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A New Sub-Packetization Bound for Minimum Storage Regenerating Codes

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***Abstract*—Codes for distributed storage systems are often designed to sustain failure of multiple storage disks. Specifically, an (n, k) MDS code stores k symbols in n disks such that the overall system is tolerant to a failure of up to $n - k$ disks. However, access to at least k disks is still required to repair a single erasure. To reduce repair bandwidth, array codes are used where the stored symbols or packets are vectors of length ℓ. MDS array codes can potentially repair a single erasure using a fraction $1/(n - k)$ of data stored in the surviving nodes. We ask the following question: for a given (n, k), what is the minimum vector-length or sub-packetization factor ℓ required to achieve this optimal fraction? For *exact recovery* of systematic disks in an MDS code of low redundancy, i.e. $k/n > 1/2$, the best known explicit codes [1] have a sub-packetization factor ℓ which is exponential in k. It has been conjectured [2] that for a fixed number of parity nodes, it is in fact necessary for ℓ to be exponential in k. In this paper, we provide new converse bounds on k for a given ℓ. We prove that $k \leq \ell^2$ for an arbitrary but fixed number of parity nodes $r = n - k$. For the practical case of 2 parity nodes, we prove a stronger result that $k \leq 4\ell$.**

I. Introduction

Maximum Distance Separable (MDS) codes are ubiquitous in distributed storage systems [3] and provide the maximum recoverability for an erasure code of a given redundancy. We define an (n, k, ℓ) storage system as consisting of n nodes or disks of capacity ℓ units, and storing a total of $k\ell$ data units. When the ℓ units in each disk constitute a symbol in an MDS (array) code, the system is immune to an erasure of up to $r = n - k$ disks. Failure of a single disk at a time occurs most frequently in practice. The objective is to quickly and efficiently recover the data in an erased disk. A naïve way is to recover the entire data by using any k of the surviving disks and reconstruct the data in the lost node. However, as can be seen in the simple example in Fig. 1, transmission of $k\ell$ units is not necessary to reconstruct the loss of only ℓ data units. Dimakis et al. [4] formalized this problem of efficient repair and proved that the bandwidth or the amount of transmitted data required to recover a single disk erasure is lower bounded by

$$\frac{n-1}{n-k}\ell \quad = \quad \frac{n-1}{r}\ell,$$

where all the surviving $n - 1$ disks transmit a fraction $1/r$ of their data.

a_1	b_1	$a_1 + b_1$	$a_2 + b_1$
a_2	b_2	$a_2 + b_2$	$a_1 + a_2 + b_2$

Fig. 1. A $(4, 2, 2)$ MDS code over $\mathbb{F}_2$ ([3], [5]). Each single erasure can be recovered using 3 data units. For example, to recover the first node (a_1, a_2), we can use b_2, $a_2 + b_2$ and $a_1 + a_2 + b_2$. To recover the second parity node $(a_2 + b_1, a_1 + a_2 + b_2)$, we can use $a_1, b_1 + b_2$ and $a_2 + b_2$, where the second unit of information can be obtained from the second systematic node by a linear combination of its stored data.

Codes which achieve this lower bound are called *optimal bandwidth MDS codes* or *minimum-storage regenerating (MSR) codes* with $n-1$ *helper* nodes. Much progress has been made recently in constructing such codes. Network coding is sufficient to obtain optimal bandwidth codes for *functional* repair, where the objective is to reconstruct a lost disk such that the MDS property is preserved in the new set of n disks. The case of *exact* repair, requiring the recovered node to exactly replicate the lost disk has proved to be more challenging. Optimal bandwidth exact repair codes were constructed in [6], [7] for the low-rate regime ($k/n < 1/2$), where the number of parity nodes exceed the number of systematic nodes. In the high-rate case ($k/n > 1/2$), optimality was proved to be achievable asymptotically [8]. Recent contributions construct finite length (finite ℓ) codes which exactly recover the systematic nodes [9], [10], [1]. Explicit finite-length optimal exact repair codes have also been constructed [9], [11], where both systematic as well as parity nodes are optimally recoverable.

A. Sub-packetization:

In this paper, we look closely at the question of disk capacity ℓ for which optimal bandwidth MDS codes exist for a given number of parity nodes r. This question is intimately connected to the concepts of array and block codes [5], [12], linear vector coding, sub-packetization [8], and symbol extension [13]. The capacity ℓ, also known as the *sub-packetization factor*, represents the minimum dimension over which all the recovery arithmetic operations are needed, independent of the field involved. For example, if $\ell = 1$, we cannot do better than reconstructing the entire file to recover a single erasure. A sub-packetization of $\ell > 1$ is required to achieve the optimal bandwidth. The disk capacity of ℓ symbols is linked to its actual "raw capacity" (or size in bits or bytes) via the field $\mathbb{F}$ over which the MDS code and the repair operations are assumed. For a given disk capacity of ℓ symbols and a given number of parity nodes r, the number of systematic nodes k which a DSS can "support" while storing an optimal bandwidth MDS code is bounded from above. Alternatively, a disk of size ℓ bits cannot store more than ℓ symbols (in any field) and therefore has a field-independent upper bound on k.

Example 1. A 1*kB Storage Disk.* Consider a storage disk of size 1 kilobyte $= 2^{13}$ bits. This disk can have a capacity of at most $\ell = 2^{13}$ symbols. For two parity nodes, the current upper bound on k is exponential in ℓ and amounts to more than 10^{2467} systematic nodes. We prove that k in fact cannot be more than $2^{15} = 32768$.

For the low-rate case, i.e, when $k/n < 1/2$, a linear sub-packetization (in terms of r) is sufficient [7]. In fact, $\ell = n-k$ when all the $n-1$ nodes aid in repair, because each disk need only contribute one unit (scalar repair) of repair bandwidth. The absence of optimal scalar linear repair codes [14] for the case of $n/k \geq 1/2$ justifies the search for vector linear repair codes. Cadambe et al. incorporated the idea of symbol-extension from interference alignment [8] and proved the existence of exact-repair MSR codes, albeit for asymptotically large ℓ. Finite-length codes were discovered by [9], [10], [1] where ℓ is exponential in the number of systematic nodes k.

Tamo et al. [2] conjecture that for a given disk capacity ℓ and an arbitrary fixed number of parity nodes r, the maximum number of systematic nodes k is of the order of $\log \ell$. For the specific practical case of two parity nodes ($r = 2$)‘, the current bounds for optimal k [2], [1] for a given value of ℓ are:

$$3\log_2 \ell \leq k \leq 1 + \ell\binom{\ell}{\ell/2}. \tag{1}$$

B. Our Contribution:

We provide new upper bounds on the maximum number of systematic nodes k possible for a given sub-packetization factor ℓ. The existence of optimal bandwidth MDS codes for the exact repair of systematic nodes can be expressed as an interesting linear algebra problem (e.g. [2]) involving subspaces and linear operators, as described in Section II. We exploit these conditions and prove in Section III that for a given number of parity nodes r, the maximum number of systematic nodes k is bounded by

$$k \quad \leq \quad \ell^2. \tag{2}$$

In other words, the sub-packetization factor should at least be $\sqrt{k}$. For the particular case of two parity nodes, we derive a stronger upper bound on k in Section IV:

$$k \quad \leq \quad 4\ell, \tag{3}$$

i.e., the sub-packetization is required to be of the order of k. We conclude in Section V.

II. Problem Setting

We define an (n, k, ℓ) MDS array code as a set of n symbol vectors (disks) of length ℓ over a field $\mathbb{F}$, such that any set of k vectors are sufficient to recover the entire data of $k\ell$ units. The first k symbols represent the systematic nodes consisting of the data vectors $v_1, \ldots, v_k$ of column-length ℓ. The remaining $r = n - k$ symbols are the parity nodes, which store linear combinations of the systematic data vectors. The vector v_{k+i} stored in parity node i is given by

$$v_{k+i} = \sum_{j=1}^{k} A_{i,j} v_j,$$

where $A_{i,j}$ is the *coding matrix* in $\mathbb{F}^{\ell\times\ell}$ corresponding to the parity node $i \in [r] := \{1, 2, \ldots, r\}$ and the systematic node $j \in [k]$. For optimal (bandwidth) repair of a failed systematic node $i \in [k]$, all other nodes transmit a fraction $1/r$ of the stored data, i.e., the *helper* node $j \neq i, j \in [n]$ transmits a vector of length ℓ/r given by $S_{i,j}v_j$, where $S_{i,j}$ is a matrix in $\mathbb{F}^{\ell/r\times\ell}$. An alternate interpretation[1] is that the vector transmitted by node j is the projection of v_j onto a subspace of dimension ℓ/r. The subspace corresponds to the subspace of the span of rows of $S_{i,j}$. It can be shown using interference alignment ideas [1] that the optimal repair of a systematic node i is possible if and only if there exist $\ell/r \times \ell$ matrices $S_{i,j}, j \neq i, j \in [n]$, which satisfy the following *subspace properties*:

$$S_{i,j} \quad \backsimeq \quad S_{i,k+t}A_{t,j}, \text{ and} \tag{4}$$
$$\sum_{t=1}^{r} S_{i,k+t}A_{t,i} \quad \backsimeq \quad \mathbb{F}^{\ell}, \tag{5}$$

for all $j \neq i, j \in [k], t \in [r]$. The equalities $\backsimeq$ in the subspace properties are defined on the row spans (or subspaces) instead of the corresponding matrices. The sum of subspaces B, C is defined as $B + C = \{b + c : b \in B, c \in C\}$. In fact, it can be shown that in (5), the sum of subspaces is a direct sum.

In general, we want to find for a given sub-packetization factor ℓ and the number of parity nodes r, the largest number of systematic nodes k for which there exists an optimal repair scheme (of systematic nodes). For $r = 2$ parity nodes, it can be shown [2] that if there exists an optimal bandwidth $(k + 3, k + 1, \ell)$ MDS code, then there exist a set of invertible matrices $\Phi_1, \ldots, \Phi_k$ of order ℓ and a corresponding set of subspaces $S_1, \ldots, S_k$ each of dimension $\ell/2$ such that for any $j \neq i, j \in [k]$,

$$S_i\Phi_j \quad \backsimeq \quad S_i, \text{ and} \tag{6}$$
$$S_i\Phi_i + S_i \quad \backsimeq \quad \mathbb{F}^{\ell}, \tag{7}$$

where the sum can again be seen as a direct sum.

[1] We alternate between interpreting $S_{i,j}$ and $S_{i,k+t}A_{t,j}$ as matrices of size $\ell/r \times \ell$ and subspaces of their row spans of dimension ℓ/r.

III. Sub-Packetization Bound for Arbitrary Number of Parity Nodes

We first prove a quadratic (in terms of ℓ) upper bound for the maximum possible number of systematic nodes k for the general case of r parity nodes.

Theorem 1: For a given disk capacity ℓ and for an arbitrary but fixed number of parity nodes r, the number of systematic nodes k is upper bounded by

$$k \quad \leq \quad \ell^2. \tag{8}$$

Proof: Consider the subspace properties (4) and (5) for the parity nodes 1 and 2. The idea[2] is to convert the given set of subspaces and matrices to another set which satisfy properties similar to the second set of subspace properties (6) and (7). Let us define for $i \in [k-1]$,

$$\Theta_i = A_{1,i}A_{2,i}^{-1}A_{2,k}A_{1,k}^{-1}.$$

We then have, using the properties (4) and (5), for $j \notin \{i,k\}$,

$$\begin{aligned} S_{j,k+1}\Theta_i &\simeq S_{j,k+1}A_{1,i}A_{2,i}^{-1}A_{2,k}A_{1,k}^{-1} \\ &\simeq S_{j,k+2}A_{2,i}A_{2,i}^{-1}A_{2,k}A_{1,k}^{-1} \\ &\simeq S_{j,k+2}A_{2,k}A_{1,k}^{-1} \\ &\simeq S_{j,k+1}A_{1,k}A_{1,k}^{-1} \\ &\simeq S_{j,k+1}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} & S_{i,k+1}\Theta_i \cap S_{i,k+1} = \{0\} \qquad (10) \\ \iff\quad & S_{i,k+1}A_{1,i}A_{2,i}^{-1} \cap S_{i,k+1}A_{1,k}A_{2,k}^{-1} = \{0\} \\ \iff\quad & S_{i,k+1}A_{1,i}A_{2,i}^{-1} \cap S_{i,k+2}A_{2,k}A_{2,k}^{-1} = \{0\} \\ \iff\quad & S_{i,k+1}A_{1,i} \cap S_{i,k+2}A_{2,i} = \{0\}, \end{aligned}$$

which is indeed true by (5). Defining $S_{i,k+1}$ as S_i, we have the properties

$$S_i\Theta_j \quad \simeq \quad S_i, \text{ and} \tag{11}$$

$$S_i\Theta_i \cap S_i \quad = \quad \{0\}, \tag{12}$$

for any $j \neq i, j \in [k-1]$, and 0 is the zero-vector.

The proof follows from the observation that the matrices $\Theta_i, i \in [k-1]$ and the identity matrix I are linearly independent. Since they lie in the ℓ^2-dimensional space of matrices $\mathbb{F}^{\ell\times\ell}$, $k \leq \ell^2$. To see that the matrices are linearly independent, suppose they are not. We then have some equation of the form, without loss of generality,

$$\Theta_1 = \sum_{i=2}^{t} \alpha_i\Theta_i + \beta\mathrm{I},\ t \leq k, \alpha_i \neq 0.$$

Then, operating Θ_1 on S_1, we have

$$S_1\Theta_1 \quad \simeq \quad S_1\left(\sum_{i=2}^{t}\alpha_i\Theta_i + \beta\mathrm{I}\right).$$

The right hand side of the above equation lies in S_1 from (11) and the fact that any subspace is invariant under the identity transformation[3]. But S_1 is non-intersecting[4] with the subspace on the left hand side $S_1\Theta_1$ from (12), which implies that $S_1\Theta_1 = \{0\}$. This further implies that $S_1 = \{0\}$, by the non-singularity of Θ_1, a contradiction to the fact that all $S_{i,k+1}$ are full rank matrices of rank ℓ/r. ■

IV. An Improved Bound for Two Parity Nodes

In this section we prove a stronger upper bound on k for the case of two parity nodes.

Theorem 2: For a given ℓ and k, if there exist invertible matrices $\Phi_i, i \in [k]$ and subspaces $S_i, i \in [k]$, satisfying the subspace conditions (6) and (7), then

$$k \leq \max\left(4\ell, 8\log_2\ell\right). \tag{13}$$

We first look at some interesting properties stemming from the intersections of the subspaces $S_i, i \in [k]$.

Let T be the set of products of pairs of matrices $\Phi_i, i \in [2t]$, where each product is of the form $\Phi_{2p-1}\Phi_{2q}$, where $2p-1, 2q \in [2t]$. Thus, the cardinality of the set $|T| = t^2$.

Lemma 1: If there exists an element in T, say $\Phi_i\Phi_j$, which can be expressed as a non-zero linear combination of other elements in T, then $S_i \cap S_j = \{0\}$. In other words, if $\Phi_i\Phi_j$ lies in the span of the rest of the elements in T, then S_i and S_j are complementary subspaces of dimension $\ell/2$.

Proof: Without loss of generality, let $(i,j) = (1,2)$. If $\Phi_1\Phi_2$ is in the span of the rest of the elements in T, then

$$\Phi_1\Phi_2 \quad = \quad \sum_{j\neq 2}\alpha_{1j}\Phi_1\Phi_j + \sum_{i\neq 1}\alpha_{i2}\Phi_i\Phi_2 + \sum_{\substack{m\neq 1\\ n\neq 2}}\alpha_{mn}\Phi_m\Phi_n,$$

where no $\alpha\,(\in\mathbb{F})$ is zero in the above summation. Applying both sides to the subspace S_2, we obtain

$$\begin{aligned} & S_2\left(\Phi_1\Phi_2 - \sum_{i\neq 1}\alpha_{i2}\Phi_i\Phi_2\right) \\ & \qquad \simeq S_2\left(\sum_{j\neq 2}\alpha_{1j}\Phi_1\Phi_j + \sum_{\substack{m\neq 1\\ n\neq 2}}\alpha_{mn}\Phi_m\Phi_n\right). \end{aligned}$$

By properties (6), (7), the left hand side of the above equation lies in $S_2\Phi_2$, whereas the right hand side lies in S_2. This is possible only if

$$S_2\left(\Phi_1\Phi_2 - \sum_{i\neq 1}\alpha_{i2}\Phi_i\Phi_2\right) = \{0\}.$$

[2]The idea behind this transformation is the same as in [2]. It can also be shown from the MDS property of the system that all the coding matrices $A_{i,j}$ have full rank, and from the subspace conditions that all the subspaces $S_{i.j}$ have full rank as well.

[3]A subspace S is invariant under a linear operator Θ if $S\Theta \subseteq S$. If S is invariant under two matrices, it is also invariant under their sum. If two subspaces are invariant under an operator, so is their intersection.

[4]We use the word *non-intersecting* in the context of subspaces to imply that they intersect only in the zero vector.

Because Φ_2 is full rank, this reduces to

$$S_2 \left(\Phi_1 - \sum_{i \neq 1} \alpha_{i2} \Phi_i \right) = \{0\}.$$

Thus, the relation also holds for a subspace of S_2:

$$\begin{aligned} (S_1 \cap S_2) \left(\Phi_1 - \sum_{i \neq 1} \alpha_{i2} \Phi_i \right) &= \{0\}, \text{ or} \qquad (14) \\ (S_1 \cap S_2) \left(\sum_{i \neq 1} \alpha_{i2} \Phi_i \right) &= (S_1 \cap S_2)\, \Phi_1. \end{aligned}$$

As before, by properties (6), (7), the right hand side of the above equation lies in $S_1\Phi_1 \cap S_2$, whereas the left hand side lies in $S_1 \cap S_2$. Note that on the left hand side, $i \neq 2$ by construction. Because $S_1\Phi_1 \cap S_1 = \{0\}$, so is $(S_1\Phi_1 \cap S_2) \cap (S_1 \cap S_2)$. Therefore, (14) is possible only if

$$\begin{aligned} (S_1 \cap S_2)\, \Phi_1 &= \{0\}, \text{ i.e.} \\ S_1 \cap S_2 &= \{0\}, \end{aligned}$$

because of the non-singularity of Φ_1. ■

Corollary 1: Consider the set T as defined above. Let us map each element $\Phi_i\Phi_j \in T$ to a corresponding subspace $S_i \cap S_j$. If no element in T maps to $\{0\}$, then T consists of t^2 linearly independent elements.

Proof: Proof follows from Lemma 1. ■

Corollary 1 indicates a situation which results when subspaces intersect non-trivially (i.e. are not non-intersecting). We now describe a result which ensues if we do find complementary subspaces amongst the given subspaces $S_i, i \in [k]$.

Lemma 2: Suppose we have a set of $n \leq k/2$ disjoint pairs of subspaces (S_i, S_j), such that $S_i \cap S_j = \{0\}, \{i, j\} \in [k]$. Then we can find a set of 2^n linearly independent matrices in $\mathbb{F}^{\ell \times \ell}$.

Proof: Without loss of generality, let these pairs be $(S_1, S_2), (S_3, S_4), \ldots, (S_{2n-1}, S_{2n})$. Let M be the following set of 2^n $\ell \times \ell$ matrices:

$$\Upsilon_{\epsilon_1\epsilon_2\ldots\epsilon_n} = \prod_{j=1}^{n} \left(\Phi_{2j-1}\Phi_{2j}\right)^{\epsilon_j}, \text{ where } \epsilon_j \in \{0,1\}\ \forall j \in [n],$$

where the product goes from left to right. For example,

$$\Upsilon_{11} = (\Phi_1\Phi_2)(\Phi_3\Phi_4).$$

We now prove by induction that no element in M lies in the (linear) span of its remaining elements. The induction is on the sets $M_1, M_2, \ldots, M_{n-1}, M_n (= M)$, where M_s is the set of the following 2^s $\ell \times \ell$ matrices:

$$\Upsilon_{\epsilon_1\epsilon_2\ldots\epsilon_s} = \prod_{j=1}^{s} \left(\Phi_{2j-1}\Phi_{2j}\right)^{\epsilon_j}, \text{ where } \epsilon_j \in \{0,1\}\ \forall j \in [s].$$

Induction Claim: For all $s \in [n]$, no element in the set M_s lies in the span of its remaining $2^s - 1$ elements.

Base case: For $s = 1$, the proof follows from the fact that $\Upsilon_0 = \mathrm{I}$, the identity matrix and $\Upsilon_1 = \Phi_1\Phi_2$ are linearly independent. If not, then

$$\begin{aligned} \mathrm{I} &= \alpha\Phi_1\Phi_2, \ \alpha \neq 0, \\ \Longrightarrow S_2 &\backsimeq \alpha S_2\Phi_1\Phi_2 \\ &\backsimeq S_2\Phi_2, \end{aligned}$$

a contradiction by (7).

Inductive step: Let the statement in the lemma be true for some s, then it is true for $s+1$. If not, let some linear combination of elements in M_{s+1} be equal to zero. Note that each element in M_{s+1} is a product ending either in $\Phi_{2s+1}\Phi_{2s+2}$ or not. We have:

$$\Psi_1^{(s)} = \Psi_2^{(s)}\Phi_{2s+1}\Phi_{2s+2}, \qquad (15)$$

where $\Psi_1^{(s)}$ and $\Psi_2^{(s)}$ are linear combinations of elements in M_s. Now, operating both sides on the subspace S_{2s+2}, we obtain

$$S_{2s+2}\Psi_1^{(s)} \backsimeq S_{2s+2}\Psi_2^{(s)}\Phi_{2s+1}\Phi_{2s+2}. \qquad (16)$$

As in Lemma 1, this is possible only if

$$S_{2s+2}\Psi_1^{(s)} = \{0\}. \qquad (17)$$

Similarly, operating both sides of (15) on the subspace S_{2s+1}, we obtain

$$S_{2s+1}\Psi_1^{(s)} \backsimeq S_{2s+1}\Psi_2^{(s)}\Phi_{2s+1}\Phi_{2s+2}. \qquad (18)$$

The left hand side of the equation is in S_{2s+1} and the right hand side is in $S_{2s+1}\Phi_{2s+1}\Phi_{2s+2}$. But,

$$\begin{aligned} S_{2s+1} \oplus S_{2s+1}\Phi_{2s+1} &\backsimeq \mathbb{F}^\ell, \text{ (cf. (7))} \\ \Longrightarrow S_{2s+1}\Phi_{2s+2} \oplus S_{2s+1}\Phi_{2s+1}\Phi_{2s+2} &\backsimeq \mathbb{F}^\ell, \\ \Longrightarrow S_{2s+1} \oplus S_{2s+1}\Phi_{2s+1}\Phi_{2s+2} &\backsimeq \mathbb{F}^\ell. \text{ (cf. (6))} \end{aligned}$$

Thus, (18) is possible only if

$$S_{2s+1}\Psi_1^{(s)} = \{0\}. \qquad (19)$$

But by our assumption, $S_{2s+1} \cap S_{2s+2} = \{0\}$. So, by (17), (19), we have $\Psi_1^{(s)} = 0$, contradicting our induction assumption. ■

We are ready to prove Theorem 2 now.

Proof of Theorem 2: We are given a set of subspaces $S_i, i \in [k]$ and the corresponding set of matrices $\Phi_i, i \in [k]$, which satisfy the subspace conditions (6) and (7). Suppose we can find at most n disjoint pairs of complementary subspaces (S_i, S_j) in the given set and no more, where $2n \leq k$. Without loss of generality, let these pairs be $(S_1, S_2), \ldots, (S_{2n-1}, S_{2n})$. Let k be an even integer for convenience. We can construct a set T as in Lemma 1, where T is the set of $((k-2n)/2)^2$ products of the form $\Phi_{2p-1}\Phi_{2q}$, where $2p-1, 2q \in \{2n+1, \ldots, k\}$. Note that we could have arranged the $(k-2n)$ matrices $\Phi_i, i \in \{2n+1, \ldots, k\}$ into any two subsets of size $(k-2n)/2$ and taken products where the first factor is in the first set and the second factor in the second set.

Observe that by Corollary 1, the set T must consist of $((k-2n)/2)^2$ linearly independent matrices. Otherwise, we can find another pair of complementary subspaces disjoint from the given n pairs. Let R be the following set of $2^n\,((k-2n)/2)^2$ $\ell \times \ell$ matrices:

$$\Upsilon^i_{\epsilon_1\epsilon_2\ldots\epsilon_n} = \Omega_i \prod_{j=1}^{n} \left(\Phi_{2j-1}\Phi_{2j}\right)^{\epsilon_j},$$

where $\epsilon_j \in \{0,1\}$ for all $j \in [n]$, $i \in [|T|] := \{1,\ldots,|T|\}$. Ω_i is the i^{th} matrix in the set T.

It can be proved that R consists of $|R|$ linearly independent $\ell \times \ell$ matrices. The proof runs along the same lines as in Lemma 2, except that the base case relies on the linear independence of the matrices in T. Notice that $S_j\Omega_i \simeq S_j$, where $S_j, j \in [2n]$, is a subspace in the list of complementary subspaces and $\Omega_i, i \in \{2n+1,\ldots,k\}$, is a matrix in the set T.

We therefore have[5] $|R| = 2^n\,((k-2n)/2)^2$ linearly independent $\ell \times \ell$ matrices, and as in Theorem 1, to satisfy dimensionality,

$$|R| \leq \ell^2. \tag{20}$$

The only missing link is that we do not really know how many complementary subspaces we can find. A simple bound can be obtained by taking the cases $n \leq k/4$ and $n > k/4$, one of which must necessarily occur. If $n \leq k/4$ and for convenience, say k is a multiple of 4 and $k \geq 8$, then $|R| \geq k^2/16$. If $n > k/4$, then $|R| \geq 2^{k/4}$. Thus,

$$\begin{aligned} \min\left(2^{k/4}, k^2/16\right) &\leq \ell^2, \text{ or,} \\ k &\leq \max\left(4\ell, 8\log_2 \ell\right). \end{aligned}$$

It can be shown that for $\ell > 7$, we have $k \leq 4\ell$.

A tighter bound can be obtained by observing that

$$\min\left(\min_{n\in\{0,1,\ldots,k/2-1\}} 2^n\,((k-2n)/2)^2, 2^{k/2}\right) \leq \ell^2,$$

which for a sufficiently large k, results in the bound $k \leq 2\ell$. ■

V. Concluding Remarks

In this paper, we make some progress towards the open problem of finding the maximum k for which an optimal bandwidth MDS code of a given sub-packetization exists. For two parity nodes, we now have

$$3\log_2 \ell \leq k \leq 4\ell.$$

We envision that tapping into the subspace conditions associated with the problem would either lead us to tighter upper bounds or in the least help us explore newer codes with more systematic nodes.

Note: The results obtained here have been improved in joint work with Itzhak Tamo for the general case of r number of parity nodes [15].

[5] Note that if $n = k/2$, we have $|R| = 2^{k/2}$ and T is an empty set.

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