

Building Consensus via Iterative Voting

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Abstract—In networked systems comprised of many agents, it is often required to reach a common operating point of all agents, termed the *network consensus*. We consider two iterative methods for reaching a ranking (ordering) consensus over a voter network, where the initial preference of every voter is of the form of a full ranking of candidates. The voters are allowed, one at a time and based on some random scheme, to change their votes to bring them “closer” to the opinions of selected subsets of peers. The first consensus method is based on changing votes one adjacent swap at a time; the second method is based on changing votes via averaging with the votes of peers, potentially leading to many adjacent swaps at a given time. For the first model, we characterize convergence points and conditions for convergence. For the second model, we prove convergence to a global ranking and derive the rate of convergence to this consensus.

I. INTRODUCTION

In the past two decades, the theory of rank aggregation and social choice has received renewed attention because of its wide-ranging applications, including recommender systems (collaborative filtering), meta-search engines, decision making among humans or automated agents, sports and other competitions, and elections [6], [9], [13], [14], [17].

Rank aggregation refers to a class of methods for collecting preferences and collectively making decisions based upon them. In particular, every voter has a vote, which may be a full (or partial) ranking of candidates based on the voter’s preference. A unique voters’ representative full (or partial) ranking of the candidates is determined based upon some decision rule. There exist many decision rules for vote aggregation, including the plurality rule, Borda’s scoring method, and the Kemeny-Young approach [2], [3].

In most voting decision rules, it is assumed that the aggregate is formed in one step, without allowing the voters to adapt their rankings to the rankings of other voters. However, in many applications, a different approach – dynamic voting over networks – may be required or may actually represent a more realistic model of social opinion aggregation. One may regard this dynamic procedure as *iterative voting*, where voters are allowed to change their preference rankings based on fully or partially observed rankings of other voters. An equilibrium point is reached when no voter seeks to change her vote. This approach is particularly useful for planning in autonomous systems, consensus decision making among humans in social networks, and modeling the evolution of preferences of individuals. Examples include *doodle* and *whenisgood* event schedulers, and consensus decision making in Wikipedia [20].

We consider the problem of reaching consensus in iterative voting systems wherein every voter assigns its initial preference in the form of a full ranking of all candidates. Information about preferences of voters is disseminated by either public announcements or by communicating along the edges of a given

connectivity graph. Based on the information shared by other network nodes, voters are allowed to change their rankings one step at a time using a random protocol, so as to increase general agreement in the group. Notice that the goal of the iterative exchange of information is to achieve consensus on the ranking of candidates, thereby eliminating the need to track quantitative information of the voters.

There exists a number of rank aggregation methods that may be considered in an iterative setting and for which voters may be assumed to communicate within a networked environment. We focus on two such models that fall into two conceptually different classes of aggregation methods, namely distance-based and score-based rules. Distance-based aggregation amounts to finding the median of a set of rankings, according to a predefined distance function. Score-based aggregation is based on the simple idea that each position in a ranking has a score associated with it, which may be used to assess the global rank of a candidate. Furthermore, to illustrate the full generality of the class of iterative voting problems, we consider two different voter communication strategies.

The first model is based on pairwise comparisons, and it falls into the family of so called Condorcet methods. Condorcet methods are aggregation rules that assign the candidate that wins every pairwise comparison the highest rank. This model can be viewed as a distributed aggregation model in which the underlying distance measure is the Kendall tau distance [2]. In this scheme, each voter starts with a preference ranking of all candidates which is *known to all other voters*. At each time instant, a voter is chosen randomly. The chosen voter then tries to identify two adjacently ranked candidates a and b in her ranking such that at least as many voters prefer b to a as they prefer a to b . If such a pair exists, the voter switches the positions of a and b in her ranking. Performing such an operation is guaranteed to decrease the sum of the pairwise Kendall tau distances between the voters. This model is combinatorial in nature, and discussed in Section II. Due to lack of space, some of the proofs are omitted.

The second model is based on a generalization of the Borda scoring method for rank aggregation. In the Borda method, for each vote, a candidate is awarded a number of points that depends on its position in the vote. The aggregate ranking is determined based on the average number of points each candidate is awarded, with ties broken arbitrarily. In our model, we assume that the voters are only allowed to communicate on edges of a given social interaction graph, in which vertices correspond to voters. At each time, an edge is chosen at random and the corresponding voters are asked to average their Borda scores. This model resembles classical gossiping over networks, with the major difference being that one requires the consensus to be achieved *only on the ranking of the can-*

didates, and not on their scores. This method is discussed in Section III.

In both models, the voters are assumed to be truthful, which eliminates the option of strategic voting studied in [17], [18].

Related Work: The results closest in theme to the one we discuss are the recent analysis by Ephrati and Rosenschein [6], who proposed a voting algorithm which is resistant to strategic voting and where agents do not have to reveal their entire preference rankings; the work by Meir et al. [17], introducing iterative voting based on the plurality rule and using game theory; and a recent extension of the aforementioned results reported in [18] and [19], including the k -approval, Veto, and Borda rules. In all of the above papers, it is assumed that a vote aggregation rule is used to obtain a representative ranking after votes cease to change. Note that it is not required to achieve consensus, i.e., to have all voters reach the same ranking of candidates. This is in stark contrast with our scheme, in which voters change their votes in order to achieve consensus.

In [16], the authors studied consensus in a network of voters, where a voter may adopt the opinion of one of her neighbors at random, or adopt the opinion of the majority of her neighbors. In our work, each opinion is represented by a *ranking of a set of candidates*, while in [16], opinions are chosen from a finite set that has no particular structure; there, it was also assumed that a vote can be changed to any other vote. The schemes described in what follows allow a voter only to make “small” changes at any time.

II. CONSENSUS VIA DISTANCE-BASED RANK AGGREGATION

Assume there are n voters and m candidates. Every voter provides a full ranking of the m candidates, i.e., a permutation σ in \mathbb{S}_m , the symmetric group of order $m!$. For $\sigma, \pi \in \mathbb{S}_m$, we let $d_\tau(\sigma, \pi)$ denote the *Kendall tau distance* between σ and π , that is, the smallest number of adjacent swaps needed to transform σ into π .

The vector of votes $\Sigma = (\sigma_1, \dots, \sigma_n)$ is called a *profile*. For every two distinct candidates $a, b \in [m]$, where $[m] = \{1, \dots, m\}$, $n_{ab}(\Sigma)$ denotes the number of voters who rank a higher than b . Hence, $n_{ab}(\Sigma) + n_{ba}(\Sigma) = n$. To simplify notation, whenever the profile Σ is clear from the context we simply write n_{ab} instead of $n_{ab}(\Sigma)$. We say that candidate a *wins* against candidate b if $n_{ab} > n_{ba}$.

The sum $\sum_{j < i} d_\tau(\sigma_i, \sigma_j)$ is the *profile potential* and is denoted by $\psi(\Sigma)$. It is easy to see that

$$\psi(\Sigma) = \sum_{1 \leq j < i \leq n} d_\tau(\sigma_i, \sigma_j) = \sum_{1 \leq a < b \leq m} n_{ab}(n - n_{ba}).$$

The vote aggregation algorithm, Algorithm 1, mimics a social process in which each voter may change her opinion regarding the relative order one pair of candidates at a time. Such a change is the smallest change that can be applied to a ranking. Furthermore, *any* change in a voter’s ranking can be viewed as a sequence of changes in the relative order of pairs of candidates. Note that changing the relative order of two candidates in a ranking is only possible if the candidates are adjacent in that ranking.

The algorithm may be succinctly described as follows: at each iteration, one of the voters is allowed to change her ranking $\sigma = \dots ab \dots$ by swapping two adjacent candidates a and b , resulting in the ranking $\sigma' = \dots ba \dots$; the swap is allowed iff there are at least as many voters that prefer b to a as there are voters that prefer a to b (equivalently, a voter is allowed to change her vote from σ to σ' if $d_\tau(\sigma, \sigma') = 1$ and the value of the potential function decreases by introducing the swap). Consensus is reached if all voters agree on the same ranking, in other words, if the potential function of the votes equals zero. In such a case, we say that the algorithm *converged*. If the potential function does not equal zero, but no voter is allowed to change her vote, we say that the algorithm *diverged*.

Algorithm 1. ITR-CONDORCET($\sigma_1, \dots, \sigma_n$)

Input: A profile of n rankings: $(\sigma_1, \dots, \sigma_n) \in (\mathbb{S}_m)^n$.

1. $\mathbf{b} = (b_1, \dots, b_n) = (0, \dots, 0) \in \{0, 1\}^n$.
2. For $i = 1, \dots, n$ do:
 - if $\exists a, b$ s.t. $\sigma_i = \dots ab \dots$ and $n_{ba} \geq n_{ab}$, set $b_i = 1$.
3. If $\mathbf{b} = (0, \dots, 0)$, Return $(\sigma_1, \dots, \sigma_n)$.
4. Choose randomly $i \in [n]$, such that $b_i = 1$, and find a pair a, b satisfying the condition in Step 2.
5. Swap a and b in σ_i to generate σ'_i .
6. Replace σ_i with σ'_i in the profile.
7. Go to step 1.

Remark 1. Note that while Algorithm 1 is not described as a distributed algorithm, it can be easily rewritten in this form: voters, one at a time and in some arbitrary order, check if they are allowed to make an adjacent swap. The process terminates once no changes to votes are allowed.

Let $\pi_1, \dots, \pi_{m!}$ be some ordering of the $m!$ permutations in \mathbb{S}_m . Given a profile $\Sigma \in (\mathbb{S}_m)^n$, we define $m!$ variables $n_1, \dots, n_{m!}$, where n_j indicates the number of voters who decided in favor of the j -th ranking, π_j . Note that $\sum_{j=1}^{m!} n_j = n$. With a slight abuse of terminology, we also call the vector $\mathbf{n} = (n_1, \dots, n_{m!})$ a *profile*, since it provides the same information as the previously defined profiles.

We say that a profile $\mathbf{n} = (n_1, \dots, n_{m!})$ has a *good move* if there exists an allowed swap as defined in Step 2 of Algorithm 1. A good move decreases the profile potential. A profile \mathbf{n} is called a *good profile* if it has a good move, and is called a *bad profile* otherwise. Note that Algorithm 1 terminates when it reaches a bad profile. A bad profile is also called a *local minimum* since it is a local minimum for the profile potential ψ . Observe that the algorithm always terminates in a finite number of steps. The algorithm converges when reaching a bad profile if all voters have the same ranking, and diverges otherwise.

Example 1. Consider the following profile

$$\Sigma_1 = (123, 231, 132).$$

In this case, both the second and third voters have a good move. The second voter may change her permutation to 213, while the third voter may change her vote to 123. Assume that the second voter changed her permutation. In the following

step, the same voters have a good move. Assume now that the third voter changed her permutation to 123. In the last step, the second voter will change her permutation to 123 as well. Hence, Algorithm 1 converges to (123, 123, 123) given the profile Σ_1 .

As another example, note that Algorithm 1 diverges on the profile $\Sigma_2 = (123, 231, 312)$.

A. Characterizing Profiles

In what follows, we characterize some useful properties of profiles that will allow us to assess when Algorithm 1 converges. We start with describing a sufficient condition for a good profile.

Lemma 1. *A profile \mathbf{n} is good if there exist two candidates a, b and two rankings σ_i, σ_j , such that σ_i is of the form $\sigma_i = \dots ab \dots$, while σ_j is of the form $\sigma_j = \dots ba \dots$.*

Proof: Suppose this is not the case, i.e., \mathbf{n} is a bad profile. Since swapping a and b in σ_i is not a good move, we have $n_{ab} > n_{ba}$. Similarly, since swapping b and a in σ_j is not a good move, we have $n_{ba} > n_{ab}$, which leads to a contradiction. ■

In particular, Lemma 1 implies that a bad profile \mathbf{n} cannot contain two votes differing in a single adjacent swap, or a vote and its reverse.

Lemma 2. *Let \mathbf{n} be a profile at a local minimum and σ a permutation of one of the voters. Then the permutation σ eliminates*

$$N(m) = \sum_{k=1}^{m-1} (-1)^{k+1} \binom{m-1}{k} (m-k)! \approx m! - \frac{m!}{e}$$

other permutations from being included in the profile.

From Lemma 2, it follows that a bad profile can have approximately at most $\frac{m!}{e}$ different permutations. We next show how to improve this upper bound.

For a given profile \mathbf{n} , we construct its *profile graph*, $G_n = (V_{G_n}, E_{G_n})$, also known as a *tournament graph*, as follows. The vertex set of the profile graph consists of all candidates, i.e., $V_{G_n} = [m]$, and there is an edge (a, b) if candidate a wins against candidate b (i.e., if $n_{ab} > n_{ba}$). For simplicity, we assume that the number of voters is odd so that it is impossible to have $n_{ab} = n_{ba}$ for any pair a, b . As a result, for any profile \mathbf{n} , the graph G_n is a complete directed graph with no inner loops or symmetric edges. The next lemma is a generalization of Lemma 1.

Lemma 3. *If there exist two candidates a, b and a voter with ranking $\sigma_i = \dots ab \dots$, while $(b, a) \in E_{G_n}$, then \mathbf{n} is a good profile.*

Proof: Since $(b, a) \in E_{G_n}$, we know that $n_{ba} > n_{ab}$. Therefore, according to the conditions of Step 2 in Algorithm 1, the i -th voter has a good move and thus \mathbf{n} is a good profile. ■

From Lemma 3 we conclude that a voter cannot reduce the potential function value if its permutation corresponds to

a Hamiltonian path on the graph G_n . Hence, given an upper bound on the number of different Hamiltonian paths in a tournament graph, one may derive an upper bound on the number of different permutations in a bad profile. Let $T(m)$ denote the maximum number of Hamiltonian paths in a tournament graph with m candidates. We hence have the following lemma.

Lemma 4. *If the number of different permutations in a profile is greater than $T(m)$, then the profile is necessarily a good profile.*

Alon [5] proved that $T(m) < cm^{3/2}m!2^{-(m-1)}$, where c is a positive constant, and Friedgut and Kahn [10] subsequently showed that $T(m) = O(m^{3/2-\xi}m!2^{-m})$, where $\xi = 0.2507 \dots$.

When combined, the above lemmas lead to the theorem below.

Theorem 5. *If n is large enough, then the fraction of good profiles is at least*

$$\frac{\sum_{\ell=T(m)+1}^{m!} \binom{m!}{\ell} \cdot \ell! \cdot S(n, \ell)}{(m!)^\ell},$$

where $S(n, \ell) = \frac{1}{\ell!} \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} t^n$. In particular, as n grows, this fraction approaches one.

A direct result of the above theorem is that the ratio of the number of bad profiles to the total number of points in $(\mathbb{S}_m)^n$ converges to 0 as $n \rightarrow \infty$.

B. Convergence Properties

In voting theory, a *Condorcet winner* is a candidate who wins against every other candidate in pairwise competition. Note that in general a Condorcet winner may not exist. In the tournament graph, a Condorcet winner corresponds to a vertex with in-degree zero. A *Condorcet loser* is defined similarly.

Lemma 6. *If the initial profile has a Condorcet winner, then after Algorithm 1 terminates, the Condorcet winner is ranked first in every vote in the resulting profile.*

Similarly, it is possible to show that if there is a Condorcet loser, after the algorithm terminates, the Condorcet loser is ranked last in every vote. We omit the proof of Lemma 6 as it is a special case of a more general lemma, Lemma 7, stated below.

A *winning set* in a profile is a set $S \subseteq [m]$ such that every candidate $a \in S$ wins against every candidate $b \notin S$.

Lemma 7. *Assume $S \subseteq [m]$ is a winning set in the initial profile. After Algorithm 1 terminates, every $a \in S$ is ranked higher than every $b \notin S$ in every vote in the resulting profile.*

Proof: Without loss of generality, assume that S is the set $\{1, \dots, i\}$, for some $i \in [m]$. At any step of Algorithm 1, assume that there is a ranking σ_j of some voter j such that at least one candidate indexed by an element in $\{1, \dots, i\}$ is ranked lower than at least one candidate indexed by an element in $\{i+1, \dots, m\}$. Then, there are two candidates $1 \leq k \leq i$ and $i+1 \leq \ell \leq m$ such that σ_j is of the form $\sigma_j = \dots k \ell \dots$. Thus, swapping k and ℓ is a good move. When the algorithm

terminates, there are no good moves, so all the candidates in S have to be ranked higher than those in $[m] \setminus S$ in every vote. ■

If the graph G_n does not have a cycle, then there exists a permutation $\pi^* = a_1 \cdots a_m$ such that $\{a_1, \dots, a_i\}$ is a winning set for each $i \in [m]$. Hence, if G_n has no cycles, then Algorithm 1 converges to π^* . Furthermore, π^* is the solution to the Kemeny-Young problem [2], [3],

$$\arg \min_{\pi \in \mathbb{S}_m} \sum_{i=1}^n d_\tau(\pi, \sigma_i).$$

In other words, π^* is the median of Σ under the Kendall tau distance. The number of moves needed for the algorithm to converge to π^* equals $\sum_{i=1}^n d_\tau(\pi^*, \sigma_i)$, since each move decreases the distance of a ranking to π^* by one.

On the other hand, whenever there is a cycle in G_n , Algorithm 1 diverges. To see this, note that a voter can swap between two adjacent candidates a and b if and only if more voters prefer b to a than a to b (remember that we assumed the number of voters is odd) and hence Algorithm 1 cannot change the directions of the edges in the tournament graph. That is, it is not possible for the algorithm to “break” a cycle. Thus, we have the following theorem.

Theorem 8. *Algorithm 1 converges if and only if the graph G_n does not have a cycle.*

It is known that if every voter draws her ranking uniformly at random from \mathbb{S}_m (sometimes called the *impartial culture* model), then the tournament graph will have a cycle with probability which approaches one as the number of candidates grows; see e.g. [4], [7]. Therefore, the same statement holds regarding the convergence of Algorithm 1, even though if n is large enough, the fraction of good states approaches one. In practical social choice scenarios, cycles occur rarely [8], [12].

It is also possible to extend Algorithm 1 so as to accommodate larger changes in votes at each step, i.e., to allow each voter in Step 2 of Algorithm 1 to change her vote to a ranking that is at Kendall tau distance at most b from her current ranking, where b is some prescribed positive integer. Note that Algorithm 1 is equivalent to setting $b = 1$ since only adjacent transpositions are allowed. An interesting result is that if $b = m - 1$, that is, voters can change their vote to a ranking of Kendall tau distance at most $m - 1$, then every profile is a good profile and thus Algorithm 1 always converges. Furthermore, this result is tight in the sense that if $b < m - 1$, then bad profiles are possible. Details of this analysis are omitted due to lack of space.

III. CONSENSUS VIA ITERATIVE BORDA SCORING

We describe next a method for iterative vote aggregation based on Borda’s method [1]. The gist of the approach is to assign scores to candidates based on their ranks: in any ranking, the first candidate receives m points, the second candidate receives $m - 1$, points and so on. The total score of a candidate equals the average of her scores over all votes cast, and ranking is performed in decreasing order of the scores. Other

scoring methods are possible as well, for example assigning score s_i to the i -th ranked candidate, where s_1, \dots, s_m is a predetermined decreasing sequence.

In this setting, consensus may be achieved through gossiping protocols [15] in a networked set of voters. The voter network $G = ([n], E)$ is a connected, undirected graph with vertices representing the voters, and an edge set E that captures the social connectivity patterns of the voters. Initially, each voter i has a preference ranking σ_i of m candidates, based on which an initial score is assigned to each candidate. Let $b_i(0)$ denote the vector of initial scores assigned by voter i . At each time instance, one picks an edge $\{x, y\} \in E$ with probability $p_{xy} > 0$, where $\sum_{\{x, y\} \in E} p_{xy} = 1$. Then, voters x and y each update their score vector to the average of the two vectors. The procedure is repeated until the correct global ranking is reached, if allowed by the distributed protocol:

Algorithm 2. ITR-BORDA

For $t = 0, 1, 2, \dots$ do:

1. Pick an edge $\{i, j\} \in E$ with probability p_{ij} .
2. Let voters i, j exchange their estimates $b_i(t), b_j(t)$.
3. Let $b_i(t+1) = b_j(t+1) = \frac{1}{2}(b_i(t) + b_j(t))$.
4. For $\ell \neq i, j$, set $b_\ell(t+1) = b_\ell(t)$.
5. Stop when *all voters agree on one ranking*.

Let $\bar{b} = \frac{1}{n} \sum_{i=1}^n b_i(0)$. As a direct consequence of the results in [11], we have the following lemma.

Lemma 9 *If $G = ([n], E)$ is connected, then $\lim_{t \rightarrow \infty} b_i(t) = \bar{b}$ holds almost surely.*

As a result, the score vector of each voter converges to the vector of the average scores of the candidates, i.e., their Borda scores. However, our goal is to find the correct ranking of candidates based on their score in \bar{b} , rather than the vector \bar{b} itself. Thus, it is not important that the estimates of the ranking vectors converges to \bar{b} , but that the estimates of the actual ranks are correct. In other words, if for some time t , for all voters i , the ranking of $b_i(t)$ matches the ranking of \bar{b} , then the society has achieved consensus over the *ranking* of the candidates. We derive next a probabilistic bound on the number of iterations needed to reach the Borda ranking in a distributed fashion.

Without loss of generality, assume that \bar{b} is ordered, i.e. $\bar{b}^1 \leq \bar{b}^2 \leq \dots \leq \bar{b}^m$, with superscripts denoting candidates. We say that t is a consensus time for the *ranking* if the ranking of $\bar{b}_i(t)$ matches the ranking of \bar{b} for all $i \in [n]$. The following result is a consequence of this definition.

Lemma 10 *If t is a consensus time for the ranking, then any $t' > t$ is a consensus time for the ranking.*

Proof: It suffice to show the result for $t' = t + 1$. Let $\{i, i'\}$ be the edge that is chosen randomly at time t . Since t is a consensus time for the ranking, we have $b_i^1(t) \leq \dots \leq b_i^m(t)$ and $b_{i'}^1(t) \leq \dots \leq b_{i'}^m(t)$, and thus we also have

$$\begin{aligned} b_i^1(t+1) &= \frac{1}{2} (b_i^1(t) + b_{i'}^1(t)) \\ &\leq \dots \leq b_i^m(t+1) = \frac{1}{2} (b_i^m(t) + b_{i'}^m(t)), \end{aligned}$$

which proves the claim.

Define the consensus time as:

$$T = \min\{t \geq 0 \mid t \text{ is a consensus time for the ranking}\}.$$

Note that for the random gossip scheme, T is a random variable and if we have an adapted process for the random choice of edges, T is a stopping time. Our goal is to provide a probabilistic bound for T .

To this end, let $r^j = \min\{\bar{b}^{j+1} - \bar{b}^j, \bar{b}^j - \bar{b}^{j-1}\}$ and let $d^j = \max_i b_i^j(0) - \min_i b_i^j(0)$. That is, r^j equals the minimum distance of the average score of j from its neighboring candidates, while d^j equals the spread of the initial scores assigned to candidate j by the voters. Then, we have the following result.

Theorem 11 *The consensus time T of the ranking satisfies*

$$P(T > t) \leq 4n\lambda_2^t(W) \sum_{j=1}^m \left(\frac{d^j}{r^j}\right)^2,$$

where $W = \sum_{\{i,i'\} \in E} P_{ii'} \left(I - \frac{1}{2}(e_i - e_{i'})(e_i - e_{i'})^T\right)$, $e_i = [0 \dots 0 \ 1 \ 0 \dots 0]^T$ is an $m \times 1$ vector with i -th element equal to one, and $\lambda_2(W)$ is the second largest eigenvalue of W .

Proof: Let $b^j(t) = (b_1^j(t), \dots, b_n^j(t))$ and let $y^j(t) = b^j(t) - \bar{b}^j$. Note that $\|y^j(t)\|^2 \leq \left(\frac{r^j}{2}\right)^2$ implies that $|b_i^j(t) - \bar{b}^j| \leq \frac{r^j}{2}$, $\forall i \in [n]$. Thus, if $\|y^j(t)\|^2 \leq \left(\frac{r^j}{2}\right)^2$ holds $\forall j \in [m]$, then

$$b_i^j(t) \leq \bar{b}^j + \frac{r^j}{2} \leq \frac{1}{2}(\bar{b}^j + \bar{b}^{j+1}),$$

where the last inequality follows from the fact that $r^j \leq \bar{b}^{j+1} - \bar{b}^j$. Similarly, we have:

$$b_i^{j+1}(t) \geq \bar{b}^{j+1} - \frac{r^{j+1}}{2} \geq \frac{1}{2}(\bar{b}^{j+1} + \bar{b}^j),$$

which follows from $r^{j+1} \leq \bar{b}^{j+1} - \bar{b}^j$. Hence,

$$\begin{aligned} b_i^1(t) &\leq \frac{1}{2}(\bar{b}^2 + \bar{b}^1) \leq b_i^2(t) \leq \frac{1}{2}(\bar{b}^3 + \bar{b}^2) \\ &\leq \dots \leq \frac{1}{2}(\bar{b}^{m-1} + \bar{b}^m) \leq b_i^m(t), \end{aligned}$$

and t is a consensus time for the algorithm. Thus,

$$\{T > t\} \subseteq \bigcup_{j=1}^m \left\{ \|y^j(t)\|^2 \geq \left(\frac{r^j}{2}\right)^2 \right\},$$

which, after invoking the union bound, leads to

$$P(T > t) \leq \sum_{j=1}^m P\left(\|y^j(t)\|^2 \geq \left(\frac{r^j}{2}\right)^2\right). \quad (1)$$

Markov's inequality gives

$$P\left(\|y^j(t)\|^2 \geq \left(\frac{r^j}{2}\right)^2\right) \leq \left(\frac{2}{r^j}\right)^2 E\left[\|y^j(t)\|^2\right].$$

Using the analysis in [11], it can be shown that

$$E\left[\|y^j(t)\|^2\right] \leq \lambda_2^t(W) \|y^j(0)\|^2 \leq n\lambda_2^t(W) (d^j)^2.$$

Combining the above two relations, we find

$$P\left(\|y^j(t)\|^2 \geq \left(\frac{r^j}{2}\right)^2\right) \leq 4n\lambda_2^t(W) \left(\frac{d^j}{r^j}\right)^2.$$

Replacing the last inequality in (1) proves the assertion. ■

Note that from [11] it follows that if G is connected, then $\lambda_2(W) < 1$, and thus the probability $P(T > t)$ decays exponentially.

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