On the K-user Cognitive Interference Channel with Cumulative Message Sharing Sum-Capacity

Diana Maamari, Daniela Tuninetti and Natasha Devroye
Department of Electrical and Computer Engineering
University of Illinois at Chicago, Chicago IL 60607, USA
Email: dmaama2, danielat, devroye @ uic.edu

***Abstract*—This paper considers the K-user cognitive interference channel with one primary and $K-1$ secondary/cognitive transmitters with a *cumulative message sharing* structure, i.e., cognitive transmitter i, $i \in [2:K]$, non-causally knows all messages of the users with index less than i. We first propose a computable outer bound valid for any memoryless channel and show the sum-rate to be achievable for the symmetric K-user Linear Deterministic Channel. Interestingly, for the K-user channel having only the K-th transmitter know all other messages is sufficient to achieve the sum-capacity, i.e., cognition at transmitters 2 to $K-1$ is not needed. Next, the sum-capacity of the symmetric Gaussian noise channel is characterized to within a constant additive and multiplicative gap, which depend on K. As opposed to other interference channel models, a single scheme suffices for both the weak and strong interference regimes. Moreover it is only required for transmitters 2 to $K-1$ to have, in addition to their own message, non-causal message knowledge of the transmitter 1's message.**

I. Introduction

The cognitive radio channel, first introduced in [1], consists of two source-destination pairs in which one of the transmitters called the *secondary transmitter* has non-causal knowledge of the message of the other transmitter known as the *primary transmitter*. This non-causal message knowledge idealizes a cognitive radio's ability to overhear other transmissions and exploit them to either cancel them out at their own receiver or aid in their transmission. For the state-of-the-art on the 2-user cognitive channel we refer the reader to [2], [3]. In this paper we extend the 2-user cognitive interference channel model to K-users. The K-user cognitive interference channel (CIFC) consists of one primary and $K-1$ secondary, or cognitive, users. We assume a *cumulative* message sharing (CMS) cognition structure introduced in [4] for the 3-user channel, and extended here to K-users, whereby user 1 is the primary user, and cognitive users i, $i \in [2:K]$, know the messages of user 1 through $i-1$. The cumulative message cognition model is inspired by the concept of overlaying, or layering multiple co-existing cognitive networks. The first "layer" consists of the primary users. Each additional cognitive layer transmits simultaneously with the previous layers (overlay) given the lower layers' codebooks. We idealize higher layers being able to obtain lower layers' messages through non-causal cognition.

Past Work. The literature on multi-user cognitive interference channels is limited, in part since the 2-user counterpart is not yet fully understood [2], [3]. The only other work on a K-CIFC with $K > 3$ is that of [5], [6] to the best of our knowledge. In [5] the channel model consists of one primary user and $K-1$ parallel cognitive users; each cognitive user only knows the primary message in addition to their own message (thus not a cumulative message structure); for this channel model the capacity in the "very strong" interference regime is obtained [5]. In [6] the sum capacity of the K-user linear deterministic cognitive interference channel with cumulative message sharing is obtained. In [4], [6]–[10] different 3-user cognitive channels are considered which differ from the one considered here either in the number of transmitter/receivers, or in the message sharing/cognition structure in all but [4], [6], [7]. In [7], several types of 3-user cognitive interference channels are proposed: that with "cumulative message sharing" (CMS) as considered here, that with "primary message sharing" where the message of the single primary user is known at both cognitive transmitters, and finally "cognitive only message sharing" (CoMS) where there are two primary users who do not know each others' message and a single cognitive user which knows both primary messages. In [6] the sum-capacity for the 3-user channel with CMS for the linear deterministic channel (LDC) is obtained – our work generalizes this to K-users and to the Gaussian noise channel, where a constant gap to capacity is obtained.

Contributions and Outline. We consider the K-user CIFC with CMS (K-CIFC-CMS), with the following contributions:
1) In Section III we obtain a novel outer bound region that reduces to the outer bound of [2] for the 2-user case (partially presented in [6] for the LDC). The bound is valid for any memoryless channel and any number of users, and does not contain auxiliary random variables and is thus computable for many channels including the Gaussian noise channel.
2) In Section IV we derive the sum-capacity for the symmetric K-user LDC exactly; the outer bound was obtained in [6] and here we show achievability. Interestingly, the scheme only requires one user to be fully cognitive of the messages of all other users, which considerably relaxes the CMS requirement.
3) In Section V we derive the sum-capacity for the symmetric K-user Gaussian noise channel to within a constant additive and multiplicative gap. The additive gap is a function of the number of users and grows as $(K-2)\log_2(K-2)$; the proposed achievable scheme can be thought of as a MIMO-broadcast scheme where only one encoding order is possible due to the CMS mechanism; interestingly, a single scheme suffices for both the weak and strong interference regimes;

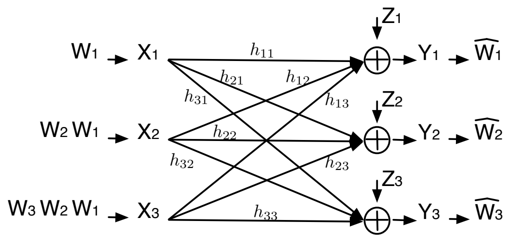

Fig. 1. The Gaussian 3-CIFC-CMS

moreover the achievable scheme only requires the first $K-1$ cognitive users to know the message of the primary user but not those of other cognitive users, again considerably relaxing the CMS requirement. The multiplicative gap is K and is achieved by having all users beam-form to the primary user.

II. ChannelModel

The K-CIFC-CMS channel consists of: channel inputs $X_i \in \mathcal{X}_i,\ i \in [1:K]$, channel outputs $Y_i \in \mathcal{Y}_i,\ i \in [1:K]$, a memoryless channel $\mathbb{P}(Y_1,\ldots,Y_K|X_1,\ldots,X_K)$, and independent messages W_i known to users $1,2,\ldots,i$, $i \in [1:K]$. A code with non-negative rate vector $(R_1,\ldots,R_K)$ and block length N is defined by: messages $W_i,\ i \in [1:K]$, uniformly distributed over $[1:2^{NR_i}]$, encoding functions $X_i^N(W_1,\ldots,W_i),\ i \in [1:K]$, decoding functions $\widehat{W}_i(Y_i^N),\ i \in [1:K]$, and probability of error $P_e^{(N)} := \max_{i\in[1:K]} \mathbb{P}[\widehat{W}_i \neq W_i]$. The capacity of the K-CIFC-CMS channel consists of all rates $(R_1,\ldots,R_K)$ such that $P_e^{(N)} \to 0$ as $N \to \infty$.

A. The Gaussian Noise Channel

The single-antenna complex-valued K-CIFC-CMS with Additive White Gaussian Noise (AWGN), shown in Fig. 1 for the case $K=3$, has input-output relationship

$$Y_\ell = \sum_{i\in[1:K]} h_{\ell i} X_i + Z_\ell,\ \ \ell \in [1:K], \tag{1}$$

where, without loss of generality, the inputs are subject to the power constraint $\mathbb{E}[|X_i|^2] \leq 1,\ i \in [1:K]$, and the noises are marginally (the joint distribution does not matter as the receivers cannot cooperate) proper-complex Gaussian random variables with parameters $Z_\ell \sim \mathcal{N}(0,1),\ \ell \in [1:K]$. The channel gains h_{ij}, $(i,j) \in [1:K]^2$, are constant. Without loss of generality we may assume the direct links $h_{ii},\ i \in [1:K]$ to be real-valued and non-negative since a receiver can compensate for the phase of one of its channel gains.

B. The Linear Deterministic Approximation

The Linear Deterministic approximation of the Gaussian Noise Channel at high SNR (LDC) was first introduced in [11] and for the K-CIFC-CMS has input-output relationship:

$$Y_\ell = \sum_{i\in[1:K]} \mathbf{S}^{m-n_{\ell i}} X_i,\ \ \ell \in [1:K], \tag{2}$$

where $m := \max_{(i,j)\in[1:K]^2}\{n_{ij}\}$, $\mathbf{S}$ is the binary shift matrix of dimension m, X_i and Y_i are binary vectors of length m, and the channel gains $n_{\ell i}$, $(\ell,i) \in [1:K]^2$, are non-negative integers. The channel in (2) can be thought of as the high SNR approximation of the channel in (1) with their parameters related as $n_{ij} = \lfloor \log(1+|h_{ij}|^2) \rfloor,\ (i,j) \in [1:K]^2$.

III. Outerbound for the K-user CIFC with CMS

We obtain a general and computable outer bound next. The sum-capacity bound is obtained by giving S_i as side information to receiver i, $i \in [1:K]$, where $S_i := [S_{i-1}, W_{i-1}, Y_{i-1}^N]$ starting with $S_1 := \emptyset$. With this "nested" side information, the mutual information terms can be expressed in terms of entropies which may be recombined in ways that can be easily single-letterized. This form of the side information allows us to extend the result from the 3-user case [6] to any number of users. Other partial sum-rate bounds are obtained with similar side information structure.

Theorem 1. *The capacity region of a general memoryless K-CIFC-CMS is contained in the region defined by $i \in [1:K]$*

$$R_i \leq I(Y_i; X_{[i:K]}|X_{[1:i-1]}), \tag{3a}$$

$$\sum_{j=i}^{K} R_j \leq \sum_{j=i}^{K} I(Y_j; X_{[j:K]}|X_{[1:j-1]}, Y_{[1:j-1]}), i \in [1:K] \tag{3b}$$

for some joint input distribution $P_{X_1,\ldots,X_K}$, where $X_\mathcal{S} := \{X_i : i \in \mathcal{S}\}$ for some index set $\mathcal{S} \subseteq [1:K]$. Moreover, each rate bound in (3b) *may be tightened with respect to the channel joint conditional distribution as long as the channel conditional marginal distributions are preserved.*

Proof: The the individual rate bounds in (3a) are trivial cut-set bounds, we therefore present the proof of (3b):

$$\begin{aligned}
N\sum_{j=i}^{K}(R_j - \epsilon_N) &\leq \sum_{j=i}^{K} I(Y_j^N; W_j) \quad \text{Fano's inequality} \\
&\leq \sum_{j=i}^{K} I(Y_j^N,\ W_{[1:i-1]},\ W_{[i:j-1]}, Y_{[i:j-1]}^N; W_j) \\
&= \sum_{j=i}^{K} I(Y_{[i:j]}^N; W_j|W_{[1:j-1]}) \quad \text{independence of messages} \\
&= \sum_{j=i}^{K}\sum_{k=i}^{j} I(Y_k^N; W_j|W_{[1:j-1]},\ Y_{[i:k-1]}^N) \quad \text{chain rule} \\
&= \sum_{k=i}^{K}\sum_{j=k}^{K} I(Y_k^N; W_j|W_{[1:j-1]},\ Y_{[i:k-1]}^N) \\
&= \sum_{k=i}^{K} I(Y_k^N; W_{[k:K]}|W_{[1:i-1]}, W_{[i:k-1]}, Y_{[i:k-1]}^N) \\
&\leq \sum_{k=i}^{K}\sum_{t=1}^{N} I(Y_{k,t}; X_{[k:K],t}|X_{[1:k-1],t}, Y_{[i:k-1],t}).
\end{aligned}$$

Finally, by introducing a time-sharing random variable and by 'conditioning reduces entropy', we arrive at (3b). ■

IV. Sum-capacity of the LDC K-CIFC-CMS

For the LDC, we show a sum-capacity achieving scheme that only requires cognition of all messages at one transmitter. For simplicity we consider the symmetric case only. In a *symmetric* LDC all direct links have the same strength $n_{ii} = n_{\rm d} \geq 0, i \in [1:K]$, and all the cross links have the same strength $n_{\ell i} = n_{\rm c} = \alpha\ n_{\rm d} \geq 0, (\ell, i) \in [1:K]^2,\ \ell \neq i$.

Theorem 2. *The sum-capacity bound in* (3b) *for* $i = 1$ *is achievable for the symmetric LDC K-CIFC-CMS.*

Proof: For the symmetric LDC K-CIFC-CMS the sum-capacity in was shown to be upper bounded by [6], [12]:

$$\frac{\sum_{k=1}^K R_k}{n_{\rm d}} \leq \begin{cases} K\max\{1,\alpha\} - \alpha & \text{for } \alpha \neq 1 \\ 1 & \text{for } \alpha = 1 \end{cases}. \quad (4)$$

The discontinuity at $\alpha = 1$ in (4) follows from the fact that when $n_{\rm d} = n_{\rm c}$ the channel reduces to a K-user MAC with sum-capacity given by $\max H(Y_1) = n_{\rm d}$.

To show the achievability of (4), let $U_j,\ j \in [1:K]$, be a vector composed of i.i.d. Bernoulli(1/2) bits. Let the transmit signals be

$$\begin{aligned} X_j &= U_j,\ j \in [1:K-1], \\ X_K &= \begin{bmatrix} I_{n_{\rm c}} & 0_{n_{\rm c}\times[n_{\rm d}-n_{\rm c}]^+} \\ 0_{[n_{\rm d}-n_{\rm c}]^+\times n_{\rm c}} & 0_{[n_{\rm d}-n_{\rm c}]^+\times[n_{\rm d}-n_{\rm c}]^+} \end{bmatrix} \left(\sum_{j=1}^{K-1} U_j\right) \\ &\quad + \begin{bmatrix} 0_{n_{\rm c}\times n_{\rm c}} & 0_{n_{\rm c}\times[n_{\rm d}-n_{\rm c}]^+} \\ 0_{[n_{\rm d}-n_{\rm c}]^+\times n_{\rm c}} & I_{[n_{\rm d}-n_{\rm c}]^+} \end{bmatrix} U_K, \end{aligned}$$

where $0_{n\times m}$ indicates the all zero matrix of dimension $n \times m$, I_n the identity matrix of dimension n, and $[x]^+ := \max\{x, 0\}$. Recall that operations are on GF(2). With these choices, it may be shown that $Y_\ell = (S^{m-n_{\rm d}} + S^{m-n_{\rm c}})X_\ell$. Then, since the matrix $S^{m-n_{\rm d}} + S^{m-n_{\rm c}}$ is full rank for $n_{\rm d} \neq n_{\rm c}$, receiver $\ell,\ \ell \in [1:K]$, decodes U_ℓ from $(S^{m-n_{\rm d}} + S^{m-n_{\rm c}})^{-1}Y_\ell = X_\ell$. Hence receiver $\ell,\ \ell \in [1:K-1]$, can decode $m = \max\{n_{\rm d}, n_{\rm c}\} = n_{\rm d}\max\{1,\alpha\}$ bits since $X_\ell = U_\ell$, while receiver K can decode the lower $[n_{\rm d}-n_{\rm c}]^+ = n_{\rm d}(\max\{1,\alpha\} - \alpha)$ bits of U_K from X_K. ■

Remark: Interestingly, with the proposed scheme, receivers from 1 to $K-1$ are interference free, while receiver K decodes $n_{\rm c}$ bits of the 'interference function' $\sum_{j=1}^{K-1} U_j$. Hence, cognition is only needed at one transmitter in all interference regimes, i.e., our sum-capacity result holds for all symmetric LDC cognitive channels where user i is cognizant of any subset (including the empty set) of the messages of users with index less than i.

V. Sum-Capacity of the Gaussian K-CIFC-CMS

In this section we derive the sum-capacity for the symmetric Gaussian channel to within a constant gap. We denote the direct link gains as $|h_{\rm d}|$, which can be taken to be real-valued and non-negative without loss of generality, and the interference link gains as $h_{\rm i}$. For this channel, we may show constant additive and multiplicative gaps which depend on K but not the channel parameters. To do so, we first evaluate our outer bound, then obtain a generic achievability scheme, and finally show the additive and multiplicative gaps.

A. Upper Bound

For the symmetric Gaussian K-CIFC-CMS with $|h_{\rm d}| \neq h_{\rm i}$ the bound in (3b) with $i = 1$ is further upper bounded as

$$\begin{aligned} &\sum_{u=1}^K R_u \leq \sum_{u=1}^K I\Big(X_u, \cdots, X_K; Y_u \Big| X_1, Y_1, \cdots, X_{u-1}, Y_{u-1}\Big) \\ &= I\Big(X_1, \cdots, X_K; |h_{\rm d}|X_1 + h_{\rm i}\sum_{i=2}^K X_i + Z_1\Big) \\ &+ \sum_{u=2}^{K-1} I\Big(X_u, \cdots, X_K; |h_{\rm d}|X_u + h_{\rm i}\sum_{i=u+1}^K X_i + Z_u \\ &\qquad \Big| X_\ell,\ h_{\rm i}\sum_{i=u}^K X_i + Z_\ell,\ \ell \in [1:u-1]\Big) \\ &+ I\Big(X_K; |h_{\rm d}|X_K + Z_K \Big| X_\ell,\ h_{\rm i}X_K + Z_\ell,\ \ell \in [1:K-1]\Big) \\ &\leq h\Big(|h_{\rm d}|X_1 + h_{\rm i}\sum_{i=2}^K X_i + Z_1\Big) - h(Z_1) \\ &+ \sum_{u=2}^{K-1} h\Big([|h_{\rm d}| - h_{\rm i}]X_u + Z_u - Z_{u-1}\Big) - h(Z_u) \\ &+ h\Big(|h_{\rm d}|X_K + Z_K \Big| h_{\rm i}X_K + \frac{1}{K-1}\sum_{\ell=1}^{K-1} Z_\ell\Big) - h(Z_K). \end{aligned}$$

Finally, by the "Gaussian maximizes entropy" principle:

$$\sum_{k=1}^K R_k \leq \log\left(1 + \Big(|h_{\rm d}| + (K-1)|h_{\rm i}|\Big)^2\right) \quad (5a)$$

$$+ (K-2)\log(2) + (K-2)\log\left(1 + \frac{\big||h_{\rm d}| - h_{\rm i}\big|^2}{2}\right) \quad (5b)$$

$$+ \log\left(1 + \frac{|h_{\rm d}|^2}{1 + (K-1)|h_{\rm i}|^2}\right). \quad (5c)$$

For $h_{\rm i} = |h_{\rm d}|$ all received signals are statistically equivalent, therefore the K-CIFC-CMS is equivalent to a K-user MAC with correlated inputs, whose sum-capacity is bounded by

$$\sum_{k=1}^K R_k \leq \log(1 + K^2|h_{\rm d}|^2).$$

B. Achievable Scheme

In this section we describe a scheme which will be used to show a constant gap to the symmetric upper bound derived in Section V-A. Inspired by the capacity achieving strategy for the Gaussian MIMO-BC, we introduce a scheme that uses Dirty Paper Coding (DPC) with encoding order $1 \to 2 \to 3 \to \cdots K$. We denote by $\mathbf{\Sigma}_\ell$ the covariance matrix corresponding to the message intended for decoder ℓ,

$\ell \in [1:K]$, as transmitted across the K antennas/transmitters. The input covariance matrix is

$$\mathrm{Cov}[X_1,\ldots,X_K] = \sum_{\ell=1}^{K} \boldsymbol{\Sigma}_\ell : \left[\sum_{\ell=1}^{K} \boldsymbol{\Sigma}_\ell\right]_{k,k} \le 1,\ k \in [1:K], \tag{6a}$$

where the constraints on the diagonal elements correspond to the input power constraints. Moreover, since message ℓ can only be broadcasted by transmitters with index larger than ℓ, we further impose

$$\left[\boldsymbol{\Sigma}_\ell\right]_{k,k} = 0 \text{ for all } 1 \le k < \ell \le K. \tag{6b}$$

The achievable rate region is then the set of non-negative rates $(R_1,\ldots,R_K)$ that satisfy, for $\mathbf{h}_\ell^\dagger := [h_{\ell,1} h_{\ell,2} \ldots h_{\ell,K}]$

$$R_\ell \le \log\left(1 + \frac{\mathbf{h}_\ell^\dagger \boldsymbol{\Sigma}_\ell \mathbf{h}_\ell}{1 + \mathbf{h}_\ell^\dagger \left(\sum_{k=\ell+1}^{K} \boldsymbol{\Sigma}_k\right) \mathbf{h}_\ell}\right), \tag{7}$$

for $\ell \in [1:K]$ and all possible $\mathrm{Cov}[X_1,\ldots,X_K]$ complying with (6), with the convention that $\sum_{k=K+1}^{K} \boldsymbol{\Sigma}_k = 0$.

In particular we consider the transmit signals

$$\begin{aligned}
X_1 &= \alpha_1 U_1,\\
X_j &= \gamma_j U_j + \beta_j U_j^{(\mathrm{ZF})} + \alpha_j U_1,\ j \in [2:K-1],\\
X_K &= \gamma_K U_K - \beta_K \sum_{j=2}^{K-1} U_j^{(\mathrm{ZF})} + \alpha_K U_1,
\end{aligned}$$

where $U_\ell, U_\ell^{(\mathrm{ZF})}$ are i.i.d. $\mathcal{N}(0,1), \ell \in [1:K]$, and the coefficients $\{\alpha_j,\beta_j,\gamma_j\}_{j\in[1:K]}$ are such that

$$\begin{aligned}
&|\alpha_1|^2 \le 1,\\
&|\gamma_j|^2 + |\beta_j|^2 + |\alpha_j|^2 \le 1,\ j \in [2:K-1],\\
&|\gamma_K|^2 + |\beta_K|^2(K-2) + |\alpha_K|^2 \le 1,
\end{aligned}$$

in order to satisfy the power constraints. Notice the negative sign for β_K, which we shall use to implement zero-forcing of the aggregate interference $\sum_{j=2}^{K-1} U_j^{(\mathrm{ZF})}$. Moreover, all transmitters cooperate in beam forming U_1 to receiver 1. These two facts can be easily seen by observing that for $\beta_1 = \ldots = \beta_K := \beta$

$$\sum_{\ell=1}^{K} X_\ell = \sum_{\ell=1}^{K} \gamma_\ell U_\ell, \quad \gamma_1 := \sum_{\ell=1}^{K} \alpha_\ell.$$

With these choices the message covariance matrices are

$$\begin{aligned}
&\boldsymbol{\Sigma}_1 = \mathbf{a}\mathbf{a}^\dagger,\ \mathbf{a} := [\alpha_1,\ldots,\alpha_K]^T,\\
&\boldsymbol{\Sigma}_j = |\gamma_j|^2\ \mathbf{e}_j\mathbf{e}_j^\dagger + |\beta|^2\ (\mathbf{e}_j - \mathbf{e}_K)(\mathbf{e}_j - \mathbf{e}_K)^\dagger,\ j \in [2:K],
\end{aligned}$$

where $\mathbf{e}_j$ indicates a length-K vector of all zeros except for a one in position j, $j \in [1:K]$, $\dagger$ indicates the Hermitian transpose, and where $\beta = \beta_1 = \ldots = \beta_K$. We next express the channel vectors $\mathbf{h}_\ell$ for the symmetric Gaussian channel as

$$\mathbf{h}_\ell = (|h_\mathrm{d}| - h_\mathrm{i})\ \mathbf{e}_\ell + h_\mathrm{i}\ \left(\sum_{k=1}^{K} \mathbf{e}_k\right),\ \ell \in [1:K].$$

By noticing that $\mathbf{h}_\ell \mathbf{e}_j^\dagger = \delta[\ell - j](|h_\mathrm{d}| - h_\mathrm{i}) + h_\mathrm{i},\ \ell \in [1:K]$, where $\delta[k]$ is the Kronecker's delta function, the following rates (see [12]) are achievable

$$R_1 = \log\left(1 + \frac{\left||h_\mathrm{d}| + |h_\mathrm{i}| \sum_{j=2}^{K} \alpha_j\right|^2}{1 + |h_\mathrm{i}|^2 \sum_{k=2}^{K} |\gamma_k|^2}\right), \tag{8a}$$

$$R_j = \log\left(1 + \frac{\left||h_\mathrm{d}| - h_\mathrm{i}\right|^2\ |\beta|^2 + |h_\mathrm{d}|^2\ |\gamma_j|^2}{1 + |h_\mathrm{i}|^2 \sum_{k=j+1}^{K} |\gamma_k|^2}\right), \tag{8b}$$

$$R_K = \log\left(1 + |h_\mathrm{d}|^2\ |\gamma_K|^2\right), \tag{8c}$$

for $j \in [2:K-1]$ and with $\alpha_1 = \exp(j\angle h_\mathrm{i})$ (notice the phase of α_1 which allows coherent combining at receiver 1 of the different signals carrying U_1, i.e., all users beamform to the primary receiver).

C. Constant Additive Gap

Theorem 3. *The sum-capacity bound in* (3b) *is achievable for the symmetric Gaussian K-CIFC-CMS to within* 6 *bits per channel use for $K = 3$ and to within $(K-2)\log_2(K-2) + 3.88$ bits per channel use for $K \ge 4$.*

Proof: We now choose the parameters in (8) so as to match the upper bound in (5). We recall that user K is the most cognitive user and can therefore 'pre-code' the whole interference seen at its receiver by using DPC; by doing so, receiver K would not have anything to treat as noise besides the Gaussian noise itself. We therefore interpret the term $\frac{1}{1+(K-1)|h_\mathrm{i}|^2} \le 1$ in (5c) as the fraction of power transmitter K dedicates to its own signal. This is as setting

$$|\gamma_K|^2 = \frac{1}{1 + (K-1)|h_\mathrm{i}|^2}$$

in (8c). This choice guarantees that (8c) exactly matches (5c). Next we match the upper bound term in (5b) to the achievable rates in (8b) by setting

$$\gamma_j = 0,\ j \in [2, K-1], \quad \frac{1}{2} = \frac{|\beta|^2}{1 + |h_\mathrm{i}|^2 |\gamma_K|^2}.$$

However, from the power constraint for user K, $|\beta|^2 \le \frac{1-|\gamma_K|^2}{K-2}$, which imposes the following condition

$$\frac{K-4}{K-2} + \left(|h_\mathrm{i}|^2 + \frac{2}{K-2}\right)|\gamma_K|^2 \le 0.$$

The above condition cannot be satisfied for $K \ge 4$; for $K = 3$ it requires that

$$|\gamma_3|^2 = \frac{1}{1 + 2|h_\mathrm{i}|^2} \le \frac{1}{|h_\mathrm{i}|^2 + 2},$$

which can be satisfied by $|h_\mathrm{i}|^2 \ge 1$. Therefore, in the following we shall assume $|h_\mathrm{i}|^2 \ge 1$ and set $\gamma_j = 0,\ j \in [2, K-1]$ and

$$|\beta|^2 = \begin{cases} \frac{1-|\gamma_K|^2}{K-2} = \frac{1}{K-2}\left(1 - \frac{1}{1+(K-1)|h_\mathrm{i}|^2}\right) & K \ge 4\\ \frac{1+|h_\mathrm{i}|^2|\gamma_3|^2}{2} = \frac{1+3|h_\mathrm{i}|^2}{2(1+2|h_\mathrm{i}|^2)} & K = 3 \end{cases},$$

which implies

$$|\alpha_K|^2 = \begin{cases} 0 & K \geq 4 \\ 1 - |\beta|^2 - |\gamma_K|^2 = \frac{-1+|h_\mathrm{i}|^2}{2(1+2|h_\mathrm{i}|^2)} & K = 3 \end{cases}.$$

Finally, for $j \in [2:K-1]$

$$|\alpha_j|^2 = 1 - |\beta_j|^2 = \begin{cases} \frac{K-3}{K-2} + \frac{1}{K-2}\,\frac{1}{1+(K-1)|h_\mathrm{i}|^2} & K \geq 4 \\ \frac{1+|h_\mathrm{i}|^2}{2(1+2|h_\mathrm{i}|^2)} & K = 3 \end{cases}.$$

The rates may then be bounded for $K \geq 4$ (using $|h_\mathrm{i}|^2 \geq 1$):

$$R_K = \log\left(1 + \frac{|h_\mathrm{d}|^2}{1+(K-1)|h_\mathrm{i}|^2}\right)$$
$$R_j \geq \log\left(1 + \frac{\big||h_\mathrm{d}| - h_\mathrm{i}\big|^2}{K-2}\,\frac{K-1}{K+1}\right),\ j \in [2:K-1],$$
$$R_1 \geq \log\left(1 + \frac{\big||h_\mathrm{d}| + |h_\mathrm{i}|\sqrt{(K-3)(K-2)}\big|^2}{2}\right),$$

and for $K = 3$

$$R_3 = \log\left(1 + \frac{|h_\mathrm{d}|^2}{1+2|h_\mathrm{i}|^2}\right),\ \ R_2 = \log\left(1 + \big||h_\mathrm{d}| - h_\mathrm{i}\big|^2 \frac{1}{2}\right)$$
$$R_1 \geq \log\left(1 + \frac{\big||h_\mathrm{d}| + |h_\mathrm{i}|\frac{1}{2}\big|^2}{2}\right).$$

By taking the difference between the upper bound in (5) and the derived achievable-rates, we find that the gap is upper bounded by, for $K \geq 4$:

$$\mathsf{GAP} \leq (K-2)\log(K-2) + \log(2\exp(2)),$$

(where we used $K\log_\mathrm{e}(1+1/K) \leq 1$), and for $K \geq 3$

$$\mathsf{GAP} \leq \log(2) + \log\left(1 + \left(|h_\mathrm{d}| + 2|h_\mathrm{i}|\right)^2\right)$$
$$- \log\left(1 + \frac{\big||h_\mathrm{d}| + |h_\mathrm{i}|\frac{1}{2}\big|^2}{2}\right) \leq 6\log(2).$$

For $|h_\mathrm{i}|^2 < 1$, set $\beta_j = \alpha_j = 0, \gamma_j = 1$ for $j \in [2:K]$ so that

$$\sum_{\ell=1}^{K} R_\ell = \sum_{\ell=1}^{K} \log\left(1 + \frac{|h_\mathrm{d}|^2}{1+(K-\ell)|h_\mathrm{i}|^2}\right).$$

The gap to the upper bound is at most

$$\mathsf{GAP} \leq (K-2)\log(2) + 2\log(K-1) + \sum_{\ell=2}^{K-1}\log\left(\frac{K-\ell}{2}\right).$$

which is smaller than the gap for $|h_\mathrm{i}|^2 \geq 1$. Details on the calculation of the gaps are found in [12]. ∎

D. Constant Multiplicative Gap

We next consider the sum-capacity to within a multiplicative gap, more relevant at low SNR than additive gaps.

Theorem 4. *The symmetric sum-capacity is achievable to within a factor K by beamforming to the primary user.*

Proof: The rate of user j in (3a) is upper bounded by $C_j := \log(1 + (|h_\mathrm{d}| + (K-j)|h_\mathrm{i}|)^2) \leq C_1,\ j \in [1:K]$; this implies that the sum-rate is upper bounded by $K \times C_1$. Consider an achievability scheme in which all users beamform to user 1: this achieves the sum-rate $R_1 + \cdots R_K = C_1$. This is to within a factor K of the upper bound, proving Th. 4. ∎

Interestingly, the achievable scheme only requires user K to be cognitive of the messages of *all* other users, while the rest of users only need to know the message of user 1 in addition to their own message; this considerably relaxes the CMS requirement.

VI. Conclusion

We presented a general outer bound region for the K-user cognitive interference channel with cumulative message sharing. This computable outer bound was used to show that the symmetric sum-capacity is exactly achievable for the linear deterministic channel and to within constant gap for the Gaussian noise channel. While the focus of this paper was on cumulative message sharing, we note that, interestingly, the *same* sum-capacity outer bound may be achieved with a different message sharing structure with less message knowledge at the cognitive transmitters.

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