Controlled Sensing for Multihypothesis Testing Based on Markovian Observations

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***Abstract*—A new model for controlled sensing for multiphypothesis testing is proposed and studied in both the sequential and fixed sample size settings. This new model, termed a *stationary Markov* model, exhibits a more complicated memory structure in the controlled observations than the existing stationary memoryless model. In the sequential setting, an asymptotically optimal sequential test using a *stationary causal Markov control* policy enjoying a *strong* asymptotic optimality condition is proposed for this new model, and its asymptotic performance is characterized. In the fixed sample size setting, bounds for the optimal error exponent for binary hypothesis testing are derived; it is conjectured that the structure of the asymptotically optimal control for the stationary Markov model will be much more complicated than that for the stationary memoryless model.**

I. Introduction

The primary concern in the topic of controlled sensing for inference is adaptively managing and controlling multiple degrees of freedom in an information-gathering system, ranging from sensing modality to the physical control of sensors, to solve a given inference task. In contrast to the traditional control systems, in which the control primarily affects the evolution of the state, in controlled sensing the control affects only the observations. The goal in controlled sensing is not to drive the sate to the desired range, but to best shape the quality of the observations to enable or facilitate solving the given inference task.

Applications of controlled sensing include, but are by no means limited to adaptive resource management in sensor and wireless networks, clinical diagnosis, and search and target tracking problems. [1], [2], [3], [4].

In this paper, our main focus will be on the basic inference problem of hypothesis testing, and our goals are first, to design asymptotically optimal tests, consisting of control policies, possibly, stopping rules, and decision rules, and second, to characterize their asymptotic performance. Special emphasis will be placed on characterizing the *structure* of asymptotically optimal control policies. We consider both the sequential setting, in which the controller can adaptively choose to stop taking observations, as well as the fixed sample size setting, in which the controller is not allowed such adaptivity.

Controlled sensing for multihypothesis testing has been previously studied for the stationary memoryless model in which conditioned on the control value at each time, the observation at that time is conditionally independent of all past observations and control values. In the sequential setting and under a certain "positivity" assumption, an asymptotically optimal sequential (Chernoff) test using a *stationary causal control* policy was proposed, first by Chernoff [5] for only binary hypothesis testing, and then by Bessler [6] for general multihypothesis testing, and its asymptotic performance was characterized. Another sequential test was proposed in [7] that can successfully be used to dispense with the posititivity assumption completely. It was also proved in [7] that this sequential test enjoys a stronger asymptotic optimality condition than the one adopted in [5], [6]. In [8], a Bayesian version of the sequential controlled sensing problem was considered in the non-asymptotic regime, and the associated dynamic programming equations [8], [9], [10] are analyzed to find the structures of the optimal tests. In the fixed sample size setting, the characterization of the structure of the asymptotically optimal control policy and its performance is a much harder problem; a complete answer is available only for the easiest case of binary hypothesis testing [11], in which the asymptotically optimal control policy was shown to be a *stationary open-loop* policy.

It is important to note that our problem of controlled sensing for hypothesis testing is fundamentally different from the feedback channel coding problem. First, the controller in the controlled sensing problem does *not* know the hypothesis, whereas the feedback encoder (which is equivalent to the controller) in the channel coding problem *knows* the message (which is equivalent to the hypothesis). Second, the distributional model of the observation in our controlled sensing problem can depend on the hypothesis in an *arbitrary* manner, whereas in the channel coding problem, the distribution of the channel output is constrained to depend on the message through a *fixed* channel that is *independent* of the message index. These two facts make the controlled sensing problem much more challenging than the feedback channel coding problem with a fixed number of messages, and therefore make the vast literature on feedback channel coding (e.g., [12], [13], [14]) not directly relevant to the controlled sensing problem.

In this work, we propose a new observation model for controlled sensing for multiphypothesis testing termed *stationary Markov* model. The memory structure in the controlled observations of this new model is more complicated than that in the existing stationary memoryless model. Our technical contributions are as follows. In the sequential setting, we propose an asymptotically optimal sequential test using a *stationary causal Markov control* policy enjoying the *strong* asymptotic optimality condition, and characterize its asymptotic performance. Note that even in the uncontrolled case, the characterization of the structure of the asymptotically optimal sequential test when the observations are *not* memoryless proved to be much harder than when they are [15]. Consequently, in the case with controlled observations, although the structure of the asymptotically optimal control was successfully characterized for the stationary memoryless model, it is not clear at the outset what the structure the asymptotically optimal control would be for the stationary Markov model with the more complicated memory structure in the controlled observations.

In the fixed sample size setting, we derive upper and lower bounds for the optimal error exponent for binary hypothesis testing. We conjecture that the structure of the asymptotically optimal control for our model will be much more complicated than that for the stationary memoryless model.

II. Preliminaries

Hereafter all random variables are denoted by capital letters and their realizations are denoted by the corresponding lowercase letters.

Consider hypothesis testing among M hypotheses with the set of hypotheses $\{0,1,\ldots,M-1\}$ being denoted by $\mathcal{M}$. The observation and control value at each time step take values in *finite* sets $\mathcal{Y}$ and $\mathcal{U}$, respectively. At each time $k=1,2,\ldots,$ conditioned on each hypothesis $i\in\mathcal{M}$, the current control value u_k and the previous observation y_{k-1}, the current observation Y_k is assumed to be conditionally independent of all earlier observations and control values $\left(y^{k-2},u^{k-1}\right)\triangleq\left(\left(y_1,\ldots,y_{k-2}\right),\left(u_1,\ldots,u_{k-1}\right)\right)$. Furthermore, conditioning on the aforementioned hypothesis and $\left(u_k,y_{k-1}\right)$, $k=1,2,\ldots,$ Y_k is assumed to be conditionally distributed according to $p_i^{u_k}\left(\cdot|y_{k-1}\right)$, where $\left\{p_i^u\left(\cdot|\cdot\right)\right\}_{i\in\mathcal{M}}^{u\in\mathcal{U}}$ is a fixed collection of transitions probabilities from $\mathcal{Y}$ to itself. We assume throughout that for every $i\in\mathcal{M}$, $u\in\mathcal{U}$, $y,\tilde{y}\in\mathcal{Y}$,

$$p_i^u\left(y|\tilde{y}\right) \;>\; 0. \tag{1}$$

The initial observation y_0 is assumed to be a constant. We shall refer to this observation model with the aforementioned (conditional) independence assumption as the *stationary Markov model*.

Controlled sensing for hypothesis testing has been previously studied for a simpler observation model in which for every $u\in\mathcal{U},\ i\in\mathcal{M}$, the $p_i^u\left(\cdot|\tilde{y}\right)$ in (1) is independent of $\tilde{y}$. Hence, for this "memoryless" model, conditioning on each hypothesis i and $u_k,\ k=1,2,\ldots,$ Y_k is conditionally independent of $\left(y^{k-1},u^{k-1}\right)$. This simpler observation model is referred to as the *stationary memoryless model*.

Our main interest here will be on *causal* control policies for which the control U_k at each time $k=2,3,\ldots,$ can be any (possibly randomized) function of past observations and past control values. In particular, each $U_k,\ k=2,3,\ldots,$ is specified by an arbitrary conditional pmf $q_k\left(u_k|y^{k-1},u^{k-1}\right)$, and U_1 is specified by a conditional pmf $q_1\left(u_1|y_0\right)$. Should all these (conditional) pmfs be point-mass distributions, i.e., the control values are deterministic functions of past observations and past control values, the resulting control policy is termed a *pure* causal control policy. For the stationary Markov model and for any $n=2,3,\ldots,$ the joint distribution function of $\left(Y^n,U^n\right)$ conditioning on each hypothesis i (and on the initial observation y_0), denoted by $p_i\left(y^n,u^n\right)$, can be written as

$$\begin{aligned} &p_i\left(y^n,u^n\right)\\ &\quad\triangleq\; q_1\left(u_1|y_0\right)\prod_{k=1}^{n}p_i^{u_k}\left(y_k|y_{k-1}\right)\prod_{k=2}^{n}q_k^{u_k}\left(y_k|y_{k-1},u_{k-1}\right). \end{aligned}$$

The entire collection of conditional pmfs $\{q_1\left(u_1|y_0\right),$ $\left\{q_k^{u_k}\left(y_k|y_{k-1},u_{k-1}\right)\right\}_{k=2}^{\infty}\}$ describes a causal control policy which will be denoted collectively by ϕ.

We consider hypothesis testing under controlled observations in both the sequential setting and the fixed sample size setting. In the sequential setting, the controller can adaptively decide, based on realization of past observations and past control values, whether to continue collecting new observations, thereby deferring making a final decision about the hypothesis until later time, or to stop taking observations and make the final decision. On the other hand, the decision is made at a fixed time horizon in the fixed sample size setting.

Let $\mathcal{F}_k$ denote the σ-field generated by $\left(Y^k,U^k\right)$. A *sequential test* $\gamma=\left(\phi,N,\delta\right)$ consists of a causal control policy ϕ, an $\mathcal{F}_k$-stopping time N denoting the (random) number of observations before the final decision, and the decision rule $\delta=\delta\left(Y^N,U^N\right)$ taking values in $\mathcal{M}$. In the fixed sample size setting, the final decision δ about the hypothesis is made at a fixed time n, i.e., $\delta=\delta\left(Y^n,U^n\right)$, and a test γ (with a fixed sample size) consists only of the control policy ϕ and the final decision rule δ, i.e., $\gamma=(\phi,\delta)$. The goal is to design a sequence of (sequential) tests to achieve the asymptotically optimal tradeoff between reliability, in terms of probabilities of error, and delay, in terms of (expected) sample sizes needed in decision making.

To describe our results pertaining to *strong* asymptotic optimality in the sequential setting, we shall need the following concept of decision risks or frequentist error probabilities [16], [15]. Specifically, let $\pi(i),\ i\in\mathcal{M}$, be a prior distribution of the hypothesis with a full support. For each $i\in\mathcal{M}$, the *probability of incorrectly deciding* i or the *risk of deciding* i is given by

$$R_i \;\triangleq\; \sum_{j\in\mathcal{M}\backslash\{i\}}\pi(j)\;\mathbb{P}_j\left\{\delta=i\right\}. \tag{2}$$

Note that for each $i \in \mathcal{M}$,

$$R_i \;=\; \sum_{j \in \mathcal{M}\setminus\{i\}} \pi(j)\; \mathbb{P}_j\left\{\delta = i\right\} \;\leq\; \max_{k \in \mathcal{M}}\; \mathbb{P}_k\left\{\delta \neq k\right\}, \quad (3)$$

where the right-side of (3) is the maximal error probability. In the sequential setting, we shall be interested in the asymptotic regime in which $\max_{i \in \mathcal{M}} R_i \;\to\; 0$ (or, equivalently, $\max_{i \in \mathcal{M}}\; \mathbb{P}_i\left\{\delta \neq i\right\} \;\to\; 0$). In this regime, our *strong* asymptotic optimality condition will be stated in terms of the relationship between the growth rates of the expected sample sizes $\mathbb{E}_i\left[N\right]$ under hypotheses $i = 0, \ldots, M-1$, relative to the corresponding risks R_i. It will be seen that in the asymptotically optimal regime, the various risks go to zero exponentially as the corresponding expected sample sizes go unbounded. Due to (3), our version of asymptotic optimality condition (in terms of risks) is stronger than the typical one in terms of the relationship between the growth rates of the various expected sample sizes relative to the maximal error probability.

In the fixed sample size setting, the sample size is not random and, hence, is the same for every hypothesis. Consequently, in this setting and for the consideration of asymptotic optimality, we will be interested in the *single* largest achievable exponent for the maximal error probability defined as

$$\beta \;\triangleq\; \limsup_{n\to\infty}\; \max_{(\phi,\delta(y^n,u^n))}\; -\frac{1}{n}\log\left(\max_{i \in \mathcal{M}}\; \mathbb{P}_i\left\{\delta \neq i\right\}\right). \quad (4)$$

Note that since the number of hypotheses is fixed, the exponent for the maximal error probability is the same as that for the Bayesian error probability (with any prior distribution of the hypothesis with a full support). Since a pure control policy suffices to minimize the Bayesian error probability, for our consideration of asymptotic optimality in the fixed sample size setting, it suffices to restrict at the outset to only *pure* causal control policies.

For two pmfs p_1 and p_2 on $\mathcal{Y}$ each with a full support, the Kullback-Leibler (KL) distance between p_1 and p_2 denoted by $D\left(p_1 \| p_2\right)$, is defined as

$$D\left(p_1 \| p_2\right) \;\triangleq\; \sum_{y \in \mathcal{Y}} p_1(y) \log\left(\frac{p_1(y)}{p_2(y)}\right). \quad (5)$$

A. Review of Results for the Stationarily Memoryless Model

In order to motivate our results for the stationary Markov model, which will be presented in the next section, and to provide some background for the results, we now review certain facts pertaining to the prior results for the stationary memoryless model.

For binary hypothesis testing in the sequential setting and under a suitable positivity assumption, an asymptotically optimal sequential test for the stationary memoryless model was proposed in [5] where the asymptotic optimality condition was formulated in terms of the maximal error probability. This sequential test was extended to the general case of multihypothesis testing in [6] under a similar positivity assumption. We shall refer to this sequential test for the stationary memoryless model that is asymptotically optimal under the positivity assumption as the *Chernoff test.* The control policy in the Chernoff test follows the separation principle between estimation and control. In particular, at each time, the control value is selected to be a (randomized) function of the Maximum Likelihood (ML) estimate of the hypothesis based on all past observations and control values; the random values of this function for the various hypotheses can be characterized off-line based on the observation model. We shall refer to this structure of the asymptotically optimal control policy in the Chernoff test as "stationary causal control."

The characterization of the structure of asymptotically optimal control in the fixed sample size setting and the associated largest achievable error exponent is much harder than that in the sequential setting, and is a longstanding open problem in decision theory and information theory. Even for the stationary memoryless model, such a characterization in the fixed sample size setting is only available for the special case of binary hypothesis testing. This is why in the fixed sample size setting, we shall only focus on binary hypothesis testing in this paper. For binary hypothesis testing in the fixed sample size setting, it was proved in [11] that for the stationary memoryless model, the asymptotically optimal control has a fixed value at all times; hence, the control value at any time is independent of the past observations, i.e., the asymptotically optimal control policy is "stationary open-loop." In particular, the common optimal control value is the one that maximizes the Chernoff information of the distributions of the observation under the two hypotheses; this optimal control value can be easily computed off-line based on the observation model.

III. Results

This section concerns the presentation of our novel results for the stationary Markov model.

A. Sequential Setting

1) The Proposed Sequential Test:

We first describe our sequential test which will be proven to be asymptotically optimal under the following positivity assumption: *For every* $u \in \mathcal{U},\; \tilde{y} \in \mathcal{Y},\; 0 \leq i < j < M-1$,

$$D\left(p_i^u\left(\cdot|\tilde{y}\right) \| p_j^u\left(\cdot|\tilde{y}\right)\right) \;>\; 0. \quad (6)$$

Our sequential test is motivated by the Chernoff test [5], [6] for the stationary memoryless model.

For every $i \in \mathcal{M}$ and every conditional distribution $q\left(u|\tilde{y}\right)$, we define p_i^q as the following transition probabilities from $\mathcal{Y}$ to itself

$$p_i^q\left(y|\tilde{y}\right) \;\triangleq\; \sum_{u \in \mathcal{U}} q\left(u|\tilde{y}\right) p_i^u\left(y|\tilde{y}\right).$$

It follows from Assumption (1) that, for every i, q, such p_i^q admits a unique stationary distribution $\mu_i^q\left(\tilde{y}\right)$.

Our proposed test for the stationary Markov model admits the following sequential description. Having fixed the control policy up to time $k-1$ and having obtained the first $k-1$ observations and control values y^{k-1}, u^{k-1}, if the controller decides to continue taking more observations, then at time k, a *randomized* control policy is adopted wherein $U_k \in \mathcal{U}$ is selected based on y^{k-1}, u^{k-1} as

$$q(u) \; = \; q\left(u \,\middle|\, \hat{i}_{k-1}, y_{k-1}\right), \tag{7}$$

where for each $i \in \mathcal{M}$,

$$q\left(\cdot|i,\cdot\right) \; = \; \underset{q(\cdot|\cdot)}{\operatorname{argmax}} \min_{j\in\mathcal{M}\setminus\{i\}} \sum_{u,\tilde{y}} \mu_i^q\left(\tilde{y}\right) q\left(u|\tilde{y}\right) D\left(p_i^u\left(\cdot|\tilde{y}\right) \| p_j^u\left(\cdot|\tilde{y}\right)\right), \tag{8}$$

and $\hat{i}_{k-1} = \underset{i\in\mathcal{M}}{\operatorname{argmax}} \; p_i\left(y^{k-1}, u^{k-1}\right)$, is the ML estimate of the hypothesis after time $k-1$. The stopping rule is defined as the first time n for which

$$\log\left(\frac{p_{\hat{i}_n}\left(y^n, u^n\right)}{\max\limits_{j\neq\hat{i}_n} p_j\left(y^n, u^n\right)}\right) \; \geq \; -\log\left(c\right) \tag{9}$$

where c is a positive real-valued parameter that will be selected to approach zero in order to drive the error probability and risks to zero. At the stopping time n, the decision rule is ML, i.e., $\delta\left(y^n, u^n\right) \; = \; \hat{i}_n$. Note that randomization is used in the causal control policy. This facilitates the simultaneous minimization of the expected stopping time under M hypotheses as the error probabilities and risks go to zero.

The difference between our proposed control policy and that of the Chernoff test is that in the former, the random control $U_k, \; k = 2, 3, \ldots$, in (7), (8) depends on both $\hat{i}_{k-1}$ as well as y_{k-1} whereas in the latter, U_k only depends on $\hat{i}_{k-1}$ (cf. [5], [6]). Note that the transitions probabilities $q\left(\cdot|i,\cdot\right), \; i \in \mathcal{M}$, in (8) are fixed and can be computed offline based on the observation model. Consequently, we shall call the control structure advocated by the control policy (7), (8) in our proposed sequential test "stationary causal Markov control," whereas the simpler control structure in the Chernoff test [5], [6] is termed stationary causal control.

2) Strong Asymptotic Optimality:

Theorem 1: For the stationary Markov model, regardless of the positivity assumption (6), there exists a sequence of sequential tests that satisfies

$$\max_{i\in\mathcal{M}} \mathbb{P}\left\{\delta\left(Y^N, U^N\right) \neq i\right\} \; \rightarrow \; 0, \tag{10}$$

as well as for each $i \in \mathcal{M}$,

$$\mathbb{E}_i\left[N\right] \leq \frac{-\log\left(\max\limits_{k\in\mathcal{M}} \mathbb{P}_k\left\{\delta \neq k\right\}\right)\left(1+o(1)\right)}{\max\limits_{q(\cdot|\cdot)} \min\limits_{j\in\mathcal{M}\setminus\{i\}} \sum\limits_{u,\tilde{y}} \mu_i^q\left(\tilde{y}\right) q\left(u|\tilde{y}\right) D\left(p_i^u\left(\cdot|\tilde{y}\right) \| p_j^u\left(\cdot|\tilde{y}\right)\right)} \tag{11}$$

$$\leq \frac{-\log\left(R_i\right)\left(1+o(1)\right)}{\max\limits_{q(\cdot|\cdot)} \min\limits_{j\in\mathcal{M}\setminus\{i\}} \sum\limits_{u,\tilde{y}} \mu_i^q\left(\tilde{y}\right) q\left(u|\tilde{y}\right) D\left(p_i^u\left(\cdot|\tilde{y}\right) \| p_j^u\left(\cdot|\tilde{y}\right)\right)}. \tag{12}$$

In particular, with (6), our proposed test achieves (10), (11), (12), as $c \rightarrow 0$. Furthermore, this asymptotic performance is optimal in the following *strong sense*: any sequence of tests with vanishing maximal risk, i.e., $\max\limits_{i\in\mathcal{M}} R_i \; \rightarrow \; 0$, satisfies for *every* $i \in \mathcal{M}$,

$$\mathbb{E}_i\left[N\right] \geq \frac{-\log\left(R_i\right)\left(1+o(1)\right)}{\max\limits_{q(\cdot|\cdot)} \min\limits_{j\in\mathcal{M}\setminus\{i\}} \sum\limits_{u,\tilde{y}} \mu_i^q\left(\tilde{y}\right) q\left(u|\tilde{y}\right) D\left(p_i^u\left(\cdot|\tilde{y}\right) \| p_j^u\left(\cdot|\tilde{y}\right)\right)}. \tag{13}$$

Remark 1: Due to (3), the converse assertion (13) in terms of risks implies the one in terms of the maximal error probability, but not vice versa; hence, the former version of the converse assertion is *stronger* than the latter one.

Remark 2: In order to completely dispense with Assumption (6), we need to make nontrivial modifications to our proposed test. The details of this new test will be too complicated to be explained in this short manuscript. The ideas behind this new test are based on, but are more complicated than, those in [7] used in the modification of the Chernoff test for the simpler stationary memoryless model.

B. Fixed Sample Size Setting

In the fixed sample size setting, it is much harder to pinpoint the structure of the asymptotically optimal test for binary hypothesis testing in the stationary Markov model than in the stationary memoryless model. In this subsection, we present some partial results related to this difficult problem.

Motivated by the results in [11] for the stationary memoryless model, it is tempting to guess that for the stationary Markov model, the asymptotically optimal fixed-sample-size test for differentiating between the two hypotheses will have the structure of "stationary Markov control," namely the control policy would follow the law $u_k = f\left(y_{k-1}\right)$, for every $k = 1, 2, \ldots$, for a fixed function $f : \mathcal{Y} \rightarrow \mathcal{U}$. So far, we only have the characterization of the error exponent achievable by the best test with such a control policy, and an upper bound that applies for any test. However, we conjecture that for the stationary Markov model, such stationary Markov control is *not* necessarily optimal for binary hypothesis testing in the fixed sample size setting. Currently, a counter-example for this conjecture is being sought.

Denote the set of all functions $f : \mathcal{Y} \rightarrow \mathcal{U}$ by $\mathcal{F}$. Note that $\mathcal{F}$ is finite, in particular, $|\mathcal{F}| \; = \; |\mathcal{U}|^{|\mathcal{Y}|}$. For such a function f and for $i = 0, 1$, we denote the transition probability from $\mathcal{Y}$ to itself: $p_i^{f(\tilde{y})}\left(y|\tilde{y}\right)$ by p_i^f.

It follows from Assumption (1) and the Perron Frobenious Theorem that for every such a function f and for every $s \in [0, 1]$, the squared matrix incarnation of $p_0^f\left(y|\tilde{y}\right)^s p_1^f\left(y|\tilde{y}\right)^{1-s}$ has a maximal right eigenvalue $\lambda_s^f \in (0, 1]$, with a corresponding normalized right eigenvector, denoted by $\nu_s^f\left(y\right), \; y \in \mathcal{Y}$, with all elements being strictly positive. We then define the corresponding stochastic matrix as

$$b_s^f\left(y|\tilde{y}\right) \; \triangleq \; \frac{p_0^f\left(y|\tilde{y}\right)^s p_1^f\left(y|\tilde{y}\right)^{1-s} \nu_s^f\left(y\right)}{\lambda_s^f \nu_s^f\left(\tilde{y}\right)} \; > \; 0. \tag{14}$$

For each $f : \; \mathcal{Y} \; \rightarrow \; \mathcal{U}$, if we adopt the stationary Markov control $u_k \; = \; f\left(y_{k-1}\right), \; k \; = \; 1, \ldots$, then the observations under hypotheses 0 and 1 will follow the stationary Markov models p_0^f and p_1^f, respectively. For such a control policy with a fixed f, it now follows from the results in [17], [18], [19] that the corresponding achievable error exponent for binary hypothesis testing in the fixed sample size setting is

$$\begin{aligned}\underline{\beta} &\triangleq \max_{s\in[0,1]} -\log\left(\lambda_s^f\right)\\ &= \sum_{\tilde{y}} \mu^f\left(\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_0^f\left(\cdot|\tilde{y}\right)\right)\\ &= \sum_{\tilde{y}} \mu^f\left(\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_1^f\left(\cdot|\tilde{y}\right)\right),\end{aligned}$$

where $s^* = s^*(f) \triangleq \underset{s\in[0,1]}{\operatorname{argmax}} -\log\left(\lambda_s^f\right)$, and μ^f is the unique stationary distribution of $b_{s^*}^f$.

We now describe the motivation behind our novel upper bound for the optimal error exponent. Associated with any model $\{p_i^u\}_{i=0,1}^{u\in\mathcal{U}}$, is its super-model with $\mathcal{U}$ being replaced by $\mathcal{F}$. In particular, for the various $f\in\mathcal{F}$, we can consider the super-model $\left\{p_i^f \triangleq p_i^{f(\tilde{y})}\left(y|\tilde{y}\right)\right\}_{i=0,1}^{\mathcal{F}}$. On the other hand, any pure causal control policy $u_k = u_k\left(y^{k-1}\right)$, $k = 2,3,\ldots$, can be thought of as

$$u_k\left(y^{k-1}\right) = u_k\left(\,\cdot\,, y^{k-2}\right) = f_k\left(y^{k-2}\right) \in \mathcal{F}.$$

In this way, we can always think of any causal control policy for the model $\{p_i^u\}_{i=0,1}^{u\in\mathcal{U}}$ as another causal control policy with a delay in the observation of one time slot for the super-model $\left\{p_i^f\right\}_{i=0,1}^{f\in\mathcal{F}}$. Our upper bound for the optimal error exponent will be expressed in terms of this super-model.

For each conditional probability distribution $r:\mathcal{Y}\to\mathcal{F}$, consider the transition probabilities $\hat{b}^r$ on $\mathcal{Y}$ defined as

$$\hat{b}^r \triangleq \sum_{f\in\mathcal{F}} r\left(f|\tilde{y}\right)\; b_{s^*(f)}^f\left(y|\tilde{y}\right).$$

It follows from (1) that for any such r, the transition probabilities $\hat{b}^r$ admits a unique stationary distribution, which will be denoted by $\hat{\mu}^r\left(\tilde{y}\right)$.

Our last result concerns the mentioned achievability bound, and a new converse bound for the optimal error exponent for binary hypothesis testing in the fixed sample size setting.

Theorem 2: For $M = 2$ and for the stationary Markov model, it holds that

$$\begin{aligned}&\max_f \max_{s\in[0,1]} -\log\left(\lambda_s^f\right)\\ &\quad= \max_f \sum_{\tilde{y}} \mu^f\left(\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_0^f\left(\cdot|\tilde{y}\right)\right)\\ &\quad= \max_f \sum_{\tilde{y}} \mu^f\left(\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_1^f\left(\cdot|\tilde{y}\right)\right)\\ &\quad\le \beta \le\\ &\max_{r(f|\tilde{y})} \max \left\{\begin{array}{l} \sum_{\tilde{y}\in\mathcal{Y},f\in\mathcal{F}} \hat{\mu}^r\left(\tilde{y}\right) r\left(f|\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_0^f\left(\cdot|\tilde{y}\right)\right),\\ \sum_{\tilde{y}\in\mathcal{Y},f\in\mathcal{F}} \hat{\mu}^r\left(\tilde{y}\right) r\left(f|\tilde{y}\right) D\left(b_{s^*}^f\left(\cdot|\tilde{y}\right)\|p_1^f\left(\cdot|\tilde{y}\right)\right)\end{array}\right\}.\end{aligned}$$

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