

Achievable rate region based on coset codes for multiple access channel with states

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Abstract—We derive a new achievable rate region for the problem of communicating over a multiple access channel with states. Our coding technique is based on the ensemble of nested coset codes and the technique of typicality decoding. Exploiting structure in this ensemble, we analyze a more efficient decoding strategy to improve upon the rate region achievable using unstructured codes. We identify examples for which the achievable rate region based on nested coset codes is strictly larger than the ones achievable using random unstructured codes.

I. INTRODUCTION

The most common technique of proving achievability of rate regions in information theory is random coding. Traditionally, the distribution induced on the ensemble of codes is such that individual codewords are mutually independent. Furthermore, in communication models with multiple terminals, codebooks associated with these terminals are mutually independent of each other. Such an analysis has proved sufficient for single user and particular multi-terminal communication problems.¹

The problem of distributed reconstruction of mod-2 sum of binary sources studied by Körner and Marton [1], proved to be the first exception. They proposed a coding technique based on nested linear codes that strictly outperforms the best known strategy based on independent unstructured codes. Recently, a similar phenomenon has been identified by Philosof and Zamir [2] for a particular multiple access channel with state information distributed at the transmitters (MAC-DSTx). Restricting their attention to a binary symmetric noiseless additive doubly dirty MAC-DSTx (BDD-MAC), they propose a partition of the two channel codes into bins using cosets of a common linear code.² They propose a coding technique, henceforth referred to as PZ-technique, that achieves the capacity of BDD-MAC and thereby prove strict sub-optimality of the best known coding technique based on independent unstructured codes.

Nevertheless ingenious, PZ-technique is very specific to the additive and symmetric nature of BDD-MAC. This technique being strictly more efficient than currently known best strategy based on independent unstructured codes raises the following question. Is there a general coding scheme for communicating

over an *arbitrary* discrete MAC-DSTx, that reduces to the PZ-technique for the BDD-MAC, and that would yield an achievable rate region strictly larger than the best known achievable rate region using unstructured independent codes even for non-additive and non-symmetric MAC-DSTx?

In this article, we propose a coding scheme based on nested coset codes [4] for communication over an arbitrary MAC-DSTx and thereby answer the above questions in the affirmative. We present our coding scheme in two stages. We begin by identifying two central elements of PZ-technique 1) decoding mod-2 sum, instead of the pair, of auxiliary codewords and 2) choosing source code of each user's code to be cosets of a common linear code to enable restrict range of this mod-2 sum of auxiliary codewords. In the first stage, presented in section III, we employ nested coset codes and analyze decoding the sum of auxiliary codewords over an *arbitrary* MAC-DSTx. This stage captures the key element of our coding scheme which is the use of joint typical encoding and decoding of nested coset codes that enable us induce arbitrary distributions over the (auxiliary) input alphabet.

The significance of the rate region proved achievable in the first stage is illustrated through an example in section III-C for which it is necessary to induce non-uniform input distributions and is more efficient to decode the sum of transmitted codewords. This example illustrates that structured-code based strategies do not hinge on the channel being additive but would benefit as long as the optimizing test channel from the auxiliary inputs to the channel output is not far from additive.

Does the rate region proposed in the first stage subsume the rate region achievable using unstructured coding technique? It is our belief that techniques based on structured codes are not in lieu of their counterparts based on unstructured codes. Indeed, the technique proposed by Körner and Marton is strictly inferior to that of Berger-Tung [5] for a class of source distributions. We therefore take the approach of Ahlswede and Han [6, Section VI] and propose a two layer coding scheme in section IV that incorporates both unstructured and structured coding techniques. We present an example to illustrate how the gluing of unstructured and structured coding techniques can yield a rate region larger than either one, and their union. We remark that in spite of our inability to compute the achievable rate region proposed in section IV, we are able to demonstrate the significance of the same through an example.

¹However, characterization of optimal performance in multi-terminal communication problems such as distributed source coding, broadcast channel, multiple description coding remain open.

²Recall that communicating over a channel with state information at transmitter involves binning of the codebooks of the two transmitters [3].

If the channel is far from additive, it may not be efficient to decode the sum, with respect to a finite field, of codewords. For example, if the MAC-DSTx is doubly dirty with field addition replaced by addition of an Abelian group, then it is natural to decode group sum of codewords. In other words, the technique of decoding sum of codewords must be generalized to decoding any arbitrary bivariate function of the auxiliary inputs. In [7], we employ codes over Abelian groups [8] to decode group sum of transmitted codewords, and thereby develop an algebraic framework for communication over an arbitrary MAC-DSTx based on the principles presented herein.

Several findings in the context of multi-terminal communication problems point to efficient strategies based on structured codes. The $K(\geq 3)$ -user Gaussian and discrete interference channels benefit from the use of structured codes. Coding techniques based on lattices [9] for the former, and nested coset codes [10] for the latter have been proven to outperform Han-Kobayashi [11] technique. Krithivasan and Pradhan [12] propose a framework based on structured codes for the distributed source coding problem that outperforms unstructured-code based techniques [5]. We have employed the same ensemble of nested coset codes to strictly enlarge the largest known achievable rate region for the general 3-user discrete broadcast channel in [13].

II. PRELIMINARIES AND PROBLEM STATEMENT

We employ notation that is now widely employed in information theory literature supplemented by the following. For any set A , $\text{cocl}(A)$ denotes closure of the convex hull of A . We let \oplus denote addition in a finite field.³ $h_b : [0, 1] \rightarrow [0, 1]$ defined as $h_b(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ denotes binary entropy function. For $j \in \{1, 2\}$, j denotes the element in $\{1, 2\} \setminus \{j\}$.

A. MAC-DSTx

Consider the two user multiple access analogue of the point to point channel with state (PTP-STx) studied by Gelfand and Pinsker [3]. Let \mathcal{X}_1 and \mathcal{X}_2 denote finite input alphabet sets and \mathcal{Y} , the output alphabet set. Transition probabilities depend on a random parameter $\mathbf{S} := (S_1, S_2)$, called state, that takes values in a finite set $\mathcal{S} := \mathcal{S}_1 \times \mathcal{S}_2$. The discrete time channel is time invariant, memoryless, and used without feedback. Let $W_{Y|XS}(y|x, s)$ be probability of observing $y \in \mathcal{Y}$ given $\mathbf{x} := (x_1, x_2) \in \mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ is input to the channel in state $\mathbf{s} := (s_1, s_2) \in \mathcal{S}$. The state at time i , \mathbf{S}_i is (i) independent of $(\mathbf{S}_t, \mathbf{X}_t, \mathbf{Y}_t) : 1 \leq t < i$, and (ii) identically distributed for all i . Let $W_S(s)$ be probability of MAC-DSTx being in state $\mathbf{s} \in \mathcal{S}$. We assume S_j^n is non causally known to encoder j . Input X_j is constrained with respect to an additive cost function $\kappa_j : \mathcal{X}_j \times \mathcal{S}_j \rightarrow [0, \infty)$. We refer the reader to [2] for standard definitions of a code, achievability, capacity. Let $\mathbb{C}(\boldsymbol{\tau}) := \text{cl}\{\mathbf{R} = (R_1, R_2) \in \mathbb{R}^2 : (\mathbf{R}, \boldsymbol{\tau}) \text{ is achievable}\}$ denote the capacity region when constrained to average cost of $\boldsymbol{\tau} = (\tau_1, \tau_2)$.

³The particular finite field is uniquely determined it's cardinality.

B. An achievable rate region based on unstructured codes

The coding technique that achieves capacity of PTP-STx [3] can be generalized to obtain the largest known inner bound to $\mathbb{C}(\boldsymbol{\tau})$. We provide a characterization of the same below.

Definition 1: Let $\mathbb{D}(\boldsymbol{\tau})$ be collection of pmfs p_{UXSY} on $\mathcal{U}^2 \times \mathcal{X} \times \mathcal{S} \times \mathcal{Y}$, where \mathcal{U} denotes U_1, U_2 and \mathcal{U}^2 is a two fold Cartesian product of a finite set \mathcal{U} , such that (i) $p_S = W_S$, (ii) $p_{Y|XSY} = p_{Y|XS} = W_{Y|XS}$, (iii) $p_{U_j|SU_j} = p_{U_j|S} = p_{U_j|S_j}$ and $p_{X_j|SU_j} = p_{X_j|S} = p_{X_j|S_j U_j}$ for all $j \in \{1, 2\}$, (iv) $p_{X_j|S_j U_j}(x_j|s_j, u_j) \in \{0, 1\}$ for all $(u_j, s_j, x_j), j \in \{1, 2\}$ and (v) $\mathbb{E}\{\kappa_j(X_j, S_j)\} \leq \tau_j$ for $j = 1, 2$. For $p_{UXSY} \in \mathbb{D}(\boldsymbol{\tau})$, let $\alpha(p_{UXSY})$ be defined as the set of rate pairs $(R_1, R_2) \in [0, \infty)^2$ that satisfy

$$\begin{aligned} R_j &< I(U_j; Y, U_j) - I(U_j; S_j) \text{ for } j \in \{1, 2\} \\ R_1 + R_2 &< I(U; Y) + I(U_1; U_2) - \sum_{j=1}^2 I(U_j; S_j), \text{ and} \\ \alpha(\boldsymbol{\tau}) &:= \text{cocl} \left(\bigcup_{p_{UXSY} \in \mathbb{D}(\boldsymbol{\tau})} \alpha(p_{UXSY}) \right). \end{aligned}$$

Theorem 1: $\alpha(\boldsymbol{\tau}) \subseteq \mathbb{C}(\boldsymbol{\tau})$.

Achievability of $\alpha(\boldsymbol{\tau})$ can be proved by employing the encoding technique proposed in [3] at each encoder and joint decoding [14], [15] at the decoder. In the sequel, we provide an illustration of this coding technique for BDD-MAC.

C. Rates achievable using unstructured codes for BDD-MAC

Philosof and Zamir characterize $\mathbb{C}(\boldsymbol{\tau})$ for BDD-MAC using PZ-technique and prove $\alpha(\boldsymbol{\tau}) \subsetneq \mathbb{C}(\boldsymbol{\tau})$ for the same. In order to identify the key elements of PZ-technique, we briefly analyze unstructured coding (this section), PZ-technique (section III-A) and set the stage for a new coding scheme.

BDD-MAC is a MAC-DSTx with binary alphabets $\mathcal{S}_j = \mathcal{X}_j = \mathcal{Y} = \{0, 1\}$, $j = 1, 2$. The state sequences are independent Bernoulli- $\frac{1}{2}$ processes, i.e., $W_S(s) = \frac{1}{4}$ for all $\mathbf{s} \in \mathcal{S}$. The channel transition is described by the relation $Y = X_1 \oplus S_1 \oplus X_2 \oplus S_2$. An additive Hamming cost is assumed on the input, i.e., $\kappa_j(1, s_j) = 1$ and $\kappa_j(0, s_j) = 0$ for any $s_j \in \mathcal{S}_j$, $j = 1, 2$ and the input is subject to a symmetric cost constraint $\boldsymbol{\tau} = (\tau, \tau)$.

We describe the test channel $p_{USXY} \in \mathbb{D}(\boldsymbol{\tau})$ that achieves $\alpha(\boldsymbol{\tau})$. For each user j , consider the test channel that achieves the Gelfand-Pinsker capacity treating the other user as noise i.e., $p_{U_j S_j X_j}(0, 1, 1) = p_{U_j S_j X_k}(1, 0, 1) = \frac{\tau}{2}$, $p_{U_j S_j X_j}(0, 0, 0) = p_{U_j S_j X_j}(1, 1, 0) = \frac{1-\tau}{2}$. Philosof and Zamir prove $p_{USX} = p_{U_1 S_1 X_1} p_{U_2 S_2 X_2}$ achieves $\alpha(\boldsymbol{\tau}) = \{\mathbf{R} : R_1 + R_2 \leq |2h_b(\tau) - 1|^+\}$, where $|\cdot|^+$ denotes upper convex envelope.

Let us take a closer look at achievability of the vertex $(2h_b(\tau) - 1, 0)$ using the above test channel. Since user 2 has no message to transmit, it picks a single bin with roughly $2^{nI(U_2; S_2)} = 2^{n(1-h_b(\tau))}$ codewords independently and uniformly from the entire space of binary vectors. User 1 picks 2^{nR_1} bins each with roughly $2^{nI(U_1; S_1)} = 2^{n(1-h_b(\tau))}$ independently and uniformly distributed binary vectors. Encoder 2 observes S_2^n and chooses a codeword, say U_2^n , that is within a Hamming distance of roughly $n\tau$ from S_2^n and

transmits $X_2^n = U_2^n \oplus S_2^n$. Encoder 1 performs a similar encoding, except that it restricts the choice of U_1^n to the bin indexed by user 1's message, and transmits $X_1^n = U_1^n \oplus S_1^n$.

What is the maximum rate R_1 at which user 1 can transmit its message? Decoder receives $Y^n = U_1^n \oplus U_2^n$ and looks for all pairs of codewords that are jointly typical with Y^n . Since any pair of binary n -length vectors are jointly typical (U_1 and U_2 are independent and uniform), the decoding rule reduces to finding all pairs of binary n -length vectors in the pair of codebooks that sum to the received vector Y^n . All bins chosen independently without structure imply that any bin of user 1's codebook when added to the user 2's codebook (a single bin) results in roughly $2^{n(2-2h_b(\tau))}$ distinct vectors. Therefore, we cannot hope to pack more than roughly $\frac{2^n}{2^{n(2-2h_b(\tau))}} = 2^{n(2h_b(\tau)-1)}$ bins in user 1's codebook. We remark that *an explosion in the range of sum of transmitted codewords severely limits achievable rate.*

We make a few observations. Effectively, communication occurs over the $(U_1, U_2) - Y$ channel and the test channel induces the Markov chain $(U_1, U_2) - U_1 \oplus U_2 - Y$. It would therefore be more efficient to communicate information over the $U_1 \oplus U_2 - Y$ channel which suggests an efficient utilization of $U_1 \oplus U_2$ -space. Having chosen codewords in each bin independently and moreover the two users' bins independently, each message pair utilizes $2^{n(2-2h_b(\tau))}$ vectors in the $U_1 \oplus U_2$ -space. In section III-A, we summarize PZ-technique, wherein the algebraic structure in the codebooks is exploited for more efficient utilization of $U_1 \oplus U_2$ -space.

III. AN ACHIEVABLE RATE REGION USING NESTED COSET CODES

A. Nested linear codes for BDD-MAC

We present PZ-technique proposed for BDD-MAC. The encoding and decoding techniques are similar to that stated in II-C except for one key difference. The bins of user 1 and 2's codebooks are cosets of a common linear code. In particular, let λ_I denote a linear code of rate roughly equal to $1 - h_b(\tau)$ that can quantize a uniform source, state S_j^n in our case, within an average Hamming distortion of τ . Since user 2 has no message to transmit, it employs λ_I as its only bin. Encoder 1 employs 2^{nR_1} cosets of λ_I within a larger linear code, called λ_O , as its bins. Note that rate of λ_O is roughly $R_1 + 1 - h_b(\tau)$. Encoding rule is as described in section II-C.

User 2's codebook when added to any bin of user 1's code results in a coset of λ_I , and therefore contains approximately at most $2^{n(1-h_b(\tau))}$ codewords. Moreover, since U_1^n lies in λ_I , user 2's codeword U_2^n and the received vector $Y^n = U_1^n \oplus U_2^n$ lie in the same coset.⁴ Since the channel is noiseless, user 1 may employ all cosets of λ_I and therefore communicate at rate $h_b(\tau)$ which is larger than $2h_b(\tau) - 1$ for all $\tau \in (0, \frac{1}{2})$.

What are the key elements of PZ-technique? Each message pair corresponds to roughly $2^{n(1-h_b(\tau))}$ vectors in $U_1 \oplus U_2$ -space, resulting in a more efficient utilization of this space. This indeed is the difference in the sum rate achievable

using independent unstructured codes and PZ-technique. We also note the *decoder does not attempt to disambiguate the pair (U_1^n, U_2^n) and restricts to decoding $U_1^n \oplus U_2^n$. This is motivated by the Markov chain $(U_1, U_2) - U_1 \oplus U_2 - Y$ induced by the test channel and the use of structured codebooks that contain the sum.*

It is instructive to investigate the efficacy of this technique if users 1 and 2 employ distinct linear codes $\lambda_{I1}, \lambda_{I2}$ of rate $1 - h_b(\tau)$ instead of a common linear code λ_I . In this case, each message of user 1 can result in $2^{2-2h_b(\tau)}$ received vectors which restricts user 1's rate to $2h_b(q) - 1$ and provides no improvement over the unstructured coding technique. We conclude that *if the bins of the MAC channel code are nontrivial, as in this case due to the presence of a state, then it maybe beneficial to endow the bins with an algebraic structure that restricts the range of a bivariate function, and enable the decoder decode this function of chosen codewords.*

B. An achievable rate region for arbitrary MAC-DSTx using nested coset codes

In this section, we present the first stage of our coding scheme that uses joint typical encoding and decoding and nested coset codes over an arbitrary MAC-DSTx. The technique proposed by Philosof and Zamir is specific to the BDD-MAC - Hamming cost constraint that induces additive test channels between the auxiliary and state random variables, and additive and symmetric nature of the channel. Moreover, linear codes only achieve the symmetric capacity, and therefore if the output were obtained by passing $(X_1^n \oplus S_1^n, X_2^n \oplus S_2^n)$ through an asymmetric MAC, linear codes though applicable, might not be optimal.

We begin with a characterization of test channels followed by achievability.

Definition 2: Let $\mathbb{D}_f(\tau) \subseteq \mathbb{D}(\tau)$ be the collection of distributions $p_{\mathbf{V} \mathbf{S} \mathbf{X} \mathbf{Y}}$ on $\mathcal{V}^2 \times \mathcal{S} \times \mathcal{X} \times \mathcal{Y}$ where \mathcal{V} is a finite field. For $p_{\mathbf{V} \mathbf{X} \mathbf{S} \mathbf{Y}} \in \mathbb{D}_f(\tau)$, let $\beta_f(p_{\mathbf{V} \mathbf{X} \mathbf{S} \mathbf{Y}})$ be defined as the set of rate pairs $(R_1, R_2) \in [0, \infty)^2$ that satisfy

$$R_1 + R_2 < \min\{H(V_1|S_1), H(V_2|S_2)\} - H(V_1 \oplus V_2|Y), \text{ and}$$

$$\beta_f(\tau) := \text{cocl} \left(\bigcup_{p_{\mathbf{V} \mathbf{X} \mathbf{S} \mathbf{Y}} \in \mathbb{D}_f(\tau)} \beta_f(p_{\mathbf{V} \mathbf{X} \mathbf{S} \mathbf{Y}}) \right).$$

Theorem 2: $\beta_f(\tau) \subseteq \mathcal{C}(\tau)$.

In the interest of brevity, we only state the coding technique and refer to [7] for a detailed proof.

As stated in section III-A, the key aspect is to employ cosets of a common linear code as bins for quantizing the state. We employ three nested coset codes - one each for the two encoders and the decoder - that share a common inner (sparser) code. We begin by describing the encoding rule. The nested coset code provided to encoder j is described through a pair of generator matrices $g_I \in \mathcal{V}^{k \times n}$ and $g_{Oj/I} \in \mathcal{V}^{l_j \times n}$ where (i) g_I and $g_{Oj/I}^T := \begin{bmatrix} g_I^T & g_{Oj/I}^T \end{bmatrix}$ are generator matrices of inner and complete (denser) codes respectively, (ii)

$$\frac{k \log |\mathcal{V}|}{n} > \log |\mathcal{V}| - \min\{H(V_1|S_1), H(V_2|S_2)\}, \quad (1)$$

$$\frac{(k+l_1+l_2) \log |\mathcal{V}|}{n} < 1 - H(V_1 \oplus V_2|Y), \quad (2)$$

⁴This is also because the channel is noiseless.

and (ii) bias vector b_j^n . Let λ_I and λ_{Oj} denote linear codes corresponding to generator matrices g_I and g_{Oj} respectively. User j 's message $M_j^{l_j} \in \mathcal{V}^{l_j}$ indexes the coset $(a^k g_I \oplus M_j^{l_j} g_{Oj/I} \oplus b_j^n : a^k \in \mathcal{V}^k)$. Encoder j observes state S_j^n and looks for a codeword in the coset indexed by the message that is jointly typical with the state sequence S_j^n according to $p_{S_j V_j}$. If it finds one such codeword, say V_j^n , a vector X_j^n is generated according $\prod_{t=1}^n p_{X_j|S_j V_j}(\cdot|S_{jt} V_{jt})$ and X_j^n is fed as input to the channel. Otherwise, it declares an error.

Now to the decoding rule. Let λ_O denote the complete code provided to the decoder, i.e., the coset code whose (i) generator matrix is $g_O^T := \begin{bmatrix} g_I^T & g_{O/I}^T \end{bmatrix}$, where $g_{O/I}^T := \begin{bmatrix} g_{O1/I}^T & g_{O2/I}^T \end{bmatrix}$ and (ii) bias vector $b_1^n \oplus b_2^n$. Having received Y^n , it lists all codewords in λ_O that are jointly typical with Y^n with respect to $p_{V_1 \oplus V_2, Y}$. If all such codewords belong to a unique coset (of λ_I in λ_O) say $(a^k g_I \oplus m_1^{l_1} g_{O1/I} \oplus m_2^{l_2} g_{O2/I} \oplus b_1^n \oplus b_2^n : a^k \in \mathcal{V}^k)$, it declares $(m_1^{l_1}, m_2^{l_2})$ as the pair of decoded messages. Otherwise, it declares an error.

We pick entries of each of the constituent generator matrices $g_I, g_{O1/I}, g_{O2/I}$ independently and uniformly from \mathcal{V} . Each codeword in the coset indexed by the message is uniformly distributed over \mathcal{V}^n and is therefore jointly typical with state sequence with probability $|\mathcal{V}|^{n(H(V_j|S_j)-1)}$.⁵ An informed reader will now recognize that if the coset is of rate at least $1 - \frac{H(V_1|S_1)}{|\mathcal{V}|}$, i.e., (1) is satisfied, then the encoder will find a codeword jointly typical with the state sequence with high probability. Thus, nesting of cosets with the inner code chosen sufficiently large enables achieve arbitrary input distributions.

The decoder decodes into a codebook that is a union of uniformly distributed cosets, the codewords of which are also uniformly distributed over \mathcal{V}^n . Any codeword other than the sum of legitimate transmitted codewords is jointly typical with received vector with probability $|\mathcal{V}|^{n(H(V_1 \oplus V_2|Y)-1)}$. Employing a union bound one can prove the rate of the union of cosets can be at most $1 - \frac{H(V_1 \oplus V_2|Y)}{|\mathcal{V}|}$. The decoder therefore makes an error with arbitrarily small probability if (2) is satisfied. From (1), (2) it can be verified that $R_1 + R_2 = \frac{l_1 + l_2}{n} \leq \min \{H(V_1|S_1), H(V_2|S_2) - H(V_1 \oplus V_2|Y)\}$ is achievable.

We conclude this section with two remarks.

Remark 1: For BDD-MAC $\beta_f(\tau) = \mathbb{C}(\tau)$.⁶ Indeed, the test channel $p_{V S X Y} \in \mathbb{D}_f(\tau)$ defined as $p_{V S X} = \prod_{j=1}^2 p_{V_j S_j X_j}$ where V_j takes values over $\mathcal{V}_j = \{0, 1\}$ with

$$p_{V_j, X_j|S_j}(x_j \oplus s_j, x_j|s_j) = \begin{cases} 1 - \tau & \text{if } x_j = 0 \\ \tau & \text{otherwise} \end{cases}$$

for each $j = 1, 2$ and $s_j \in \{0, 1\}$ achieves $\mathbb{C}(\tau) = \{(R_1, R_2) : R_1 + R_2 \leq h_b(\tau)\}$.

We have thus presented a coding scheme based on decoding the sum of codewords chosen by the encoders and presented an achievable rate region for an arbitrary MAC-DSTx. One might attempt a generalization of PZ-technique using modulo-lattice transformation proposed by Haim, Kochman and Erez

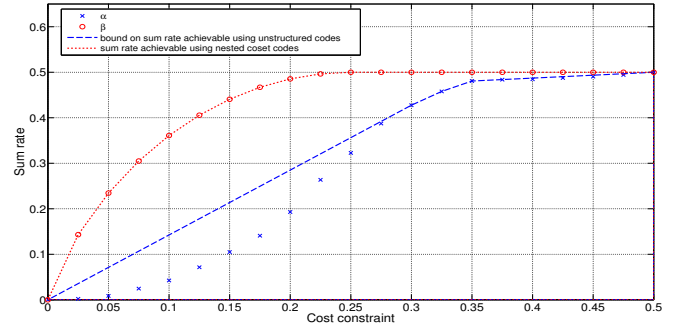


Fig. 1. Bounds on sum rate for example 1

[16]. The rate region proposed herein subsumes that achievable through modulo-lattice transformation using test channels identified through the virtual channel in a natural way.

C. Examples

A key element of the coding framework proposed herein lies in characterizing achievable rate regions for arbitrary test channels, i.e., test channels that are not restricted to be uniform or additive in nature using structured codes. Example 1 illustrates that such test channels indeed optimize the achievable rate region for certain MAC-DSTx.

A few remarks on our study of example 1 are in order. The example needing to be non-additive lends it considerably hard to provide analytical upper bounds for the rate region achievable using unstructured codes.⁷ We therefore resort to computation. Limited by currently available computational resources, we sample the space of probability distributions with binary auxiliary alphabet with a step size of 0.015 in each dimension followed by convex hull operation. The resulting bound on the sum rate achievable using unstructured codes (without time sharing) is marked with blue crosses (denoted α in the legend) in the plots. The resulting upper bound is obtained as an upper convex envelope. Similarly, sum rate achievable using nested coset codes is marked with red circles (denoted β in the legend) in the plots.

Example 1: As in BDD-MAC, we assume the alphabet sets to be binary $\mathcal{S}_j = \mathcal{X}_j = \{0, 1\}$, $j = 1, 2$, (ii) uniform and independent states, i.e., $W_S(s) = \frac{1}{4}$ for all $s \in \mathcal{S}$, (iii) a Hamming cost function $\kappa_j(1, s_j) = 1$ and $\kappa_j(0, s_j) = 0$ for any $s_j \in \mathcal{S}_j$, $j = 1, 2$. Let $Y = (X_1 \vee S_1) \oplus (X_2 \vee S_2)$, where \vee denotes logical OR operator. Having studied the BDD-MAC it is natural to conjecture that the test channel that optimizes the sum rate achievable using linear codes to be $p_{U_j X_j|S_j}(0, 0|0) = 1 - 2\tau$, $p_{U_j X_j|S_j}(1, 1|0) = 2\tau$, $p_{U_j X_j|S_j}(1, 0|1) = 1$, for $j = 1, 2$ when the cost constraint $\tau \in [0, \frac{1}{4}]$. Indeed, our numerical computation asserts this. In other words, the sum rate achievable using nested coset codes for a cost $\tau \in (0, \frac{1}{4})$ is $\frac{h_b(2\tau)}{2}$ and 0.5 for $\tau \in [0.25, 0.5]$. The sum rate achievable using unstructured codes and nested coset codes are plotted

⁵Here the entropy is evaluated with respect to base $|\mathcal{V}|$.

⁶This follows from characterization of $\mathbb{C}(\tau)$ for BDD-MAC in [2].

⁷We recognize that the analytical upper bound derived in [2] is a key element of the findings therein.

in figure 1. We highlight significant gains achievable using nested coset codes.

A preliminary look at this channel may lead the reader to conclude that PZ-technique appropriately modified can achieve the same sum rate as that achievable using nested coset codes, since the above test channel is additive, i.e., $U_j = S_j \oplus X_j$ for $j = 1, 2$ and $Y = U_1 \oplus U_2$. However, a careful analysis will reveal the significance of the coding framework proposed herein. The induced pmf on U_j , $p_{U_j}(1) = \frac{1}{2} + 2\tau$ for $\tau \in (0, \frac{1}{4})$ is *not* uniform, and the PZ-technique of choosing a codeword in the indexed bin with an average Hamming distance of τ does not yield the sum rate guaranteed by nested coset codes. Nesting of codes enables achieving non-uniform distributions that are necessary as exemplified herein.

IV. A UNIFIED ACHIEVABLE RATE REGION

In this section, we put together the techniques of unstructured and structured random coding to derive an achievable rate region for a general MAC-DSTx. Our approach is similar to that proposed in [6, Section VI] for the problem of reconstructing mod-2 sum of distributed binary sources. We begin with a characterization of valid test channels.

Definition 3: Let $\mathbb{D}_{sf}(\tau) \subseteq \mathbb{D}(\tau)$ be the collection of distributions p_{UVSXY} on $(\mathcal{U} \times \mathcal{V})^2 \times \mathcal{S} \times \mathcal{X} \times \mathcal{Y}$ where \mathcal{U} is a finite set and \mathcal{V} is a finite field. For $p_{UVSXY} \in \mathbb{D}_{sf}(\tau)$, let $\mathcal{R}_{sf}(p_{UVSXY})$ be defined as the set of rate pairs $(R_1, R_2) \in [0, \infty)^2$ that satisfy

$$R_j \leq I(U_j; U_j Y) - I(U_j; S_j) + \rho \text{ for } j \in \{1, 2\}$$

$$R_1 + R_2 \leq I(U; Y) + I(U_1; U_2) - \sum_{j=1}^2 I(U_j; S_j) + \rho,$$

where

$$\rho := \min\{H(V_1|U_1, S_1), H(V_2|U_2, S_2)\} - H(V_1 \oplus V_2|U, Y).$$

Let

$$\mathcal{R}_{sf}(\tau) := \text{cocl} \left(\bigcup_{p_{UVSXY} \in \mathbb{D}_{sf}(\tau)} \mathcal{R}_{sf}(p_{UVSXY}) \right).$$

Theorem 3: $\mathcal{R}_{sf}(\tau) \subseteq \mathcal{C}(\tau)$.

Proof of achievability follows by a standard analysis of sequential encoding and decoding. The reader is referred to [7] for an outline of the coding technique and a proof.

Remark 2: $\alpha(\tau) \subsetneq \mathcal{R}_{sf}(\tau)$.

We conclude with an illustrative example.

Example 2: Assume (i) $\mathcal{S}_j = \mathcal{X}_j = \{0, 1\}$, $j = 1, 2$, (ii) uniform and independent states, i.e., $W_S(s) = \frac{1}{4}$ for all $s \in \mathcal{S}$, (iii) a Hamming cost function $\kappa_j(1, s_j) = 1$ and $\kappa_j(0, s_j) = 0$ for any $s_j \in \mathcal{S}_j$, $j = 1, 2$. The channel transition is described as $W_{Y|XS}(y|x, s) = W_{Y|g(X, S)}^*(y|g(x, s))$, where $g(x, s) = [s_1 \wedge (s_1 \oplus x_1)] \oplus [s_2 \wedge (s_2 \oplus x_2)]$, \wedge denotes logical AND, and $W_{Y|g(X, S)}^*(1|0) = 0.02$, $W_{Y|g(X, S)}^*(0|1) = 0.04$.

The bounds on the sum rate achievable with unstructured and nested coset codes are plotted in figure 2. The above plots unequivocally indicate $\mathcal{R}_{sf}(\tau)$ to be strictly larger than $\alpha(\tau) \cup \beta_f(\tau)$ and in particular either one of $\alpha(\tau)$, $\beta_f(\tau)$.

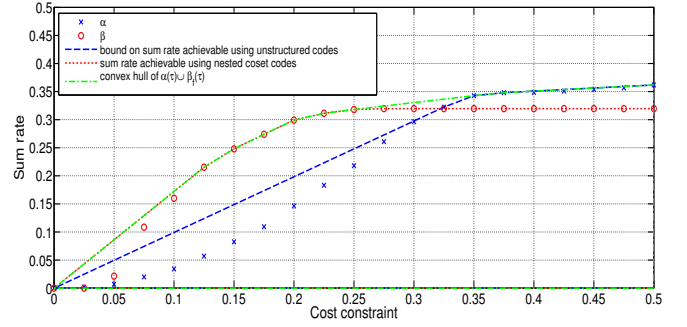


Fig. 2. Bounds on sum rate for example 2

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