Large Deviation Approach to the Outage Optical MIMO Capacity

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***Abstract*—MIMO processing techniques in fiber optical communications have been proposed as a promising approach to meet increasing demand for information throughput. In this context, the multiple channels correspond to the multiple modes and/or multiple cores in the fiber. The lack of back-scattering necessitates the modeling of the transmission coefficients between modes as elements of a unitary matrix. Also, due to the scattering between modes the channel is modeled as a random Haar unitary matrix between N_t transmitting and N_r receiving modes. In this paper, we apply a large-deviations approach from random matrix theory to obtain the outage capacity for Haar matrices in the low outage limit, which is appropriate for fiber-optical communications. This methodology is based on the Coulomb gas method for the eigenvalues of a matrix developed in statistical physics. By comparing our analytic results to simulations, we see that, despite the fact that this method is nominally valid for large number of modes, our method is quite accurate even for small to modest number of channels.**

***Index Terms*—Optical fiber transmission, MIMO, outage capacity, random matrix theory**

I. Introduction

THE ongoing exponential growth in wire-line data traffic is primarily driven by high-bandwidth digital applications, such as video-on-demand, cloud computing and telepresence. As a result, it is expected that the currently deployed infrastructure will soon reach its limits, leading to the so-called "capacity crunch" [1]. To counter this trend, scientists have been working towards exhausting all available degrees of freedom of fiber-optical transmission, including the bandwidth (through WDM modulation), available power (subject to power constraints imposed by non-linearities), and polarization diversity [2]. One additional possibility to increase throughput is spatial modulation, which would allow multiple transmission within the same fiber or fiber bundle. This can be achieved by designing multi-mode (MMF) and/or multi-core fibers (MCF). While such fibers may be constructed to have small cross talk between the propagating modes [3], in the typical situation cross-talk between fiber modes cannot be neglected, for reasons related to increased length of the fiber segment [4], as well as the twist and the bending of the fiber [5], [6]. In addition, the increase of the number of modes necessarily will bring the modes closer in space, resulting to higher cross-talk, which can lead to the power being spread equally between modes of the fiber [7]. As a result, strong coupling between cores needs to be addressed with a different approach.

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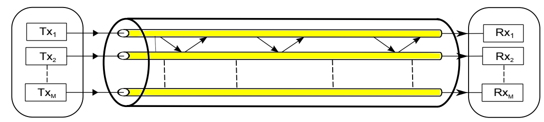


Fig. 1: Optical SDM using M parallel transmission paths in MCF. There is crosstalking between adjacent cores.

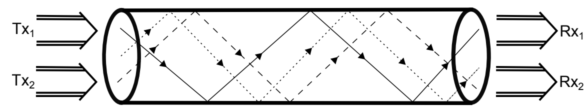


Fig. 2: Light beam scattering paradigm in MMF.

Recently, [2], [8], [9] it has been proposed to use techniques developed in the context of wireless communications between multiple transmitting and receiving antennas (MIMO). As it is well known in wireless multi-antenna systems the crosstalk is significant and yet sophisticated techniques developed there can be used to reach large information throughputs. Of course, the application of MIMO methodology to optical fiber multi-core systems has certain differences which need to be addressed. First, while the channel matrix for wireless MIMO is typically assumed to be complex Gaussian, this is not usually the case for fibers. It has been suggested instead that the channel matrix should be a subset of a unitary matrix [2], [10]. Second, due to the existence of non-linearities at high powers, one should specifically have in mind low to moderate powers per channel. Hence, the asymptotic behavior appearing in the limit of large powers [11] is not valid. Finally, the practical metric for the performance is not the ergodic mutual information, but, rather, instead the outage capacity at very low outage (e.g. 10^{-4}) [2], due to the fact that feedback from the receiver to the transmitter is almost always impossible.

In this paper, we analyze the outage capacity of an optical MIMO channel. As mentioned above, this is the relevant information transmission metric for fiber-optical coupled multi-core channels when the crosstalk varies over time slowly enough, so as not to be able to code around it. We obtain analytical expressions, which are valid technically in the limit of large channel numbers, but also works well over smaller channel numbers. It is particularly suited to obtain outage mutual information for very low outages with finite SNR. Essentially, it amounts to calculating the rate function of the logarithm of the average moment generating function of the mutual information. The methodology we use is based on the so-called Coulomb gas approach which was developed in the physics literature in the context of random matrix theory [12] in the 60's. It is quite intuitive because it interprets the eigenvalues as point charges on a line repelling each other logarithmically. The Coulomb gas method has seen recently a renewed interest in its use to obtain large deviations results for random matrix problems [13], [14] and also in communications [15], [16]. We will follow the basic steps discussed in more details in [15].

A. *Outline*

In the next section we will define the system model, and show that that the appropriate channel matrix is a random Haar unitary matrix. In Section III we introduce the mathematical methodology of the Coulomb gas and provide our analytic results. In addition, we compare to numerical simulations. Finally, in Section IV we conclude.

II. System Model

In this paper we consider a single-segment N-channel lossless optical fiber system, with $N_t \leq N$ transmitting channels excited and $N_r \leq N$ receiving channels coherently excited in the input (left) and output (right) side of the fiber. The propagation through the fiber may be analyzed through its $2N \times 2N$ scattering matrix given by [2], [10]

$$\mathbf{S} = \begin{bmatrix} \mathbf{r}_t & \mathbf{t} \\ \mathbf{t}^T & \mathbf{r}_r \end{bmatrix} \tag{1}$$

Due to time-reversal symmetry this matrix is a complex unitary *symmetric* matrix (with $\mathbf{S} = \mathbf{S}^T$) [17]. The $N \times N$ blocks on the diagonal $\mathbf{r}_t$, $\mathbf{r}_r$ represent the reflection matrices of the left ingoing and right ingoing channels. In the case of optical fiber transmission we are interested only in transmission from the left to right. Also proper manipulation of the input through the use of optical isolators results to no backscattering, i.e. $\mathbf{r}_t = \mathbf{r}_r = 0$. Assuming strong scattering between right-moving channels, as discussed above, we may model $\mathbf{t}$ as a complex Haar-distributed matrix with $\mathbf{t}^\dagger\mathbf{t} = \mathbf{t}\mathbf{t}^\dagger = \mathbf{I}_N$. We are particularly interested in a segment of this matrix corresponding to the N_t columns and the N_r rows, which are coupled to the transceiver. We denote this $N_r \times N_t$ matrix $\mathbf{U}$ and without loss of generality we assume it is the upper left corner of $\mathbf{t}$. As a result, the corresponding MIMO channel for this system reads

$$\mathbf{y} = \mathbf{U}\mathbf{x} + \mathbf{z} \tag{2}$$

with coherent detection and channel state information only at the receiver [18], [19]. $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are the $N_t \times 1$ input, the $N_r \times 1$ output signal vectors and the $N_r \times 1$ unit variance noise vector, respectively, all assumed for simplicity to be complex Gaussian. We also assume no differential delays between channels, which effectively leads to frequency flat fading [2]. We also assume no mode-dependent loss. As a result, the mutual information can be expressed as

$$\begin{aligned} I_N(\mathbf{U}) &= \log\det(I + \rho\mathbf{U}^\dagger\mathbf{U}) \\ &= \sum_{k=1}^{N_t} \log(1 + \rho\lambda_k) \end{aligned} \tag{3}$$

where "log" is the natural logarithm, ρ is the average total signal-to-noise ratio, λ_k are the eigenvalues of the matrix $\mathbf{U}^\dagger\mathbf{U}$ and we assume for concreteness $N_t \leq N_r$.

III. Methodology

We are now able to define the main problem we address here, namely the calculation of

$$\begin{aligned} P_{out}(r) &= Prob(I_N/N_t < r) && (4) \\ &= E_{\mathbf{U}}\left[\Theta(N_t r - I_N(\mathbf{U}))\right] && (5) \end{aligned}$$

where $\Theta(x)$ is the indicator (step) function. We will first analyze the density of r i.e.

$$P(r) = P'_{out}(r) = E_{\mathbf{U}}\left[\delta(r - I_N(\mathbf{U})/N_t)\right] \tag{6}$$

Due to space constraints, we will focus on the case $N_t + N_r < N$ and expand on other parameter regions in a longer version. It is useful to define $\beta = N_r/N_t > 1$, $N_0 = (N - N_t - N_r)/N_t$ and $n_0 = N_0/N_t$ The first step is to express the joint distribution of eigenvalues of $\mathbf{U}^\dagger\mathbf{U}$ as derived initially in [20] and more recently in this context [10]

$$\begin{aligned} P_\lambda(\lambda_1 ... \lambda_{N_t}) &\propto \prod_{n<m} |\lambda_n - \lambda_m|^2 \prod_k \lambda_k^{|N_t - N_r|}(1 - \lambda_k)^{N_0} \\ &= A_N e^{-N_t^2 E(\lambda)} \end{aligned} \tag{7}$$

where A_N is a normalization constant and $E(\lambda)$ is an energy function of the eigenvalues λ_i. $E(\lambda)$ represents the potential energy of N_t unit charges bound to move on a the unit interval $x \in (0,1)$, while repelling each other and from the boundaries logarithmically. It is reasonable to assume that when N is large then they will coalesce to a smooth density $p(x)$, which will be such that the energy $E(\lambda) = \mathcal{E}[p]$ will be minimum. Since the above equation is in fact a probability distribution, this will correspond to the most probable eigenvalue distribution. In this continuum limit we can rewrite the minimum energy of the eigenvalues/coulomb gas charges as

$$\begin{aligned} \mathcal{E}_0 &= \sup_c \inf_p \mathcal{L}_0[c, p] \qquad (8) \\ \mathcal{L}_0 &= -n_0 \int p(x)\log(1 - x)dx \\ &\quad -(\beta - 1)\int p(x)\log(x)dx \\ &\quad -\iint p(x)p(y)\log|x - y|dydx \\ &\quad -c\left(\int p(x)dx - 1\right) \end{aligned}$$

where we have added a Lagrange multiplier c to ensure that $p(x)$ is properly normalized, while implicitly we assume that $p(x)$ is continuous in $x \in (0,1)$. The minimization of $\mathcal{L}_0$ with respect to p can be done by taking its functional derivative and setting it to zero. More technical details about this can be found in [15]. Following this methodology, we can show that the local minimum of $\mathcal{L}_0$ is unique in the space of continuous $L^{1+\epsilon}$-integrable functions of $p(x)$, where $x \in (0,1)$.

As we will see from the analysis later on, the density function minimizing $\mathcal{L}_0$ is

$$p_0(x) = \frac{\sqrt{(x-a)(b-x)}}{2\pi x(1-x)} \tag{9}$$

where $a, b = (\sqrt{1+n_0} \pm \sqrt{\beta(n_0+\beta)})^2/(n_0+1+\beta)$ obtained using other methods in [20], [21]. We now move on to incorporate the constraint on the rate r by introducing a new Lagrange multiplier so that

$$\mathcal{E}(r) = \sup_{c,k} \inf_{p} \mathcal{L}[c,k,p] \tag{10}$$

$$\mathcal{L} = \mathcal{L}_0 - k\left(\int p(x)\log(1+\rho x)dx - r\right)$$

The minimum of the above function is also unique since we can extract $\mathcal{E}_0$ from $\mathcal{E}(r)$ maximizing over $\mathcal{L}$ while keeping k = 0 [15]. The extra term above, proportional to k, has the intuitive explanation of an additional force acting on the charges-eigenvalues towards the left or the right, depending on the sign of k. Taking the functional derivative with respect to $p(x)$ and setting to zero and differentiating with respect to x, provides us with the following integral equation representing a balance of forces on the charge located at x.

$$2\mathcal{P}\int_a^b \frac{p(x')}{x-x'}dx' = \frac{n_0}{1-x} - \frac{\beta-1}{x} - \frac{k\rho}{1+\rho x} \tag{11}$$

where $\mathcal{P}$ represents the Cauchy principle value of the integral and $a, b \in (0,1)$ the limits of support of p. Following Tricomi's theorem [15], [22] this integral equation may be solved to give a unique solution. For the values $\beta > 1$ and $n_0 > 0$ we get

$$p(x) = \frac{\sqrt{(x-a)(b-x)}}{2\pi(1+\rho x)}\left(\frac{n_0(\rho+1)}{(1-x)\sqrt{(1-a)(1-b)}} + \frac{\beta-1}{x\sqrt{ab}}\right) \tag{12}$$

with the additional constraint (stemming from the continuity constraints $p(a) = p(b) = 0$)

$$\frac{n_0}{\sqrt{(1-a)(1-b)}} = \frac{\beta-1}{\sqrt{ab}} + \frac{k\rho}{\sqrt{(1+\rho a)(1+\rho b)}} \tag{13}$$

The parameters a, b, k can be evaluated uniquely from the above equation in addition to the normalization constraint $\int dx p(x) = 1$

$$n_0 + \beta + 1 + k = \frac{\beta-1}{\sqrt{ab}} + \frac{k(1+\rho)}{\sqrt{(1+\rho a)(1+\rho b)}} \tag{14}$$

and the rate constraint $\int dx p(x)\log(1+\rho x) = r$

$$\begin{aligned} r = \log\frac{\Delta}{z} &+ \frac{k\Delta}{2\sqrt{(z+a)(z+b)}}\left[G\left(\frac{a+z}{\Delta}, \frac{a+z}{\Delta}\right)\right. \\ &\left.- G\left(\frac{a+z}{\Delta}, -\frac{1-a}{\Delta}\right)\right] + \frac{(\beta-1)\Delta}{2\sqrt{ab}}\left[G\left(\frac{a+z}{\Delta}, \frac{a}{\Delta}\right)\right. \\ &\left.- G\left(\frac{a+z}{\Delta}, -\frac{1-a}{\Delta}\right)\right] \end{aligned} \tag{15}$$

where $\Delta = b - a$ and $z = \frac{1}{\rho}$. The function $G(x,y)$ defined for $x > 0$ and $y > 0$ or $y < -1$ is given by [15]

$$\begin{aligned} G(x,y) &= \frac{1}{\pi}\int_0^1 \sqrt{t(1-t)}\frac{\log(t+x)}{t+y}dt \\ &= -2\text{sgn}(y)\sqrt{|y(1+y)|}\log\left[\frac{\sqrt{|x(1+y)|}+\sqrt{|y(1+x)|}}{\sqrt{|1+y|}+\sqrt{|y|}}\right] \\ &+ |1+2y|\log\left[\frac{\sqrt{|1+x|}+\sqrt{|x|}}{2}\right] - \frac{1}{2}\left(\sqrt{|1+x|}-\sqrt{|x|}\right)^2 \end{aligned} \tag{16}$$

We may now integrate over $p(x)$ and obtain an expression for $\mathcal{E}(r)$ as follows

$$\begin{aligned} \mathcal{E}(r) = &-\frac{\log\Delta}{2}(n_0+\beta+1) \\ &- \frac{k\Delta}{2z\sqrt{(z+a)(z+b}}\left[\frac{n_0}{2}\left(G\left(\tilde{\phi},\tilde{\phi}\right) - G\left(\tilde{\phi},-\tilde{y}\right)\right)\right. \\ &+ \frac{\beta-1}{2}\left(G(w,y) - G(w,-\phi)\right) \\ &\left.+ G(0,y) - G(0,-\phi)\right] \\ &- \frac{(\beta-1)\Delta}{2\sqrt{ab}}\left[\frac{n_0}{2}\left(G\left(\tilde{\phi},\tilde{\phi}\right) - G\left(\tilde{\phi},-\tilde{w}\right)\right)\right. \\ &+ \frac{(\beta-1)}{2}\left(G(w,w) - G(w,\phi)\right) \\ &\left.+ G(0,w) - G(0,-\phi)\right] \\ &+ \frac{k}{2}\left(r - \log\left(1+\frac{a}{z}\right)\right) \\ &- \frac{n_0}{2}\log(1-a) - \frac{(\beta-1)}{2}\log a \end{aligned} \tag{17}$$

where $\phi = \frac{1-a}{\Delta}, \tilde{\phi} = \phi - 1, w = \frac{a}{\Delta}, \tilde{w} = w + 1, y = \frac{z+a}{\Delta}, \tilde{y} = y + 1$. $\mathcal{E}_0$ can be evaluated, e.g. from the above by setting $k = 0$.

A. Probability Distributions $P(r)$ and $P_{out}(r)$

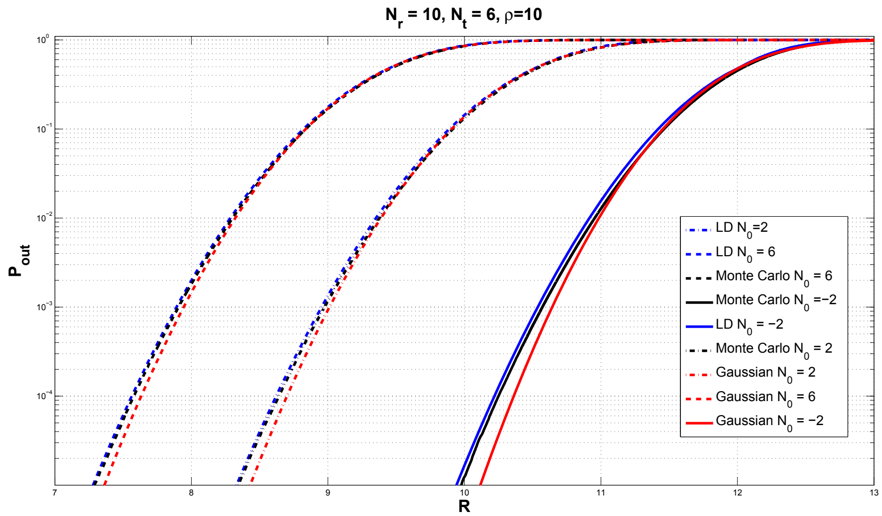
We may now obtain $P(r)$ from the above to leading exponential order, by properly normalizing $e^{-N_t^2\mathcal{E}(r)}$ using the fact that due to exponential nature in N the bulk of the distribution will be centered around the minimum of $\mathcal{E}(r)$ with respect to r which occurs at $k = 0$, which corresponds to Gaussian approximation. Hence

$$P(r) \approx N_t \frac{e^{-N_t^2(\mathcal{E}(r)-\mathcal{E}_0)}}{\sqrt{2\pi v_{erg}}} \tag{18}$$

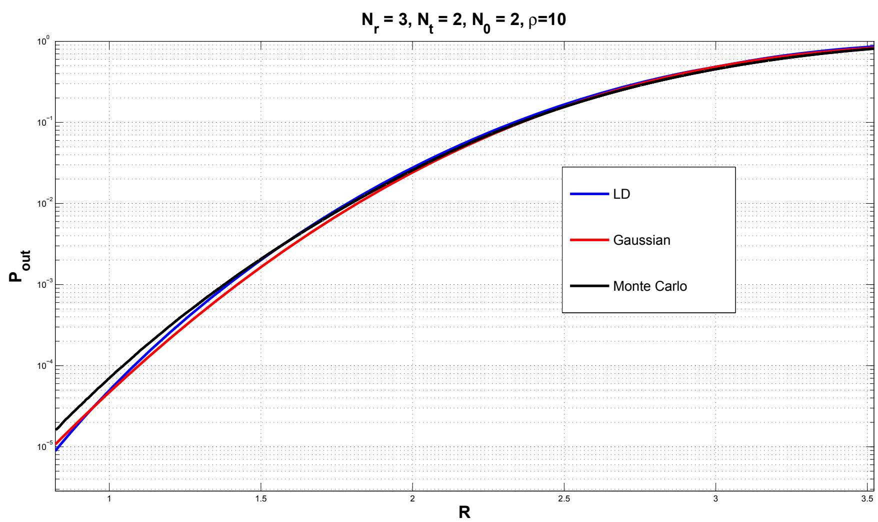
where $v_{erg} = 1/\mathcal{E}''(r_{erg})$ is the variance at the peak of the distribution, and r_{erg} [23] is the solution of (15) for $k = 0$ corresponding to the ergodic rate.

The outage probability may also be obtained in a similar fashion [15]. For $r < r_{erg}$ the outage probability is

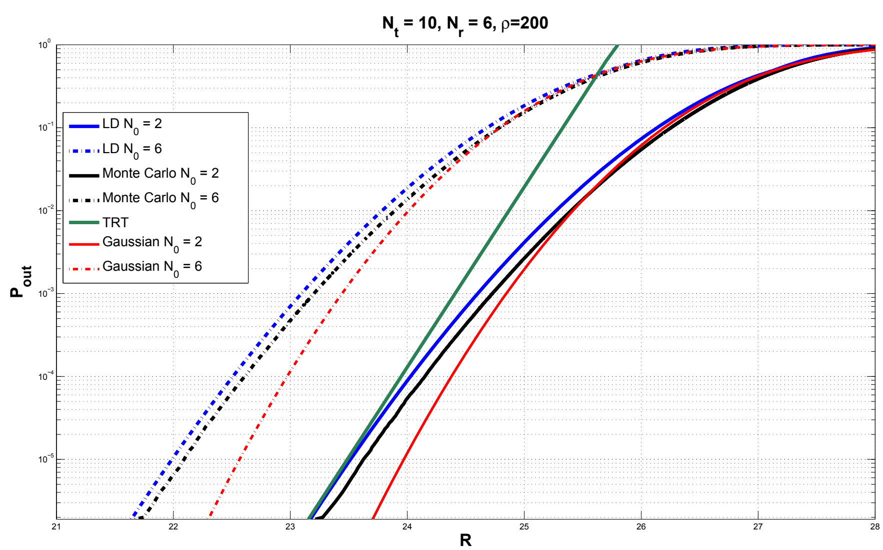
$$P_{out}(r) \approx \frac{e^{-N^2[\mathcal{E}_1(r)-\mathcal{E}_0-\frac{\mathcal{E}_1'(r)^2}{2\mathcal{E}_1''(r)}]}Q\left(\frac{N|\mathcal{E}_1'(r)|}{\sqrt{\mathcal{E}_1''(r)}}\right)}{\sqrt{\mathcal{E}_1''(r)v_{erg}}} \tag{19}$$



(a) Outage probability curves for different values of N_0 We observe that, generally,the Gaussian curves fail to follow the respective Monte Carlo, while the LD curves are closer to them.



(b) For small values of ρ and N_r, N_t, N_0 the discrepancy between the different methods is almost insignificant.



(c) At the extreme case of very high SNR, the discrepancy between the different methods is large. The TRT curve can successfully follow the LD and Monte Carlo curve only for low outage probabilities .

Fig. 3: Simulation results

and for $r > r_{erg}$ it is

$$P_{out}(r) \approx 1 - \frac{e^{-N^2[\mathcal{E}_1(r)-\mathcal{E}_0-\frac{\mathcal{E}_1'(r)^2}{2\mathcal{E}_1''(r)}]} Q\left(\frac{N|\mathcal{E}_1'(r)|}{\sqrt{\mathcal{E}_1''(r)}}\right)}{\sqrt{\mathcal{E}_1''(r)v_{erg}}} \quad (20)$$

where $\mathcal{E}_1'(r)$ and $\mathcal{E}_1''(r)$ are the first and second derivative of $\mathcal{E}_1(r)$ with respect to r and $Q(x) = \int_x^\infty dte^{-t^2/2}/\sqrt{2\pi}$.

IV. Numerical Simulations

We have performed a series of numerical simulations and have compared the *Large Deviation* (LD) approach to *Gaussian* approximation and *Monte Carlo* simulation. We start with the case of small ρ and for different values of N_t, N_r and N_0. In the figures we plot the outage probability according to the LD, Gaussian and Monte Carlo approach, for $r < r_{erg}$. The curves are for different values of ρ and different realisations of MIMO systems. It is visible that the P_{out} from the Gaussian simulation induces significant deviation from the reality (Monte Carlo) at the "tails" and the LD approach shows much better behavior. For comparison in large SNR (ρ = 200) we have inserted the *Throughput Reliability Tradeoff* (TRT) curve. [24] This is a piecewise linear function (with $k < R/\log(\rho) \leq k+1$ of the outage probability, which has asymptotically the correct $\log(\rho)$ slope but not the offset.

$$\begin{aligned} \log P_{out} &\approx c(k)R - g(k)\log\rho \quad (21)\\ c(k) &= M + N - 2k - 1 \\ g(k) &= MN - k(k+1) \end{aligned}$$

V. Conclusion

In this paper we have used a large deviations approach first introduced in physics [12] to calculate the outage capacity for the optical MIMO channel in the limit of large channel numbers. Our method is especially applicable for the tails of the distribution, which is relevant for low outage requirements due to the absence of feedback and finite SNR. Our analytical results agree very well with numerical experiments. Additionally the method provides the distribution of eigenvalues constrained on the transmission rate and SNR. Although the channel assumptions taken here (random unitary without losses) are somewhat idealized, this result gives an analytic metric to compare with other more complicated channel models.

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