\log q - (1-q)\log(1-q)$ denote the binary entropy function and let

$$f(x,y,z) = [1-y+y/x]H(\frac{y}{y+x(1-y)}) - (y/x)L(\frac{xz}{y}).$$

We now state and prove the asymptotic version of Theorem 4.4.

Theorem 5.1: With the above notation, we have

$$f(r, \eta, \delta) \leq R \leq f(r, \eta, \delta/2).$$

Proof: The result follows from Theorem 4.4 by standard entropic estimates for binomial coefficients for the numerator and the result on large alphabet Lee balls from [8] for the denominators. ■

In Fig. 1 and 2, the graphs of the asymptotic lower bound curve $f(r, \eta, \delta)$ with different parameters η and $r = 2$ show that the rate R is high when η is around 0.5.

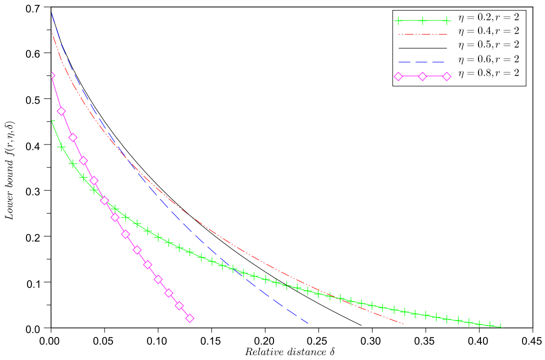

Fig. 1. Graphs of $f(r, \eta, \delta)$ for $r = 2$ and $\eta = 0.2, 0.4, 0.5, 0.6, 0.8$

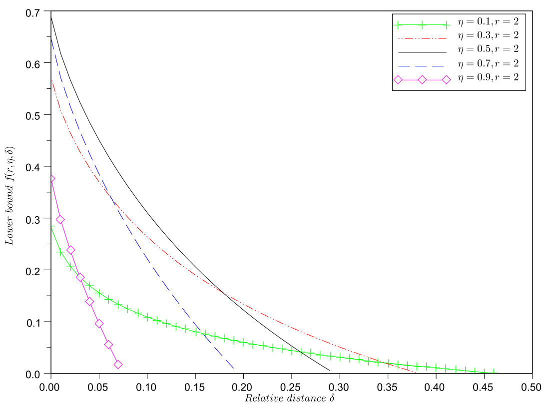

Fig. 2. Graphs of $f(r, \eta, \delta)$ for $r = 2$ and $\eta = 0.1, 0.3, 0.5, 0.7, 0.9$

VI. Conclusion

In this work, we have approached a problem of binary coding for the Levenshtein distance by using lattices for the L^1 metric. These lattices are obtained by Construction A applied to binary and quaternary codes. Finding the densest lattice for the L^1-metric in a given dimension is still an open problem. Therefore it is worth varying codes, alphabets and use other constructions to improve the constructions of $(n, d, N, r)-$sets.

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