Asymptotic Analysis of LDPC Codes with Depth-2 Connectivity Distributions over BEC

Min Jang, Jin Whan Kang, Jong-Hwan Kim, and Sang-Hyo Kim†
College of Information & Communication Engineering
Sungkyunkwan University
Seoul, Republic of Korea
Email: minycloud@gmail.com, {jinhwan, sas2323, iamshkim†}@skku.edu

***Abstract*—This paper deals with the randomness of depth-2 connectivity of LDPC ensembles. We study how the randomness of LDPC graphs affect asymptotic performance over binary erasure channel (BEC). First, a new attribute, degree-spectrum distribution, is defined to specify the connectivity of more detailed ensembles than those defined by degree distribution. With this, specified ensembles become subsets of conventional ensemble, and we develop a new density evolution method. Some extreme cases of specified ensembles are given, and we prove that the random specified ensemble is better than other regularized types.**

I. Introduction

In low-density parity-check (LDPC) coding theory, it is well known that the "irregularity" of parity check matrices, i.e. irregular degree distribution of corresponding graphs, promotes the better asymptotic performance. Many researchers have tried to find good irregular degree distribution [1]–[5], but the irregular LDPC ensemble is not the most general possible framework. It only evaluates the average performance of countless instances of the ensemble, and this approach does not guarantee the best performance.

A notable study to break through the limitation of the degree distribution optimization framework is a generalization of ensembles [6], [7]. Richardson and Urbanke introduced the multi-edge type ensemble [6] that gives an almost complete description of code structure including the irregularity of degree distribution. In addition, Kasai *et al.* defined the ensembles with the joint degree distributions [7]. In the ensembles, edges are classified by considering both connected variable node degree and check node degree, and a new density evolution was developed to deals with the ensembles defined by joint degree distribution. These ensembles enable a more comprehensive analysis and design of LDPC ensembles.

In this paper, we are interested in a special way of ensemble generalization, *specified ensemble*. Based on the empirical results provided in [8] that show the performance variations among instances in the same conventional irregular ensemble, we analyze the asymptotic performance of specified ensembles whose connectivity of nodes is defined with more detail than that of the conventional ensemble. We first define *degree-spectrum distribution* from the edge perspective and develop a modified density evolution. While it is assumed that the messages at each iteration of BP decoding obey the identical distribution in the analysis of the conventional LDPC ensemble, we allow messages from variable nodes to have different probability densities depending on the degree of the node. Moreover, we take the expectation of messages from the check nodes with degree-spectrum distribution.

We present some extreme examples of specified ensembles defined by degree-spectrum distributions. The first one is the *random ensemble* whose degree-spectrum distribution is an expectation of the random graph with given degree distribution pair. We show that the noise threshold of the random ensemble coincides with that of conventional ensemble with the same degree distribution pair. Other types of specified ensembles are *single-edge type ensemble* and *single-node type ensemble*, which are more regularized than the random type. The most practical construction schemes for LDPC codes, such as the progressive-edge-growth (PEG) algorithm [9], make constructed code fall into near single-node type ensemble [8]. We will verify that the noise threshold of the random type is always greater than or equal to that of regularized ensemble over the binary erasure channel (BEC), so that we convincingly demonstrate that the empirical results in [8] are theoretically supported.

II. Preliminaries

A. Binary LDPC Ensembles

A binary LDPC ensemble $\mathcal{C}(\lambda(x), \rho(x))$ is defined by its degree distribution pair, $\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1}$, where λ_i and ρ_j are the fractions of edges which connect to degree-i variable nodes and degree-j check nodes, respectively. In degree distribution polynomials, d_v (d_c) indicates the maximum degree of variable node (check node) in the ensemble.

Density evolution is a technique for analyzing the asymptotic performance of LDPC ensembles [1]. Let us recall the simple density evolution over BEC. Let p_0 be the erasure probability of the channel. It is shown in many papers including [10] that the expected fraction of erasure messages sent from variable nodes at iteration ℓ is as follows:

$$p_\ell = p_0 \lambda(1 - \rho(1 - p_{\ell-1})) = p_0 \lambda(q_\ell), \tag{1}$$

where q_ℓ indicates the expected fraction of erasure messages from check nodes at the ℓ-th iteration. In this way, we can check the evolution of average error probability with respect to the iterations of BP decoding.

The ensemble defined by degree distribution, however, is too general, and the density evolution just give us an average asymptotic performance. If we take a subset from the conventional ensemble by restricting some characteristics, the performance may be apart from the average. In this respect, there are some generalization tools for specifying ensembles [6], [7]. In their frameworks, edges are classified by the connected node characteristics, so the framework introduces the multi-edge type ensemble. The framework enables us to design a better code under a special conditions, such as extremely high or low rate.

B. Specified Ensembles with Degree-Spectrum Distributions

We now define the degree-spectrum distribution. First, we clarify the terminology on degree-spectrum to avoid ambiguity. *Variable node degree-spectrum* is defined as the set of mapping functions for variable node degrees, and *check node degree-spectrum* is defined as the set of element functions that are related to the degree of check nodes. Because a concentrated form of check node degree distribution $\rho(x)$ at two consecutive degrees is preferred for good performance [4], we only consider the variable node degree-spectrum that can considerably vary with code constructions.

Let us consider an LDPC graph $\mathrm{G}(n, \lambda(x), \rho(x))$. Let $\mathbf{D}_v = \{d \in \mathbb{N} : 2 \le d \le d_v\}$ be the set of variable node degrees, and let $\mathbf{N}_v = \{n \in \mathbb{Z} : 0 \le u \le d_c - 1\}$ denote the set of the number of neighbor variable nodes, where $\mathbb{N}$ and $\mathbb{Z}$ refer to sets of all natural numbers and integers, respectively.

Assume that an edge e connects a variable node v to a check node c. Let $\mathbf{V}_c$ denote the set of all neighbor variable nodes of c, and let $\mathbf{E}_c$ be the set of all edges emanating from c. Then, each edge of $\mathbf{E}_c \setminus \{e\}$ connects the check node c to each variable node in $\mathbf{V}_c \setminus \{v\}$. The function $\pi_e : \mathbf{D}_v \to \mathbf{N}_v$ is a *degree-spectrum* of the edge e, and the π_e is the set of *degree-spectrum mappers* $\pi_e(d)$ for $d \in \mathbf{D}_v$. The equality $\pi_e(d) = u$ indicates that the number of degree-d variable nodes in $\mathbf{V}_c \setminus \{v\}$ is u.

The degree-spectrums of the edges in a graph are constrained by the degree distribution pair. First, the sum of the degree-spectrum mappers for an edge e is expressed as follows,

$$\sum_{d=2}^{d_v} \pi_e(d) = \deg(c_e) - 1, \tag{2}$$

where c_e is the check node to which the edge e is connected. Consequently,

$$\rho_j = \frac{1}{|\mathbf{E}|} \sum_{e \in \mathbf{E}} \mathbf{I}\left(\sum_{d=2}^{d_v} \pi_e(d) + 1 = j \right), \tag{3}$$

where $\mathbf{I}(\cdot)$ is the indicator function which returns 1 if the inner statement is true, and returns 0 otherwise.

We will consider the statistics of the degree-spectrums in a whole LDPC graph. Two degree-set spectrums, π_i and π_j are said to be identical to each other if $\pi_i(d) = \pi_j(d)$ holds for all $d \in \mathbf{D}_v$; otherwise, two degree-spectrums are distinct. Let d_π be the number of distinct degree-spectrums in the graph. Then, the sample space of degree-spectrum, $\mathbb{P} = \{\mathcal{P}_i : i = 1, \ldots, d_\pi\}$ is defined as the set of all distinct degree-spectrums, and $\mathcal{P}_i = \mathcal{P}_j$ if and only if $i = j$. The fractions of degree-spectrums are given as

$$\Omega_i = \frac{|\{\pi_e = \mathcal{P}_i : e \in \mathbf{E}\}|}{|\mathbf{E}|}, \ i = 1, \ldots, d_\pi. \tag{4}$$

Degree-spectrum distribution $\{\mathcal{P}, \Omega\} \equiv \{\mathcal{P}_i, \Omega_i\}_{i=1,\ldots,d_\pi}$ is the collection of sample and fraction pairs of all degree-spectrums in the graph.

Referring to (3), $\rho(x)$ can be derived from $\{\mathcal{P}, \Omega\}$ as

$$\rho_j = \sum_{i=1}^{d_\pi} \Omega_i \mathbf{I}\left(\sum_{d=2}^{D_v} \mathcal{P}_i(d) + 1 = j \right). \tag{5}$$

Therefore, we can omit $\rho(x)$ and write the specified ensemble in the form of $\mathcal{C}(\lambda(x), \{\mathcal{P}, \Omega\})$. It is certain that $\mathcal{C}(\lambda(x), \{\mathcal{P}, \Omega\}) \subset \mathcal{C}(\lambda(x), \rho(x))$, where $\rho(x)$ is derived from $\{\mathcal{P}, \Omega\}$.

Note that the degree-spectrum distribution is a special way to define multi-edge type LDPC ensemble [6], [7]. In our framework, edges are divided into more types according to the other edges connected to the same check node. This approach is a special case of defining multi-edge type ensembles in [6]. In addition, although the expressions are different, joint degree distribution defined in [7] and degree-spectrum distribution are closely related. However, our eventual goal is to verify that the random connections in terms of depth-2 connectivity make a positive influence on specified ensembles.

III. Density Evolution

In this paper, we only consider BEC, but our analysis can be extended to more general channels. Let us begin with the assumption that the messages from a check node to variable nodes are identically distributed, and we have $q_{\ell-1}$, the expected fraction of check node erasure messages at iteration $\ell - 1$. Then, the message passed from a certain variable node v at the ℓ-th iteration, call it $p_{\ell,v}$, is equal to $p_0 (q_{\ell-q})^{\deg(v)-1}$. According to the definition of the degree-spectrum, the message from c_e to v_e via an edge e is given as

$$q_{\ell,e} = 1 - \prod_{k=2}^{d_v} \left(1 - p_0 (q_{\ell-1})^{k-1}\right)^{\pi_e(k)}. \tag{6}$$

For a degree-set mapper $\mathcal{P}$, we have

$$q_{\ell,\mathcal{P}} = 1 - \prod_{k=2}^{d_v} \left(1 - p_0 (q_{\ell-1})^{k-1}\right)^{\mathcal{P}(k)}. \tag{7}$$

Given the degree-set distribution $\{\mathcal{P}, \Omega\}$, the expectation of check node messages at the ℓ-th iteration is given as

$$\begin{aligned} q_\ell = \mathbb{E}_{\mathcal{P}}[q_{\ell,\mathcal{P}}] &= \mathbb{E}_{\mathcal{P}}\left[1 - \prod_{k=2}^{d_v} \left(1 - p_0 (q_{\ell-1})^{k-1}\right)^{\mathcal{P}(k)}\right] \\ &= 1 - \sum_{i=1}^{d_\pi} \Omega_i \left(1 - \prod_{k=2}^{d_v} \left(1 - p_0 (q_{\ell-1})^{k-1}\right)^{\mathcal{P}_i(k)}\right), \end{aligned} \tag{8}$$

and we define the degree-spectrum polynomial as

$$\Omega(x)=\sum_{i=1}^{d_\pi}\Omega_i\left(1-\prod_{k=2}^{d_v}\left(1-p_0\left(x\right)^{k-1}\right)^{\mathcal{P}_i(k)}\right). \quad (9)$$

Let $q_0=1$, and then, the expectation of erasure messages from variable nodes at iteration ℓ+1 can be obtained recursively as follows:

$$p_{\ell+1}=p_0\lambda\left(q_\ell\right) \quad (10)$$

and

$$q_\ell=1-\Omega\left(q_{\ell-1}\right). \quad (11)$$

In (10), it is assumed that every edge is evenly connected to variable nodes according to the variable node degree distribution $\lambda(x)$, and this assumption is essential for our framework. We refer to the assumption as the *equitable edge-connection condition*, and specified ensembles that we will deal with in Section IV are compliant with this condition.

In order to determine the noise threshold of specified ensembles, we just need to check whether an additional iterative decoding reduces the erasure probability or not. Instead of the recursive calculation in (11), it is simple to directly determine $p_{\ell+1}$ from p_ℓ.

Lemma 1: Given an initial erasure probability $0\le p_0\le 1$ and an expected erasure probability in the ℓ-th iteration $0\le p_\ell\le p_0$, $q_{\ell-1}$ is the unique real non-negative solution of

$$\lambda(x)-\frac{p_\ell}{p_0}=\sum_{i=2}^{d_v}\lambda_i x^{i-1}-\frac{p_\ell}{p_0}=0, \quad (12)$$

and $0\le q_{\ell-1}\le 1$. One can obtain $p_{\ell+1}$ from (10) and (11) with the solution $q_{\ell-1}$.

Proof: According to (10), $q_{\ell-1}$ is a solution of $\lambda(x)-p_\ell/p_0=0$. The fractions $\lambda_i\ge 0$ for all i, so $\lambda(x)-p_\ell/p_0$ is an increasing function for $x\ge 0$. Since $\lambda(0)-p_\ell/p_0\le 0$ and $\lambda(1)-p_\ell/p_0\ge 0$, the equation (12) has only one real non-negative solution in the range $[0,1]$. □

IV. Some Specified Ensembles

We introduce some specified ensembles that share the same degree distribution pair – random multi-edge type ensemble, single-edge type ensemble, and single-node type ensemble.

A. Random Multi-Edge Type Ensemble

In the analysis of the conventional ensemble $\mathcal{C}(\lambda(x),\rho(x))$, it is assumed that variable nodes are connected randomly to check nodes. Therefore, it is intuitively certain that $\mathcal{C}(\lambda(x),\rho(x))=\mathcal{C}(\lambda(x),\{\tilde{\mathcal{P}},\tilde{\Omega}\})$, where $\{\tilde{\mathcal{P}},\tilde{\Omega}\}$ is the expected degree-spectrum distribution of the random graphs constrained by $\lambda(x)$ and $\rho(x)$. It is possible to obtain $\{\tilde{\mathcal{P}},\tilde{\Omega}\}$ similarly to the calculation of the expected degree-set/sum distributions in [8], but this is beyond the scope of this paper.

Let $p_{\ell,\mathrm{RD}}$ and $q_{\ell,\mathrm{RD}}$ be the expected erasure probability of messages from the variable nodes and that from the check nodes in the random multi-edge type ensemble at iteration ℓ, respectively. Then we have the following lemma.

Lemma 2: The density evolution of $\mathcal{C}(\lambda(x),\{\tilde{\mathcal{P}},\tilde{\Omega}\})$ follows:

$$p_{\ell+1,\mathrm{RD}}=p_0\lambda\left(1-\rho\left(\sum_{i=2}^{d_v}\lambda_i\left(1-p_0\left(q_{\ell-1}\right)^{i-1}\right)\right)\right), \quad (13)$$

where $\rho(x)$ is the conventional check node degree distribution corresponding to $\{\tilde{\mathcal{P}},\tilde{\Omega}\}$.

Proof: In the analysis of a conventional ensemble $\mathcal{C}\left(\lambda(x),\rho(x)\right)$ which is identical to $\mathcal{C}(\lambda(x),\{\tilde{\mathcal{P}},\tilde{\Omega}\})$, the expected erasure probability of messages from the variable nodes at the ℓ-th iteration of BP decoding is given as

$$p_{\ell,\mathrm{RD}}=p_0\lambda\left(1-\rho\left(1-p_{\ell-1,\mathrm{RD}}\right)\right)=p_0\mathbb{E}_i\left[\left(q_{\ell-1,\mathrm{RD}}\right)^{i-1}\right] \quad (14)$$

Hence, $p_{\ell+1,\mathrm{RD}}$ can be expressed as

$$p_{\ell+1,\mathrm{RD}}=p_0\lambda\left(1-\rho\left(\mathbb{E}_i\left[1-p_0\left(q_{\ell-1}\right)^{i-1}\right]\right)\right). \quad (15)$$

□

According to Lemma 2 and (10), then we have

$$q_{\ell,\mathrm{RD}}=1-\rho\left(\sum_{i=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{i-1}\right)\right), \quad (16)$$

where $q_{\ell-1}$ is obtained from the given p_ℓ by Lemma 1.

B. Single-Edge Type LDPC Ensemble

Regardless of feasibility, consider an ensemble whose edges have the same type of degree-spectrum. More precisely, in this ensemble, the edges connected to check nodes with the same degree have the same degree-spectrum, so, in the graph, there are as many degree-spectrum types as the number of distinct check node degrees. Technically, this ensemble has more than one edge type, but we call it a *single-edge type ensemble* for simplicity, and it is denoted by $\mathcal{C}\left(\lambda(x),\{\mathcal{P},\Omega\}_{\mathrm{SE}}\right)$.

By the definition of a single-edge type ensemble,

$$\frac{\mathcal{P}_j\left(d\right)}{j-1}=\lambda_d,\qquad j=2,\cdots,d_c,$$

where $\mathcal{P}_j$ is the degree-spectrum of the edges connected to degree-j check nodes. Hence, the degree-spectrum distribution is given as

$$\{\mathcal{P},\Omega\}_{\mathrm{SE}}=\{\mathcal{P}_j(d)=\lambda_d\left(j-1\right),\ \Omega_j=\rho_j\}_{j=2,\cdots,d_c}.$$

For the single-edge type ensemble, the expectation of the erasure messages from the check nodes in (8) becomes

$$\begin{aligned}q_{\ell,\mathrm{SE}}&=1-\Omega\left(q_{\ell-1}\right)\\&=1-\sum_{j=2}^{d_c}\rho_j\prod_{k=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{k-1}\right)^{\lambda_k(j-1)}\\&=1-\rho\left(\prod_{k=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{k-1}\right)^{\lambda_k}\right),\end{aligned} \quad (17)$$

and $p_{(\ell+1),\mathrm{SE}} = p_0\lambda(q_{\ell,\mathrm{SE}})$ by (10) and (11).

Theorem 3 [Random vs. Single-Edge]: For $0 \le p_0 \le 1$ and $0 \le p_\ell \le 1$, $p_{(\ell+1),\mathrm{RD}}$ is always less than or equal to $p_{(\ell+1),\mathrm{SE}}$, that is,

$$\begin{aligned} & p_0\lambda\left(1-\rho\left(\sum_{i=2}^{d_v}\lambda_i\left(1-p_0\left(q_{\ell-1}\right)^{i-1}\right)\right)\right) \\ & \le p_0\lambda\left(1-\rho\left(\prod_{k=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{k-1}\right)^{\lambda_k}\right)\right). \end{aligned} \tag{18}$$

Proof: Inequality (18) follows directly from weighted arithmetic mean and geometric mean (AM-GM) inequality. □

Corollary 4: Equality in (18) holds for only the regular variable node degree distribution $\lambda(x) = x^{d_v-1}$. It is easily proved with the equality condition of weighted AM-GM inequality, and this corollary means that the random edge-type ensemble and the single-edge type ensemble are equal to each other if regular variable node degree distribution is given.

C. Single-Node Type LDPC Ensemble

In practice, it is impossible to construct a member graph in a single-edge type ensemble due to the equitable edge-connection condition unless the variable node degree distribution $\lambda(x)$ is regular. Nevertheless, the analysis of single-edge type helps us to readily approach similar feasible ensemble, *single-node type ensemble*.

In the single-node type ensemble $\mathcal{C}(\lambda(x), \{\mathcal{P},\Omega\}_{\mathrm{SN}})$, unlike in the single-edge type ensemble, the degree-spectrum distribution is specified with a node perspective constraint. Every check node with the same degree j is connected to the same number of degree-i variable nodes. For an edge e that connects v_e and c_e, the degree-spectrum mappers are given as

$$\pi_e(d) = \begin{cases} \lambda_d \mathrm{deg}(c_e) & d \ne \mathrm{deg}(v_e) \\ \lambda_d \mathrm{deg}(c_e) - 1 & d = \mathrm{deg}(v_e). \end{cases}$$

Let $\mathcal{P}_{i,j}$ denote the degree-spectrum of the edges between the degree-i variable nodes and the degree-j check nodes, then, we have

$$\mathcal{P}_{i,j}(d) = \begin{cases} \lambda_d j, & d \ne i \\ \lambda_d j - 1, & d = i, \end{cases} \tag{19}$$

and

$$\Omega_{i,j} = \lambda_i \rho_j. \tag{20}$$

Therefore, the degree-spectrum distribution in the single-node type ensemble is given as

$$\{\mathcal{P},\Omega\}_{\mathrm{SN}} = \{\mathcal{P}_{i,j}, \lambda_i\rho_j\}_{i=2,\cdots,d_v, j=2,\cdots,d_c}$$

The single-node type ensembles are only practical if $\lambda_d j \in \mathbb{Z}$ for all d and j, so the single-node type ensembles are imaginary in the majority of cases. However, the analysis of those imaginary single-node type ensembles theoretically help us estimate the performance of practical codes which are close to the ensembles. In addition, we will use practical single-node type ensembles with integer valued degree-spectrums and numerically analyze them in Section IV-D.

With the degree-spectrum distribution given as (19) and (20), the expected erasure probability of check node messages $q_{\ell,\mathrm{SN}}$ is written as

$$\begin{aligned} q_{\ell,\mathrm{SN}} = 1 - & \left(\sum_{i=2}^{d_v}\lambda_i\left(1-p_0\left(q_{\ell-1}\right)^{i-1}\right)^{-1}\right) \\ \cdot & \left(\prod_{k=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{k-1}\right)^{\lambda_k}\right)\rho\left(\prod_{k=2}^{d_v}\left(1-p_0\left(q_{\ell-1}\right)^{k-1}\right)^{\lambda_k}\right), \end{aligned} \tag{21}$$

and $p_{\ell+1,\mathrm{SN}} = p_0\lambda(q_{\ell,\mathrm{SN}})$.

Theorem 5 [Single-Node vs. Single-Edge]: Given any $0 \le p_0 \le 1$ and $0 \le p_\ell \le 1$, $p_{\ell+1,\mathrm{SN}} \le p_{\ell+q,\mathrm{SE}}$.

Proof: We only compare $q_{\ell,\mathrm{SE}}$ and $q_{\ell,\mathrm{SN}}$, which are calculated from the same p_ℓ, because $p_{\ell+1,\mathrm{SN}} = p_0\lambda(q_{\ell,\mathrm{SN}})$ and $p_{\ell+1,\mathrm{SE}} = p_0\lambda(q_{\ell,\mathrm{SE}})$. For simplicity, let us substitute x_i for $\sum_i \lambda_i(1-p_0(q_{\ell-1})^{i-1})$. The only difference between (17) and (21) is that $\sum_i \lambda_i x_i^{-1}\prod_k x_k^{\lambda_k}$ is multiplied to $\rho(x_k^{\lambda_k})$ in (21). According to Jensen's inequality for a strictly concave function $\log(\cdot)$,

$$\begin{aligned} \sum_{i=2}^{d_v}\lambda_i x_i^{-1}\prod_{i=2}^{d_v}x_i^{\lambda_k} &= \exp\left(\log\mathbb{E}_i\left[x_i^{-1}\right] + \mathbb{E}_i\left[\log x_i\right]\right) && (22) \\ &\ge \exp\left(\mathbb{E}_i\left[\log x_i^{-1}\right] + \mathbb{E}_i\left[\log x_i\right]\right) && (23) \\ &= \exp\left(\mathbb{E}_i\left[\log\left(x_i^{-1}x_i\right)\right]\right) = 1. \end{aligned}$$

Therefore, $q_{\ell,\mathrm{SN}} \le q_{\ell,\mathrm{SN}}$, and $p_{\ell+1,\mathrm{SN}} \ge p_{\ell+1,\mathrm{SN}}$ due to the monotonicity of the expectation function. □

Corollary 6: Given the regular variable node degree distribution, the single-edge type ensemble and the single-node type ensemble are the same. According to Corollary 4, the random multi-edge type ensemble is also equal to the single-node type ensemble if a regular variable node degree distribution is given.

Conjecture 7: In Theorems 3 and 5, we prove that the single-edge type ensemble is worse than both the random multi-edge type ensemble and the single-node type ensemble, but it is not clarified that the random multi-edge type ensemble is superior to the single-node type ensemble. The superiority of random multi-edge type over single-node type remains as an open problem, and one can prove it by showing that $q_{\ell,\mathrm{RD}}$ in (16) is always less than or equal to $q_{\ell,\mathrm{SN}}$ in (21). Instead, we just conjecture that the capacity of the single-node type is worse than that of the random multi-edge type because the single-node type is a slight modification of the single-edge type. We will demonstrate that a random multi-edge type ensemble is better than a single-node type ensemble by several examples in the next subsection.

D. Threshold Analysis

In this section, we evaluate the thresholds of the specified ensembles over BEC. We use the graphical threshold determination method in [6] based on fixed point characterization. We consider four half-rate degree distribution pairs given in Table I. The first degree distribution is a regular one with variable node degree $d_v = 3$ and check node degree $d_c = 6$. The others are irregular degree distributions with maximum variable node degree 6, 20, and 30, respectively [6], [11], [12].

For each degree distribution pair, four specified ensembles are addressed. Random multi-edge type, single-edge type, and single-node type ensembles were introduced in the previous subsections. Moreover, we introduce the *practical single-node type ensemble*, which is a modification of the single-node type. As stated in Section IV-C, degree-spectrum $\mathcal{P}_{i,j}(d) = \lambda_d j$ may not be an integer number. In order to make a practical ensemble, we strictly have each degree-spectrum consist of only integers. Consequently, the practical ensembles have more types of degree-spectrums than the single-node type ensembles, so they are situated between the single-node type and random type.

The resulting threshold values are given in Table I. Specified ensembles with $\lambda(x)$ are identical to each other, and they have the same threshold value according to Corollary 4 and 6. For irregular ensembles,threshold values of the random multi-edge type ensembles are always the greatest, and $\epsilon_{\rm SE} \leq \epsilon_{\rm SN} \leq \epsilon_{\rm Practical-SN} \leq \epsilon_{\rm RD}$ for all our examples. The results support Theorem 3, 5, and Conjecture 7.

A glance at the results shows that, although the ensembles have the same degree distribution pair, they have different thresholds depending on the degree-spectrum distribution. According to the interpretations of the construction schemes of LDPC codes such as PEG algorithm in [8], the schemes construct codes with degree-spectrum distribution near that of the single-node type. Thus, the codes made by conventional construction schemes actually have lower thresholds than expected. Based on the results in this paper, we can consider the degree-spectrum distribution as a design factor of finite-length LDPC codes in order to improve the performance at the waterfall region.

TABLE I
DEGREE DISTRIBUTIONS AND THRESHOLDS OF HALF-RATE ENSEMBLES

	Reg-3	Irreg-6 [11]	Irreg-20 [6]	Irreg-30 [12]
λ_2		0.2895	0.1063	0.2633
λ_3	1	0.3158	0.4867	0.1802
λ_6		0.3947		
λ_7				0.2700
λ_{11}			0.0104	
λ_{20}			0.3967	
λ_{30}				0.2865
$\epsilon_{\rm SE}$	0.4294	0.4527	0.4471	0.4606
$\epsilon_{\rm SN}$	0.4294	0.4543	0.4503	0.4648
$\epsilon_{\rm Practical-SN}$	0.4294	0.4576	0.4643	0.4686
$\epsilon_{\rm RD}$	0.4294	**0.4610**	**0.4741**	**0.4955**

V. Conclusion

Beyond degree distribution pair $\lambda(x)$ and $\rho(x)$, the degree-spectrum distribution $\{\mathcal{P}, \Omega\}$ was defined. Density evolution that evaluates the performance of specified ensembles defined by the degree-spectrum distribution was derived. We introduced several specified ensembles from the most regularized single-edge type to the random multi-edge type. Over BEC, the inequalities between random multi-edge type and other regularized types have been proved. Moreover, some numerical results, which support the inequalities, are given. We expect that our results will extend to more general channels.

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