Generalized Degrees of Freedom for Network-Coded Cognitive Interference Channel

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***Abstract*—We study a two-user cognitive interference channel (CIC) where one of the transmitters (primary) has knowledge of a linear combination (over an appropriate finite-field) of the two information messages. We refer to this channel model as Network-Coded CIC, since the linear combination may be the result of some linear network coding scheme implemented in the backbone wired network. In this paper, we characterize the generalized degrees of freedom (GDoF) for the Gaussian Network-Coded CIC. For achievability, we use the novel Precoded Compute-and-Forward (PCoF) and Dirty Paper Coding (DPC), based on nested lattice codes. Through the GDoF characterization, we show that knowing "mixed data" (a linear combination of the information messages) provides an *unbounded* spectral efficiency gain over the classical CIC counterpart, if the ratio (in dB) of signal-to-noise (SNR) to interference-to-noise (INR) is larger than ceratin threshold. For example, when $\mathsf{SNR} = \mathsf{INR}$, the Network-Coded cognition yields a 100% gain over the classical Gaussian CIC.**

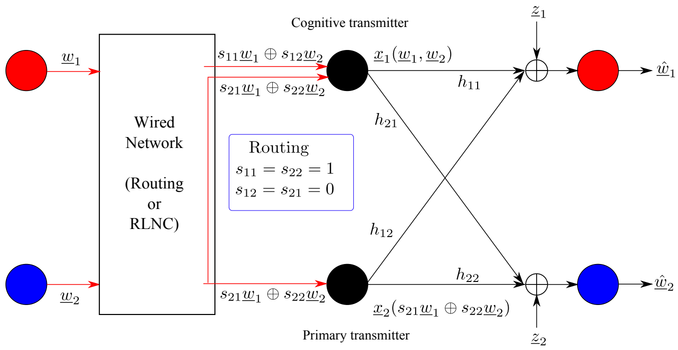

Fig. 1. Application of cognitive interference channel (CIC).

I. Introduction

Transmitters or receivers, in many practical communication systems, are not isolated, and they can share certain amount of information (i.e., information messages, channel state information, and so on). For example, in a cloud base station architecture, small base stations (BSs) are spatially distributed over a certain area, and connected to the infrastructure networks via wired backhaul [1]. Cooperation among transmitters or receivers can mitigate interferences by forming distributed MIMO systems. One special case of particular interest is the two-user Cognitive Interference Channel (CIC), where one of the transmitters (referred to as "cognitive") has knowledge of both information messages to the two users, while the other (referred to as "primary") has knowledge of the message destined to its intended receiver only. This model is relevant under certain assumptions on the underlying wired backbone network connecting the two transmitters. For example, in the case of *unidirectional cooperation*, the primary transmitter sends its message to the cognitive transmitter via an a wired link of infinite capacity. Another example is the asymmetric situation shown in Fig. 1, where one transmitter (cognitive) has larger wired backhaul capacity, and therefore is able to observe both messages. The CIC has been extensively investigated in the literature. The capacity region of the strong interference regime was characterized in [2]. When the interference at the primary receiver is weak, the capacity region was characterized in [3]–[5]. Recently, the capacity region for Gaussian CIC was approximately characterized within 1.87 bits, regardless of channel parameters [5].

In wired networks, routing is generally optimal only for the single-source single-destination case [6]. In the more general case of multiple sources and multiple destinations (multi-source multicasting), linear network coding is known to achieve the min-cut max-flow bound [7]. In practice, random linear network coding (RLNC) is of particular interest for its simplicity. In this case, intermediate nodes forward linear combinations of the incoming messages by randomly and independently choosing the coefficients from an appropriate finite-field [8]. Assuming that RLNC is used in the backbone wired network, we introduce the Network-Coded CIC as a generalization of the classical CIC where the primary transmitter knows a linear combination of the information messages (referred to as "mixed data"). This is motivated in Fig. 1 by introducing RLNC instead of just routing in the backbone network. Since delivering mixed data at the primary transmitter has the same cost (in terms of backhaul capacity) than delivering a single message, a natural question arises: *Does mixed data at the primary transmitter provide capacity increase "for free" for cognitive interference channel?*

Our main contribution is to approximately characterize the sum capacity of Gaussian Network-Coded CIC with respect to the sum Generalized Degrees of Freedom (GDoF) [9]. This is enabled by properly using a novel Scaled Precoded Compute-and-Forward (PCoF) and Dirty Paper Coding (DPC). As a

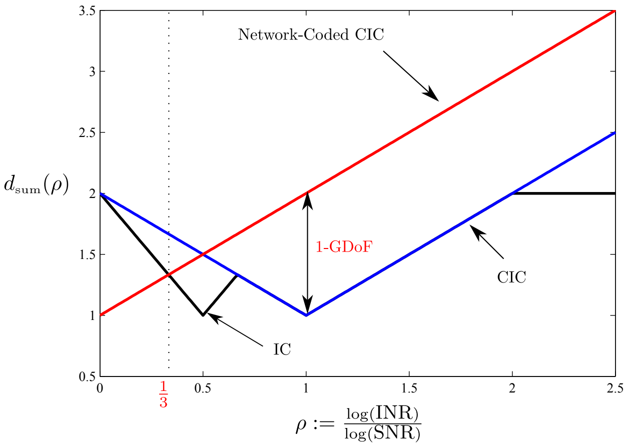

Fig. 2. The generalized degrees-of-freedom (GDoF) of the two-user Gaussian Network-Coded cognitive interference channel (CIC).

consequence of the GDoF analysis, we show that Network-Coded cognition can provide *multiplicative* gain in cognitive interference channels. As shown in Fig. 2, the sum GDoF of the Network-Coded CIC is larger than the sum GDoF of the standard IC when $\rho = \frac{\log \mathsf{INR}}{\log \mathsf{SNR}}$, the ratio of the interference power over the direct link power (expressed in dB), is larger than $1/3$, and it is larger than the sum GDoF of the standard CIC when $\rho > 1/2$. It is also interesting to notice that for $\rho = 1$, Network-Coded cognition provides 100% gain over classical cognition and provides the same two-degrees of freedom of full-cooperation (two-user vector broadcast channel).

II. Preliminaries

In this section we provide some basic definitions and results which will be extensively used in the sequel.

A. System Model

A two-user Gaussian Network-Coded CIC consists of a Gaussian interference channel where transmitter 1 (the cognitive transmitter) knows both user 1 and user 2 information messages (or, equivalently, two linearly independent linear combinations thereof) and transmitter 2 (the primary transmitter) knows only one linear combination of the messages. Without loss of generality, we assume that the cognitive transmitter knows $(\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2)$, and the primary transmitter has $\underline{\mathbf{w}}_1 \oplus \underline{\mathbf{w}}_2$, where $\underline{\mathbf{w}}_k \in \mathbb{F}_q^r$ denotes the information message desired at receiver k, at rate R_k bit/symbol, for $k = 1, 2$. We assume that if $R_1 \neq R_2$ then the lowest rate message is zero-padded such that both messages have a common length, given by $r = \max\{nR_1, nR_2\}$, where n denotes the coding block length. A block of n channel uses of the discrete-time complex baseband two-user IC is described by

$$\underline{\mathbf{y}}_1 = h_{11}\underline{\mathbf{x}}_1 + h_{12}\underline{\mathbf{x}}_2 + \underline{\mathbf{z}}_1 \quad (1)$$
$$\underline{\mathbf{y}}_2 = h_{21}\underline{\mathbf{x}}_1 + h_{22}\underline{\mathbf{x}}_2 + \underline{\mathbf{z}}_2 \quad (2)$$

where $\underline{\mathbf{z}}_k \in \mathbb{C}^{n\times 1}$ contains i.i.d. Gaussian noise samples $\sim \mathcal{CN}(0,1)$ and $h_{ij} \in \mathbb{C}$ denotes the channel coefficients, assumed to be constant over the whole block of length n and known to all nodes. Also, the 2×2 channel matrix is assumed to have full-rank (i.e., $h_{11}h_{22} - h_{12}h_{21} \neq 0$). We have a common per-user power constraint, given by $\frac{1}{n}\mathbb{E}[\|\underline{\mathbf{x}}_k\|^2] \leq \mathsf{SNR}$ for $k = 1, 2$, where $\|\cdot\|$ denotes the ℓ_2-norm. Each receiver k observes the channel output $\underline{\mathbf{y}}_k$ and produces an estimate $\hat{\underline{\mathbf{w}}}_k$ of the desired message $\underline{\mathbf{w}}_k$. A rate pair (R_1, R_2) is achievable if there exists a family of codes satisfying the power constraint, such that the average decoding error probability satisfies $\lim_{n\to\infty} \mathbb{P}(\hat{\underline{\mathbf{w}}}_k \neq \underline{\mathbf{w}}_k) = 0$, for $k = 1, 2$.

B. Nested Lattice Codes

Let $\mathbb{Z}[j]$ be the ring of Gaussian integers and p be a prime. Let $\oplus$ denote the addition over $\mathbb{F}_q$ with $q = p^2$, and let $g : \mathbb{F}_q \to \mathbb{C}$ be the natural mapping of $\mathbb{F}_q$ onto $\{a + jb : a, b \in \mathbb{Z}_p\} \subset \mathbb{C}$. We recall the nested lattice code construction given in [11]. Let $\Lambda = \{\underline{\boldsymbol{\lambda}} = \underline{\mathbf{z}}\mathbf{T} : \underline{\mathbf{z}} \in \mathbb{Z}^n[j]\}$ be a lattice in $\mathbb{C}^n$, with full-rank generator matrix $\mathbf{T} \in \mathbb{C}^{n\times n}$. Let $\mathcal{C} = \{\underline{\mathbf{c}} = \underline{\mathbf{w}}\mathbf{G} : \mathbf{w} \in \mathbb{F}_q^r\}$ denote a linear code over $\mathbb{F}_q$ with block length n and dimension r, with generator matrix $\mathbf{G}$. The lattice Λ_1 is defined through "construction A" (see [10] and references therein) as

$$\Lambda_1 = p^{-1}g(\mathcal{C})\mathbf{T} + \Lambda, \quad (3)$$

where $g(\mathcal{C})$ is the image of $\mathcal{C}$ under the mapping g (applied component-wise). It follows that $\Lambda \subseteq \Lambda_1 \subseteq p^{-1}\Lambda$ is a chain of nested lattices, such that $|\Lambda_1/\Lambda| = p^{2r}$ and $|p^{-1}\Lambda/\Lambda_1| = p^{2(n-r)}$.

For a lattice Λ and $\underline{\mathbf{r}} \in \mathbb{C}^n$, we define the lattice quantizer $Q_\Lambda(\underline{\mathbf{r}}) = \operatorname{argmin}_{\underline{\boldsymbol{\lambda}}\in\Lambda} \|\underline{\mathbf{r}} - \underline{\boldsymbol{\lambda}}\|^2$, the Voronoi region $\mathcal{V}_\Lambda = \{\underline{\mathbf{r}} \in \mathbb{C}^n : Q_\Lambda(\underline{\mathbf{r}}) = \underline{\mathbf{0}}\}$ and $[\underline{\mathbf{r}}] \mod \Lambda = \underline{\mathbf{r}} - Q_\Lambda(\underline{\mathbf{r}})$. For Λ and Λ_1 given above, we define the lattice code $\mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda$ with rate $R = \frac{1}{n}\log|\mathcal{L}| = \frac{r}{n}\log q$. Construction A provides a *natural labeling* of the codewords of $\mathcal{L}$ by the information messages $\underline{\mathbf{w}} \in \mathbb{F}_q^r$. Notice that the set $p^{-1}g(\mathcal{C})\mathbf{T}$ is a *system of coset representatives* of the cosets of Λ in Λ_1. Hence, the natural labeling function $f : \mathbb{F}_q^r \to \mathcal{L}$ is defined by $f(\underline{\mathbf{w}}) = p^{-1}g(\underline{\mathbf{w}}\mathbf{G})\mathbf{T} \mod \Lambda$.

C. Compute-and-Forward

We recall here the CoF scheme of [11]. Consider the two-user Gaussian multiple access channel defined by

$$\underline{\mathbf{y}} = \sum_{k=1}^{2} h_k\underline{\mathbf{x}}_k + \underline{\mathbf{z}}, \quad (4)$$

where $\mathbf{h} = [h_1, h_2]^{\mathsf{T}}$ and the elements of $\underline{\mathbf{z}}$ are i.i.d. $\sim \mathcal{CN}(0,1)$. All users make use of the same nested lattice codebook $\mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda$, where Λ has *second moment* $\sigma_\Lambda^2 \triangleq \frac{1}{n\mathrm{Vol}(\mathcal{V})}\int_{\mathcal{V}} \|\underline{\mathbf{r}}\|^2 d\underline{\mathbf{r}} = \mathsf{SNR}$. Each user k encodes its information message $\underline{\mathbf{w}}_k \in \mathbb{F}_q^r$ into the corresponding codeword $\underline{\mathbf{t}}_k = f(\underline{\mathbf{w}}_k)$ and produces its channel input according to

$$\underline{\mathbf{x}}_k = [\underline{\mathbf{t}}_k + \underline{\mathbf{d}}_k] \mod \Lambda, \quad (5)$$

where the *dithering sequences* $\underline{\mathbf{d}}_k$'s are mutually independent across the users, uniformly distributed over $\mathcal{V}_\Lambda$, and known to the receiver. The decoder's goal is to recover a linear combination $\underline{\mathbf{v}} = [\sum_{k=1}^{2} a_k \underline{\mathbf{t}}_k] \mod \Lambda$ with *integer coefficient vector* $\mathbf{a} = [a_1, a_2]^{\mathsf{T}} \in \mathbb{Z}^2[j]$. Since Λ_1 is a $\mathbb{Z}[j]$-module (closed under linear combinations with Gaussian integer coefficients), then $\underline{\mathbf{v}} \in \mathcal{L}$. Letting $\hat{\underline{\mathbf{v}}}$ be the decoded codeword (for some decoding function which in general depends on $\mathbf{h}$ and $\mathbf{a}$), we say that a computation rate R is achievable for this setting if there exists sequences of lattice codes $\mathcal{L}$ of rate R and increasing block length n, such that the decoding error probability satisfies $\lim_{n\to\infty} \mathbb{P}(\hat{\underline{\mathbf{v}}} \neq \underline{\mathbf{v}}) = 0$.

In the scheme of [11], the receiver computes

$$\begin{aligned} \hat{\underline{\mathbf{y}}} &= \left[\alpha \underline{\mathbf{y}} - \sum_{k=1}^{2} a_k \underline{\mathbf{d}}_k \right] \mod \Lambda \\ &= [\underline{\mathbf{v}} + \underline{\mathbf{z}}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \alpha)] \mod \Lambda \end{aligned} \tag{6}$$

where

$$\underline{\mathbf{z}}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \alpha) = \sum_{k=1}^{2} (\alpha h_k - a_k) \underline{\mathbf{x}}_k + \alpha \underline{\mathbf{z}} \tag{7}$$

denotes the *effective noise*, including the non-integer self-interference (due to the fact that $\alpha h_k \notin \mathbb{Z}[j]$ in general) and the additive Gaussian noise term. The scaling, dither removal and modulo-Λ operation in (6) is referred to as the *CoF receiver mapping* in the following. From [11], we know that by applying lattice decoding to $\hat{\underline{\mathbf{y}}}$ given in (6) there exist sequences of lattice codes $\mathcal{L}$ of rate R and increasing block length n such that $\underline{\mathbf{v}}$ can be decoded successfully with arbitrarily high probability as $n \to \infty$, provided that [1]

$$R < \log^+ \left(\frac{\mathsf{SNR}}{\sigma^2_{\text{eff}}} \right), \tag{8}$$

where the expression in the right-hand side of (8) is the *computation rate* for the modulo-Λ additive noise channel with given SNR and effective noise variance

$$\sigma^2_{\text{eff}} = \sum_{k=1}^{2} |\alpha h_k - a_k|^2 + |\alpha|^2. \tag{9}$$

III. An achievable rate region for Gaussian Network-Coded CIC

Using the fact that the cognitive transmitter has non-causal information of the primary transmitter signal, it can totally eliminate the known interference at its own intended receiver by using DPC. Also, we can remove the interference at the receiver 2 using Scaled PCoF. Using CoF decoding, the receiver 2 can reliably decode an integer linear combination of the lattice codewords sent by transmitters. The "interference" in the finite-field domain can be completely eliminated by precoding over the finite-field at the cognitive transmitter. It is well known that the performance of CoF is deteriorated by the non-integer penalty (i.e., the residual "self-interference" due to the fact that the channel coefficients take on non-integer values in practice). In order to eliminate this penalty, the primary transmitter scales its signal by some constant $\beta \in \mathcal{P}$ to create more favorable channel for CoF receiver mapping where $\mathcal{P} = \{\beta \in \mathbb{C} : |\beta| \leq 1\}$.

We let $\mathbf{a} = [a_1, a_2] \in \mathbb{Z}[j]^2$ denote the integer coefficients vector used at receiver 2 for the modulo-Λ receiver mapping (6), and we let $q_\ell = g^{-1}([a_\ell] \mod p\mathbb{Z}[j])$. For the time being, it is assumed that $q_1, q_2 \neq 0$ over $\mathbb{F}_q$. The proposed achievability scheme proceeds as follows

- The primary transmitter produces the lattice codeword $\underline{\mathbf{v}}_2 = f(\underline{\mathbf{w}}_1 \oplus \underline{\mathbf{w}}_2)$ and produces the channel input with power scaling factor $\beta \in \mathcal{P}$:

$$\underline{\mathbf{x}}'_2 = \beta \underline{\mathbf{x}}_2 \tag{10}$$

  where $\underline{\mathbf{x}}_2 = [\underline{\mathbf{v}}_2 + \underline{\mathbf{d}}_2] \mod \Lambda$.
- The cognitive transmitter produces the precoded message $b\underline{\mathbf{w}}_1$ where $m \in \mathbb{F}_q$ is given by

$$q_1 m \oplus q_2 = 0 \Rightarrow m = (q_1)^{-1}(-q_2) \tag{11}$$

  where $(q_1)^{-1}$ denotes the multiplicative inverse of q_1 and $(-q_2)$ denotes the additive inverse of q_2. Then, it uses DPC for the known interference signal $h_{12}\underline{\mathbf{x}}'_2$ and forms:

$$\underline{\mathbf{x}}_1 = [\underline{\mathbf{v}}_1 - \alpha_1 (h_{12}/h_{11}) \underline{\mathbf{x}}'_2 + \underline{\mathbf{d}}_1] \mod \Lambda, \tag{12}$$

  where $\underline{\mathbf{v}}_1 = f(m\underline{\mathbf{w}}_1)$. The known interference signal is in fact generated by using the knowledge of the message $\underline{\mathbf{w}}_1 \oplus \underline{\mathbf{w}}_2$, the dense lattice codebooks, and the dithering sequence $\underline{\mathbf{d}}_2$ The $\mathbf{d}_\ell$'s are mutually independent across the transmitters, uniformly distributed over $\mathcal{V}_\Lambda$, and known to all nodes.

Because of linearity, the precoding and the encoding over the finite-field commute. Therefore, we can write

$$\begin{aligned} \underline{\mathbf{v}}_1 &= g(m)\underline{\mathbf{t}}_1 \mod \Lambda & (13) \\ \underline{\mathbf{v}}_2 &= \underline{\mathbf{t}}_1 + \underline{\mathbf{t}}_2 \mod \Lambda & (14) \end{aligned}$$

where $\underline{\mathbf{t}}_1 = f(\underline{\mathbf{w}}_1)$ and $\underline{\mathbf{t}}_2 = f(\underline{\mathbf{w}}_2)$. Receiver 1 performs the inflated modulo-lattice mapping as $\hat{\underline{\mathbf{y}}}_1 = [\alpha_1 \underline{\mathbf{y}}_1 / h_{11} - \underline{\mathbf{d}}_1] \mod \Lambda$. This results in the mod-Λ additive noise channel given by:

$$\hat{\underline{\mathbf{y}}}_1 = [\underline{\mathbf{v}}_1 - (1 - \alpha_1)\underline{\mathbf{u}}_1 + (\alpha_1/h_{11})\underline{\mathbf{z}}_1] \mod \Lambda$$

where $\underline{\mathbf{u}}_1$ is uniformly distributed on $\mathcal{V}_\Lambda$ and is independent of $\underline{\mathbf{z}}_1$ and $\underline{\mathbf{v}}_1$ by the Crypto Lemma [12]. From standard DPC results [13], choosing

$$\alpha_1 = \alpha_{1,\text{MMSE}} \triangleq \frac{\mathsf{SNR}|h_{11}|^2}{1 + \mathsf{SNR}|h_{11}|^2}, \tag{15}$$

the coding rate R_1 is achievable if

$$R_1 \leq \log(1 + |h_{11}|^2 \mathsf{SNR}). \tag{16}$$

Letting $\tilde{\mathbf{h}}(\beta) = [h_{21}, \beta \tilde{h}_{22}]$ with $\tilde{h}_{22} = h_{22} - \alpha_{1,\text{MMSE}} h_{12} h_{21} / h_{11}$, receiver 2 applies the CoF receiver mapping in (6) with integer coefficients vector $\mathbf{a} = (a_1, a_2) \in$

[1]We define $\log^+(x) \triangleq \max\{\log(x), 0\}$.

$\mathbb{Z}[j]^2$ and scaling factor $\alpha_2 = a_1/h_{21}$, yielding

$$\begin{aligned}
\hat{\underline{\mathbf{y}}}_2 &= [\alpha_2 \underline{\mathbf{y}}_2 - a_1 \underline{\mathbf{d}}_1 - a_2 \underline{\mathbf{d}}_2] \mod \Lambda \\
&= [a_1 \underline{\mathbf{v}}_1 + a_2 \underline{\mathbf{v}}_2 + \alpha_2 (h_{21} \underline{\mathbf{x}}_1 + h_{22} \underline{\mathbf{x}}_2' + \underline{\mathbf{z}}_2) \\
&\quad -a_1[\underline{\mathbf{v}}_1 + \underline{\mathbf{d}}_1] - a_2[\underline{\mathbf{v}}_2 + \underline{\mathbf{d}}_2]] \mod \Lambda \\
&= [a_1 \underline{\mathbf{v}}_1 + a_2 \underline{\mathbf{v}}_2 + (\alpha_2 h_{21} - a_1)[\underline{\mathbf{v}}_1 + \underline{\mathbf{d}}_1] \\
&\quad +(\alpha_2 \beta \tilde{h}_{22} - a_2)\underline{\mathbf{x}}_2 + \alpha_2 h_{21} \underline{\boldsymbol{\lambda}} + \alpha_2 \underline{\mathbf{z}}_2] \mod \Lambda \\
&\overset{(a)}{=} \left[\mathbf{a}^{\mathsf{T}} \begin{bmatrix} \underline{\mathbf{v}}_1 \\ \underline{\mathbf{v}}_2 \end{bmatrix} + (\alpha_2 \beta \tilde{h}_{22} - a_2)\underline{\mathbf{u}}_2 + \alpha_2 \underline{\mathbf{z}}_2 \right] \mod \Lambda \\
&= \left[\left(\mathbf{a}^{\mathsf{T}} \begin{bmatrix} g(m) & 0 \\ 1 & 1 \end{bmatrix} \mod p\mathbb{Z}[j] \right) \begin{bmatrix} \underline{\mathbf{t}}_1 \\ \underline{\mathbf{t}}_2 \end{bmatrix} \right. \\
&\quad \left. + \underline{\mathbf{z}}_{\text{eff}}(\tilde{\mathbf{h}}(\beta), \mathbf{a}) \right] \mod \Lambda \\
&\overset{(b)}{=} [([a_2] \mod p\mathbb{Z}[j])\underline{\mathbf{t}}_2 + \underline{\mathbf{z}}_{\text{eff}}(\tilde{\mathbf{h}}(\beta), \mathbf{a})] \mod \Lambda
\end{aligned}$$

where $\underline{\mathbf{u}}_2$ is uniformly distributed on $\mathcal{V}_\Lambda$ and is independent of $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$, and $\underline{\mathbf{z}}_2$ by the independence and uniformity of dithering and by the Crypto Lemma [12], $\underline{\boldsymbol{\lambda}} = Q_\Lambda(\underline{\mathbf{v}}_1 - \alpha_{1,\text{MMSE}}(h_{12}/h_{11})\underline{\mathbf{x}}_2' + \underline{\mathbf{d}}_1)$, (a) is due to the fact that $\alpha_2 h_{21} \underline{\boldsymbol{\lambda}} = a_1 \underline{\boldsymbol{\lambda}} \in \Lambda$, and (b) follows from the fact that the m is chosen to satisfy the (11), i.e., $a_1 g(m) + a_2 \mod p\mathbb{Z}[j] = 0$. Furthermore, we define the effective noise:

$$\underline{\mathbf{z}}_{\text{eff}}(\tilde{\mathbf{h}}(\beta), \mathbf{a}) = (a_1 \beta \tilde{h}_{22}/h_{21} - a_2)\underline{\mathbf{u}}_2 + (a_1/h_{21})\underline{\mathbf{z}}_2. \quad (17)$$

Receiver 2 decodes $\underline{\mathbf{t}}_2$ by applying lattice decoding to $\hat{\underline{\mathbf{y}}}_2$. This yields the achievable region given by:

Theorem 1: Scaled PCoF and DPC applied to Gaussian Network-Coded CIC with $\mathbf{H} = [h_{ij}] \in \mathbb{C}^{2\times 2}$ achieves the rate pairs (R_1, R_2) such that

$$R_1 \le \log(1 + |h_{11}|^2 \mathsf{SNR}) \quad (18)$$

$$R_2 \le \log^+ \left(\frac{\mathsf{SNR}}{\sigma^2_{\text{eff}}(\beta)} \right) \quad (19)$$

for any $\mathbf{a} \in \mathbb{Z}[j]^2$ with $a_1, a_2 \neq 0$ and any $\beta \in \mathcal{P}$, where

$$\sigma^2_{\text{eff}}(\beta) = \left| a_1 \frac{\beta \tilde{h}_{22}}{h_{21}} - a_2 \right|^2 \mathsf{SNR} + \left| \frac{a_1}{h_{21}} \right|^2. \quad (20)$$

■

IV. Generalized Degrees of Freedom

In the high SNR regime, a useful proxy for the performance of wireless networks is provided by the Generalized Degrees-of-Freedom (GDoF), which characterize the capacity pre-log factor in different relative scaling regimes of the channel coefficients, as SNR grows to infinity [9]. In this section we study the symmetric GDoFs. In particular, we consider the following channel model:

$$\underline{\mathbf{y}}_1 = h_{11}\sqrt{\mathsf{SNR}}\underline{\mathbf{x}}_1 + h_{12}\sqrt{\mathsf{INR}}\underline{\mathbf{x}}_2 + \underline{\mathbf{z}}_1 \quad (21)$$

$$\underline{\mathbf{y}}_2 = h_{21}\sqrt{\mathsf{INR}}\underline{\mathbf{x}}_1 + h_{22}\sqrt{\mathsf{SNR}}\underline{\mathbf{x}}_2 + \underline{\mathbf{z}}_2 \quad (22)$$

where $h_{ij} \in \mathbb{C}$ are bounded non-zero constants independent of $\mathsf{SNR}, \mathsf{INR}$, $\underline{\mathbf{z}}_k$ is the i.i.d. Gaussian noise $\sim \mathcal{CN}(0,1)$, and $\frac{1}{n}\mathbb{E}[\|\underline{\mathbf{x}}_k\|^2] \le 1$ for $k = 1, 2$. The channel is parameterized by SNR and INR, both growing to infinity such that $\mathsf{INR} = \mathsf{SNR}^\rho$ as $\mathsf{SNR} \to \infty$, where $\rho \ge 0$ defines the relative strength of the direct and interference paths. Letting $C_{\text{sum}}(\mathsf{SNR}, \rho)$ denote the sum capacity for given SNR and ρ, the sum symmetric GDoF is defined by

$$d_{\text{sum}}(\rho) = \lim_{\mathsf{SNR}\to\infty} \frac{C_{\text{sum}}(\mathsf{SNR}, \rho)}{\log \mathsf{SNR}}. \quad (23)$$

The main result of this section is given by:

Theorem 2: For the Gaussian Network-Coded CIC, the sum symmetric GDoF is given by

$$d_{\text{sum}}(\rho) = 1 + \rho. \quad (24)$$

Proof: See Appendix A. ■

In order to demonstrate the benefit gain of the mixed message at the primary transmitter, we compare the sum GDoF of Gaussian IC and Gaussian CIC. The GDoF of Gaussian IC is computed in [9]. Also, from the constant gap result in [5], we can immediately compute the sum GDoF of Gaussian CIC. The sum symmetric GDoF of these three channel models are shown in Fig. 2.

V. Concluding Remarks

We investigated a two-user cognitive interference channel (CIC), in the case where the "primary" transmitter knows a linear combination of the information messages. The proposed combination of Scaled PCoF and Dirty Paper Coding, based on nested lattice codes, allowed us to characterize the sum generalized degrees-of-freedom of the Gaussian Network-Coded CIC. In particular, our result shows the surprising fact that, in certain regimes of the SNR/INR scaling region, network-coded cognition yields an unbounded gain (i.e., multiplicative gain) in the Gaussian CIC, with respect to the classical cognitive transmitter model.

Appendix

A. Achievable scheme

We use the achievable rates given in Theorem 1. It is immediately shown that the achievable GDoF of message 1 (cognitive user), obtained by

$$d_1(\rho) = \lim_{\mathsf{SNR}\to\infty} \frac{\log(1 + |h_{11}|^2 \mathsf{SNR})}{\log \mathsf{SNR}} = 1. \quad (25)$$

In this proof, we show that message 2 (primary user) achieves GDoF equal to ρ, by carefully choosing the power scaling factor $\beta \in \mathcal{P}$. The effective channel for Scaled PCoF is given by $\tilde{\mathbf{h}}(\beta) = [h_{21}\sqrt{\mathsf{INR}}, \beta(h_{22}\sqrt{\mathsf{SNR}} - \alpha_{1,\text{MMSE}}(h_{12}h_{21}/h_{11})\mathsf{SNR}^{\rho-\frac{1}{2}})]$ and can be rewritten as

$$\tilde{\mathbf{h}}(\beta) = \mathsf{SNR}^{\rho/2}[h_{21}, \beta\tilde{h}_{22}] \quad (26)$$

where $\tilde{h}_{22} = h_{22}\mathsf{SNR}^{(1-\rho)/2} - h\mathsf{SNR}^{(\rho-1)/2}$ and $h = \alpha_{1,\text{MMSE}}(h_{12}h_{21}/h_{11})$. Here, we choose $\beta = \beta^\star \triangleq h_{21}/(\tilde{h}_{22}\gamma)$, where $\gamma \ge 1$ is an integer with $\gamma = \lceil \|h_{21}/\tilde{h}_{22}\| \rceil \in \mathbb{Z}_+$. This produces a kind of "aligned" channel:

$$\tilde{\mathbf{h}} = \mathsf{SNR}^{\rho/2}[h_{21}, h_{21}/\gamma]. \quad (27)$$

Letting $a_1 = \gamma$ and $a_2 = 1$, the effective noise in (17) is obtained by

$$\mathbf{z}_{\text{eff}}(\tilde{\mathbf{h}}, \mathbf{a}) = (\gamma/(h_{21}\mathsf{SNR}^{\rho/2}))\underline{\mathbf{z}}_2. \quad (28)$$

This shows that non-integer penalty is completely eliminated. Also, we can use the zero forcing precoding over $\mathbb{F}_q$ since the integer coefficients a_1 and a_2 are non-zero. From this, we have the lower bound on the achievable rate of Scaled PCoF:

$$\max_{\beta} R_2(\beta) \geq R_2(\beta^*) = \rho \log(|h_{21}|^2\mathsf{SNR}) - 2\log(\gamma). \quad (29)$$

The lower and upper bounds on γ is given by

$$1 \leq \gamma \leq 1 + \left|\frac{h_{21}}{h_{22}\mathsf{SNR}^{(1-\rho)/2} - h\mathsf{SNR}^{(\rho-1)/2}}\right| \quad (30)$$

where γ converges to a const as $\mathsf{SNR} \to \infty$. Finally, the achievable GDoF of the primary transmitter is derived as

$$d_2(\rho) \geq \lim_{\mathsf{SNR},\mathsf{INR}\to\infty} \frac{R_2(\beta^*)}{\log \mathsf{SNR}} = \rho. \quad (31)$$

From (25) and (31), the achievable sum GDoF is $1 + \rho$.

B. Converse

For given rates R_1 and R_2, we define $R_{\min} = \min\{R_1, R_2\}$ and $R_\Delta = \max\{R_1, R_2\} - R_{\min}$. If $R_1 > R_2$ then $W_1 = (M_1, M_\Delta)$ and $W_2 = (M_2, \mathbf{0})$. In the reverse case, we have that $W_1 = (M_1, \mathbf{0})$ and $W_2 = (M_2, M_\Delta)$. In both cases, the primary transmitter knows the linear combination, $W_1 \oplus W_2 = (M_1 \oplus M_2, M_\Delta)$. From the well-known Crypto Lemma [12], the $M_1 \oplus M_2$ is mutually statistically independent of M_1, as well as $M_1 \oplus M_2$ is mutually statistically independent of M_2. In this proof, we derive the upper bounds on $R_{\min}$ and $R_{\max} = R_{\min} + R_\Delta$. First, we derive the upper bound on the minimum rate $R_{\min}$:

$$\begin{aligned}
nR_{\min} &= H(M_1) = H(M_1|M_1 \oplus M_2, M_\Delta) \\
&= H(M_1|M_1 \oplus M_2, M_\Delta) \\
&\quad -H(M_1|Y_1^n, M_1 \oplus M_2, M_\Delta) \\
&\quad +H(M_1|Y_1^n, M_1 \oplus M_2, M_\Delta) \\
&\overset{(a)}{\leq} I(M_1; Y_1^n|M_1 \oplus M_2, M_\Delta) + n\epsilon_n \\
&\leq h(Y_1^n|X_2^n) - h(Y_1^n|X_1^n, X_2^n) + n\epsilon_n \\
&= I(X_1^n; Y_1^n|X_2^n) + n\epsilon_n \\
&\leq n\log(1 + |h_{11}|^2\mathsf{SNR}) + n\epsilon_n
\end{aligned}$$

where (a) follows from the Fano's inequality and data processing inequality as

$$H(M_1|Y_1^n, M_1 \oplus M_2, M_\Delta) \leq H(M_1|Y_1^n) \leq H(M_1|\hat{M}_1) \leq n\epsilon_n.$$

In the same manner, we get:

$$\begin{aligned}
nR_{\min} &= H(M_2) = H(M_2|M_1 \oplus M_2, M_\Delta) \\
&\leq n\log(1 + |h_{21}|^2\mathsf{INR}) + n\epsilon_n.
\end{aligned}$$

From the above, we have the upper bound on $R_{\min}$ as

$$R_{\min} \leq \min\{\log(1 + |h_{11}|^2\mathsf{SNR}), \log(1 + |h_{21}|^2\mathsf{INR})\}. \quad (32)$$

An obvious upper bound on R_1 and R_2 are given by

$$\begin{aligned}
nR_1 &\leq I(X_1^n, X_2^n; Y_1^n) + \epsilon_n \\
&\leq n\log(1 + |h_{11}|^2\mathsf{SNR} + |h_{12}|^2\mathsf{INR}) + n\epsilon_n \\
nR_2 &\leq I(X_1^n, X_2^n; Y_2^n) + \epsilon_n \\
&\leq n\log(1 + |h_{21}|^2\mathsf{INR} + |h_{22}|^2\mathsf{SNR}) + n\epsilon_n.
\end{aligned}$$

Since $R_{\min} + R_\Delta = \max\{R_1, R_2\}$, we have:

$$\begin{aligned}
R_{\max} \leq\; &\max\{\log(1 + |h_{11}|^2\mathsf{SNR} + |h_{12}|^2\mathsf{INR}), \\
&\log(1 + |h_{21}|^2\mathsf{INR} + |h_{22}|^2\mathsf{SNR})\}. \quad (33)
\end{aligned}$$

Using (32), (33), and $\mathsf{INR} = \mathsf{SNR}^\rho$, we have the upper bounds in the asymptotic case:

$$\lim_{\mathsf{SNR}\to\infty} \left(\frac{R_{\min}}{\log \mathsf{SNR}} + \frac{R_{\max}}{\log \mathsf{SNR}}\right) \leq \min\{1, \rho\} + \max\{1, \rho\},$$

yielding the upper bound on the sum symmetric GDoF as

$$d_{\text{sum}} = \lim_{\mathsf{SNR}\to\infty} \frac{R_{\min} + R_{\max}}{\log \mathsf{SNR}} \leq 1 + \rho. \quad (34)$$

This completes the proof.

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