Quality Sensitive Price Competition in Spectrum Oligopoly

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***Abstract*—We investigate a spectrum oligopoly where primary users allow secondary access in lieu of financial remuneration. Transmission qualities of the licensed bands fluctuate randomly. Each primary needs to select the price of its channel with the knowledge of its own channel state but not that of its competitors. Secondaries choose among the channels available on sale based on their states and prices. We formulate the price selection as a non-cooperative game and prove that a symmetric Nash equilibrium (NE) strategy profile exists uniquely. We explicitly compute this strategy profile and analytically and numerically evaluate its efficiency. Our structural results provide certain key insights about the unique symmetric NE.**

I. Introduction

Recent investigations augur that demand for mobile broadband – driven by the large scale proliferation of wireless industry – will surpass the availability of wireless spectrum in imminent future. Yet, as recent measurements suggest, the licensed bands remain largely under-utilized. A reasonable conjecture therefore is that unlicensed access of idle (but licensed) spectrum bands, commonly referred to as secondary spectrum access, would avert the impending crisis. Recently, FCC has legalized the access of TV white space spectrum, and the advent of cognitive radios together with the design of a plethora of sophisticated algorithms have enabled intelligent selection of bands. Large-scale secondary spectrum access can not however be realized only through the availability of the enabling technology and the regulatory progress: secondary access must also be rendered profitable for the license holders. Accordingly, we investigate a spectrum oligopoly where license holders (hitherto referred to as primaries) allow unlicensed users (hitherto referred to as secondaries), in lieu of financial remuneration, access to the channels (licensed bands) that are not in use. Different channels offer different transmission rates to the secondaries depending on their states which evolve randomly and reflect the usage levels of the primaries as also transmission quality fluctuations owing to fading. Each primary quotes a price for the channel that it offers and secondaries select among the available channels depending on the states and the prices quoted. Thus, if a primary quotes a high price, it will earn a large profit if it sells its channel, but may not be able to sell at all; on the other hand a low price will enhance the probability of a sale but also fetch lower profits in the event of a sale.

Price selection in oligopolies may be modeled as a non-cooperative game and has naturally been extensively investigated in economics [6]; we defer an extensive review to our technical report [1] owing to space constraints. The existing literature in economics as also in the specific application context of cognitive radios largely ignore the uncertainty of competition which distinguishes spectrum markets from standard oligopolies: a primary knows the state of its channel but does not those of its competitors before deciding the price for its channel. The papers that consider uncertainty of competition, namely [2]–[5], assume that the commodity on sale can be in one of two states: available or otherwise. This assumption does not capture different transmission qualities offered by the available channels. The consideration of the latter significantly complicates the analysis of the game. A primary may now need to employ different pricing strategies for different states, while in the former case a single pricing strategy will suffice as a price need not be quoted for an unavailable commodity. Our investigation seeks to contribute in this space.

We have modeled the price selection as a game with primaries as the players (Section II) and seek a Nash Equilibrium (NE) pricing strategy. We consider that the preference of the secondaries can be captured by a penalty function which associates a penalty value to each channel that is available for sale depending on its state and price quoted. Given the state of a channel, there is a one-to-one correspondence between the price quoted and the penalty perceived by a secondary. Thus, the strategy for selection of a price for a channel in a given state may be equivalently represented as a strategy for selection of penalty that the channel offers to a secondary. Since prices and therefore the penalties take real values, the strategy set of the players are continuous; also the payoff functions for the primaries turn out to be discontinuous. Thus, classical results do not guarantee the existence, let alone the uniqueness, of an NE. In addition, existing literature does not provide algorithms for computing an NE unlike when the strategy set is finite. Starting from a general set of strategy profiles for the primaries which allows for selecting penalties using arbitrary probability distributions, we show that for a large class of penalty functions, there exists a unique symmetric NE strategy profile, which we explicitly compute (Section III). Our analysis reveals several interesting insights about the structure of the symmetric NE. First, we learn that if a channel in state i provides a higher transmission rate to a secondary than that in state j, then the symmetric NE strategy profile selects the penalties for i, j respectively from ranges $[L_i, U_i], [L_j, U_j]$ where $U_i \leq L_j$. Thus, a secondary will always prefer a channel in state i to a channel in state j considering both the prices and the states. This negates the intuition that prices ought to be selected for the states so as to render them equally preferable to a secondary - symmetric NE strategy profiles in fact price the channels so as to retain the preference order provided by the states. The analysis also reveals that the unique symmetric NE strategy profile consists "nice" cumulative distributions in that they are continuous and strictly increasing; the former rules out pure strategy symmetric NEs and the latter ensures that the support sets are contiguous. Finally, utilizing the explicit computation algorithm for the symmetric NE strategies, we analytically and numerically investigate the reduction in expected profit suffered under the unique symmetric NE pricing strategies as compared to the maximum possible value allowing for collusion among primaries (Section IV).

II. System Model

We consider a spectrum market with n primaries and m secondaries. We will initially consider the case that the primaries know m, later generalize our results for random, apriori unknown m. Each primary has access to a channel which can be in states $0, 1, \ldots, n$, where state i provides a lower transmission rate to a secondary than state j if $i < j$ and state 0 arises when the channel is not available for sale and provides 0 transmission rate. Different channels constitute disjoint frequency bands leased by the primaries. A channel is in state $i \geq 1$ w.p. q_i and in state 0 w.p. $1-q$ where $q = \sum_{i=1}^{n} q_i$, independent of the states of other channels. If a primary quotes a price p for a channel in state i, then the channel offers a penalty $g_i(p)$ to a secondary. Each $g_i(\cdot)$ is continuous, strictly increasing in its argument, and therefore invertible. We denote $f_i(\cdot)$ as the inverse of $g_i(\cdot)$; clearly $f_i(\cdot)$ is continuous and strictly increasing in its argument as well. Also, $g_i(p) > g_j(p)$ and $f_i(x) < f_j(x)$ for each x, p and $i < j$. No secondary buys a channel whose penalty is higher than v, and as the name suggests a secondary prefers a channel with a lower penalty (a secondary's preference depends entirely on the penalty). Each primary also incurs a transition cost $c > 0$ for an available channel, and therefore never selects a price lower than c. We assume that

$$\frac{f_j(y) - c}{f_k(y) - c} < \frac{f_j(x) - c}{f_k(x) - c} \quad \text{for all } x > y > g_j(c), j < k \qquad (1)$$

A large class of penalty functions $g_i(\cdot)$ satisfy the above property required of the corresponding inverses, e.g., $g_i(p) = \zeta\left(p - h(i)\right), g_i(p) = \zeta\left(p/h(i)\right)$ where $\zeta(\cdot)$ is continuous, $\zeta(\cdot)$ and $h(\cdot)$ are strictly increasing, $g_i(p) = p^r - h(i), g_i(p) = p^r h(i), g_i(p) = \exp(p) - h(i)$ for $r > 0$ and a strictly increasing $h(\cdot)$. In addition, $g_i(\cdot)$ such that the inverses are of the form $f_i(x) = h_1(x) + h_2(i), f_i(x) = h_1(x)h_2(i)$, where $h_1(\cdot)$ is continuous, $h_1(\cdot)$ and $h_2(\cdot)$ are strictly increasing, satisfy the above assumption.

If primary i quotes a price p for its channel then its profit is

$$\begin{cases} p - c & \text{if the primary sells its channel} \\ 0 & \text{otherwise} \end{cases}$$

Note that if Y is the number of channels offered for sale for which the penalties are upper bounded by v, then those with $\min(Y, m)$ lowest penalties are sold since secondaries select channels in increasing order of penalties. The ties among channels with identical penalties are broken randomly and symmetrically among the primaries. Also, note that utilities of primaries are not continuous functions of their actions.

Each primary selects the penalty for its channel with the knowledge of the state of the channel, but without knowing the states of the other channels; a primary however knows $l, m, n, q_1, \ldots, q_n$. Note that the choice of the penalty uniquely determines the price since there is a one-to-one correspondence between the two given the state of a channel. Primary i chooses its penalty using an arbitrary cumulative distribution function $\psi_{i,j}(.)$ when its channel is in state $j \geq 1$. If $j = 0$ (i.e., the channel is unavailable), i chooses a penalty of $v+1$: this is equivalent to considering that such a channel is not offered for sale as no secondary buys a channel whose penalty exceeds v. For $j \geq 1$, each primary selects its price so as to maximize its expected profit. Thus, if $m \geq l$, primaries select the highest penalty for each state $1, \ldots, n$, since all available channels will be sold. So, we consider $m < l$. $S_i = (\psi_{i,1}, ..., \psi_{i,n})$ denotes the strategy of primary i, and $(S_1, ..., S_l)$ denotes the strategy profile of all primaries (players).

Definition 1. *S_{-i} denotes the strategy profile of primaries other than i. $E\{u_{i,j}(\psi_{i,j}, S_{-i})\}$ denotes the expected profit when primary i's channel is in state j and it uses strategy $\psi_{i,j}(\cdot)$ and other primaries use strategy S_{-i}.*

Definition 2. *A* Nash equilibrium *$(S_1, \ldots, S_n)$ is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy [6]. So, with $S_i = (\psi_{i,1}, ..., \psi_{i,n})$, $(S_1, \ldots, S_n)$, is a Nash equilibrium (NE) if for each primary i and channel state j*

$$E\{u_{i,j}(\psi_{i,j}, S_{-i})\} \geq E\{u_{i,j}(\tilde{\psi}_{i,j}, S_{-i})\} \;\forall\; \tilde{\psi}_{i,j}. \qquad (2)$$

An NE $(S_1, \ldots, S_n)$ is a symmetric NE *if $S_i = S_j$ for all i, j.*

The above game is a symmetric one since primaries have the same action sets, payoff functions and their channels are statistically identical. We therefore consider only symmetric NEs[1]. Clearly, for any symmetric NE, we can represent the

[1]For a symmetric game, an asymmetric NE is rarely realized. For example. for two players, if (S_1, S_2) is an NE, (S_2, S_1) is also an NE. The realization of such an NE is possible only when each player knows whether the other uses S_1 or S_2. This complication is somewhat alleviated for a symmetric NE as all players play the same strategy; this complication is eliminated when there is a unique symmetric NE. We prove that the latter is indeed the case for the game we consider.

strategy of any primary as $S = (\psi_1(.), \psi_2(.), ..., \psi_n(.))$ where we drop the index corresponding to the primary.

Let $\phi_j(x)$ denote the expected profit of a primary whose channel is in state j and who selects a penalty x and $r(x)$ denote the probability that a channel quoted at penalty x is sold. Note that the dependence of $\phi_j(x), r(x)$ on the strategy profile of the primaries is not explicitly indicated to ensure notational simplicity. Also, note that $r(x)$ does not depend on the state of the channel since secondaries select the channels based only on the penalties. Next,

$$\phi_j(x) = (f_j(x) - c)r(x). \tag{3}$$

(recall that the inverse of the penalty function $g_j(\cdot)$, $f_j(\cdot)$, provides the price that corresponds to penalty x and channel state j).

Definition 3. *A* best response *penalty for a channel in state $j \geq 1$ is x if and only if*

$$\phi_j(x) = \sup_{y \in \Re} \phi_j(y).$$

Let $u_{j,max} = \phi_j(x)$ for a best response x for state j, $j \geq 1$ i.e., $u_{j,max}$ is the maximum expected profit that a primary earns when its channel is in state j, $j \geq 1$.

III. A SYMMETRIC NE: EXISTENCE, UNIQUENESS AND COMPUTATION

First, we identify key structural properties of a symmetric NE (should it exist). Next we show that the above properties leads to a unique strategy profile which we explicitly compute - thus the symmetric NE is unique should it exist. We finally prove that the strategy profile resulting from the structural properties above is indeed a symmetric NE thereby establishing existence.

A. Structure of a symmetric NE

We start with by providing some important properties that any symmetric NE $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ must satisfy.

Theorem 1. *$\psi_i(.), i \in \{1, .., n\}$ is a continuous cumulative distribution.*

The above theorem rules out any pure strategy symmetric NE.

Definition 4. *We denote the lower and upper endpoints of the support set[2] of $\psi_i(.)$ as L_i and U_i respectively i.e.*

$$L_i = \inf\{x : \psi_i(x) > 0\}$$

$$U_i = \inf\{x : \psi_i(x) = 1\}$$

We next show that the support sets are ordered in increasing order of the state indices.

Theorem 2. *$U_i \leq L_j$, if $j < i$*

[2]The support set of a cumulative distribution is the smallest closed set such that the probability of its complement is 0.

We finally rule out any "gaps" inside the support sets and between the support sets for different $\psi_i(\cdot)$, $i = 1, .., n$. This also establishes that $\psi_i(\cdot)$ is strictly increasing in $[L_i, U_i]$.

Theorem 3. *The support set of $\psi_i(.), i = 1, .., n$ is $[L_i, U_i]$ and $U_i = L_{i-1}$ for $i = 2, .., n$, $U_1 = v$.*

Remark: The structure of the symmetric NE identified in Theorems 1 to 3 provide several interesting insights:

- Theorem 2 implies that the primaries select the highest penalties for the worst states. The primaries therefore do not strive to render all states equally preferable to the secondaries through price selection.
- Theorems 1 and 3 reveal that the symmetric NE strategy profile consists of "well-behaved" distribution functions.

We outline the proofs of the theorems below and defer the details to our technical report [1]. We start with an observation that we will use throughout:

Observation 1. *Any point $y \leq g_j(c)$ can not be a best response (definition 3) for channel state j.*

The observation is evident as the profit $\phi_j(\cdot)$ of a primary is non-positive if the selected penalty is $\leq g_j(c)$. But, $\phi_j(x) > 0$ for $g_j(c) < x \leq v$ as $0 < \sum_{i=1}^{n} q_i < 1$.

1) Outline of proof of Theorem 1: Otherwise if, $\psi_j(.)$ has a jump at $x > g_j(c)$, then all primaries select x as their penalties with positive probability whenever their channel states are j. Thus, a primary can increase its expected profit by choosing a penalty slightly below x. That is, $r(x - \epsilon) \geq r(x) + \gamma$ for all $\epsilon > 0$ and some $\gamma > 0$. Since $f_j(\cdot)$ is continuous, (3) reveals that $\phi_j(x - \epsilon) > \phi_j(x)$ for all small but positive ϵ. Thus, x is not a best response which rules out its selection with positive probability by an NE strategy profile.

2) Outline of proof of Theorem 2: We introduce some terminologies that we will use throughout the paper.

Definition 5. *Let X_m be the mth smallest offered penalty offered by primaries $i = 2, \ldots, l$, and let $F(\cdot)$ denote the distribution function of X_m.*

For a symmetric strategy profile, $F(\cdot)$ would remain the same if we had considered any $l - 1$ primaries rather than $2, \ldots, l$. Let $c_{min} = \min_{i \in \{1, \cdot, n\}} g_i(c)$. The following lemma and observation are direct consequences of Theorem 1:

Lemma 1. *$F(\cdot)$ is continuous in $[c_{min}, v]$ and if $\sum_{j=1}^{n} \psi_j(y) > \sum_{j=1}^{n} \psi_j(x)$, then $F(y) > F(x)$*

Observation 2. *Every element in the support set of $\psi_i(\cdot)$ is a best response; thus, so are L_i, U_i.*

Lemma 1 implies that $P(X_m = x) = 0$ for each x and thus $r(x) = 1 - F(x)$. Hence,

$$\phi_j(x) = (f_j(x) - c)(1 - F(x)) \tag{4}$$

The proof of Theorem 2 proceeds as follows. from Observation 2 it is sufficient to show that for any x, y such that $c_{min} < x < y \leq v$, if x is a best response when the state of the channel is j, then y can not be a best response when the state of the channel is i for $i > j$. If not consider $y > x$ such that x, y are the best responses when channel states are respectively j, i. Now, from Observation 1 $f_i(y) > c, f_j(x) > c$. Also,

$$\begin{aligned} u_{i,max} &= (f_i(y) - c)(1 - F(y)) \quad (5) \\ \phi_j(y) &= (f_j(y) - c)(1 - F(y)) \\ &= u_{i,max} \cdot \frac{f_j(y) - c}{f_i(y) - c} \quad \text{(from(5))} \\ u_{j,max} &\geq u_{i,max} \cdot \frac{f_j(y) - c}{f_i(y) - c} \quad (6) \end{aligned}$$

Next,

$$\begin{aligned} u_{j,max} &= (f_j(x) - c)(1 - F(x)) \\ \phi_i(x) &= (f_i(x) - c)(1 - F(x)) \\ &= u_{j,max} \cdot \frac{f_i(x) - c}{f_j(x) - c} \quad (7) \end{aligned}$$

Using (6) and (7), we obtain-

$$\phi_i(x) \geq u_{i,max} \cdot \frac{(f_j(y) - c)(f_i(x) - c)}{(f_i(y) - c)(f_j(x) - c)} \quad (8)$$

But, then, since $y > x$, $i > j$, (1) implies that $\phi_i(x) > u_{i,max}$ which contradicts the definitions of $u_{i,max}$ and $\phi_i(x)$. □

3) Proof Of Theorem 3: Otherwise, using Theorem 2, it follows that there exists an interval $(x, y) \subseteq [L_n, v]$ such that no primary offers penalty in the interval (x, y) with positive probability. So, we must have $\tilde{a}$ such that

$$\tilde{a} = \inf\{b \leq x : \psi_j(b) = \psi_j(x), \forall j\}$$

By definition of $\tilde{a}$, $\tilde{a}$ is a best response for at least one state i. But, as primaries do not offer penalty in the range $(\tilde{a}, y)$, so from (4), $\phi_i(z) > \phi_i(\tilde{a})$ for each $z \in (\tilde{a}, y)$. This is because $F(y) = F(\tilde{a})$ and $f_i(\tilde{a}) < f_i(z)$. Thus, $\tilde{a}$ can not be a best response for state i. □

B. Computation and Uniqueness of a Symmetric NE

We now show that the structural properties of a symmetric NE identified in Theorems 1, 2, 3 are satisfied by a unique strategy profile, which we explicitly compute. This proves the uniqueness of a symmetric NE subject to existence. We start with the following definitions.

$$\begin{aligned} w(x) &= \sum_{i=m}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-i-1} \quad (9) \\ w_i &= w(\sum_{j=i}^{n} q_j) \quad \text{for} i = 1, ..., n, \quad w_{n+1} = 0 \quad (10) \end{aligned}$$

Clearly, for $x \in [0, 1]$, $w(x)$ is the probability of at least m successes out of l-1 independent Bernoulli trials, each of which occurs with probability x. Note that $w(\cdot)$ is continuous and strictly increasing in $[0, 1]$, so its inverse exists. Note that $w_i > w_j$ if $i < j, i, j \in \{1, ..., n\}$ as w_i is the success probability of at least m successes out of (l-1) independent Bernoulli Events, where each of which occurs with probability $\sum_{j=i}^{n} q_j$.

Lemma 2. *For* $1 \leq i \leq n$,

$$\begin{aligned} u_{i,max} &= p_i - c \\ \textit{where,} p_i &= c + (f_1(v) - c)(1 - w_1) \\ &\quad + \sum_{j=1}^{i-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \quad (11) \\ \textit{and } L_i &= g_i(\frac{p_i - c}{1 - w_{i+1}} + c) \quad (12) \end{aligned}$$

Using (11) and (12) $u_{i,max}, L_i$ can be computed recursively starting from $i = 1$.

Proof. We first prove (11) using induction. From theorem 3, $\psi_i(\cdot)$'s support set is $[L_i, L_{i-1}], i = \{2, ..., n\}$ and $[L_1, v]$ for $i = 1$. As v is the best response for channel state 1,

$$u_{1,max} = (f_1(v) - c)(1 - w(\sum_{i=1}^{n} q_i)) = p_1 - c \quad (13)$$

Thus, (11) holds for $i = 1$. Let, (11) be true for $i = t < n$. We have to show that (11) is satisfied for $i = t + 1$ assuming that it is true for $i = t$. Thus,by induction hypothesis,

$$\begin{aligned} u_{t,max} &= p_t - c = (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \quad (14) \end{aligned}$$

Now, L_t is a best response for state t, and

$$\phi_t(L_t) = (f_t(L_t) - c)(1 - w_{t+1}) = p_t - c \quad (15)$$

Now, as L_t is also a best response for state t+1,

$$\phi_{t+1}(L_t) = (f_{t+1}(L_t) - c)(1 - w_{t+1}) = u_{t+1,max} \quad (16)$$

Using (15), (14) in (16) we obtain-

$$\begin{aligned} u_{t+1,max} &= (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \\ &+ (f_{t+1}(L_t) - f_t(L_t))(1 - w_{t+1}) \\ &= (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \\ &= p_{t+1} - c \end{aligned}$$

Thus, $u_{t+1,max} = p_{t+1} - c$ and it satisfies (11). Thus, (11) follows from mathematical induction.

(12) follows since $f_i(L_i) - c)(1 - w_{i+1}) = p_i - c$ and $g_i(\cdot)$ is the inverse of $f_i(\cdot)$. □

Lemma 3. *A symmetric NE strategy profile* $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ *comprises of:*

$$\begin{aligned} \psi_i(x) = & 0, \textit{if } x < L_i \\ & \frac{1}{q_i}(w^{-1}(\frac{f_i(x) - p_i}{f_i(x) - c}) - \sum_{j=i+1}^{n} q_j), \textit{if } L_{i-1} \geq x \geq L_i \\ & 1, \textit{if } x > L_{i-1} \end{aligned}$$

Proof. L_i, L_{i-1} are the end-points of the support set of $\psi_i(\cdot)$ from definition, and their values have been computed in lemma 2. We should have for $x < L_i$, $\psi_i(x) = 0$ and for $x > L_{i-1}$,$\psi_i(x) = 1$ From theorem 3, every point $x \in [L_i, L_{i-1}]$ is a best response for state i, and hence,

$$(f_i(x) - c)(1 - w(\sum_{j=i+1}^{n} q_j + q_i.\psi_i(x))) = u_{i,max} = p_i - c.$$

Thus, the expression for $\psi_i(\cdot)$ follows. We conclude the proof noting that the range of $w(.)$ is $[0,1]$, and $\frac{p_i - c}{f_i(x) - c} < 1$ for $x \in [L_i, L_{i-1}]$: so $w^{-1}(.)$ is defined at $1 - \frac{p_i - c}{f_i(x) - c}$. □

It can be shown that $\psi_i(.)$ as defined in Lemma 3 is a strictly increasing and continuous distribution function [1].

C. Existence of a symmetric NE

We prove in [1] that

Theorem 4. *$(\psi_1(\cdot), \ldots, \psi_n(\cdot))\, j = 1, .., n$ as defined in lemma 3 is a symmetric NE.*

a) Remark: Note that all our results readily generalize to allow for random number of secondaries (M) with probability mass functions (p.m.f.) $\Pr(M = m) = \gamma_m$. A primary does not have the exact realization of number of secondaries, but it knows the p.m.f. . We only have to redefine $w(x)$ as-

$$\sum_{k=0}^{\max(M)} \gamma_k \sum_{i=k}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-1-i} \tag{17}$$

and $w_{n+1} = \gamma_0$.

IV. Performance evaluation of the symmetric NE

Definition 6. *Let R_{NE} denote the total expected profit at Nash equilibrium. Then,*

$$R_{NE} = l.\sum_{i=1}^{n} (q_i.(p_i - c)) \tag{18}$$

Lemma 4. *Let $c_j = g_j(c), j = 1, .., n$.*

1) *If $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$, then $R_{NE} \to l.\sum_{j=1}^{n} q_j.(f_j(v) - c)$ as $l \to \infty$.*
2) *If $(l-1)(\sum_{j=i-1}^{n} q_j - \epsilon) \geq m \geq (l-1)(\sum_{j=i}^{n} q_j + \epsilon), i \in \{2, .., n\}$, for some $\epsilon > 0$ $R_{NE} \to l.\sum_{j=i}^{n} q_j.(f_j(c_{i-1}) - c)$ as $l \to \infty$.*
3) *If $m \leq (l-1)(q_n - \epsilon)$ for some $\epsilon > 0$, then $R_{NE} \to 0$.*

We prove the above lemma in our technical report [1]. Note that, for $j > i$, $c_j < c_i$ (as $g_j(c) < g_i(c)$), thus, asymptotically R_{NE} decreases as m decreases. This is expected as competition increases with decrease in m, and thus prices are chosen progressively closer to the lower limit, that of the transition cost, c.

Definition 7. *Let R_{OPT} be the maximum expected profit earned through collusive selection of prices by the primaries.* Efficiency *η is defined as $\frac{R_{NE}}{R_{OPT}}$.*

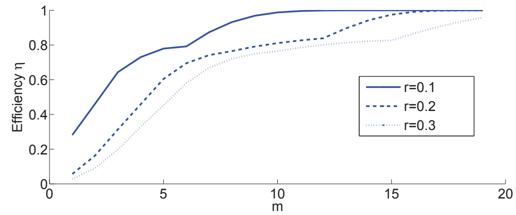

Fig. 1: Efficiency versus m for three different sets of values of probabilities for $l = 20$ and $n = 3$, $q_1 = q_2 = q_3 = r$, $v = 100, c = 1$, $g_i(x) = x^{10} - i^7$.

Efficiency is a measure of the reduction in the expected profit owing to competition. The following asymptotic limits on the efficiency can be obtained through an application of the weak law of large numbers [1]:

Lemma 5.
1) *If $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$, then $\eta \to 1$ as $l \to \infty$.*
2) *If $m \leq (l-1)(q_n - \epsilon)$ for some $\epsilon > 0$, then $\eta \to 0$ as $l \to \infty$.*

The lemma does not characterize the asymptotic limits for η for $m \in [(l-1)\sum_{j=1}^{n} q_j, (l-1)q_n]$. However, our numerical computation reveals that η increases from 0 to 1 with increase in m (figure 1) - the variation is largely monotonic barring for a few discrepancies owing to n, m being finite. Intuitively, demand increases with increase in m; thus primaries set their penalties close to the highest possible value for all states which leads to higher efficiency. On the other hand, when m decrease, competition becomes intense and primaries chooses prices close to c and expected profits under the symmetric NE decreases: R_{NE} is very small as lemma 4 reveals. But, if primaries collude, the decrease in R_{OPT} with decrease in m is slower, which leads to lower efficiency for low m.

V. Future Work

Generalizations that allow for different channel statistics for primaries and different utility functions for different secondaries constitute interesting directions for future research.

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