Universal Multiple Access via Spatially Coupling Data Transmission

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***Abstract*—We consider a signaling format where information is modulated via a superposition of independent data streams. Each data stream is formed by replication and permutation of encoded information bits. The relations between data bits and modulation symbols transmitted over the channel can be represented in the form of a sparse graph. The modulated streams are transmitted with time offsets enabling spatial coupling of the sparse modulation graphs. We prove that iterative demodulation based on symbol estimation and interference cancellation followed by parallel error correction decoding is a universal multiple access technique which achieves the entire capacity region of the additive white Gaussian noise (AWGN) multiple access channel.**

I. Introduction

Recently, the technique of spatial graph coupling attracted significant interest in many areas of communications. The method was first introduced to construct low-density parity-check convolutional codes (LDPCCCs) [1] that exhibit the so called *threshold saturation* behavior [2][3] which occurs when the limit (threshold) of the suboptimal iterative decoding of LDPCCCs asymptotically achieves the optimum maximum a posteriori probability (MAP) decoding threshold [3] of LDPC block codes with the same structure. The idea of constructing graph structures from connected identical copies of a single graph has since been applied to compressed sensing [20], image recognition, quantum coding [11], multi-user detection [13][19], and other fields.

In this work we consider a communication format in which a sequence of modulated symbols at the transmitter is formed as the sum of equal-power redundant independent data streams. A single data stream is constructed by encoding an information sequence with a binary error correction code, replicating each encoded bit a number of times and permuting the replicated bits. The main feature discussed in the paper is the data stream coupling accomplished by linear superposition of the data streams (in the real or complex domain) transmitted with time offsets by a single or multiple transmitters. The receiver needs to carry out two tasks: suppression of the inter-stream interference and error correction decoding. We prove that these tasks can be efficiently accomplished by iterative data stream layering and subsequent error correction decoding performed for all data streams in parallel.

The proposed modulation format is inspired by the multi-user modulation methods [8][9][10] in which parts of the users' direct-sequence spread data is also permuted, and has features in common with multi-level coding [5], bit interleaved coded modulation [6], superposition modulation [7], and repeat-accumulate codes [18]. Besides the similarities there are also some important differences. Contrary to higher order modulations spatially coupling modulation uses data streams of equal power which can be encoded by the same error-correction code and do not require multiple rate optimization. Contrary to bit-interleaved coded modulation, each data bit is not only permuted but also replicated. This allows to perform iterative data demodulation at the receiver prior to error correction decoding. Finally, while repeat-accumulate codes include differential encoding and utilize modulo-two bit addition, creating a convolutional outer code, spatially coupling modulation utilizes transmission of (pseudo-)random real domain sums of the replicated bits.

To find the data rate achieved by the spatially coupling modulation we study a system of recurrent equations describing the evolution of the noise-and-interference power experienced by the data streams throughout the demodulation iterations. We consider equal power and equal rate data streams and prove that the achievable data rate is within a small gap from the AWGN channel capacity and the gap disappears asymptotically as the signal-to-noise ratio (SNR) grows. This result is a counterpart to the result in [13] showing that spatially coupling modulation achieves the AWGN channel capacity with modified successive interference cancellation decoding. In this paper we show that it is sufficient to perform demodulation and decoding sequentially. No feedback loop is required between the two processes and the iterative demodulation can be efficiently pipelined. Data streams can originate at distinct terminals and, thus, the result proves that the capacity of the Gaussian multiple access channel (MAC) is achieved for equal power and equal rate data streams, a scenario which is the most difficult for multiple user detection [16].

Finally, we consider a two-user MAC in which the users can transmit using arbitrary power levels and each user employs spatially coupling transmission. We then prove that the entire capacity region of the two-user Gaussian MAC can be achieved by spatially coupling data transmission which is, therefore, a *universal multiple access technique*. It is interesting to note that for a particular case of two-user Gaussian MAC with binary inputs the universality has been demonstrated numerally in [12] using spatially coupled LDPC codes.

II. System Model

We consider a modulation technique in which the signal transmitted over the channel is formed by a superposition of

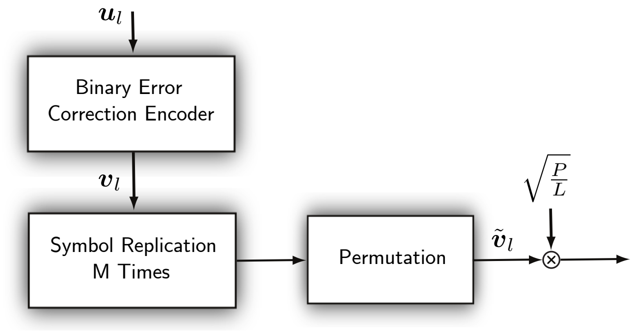

Fig. 1. Modulation of one data stream.

L independently modulated data streams. These streams may initiate at multiple terminals and superimpose at the receiver.

A schematic diagram of the modulator for the lth data stream is given in Fig. 1. First a binary information sequence $\boldsymbol{u}_l = u_{1,l}, u_{2,l}, u_{3,l}, \cdots, u_{K,l}$ enters a binary forward error correction encoder of rate $R = K/N$. Then each bit of the encoded binary sequence $\boldsymbol{v}_l = v_{1,l}, v_{2,l}, v_{3,l}, \cdots, v_{N,l}$, where $v_{j,l} \in \{-1, 1\}$, $j = 1, 2, \cdots, N$ is replicated M times and permuted producing a sequence $\tilde{\boldsymbol{v}}_l = \tilde{v}_{1,l}, \tilde{v}_{2,l}, \cdots, \tilde{v}_{MN,l}$[1].

We consider the case where L modulated data streams add up with time offsets defined as follows. Transmission of the first data stream starts at time $t = 1$. For the first τ_2 symbol time intervals the modulated signal consists of a single data stream. After τ_2 symbols transmission of the second data stream is initiated. For time instances $t \in [\tau_2 + 1, \tau_2 + \tau_3]$ the transmitted signal consists of a superposition (a sum) of two data streams. Then after a delay of τ_3 symbols the third data stream is also added and so on. Finally, at time $t = m$ the Lth data stream is added to the system. This process is illustrated in Figure 2 for the case of $L = 5$ streams and $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 1$. In each data stream a transmitted block of MN data symbols is immediately followed by the next block. We call this process of transmission initialization *stream coupling*.

The modulated signal $\boldsymbol{s} = (s_1, s_2, s_3, \cdots)$ is computed as the superposition

$$s_t = \sum_{l=1}^{L(t)} \tilde{v}_{t,l} \qquad t = 1, 2, \cdots , \tag{1}$$

where $L(t) \leq L$ denotes the number of data streams in the system at time t. Clearly $L(t) = L$ for $t \geq m$.

Each data stream is multiplied by the power normalizing amplitude $\sqrt{P/L}$ and transmitted over the channel. We start with considering equal power data streams. A generalization to unequal power case is discussed in Section IV. We consider transmission over a real-valued AWGN channel. Thus, the received signal equals

$$\boldsymbol{y} = \sqrt{\frac{P}{L}}\boldsymbol{s} + \boldsymbol{n} , \tag{2}$$

where $\boldsymbol{n} = (n_1, n_2, \cdots)$ is a noise vector with standard iid Gaussian components of mean zero and variance σ^2. The power of the modulated signal $s_t\sqrt{P/L}$ equals P since s_t in (1) is a sum of L independent binary random variables (for $t \geq m$) assuming the values -1 and 1 with probability $1/2$. Without loss of generality we can assume that $P = 1$. Therefore, the total normalized signal-to-noise ratio (SNR) is $1/\sigma^2$. The total transmit data rate equals $RL/M = \alpha R$ information bits per channel use. The ratio $\alpha = L/M$ is called the *modulation load*.

We notice that the proposed modulation format allows for a graph representation in which encoded data bits $v_{t,l}$ are represented by "variable" nodes which are connected to "channel" nodes representing modulation symbols s_t.

The received signal $\boldsymbol{y}$ contains M replicas of each transmitted bit $v_{j,l}$, $j = 1, 2, \cdots, N$, $l = 1, 2, \cdots, L$. Let $\mathcal{T}(j, l)$ denote the set of indices t such that the signal s_t, and therefore y_t, contains $v_{j,l}$. Moreover, by $\mathcal{J}(t)$ we denote the set of all index pairs (j, l) such that $v_{j,l}$ is included in y_t. For each bit $v_{j,l}$ we use the set of received signals $\{y_t\}_{t\in\mathcal{T}(j,l)}$ to form a vector $\boldsymbol{y}_{j,l}$. Since each y_t, $t \in \mathcal{T}(j, l)$ contains $v_{j,l}$ we have

$$\boldsymbol{y}_{j,l} = \boldsymbol{h}v_{j,l} + \boldsymbol{\xi}_{j,l} \tag{3}$$

where $\boldsymbol{h} = \sqrt{1/L}(1, 1, \cdots, 1)$. The vector $\boldsymbol{\xi}_{j,l} = (\xi_{j,l,t_1}, \xi_{j,l,t_2}, \cdots, \xi_{j,l,t_M})$, where $(t_1, t_2, \cdots, t_M) = \mathcal{T}(j, l)$ is the noise-and-interference vector with respect to the signal $v_{j,l}$. The components of this noise-and-interference vector are

$$\xi_{j,l,t} = \sqrt{\frac{1}{L}} \sum_{\substack{(j',l')\in\mathcal{J}(t) \\ \text{s.t. } (j',l')\neq(j,l)}} v_{j',l'} + n_t, \quad t = t_1, \cdots, t_M. \tag{4}$$

The Central Limit Theorem implies that the vector $\boldsymbol{\xi}_{j,l}$ converges to a Gaussian random vector with independent zero-mean components and covariance matrix $\boldsymbol{R}_{j,l} = \text{diag}(\sigma^2_{t_1}, \sigma^2_{t_2}, \cdots, \sigma^2_{t_M})$ as L increases. Here σ^2_t denotes the variance of $\xi_{j,l,t}$, $t \in \mathcal{T}(j, l)$. For $t > m$ the cardinality of the set $\mathcal{J}(t)$ is $|\mathcal{J}(t)| = L$ giving the variance $\sigma^2_t = (L - 1)/L + \sigma^2$ according to (4). However, since $|\mathcal{J}(t)| = L(t) < L$ for some $t \leq m$, and since the variances are influenced by the interference cancellation throughout the demodulation iterations, we prefer to use the expression based on σ^2_t.

We now perform a linear filtering on $\boldsymbol{y}_{j,l}$ (3) to form an SNR-optimal linear estimate of $v_{j,l}$, given by

$$z_{j,l} = \boldsymbol{w}_{j,l}\boldsymbol{y}_{j,l}, \qquad \text{where} \qquad \boldsymbol{w}_{j,l} = \boldsymbol{h}^*\boldsymbol{R}^{-1}_{j,l} . \tag{5}$$

The resulting SNR of the signal $z_{j,l}$ equals

$$\gamma_{j,l} = \boldsymbol{h}^*\boldsymbol{R}^{-1}_{j,l}\boldsymbol{h} = \frac{1}{L} \sum_{t\in\mathcal{T}(j,l)} \frac{1}{\sigma^2_t} . \tag{6}$$

Since $v_{j,l} \in \{1, -1\}$, and takes each of the two values with probability $1/2$, we can form a conditional expectation

[1]In a practical implementation of the technique each stream $\tilde{\boldsymbol{v}}_l$ may additionally be multiplied by a pseudo-random signature sequence $\tilde{\boldsymbol{q}}_l = \tilde{q}_{1,l}, \tilde{q}_{2,l}, \cdots, \tilde{q}_{MN,l}$ where $q_{i,l} \in \{-1, 1\}$ to facilitate convergence of the iterative demodulation and ensure that the modulated streams are uncorrelated. In this case the corresponding matched filters are included at the receiver.

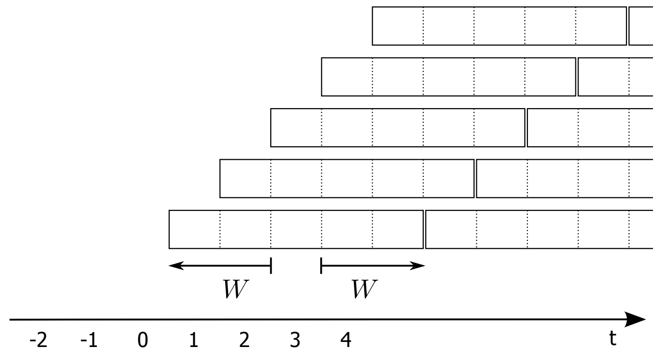

Fig. 2. Coupling of $L = 5$ modulated data streams with time offsets $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 1$. In this figure each data stream consists of blocks of $2W + 1$ sections, where $W = 2$.

estimate $\hat{v}_{j,l}$ of $v_{j,l}$ as

$$\hat{v}_{j,l} = \mathbb{E}(v_{j,l}|z_{j,l}) = \tanh(z_{j,l}) \ . \tag{7}$$

Once the estimates $\hat{v}_{j,l}$ are generated for all data bits $v_{j,l}$, $j = 1, 2, \cdots, N$, $l = 1, 2, \cdots, L$, the next demodulation iteration starts with an interference cancellation step performed by computing $\boldsymbol{y}^{(1)}_{j,l}$ with components

$$y^{(1)}_{j,l,t} = y_t - \sqrt{\frac{1}{L}} \sum_{\substack{(j',l') \in \mathcal{J}(t) \\ \text{s.t. } (j',l') \neq (j,l)}} \hat{v}_{j',l'}, \quad t \in \mathcal{T}(j,l) \ . \tag{8}$$

Therefore,

$$\boldsymbol{y}^{(1)}_{j,l} = \boldsymbol{h} v_{j,l} + \boldsymbol{\xi}^{(1)}_{j,l} \ , \tag{9}$$

where the components of $\boldsymbol{\xi}^{(1)}_{j,l}$ equal

$$\xi^{(1)}_{j,l,t} = \sqrt{\frac{1}{L}} \sum_{\substack{(j',l') \in \mathcal{J}(t) \\ \text{s.t. } (j',l') \neq (j,l)}} (v_{j',l} - \hat{v}_{j',l'}) + n_t, \ \ t \in \mathcal{T}(j,l).$$

We then proceed with calculating new estimates $\hat{v}^{(1)}_{j,l}$ repeating the same procedure that lead to (5) and (7), and the estimation and interference cancellation steps are repeated for a number of iterations. To avoid reusing the same information throughout the iterations, we compute M extrinsic vectors $\boldsymbol{y}^{(i)}_{j,l,t}$, $t \in \mathcal{T}(j,l)$ for each bit $v_{j,l}$ at iteration i. The vectors $\boldsymbol{y}^{(i)}_{j,l,t}$ are used to form M estimates $z^{(i)}_{j,l,t}$ according to (5) and corresponding $\hat{v}^{(i)}_{j,l,t}$ estimates according to (7), one for each replica of $v_{j,l}$ present in the signal $\boldsymbol{s}$.

We consider a two-stage decoding schedule. At the first stage I iterative demodulation iterations are performed as described above. The second stage comprises simultaneous decoding by the forward error correction codes used to encode the information into the data streams.

III. Performance Analysis

To simplify the derivation of the equations describing the evolution of the noise-and-interference power throughout the demodulation iterations we assume that each modulated data block of MN symbols consists of $2W + 1$ equal length subblocks. Assume that the time t is measured in subblocks. We also assume that L data streams are split into $2W+1$ equal size groups with $L/(2W+1)$ data streams in each group. The transmission of the first group of streams starts at $t = 1$, the transmission of the second group starts at $t = 2$ and so on. This situation is illustrated in Fig. 2 for the case of $W = 2$.

The noise-and-interference power at iteration i for symbols $v_{j,l}$ transmitted at time t is given by

$$x^t_i = \frac{1}{2W+1} \sum_{j=-W}^{W} g_{\mathrm{m}} \left(\frac{1}{\alpha} \frac{1}{2W+1} \sum_{l=-W}^{W} \frac{1}{x^{t+j+l}_{i-1}} \right) + \sigma^2 \ , \tag{10}$$

where $\quad g_{\mathrm{m}}(a) \stackrel{\text{def}}{=} \mathbb{E}\left[\left(1 - \tanh\left(a + \xi\sqrt{a}\right)\right)^2 \right], \ \xi \sim \mathcal{N}(0,1),$ and $\mathcal{N}(0,1)$ denotes a standard normal random variable. The recursion (10) was derived in [4] for a block-coupled multi-user system and it can be explained as follows.

By definition the function $g_{\mathrm{m}}(a)$ is a mean-squared error $\mathbb{E}|v_{j,l} - \hat{v}_{j,l}|^2$ of bit estimates $\hat{v}_{j,l}$ (obtained using the localy best estimator (7)) for a data stream with signal-to-noise ratio $a = \gamma_{j,l}$. The interference cancellation operation implies that the noise-and-interference power at time t consists of the mean square error contributions of data streams transmitted at times $t - 2W, t - 2W + 1, \cdots, t$ and the noise power σ^2, see Fig. 2. This gives the outer summation in (10). The M replicas of each data bit in a data stream transmitted at time t experience SNRs $1/(\alpha x^t_{i-1})$, $1/(\alpha x^{t+1}_{i-1}), \cdots \ 1/(\alpha x^{t+2W}_{i-1})$. We assume that $M \gg W$ and the replicas of each data bit are uniformly distributed through the $2W + 1$ sections of the data stream. Therefore, the SNR of the data stream is given by the argument of the function $g_{\mathrm{m}}(\cdot)$ in (10) as a result of the optimal combining of the replicas according to (7).

We assume that transmission starts at time $t = 1$. At every time instant t, the modulation load is increased by $\alpha/(2W+1)$ for $t = 1, 2, \cdots, 2W + 1$. As a result, the initial conditions for recursion (10) can be formulated as

$$x^t_0 = 0 \quad t \leq 0 \ , \tag{11}$$

$$x^t_0 = \frac{t}{2W+1} + \sigma^2; \quad t \in [1, 2W + 1] \tag{12}$$

$$x^t_0 = 1 + \sigma^2 \quad t > 2W + 1 \ . \tag{13}$$

where we assume that initially the MMSE of the data bits equals 1 since initially all data bits are unknown.

Following I demodulation iterations the residual noise-and-interference power of a data bit in a block transmitted at time t equals x^t_I. The SNR for the bits of a data stream transmitted at time t equals

$$\gamma^t_I = \frac{1}{\alpha(2W+1)} \sum_{j=0}^{2W} \frac{1}{x^{t+j}_I}$$

(see the argument of $g_{\rm m}(\cdot)$ in (10)). The subsequent error correction decoding performed at the second stage of the two-stage reception process is successful for the block transmitted at time t iff

$$\gamma_I^t > \theta, \tag{14}$$

where θ is the decoding threshold SNR of the external forward error correction codes used.

We note that the uncoupled system in which all data streams are transmitted simultaneously corresponds to the case $W = 0$ in (10). The evolution of the noise-and-interference power throughout the demodulation iterations for the uncoupled system is given by

$$x_i = g_{\rm m}\left(\frac{1}{\alpha x_{i-1}}\right) + \sigma^2 \ . \tag{15}$$

In the uncoupled case the modulation load equals α at all time instants t and all the data bits experience the same SNR equal to $1/(\alpha x_i)$ at iteration i, irrespective of t.

The uncoupled system (15) always converges to the largest root $x^{(3)}$ of

$$x = g_{\rm m}\left(\frac{1}{\alpha x}\right) + \sigma^2 \ , \tag{16}$$

while the coupled system (10) can for the same α converge to the smallest root $x^{(1)}$ of (16) which is, in turn, close to σ^2. Therefore, the coupled system is often capable of canceling all inter-stream interference while the uncoupled system is only capable of doing it for small loads α, for which $x^{(3)} = x^{(1)}$.

If the system (10) converges to the smallest root $x^{(1)}$ of (16) after a possibly infinite number of iterations, the SNR of the demodulated data bits equals $M/(Lx^{(1)}) = 1/(\alpha x^{(1)})$, since each bit has power $1/L$ and is replicated M times. Each replica experiences noise-and-interference power $x^{(1)}$ and the replicas are combined. The total communication rate (sum-rate) achievable by the system is, therefore,

$$\mathcal{R}(\alpha, \sigma^2) = \frac{L}{M}\mathcal{C}_{\rm BIAWGN}\left(\frac{M}{Lx^{(1)}}\right) = \alpha\mathcal{C}_{\rm BIAWGN}\left(\frac{1}{\alpha x^{(1)}}\right) \ ,$$

where $\mathcal{C}_{\rm BIAWGN}(a)$ denotes the capacity of the binary-input AWGN (BIAWGN) channel with SNR equal to a. The above expression assumes that the error correction codes used for each data stream are optimal for the BIAWGN channel with SNR $1/(\alpha x^{(1)})$ and have rate $R = \mathcal{C}_{\rm BIAWGN}\left(1/(\alpha x^{(1)})\right)$.

Here we recall that system (10) operates with total signal power 1 and noise power σ^2. The capacity of the (real-valued) AWGN channel for these parameters equals

$$\mathcal{C}(\sigma^2) \stackrel{\rm def}{=} \frac{1}{2}\log_2\left(1 + \frac{1}{\sigma^2}\right) \ .$$

We are now ready to state the main result of the paper. Let $\bar{\alpha}$ denote the maximum load for which the coupled system converges to $x^{(1)}$, i.e., for any $\alpha \le \bar{\alpha}$ the solution of the coupled system (10) satisfies $\lim_{i\to\infty} x_i^t = x^{(1)}$ for any $t > 0$. Note that $\bar{\alpha}$ is a function of σ^2 and W.

Theorem 1. *There exists a $\bar{W} > 0$ such that for any $W > \bar{W}$*

$$\mathcal{R}(\bar{\alpha}, \sigma^2) = \mathcal{C}(\sigma^2) - \frac{\frac{3}{2\ln 2}}{\mathcal{C}(\sigma^2)} + o\left(\frac{1}{\mathcal{C}(\sigma^2)}\right) \tag{17}$$

as $\sigma^2 \to 0$, and, therefore,

$$\lim_{\sigma^2\to 0}\left(\mathcal{C}(\sigma^2) - \mathcal{R}(\bar{\alpha}, \sigma^2)\right) = 0 \ .$$

Proof. For complete proof see [21]. First we represent the system (15) (shifted by $x^{(1)}$) in the form $x_i = f(g(x_{i-1}), \alpha)$

$$\text{where} \quad f(x,\alpha) \stackrel{\rm def}{=} g_{\rm m}\left(\frac{1}{\alpha}\left(\frac{1}{x^{(1)}} - x\right)\right) + \sigma^2 - x^{(1)} \ ,$$

$$g(x) \stackrel{\rm def}{=} \frac{1}{x^{(1)}} - \frac{1}{x^{(1)} + x} \ ,$$

and construct a potential function $U(x, \alpha)$ characterizing (10) according to the method introduced in [17] and [19]

$$\begin{aligned} U(x,\alpha) &\stackrel{\rm def}{=} \int_0^x (z - f(g(z),\alpha))\, g'(z) dz \qquad (18)\\ &= \ln\frac{x + x^{(1)}}{x^{(1)}} - \frac{\sigma^2 x}{x^{(1)}(x + x^{(1)})} - \alpha\int_{\frac{1}{\alpha(x+x^{(1)})}}^{\frac{1}{\alpha x^{(1)}}} g_{\rm m}(y) dy \end{aligned}$$

The potential function characterizes the convergence of the coupled system in a sense that $\min_x U(x,\alpha) > 0$ implies convergence of (10) to $x^{(1)}$ (see [15]).

The following proof steps involve an analysis of $U(x, \alpha)$ and estimation of $\bar{\alpha}$, such that $\min_x U(x,\alpha) > 0$ for all $\alpha < \bar{\alpha}$ and $U(x, \bar{\alpha}) = 0$ for some value of the argument x.

Finally, we compute $\mathcal{R}(\bar{\alpha}, \sigma^2)$ using a relationship between the MMSE given by $g_{\rm m}(\cdot)$ and the capacity $\mathcal{C}_{\rm BIAWGN}(\cdot)$ found in [14] and prove that $\mathcal{R}(\bar{\alpha}, \sigma^2)$ satisfies (17). □

IV. Universal Multiple Access

Consider now two users, one transmitting L_1 data streams with total power P_1 and the other, transmitting L_2 streams with total power P_2. The power of the AWG channel noise equals σ^2. Both users employ repetition factor M. Thus, the modulation loads of the users equal $\alpha_1 = L_1/M$ and $\alpha_2 = L_2/M$ respectively. Generalization of the approach to a higher number of users is straightforward.

The noise-and-interference power at iteration i for symbols transmitted at time t is given by

$$\begin{aligned} x_i^t = &\frac{P_1}{2W+1}\sum_{j=-W}^{W} g_{\rm m}\left(\frac{1}{\alpha_1}\frac{P_1}{2W+1}\sum_{l=-W}^{W}\frac{1}{x_{i-1}^{t+j+l}}\right) + \sigma^2 \\ &+ \frac{P_2}{2W+1}\sum_{j=-W}^{W} g_{\rm m}\left(\frac{1}{\alpha_2}\frac{P_2}{2W+1}\sum_{l=-W}^{W}\frac{1}{x_{i-1}^{t+j+l}}\right) \end{aligned} \tag{19}$$

where W is the coupling window parameter introduced above. The potential function (18) of the system (19) equals

$$\begin{aligned} U(x,\alpha_1,\alpha_2) = &\ln\frac{x + x^{(1)}}{x^{(1)}} - \frac{\sigma^2 x}{x^{(1)}(x + x^{(1)})} \qquad (20)\\ &- \alpha_1\int_{\frac{P_1}{\alpha_1(x+x^{(1)})}}^{\frac{P_1}{\alpha_1 x^{(1)}}} g_{\rm m}(y) dy - \alpha_2\int_{\frac{P_2}{\alpha_2(x+x^{(1)})}}^{\frac{P_2}{\alpha_2 x^{(1)}}} g_{\rm m}(y) dy \ , \end{aligned}$$

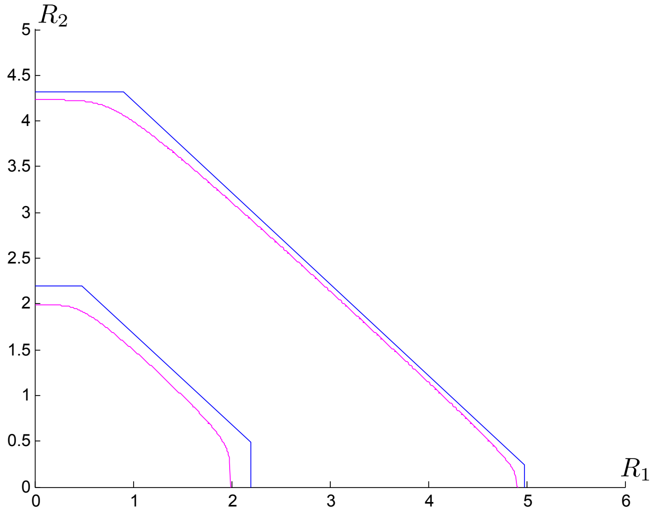

Fig. 3. Capacity regions for two two-user MACs (blue) and the corresponding regions achievable by the coupled transmission (magenta).

where $x^{(1)}$ is the smallest root of

$$x = P_1 g_{\mathrm{m}} \left(\frac{P_1}{\alpha_1 x} \right) + P_2 g_{\mathrm{m}} \left(\frac{P_2}{\alpha_2 x} \right) + \sigma^2 \ . \qquad (21)$$

The capacity regions of two multiple access channels (blue polytopes) and the corresponding rate regions achievable by the coupling transmission (magenta curves) are given in Fig. 3. The inner region corresponds to a channel with $P_1 = P_2 = 0.2$ and the outer one is the channel with $P_1 = 10$ and $P_2 = 4$. The noise power $\sigma^2 = 0.01$ in both cases. We can see that the regions achievable by the coupling transmission are close to the MAC capacity regions. The achievable region becomes closer to the capacity region as the SNRs of the users increase.

Theorem 2. *Consider arbitrary user powers P_1 and P_2. Then for any $\epsilon > 0$ there exists a sufficiently small σ^2 and a sufficiently large W such that for any point $(\mathcal{C}_1(\sigma^2), \mathcal{C}_2(\sigma^2))$ of the capacity region of the MAC with AWGN power σ^2 there exist modulation loads $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that the achievable rate point $(\mathcal{R}_1(\bar{\alpha}_1, \bar{\alpha}_2, \sigma^2), \mathcal{R}_2(\bar{\alpha}_1, \bar{\alpha}_2, \sigma^2))$ satisfies*

$$\frac{\mathcal{R}_1(\bar{\alpha}_1, \bar{\alpha}_2, \sigma^2)}{\mathcal{C}_1(\sigma^2)} > 1 - \epsilon \quad \textit{and} \quad \frac{\mathcal{R}_2(\bar{\alpha}_1, \bar{\alpha}_2, \sigma^2)}{\mathcal{C}_2(\sigma^2)} > 1 - \epsilon \ .$$

Proof. The proof follows the steps of Theorem 1. An analysis of the function $U(x, \alpha_1, \alpha_2)$ (20) revels the region of load points $(\bar{\alpha}_1, \bar{\alpha}_2)$ for which the potential function is positive. Computation of the rates achievable at the region's boundary gives the desired result. □

V. Conclusion

We consider modulation of information in a form a superposition of independent equal power and equal rate data streams. Each stream is formed by repetition and permutation of data and the streams are added up with an offset initiating the effect of "stream coupling". We prove that the proposed system used with iterative demodulation followed by external error control decoding achieves the capacity of the AWGN channel and the entire capacity region of the Gaussian multiple access channel.

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