Second-Order Slepian-Wolf Coding Theorems for Non-Mixed and Mixed Sources

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***Abstract*—The *second-order* achievable rate region in Slepian-Wolf source coding systems is investigated. The concept of *second-order* achievable rates, which enables us to make a finer evaluation of achievable rates, has already been introduced and analyzed for *general* sources in the single-user source coding problem. Accordingly, in this paper, we first define the *second-order* achievable rate region for the Slepian-Wolf coding system and establish the source coding theorem for *general* sources in the *second-order* sense. Moreover, we compute the explicit *second-order* achievable rate region for i.i.d. correlated sources with *countably infinite* alphabets and *mixed* correlated sources, respectively, using the relevant asymptotic normality.**

I. Introduction

We establish the *second-order* source coding theorems for the Slepian-Wolf coding system [1]. In the single-user source coding system, Han and Verdú [2] has shown the source coding theorem for *general* sources using the *information spectrum* methods. Since the class of *general* sources is quite large, their results are very fundamental and useful. On the other hand, there are several researches concerning a finer evaluation of achievable rates called rates of the *second-order*. Hayashi [3] has given the second-order optimal achievability theorem for the fixed-length source coding problem for *general* sources, and actually computed it for i.i.d. sources by invoking the asymptotic normality. Nomura and Han [4] have computed the second-order optimal rate for *mixed* sources, which is a typical case of *nonergodic* source, again by invoking the relevant asymptotic normality.

In the area of multi-user source coding problems, the Slepian-Wolf coding problem for two correlated sources is one of typical problems. In this paper, we shall determine the *second-order* achievable rate region for *general* correlated sources with *countably infinite* alphabets in the Slepian-Wolf coding problem. Furthermore, for i.i.d. correlated sources with *countably infinite* alphabets and mixed i.i.d. correlated sources with *finite* alphabets, we apply our fundamental results to derive the explicit *second-order* achievable rate region. In the *second-order* analysis for i.i.d. correlated sources and mixed i.i.d. correlated sources, the multivariate normal distribution function due to the central limit theorem plays the key role, while in the previous literature on the *second-order* achievable rates (cf. [3], [4]), only one-dimensional normal distribution functions were enough to consider. Recently, Tan and Kosut [5] has independently and first established the *second-order* source coding theorem for non-mixed i.i.d. correlated sources, which is derived partly via the method of information spectra in addition to the standard multi-dimensional central limit theorem. It should be noted that the result in [5] heavily relies on the method of *types*. Although the coding/decoding method used in [5] is *universal*, it requires the assumption of *finiteness* of alphabets. The analyses here are based wholly on information spectrum methods to invoke the multi-dimensional normal distribution functions.

II. Preliminaries

A. Correlated Sources

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be alphabets of two correlated sources, where $\mathcal{X}_1$ and $\mathcal{X}_2$ may be *countably infinite*. Let $(\mathbf{X}_1, \mathbf{X}_2) = \{(X_1^n, X_2^n)\}_{n=1}^\infty$ denote a general correlated source pair, i.e., (X_1^n, X_2^n) taking values in $\mathcal{X}_1^n \times \mathcal{X}_2^n$ is a pair of correlated source variables of block length n, and we write as

$$(X_1^n, X_2^n) = ((X_{11}, X_{21}), (X_{12}, X_{22}), \cdots, (X_{1n}, X_{2n})),$$

and let $\mathbf{x}_j = x_{j1}, x_{j2}, \cdots, x_{jn}$ be a realization of random variable X_j^n $(j = 1, 2)$. The probability distribution of $(\mathbf{x}_1, \mathbf{x}_2)$ is denoted by $P_{X_1^n X_2^n}(\mathbf{x}_1, \mathbf{x}_2)$. In particular, in the case that the pair of correlated sources has an i.i.d. property, it holds that $P_{X_1^n X_2^n}(\mathbf{x}_1, \mathbf{x}_2) = \prod_{i=1}^n P_{X_1 X_2}(x_{1i}, x_{2i})$, with generic correlated random variable (X_1, X_2), where we use the convention that $P_Z(\cdot)$ denotes the probability distribution of Z, and $P_{Z|W}(\cdot|\cdot)$ denotes the conditional probability distribution of Z given W.

B. ε-Achievable Rate Region

The fixed-length codes for correlated sources are characterized by a pair of encoders $(\phi_n^{(1)}, \phi_n^{(2)})$ and a decoder ψ_n. The encoders are mappings such as $\phi_n^{(1)} : \mathcal{X}_1^n \to \mathcal{M}_n^{(1)}$, $\phi_n^{(2)} : \mathcal{X}_2^n \to \mathcal{M}_n^{(2)}$, where

$$\mathcal{M}_n^{(1)} = \{1, 2, \cdots, M_n^{(1)}\}, \quad \mathcal{M}_n^{(2)} = \{1, 2, \cdots, M_n^{(2)}\}$$

denote the code sets. The decoder is defined as a mapping $\psi_n : \mathcal{M}_n^{(1)} \times \mathcal{M}_n^{(2)} \to \mathcal{X}_1^n \times \mathcal{X}_2^n$.

Then, the error probability is given by

$$\varepsilon_n = \Pr\{(X_1^n, X_2^n) \neq \psi_n(\phi_n^{(1)}(X_1^n), \phi_n^{(2)}(X_2^n))\}.$$

We call such a pair of encoders $(\phi_n^{(1)}, \phi_n^{(2)})$ and decoder ψ_n along with error probability ε_n an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code.

Definition 2.1: A rate pair (R_1, R_2) is called an ε-achievable rate pair if there exists an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code satisfying $\limsup_{n\to\infty} \varepsilon_n \le \varepsilon$ and

$$\limsup_{n\to\infty} \frac{1}{n}\log M_n^{(1)} \le R_1, \quad \limsup_{n\to\infty} \frac{1}{n}\log M_n^{(2)} \le R_2.$$

Then, the ε-achievable rate region is defined as the set of all ε-achievable rate pairs:

Definition 2.2: (ε-Achievable Rate Region)

$$R(\varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \{(R_1, R_2) | (R_1, R_2) \text{ is } \varepsilon\text{-achievable}\}.$$

The following *first-order* achievable rate theorem reveals the rate region for i.i.d. correlated sources with *countably infinite* alphabets.

Theorem 2.1 (Cover [6]): For any i.i.d. correlated sources with countably infinite alphabets, the 0-achievable rate region is given as the set of (R_1, R_2) satisfying

$$\begin{aligned} R(0|\mathbf{X}_1, \mathbf{X}_2) = \{&(R_1, R_2) | R_1 \ge H(X_1|X_2), \\ &R_2 \ge H(X_2|X_1), R_1 + R_2 \ge H(X_1X_2)\}, \end{aligned}$$

where $H(X_1|X_2)$ and $H(X_2|X_1)$ denote conditional entropies and $H(X_1X_2)$ denotes joint entropy.

C. (a_1, a_2, ε)-*Achievable Rate Region*

We define the *second-order* achievable rate pair as follows.

Definition 2.3: A rate pair (L_1, L_2) is called a *second-order* (a_1, a_2, ε)-achievable rate pair if there exists an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code satisfying $\limsup_{n\to\infty} \varepsilon_n \le \varepsilon$ and

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}\log \frac{M_n^{(1)}}{e^{na_1}} \le L_1, \quad \limsup_{n\to\infty} \frac{1}{\sqrt{n}}\log \frac{M_n^{(2)}}{e^{na_2}} \le L_2.$$

Moreover, let us define the set of (a_1, a_2, ε)-achievable rate pairs given (a_1, a_2) as

Definition 2.4 ((a_1, a_2, ε)-achievable rate region):

$$\begin{aligned} &L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) \\ &= \{(L_1, L_2) | (L_1, L_2) \text{ is } (a_1, a_2, \varepsilon)\text{-achievable}\}. \end{aligned}$$

In the next section, we shall determine the (a_1, a_2, ε)-achievable rate region for *general* sources, by using the following fundamental lemmas, that were originally invoked to derive the *first-order* achievable rate region [7], where $\{z_n\}_{n=1}^\infty$ is a sequence of arbitrary numbers such that $z_i > 0$ $(\forall i = 1, 2, \cdots)$.

Lemma 2.1 (Han [7]): Let $M_n^{(1)}$ and $M_n^{(2)}$ be arbitrarily given positive integers. Then, for all $n = 1, 2, \cdots$, there exists an $(n, M_n^{(1)}, M_n^{(2)}, \epsilon_n)$ code satisfying

$$\begin{aligned} \varepsilon_n \le \Pr\Bigg\{ & z_n P_{X_1^n|X_2^n}(X_1^n|X_2^n) \le \frac{1}{M_n^{(1)}} \\ & \text{or } z_n P_{X_2^n|X_1^n}(X_2^n|X_1^n) \le \frac{1}{M_n^{(2)}} \\ & \text{or } z_n P_{X_1^nX_2^n}(X_1^n, X_2^n) \le \frac{1}{M_n^{(1)}M_n^{(2)}} \Bigg\} + 3z_n. \ \blacksquare \end{aligned}$$

Lemma 2.2 (Han [7]): Any $(n, M_n^{(1)}, M_n^{(2)}, \epsilon_n)$ code satisfies

$$\begin{aligned} \varepsilon_n \ge \Pr\Bigg\{ & P_{X_1^n|X_2^n}(X_1^n|X_2^n) \le \frac{z_n}{M_n^{(1)}} \\ & \text{or } P_{X_2^n|X_1^n}(X_2^n|X_1^n) \le \frac{z_n}{M_n^{(2)}} \\ & \text{or } P_{X_1^nX_2^n}(X_1^n, X_2^n) \le \frac{z_n}{M_n^{(1)}M_n^{(2)}} \Bigg\} - 3z_n. \ \blacksquare \end{aligned}$$

III. (a_1, a_2, ε)-ACHIEVABLE RATE REGION FOR GENERAL CORRELATED SOURCES

We shall determine the (a_1, a_2, ε)-achievable rate region for *general* correlated sources with countably infinite alphabets. Let us define the function $\overline{F}_n(L_1, L_2|a_1, a_2)$ by

$$\begin{aligned} &\overline{F}_n(L_1, L_2|a_1, a_2) \\ &= \Pr\Bigg\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - na_1}{\sqrt{n}} < L_1, \\ &\qquad \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - na_2}{\sqrt{n}} < L_2, \\ &\qquad \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - n(a_1 + a_2)}{\sqrt{n}} < L_1 + L_2 \Bigg\}. \end{aligned}$$

Notice here that $\overline{F}_n(L_1, L_2|a_1, a_2)$ is a multivariate cumulative distribution function. Now, we have

Theorem 3.1:

$$\begin{aligned} &L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) \\ &= \mathrm{Cl}\left(\left\{ (L_1, L_2) \middle| \liminf_{n\to\infty} \overline{F}_n(L_1, L_2|a_1, a_2) \ge 1 - \varepsilon \right\}\right), \end{aligned}$$

where $\mathrm{Cl}(\cdot)$ denotes the closure operation.

Proof: 1) Direct Part:

For any fixed (L_1, L_2) satisfying

$$\liminf_{n\to\infty} \overline{F}_n(L_1, L_2|a_1, a_2) \ge 1 - \epsilon, \tag{1}$$

set $M_n^{(1)} = e^{na_1 + L_1\sqrt{n} + 2\sqrt[4]{n}\gamma}$, $M_n^{(2)} = e^{na_2 + L_2\sqrt{n} + 2\sqrt[4]{n}\gamma}$, where $\gamma > 0$ is an arbitrary small constant. It is obvious that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}\log \frac{M_n^{(1)}}{e^{na_1}} \le L_1, \ \limsup_{n\to\infty} \frac{1}{\sqrt{n}}\log \frac{M_n^{(2)}}{e^{na_2}} \le L_2,$$

hold. Thus, in this direct part it suffices to show the existence of an $(n, M_n^{(1)}, M_n^{(2)}, \epsilon_n)$ code such that $\limsup_{n\to\infty} \varepsilon_n \le \varepsilon$.

Lemma 2.1 with $z_n = e^{-\sqrt[4]{n}\gamma}$ implies that there exists an $(n, M_n^{(1)}, M_n^{(2)}, \epsilon_n)$ code such that

$$\begin{aligned} \varepsilon_n \le \Pr\Bigg\{ & \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - na_1}{\sqrt{n}} \ge L_1 + \frac{\gamma}{\sqrt[4]{n}} \\ & \text{or } \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - na_2}{\sqrt{n}} \ge L_2 + \frac{\gamma}{\sqrt[4]{n}} \\ & \text{or } \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - n(a_1 + a_2)}{\sqrt{n}} \\ & \qquad \ge L_1 + L_2 + \frac{3\gamma}{\sqrt[4]{n}} \Bigg\} + 3e^{-\sqrt[4]{n}\gamma} \\ \le 1 - \overline{F}_n & \left(L_1 + \frac{\gamma}{\sqrt[4]{n}}, L_2 + \frac{\gamma}{\sqrt[4]{n}} \middle| a_1, a_2 \right) + 3e^{-\sqrt[4]{n}\gamma}, \end{aligned}$$

where the last inequality is derived from De Morgan's law. Thus, we have

$$\varepsilon_n \leq 1 - \overline{F}_n \left(L_1 + \frac{\gamma}{\sqrt[4]{n}}, L_2 + \frac{\gamma}{\sqrt[4]{n}} \middle| a_1, a_2 \right) + 3e^{-\sqrt[4]{n}\gamma}$$
$$\leq 1 - \overline{F}_n \left(L_1, L_2 \middle| a_1, a_2 \right) + 3e^{-\sqrt[4]{n}\gamma}.$$

By taking $\limsup_{n\to\infty}$, we have

$$\limsup_{n\to\infty} \varepsilon_n \leq 1 - \liminf_{n\to\infty} \overline{F}_n \left(L_1, L_2 \middle| a_1, a_2 \right) \leq \epsilon,$$

where the last inequality follows from (1). Thus, the direct part has been proved.

2) Converse Part:

It suffices to follow the above arguments in the inverse direction, using Lemma 2.2 instead of Lemma 2.1. The details are omitted. ∎

IV. (a_1, a_2, ε)-ACHIEVABLE RATE REGION FOR I.I.D. CORRELATED SOURCES

In this and subsequent sections, we try to compute the (a_1, a_2, ε)-achievable rate region for several typical cases on the basis of Theorem 3.1. In this section we focus on i.i.d. correlated sources. In this case, the polygon $R(0|\mathbf{X}_1, \mathbf{X}_2)$ has the following boundary points:

Case I (Corner Points):

$$a_1 = H(X_1|X_2) \quad \text{and} \quad a_2 = H(X_2);$$
$$a_1 = H(X_1) \quad \text{and} \quad a_2 = H(X_2|X_1),$$

Case II (Non-Corner Points): For $0 < \forall\lambda < 1$,

$$a_1 = \lambda H(X_1) + (1-\lambda)H(X_1|X_2),$$
$$a_2 = (1-\lambda)H(X_2) + \lambda H(X_2|X_1),$$

Case III (Full Side Points):

$$a_1 = H(X_1|X_2) \quad \text{and} \quad a_2 > H(X_2);$$
$$a_1 > H(X_1) \quad \text{and} \quad a_2 = H(X_2|X_1).$$

In each of these cases, we shall compute the (a_1, a_2, ε)-achievable rate region on the basis of Theorem 3.1 to obtain Theorem 4.1.

Remark 4.1: It is not difficult to check that (L_1, L_2) can take arbitrary values in $\mathbb{R}^2$ if (a_1, a_2) is an internal point of the polygon, while the set of all achievable rate pairs (L_1, L_2) reduces to the empty set $\emptyset$ if (a_1, a_2) is outside the polygon. Thus, we may focus only on the above cases. ∎

First, we define

$$\Phi(T_1, T_2, T_3)$$
$$\equiv \lim_{n\to\infty} \Pr\left\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < T_1, \right.$$
$$\frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - nH(X_2|X_1)}{\sqrt{n}} < T_2,$$
$$\left. \frac{-\log P_{X_1^n X_2^n}(X_1^n X_2^n) - nH(X_1 X_2)}{\sqrt{n}} < T_3 \right\},$$

then, by means of the multi-dimensional central limit theorem, we see that $\Phi(T_1, T_2, T_3)$ specifies a three-dimensional normal cumulative distribution function; more specifically,

$$\Phi(T_1, T_2, T_3) \equiv \int_{-\infty}^{T_1} dy_1 \int_{-\infty}^{T_2} dy_2 \int_{-\infty}^{T_3} dy_3$$
$$\times \frac{1}{(\sqrt{2\pi})^3 \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} \mathbf{y} \Sigma^{-1} \mathbf{y}^{\mathrm{T}} \right),$$

where $\mathbf{y} = (y_1, y_2, y_3)$ is a three-dimensional row vector, and $\Sigma = (\sigma_{ij}^2) \quad (i, j = 1, 2, 3)$ denotes the covariance matrix:

$$\Sigma = \mathrm{Var}\left(-\log P_{X_1|X_2}(X_1|X_2), -\log P_{X_2|X_1}(X_2|X_1), \right.$$
$$\left. -\log P_{X_1 X_2}(X_1 X_2) \right).$$

We then consider the marginal cumulative distributions $\Phi_{13}(T_1, T_3)$, $\Phi_1(T_1)$, $\Phi_3(T_3)$ of $\Phi(T_1, T_2, T_3)$ such that

$$\Phi_{13}(T_1, T_3) \equiv \lim_{T_2\to\infty} \Phi(T_1, T_2, T_3),$$
$$\Phi_1(T_1) \equiv \lim_{T_2, T_3\to\infty} \Phi(T_1, T_2, T_3),$$
$$\Phi_3(T_3) \equiv \lim_{T_1, T_2\to\infty} \Phi(T_1, T_2, T_3).$$

Now, we have the following theorem:

Theorem 4.1: With any i.i.d. correlated sources (with *countably infinite* alphabets), we have for all $0 \leq \varepsilon < 1$:

Case I $a_1 = H(X_1|X_2)$ and $a_2 = H(X_2)$ (without loss of generality):

$$L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \{(L_1, L_2) \,|\, \Phi_{13}(L_1, L_1 + L_2) \geq 1 - \varepsilon\}.$$

Case II for correlated X_1, X_2 and $0 < \forall\lambda < 1$:

$$L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \{(L_1, L_2) \,|\, \Phi_3(L_1 + L_2) \geq 1 - \varepsilon\}.$$

Case III $a_1 = H(X_1|X_2)$ and $a_2 > H(X_2)$ (without loss of generality):

$$L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \{(L_1, L_2) \,|\, \Phi_1(L_1) \geq 1 - \varepsilon\}.$$

Proof: Applying Lemmas 2.1 and 2.2 to this special case and invoking the multi-dimensional central limit theorem are enough. ∎

Remark 4.2: Here, in view of the definitions of first-order and second-order achievable rates, any achievable rates can be expressed as in the form of (first-order achievable rate) + $\frac{1}{\sqrt{n}}$(second-order achievable rate) + $o\left(\frac{1}{\sqrt{n}}\right)$. Thus, the two-dimensional set $\mathcal{A}_n(\varepsilon)$ of all achievable rate pairs $\left(R_1^{(n)}, R_2^{(n)}\right)^{\mathrm{T}}$ in the finite blocklength regime turns out to be expressed, up to the second-order, as

$$\mathcal{A}_n(\varepsilon)$$
$$= \bigcup_{(a_1, a_2)^{\mathrm{T}} \in R(0|\mathbf{X}_1, \mathbf{X}_2)} \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{1}{\sqrt{n}} L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) \right\},$$

where the first-order Slepian-Wolf region with $\varepsilon = 0$ is depicted with solid line in Fig. 1. With the aid of Theorem 4.1, the boundary $\partial\mathcal{A}_n(\varepsilon)$ of $\mathcal{A}_n(\varepsilon)$ is also depicted in Fig. 1 with broken lines, i.e., the boundary $\partial\mathcal{A}_n(\varepsilon)$ for the case of $\varepsilon > \frac{1}{2}$ is depicted there. It should be noted that if $\varepsilon > \frac{1}{2}$ then the second-order achievable rates L_1, L_2 can be negative, i.e., in all of Cases I, II and III in Theorem 4.1 with $\varepsilon > \frac{1}{2}$ the optimal achievable rates $\left(R_1^{(n)}, R_2^{(n)}\right)^{\mathrm{T}}$ in the finite blocklength regime approach (when n becomes large) the optimal first-order achievable rate region $R(0|\mathbf{X}_1, \mathbf{X}_2)$ *from outside*.

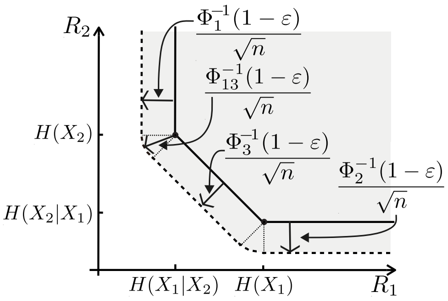

Fig. 1. Achievable rate region $\mathcal{A}_n(\varepsilon)$ ($\varepsilon > 1/2$)

Remark 4.3: Tan and Kosut [5] defined with *finite* source alphabets the region $\mathcal{R}_{\rm in}(n,\varepsilon) \subset \mathbb{R}^2$ to be the set of rate pairs (R_1, R_2) such that

$$\mathbf{R} \geq \mathbf{H} + \frac{1}{\sqrt{n}}\mathscr{S}(\Sigma,\varepsilon), \tag{2}$$

where "$\mathbf{R} \geq S$" means componentwise inequality for some element of S, and $\mathbf{R} = (R_1, R_2, R_1 + R_2)^{\rm T}$, $\mathbf{H} = (H(X_1|X_2), H(X_2|X_1), H(X_1X_2))^{\rm T}$, $\mathscr{S}(\Sigma,\varepsilon) = \left\{\mathbf{z} \in \mathbb{R}^3 \,|\Pr\left\{\mathbf{Z} \leq \mathbf{z}\right\} \geq 1-\varepsilon\right\}$ with $\mathbf{Z} \sim \mathcal{N}(\mathbf{0},\Sigma)$; then they proved that $\mathcal{R}_{\rm in}(n,\varepsilon)$ is a set of achievable rate pairs up to the second-order. It is not difficult to check that $\mathcal{R}_{\rm in}(n,\varepsilon) = \mathcal{A}_n(\varepsilon)$ for *finite* source alphabets. ■

Theorem 4.1 can be equivalently restated as the following corollary:

Corollary 4.1: For an i.i.d. correlated sources with *countably infinite* alphabets, the second-order (a_1, a_2, ε)-achievable rate region is given as the two-dimensional set:

$$\begin{aligned} &L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) \\ &= \Big\{(L_1, L_2 \,\Big|\lim_{n\to\infty} \Phi(\sqrt{n}(a_1 - H(X_1|X_2)) + L_1, \\ &\qquad \sqrt{n}(a_2 - H(X_2|X_1)) + L_2, \\ &\qquad \sqrt{n}(a_1 + a_2 - H(X_1X_2)) + L_1 + L_2) \geq 1-\varepsilon\Big\}. \end{aligned} \tag{3}$$

Remark 4.4: Compared with Theorem 4.1, it is observed that the description of Corollary 4.1 needs only a *single* equation (3) but not any classifications of pairs (a_1, a_2). This is a great advantage of the information spectrum methods. Thus, we call (3) the *canonical representation* for $L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2)$. This point of view is inherited also to Section VI, which enables us to successfully establish the second-order achievable rate region for *mixtures* of correlated i.i.d. sources. This approach is completely different from that of Tan and Kosut [5].

V. Comparison with the Koshelev Bound and Numerical Example

In this section, we compare the region (Theorem 4.1) so far derived with the *modified* Koshelev bound ([8]), i.e., the Gallager type bound for the Slepian-Wolf source coding system. Let

$$E_1(s_1) \equiv -\log \sum_{x_2\in\mathcal{X}_2}\left(\sum_{x_1\in\mathcal{X}_1} P_{X_1X_2}(x_1,x_2)^{\frac{1}{1+s_1}}\right)^{1+s_1}$$

$$E_2(s_2) \equiv -\log \sum_{x_1\in\mathcal{X}_1}\left(\sum_{x_2\in\mathcal{X}_2} P_{X_1X_2}(x_1,x_2)^{\frac{1}{1+s_2}}\right)^{1+s_2}$$

$$E_3(s_3) \equiv -\log\left(\sum_{(x_1,x_2)\in\mathcal{X}_1\times\mathcal{X}_2} P_{X_1X_2}(x_1,x_2)^{\frac{1}{1+s_3}}\right)^{1+s_3}.$$

Then, we have the following lemma, which is a stronger version of the original Koshelev bound [8] (Notice that, on the contrary to here, s_1, s_2, s_3 are constrained so as to be $0 \leq s_1 = s_2 = s_3 \leq 1$ in [8]. Tan and Kosut [5] first compared their result with this original Koshelev bound with detailed computations.):

Lemma 5.1: Let $R_1 = \frac{1}{n}\log M_n^{(1)}$ and $R_2 = \frac{1}{n}\log M_n^{(2)}$, then there exists an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code satisfying

$$\begin{aligned} \varepsilon_n \leq &\min_{0\leq s_1\leq 1} \exp\left[-n\left(R_1 s_1 - E_1(s_1)\right)\right] \\ &+ \min_{0\leq s_2\leq 1} \exp\left[-n\left(R_2 s_2 - E_2(s_2)\right)\right] \\ &+ \min_{0\leq s_3\leq 1} \exp\left[-n\left((R_1+R_2)\, s_3 - E_3(s_3)\right)\right]. \end{aligned}$$

Proof: Omitted. ■

We call this bound merely the Koshelev bound for simplicity.

In this section, we calculate the Koshelev bound for comparison in the case that $R_1 = H(X_1|X_2) + \frac{L_1}{\sqrt{n}}$ and $R_2 = H(X_2) + \frac{L_2}{\sqrt{n}}$, i.e., Case I in the previous section. Then, from Lemma 5.1 and the similar argument to that of Hayashi [9, Sect. V], we have for $L_1 \geq 0, L_1 + L_2 \geq 0$

$$\varepsilon_n \leq \exp\left[\frac{-L_1^2}{2\sigma_{11}^2}\right] + \exp\left[\frac{-(L_1+L_2)^2}{2\sigma_{33}^2}\right]. \tag{4}$$

Next, we give numerical examples. Let us consider i.i.d. correlated sources with binary alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \{0,1\}$, whose joint probabilities are given by $P_{X_1X_2}(0,0) = 0.5$, $P_{X_1X_2}(0,1) = 0.25$, $P_{X_1X_2}(1,0) = 0.15$, $P_{X_1X_2}(1,1) = 0.1$. The entropies in bits are $H(X_1|X_2) = 0.809$, $H(X_2) = 0.934$ and $H(X_1X_2) = 1.743$ and the covariance matrix is $\Sigma_{13} = \begin{pmatrix} \sigma_{11}^2 & \sigma_{13}^2 \\ \sigma_{31}^2 & \sigma_{33}^2 \end{pmatrix} = \begin{pmatrix} 0.475 & 0.492 \\ 0.492 & 0.690 \end{pmatrix}$. In this case, we compute the error probabilities given by the Koshelev bound i.e., the right hand side on (4) and the second-order evaluation:

$$1 - \Phi_{13}(L_1, L_1 + L_2). \tag{5}$$

Fig. 2 illustrates the graphs of these functions with $L_1 = L_2$. We find that there is a difference between the second-order evaluation and the Koshelev bound, which shows that the former approach outperforms the latter approach. We obtain similar results also for other (a_1, a_2) on boundary points.

VI. (a_1, a_2, ε)-achievable rate region for Mixed correlated sources

In this section, we establish the second-order achievable rate region for the mixed correlated sources. Recall that the mixed sources are typical cases of *nonergodic* sources where the asymptotic normality of self-information vector does *not* hold. We consider the following mixed correlated source:

$$P_{X_1^nX_2^n}(\mathbf{x}_1,\mathbf{x}_2) = \int_\Lambda P_{X_1^{(\theta)n}X_2^{(\theta)n}}(\mathbf{x}_1,\mathbf{x}_2)w(d\theta), \tag{6}$$

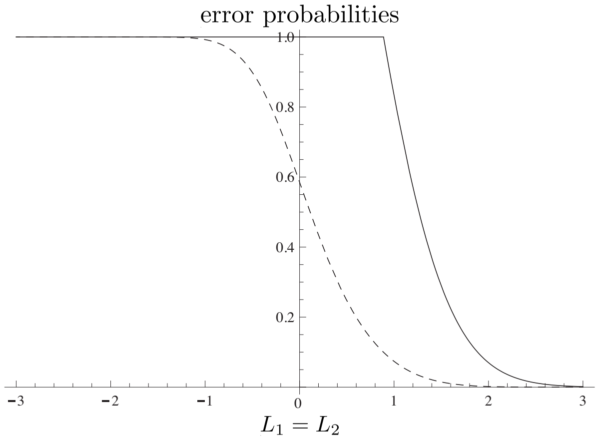

Fig. 2. The graphs of second-order evaluation (5) (broken line) and the Koshelev bound (4) (solid line) with $L_1 = L_2$ (Case I).

where $(\mathbf{X}_1, \mathbf{X}_2) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}$ and $w(d\theta)$ is an *arbitrary probability measure* on the parameter space Λ, and $\left(\mathbf{X}_1^{(\theta)}, \mathbf{X}_2^{(\theta)}\right) = \left\{\left(X_1^{(\theta)n}, X_2^{(\theta)n}\right)\right\}_{n=1}^{\infty}$ $(\theta \in \Lambda)$ are i.i.d. correlated sources with *finite* alphabets, and also the integrand on the right-hand side is assumed to be a measurable function of θ. In this case, we also define the normal cumulative distribution function for each $\theta \in \Lambda$ as follows.

$$\Phi^{(\theta)}(T_1, T_2, T_3) \equiv \int_{-\infty}^{T_1} dy_1 \int_{-\infty}^{T_2} dy_2 \int_{-\infty}^{T_3} dy_3 \times \frac{1}{(\sqrt{2\pi})^3 \sqrt{\det \Sigma_\theta}} \exp\left(-\frac{1}{2}\mathbf{y}\Sigma_\theta^{-1}\mathbf{y}^{\mathrm{T}}\right),$$

where $\mathbf{y} = (y_1, y_2, y_3)$, and the covariance matrix $\Sigma_\theta = (\sigma_{ij}^2(\theta))$ $(i, j = 1, 2, 3,\ \theta \in \Lambda)$ are defined in a similar manner to Section IV.

Then, the following theorem holds.

Theorem 6.1: For the mixed correlated source with *finite* alphabet defined by (6), the second-order (a_1, a_2, ε)-achievable rate region is given as the set:

$$L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \left\{(L_1, L_2) \middle| \int_\Lambda \lim_{n\to\infty} \Phi^{(\theta)}\left(\sqrt{n}\left(a_1 - H\left(X_1^{(\theta)} \middle| X_2^{(\theta)}\right)\right) + L_1, \sqrt{n}\left(a_2 - H\left(X_2^{(\theta)} \middle| X_1^{(\theta)}\right)\right) + L_2, \sqrt{n}\left(a_1 + a_2 - H\left(X_1^{(\theta)} X_2^{(\theta)}\right)\right) + L_1 + L_2\right) w(d\theta) \geq 1 - \varepsilon\right\}.$$

Proof: Omitted to save space. ■

An immediate consequence of Theorem 6.1 is the following compact formula *not* including $\lim_{n\to\infty}$, where $\Phi_i^{(\theta)}$, $\Phi_{ij}^{(\theta)}$ are the marginal commutative distribution functions of $\Phi^{(\theta)}$ indicated by i, ij, respectively (cf. Φ_i, Φ_{ij} in Section IV):

Theorem 6.2:

$$L(a_1, a_2, \varepsilon|\mathbf{X}_1, \mathbf{X}_2) = \left\{(L_1, L_2) | \Phi^\Lambda(a_1, a_2; L_1, L_2) \geq 1 - \varepsilon\right\},$$

where

$$\begin{aligned}
\Phi^\Lambda&(a_1, a_2; L_1, L_2) \\
&= \int_{\Lambda_0(a_1,a_2)} w(d\theta) + \int_{\Lambda_1(a_1,a_2)} \Phi_1^{(\theta)}(L_1) w(d\theta) \\
&+ \int_{\Lambda_2(a_1,a_2)} \Phi_2^{(\theta)}(L_2) w(d\theta) \\
&+ \int_{\Lambda_3(a_1,a_2)} \Phi_3^{(\theta)}(L_1 + L_2) w(d\theta) \\
&+ \int_{\Lambda_4(a_1,a_2)\setminus\Lambda_5(a_1,a_2)} \Phi_{13}^{(\theta)}(L_1, L_1 + L_2) w(d\theta) \\
&+ \int_{\Lambda_5(a_1,a_2)\setminus\Lambda_4(a_1,a_2)} \Phi_{23}^{(\theta)}(L_2, L_1 + L_2) w(d\theta) \\
&+ \int_{\Lambda_4(a_1,a_2)\cap\Lambda_5(a_1,a_2)} \Phi_{12}^{(\theta)}(L_1, L_2) w(d\theta);
\end{aligned}$$

and

$$\Lambda_0(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 > H\left(X_1^{(\theta)} \middle| X_2^{(\theta)}\right), a_2 > H\left(X_2^{(\theta)} \middle| X_1^{(\theta)}\right), a_1 + a_2 > H\left(X_1^{(\theta)} X_2^{(\theta)}\right)\right\},$$

$$\Lambda_1(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 = H\left(X_1^{(\theta)} \middle| X_2^{(\theta)}\right), a_2 > H\left(X_2^{(\theta)}\right)\right\},$$

$$\Lambda_2(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 > H\left(X_1^{(\theta)}\right), a_2 = H\left(X_2^{(\theta)} \middle| X_1^{(\theta)}\right)\right\},$$

$$\Lambda_3(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 > H\left(X_1^{(\theta)} \middle| X_2^{(\theta)}\right), a_2 > H\left(X_2^{(\theta)} \middle| X_1^{(\theta)}\right), a_1 + a_2 = H\left(X_1^{(\theta)} X_2^{(\theta)}\right)\right\},$$

$$\Lambda_4(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 = H\left(X_1^{(\theta)} \middle| X_2^{(\theta)}\right), a_2 = H\left(X_2^{(\theta)}\right)\right\},$$

$$\Lambda_5(a_1, a_2) = \left\{\theta \in \Lambda \middle| a_1 = H\left(X_1^{(\theta)}\right), a_2 = H\left(X_2^{(\theta)} \middle| X_1^{(\theta)}\right)\right\}.$$

Proof: It suffices to scrutinize in Theorem 6.1 the situation with $n \to \infty$ and take account of Theorem 4.1. ■

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