On The Effect of Self-Interference in Gaussian Two-Way Channels With Erased Outputs

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Abstract*—It is well-known that the so-called Shannon Achievable Region (SAR) in a collocated two-user Gaussian Two-Way Channel (GTWC) does not depend on the self-interference that is due to the leakage of the signal transmitted by each user at its own receiver. This is simply because each user can completely remove its self-interference. In this paper, we study a class of GTWCs where each user is unable to cancel the self interference due to random erasures at its receiver. The mixture of the intended signal for each user and its self-interference is erased independently from transmission slot to transmission slot. It is assumed that both users adopt PAM constellations for transmission purposes. Due to the fact that both users are unaware of the erasure pattern, the noise plus interference at each user is mixed Gaussian. To analyze this setup, a sequence of upper and lower bounds are developed on the differential entropy of a general mixed Gaussian random variable where it is shown that the upper and lower bounds meet as the sequence index increases. Utilizing such bounds, it is shown that the achievable rate for each user is monotonically increasing in terms of the level of self-interference and eventually saturates as self-interference grows to infinity. This saturation effect is justified analytically by showing that as self-interference increases, each user is enabled to extract the erasure pattern at its receiver. Treating the erasure pattern as side information, both users are able to cancel self-interference and decode the useful information at higher transmission rates.

Notation- Before proceeding, let us provide a list of notations adopted in this paper. For any $x \in [0,1]$, we define $\overline{x} \triangleq 1-x$. Random variables are shown in bold, e.g., $\boldsymbol{x}$ with realization x. Sets are shown by calligraphic letters such as $\mathcal{X}$. Probability Density Function (PDF) of a random variable $\boldsymbol{x}$ is shown by $p_{\boldsymbol{x}}(\cdot)$. The probability of an event $\mathcal{E}$ and the expectation of a random variable $\boldsymbol{x}$ are denoted by $\mathbb{P}(\mathcal{E})$ and $\mathbb{E}[\boldsymbol{x}]$, respectively. The entropy of a discrete random variable $\boldsymbol{x}$ is shown by $H(\boldsymbol{x})$ and the differential entropy of a continuous random variable $\boldsymbol{x}$ is denoted by $h(\boldsymbol{x})$. The mutual information between $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $I(\boldsymbol{x};\boldsymbol{y})$. A Gaussian random variable with mean μ and variance σ^2 is shown by $\mathrm{N}(\mu,\sigma^2)$. The PDF of such a random variable is shown by $\varphi(\cdot;\mu,\sigma^2)$. The so-called Q-function is defined by $Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}\mathrm{d}u$. A Bernoulli random variable with parameter $p \in (0,1)$ is denoted by $\mathrm{Ber}(p)$. The acronym i.i.d. stands for independent and identically distributed and $O(\cdot)$ is the so-called big-O notation.

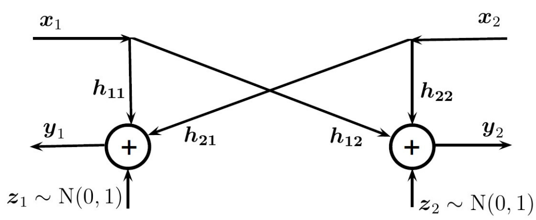

Fig. 1. Gaussian Two-Way Channel (GTWC)

I. Introduction

A. Prior Art and Motivation

The collocated two-way channel introduced by Shannon [1] is the first proposed scenario in which two users attempt to transmit and receive information through a common channel simultaneously. Each user is represented by a node that can transmit and receive at the same time. The Gaussian Two-Way Channel (GTWC) shown in Fig. 1 is a particular example of such a scenario where each user attempts at transmitting information to the other user. A feature of this channel is that the signals transmitted by node 1 (node 2) to node 2 (node 1) also leak into the receiver of node 1 (node 2). This phenomenon is referred to as *self-interference*.

Denoting the signal received at node k by $\boldsymbol{y}_k$,

$$\begin{aligned} \boldsymbol{y}_1 &= h_{1,1}\boldsymbol{x}_1 + h_{2,1}\boldsymbol{x}_2 + \boldsymbol{z}_1 \\ \boldsymbol{y}_2 &= h_{1,2}\boldsymbol{x}_1 + h_{2,2}\boldsymbol{x}_2 + \boldsymbol{z}_2 \end{aligned}, \tag{1}$$

where $h_{k,l}$ is the channel coefficient from transmitter k to receiver l, $\boldsymbol{x}_k$ is the signal transmitted by user k and $\boldsymbol{z}_k$ is the additive ambient noise at node k. It is assumed that $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are i.i.d. $\mathrm{N}(0,1)$ random variables and $\boldsymbol{x}_k$ satisfies the power constraint $E[\boldsymbol{x}_k^2] \leq P_k$. Moreover, the channel coefficients are known to both nodes.

The codebook of user k consists of $2^{\lfloor nR_k \rfloor}$ codewords of length n where each codeword represent a message intended for the other user. As n grows, R_k represents the transmission rate for user k. The so-called Shannon Achievable Region (SAR) [1] for a general memoryless two-way channel is the set of all (R_1, R_2) that satisfy

$$\begin{aligned} R_1 &\leq I(\boldsymbol{x}_1; \boldsymbol{y}_2 | \boldsymbol{x}_2) \\ R_2 &\leq I(\boldsymbol{x}_2; \boldsymbol{y}_1 | \boldsymbol{x}_1) \end{aligned}, \tag{2}$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independent random variables. By (2), SAR for GTWC does not depend on the coefficients $h_{1,1}$ and $h_{2,2}$ that represent self-interference. This is simply because users are able to cancel self-interference completely. In fact,

$$I(\boldsymbol{x}_1; \boldsymbol{y}_2 | \boldsymbol{x}_2) = I(\boldsymbol{x}_1; \boldsymbol{y}_2 - h_{2,2}\boldsymbol{x}_2 | \boldsymbol{x}_2) = I(\boldsymbol{x}_1; h_{1,2}\boldsymbol{x}_1 + \boldsymbol{z}_2) \tag{3}$$

and similarly,

$$I(\boldsymbol{x}_2; \boldsymbol{y}_1 | \boldsymbol{x}_1) = I(\boldsymbol{x}_2; h_{2,1}\boldsymbol{x}_2 + \boldsymbol{z}_1). \tag{4}$$

As such, the channel from each transmitter to its intended receiver converts to an additive white Gaussian noise channel. This implies that random Gaussian codewords achieve the largest value for $I(\boldsymbol{x}_1; \boldsymbol{y}_2 | \boldsymbol{x}_2)$ and $I(\boldsymbol{x}_2; \boldsymbol{y}_1 | \boldsymbol{x}_1)$. In fact, selecting $\boldsymbol{x}_i$ as $\mathrm{N}(0, P_i)$, SAR is given by

$$\begin{aligned} R_1 &\leq \tfrac{1}{2}\log\left(1 + h_{1,2}^2 P_1\right) \\ R_2 &\leq \tfrac{1}{2}\log\left(1 + h_{2,1}^2 P_2\right) \end{aligned}. \tag{5}$$

It is shown in [2] that (5) is indeed the capacity region of a two-user GTWC.

In the following, we consider a modification of GTWC motivated by the physics of the underlying system. This modification results in a nonlinear channel in the sense that each node can not simply remove self-interference by subtraction. Hence, in contrast to (3) and (4), $I(\boldsymbol{x}_1; \boldsymbol{y}_2 | \boldsymbol{x}_2)$ and $I(\boldsymbol{x}_2; \boldsymbol{y}_1 | \boldsymbol{x}_1)$ depend on $\boldsymbol{x}_2$ and $\boldsymbol{x}_1$, respectively. Assuming user i can adjust $h_{i,i}$ in the interval $[\alpha_i, \infty)$ for some $\alpha_i > 0$, our goal is to design $h_{i,i}$ in oder to achieve the largest SAR.

B. Contribution

We study a GTWC with discrete inputs and erased outputs as shown in Fig. 2. The random codebooks of both users are generated over the 2-PAM constellation. The mixture of signals received by each node is randomly erased from transmission slot to transmission slot before being added to the ambient noise. The point to point Gaussian erasure channel was first introduced in [3] to model channels with impulsive noise on top of Gaussian additive noise at the receiver side. In contrast to [3] where the erasure pattern is known to the receiver a priori, in this paper both nodes are unaware of the erasure pattern due to the random nature of erasures. In return, this makes each receiver be unaware of the presence of self-interference in the received signal. Therefore, subtracting self-interference does not necessarily remove self-interference. It is notable that the received signal by each receiver is a mixed Gaussian random variable and hence, there is no closed formula for the upper bounds in (2). In section III, we develop a sequence of upper and lower bound on the differential entropy of a mixed Gaussian random variable. The bounds have the property that as the sequence index increases, the upper bounds decrease and the lower bounds increase and they meet as the sequence index tends to infinity. To our knowledge, this is the first time that arbitrarily tight bounds are developed on the differential entropy of a mixed Gaussian random variable.

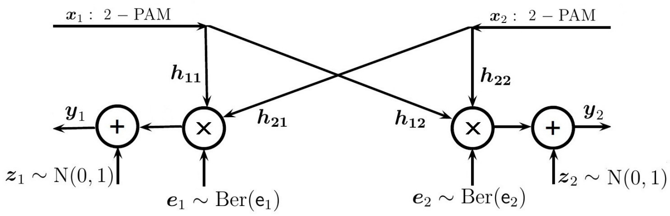

Fig. 2. GTWC with discrete inputs and erased outputs

In section III, using the proposed bounds and through an example, we show that $I(\boldsymbol{x}_1; \boldsymbol{y}_2 | \boldsymbol{x}_2)$ and $I(\boldsymbol{x}_2; \boldsymbol{y}_1 | \boldsymbol{x}_1)$ increase and eventually saturate as $h_{2,2}$ and $h_{1,1}$ increase, respectively. In fact, it is verified[1] that each node can use self-interference as a *training* tool in order to understand the erasure pattern. If self-interference is sufficiently large and the received signal is small in magnitude, the node understands that erasure has occurred. On the other hand, if the received signal appears to be at the same order of magnitude as self-interference, the node understands that erasure has not occurred and simply cancels self-interference.

II. Tight bounds on the differential entropy of mixed Gaussian random variables

The following is the main result of this section:

Theorem 1: Let $\boldsymbol{x}$ be a random variable with PDF $p_{\boldsymbol{x}}(x) = \sum_{m=1}^{M} \frac{p_m}{\sqrt{2\pi P}} e^{-\frac{(x-x_m)^2}{2P}}$, where $x_1 < \cdots < x_M$ and $(p_m)_{m=1}^{M}$ is a discrete probability sequence. Define

$$a_m = p_m e^{-\frac{x_m^2}{2P}},\ b_m = \frac{x_m}{P},\ m = 1, \cdots, M, \tag{6}$$

$$c_m^+ = \frac{a_m}{a_M},\ d_m^+ = b_m - b_M,\ m = 1, \cdots, M-1, \tag{7}$$

$$c_m^- = \frac{a_m}{a_1},\ d_m^- = b_1 - b_m,\ m = 2, \cdots, M, \tag{8}$$

$$\eta^+(x) = \ln\left(1 + \sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right),\ x \in \mathbb{R}_+, \tag{9}$$

$$\eta^-(x) = \ln\left(1 + \sum_{m=1}^{M-1} c_m^- e^{d_m^- x}\right),\ x \in \mathbb{R}_+. \tag{10}$$

Let $x_*^+ \geq 0$ be the unique root of $\eta^+(x) = \ln 2$ for $a_M < \frac{1}{2}$ and $x_*^+ = 0$ otherwise. Also, let $x_*^- \geq 0$ be the unique root of $\eta^-(x) = \ln 2$ for $a_1 < \frac{1}{2}$ and $x_*^- = 0$ otherwise. Then for any $N_1^{\mathrm{lb}}, N_1^{\mathrm{ub}}, N_2 \geq 0$,

$$\beta + \alpha_{2N_1^{\mathrm{lb}}+1, N_2} \leq h(\boldsymbol{x}) \leq \beta + \alpha_{2N_1^{\mathrm{ub}}, N_2}, \tag{11}$$

[1] See Proposition 1.

$$\alpha_{N,N'} \triangleq \log e \sum_{\mathsf{s}\in\{+,-\}} \sum_{n=1}^{N} \frac{(-1)^n}{n} \sum_{\substack{j_1,\cdots,j_{M-1}\geq 0\\ j_1+\cdots+j_{M-1}=n}} \frac{n!}{\prod_{m=1}^{M-1} j_m!} \theta^{\mathsf{s}}(j_1,\cdots,j_{M-1}) \prod_{m=1}^{M-1} (c_m^{\mathsf{s}})^{j_m}$$
$$- \log e \sum_{\mathsf{s}\in\{+,-\}} \sum_{n=0}^{N'-1} \sum_{m=1}^{M} p_m \eta^{\mathsf{s}}(x_n^{\mathsf{s}}) \left(Q\left(\frac{x_n^{\mathsf{s}} - x_m}{\sqrt{P}}\right) - Q\left(\frac{x_{n+1}^{\mathsf{s}} - x_m}{\sqrt{P}}\right)\right). \tag{12}$$

$$\beta \triangleq \frac{1}{2}\log(2\pi P) + \frac{\log e}{2P}\left(P + \sum_{m=1}^{M} p_m x_m^2\right) - \log a_M \sum_{m=1}^{M} p_m Q\left(-\frac{x_m}{\sqrt{P}}\right) - \log a_1 \sum_{m=1}^{M} p_m Q\left(\frac{x_m}{\sqrt{P}}\right)$$
$$- \log e\, b_M \sum_{m=1}^{M} p_m \left(\sqrt{\frac{P}{2\pi}} e^{-\frac{x_m^2}{2P}} + x_m Q\left(-\frac{x_m}{\sqrt{P}}\right)\right) - \log e\, b_1 \sum_{m=1}^{M} p_m \left(x_m Q\left(\frac{x_m}{\sqrt{P}}\right) - \sqrt{\frac{P}{2\pi}} e^{-\frac{x_m^2}{2P}}\right). \tag{13}$$

$$\theta^{\pm}(j_1,\cdots,j_{M-1}) \triangleq \sum_{s=1}^{M} a_s\, e^{\frac{1}{2}P\left(\pm b_s + \sum_{m=1}^{M-1} j_m d_m^{\pm}\right)^2} Q\left(\frac{1}{\sqrt{P}}\left(x_*^{\pm} - P\left(\pm b_s + \sum_{m=1}^{M-1} j_m d_m^{\pm}\right)\right)\right). \tag{14}$$

$$x_n^{+} \triangleq \frac{n x_*^{+}}{N_2},\ x_n^{-} \triangleq \frac{n x_*^{-}}{N_2},\ n = 0,\cdots,N_2. \tag{15}$$

where $\alpha_{N,N'}$ and β are given in (12) and (13), respectively. Moreover, the error terms $\alpha_{2N_1^{\mathrm{ub}},N_2} + \beta - h(\boldsymbol{x})$ and $\alpha_{2N_1^{\mathrm{lb}}+1,N_2} + \beta - h(\boldsymbol{x})$ scale like $O\left(\frac{1}{N_1^{\mathrm{ub}}}\right) + O\left(\frac{1}{N_2}\right)$ and $O\left(\frac{1}{N_1^{\mathrm{lb}}}\right) + O\left(\frac{1}{N_2}\right)$, respectively.

Proof: See appendix A. ■

III. GTWC WITH DISCRETE INPUTS AND ERASED OUTPUTS

Let us consider a two-way channel where the received signals at nodes 1 and 2 are given by

$$\begin{aligned} \boldsymbol{y}_1 &= \boldsymbol{e}_1(h_{1,1}\boldsymbol{x}_1 + h_{2,1}\boldsymbol{x}_2) + \boldsymbol{z}_1 \\ \boldsymbol{y}_2 &= \boldsymbol{e}_2(h_{1,2}\boldsymbol{x}_1 + h_{2,2}\boldsymbol{x}_2) + \boldsymbol{z}_2 \end{aligned}. \tag{16}$$

Here, $\boldsymbol{x}_k \in \mathcal{X}_k \triangleq \{-\sqrt{P_k}, \sqrt{P_k}\}$ and $\mathbb{P}(X_k = x) = \frac{1}{2}$, $x \in \mathcal{X}_k$. Also, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are independent $\mathrm{N}(0,1)$ random variables and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are independent $\mathrm{Ber}(\mathsf{e}_1)$ and $\mathrm{Ber}(\mathsf{e}_2)$ random variables, respectively, for known $\mathsf{e}_1, \mathsf{e}_2 \in (0,1)$ at both nodes. We emphasize that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unknown to both ends. The gain $h_{k,k}$ can be set at any value in $[\alpha_k, \infty)$ by user k. Let us define

$$R_1^* \triangleq I(\boldsymbol{x}_2; \boldsymbol{y}_1|\boldsymbol{x}_1),\ R_2^* \triangleq I(\boldsymbol{x}_1; \boldsymbol{y}_2|\boldsymbol{x}_2). \tag{17}$$

In this case, SAR is given by

$$\begin{aligned} R_1 &\leq \sup\nolimits_{h_{2,2}\geq\alpha_2} R_1^* \\ R_2 &\leq \sup\nolimits_{h_{1,1}\geq\alpha_1} R_2^* \end{aligned}. \tag{18}$$

We have

$$R_1^* \triangleq h(\boldsymbol{y}_1|\boldsymbol{x}_1) - h(\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{19}$$

where

$$h(\boldsymbol{y}_1|\boldsymbol{x}_1) = \frac{1}{2}\sum_{x_1\in\mathcal{X}_1} h(\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}\boldsymbol{x}_2) + \boldsymbol{z}_1) \tag{20}$$

and

$$h(\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{1}{4}\sum_{x_1\in\mathcal{X}_1, x_2\in\mathcal{X}_2} h(\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}x_2) + \boldsymbol{z}_1). \tag{21}$$

To calculate $h(\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}\boldsymbol{x}_2) + \boldsymbol{z}_1)$ in (20), note that $\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}\boldsymbol{x}_2) + \boldsymbol{z}_1$ is a mixed Gaussian random variable where all its Gaussian components have unit variance and means 0, $h_{1,1}x_1 - h_{2,1}\sqrt{P_2}$ and $h_{1,1}x_1 + h_{2,1}\sqrt{P_2}$ with corresponding probabilities $1-\mathsf{e}_1$, $\frac{\mathsf{e}_1}{2}$ and $\frac{\mathsf{e}_1}{2}$. Similarly, in computing $h(\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}x_2) + \boldsymbol{z}_1)$ in (21), note that $\boldsymbol{e}_1(h_{1,1}x_1 + h_{2,1}x_2) + \boldsymbol{z}_1$ is a mixed Gaussian random variable where both its Gaussian components have unit variance and means 0, $h_{1,1}x_1 + h_{2,1}x_2$ with corresponding probabilities $1-\mathsf{e}_1$ and e_1. As such, Theorem 1 can be utilized to derive arbitrarily tight upper and lower bounds on R_1^*.

For example, let us consider a scenario where $\alpha_1 = \alpha_2 = 1$, $h_{1,2} = h_{2,1} = 1$ and $P_1 = P_2 = 2$dB. Let us focus on a particular setting where $h_{1,1} = h_{2,2} = a$ and $a \geq 1$. Fig. 3 presents plots of lower and upper bounds on the sum rate $R_1^* + R_2^*$ in terms of a in a scenario where $(\mathsf{e}_1, \mathsf{e}_2) = (0.4, 0.2)$. Setting $N^{(\mathrm{lb})} = 5$ and $N^{(\mathrm{ub})} = 6$ guarantees a uniform distance of less than 0.01 between the upper and lower bounds on $R_1^* + R_2^*$. Moreover, it is seen that $R_1^* + R_2^*$ is an increasing function of a and eventually saturates as a grows to infinity. This example motivates us to study the behaviour of $R_1^* + R_2^*$ in the large self-interference regime where $h_{1,1}$ and

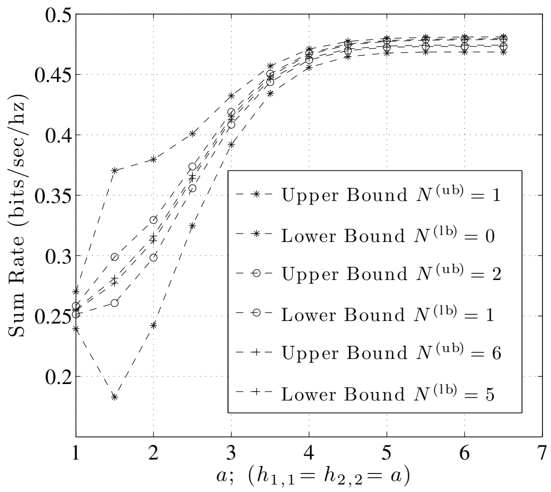

Fig. 3. Plots of lower and upper bounds on the sum rate $R_1^*+R_2^*$ in a scenario where $h_{1,2}=h_{2,1}=1$ and $P_1=P_2=2$dB. It is seen that the bounds meet as the indices N_1^{lb} and N_1^{ub} increase.

$h_{2,2}$ tend to infinity. Intuitively, if $h_{1,1}$ is sufficiently large, user 1 is capable of recognizing the realization of $\boldsymbol{e}_1$. As such, the achievable information rate by user 1 saturates on $I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{e}_1)$ which is larger than $I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1)$.[2] In fact, we have the following Proposition:

Proposition 1:

$$\lim_{h_{1,1}\to\infty} I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1)=I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{e}_1). \tag{22}$$

Proof: let us define

$$\boldsymbol{u}_1(x_1)\triangleq \boldsymbol{e}_1(h_{1,1}x_1+h_{2,1}\boldsymbol{x}_2)+\boldsymbol{z}_1,\ x_1\in\mathcal{X}_1 \tag{23}$$

and

$$\boldsymbol{v}_1(x_1,x_2)\triangleq \boldsymbol{e}_1(h_{1,1}x_1+h_{2,1}x_2)+\boldsymbol{z}_1,\ x_k\in\mathcal{X}_k,\ k=1,2. \tag{24}$$

Then

$$R_1^*=\frac{1}{2}\sum_{x_1\in\mathcal{X}_1}h(\boldsymbol{u}_1(x_1))-\frac{1}{4}\sum_{x_1\in\mathcal{X}_1,x_2\in\mathcal{X}_2}h(\boldsymbol{v}_1(x_1,x_2)). \tag{25}$$

Note that

$$p_{\boldsymbol{u}_1(x_1)}(u)=\bar{\mathsf{e}}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0)+\mathsf{e}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1), \tag{26}$$

where

$$\begin{array}{c} p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0)=\varphi(u;0,1) \\ p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1)=\frac{1}{2}\sum_{x_2\in\mathcal{X}_2}\varphi(u;h_{1,1}x_1+h_{2,1}x_2,1) \end{array}. \tag{27}$$

Let us define the two-variable function $\varepsilon(u,h_{1,1})$ by

$$\begin{aligned}\varepsilon(u,h_{1,1})&\triangleq p_{\boldsymbol{u}_1(x_1)}(u)\log p_{\boldsymbol{u}_1(x_1)}(u)\\ &-\bar{\mathsf{e}}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0)\log(\bar{\mathsf{e}}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0))\\ &-\mathsf{e}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1)\log(\mathsf{e}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1)).\end{aligned} \tag{28}$$

[2]Note that $I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{e}_1)=H(\boldsymbol{x}_2)-H(\boldsymbol{x}_2|\boldsymbol{y}_1,\boldsymbol{x}_1,\boldsymbol{e}_1)$ which is larger than $H(\boldsymbol{x}_2)-H(\boldsymbol{x}_2|\boldsymbol{y}_1,\boldsymbol{x}_1)=I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1)$ due to the fact that conditioning reduces differential entropy.

Then for any $x_1\in\mathcal{X}_1$,

$$\begin{aligned}h(\boldsymbol{u}_1(x_1))&=-\int_{-\infty}^{\infty}p_{\boldsymbol{u}_1(x_1)}(u)\log p_{\boldsymbol{u}_1(x_1)}(u)\mathrm{d}u\\ &=-\int_{-\infty}^{\infty}\varepsilon(u,h_{1,1})\mathrm{d}u\\ &-\int_{-\infty}^{\infty}\bar{\mathsf{e}}_1\left(p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0)\log(\bar{\mathsf{e}}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0))\right)\mathrm{d}u\\ &-\int_{-\infty}^{\infty}\mathsf{e}_1\left(p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1)\log(\mathsf{e}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1))\right)\mathrm{d}u\\ &=-\int_{-\infty}^{\infty}\varepsilon(u,h_{1,1})\mathrm{d}u+h_b(\mathsf{e}_1)+\bar{\mathsf{e}}_1 h(\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1=0)\\ &+\mathsf{e}_1 h(\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1=1)\\ &=-\int_{-\infty}^{\infty}\varepsilon(u,h_{1,1})\mathrm{d}u+h_b(\mathsf{e}_1)+\bar{\mathsf{e}}_1 h(\boldsymbol{z}_1)\\ &+\mathsf{e}_1 h(h_{2,1}\boldsymbol{x}_2+\boldsymbol{z}_1),\end{aligned} \tag{29}$$

where the last step is by the fact that $h(\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1=0)=h(\boldsymbol{z}_1)$ and $h(\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1=1)=h(h_{1,1}x_1+h_{1,2}\boldsymbol{x}_2+\boldsymbol{z}_1)=h(h_{1,2}\boldsymbol{x}_2+\boldsymbol{z}_1)$ as additive constants do not change differential entropy. Invoking dominated convergence [4], it is shown in appendix B that $\lim_{h_{1,1}\to\infty}\int_{-\infty}^{\infty}\varepsilon(u,h_{1,1})\mathrm{d}u=0$. Hence, we conclude that

$$\lim_{h_{1,1}\to\infty}h(\boldsymbol{u}_1(x_1))=h_b(\mathsf{e}_1)+\bar{\mathsf{e}}_1 h(\boldsymbol{z}_1)+\mathsf{e}_1 h(h_{2,1}\boldsymbol{x}_2+\boldsymbol{z}_1), \tag{30}$$

for any $x_1\in\mathcal{X}_1$. Following similar lines that led us to (30),

$$\lim_{h_{1,1}\to\infty}h(\boldsymbol{v}_1(x_1,x_2))=h_b(\mathsf{e}_1)+h(\boldsymbol{z}_1), \tag{31}$$

for any $x_k\in\mathcal{X}_k,\ k=1,2$. By (25), (30) and (31),

$$\begin{aligned}\lim_{h_{1,1}\to\infty}R_1^*&=\mathsf{e}_1 h(h_{2,1}\boldsymbol{x}_2+\boldsymbol{z}_1)+\bar{\mathsf{e}}_1 h(\boldsymbol{z}_1)-h(\boldsymbol{z}_1)\\ &=\mathsf{e}_1\left(h(h_{2,1}\boldsymbol{x}_2+\boldsymbol{z}_1)-h(\boldsymbol{z}_1)\right)\\ &=I(\boldsymbol{x}_2;\boldsymbol{y}_1|\boldsymbol{x}_1,\boldsymbol{e}_1).\end{aligned} \tag{32}$$

■

Similarly, one can see that $\lim_{h_{2,2}\to\infty}R_2^*=I(\boldsymbol{x}_1;\boldsymbol{y}_2|\boldsymbol{x}_2,\boldsymbol{e}_2)$. This verifies our earlier claim that both R_1^* and R_2^* saturate as self-interference increases and saturation rates correspond to having full side information on erasure patterns at the nodes.

APPENDIX A

One can write $p(\cdot)$ as $p(x)=\mathrm{g}(x;P)\sum_{m=1}^{M}a_m e^{b_m x}$, where $\mathrm{g}(x;P)=\frac{1}{\sqrt{2\pi P}}e^{-\frac{x^2}{2P}}$. It is easy to see that

$$\begin{aligned}\int p(x)\ln p(x)\mathrm{d}x&=-\frac{1}{2}\ln(2\pi P)-\frac{E[X^2]}{2P}+\Pr\{X>0\}\ln a_M\\ &+\Pr\{X<0\}\ln a_1+b_M E[X\mathbf{1}(X>0)]+b_1 E[X\mathbf{1}(X<0)]\\ &+\int_0^{\infty}p(x)\eta^+(x)\mathrm{d}x+\int_0^{\infty}p(-x)\eta^-(x)\mathrm{d}x.\end{aligned} \tag{33}$$

Straightforward calculations show that all term except the last two terms in (33) represent $\frac{\beta}{\log e}$ where β is given in (13). Next, let us consider the term $\int_0^\infty p(x)\eta^+(x)\mathrm{d}x$. We write

$$\int_0^\infty p(x)\eta^+(x)\mathrm{d}x = \int_0^{x_*^+} p(x)\eta^+(x)\mathrm{d}x + \int_{x_*^+}^\infty p(x)\eta^+(x)\mathrm{d}x. \tag{34}$$

We treat the integrals in (34) separately.

1- For $x > x_*^+$, we have $\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x} < 1$. Applying Leibniz test[3] for alternating series and for any $N_1^{\mathrm{lb}} \geq 0$, $\eta^+(x) \leq \sum_{n=1}^{2N_1^{\mathrm{lb}}+1} \frac{(-1)^{n-1}}{n} \left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)^n$ and the difference between the right and left side in this inequality is less than or equal to $\frac{1}{2(N_1^{\mathrm{lb}}+1)} \left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)^{2(N_1^{\mathrm{lb}}+1)} \leq \frac{1}{N_1^{\mathrm{lb}}}$. One may calculate the terms $\int_{x_*^+}^\infty p(x)\left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)^n \mathrm{d}x$ for any $n \geq 1$ by expanding $\left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)^n$ according to the multinomial identity. This results in appearance of the term involving $\theta^{\mathrm{s}}(j_1, \cdots, j_{M-1})$ in (12).

2- Note that $\eta^+(\cdot)$ is a decreasing function on $[0,\infty)$. Let us show that $\eta^+(\cdot)$ is also convex on $[0,\infty)$. One can write

$$\begin{aligned}\left(1+\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\eta^+(x) &= \sum_{m=1}^{M-1} c_m^+ (d_m^+)^2 e^{d_m^+ x} \\ &+ \left(\sum_{m=1}^{M-1} c_m^+ (d_m^+)^2 e^{d_m^+ x}\right)\left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right) \\ &- \left(\sum_{m=1}^{M-1} c_m^+ d_m^+ e^{d_m^+ x}\right)^2 .\end{aligned} \tag{35}$$

Using Cauchy-Schwartz inequality and noting that $c_m^+ > 0$, the term $\left(\sum_{m=1}^{M-1} c_m^+ d_m^+ e^{d_m^+ x}\right)^2 = \left(\sum_{m=1}^{M-1}\left((c_m^+)^{\frac{1}{2}} d_m^+ e^{\frac{1}{2}d_m^+ x}\right)\left((c_m^+)^{\frac{1}{2}} e^{\frac{1}{2}d_m^+ x}\right)\right)^2$ is less than or equal to $\left(\sum_{m=1}^{M-1} c_m^+ (d_m^+)^2 e^{d_m^+ x}\right)\left(\sum_{m=1}^{M-1} c_m^+ e^{d_m^+ x}\right)$. Applying this in (35) yields $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\eta^+(x) > 0$ for any real x, i.e., $\eta^+(\cdot)$ is convex. We partition the interval $[0, x_*^+]$ into N_2 subintervals with end-points $x_n^+ = \frac{n x_*^+}{N_2}$ for $n = 0, \cdots, N_2$. Define $\underline{\eta}(x) \triangleq \sum_{n=0}^{N_2-1} \eta^+(x_{n+1}^+)\mathbf{1}(x_n^+ \leq x < x_{n+1}^+)$ and $\overline{\eta}(x) \triangleq \sum_{n=0}^{N_2-1} \eta^+(x_n^+)\mathbf{1}(x_n^+ \leq x < x_{n+1}^+)$. for $x \in [0, x_*^+]$. Then $\underline{\eta}(x) \leq \eta^+(x) \leq \overline{\eta}(x)$, $x \in [0, x_*^+]$ and we get $\int_0^{x_*^+} p(x)\eta^+(x)\mathrm{d}x \leq \int_0^{x_*^+} p(x)\overline{\eta}(x)\mathrm{d}x = \sum_{n=0}^{N_2-1}\eta^+(x_n^+)\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$. Similarly, $\int_0^{x_*^+} p(x)\eta^+(x)\mathrm{d}x \geq \int_0^{x_*^+} p(x)\underline{\eta}(x)\mathrm{d}x = \sum_{n=0}^{N_2-1}\eta(x_{n+1}^+)\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$. The term $\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$ can be expressed as $\sum_{m=1}^M p_m \left(Q\left(\frac{x_n^+ - x_m}{\sqrt{P}}\right) - Q\left(\frac{x_{n+1}^+ - x_m}{\sqrt{P}}\right)\right)$. To bound the difference $\int_0^{x_*^+} p(x)\overline{\eta}(x)\mathrm{d}x - \int_0^{x_*^+} p(x)\eta^+(x)\mathrm{d}x$, note that $\overline{\eta}(x) - \eta^+(x) \leq \eta^+(x_n) - \eta^+(x_{n+1})$ for $x \in [x_n^+, x_{n+1}^+]$. Hence, $0 \leq \int_0^{x_*^+} p(x)\overline{\eta}(x)\mathrm{d}x - \int_0^{x_*^+} p(x)\eta^+(x)\mathrm{d}x \leq \sum_{n=0}^{N_2-1}(\eta^+(x_n) - \eta^+(x_{n+1}))\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$. The upper bound $\sum_{n=0}^{N_2-1}(\eta^+(x_n^+) - \eta^+(x_{n+1}^+))\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$ tends to 0 as N_2 grows to infinity. To see this note that by the Mean Value Theorem, there are $y_n \in (x_n^+, x_{n+1}^+)$ and $z_n \in (x_n^+, x_{n+1}^+)$ such that $\eta^+(x_n^+) - \eta^+(x_{n+1}^+) = -\frac{\mathrm{d}}{\mathrm{d}x}\eta(y_n)(x_{n+1}^+ - x_n^+) = -\frac{x_*^+ \frac{\mathrm{d}}{\mathrm{d}x}\eta^+(y_n)}{N_2}$ and $\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x = p(z_n)(x_{n+1}^+ - x_n^+) = \frac{x_*^+ p(z_n)}{N_2}$. Therefore, $\sum_{n=0}^{N_2-1}(\eta^+(x_n^+) - \eta^+(x_{n+1}^+))\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x$ is equal to $\sum_{n=0}^{N_2-1}\frac{(x_*^+)^2\left(-\frac{\mathrm{d}}{\mathrm{d}x}\eta(y_n)\right)p(z_n)}{N_2^2}$. Since $p(\cdot)$ and $-\frac{\mathrm{d}}{\mathrm{d}x}\eta^+(\cdot)$ are continuous on $[0, x_*^+]$, there are constants $k_1, k_2 \in \mathbb{R}_+$ such that $-\frac{\mathrm{d}}{\mathrm{d}x}\eta^+(y_n) \leq k_1$ and $p(z_n) \leq k_2$ for all values of n. This yields $\sum_{n=0}^{N_2-1}(\eta^+(x_n^+) - \eta^+(x_{n+1}^+))\int_{x_n^+}^{x_{n+1}^+} p(x)\mathrm{d}x \leq \frac{k_1 k_2 (x_*^+)^2}{N_2}$ which approaches 0 as N_2 tends to infinity.

Appendix B

For notational simplicity, let us define $f(u) = \bar{\mathsf{e}}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|0)$ and $g(u, h_{1,1}) = \mathsf{e}_1 p_{\boldsymbol{u}_1(x_1)|\boldsymbol{e}_1}(u|1)$. Note that by (27), $f(u)$ does not depend on $h_{1,1}$. Then

$$\begin{aligned}\frac{\varepsilon(u, h_{1,1})}{\log e} &= (f(u) + g(u, h_{1,1}))\ln(f(u) + g(u, h_{1,1})) \\ &\quad - f(u)\ln(f(u)) - g(u, h_{1,1})\ln(g(u, h_{1,1})).\end{aligned} \tag{36}$$

Since the function $x\ln(x)$ is smooth for $x > 0$ with derivative $1 + \ln(x)$, the mean value theorem implies there exists ξ (depending on u and $h_{1,1}$) such that

$$g(u, h_{1,1}) < \xi < f(u) + g(u, h_{1,1}) \tag{37}$$

and

$$\begin{aligned}&(f(u) + g(u, h_{1,1}))\ln(f(u) + g(u, h_{1,1})) \\ &- g(u, h_{1,1})\ln(g(u, h_{1,1})) = (1 + \ln(\xi))f(u).\end{aligned} \tag{38}$$

Note that $f(u) \leq \frac{\bar{\mathsf{e}}_1}{\sqrt{2\pi}}$ and $g(u, h_{1,1}) \leq \frac{\mathsf{e}_1}{\sqrt{2\pi}}$. Using these bounds together with (36), (37) and (38), $\frac{\varepsilon(u, h_{1,1})}{\log e} \leq \left(1 + \frac{1}{2}\ln\frac{1}{2\pi}\right)f(u) - f(u)\ln f(u)$. The right side of this inequality does not depend on $h_{1,1}$ and it is an integrable function as $f(u)$ represents a Gaussian pulse. Moreover, $\lim_{h_{1,1}\to\infty}\varepsilon(u, h_{1,1}) = 0$ for any $u \in \mathbb{R}$. Then one can use dominated convergence to exchange the limit and integral, i.e., $\lim_{h_{1,1}\to\infty}\int_{-\infty}^{\infty}\varepsilon(u, h_{1,1})\mathrm{d}u = \int_{-\infty}^{\infty}\lim_{h_{1,1}\to\infty}\varepsilon(u, h_{1,1})\mathrm{d}u = 0$ as desired.

References

[1] C. E. Shannon, "Two-way communication channels", *in Proc.* 4th Berkeley Symp. Math. Satist. Probab., vol. 1, 1961, pp. 611-644.

[2] T. S. Han, "A general coding theorem for the two-way channel", *IEEE Trans. on Inf. Theory*, vol. IT-30, pp. 35-44, Jan. 1984.

[3] A. Tulino, S. Verdu, G. Caire, S. Shamai "The Gaussian Erasuare Channel," *ISIT 2007. IEEE International Symposium on Information Theory*, pp.1721 - 1725, 24-29 June 2007.

[4] R. M. Dudley, "Real analysis and probability", *Cambridge University Press*, 2005.

[3] An alternating series $\sum_{n=1}^\infty (-1)^{n-1}a_n$ where $a_n > 0$ converges if a_n is decreasing and $\lim_{n\to\infty} a_n = 0$. Moreover, for any $N \in \mathbb{N}$, $\sum_{n=1}^{2N}(-1)^{n-1}a_n \leq \sum_{n=1}^{\infty}(-1)^{n-1}a_n \leq \sum_{n=1}^{2N+1}(-1)^{n-1}a_n$.