Two-Dimensional Visual Search

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***Abstract*—Consider the problem of sequentially searching for a single target in an image. Let the image be divided into $M \times M$ equal sized segments where M determines the resolution of the search. The goal is to find the segment that contains the target quickly and accurately. In each step, the player can visually inspect an allowable combination of the segments, and the outcome of the inspection is noisy.**

In this paper, a lower bound on the optimal total cost is derived. Furthermore, two heuristic policies are considered: 1) A policy that visually inspects a segment with the highest probability of having the target; and 2) a policy that in each step inspects a combination that maximizes the Extrinsic Jensen–Shannon divergence. Via numerical and asymptotic analysis, the performance of the above policies are investigated.

I. Introduction

This paper considers the problem of sequentially looking for a target in an image, where the image is divided into $M \times M$ equal sized segments and the goal is to find the segment that contains the target in a speedy manner while accounting for the penalty of wrong declarations. The visual search defined above is closely related to the problems of *fault detection*, *whereabouts search*, and *group testing*. In fault detection [1], the objective is to determine the faulty component in a system known to have one failed component. Similarly, in whereabouts search [2], the goal is to find an object which is hidden in one of N boxes. In group testing, the goal is to locate the non-zero element of a vector in $\mathbb{R}^N$ with a possible noisy linear measurement of the vector [3].

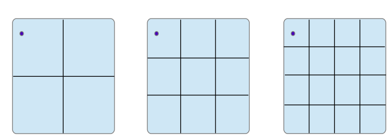
One possible search strategy for these problems is the maximum likelihood policy. In case of fault detection/whereabouts search, this policy is equivalent to one that inspects a segment with the highest probability of having the faulty component/hidden object; while in case of group testing, it is equivalent of measuring the most likely non-zero element of the vector. However, as the number of segments or the dimension of vectors, N, increases, the scheme becomes impractical. In such a case, it is more intuitive to initially inspect larger areas and narrow down the search to single segments only after we have collected sufficient information supporting the presence of the target in those segments [4], [5]. Following this intuition, we propose a heuristic policy that in each step inspects a combination of the segments that maximizes the Extrinsic Jensen–Shannon divergence [6]. Via numerical and asymptotic analysis, the performance of this policy is compared against that of the maximum likelihood policy.

The remainder of this paper is organized as follows. In Section II, we formulate the two-dimensional visual search problem. Section III provides heuristic policies whose performance will be investigated analytically and numerically in the subsequent sections. In particular, we analyze the performance of the proposed policies in Section IV-A, and discuss their complexity in Section IV-B. In Section V, we compare the performance of the proposed heuristics in a numerical example. Finally, we conclude the paper and discuss future work in Section VI.

Notation: For any set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. All logarithms are in base 2. The entropy function on a vector $\boldsymbol{\rho} = [\rho_1, \rho_2, \dots, \rho_N] \in [0,1]^N$ is defined as $H(\boldsymbol{\rho}) = \sum_{i=1}^{N} \rho_i \log \frac{1}{\rho_i}$, with the convention that $0 \log \frac{1}{0} = 0$. The Kullback–Leibler (KL) divergence between two probability density functions q and q' on space $\mathcal{Z}$ is defined as $D(q||q') = \int_{\mathcal{Z}} q(z) \log \frac{q(z)}{q'(z)} dz$, with the convention $0 \log \frac{a}{0} = 0$ and $b \log \frac{b}{0} = \infty$ for $a, b \in [0,1]$ with $b \neq 0$. Finally, let $N(m, \sigma^2)$ denote a normal distribution with mean m and variance σ^2.

II. Problem Formulation

In this section, we formulate the two-dimensional visual search problem. Let the image be divided into $M \times M$ equal sized segments where M determines the resolution of the search. In other words, the location of the target is quantized to the index of the segment containing it (hence, we use the phrases "location of the target" and "the segment containing the target" interchangeably). We can think of this segmentation in a matrix form. In particular, let G_{ij} denote the segment in the i^{th} row and j^{th} column of the matrix, and $\Omega := \bigcup_{i=1}^{M} \bigcup_{j=1}^{M} \{G_{ij}\}$ be the set of all segments. The goal is to locate the target quickly and accurately via sequentially inspecting the image.



(a) (b) (c)

Fig. 1. Two-dimensional visual search under different resolutions.

Let $a \subset \Omega$ be a subset of locations that can be simultaneously inspected, referred to as the inspection region hereafter, and let $\mathcal{A} = 2^\Omega$ be the collection of all allowable inspection regions. Generalizing the model in [4], we assume that the outcome of an inspection only depends on the size of the inspection region and the resolution of the search, i.e., the outcome of inspecting region a, where $|a| = n$, $1 \le n \le M^2$, is a $\mathcal{Z}$-valued random variable with probability density function $f_{n,M}$ if the target location is included in the inspection region; otherwise, it is distributed as $\bar{f}_{n,M}$. We assume that observation densities are known and the observations are conditionally independent over time. Let θ be the location of the target, i.e., θ is the random variable that takes the value $\theta = G_{ij}$ on the event that G_{ij} contains the target. Let τ be the stopping time at which the player retires and declares its estimated location $\hat{\theta}$ with the probability of error as

$$\mathrm{Pe} := P(\hat{\theta} \neq \theta).$$

The objective is to find a sequence of inspection regions $A(0), A(1), \ldots, A(\tau-1)$ and a stopping time τ that collectively minimize the expected total cost

$$\mathbb{E}[\tau] + L\mathrm{Pe}, \tag{1}$$

where L, $L > 1$ denotes the penalty of making a wrong declaration.[1] Here we consider a uniform prior, i.e., initially $P(\theta = \omega) = 1/M^2$ for all $\omega \in \Omega$, and hence the expectation is always taken with respect to this uniform prior on θ as well as the distributions of the sequence of inspection regions and outcomes. It is clear that in this setting, the optimum declaration is the maximum a posteriori (MAP) estimate,[2] i.e.,

$$\hat{\theta} := \arg\max_{\omega \in \Omega} P(\theta = \omega | A^{\tau-1}, Z^{\tau-1}),$$

where $A^{\tau-1} := [A(0), A(1), \ldots, A(\tau-1)]$ and $Z^{\tau-1} := [Z(0), Z(1), \ldots, Z(\tau-1)]$ represent respectively the vector of selected inspection regions and observation outcomes up to time τ.

Remark 1. It is clear from Fig. 1 that a reasonable observation model should account for the equivalence between inspecting one of the segments in (a) and inspecting the 4 corner segments in (c) simultaneously. In other words, a reasonable assumption is one under which the outcome of inspecting region a, $|a| = n$, depends on the ratio between the size of the inspection region, n, and the total number of segments, i.e., $f_{n,M} = h_{\frac{n}{M^2}}$ and $\bar{f}_{n,M} = \bar{h}_{\frac{n}{M^2}}$ where $\{h_r | r \in [0,1]\}$ and $\{\bar{h}_r | r \in [0,1]\}$ are parametric families of distributions.

III. Heuristics

In this section, we introduce heuristic policies for selecting inspection regions. Before we proceed, we need the following notations.

The two-dimensional visual search defined above is a Markov decision problem (MDP) whose information state at time t is the belief vector $\boldsymbol{\rho}(t) = [\rho_{11}(t), \rho_{12}(t), \ldots, \rho_{MM}(t)]$ where $\rho_{ij}(t) := P(\theta = G_{ij}|A^{t-1}, Z^{t-1})$. Accordingly, the information state space is $\mathbb{P}(\Omega) := \{\boldsymbol{\rho} \in [0,1]^{M^2} : \sum_{i=1}^M \sum_{j=1}^M \rho_{ij} = 1\}$ and the dynamics of the information state is dictated by the Bayes' rule.

Let d_{ij}, $1 \le i,j \le M$, represent an action under which the player retires and declares that G_{ij} contains the target. A Markov stationary deterministic policy π is defined as a mapping $\pi : \mathbb{P}(\Omega) \to \mathcal{A} \cup \bigcup_{i=1}^M \bigcup_{j=1}^M \{d_{ij}\}$ based on which inspection regions $A(t)$, $t = 0, 1, \ldots, \tau-1$ and stopping time τ are selected (the choice of any of the retire-declare actions marks the stopping time τ).

A. Maximum Likelihood

In this subsection, we define the maximum likelihood policy π_{ML}. This policy, at each step, visually inspects a segment with the highest belief.

Definition 1. Policy π_{ML} is a stationary deterministic Markov policy defined as:

$$\pi_{ML}(\boldsymbol{\rho}) := \begin{cases} d_{ij} & \text{if } \rho_{ij} \ge 1 - L^{-1} \\ \{G_{ij}\} & \text{if } 1 - L^{-1} > \rho_{ij} \ge \rho_{kl}, \ \forall k, l \end{cases}.$$

B. Extrinsic Jensen–Shannon Divergence

Definition 2 (see [6])**.** The Extrinsic Jensen–Shannon (EJS) divergence among probability density functions $q_1, q_2, \ldots, q_N$ with respect to weight vector $\boldsymbol{w} = [w_1, w_2, \ldots, w_N] \in [0,1]^N$, $\sum_{i=1}^N w_i = 1$, is defined as

$$EJS(\boldsymbol{w}; q_1, q_2, \ldots, q_N) := \sum_{i=1}^N w_i D(q_i || \sum_{k \neq i} \frac{w_k}{1 - w_i} q_k).$$

Let q_{ij}^a denote the outcome of inspecting region $a \in \mathcal{A}$ when G_{ij} contains the target, i.e.,

$$q_{ij}^a(\cdot) := \begin{cases} f_{|a|,M}(\cdot) & \text{if } G_{ij} \in a \\ \bar{f}_{|a|,M}(\cdot) & \text{if } G_{ij} \notin a \end{cases}, \ 1 \le i,j \le M. \tag{2}$$

Given a belief vector $\boldsymbol{\rho} \in \mathbb{P}(\Omega)$ and inspection region $a \in \mathcal{A}$, we define

$$EJS(\boldsymbol{\rho}, a) := EJS(\boldsymbol{\rho}; q_{11}^a, q_{12}^a, \ldots, q_{MM}^a), \tag{3}$$

which together with (2) implies that

$$\begin{aligned} EJS(\boldsymbol{\rho}, a) = &\sum_{i=1}^M \sum_{j=1}^M \rho_{ij} \times \qquad (4) \\ &\Big[D(f_{|a|,M} || \frac{\eta^a - \rho_{ij}}{1 - \rho_{ij}} f_{|a|,M} + \frac{\bar{\eta}^a}{1 - \rho_{ij}} \bar{f}_{|a|,M}) \mathbf{1}_{\{G_{ij} \in a\}} \\ &+ D(\bar{f}_{|a|,M} || \frac{\bar{\eta}^a - \rho_{ij}}{1 - \rho_{ij}} \bar{f}_{|a|,M} + \frac{\eta^a}{1 - \rho_{ij}} f_{|a|,M}) \mathbf{1}_{\{G_{ij} \notin a\}} \Big], \end{aligned}$$

where $\eta^a = \sum_{k=1}^M \sum_{l=1}^M \rho_{kl} \mathbf{1}_{\{G_{kl} \in a\}}$, and $\bar{\eta}^a = 1 - \eta^a$.

We are now ready to introduce the next heuristic policy:

Definition 3. Policy π_{EJS} is a stationary deterministic Markov policy defined as:

$$\pi_{EJS}(\boldsymbol{\rho}) := \begin{cases} d_{ij} & \text{if } \rho_{ij} \ge 1 - L^{-1} \\ \arg\max_{a \in \mathcal{A}} EJS(\boldsymbol{\rho}, a) & \text{otherwise} \end{cases}.$$

[1] This problem can be interpreted as the Lagrangian relaxation of one with the objective to minimize $\mathbb{E}[\tau]$ subject to $\mathrm{Pe} \le \epsilon$ where $\epsilon > 0$ denotes the desired probability of error.

[2] The probability of error is minimized under MAP declaration.

IV. Main Results

In this section, we analyze the performance of the proposed heuristics and provide the main results of the paper. We have the following assumptions:

Assumption 1. *For all M, $n \leq M^2$, and $z \in \mathcal{Z}$,*

$$f_{n,M}(z) = \bar{f}_{n,M}(b - z) \ \text{ for some } \ b \in \mathbb{R}.$$

Assumption 1 implies that given a fixed inspection area, the visual samples provide identical information regarding the presence of the target or its absence. This assumption is satisfied if, for instance, observation outcomes are modeled as a signal plus noise, where the signal component appears only if the target is included in the inspection region, and the noise distribution is symmetric with respect to its mean value.

Assumption 2. *For all M, $n < M^2$, $\alpha \in [0, 1]$, and $\bar{\alpha} = 1-\alpha$,*

$$D(f_{n,M}||\alpha f_{n,M} + \bar{\alpha}\bar{f}_{n,M}) \geq D(f_{n+1,M}||\alpha f_{n+1,M} + \bar{\alpha}\bar{f}_{n+1,M}).$$

Assumption 2 implies that the visual samples do not become more informative if the size of the inspection region increases.

Assumption 3. *For some $\gamma > 1$,*

$$\sup_M \max_{n \leq M^2} \int_{\mathcal{Z}} f_{n,M}(z) \left| \log \frac{f_{n,M}(z)}{\bar{f}_{n,M}(z)} \right|^{\gamma} dz < \infty.$$

Assumption 3 restricts the amount of information corresponding to each inspection region, and ensures that no observation is noise-free. This assumption is important in our asymptotic analysis, and is applied to limit the *excess over the boundary* at the stopping time (see [7], [8] for more details).

A. Lower and Upper Bounds

Theorem 1 (Lower Bound)**.** *Under Assumptions 1, 2, and 3, the optimal expected total cost is lower bounded as*

$$\min\{\mathbb{E}[\tau] + L\text{Pe}\} \geq \left(\frac{\log M^2}{C_{\max}} + \frac{\log L}{D_{\max}} \right) (1 - o(1)), \quad (5)$$

where $o(1) \to 0$ as $LM \to \infty$, and

$$C_{\max} := \lim_{M\to\infty} D(f_{1,M}||\frac{1}{2}f_{1,M} + \frac{1}{2}\bar{f}_{1,M}),$$
$$D_{\max} := \lim_{M\to\infty} D(f_{1,M}||\bar{f}_{1,M}).$$

Theorem 2 (Upper Bound)**.** *Under Assumptions 1, 2, and 3, the expected total cost under π_{ML} and π_{EJS}, denoted by $V_{\pi_{ML}}$ and $V_{\pi_{EJS}}$ respectively, are upper bounded as*

$$V_{\pi_{ML}} \leq \left(\frac{\log M^2}{\frac{1}{M^2} D_{\max}} + \frac{\log L}{D_{\max}} \right) (1 + o(1)), \quad (6)$$

$$V_{\pi_{EJS}} \leq \left(\frac{\log M^2}{C_{\min}} + \frac{\log L}{D_{\max}} \right) (1 + o(1)), \quad (7)$$

where $o(1) \to 0$ as $LM \to \infty$, and

$$C_{\min} := \lim_{M\to\infty} D(f_{\lfloor M^2/2 \rfloor,M}||\frac{1}{2}f_{\lfloor M^2/2\rfloor,M} + \frac{1}{2}\bar{f}_{\lfloor M^2/2\rfloor,M}).$$

Next we provide examples for which Assumptions 1–3 hold.

Example 1. Consider the case where the observation densities are independent of the resolution and the size of the inspection region, i.e., for al M and $n \leq M^2$, $f_{n,M} = f$ and $\bar{f}_{n,M} = \bar{f}$ where $f(z) = \bar{f}(b - z)$ for some $b \in \mathbb{R}$ and $\int_{\mathcal{Z}} f(z) |\log \frac{f(z)}{\bar{f}(z)}|^{\gamma} dz < \infty$ for some $\gamma > 1$. Then Assumptions 1, 2, and 3 hold and $C_{\min} = C_{\max} = D(f||\frac{1}{2}f + \frac{1}{2}\bar{f})$ and $D_{\max} = D(f||\bar{f})$. In this case, the upper bound associated with π_{EJS} matches the lower bound obtained asymptotically.

Example 2. Let $\{p_r\}_{r\in[0,1]}$ be an index family satisfying $0 < p_0 \leq p_r \leq p_{r'} \leq p_1 < \frac{1}{2}$ for any $r \leq r'$. Consider the case of Bernoulli noise where for all M and $n \leq M^2$, we have $f_{n,M} = \mathcal{B}(1-p_{\frac{n}{M^2}})$ and $\bar{f}_{n,M} = \mathcal{B}(p_{\frac{n}{M^2}})$. Then Assumptions 1, 2, and 3 hold and $C_{\min} = 1 - H([p_{\frac{1}{2}}, 1 - p_{\frac{1}{2}}])$, $C_{\max} = 1 - H([p_0, 1 - p_0])$, and $D_{\max} = (1 - 2p_0) \log \frac{1-p_0}{p_0}$.

Example 3. Let $\{\sigma_r^2\}_{r\in[0,1]}$ be an index family satisfying $0 < \sigma_0^2 \leq \sigma_r^2 \leq \sigma_{r'}^2 \leq \sigma_1^2 < \infty$ for any $r \leq r'$. Consider the case of Gaussian noise where for all M and $n \leq M^2$, $f_{n,M} = N(1, \sigma^2_{\frac{n}{M^2}})$ and $\bar{f}_{n,M} = N(0, \sigma^2_{\frac{n}{M^2}})$. Then Assumptions 1, 2, and 3 hold and $C_{\min} = D(N(1, \sigma^2_{\frac{1}{2}})||\frac{1}{2}N(1, \sigma^2_{\frac{1}{2}}) + \frac{1}{2}N(0, \sigma^2_{\frac{1}{2}}))$, $C_{\max} = D(N(1, \sigma_0^2)||\frac{1}{2}N(1, \sigma_0^2) + \frac{1}{2}N(0, \sigma_0^2))$, and $D_{\max} = \frac{\log e}{2\sigma_0^2}$.

B. Inspection Constraints and Computational Complexity

In this section, we discuss the computational complexity of π_{ML} and π_{EJS}. Policy π_{ML} has complexity of order $O(M^2)$ in each step since its main computational burden is to find a segment of the $M \times M$ grid with the highest probability of having the target. The main computational burden in π_{EJS} is the action selection step which requires maximizing the EJS divergence over the entire space of inspection regions $\mathcal{A}$. This means that at each step the computational complexity of π_{EJS} is of order $O(|\mathcal{A}|)$ where $|\mathcal{A}|$ is the number of allowable inspection regions.

TABLE I
Computational Complexity of π_{EJS}

$\mathcal{A}$	$\lvert\mathcal{A}\rvert$	Complexity
all combinations	2^{M^2}	$O(2^{M^2})$
rectangles	$\frac{M^2(M+1)^2}{4}$	$O(M^4)$
squares	$\frac{M(M+1)(2M+1)}{6}$	$O(M^3)$
thresholds	$2M^2$	$O(M^2)$

This provides a recipe for trading off performance for complexity by only considering a smaller number of inspection regions. Table I shows the complexity of π_{EJS} when the inspection regions are constrained. In particular, if the inspection regions are always chosen to be contiguous and of certain shape, such as rectangles, etc., the resulting heuristic necessarily has complexity comparable with that of π_{ML}. Constraining the allowable set of inspection regions not only results in a reduction in computational complexity, it also captures realistic limitations in humans' visual abilities.

However, a potential drawback of constraining the allowable inspection region is whether similar performance bounds as (7) can be obtained. The proof of Theorem 2 shows that π_{EJS} can achieve upper bound (7) so long as $\mathcal{A}$ contains the single segments and the 2-D threshold-like regions (see Appendix A for definition of the latter). An important area of future work is to show that constraining the inspection region to that of rectangles or squares is of no consequence to upper bound (7).

V. Numerical Example

In this section, we compare the performance of π_{ML} and π_{EJS} numerically. We consider the case where the inspection space $\mathcal{A}$ is constrained to squares in the $M \times M$ grid. The observation kernels are of the form $f_{n,M} = N(1, 0.4 + 0.1\frac{n-1}{M^2})$ and $\bar{f}_{n,M} = N(0, 0.4 + 0.1\frac{n-1}{M^2})$.

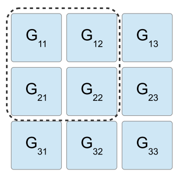


Fig. 2. 3×3 Grid. The inspection space of all squares in this grid contains 9 single segments, 4 squares of size 2×2 (one of which is shown in the figure), and one square of size 3×3.

Fig. 3 shows that when $M = 2$, the performance of the candidate policies is quite similar. This is intuitive since in this case the inspection space contains only the single segments (note that the inspection space contains also the combination of all 4 segments which does not provide any information and whose EJS is restricted to 0, hence can be ignored). However, for larger M, the performance gap of the two policies increases which signifies the superiority of π_{EJS} over π_{ML}. Note that the expected total cost corresponding to both policies has the same slope in L, which verifies the results of Section IV.

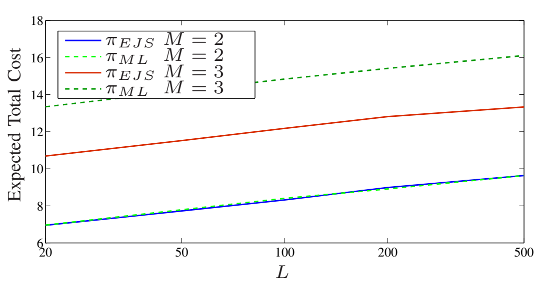


Fig. 3. Expected total cost for different values of L and M.

VI. Discussion and Future Work

We considered the problem of two-dimensional visual search. A lower bound on the optimal expected total cost was derived, and the performance of two heuristic policies π_{ML} and π_{EJS} were investigated. Through some simple, yet, realistic examples the obtained bounds have been evaluated and shown to be asymptotically tight in certain settings. Extending the results to when visual inspection is restricted to contiguous segments of certain shapes, such as rectangles or squares, that would account for the limitations in visual inspection abilities of humans is an area of future work.

In general, the obtained performance bounds are asymptotically loose. The full characterization of the optimal inspection strategy remains an important area of future work.

Appendix

A. Performance Bounds

In this appendix, we give a sketch of the proof of Theorems 1 and 2. The details are omitted in the interest of brevity.

Proof of Theorem 1: Following similar lines as those in the proof of Proposition 2 in [9] (with a slight modification in Claim 3 to relax Assumption 2 in [9]), we obtain

$$\min\{\mathbb{E}[\tau] + L\mathrm{Pe}\} \geq \Big(\frac{\log M^2}{I_{\max}} + \frac{\log L}{D_{\max}}\Big)(1 - o(1)), \tag{8}$$

where $o(1) \to 0$ as $LM \to \infty$, and

$$I_{\max} := \lim_{M\to\infty} \max_{a\in\mathcal{A}} \max_{\boldsymbol{\rho}\in\mathbb{P}(\Omega)} \sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij} D(q^a_{ij} || \sum_{k=1}^{M}\sum_{l=1}^{M} \rho_{kl} q^a_{kl}). \tag{9}$$

What remains is to show that $I_{\max} \leq C_{\max}$. From (2) and for any $a \in \mathcal{A}$ and $\boldsymbol{\rho} \in \mathbb{P}(\Omega)$,

$$\begin{aligned}
&\sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij} D(q^a_{ij} || \sum_{k=1}^{M}\sum_{l=1}^{M} \rho_{kl} q^a_{kl}) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij} \Big[\mathbf{1}_{\{G_{ij}\in a\}} D(f_{|a|,M} || \eta^a f_{|a|,M} + \bar{\eta}^a \bar{f}_{|a|,M}) \\
&\qquad\qquad + \mathbf{1}_{\{G_{ij}\notin a\}} D(\bar{f}_{|a|,M} || \eta^a f_{|a|,M} + \bar{\eta}^a \bar{f}_{|a|,M})\Big] \\
&= \eta^a D(f_{|a|,M} || \eta^a f_{|a|,M} + \bar{\eta}^a \bar{f}_{|a|,M}) \\
&\qquad + \bar{\eta}^a D(\bar{f}_{|a|,M} || \eta^a f_{|a|,M} + \bar{\eta}^a \bar{f}_{|a|,M}) \\
&\overset{(a)}{\leq} \eta^a D(f_{1,M} || \eta^a f_{1,M} + \bar{\eta}^a \bar{f}_{1,M}) \\
&\qquad + \bar{\eta}^a D(\bar{f}_{1,M} || \eta^a f_{1,M} + \bar{\eta}^a \bar{f}_{1,M}) \\
&\overset{(b)}{\leq} D(f_{1,M} || \frac{1}{2} f_{1,M} + \frac{1}{2} \bar{f}_{1,M}),
\end{aligned} \tag{10}$$

where (a) and (b) follow from Assumptions 2 and 1.

Combining (8)–(10), we have the assertion of the theorem. ■

Proof of Theorem 2: Consider a policy π that selects d_{ij} (hence marks the stopping time τ) as $\rho_{ij}(t) \geq 1 - L^{-1}$ and at each time $t < \tau$ selects inspection region $A(t)$ such that

$$\begin{cases} EJS(\boldsymbol{\rho}(t), A(t)) \geq K_1 & \text{if } \rho_{ij}(t) \leq \tilde{\rho},\ \forall i,j \\ EJS(\boldsymbol{\rho}(t), A(t)) \geq \tilde{\rho} K_2 & \text{otherwise} \end{cases}, \tag{11}$$

where $\tilde{\rho} := 1 - \frac{1}{1+\max\{\log M^2, \log L\}}$. Then under policy π and MAP declaration, $\mathrm{Pe} = \mathbb{E}[1 - \max_{i\in\Omega} \rho_i(\tau)] \leq L^{-1}$ and hence, $V_\pi = \mathbb{E}[\tau] + L\mathrm{Pe} \leq \mathbb{E}[\tau] + 1$. What remains is to find an upper bound on $\mathbb{E}[\tau]$. Define function $U(\cdot)$ as

$$U(t) := \sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij}(t) \log \frac{\rho_{ij}(t)}{1 - \rho_{ij}(t)} - \log \frac{\tilde{\rho}}{1 - \tilde{\rho}}, \tag{12}$$

and let $\mathcal{F}(t)$ denote the history of inspected regions and outcomes up to time t. It is straightforward to show that the sequence $\{U(t)\}_t$ forms a submartingale with respect to the filtration $\{\mathcal{F}(t)\}_t$ and with the following property:

$$\mathbb{E}\big[U(t+1)|\mathcal{F}(t)\big] \geq U(t) + EJS(\boldsymbol{\rho}(t), A(t)). \tag{13}$$

Let $\tau' = \min\{t : U(t) \geq \log L\}$. By construction, $\tau \leq \tau'$. From (11), (13), and by Doob's stopping theorem, $\mathbb{E}[\tau'] \leq \big(\frac{\log M^2}{K_1} + \frac{\log L}{K_2}\big)(1+o(1))$ where $o(1) \to 0$ as $LM \to \infty$.

In the rest of the proof, we characterize K_1 and K_2 for policies π_{ML} and π_{EJS}. To do so, we have to analyze $EJS(\boldsymbol{\rho}, \pi_{ML}(\boldsymbol{\rho}))$ and $EJS(\boldsymbol{\rho}, \pi_{EJS}(\boldsymbol{\rho}))$ for all $\boldsymbol{\rho} \in \mathbb{P}(\Omega)$.

Let $\eta_{ij}^a = \sum_{k=1}^M \sum_{l=1}^M \rho_{kl}[\mathbf{1}_{\{G_{kl}, G_{ij} \in a\}} + \mathbf{1}_{\{G_{kl}, G_{ij} \notin a\}}]$, and $\bar{\eta}_{ij}^a = 1 - \eta_{ij}^a$. Under Assumption 1, we can rewrite (4) as

$$EJS(\boldsymbol{\rho}, a) = \sum_{i=1}^M \sum_{j=1}^M \rho_{ij} D(f_{|a|,M} || \frac{\eta_{ij}^a - \rho_{ij}}{1-\rho_{ij}} f_{|a|,M} + \frac{\bar{\eta}_{ij}^a}{1-\rho_{ij}} \bar{f}_{|a|,M}).$$

Consider a belief vector $\boldsymbol{\rho}$ at which $\rho_{ij} \geq \rho_{kl}$ for all $1 \leq k, l \leq M$. By Definition 1, $\pi_{ML}(\boldsymbol{\rho}) = \{G_{ij}\}$. We have,

$$EJS(\boldsymbol{\rho}, \pi_{ML}(\boldsymbol{\rho})) = EJS(\boldsymbol{\rho}, \{G_{ij}\}) \geq \rho_{ij} D(f_{1,M} || \bar{f}_{1,M}),$$

and hence,

$$EJS(\boldsymbol{\rho}, \pi_{ML}(\boldsymbol{\rho})) \geq \begin{cases} \tilde{\rho} D(f_{1,M} || \bar{f}_{1,M}) & \text{if } \rho_{ij} \geq \tilde{\rho} \\ \frac{1}{M^2} D(f_{1,M} || \bar{f}_{1,M}) & \text{otherwise} \end{cases}. \tag{14}$$

Let $\mathcal{A}_s$ denote the collection of all single segments, and let $\mathcal{A}_{th}$ be the collection of all 2-D threshold-like regions, i.e., for all $1 \leq i, j \leq M$, $\bigcup_{k=1}^{i-1} \bigcup_{l=1}^M \{G_{kl}\} \cup \bigcup_{l=1}^{j-1} \{G_{il}\} \in \mathcal{A}_{th}$ and $\bigcup_{l=j}^M \{G_{il}\} \cup \bigcup_{k=i+1}^M \bigcup_{l=1}^M \{G_{kl}\} \in \mathcal{A}_{th}$. Let

$$a^* := \operatorname*{arg\,min}_{a \in \mathcal{A}_{th} : |a| \leq \lfloor M^2/2 \rfloor} |1 - 2\eta^a|.$$

Next we show that

$$EJS(\boldsymbol{\rho}, a^*) \geq D(f_{\lfloor M^2/2 \rfloor, M} || \frac{1}{2} f_{\lfloor M^2/2 \rfloor, M} + \frac{1}{2} \bar{f}_{\lfloor M^2/2 \rfloor, M}),$$

which together with Definition 3, (14), and the fact that

$$EJS(\boldsymbol{\rho}, \pi_{EJS}(\boldsymbol{\rho})) \geq \max\{EJS(\boldsymbol{\rho}, \pi_{ML}(\boldsymbol{\rho})), EJS(\boldsymbol{\rho}, a^*)\}$$

completes the proof (which holds so long as $\mathcal{A}_s \cup \mathcal{A}_{th} \subseteq \mathcal{A}$).

We write the proof for the case $a^* = \bigcup_{k=1}^{i-1} \bigcup_{l=1}^M \{G_{kl}\} \cup \bigcup_{l=1}^{j-1} \{G_{il}\}$ and $\eta^{a^*} \leq \frac{1}{2}$ which implies that $1 - 2\eta^{a^*} \leq \rho_{ij}$. Other (three) cases can be proved in a similar way.

$$\begin{aligned}
& EJS(\boldsymbol{\rho}, a^*) \\
&= \sum_{k=1}^M \sum_{l=1}^M \rho_{kl} D(f_{|a^*|,M} || \frac{\eta_{kl}^{a^*} - \rho_{kl}}{1-\rho_{kl}} f_{|a^*|,M} + \frac{\bar{\eta}_{kl}^{a^*}}{1-\rho_{kl}} \bar{f}_{|a^*|,M}) \\
&\overset{(a)}{\geq} \eta^{a^*} D(f_{|a^*|,M} || \eta^{a^*} f_{|a^*|,M} + \bar{\eta}^{a^*} \bar{f}_{|a^*|,M}) \\
&\quad + \rho_{ij} D(f_{|a^*|,M} || \frac{1}{2} f_{|a^*|,M} + \frac{1}{2} \bar{f}_{|a^*|,M}) \\
&\quad + (\bar{\eta}^{a^*} - \rho_{ij}) D(f_{|a^*|,M} || \bar{\eta}^{a^*} f_{|a^*|,M} + \eta^{a^*} \bar{f}_{|a^*|,M}) \\
&\overset{(b)}{\geq} D(f_{|a^*|,M} || \frac{1}{2} f_{|a^*|,M} + \frac{1}{2} \bar{f}_{|a^*|,M}) \\
&\overset{(c)}{\geq} D(f_{\lfloor M^2/2 \rfloor, M} || \frac{1}{2} f_{\lfloor M^2/2 \rfloor, M} + \frac{1}{2} \bar{f}_{\lfloor M^2/2 \rfloor, M}),
\end{aligned}$$

where (a) follows from the fact that $\eta_{kl}^{a^*} = \eta^{a^*} \mathbf{1}_{\{G_{kl} \in a^*\}} + \bar{\eta}^{a^*} \mathbf{1}_{\{G_{kl} \notin a^*\}}$, $\frac{\eta_{kl}^{a^*} - \rho_{kl}}{1-\rho_{kl}} \leq \eta_{kl}^{a^*}$, and $\frac{\eta_{ij}^{a^*} - \rho_{ij}}{1-\rho_{ij}} \leq \frac{1}{2}$, (b) follows from Jensen's inequality and since $(\eta^{a^*})^2 + \frac{1}{2}\rho_{ij} + (\bar{\eta}^{a^*} - \rho_{ij})\bar{\eta}^{a^*} \leq \frac{1}{2}$, and (c) follows from Assumption 2. ∎

B. Examples

It is straightforward to show that all assumptions hold in Example 1. In this appendix, we provide the proof of Example 2. The proof of Example 3 follows a similar approach.

Proof of Example 2: Assumption 1 holds trivially since $\bar{f}_{n,M}(z) = p_r \delta(z-1) + (1-p_r)\delta(z)$ and $f_{n,M}(z) = \bar{f}_{n,M}(1-z)$, where $r = \frac{n}{M^2}$ and $\delta(\cdot)$ denotes the Dirac delta function.

Next we show that Assumption 2 holds. Let $r = \frac{n}{M^2}$ and $r' = \frac{n+1}{M^2}$. The case $\alpha = 0$ is trivial. For $\alpha \in (0,1]$, we obtain

$$\begin{aligned}
& D(f_{n,M} || \alpha f_{n,M} + \bar{\alpha} \bar{f}_{n,M}) \\
&= p_r \log \frac{p_r}{\alpha p_r + \bar{\alpha}(1-p_r)} + (1-p_r) \log \frac{1-p_r}{\alpha(1-p_r) + \bar{\alpha} p_r} \\
&= -p_r \log \big(\alpha + \bar{\alpha} \frac{1-p_r}{p_r}\big) - (1-p_r) \log \big(\alpha + \bar{\alpha} \frac{p_r}{1-p_r}\big) \\
&= -\log \alpha - H_\alpha(p_r) \\
&\overset{(a)}{\geq} -\log \alpha - H_\alpha(p_{r'}) \\
&= D(f_{n+1,M} || \alpha f_{n+1,M} + \bar{\alpha} \bar{f}_{n+1,M}),
\end{aligned}$$

where

$$H_\alpha(p) := p \log \big(1 + \frac{\bar{\alpha}}{\alpha} \frac{1-p}{p}\big) + (1-p) \log \big(1 + \frac{\bar{\alpha}}{\alpha} \frac{p}{1-p}\big),$$

and inequality (a) holds since $p_r \leq p_{r'}$ and $H_\alpha(p)$ is concave and symmetric in p for any $\alpha \in (0,1]$.

Finally, Assumption 3 holds since for any $\gamma > 1$,

$$\sup_M \max_{n \leq M^2} \int_{\mathcal{Z}} f_{n,M}(z) \Big| \log \frac{f_{n,M}(z)}{\bar{f}_{n,M}(z)} \Big|^\gamma dz \leq \Big| \log \frac{1-p_0}{p_0} \Big|^\gamma.$$

∎

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