Single-track Gray Codes with Non-k-spaced Heads

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***Abstract*—Single-track Gray codes are cyclic Gray codes with an extra property that every column of all the codewords is a cyclic shift of the first column. A subclass of single-track Gray codes with non-*k*-spaced heads is introduced, which does not exist for the length that is lower than 6 and can not be obtained by any transformation from the known *k*-spaced codes. A basic construction for one category of such non-*k*-spaced heads single-track Gray codes with a subcycle of 2 is given, and the rules of choosing seed codes are discussed. Moreover, the conditions that different subclasses of single-track Gray codes can transform into each other are also introduced.**

I. Introduction

Reflected Gray codes were found by F. Gray [1] in 1953, and in 1958 a general definition of Gray codes was introduced by E. N. Gilbert [2] as an ordered list of distinct binary n-tuples, called the codewords, with the property that every two adjacent codewords differ in exactly one component which is call *mono-difference*. A typical application of Gray codes is in absolute angle measurement. Even now, most absolute rotary encoders still use reflected Gray codes such that every codeword is etched in n tracks on the coding disc. However, as the resolution of the encoder increases, more tracks are required. This places a physical limitation on the size of any devices making use of such a code.

One solution to this problem is to use a single track absolute encoder developed by N. B. Spedding in 1994 [6]. In this design, the number of tracks are reduced from n to 1 and various patterns can be adopted. In 1996, these codes were defined as single-track Gray codes by A. P. Hiltgen, K. G. Paterson and M. Brandestini [7]. Single-track Gray codes have an extra property compared with Gray codes, that is when taking the P length n codewords as a matrix each column is a cyclic shift of the first column, which is called *single-track property*. Let the codewords of a length n period P Gray code C be

$$C = [W_0, W_1, \cdots, W_{P-1}]^{\mathrm{T}}$$

where $W_i = [w_i^0, w_i^1, \cdots, w_i^{n-1}]$ for $0 \le i < P$. C is a single-track Gray code if and only if the sequence $W^j = w_0^j, w_1^j, \cdots, w_{P-1}^j$, called component sequence j, is a cyclic shift by some k_j of component sequence 0, *i.e.*,

$$w_0^j, w_1^j, \cdots, w_{P-1}^j = w_{k_j}^0, w_{k_j+1}^0, \cdots, w_{k_j+P-1}^0$$

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where $0 \le j < n$ and subscripts are reduced modulo P. Thus when such a code with this single-track property applied in an absolute encoder only one track is adequate to recover all the codeword information by n reading heads arranged around that single track at the positions $k_0, k_1, k_2, \cdots, k_{n-1} (k_0 = 0)$. The *Head interval* of code C is defined as

$$H_C = [h_0, h_1, \cdots, h_{n-1}] = [k_1 - k_0, k_2 - k_1, \cdots, k_0 - k_{n-1}],$$

where $k_0 - k_{n-1}$ is reduced modulo P. Taking a length 3 period 6 single-track Gray code for example, its component sequence 0 is 000111 and its head interval is $1, 1, 4$. For conciseness, if consecutive '0's or '1's are called one section, the binary sequence 000111 can be presented by the length of sections as $3, 3$. If $h_j \equiv k$ for every $0 \le j \le n-2$, a single-track Gray code C is defined as having k-spaced heads in [9], otherwise we say that C has non-k-spaced heads. The constructions of single-track Gray codes can be classified into necklace orderings and self-dual necklace orderings [7][8][9], which both have k-spaced heads. A survey of single-track Gray code was given in [10] which revealed that the only code with non-k-spaced heads would be obtained from self-dual necklace ordering by complementing any subset of their columns. Some new codes with non-k-spaced heads were found after we had done the exhaustive searching on length 6 single-track Gray codes. New codes with all possible head intervals are shown in TABLE I, including some equivalent transformations of self-dual necklace orderings that are mentioned above.

TABLE I
Length 6 Single-track Gray Codes with Non-k-spaced Heads

Period	Component sequence 0	H_C	$o(H_C)^*$
12	6, 6	1, 3, 1, 3, 1, 3	2
24	14, 4, 2, 4	3, 5, 3, 5, 3, 5	2
	9, 6, 3, 6	2, 2, 8, 2, 2, 8	3
36	14, 5, 6, 5, 2, 4	3, 9, 3, 9, 3, 9	2
	11, 2, 5, 11, 2, 5	3, 9, 3, 9, 3, 9	2
48	15, 8, 6, 5, 3, 3, 2, 6	4, 12, 4, 12, 4, 12	2
	14, 6, 2, 4, 2, 6, 6, 8	3, 13, 3, 13, 3, 13	2
	12, 6, 5, 2, 5, 5, 2, 11	4, 4, 16, 4, 4, 16	3
60	12, 2, 4, 6, 4, 8, 6, 4, 4, 10	9, 11, 9, 11, 9, 11	2

* $o(H_C)$ and this column will be introduced in Section II.

A basic construction of single-track Gray codes with non-

k-spaced heads shall be introduced in Section II. In Section III, the selection rules of seed codes in the basic construction are investigated from the three properties that a single-track Gray code must have, single-track property, mono-difference and distinctness. A necessary and sufficient condition of a binary sequence having both single-track property and mono-difference is given. In Section IV, we present the conditions that different subclasses of single-track Gray codes may transform into each other and also discuss the relationship between the codes with k-spaced heads and non-k-spaced heads.

II. Basic Construction

In this section a category of length n period P single-track Gray code C with non-k-spaced heads is introduced, where the head interval $H_C = k, 2P/n-k, k, 2P/n-k, \cdots$ is periodic.

Let $W=[w_0, w_1, \cdots, w_{n-1}]$ be a length n word. The cyclic order of W is defined as

$$o(W) = \min\left\{ i \,\middle|\, E^i W = W, i \geq 1 \right\},$$

Definition 1. *A length n period P code C has single-track property. If any permutation of its n columns satisfies that the head interval is periodic, i.e. $1 < o(H_C) < n$, we say that C has* periodic heads *with subcycle $o(H_C)$.*

Taking a length $n=6$ single-track Gray codes for instance, the possible cyclic order of H_C are $1, 2, 3$, and 6. If a length 6 single-track Gray code C satisfies $o(H_C)=1$, C must be a necklace ordering. On the other hand, if C satisfies $o(H_C)=6$, C must be a self-dual necklace ordering. Therefore, the possible subcycles of head interval are 2 and 3 with the examples shown in TABLE I. Single-track Gray codes having periodic heads with subcycle $o(H_C)=2$ arise in every possible period, while the codes with subcycle $o(H_C)=3$ only appear in period 24 and 48. All the factors of n except 1 and n can be the possible subcycle of a length n codes with periodic heads, while the discussions in this section will mainly focus on the even length codes with subcycle $o(H_C)=2$.

A cyclic shift operator E acting on a length n codeword W is defined by

$$EW = E[w_0, w_1, \cdots, w_{n-1}] = [w_1, \cdots, w_{n-1}, w_0].$$

Let W_1 and W_2 be two length $n/2$ *sub-codewords*,

$$W_1 = [w_1^0, w_1^1, \cdots, w_1^{n/2-1}], W_2 = [w_2^0, w_2^1, \cdots, w_2^{n/2-1}].$$

We define $W_1 \cdot W_2$ as

$$W_1 \cdot W_2 = [w_1^0, w_1^1, \cdots, w_1^{n/2-1}, w_2^0, w_2^1, \cdots, w_2^{n/2-1}].$$

Two length n codewords $W_1 \cdot W_2$ and $W_3 \cdot W_4$ are regarded as being *equivalent* if there is an integer l satisfies that $E^l W_1 = W_2$ and $E^l W_3 = W_4$, otherwise they are *nonequivalent*. The equivalent class under E acting on both length $n/2$ sub-codewords, $E^i W_1 \cdot E^i W_2$ for $0 \leq i < n/2$, is called a *half-necklace* which can be represented by any word in that class.

The cyclic order of $W_1 \cdot W_2$ is defined as

$$o(W_1 \cdot W_2) = \text{lcm}\{o(W_1), o(W_2)\},$$

where lcm indicates the operation of the least common multiple and $W_1 \cdot W_2$ has full order if $o(W_1 \cdot W_2) = n/2$.

Theorem 1. *Let $S = [S_0, S_1, \cdots, S_{r-1}]$ be r length $n/2$ binary sub-codewords. With some l relatively prime to $n/2$ and some $0 < k < r/2$, r nonequivalent full order length n half-necklaces SC, called seed codes, are constructed from S*

$$\begin{aligned} SC = [&S_0 \cdot S_k, S_1 \cdot S_{k+1}, \cdots, S_{r-k-1} \cdot S_{r-1}, \\ &S_{r-k} \cdot E^l S_0, \cdots, S_{r-1} \cdot E^l S_{k-1}]. \end{aligned}$$

These seed codes also have mono-difference which holds for $S_{r-1} \cdot E^l S_{k-1}$ and $E^l S_0 \cdot E^l S_k$ with the same l as well. The codeword list,

$$\begin{array}{ll} S_0 \cdot S_k, \cdots, S_{r-k} \cdot E^l S_0, & \cdots, S_{r-1} \cdot E^l S_{k-1}, \\ E^l S_0 \cdot E^l S_k, \cdots, E^l S_{r-k} \cdot E^{2l} S_0, & \cdots, E^l S_{r-1} \cdot E^{2l} S_{k-1}, \\ \cdots & \cdots \\ E^{(\frac{n}{2}-1)l} S_0 \cdot E^{(\frac{n}{2}-1)l} S_k, \cdots, E^{(\frac{n}{2}-1)l} S_{r-k} \cdot S_0, & \cdots, E^{(\frac{n}{2}-1)l} S_{r-1} \cdot S_{k-1} \end{array}$$

constitutes a length n period $nr/2$ single-track Gray code C with periodic head interval $H_C = k, r-k, k, r-k, \cdots$ in subcycle 2.

Proof. Since l is relatively prime to $n/2$, the integers $0, l, 2l, \cdots, (n/2-1)l$ are distinct modulo $n/2$. Based on the full order, nonequivalent and Gray code properties of the r length n half-necklace seed codes in SC, we know that the codeword list in the statement of theorem do form a cyclic Gray code. Then we will prove that the codewords have single-track property with periodic head interval $k, r-k, k, r-k, \cdots$.

Write the seed codes SC in vertical order to form an $r \times n$ matrix. Let $D_0, D_1, \cdots, D_{n/2-1}$ be column 0 to column $n/2-1$ and $G_0, G_1, \cdots, G_{n/2-1}$ be column $n/2$ to column $n-1$. The component sequence 0 and $n/2$ of code C are $C^0 = D_0 D_l D_{2l} \cdots D_{(n/2-1)l}$ and $C^{n/2} = G_0 G_l G_2 l \cdots G_{(n/2-1)l}$ where the subscripts are reduced modulo $n/2$. Since the codeword list of C is based on the half-necklace seed codes SC, component sequence j in the first half columns can be represented as $C^j = D_j D_{j+l} D_{j+2l} \cdots D_{j+(n/2-1)l}$ for $0 \leq j < n/2$ and the subscripts are reduced modulo $n/2$. Since l is relatively prime to $n/2$, the subscripts of component sequence j are distinct and there must be an i that satisfies $il (\bmod n/2) = j$. Therefore, component sequence j is a cyclic shift of component sequence 0 by $il \times r$, *i.e.*, $C^j = E^{il \times r} C^0$, where superscript is reduced modulo $nr/2$. In the same way, we can prove that the other half columns of code C also have single-track property with the same i, *i.e.*, $C^{j+n/2} = E^{il \times r} C^{n/2}$. The component sequence $n/2$ is a shifted equivalence of component sequence 0 by k from the way that all the codewords are ordered, *i.e.*, $C^{n/2} = E^k C^0$. Therefore, every column in C is a cyclic shift of the first one,

$$C^j = E^{il \times r} C^0 \text{ and } C^{j+n/2} = E^{il \times r + k} C^0,$$

which means C has single-track property. And if we rearrange the component sequences $C^0, C^1, \cdots, C^{n-1}$ as

$$C^0, C^{n/2}, C^1, C^{n/2+1}, \cdots, C^{n/2-1}, C^{n-1},$$

the periodic head interval $k, r-k, k, r-k, \cdots$ with subcycle 2 can be obtained. $\square$

A general construction of non-k-spaced head with other subcycle will be put forward if the every seed code is based on $o(H_C)$ length $n/o(H_C)$ sub-codewords combined in the same way. The equivalent class under E, like $E^iW_1{\cdot}E^iW_2\cdots E^iW_{o(H_C)}$ for $0 \le i < n/o(H_C)$, is called $o(H_C)$-*combined-necklace*. For example, choosing $l=1$, $k=2$ and 12 sub-codewords as $S=[01,01,01,00,00,00,01,01,01,11,11,11]$, a length 6 period 24 single-track Gray code with periodic head interval $2,2,8,2,2,8$ can be obtained based on a combined-necklace ordering with full order 3-combined necklaces.

III. Construction of Seed Codes

An even length n period P single-track Gray code C with periodic heads, having single-track property, mono-difference and distinctness, can be constructed by the half-necklace ordering introduced in Theorem 1 with appropriate parameters l, k and $S=[S_0,S_1,\cdots,S_{2P/n-1}]$. According to the proof of Theorem 1, as long as l is relatively prime to $n/2$ the half-necklace listed as in Theorem 1 has the single-track property. Therefore, in this section we are going to discuss the rest two properties, mono-difference and distinctness, which are determined by the choice of the seed codes

$$SC=[S_0S_k,\cdots,S_{\frac{2P}{n}-k-1}S_{\frac{2P}{n}-1},\ S_{\frac{2P}{n}-k}(E^lS_0),\cdots,S_{\frac{2P}{n}-1}(E^lS_{k-1})].$$

A. Mono-difference

A subtraction matrix $Sub(C)$ is defined by taking the absolute value of the difference between two adjacent codewords including the last one and the first one. For simplicity, we define D_j as $|S_j-S_{j+1}|$ that

$$D_j=\left[\left|s_j^0-s_{j+1}^0\right|,\left|s_j^1-s_{j+1}^1\right|,\cdots,\left|s_j^{\frac{n}{2}-1}-s_{j+1}^{\frac{n}{2}-1}\right|\right],$$

where $0\le j\le 2P/n-2$ and $D_{\frac{2P}{n}-1}$ is defined as $|S_{\frac{2P}{n}-1}-E^lS_0|$ that

$$D_{\frac{2P}{n}-1}=\left[\left|s^0_{\frac{2P}{n}-1}-s_0^l\right|,\left|s^1_{\frac{2P}{n}-1}-s_0^{l+1}\right|,\cdots,\left|s^{\frac{n}{2}-1}_{\frac{2P}{n}-1}-s_0^{l+\frac{n}{2}-1}\right|\right].$$

where the superscripts are reduced modulo $n/2$. The first $2P/n$ codewords in $Sub(C)$, represented as $Sub(C)\_S$, is called seed matrix of $Sub(C)$,

$$Sub(C)\_S=[D_0D_k,\cdots,D_{2P/n-k-1}D_{2P/n-1},\ D_{2P/n-k}(E^lD_0),\cdots,D_{2P/n-1}(E^lD_{k-1})]^{\mathrm{T}}.$$

The weight of D_j is defined as $|D_j|=\sum_{i=0}^{n/2-1}\left|s_j^i-s_{j+1}^i\right|$ which indicates the number of components that change between the two adjacent codewords.

Theorem 2. *An even length n period P single-track binary sequence C with head interval $k,2P/n-k,k,2P/n-k,\cdots$ has the mono-difference if and only if $2P/nb$ is even and each length $n/2$ sub-codeword D_j for $0\le j<2P/n$ in $Sub(C)\_S$ satisfies the following condition for every $0\le i<b$,*

$$\left\{|D_j|\in\{0,1\}\left|\begin{array}{l}|D_i|=|D_{i+2k}|=\cdots=\left|D_{i+(\frac{2P}{nc}-2)k}\right|\\ |D_{i+k}|=|D_{i+3k}|=\cdots=\left|D_{i+(\frac{2P}{nc}-1)k}\right|\\ |D_{i+k}|=1-|D_i|\end{array}\right.\right\},$$

where $b=\gcd(k,2P/n(\mathrm{mod}\,k))$, gcd *denotes the greatest common divisor and subscripts are reduced modulo $2P/n$.*

Proof. Suppose that the binary sequence C has the mono-difference, then every codeword in $Sub(C)\_S$ has a weight of 1. Without loss of generality, we let $|D_k|=1$, then $|D_0|=0$. Since C has single-track property with head interval $k,2P/n-k,k,2P/n-k,\cdots$, we can obtain the values of one group as

$$|D_k|=|D_{3k}|=|D_{5k}|=\cdots=1$$

$$|D_0|=|D_{2k}|=|D_{4k}|=\cdots=0.$$

If $2P/n(\mathrm{mod}\,k)=0$, k groups would be needed to comprise all $|D_j|$ in $Sub(C)\_S$. Otherwise, the number of groups depends on the greatest common divisor between k and $2P/n(\mathrm{mod}\,k)$. Since $\gcd(k,0)=k$, generally speaking, $|D_0|$, $|D_1|,\cdots,|D_{2P/n-1}|$ in $Sub(C)\_S$ can be divided into b individual groups, where $b=\gcd(k,2P/n(\mathrm{mod}\,k))$. The value of $|D_j|$ only can be 0 or 1 for C has the mono-difference. Therefore, there are $2P/nb$ elements in each group which also can be divided into two parts having different values. One codeword in $Sub(C)\_S$ is combined by D_j and D_{j+k} respectively from the two parts, so the number of the elements in both parts must be the same as P/nb. Thus $2P/nb$ is even, or there would be some overlap between the two parts which is a contradiction to the mono-difference.

On the other hand, suppose that D_j in $Sub(C)\_S$ satisfies the condition in Theorem 2 for every $0\le i<b$. We know $b|2$ for $b=\gcd(k,2P/n(\mathrm{mod}\,k))$, then $2P/n$ length $n/2$ sub-codewords D_j are separated equally into b groups. Since C has single-track property with head interval $k,2P/n-k,k,2P/n-k,\cdots$ and $2P/nb$ is even, every two sub-codewords D_j and D_{j+k} from one group form a length n codeword in $Sub(C)\_S$. Since $|D_{j+k}|+|D_j|=1$ and the cyclic shift operator E does not influence the weight of a codeword, every codeword in $Sub(C)$ has a weight of 1. So C has the mono-difference. $\square$

If the parameter k satisfies that $2P/nb$ is even, there must exist at least one $Sub(C)\_S$ that satisfies the assignment rules in Theorem 2. Taking length 6 period 36 codes for example, the possible choices of k are $1,2,3,4,5$, and the corresponding values of b are $1,2,3,4,1$. Thus the values of $2P/nb$ are even except when $k=4$, and this is why length 6 period 36 binary codes have the mono-difference only when $k=1,2,3$, and 5. In Theorem 2, each one from b groups has two choices to assign its two parts, thus there are 2^b situations can satisfy the mono-difference. As one half of the situations are the complementary of the other half, 2^{b-1} situations are exhaustive.

In each situation, half of the D_j in $Sub(C)\_S$ are identified as having a weight of 0, which means when $|D_{j_0}| = 0$ the two length $n/2$ sub-codewords S_{j_0} and S_{j_0+1} in S are equal, so the number of the undetermined sub-codewords in $S = [S_0, S_1, \cdots, S_{2P/n-1}]$ will be halved when mono-difference is under consideration. The other half elements in $Sub(C)\_S$ are identified as having a weight of 1, which means when $|D_{j_1}| = 1$ the two corresponding length $n/2$ sub-codewords satisfy S_{j_1} and S_{j_1+1} differ in only one component. Suppose a length 6 period 24 binary code has a periodic head interval $1, 7, 1, 7, 1, 7$, thus $b = 1$. The only one situation of $Sub(C)\_S$ is $|D_0| = |D_2| = |D_4| = |D_6| = 0$, and $|D_1| = |D_3| = |D_5| = |D_7| = 1$. Then the codewords in S satisfy that $S_0 = S_1$, $S_2 = S_3$, $S_4 = S_5$, $S_6 = S_7$ and any two adjacent sub-codewords from $S_1, S_2, S_3, S_4, S_5, S_6, S_7, E^l S_0$ differ in exactly one component. Therefore S_1, S_3, S_5, and S_7 can be eliminated from S so that

$$S = [S_0, S_0, S_2, S_2, S_4, S_4, S_6, S_6]$$

which require any two adjacent sub-codewords from $S_0, S_2, S_4, S_6, E^l S_0$ differ in exactly one component.

If we regard $S = [S_0, S_1, \cdots, S_{2P/n-1}]$ as an m-ary sequence of period $2P/n$ where $m = 2^{n/2}$ and use the section description as mentioned in Section I, then $S = [S_0, S_0, S_2, S_2, S_4, S_4, S_6, S_6]$ can be described as $2, 2, 2, 2$ where the number of the sections represents the number of the undetermined sub-codewords in S which must be P/n. When the length 6 period 24 binary code has a head interval $2, 6, 2, 6, 2, 6$ and $b = 2$, after the eliminations we obtain S in each situation as follows

$$S = [S_0, S_0, S_0, S_3, S_4, S_4, S_4, S_7]$$

$$S = [S_0, S_0, S_2, S_3, S_3, S_3, S_6, E^l S_0]$$

which are regarded as the same as $3, 1, 3, 1$. So any four length 3 sub-codewords S'_0, S'_1, S'_2, S'_3, where adjacent codewords differ in exactly one component, can construct a length 6 period 24 binary code with mono-difference and a periodic head interval $2, 6, 2, 6, 2, 6$ by reproducing every codewords as much as $3, 1, 3, 1$ times to obtain $S = [S'_0, S'_0, S'_0, S'_1, S'_2, S'_2, S'_2, S'_3]$. The section description of S after the elimination is called *mono-difference distributing pattern* of b. There is only one distributing pattern when $b = 1$ and $b = 2$ as we discussed above, and in TABLE II all the mono-difference distributing patterns are ordered for some different values of b. The number of the distributing patterns in S is the same as the number of binary self-dual necklaces with $2b$ beads. (This is the sequence A007147 in OEIS.) The patterns are the same as the head interval of length b period $2b$ single-track Gray code with self-dual necklace ordering and its equivalences. Therefore, P/n length $n/2$ sub-codewords with mono-difference distributed into $2P/n$ positions in S according to one of the distributing patterns of b can obtain a length n period P binary code with mono-difference and a periodic head interval $k, 2P/n - k, k, 2P/n - k, \cdots$.

TABLE II
MONO-DIFFERENCE DISTRIBUTING PATTERNS

b	Number of patterns	Patterns
1	1	$2, 2, 2, 2, \cdots$
2	1	$3, 1, 3, 1, \cdots$
3	2	$4, 1, 1, 4, 1, 1, \cdots$
		$2, 2, 2, 2, 2, 2, \cdots$
4	2	$1, 1, 1, 5, 1, 1, 1, 5, \cdots$
		$1, 2, 3, 2, 1, 2, 3, 2, \cdots$
5	4	$1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \cdots$
		$1, 1, 2, 4, 2, 1, 1, 2, 4, 2, \cdots$
		$1, 2, 1, 3, 3, 1, 2, 1, 3, 3, \cdots$
		$2, 2, 2, 2, 2, 2, 2, 2, \cdots$
6	5	$1, 1, 1, 1, 1, 7, 1, 1, 1, 1, 1, 7, \cdots$
		$1, 1, 1, 2, 5, 2, 1, 1, 1, 2, 5, 2, \cdots$
		$1, 1, 2, 1, 4, 3, 1, 1, 2, 1, 4, 3, \cdots$
		$1, 2, 2, 3, 2, 2, 1, 2, 2, 3, 2, 2, \cdots$
		$3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, \cdots$

B. Distinctness

For a length 6 period 24 binary sequence with periodic head interval $k, 2P/n - k, k, 2P/n - k, \cdots$, the options that satisfy the mono-difference are $k = 1, 2, 3$. However only $k = 3$ can guarantee the 24 codewords are distinct. Some ideas about the relationship between the distinctness and parameter k will be discussed based on the results of single-track Gray codes of length 6, 8, and 10.

As mentioned above, a length 6 period 24 binary sequence with periodic head interval $1, 7, 1, 7, 1, 7$ and Gray code property has only one mono-difference distributing pattern for S, thus

$$SC = [S_0S_0, S_0S_2, S_2S_2, S_2S_4, S_4S_4, S_4S_6, S_6S_6, S_6(E^l S_0)].$$

There are four special codewords needed in SC, which are S_0S_0, S_2S_2, S_4S_4, and S_6S_6. These codes combining with the same length $n/2$ codeword are defined as S_iS_i word. For length 6, only two full order half-necklaces, 001001 and 011011, are S_iS_i words. For length 8 and 10 the numbers of S_iS_i words which are the full order half-necklaces are three and six. S_iS_i words are enough for SC until length 10 and that is the reason why single-track Gray code with periodic heads $1, 7, 1, 7, \cdots$ exists until length 10. As n increases, the range of available k increases, and part of the exhaustive searching result are shown in TABLE III, where "$\surd$" indicates the existence of single-track Gray codes. However the certain relationship between the distinctness and parameters n, P and k are still unsolved. When $k = 3$, then $b = 1$ and the mono-difference distributing pattern in S is as the same as $k = 1$, thus

$$SC = [S_0S_2, S_0S_4, S_2S_4, S_2S_6, S_4S_6, S_4(E^l S_0), S_6(E^l S_0), S_6(E^l S_2)],$$

where no S_iS_i word is needed. Although S_iS_i word is not a sufficient condition of the distinctness, SC with no S_iS_i word is easier to get distinct.

TABLE III
THE EXISTENCE OF SINGLE-TRACK GRAY CODES WITH SUBCYCLE 2

P/n	n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
2	6	√						
	8	√						
	10	√						
4	6			√				
	8		√	√				
	10	√	√	√				
6	6			√				
	8			√		√		
	10	√	√	√		√		
8	6			√	√			
	8			√	√	√	√	√

IV. THE TRANSFORMATION AND CLASSIFICATION OF SINGLE-TRACK GRAY CODES

A. Transformation between non-k-spaced and k-spaced hands

The single-track Gray codes with periodic head interval can transform into the codes with k-spaced heads. Two conditions will be discussed on transforming into necklace orderings and self-dual necklace orderings.

A length n period P single-track Gray code C, having a periodic head interval with subcycle $o(H_c) = 2i + 1$ for $i > 0$, can also be a necklace ordering after rewriting the codewords matrix with the component sequence 0 and n P/n-spaced reading heads. For example, when $o(H_c) = 3$, every length 6 period 24 single-track Gray code with head interval $2, 2, 8, 2, 2, 8$ is also a necklace ordering with 4-spaced heads.

When $P/2n$ is odd and $n/o(H_c) = 2i + 1$ for $i > 1$ all the self-dual necklace orderings can also be single-track Gray codes having periodic head interval with subcycle $o(H_c) = 2$. For example, all self-dual necklace orderings of length 6 period 36 can also be single-track Gray codes with periodic head interval $3, 9, 3, 9, 3, 9$.

B. Classification

A length n period P single-track Gray code C constructed by $o(H_C)$combined-necklace ordering is called odd-subcycle code if $o(H_c) = 2j + 1$ for $j \geq 1$ and even-subcycle code if $o(H_c) = 2j$ for $i \geq 1$. As shown in Fig.1, we represent the relationship between the known single-track Gray codes with a Venn diagram, including the exhaustive result of length 6, the exhaustive result of length 8 for some periods, and some codes of length 9 to length 20. The codes that have been found so far are all involved in Fig.1. Four circles are represented four sets $\boldsymbol{N}$, $\boldsymbol{S}$, $\boldsymbol{O}$ and $\boldsymbol{E}$, which are necklace orderings, self-dual necklace orderings, odd-subcycle codes and even-subcycle codes respectively. The intersection of any two sets in Fig.1 indicates that the single-track Gray codes in those two sets can transform into each other.

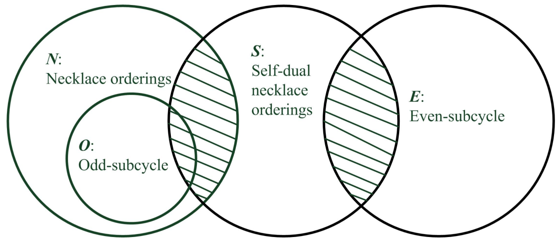

Fig. 1. Relationship between different classes of single-track Gray codes

V. CONCLUSION

A subclass of single-track Gray codes with non-k-spaced periodic heads exists when length n is a non-prime number and $n \geq 6$. The codes with subcycle 2 can be obtained by the basic construction (Theorem 1) which is based on half-necklace that applies operator E on both halves of the codeword independently. The necessary and sufficient condition (Theorem 2) for mono-difference of the codes having periodic heads is feasible. Therefore, according to Theorem 1 and Theorem 2 all the binary sequences having both mono-difference and single-track property with periodic heads can be found straightforwardly. Since the distinctness is not guaranteed in Theorem 1 and 2, a computer program is still needed to check the distinctness of every codeword list. Similarly, the codes with subcycle $o(H_C)$ can be obtained by the construction based on $o(H_C)$-combined-necklace that applies operator E on all length $n/o(H_C)$ sub-codewords.

A length 10 period 1000 [10] single-track Gray code is still under consideration for the least length of the known codes is 20, which means the cost of an absolute encoder will be doubled. From Fig.1 we know that this code may exist in the area that $\overline{\boldsymbol{S}} \cap \boldsymbol{E}$ which indicates the subcycle $o(H_C)$ and $n/o(H_C)$ are both even. For $n = 10 = 2 \times 5$, there is no possibility of making $o(H_C)$ and $n/o(H_C)$ both even. Although the relationship in Fig.1 is only a conjecture of length 10 period 1000 single-track Gray codes, it is still instructive for the existence of such codes.

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