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Time-Asynchronous Gaussian Multiple Access Channel with Correlated Sources

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***Abstract*—We study the transmission of a pair of correlated sources over a Gaussian multiple access channel with *weak* time asynchronism between the encoders. In particular, we assume that the maximum possible offset $\mathsf{d}_{\max}(n)$ between the transmitters grows without bound as the block length $n \to \infty$ while the ratio $\mathsf{d}_{\max}(n)/n$ of the maximum possible offset to the block length asymptotically vanishes. For such a joint source-channel coding problem, we derive the capacity region and also show that separate source and channel coding achieves optimal performance. Specifically, we first derive an outer bound on the source entropy content as our main result. Then, using Slepian-Wolf source coding combined with the channel coding introduced in [1], we show that the thus achieved inner bound matches the outer bound.**

***Index Terms*—Multiple-access channel, time asynchronism, joint source-channel coding, correlated sources.**

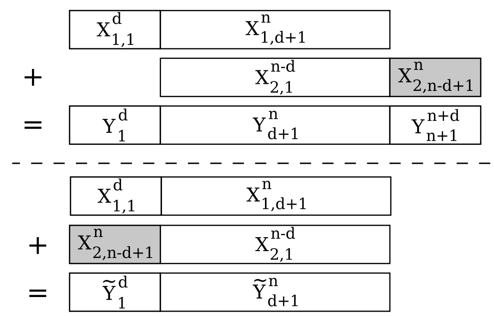

Fig. 1. Original time asynchronous MAC (top) and the cyclic MAC (bottom).

I. Introduction

Time synchronization between different nodes of a network is an essential assumption made to analyze and design communication systems. However, in practice, it is very difficult to exactly synchronize separate nodes either in time or frequency. As an example, in systems with different transmitters, the different transmitters must use their own locally generated clock. However, the initialization might be different for each clock or the frequencies at the local signal generators may not be perfectly matched [2]. Nonetheless, fundamental limits of communication in the presence of time asynchronism should be explicitly considered as a tool to better understand and tackle real-world challenges in the context of multiuser information theory.

The problem of finding the capacity of a multiple access channel (MAC) with no time synchronization between the encoders is considered in [1], [2], [3], and [4] from a channel coding perspective. In [5], a frame asynchronous MAC with memory is considered and it is shown that the capacity region can be drastically reduced in the presence of frame asynchronism. In [6], an asynchronous MAC is also considered, but with symbol asynchronism. All of these works restrict themselves to the study of channel coding only and disregard source-channel communication over an asynchronous MAC.

Specifically, [1] considers a MAC with no common time base between encoders. There, the encoders transmit with an unknown offset with respect to each other, and the offset is bounded by a maximum value $\mathsf{d}_{\max}(n)$ that is a function of coding block length n. Using a time-sharing argument, it is shown that the capacity region is the same as the capacity of the ordinary MAC as long as $\mathsf{d}_{\max}(n)/n \to 0$. On the other hand, [2] considers a *totally asynchronous* MAC in which the coding blocks of different users can potentially have no overlap at all, and thus have several block lengths of shifts between themselves. Moreover, the encoders have different clocks that are referenced with respect to a standard clock. For such a scenario, in [2], it is shown that the capacity region differs from that of the synchronous MAC only by the lack of the convex hull operation. In [7], Poltyrev also considers a model with arbitrary delays, known to the receiver (as opposed to [2]). Among other related works is the recent paper [3] that finds a single letter capacity region for the case of a 3 sender MAC, 2 of which are synchronized with each other and both asynchronous with respect to the third one.

The problem of joint source-channel coding for multiuser channels is open in general. There are several works, however, to address joint source-channel coding for MAC [8], networks with independent sources [9], and Gaussian networks with phase uncertainty among the nodes [10]. In particular, [8] shows that for the transmission of a pair of correlated discrete memoryless sources over a MAC, joint source-channel coding outperforms separation. Later, [11] showed that the sufficient conditions of [8] are not necessary for reliable communication and therefore not optimal. For a Gaussian MAC with unknown phase shifts at transmitters, however, [12] derives necessary conditions that match with the separation-based sufficient conditions. In [10], [13], we have shown the optimality of separate source-channel coding for several other phase-asynchronous Gaussian networks. Also, [14], [15], and [16] are examples of works that consider the problem of lossy joint source-channel coding for different channels.

In this paper, we study the communication of two correlated sources over a time-asynchronous MAC (TA-MAC) where the encoders cannot synchronize the starting times of their codewords. Rather, they transmit with an unknown time shift d with respect to each other. The shift d is bounded by $0 \leq d \leq \mathsf{d}_{\max}(n)$, where n is the codeword block length. The offset d is unknown to the transmitters as a practical assumption and known to the receiver side. We further assume that the maximum possible offset $\mathsf{d}_{\max}(n) \to \infty$ as $n \to \infty$ while $\mathsf{d}_{\max}(n)/n \to 0$, i.e., the asynchronism can be considered to be *weak*. For this problem, we derive *matching* necessary and sufficient conditions for reliable communication where the sufficient conditions are derived based on separate source-channel coding. Indeed, we show that due to time asynchronism, in the frequency domain, any correlation at the transmitters is essentially lost at the receivers (and vanishes as $n \to \infty$). Thus, there is no loss for the transmitters to employ independent Gaussian codebooks. This is different from [12] where correlation between transmissions from the source nodes can be power destructive for some adversarial phase shifts and it should thus be avoided.

The rest of this paper is organized as follows. In Section II, we present the problem statement and preliminaries along with a key lemma that is useful in the derivation of the converse. In Section III, the converse part of the capacity theorem is proved in which the capacity region of the ordinary MAC is proved to be an outer bound on the triple $(H(U_1|U_2), H(U_2|U_1), H(U_1, U_2))$. Then, using separate source and channel coding and the results of [1], it is shown in Section IV that the achievable region matches the outer bound. Finally, Section V concludes the paper.

II. Problem Statement and a Key Lemma

Notations: We denote random variables by upper case letters, their realizations by lower case letters, and their alphabet by calligraphic letters. For integers $b \geq a > 0$, Y_a^b denotes the $b-a+1$-tuple $(Y_a, \cdots, Y_b)$, and Y^b is a shorthand for Y_1^b. Based on this, we also denote $(X_{1,a}, \cdots, X_{1,b})$ by $X_{1,a}^b$, and likewise $(X_{2,a}, \cdots, X_{2,b})$ by $X_{2,a}^b$. Without confusion, X_1^n, X_2^n denote the length-n MAC input codewords $(X_{11}, \cdots, X_{1n})$, $(X_{21}, \cdots, X_{2n})$ respectively. The discrete Fourier transforms (DFT) of X_1^n, X_2^n are denoted by $\hat{X}_1^n = \mathrm{DFT}(X_1^n)$, and $\hat{X}_2^n = \mathrm{DFT}(X_2^n)$ respectively. Furthermore, $\mathbb{E}\|X_{ji}\|^2 \triangleq P_{ji}$, $\mathbb{E}\|\hat{X}_{ji}\|^2 \triangleq \hat{P}_{ji}$ denote the powers of X_{ji} and $\hat{X}_{ji}$ respectfully.

Consider two finite alphabet sources $\{U_{1i}, U_{2i}\}$ with correlated output random variables that are drawn according to a distribution $p(u_1, u_2)$. The sources are memoryless, i.e., (U_{1i}, U_{2i})'s are independent and identically distributed (i.i.d). Both of the sources are to be transmitted to a destination through a continuous alphabet, discrete-time memoryless multiple-access channel (MAC) $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}, p(y|x_1, x_2))$. The transmitters use different time references as a practical assumption and thus they start transmitting with an offset of d symbols. Hence, the probabilistic characterization of the time-asynchronous Gaussian MAC, referred to as Gaussian TA-MAC throughout the paper, is described by the relationship

$$Y_i = X_{1i} + X_{2(i-d)} + Z_i, \tag{1}$$

where $X_{1i}, X_{2i}, Y_i \in \mathbb{C}$, $Z_i \sim \mathcal{CN}(0, N)$ is circularly symmetric complex Gaussian noise, and $X_{ji} = 0$ for $i \notin \{1, \cdots, n\}$.

We define a joint source-channel code and the notion of reliable communication for a Gaussian TA-MAC in the sequel.

Definition 1: *Code*: A block joint source-channel code of length n for the Gaussian TA-MAC with correlated sources (U_1, U_2) is defined by

1) Two encoding functions

$$(f_{i1}, f_{i2}, \cdots, f_{in}) = f_i^n : \mathcal{U}_i^n \to \mathbb{C}^n, \quad i = 1, 2$$

that map the source outputs to the codewords. The sets of codewords are denoted by the codebook $\mathcal{C} = \{(f_1^n(u_1^n), f_2^n(u_2^n)) : u_1^n \in \mathcal{U}_1^n, u_2^n \in \mathcal{U}_2^n\}$.

2) Power constraints P_1 and P_2 on the codewords, i.e.,

$$\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}|X_{ji}|^2] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}|\hat{X}_{ji}|^2] \leq P_j, \tag{2}$$

for $j = 1, 2$, where $\mathbb{E}$ represents the expectation operation.

3) A decoding function $g_d^n : \mathbb{C}^{n+d} \to \mathcal{U}_1^n \times \mathcal{U}_2^n$.

Definition 2: We say the source $\{U_{1i}, U_{2i}\}_{i=1}^n$ of i.i.d. discrete random variables with joint probability mass function $p(u_1, u_2)$ *can be reliably sent* (or is achievable) over a TA-MAC, if there exists a sequence of block codes $\{x_n^1(U_1^n), x_n^2(U_2^n)\}$ and decoders g_d^n such that the output sequences U_1^n and U_2^n of the source can be estimated with arbitrarily asymptotically small probability of error over all delays $0 \leq d \leq \mathsf{d}_{\max}(n)$ at the receiver side from the received sequence Y^{n+d}, i.e.,

$$\sup_{0 \leq d \leq \mathsf{d}_{\max}(n)} P_e^n(d) \longrightarrow 0, \quad \text{as} \quad n \to \infty, \tag{3}$$

where

$$P_e^n(d) = P[g_d(Y^{n+d}) \neq (U_1^n, U_2^n)|d], \tag{4}$$

is the error probability for a given offset d. □

We now present a key lemma that plays an important role in the derivation of our results. In order to state the lemma, we first need to define the notion of *cyclic* MAC as follows:

Definition 3: A cyclic MAC, corresponding to the original TA-MAC defined by (1), is a MAC with output $\tilde{Y}_1^n$

$$\tilde{Y}_i = X_{1i} + X_{2(i-d \mod n)} + Z_i. \tag{5}$$

In particular, as shown in Fig. 1, the tail of the second codeword is cyclicly shifted to the beginning of the block and the output is the n-tuple that results by adding the first codeword X_1^n and the cyclic version of X_2^n. □

The following lemma implies that the mutual information rate between the inputs and the output in the original Gaussian TA-MAC and the cyclic MAC are asymptotically the same, i.e., their absolute difference asymptotically vanishes. Thus, in the analysis of problem in Section III, we can replace the original TA-MAC with the corresponding cyclic channel with delay d.

Lemma 4: For a Gaussian TA-MAC,

$$\begin{aligned}&\frac{1}{n}\left|I(X_1^n, X_2^n; Y^{n+d}|D=d) - I(X_1^n, X_2^n; \tilde{Y}^n|D=d)\right| \\ &\leq \epsilon_n, \qquad \forall\ 0 \leq d \leq \mathsf{d}_{\max}(n),\end{aligned} \tag{6}$$

where ϵ_n does not depend on d and $\epsilon_n \to 0$, as $n \to \infty$. □

Proof: The case $d = 0$ is trivial as $Y^n = \tilde{Y}^n$. For $d \neq 0$, we first bound the original mutual information $I(X_1^n, X_2^n; Y^{n+d}|D=d)$ as follows:

$$I(X_{1,d+1}^n, X_{2,1}^{n-d}; Y_{d+1}^n|D=d) \leq I(X_1^n, X_2^n; Y^{n+d}|D=d), \tag{7}$$

$$\begin{aligned}&I(X_1^n, X_2^n; Y^{n+d}|D=d) \\ &= h(Y^{n+d}|D=d) - h(Y^{n+d}|X_1^n, X_2^n, D=d) \\ &\leq h(Y_1^d|D=d) + h(Y_{d+1}^n|D=d) + h(Y_{n+1}^{n+d}|D=d)\end{aligned}$$

$$-\sum_{i=1}^{n+d} h(Z_i)$$

$$\leq \sum_{i=1}^{d} [h(Y_i|D=d) - h(Z_i)] + h(Y_{d+1}^n|D=d)$$

$$- \sum_{i=d+1}^{n} h(Z_i) + \sum_{i=n+1}^{n+d} [h(Y_i|D=d) - h(Z_i)]$$

$$= \sum_{i=1}^{d} I(X_{1i}; Y_i|D=d) + I(X_{1,d+1}^n, X_{2,1}^{n-d}; Y_{d+1}^n|D=d)$$

$$+ \sum_{i=n+1}^{n+d} I(X_{2(i-d)}; Y_i|D=d). \quad (8)$$

By following steps similar to those that resulted in (7), and (8), we can lower bound and upper bound $I(X_1^n, X_2^n; \tilde{Y}^n|D=d)$ as

$$I(X_{1,d+1}^n, X_{2,1}^{n-d}; \tilde{Y}_{d+1}^n|D=d) \leq I(X_1^n, X_2^n; \tilde{Y}^n|D=d), \quad (9)$$

$$I(X_1^n, X_2^n; \tilde{Y}^n|D=d) \leq \sum_{i=1}^{d} I(X_{1i}, X_{2(n-d+i)}; \tilde{Y}_i) + I(X_{1,d+1}^n, X_{2,1}^{n-d}; \tilde{Y}_{d+1}^n|D=d). \quad (10)$$

respectively.

Combining (7), (8), (9), and (10), and noting that $\tilde{Y}_i = Y_i$, for $d+1 \leq i \leq n$, we can now bound the absolute difference between mutual information terms in (6) as

$$\frac{1}{n}\left|I(X_1^n, X_2^n; Y^{n+d}|D=d) - I(X_1^n, X_2^n; \tilde{Y}^n|D=d)\right|$$

$$\leq \frac{1}{n}\sum_{i=1}^{d} I(X_{1i}; Y_i) + \frac{1}{n}\sum_{i=n+1}^{n+d} I(X_{2(i-d)}; Y_i) + \frac{1}{n}\sum_{i=1}^{d} I(X_{1i}, X_{2(n-d+i)}; \tilde{Y}_i). \quad (11)$$

But the three terms in the right hand side of (11) can also be bounded as follows:

$$\frac{1}{n}\sum_{i=1}^{d} I(X_{1i}; Y_i) \leq \frac{1}{n}\sum_{i=1}^{d} \log(1 + \frac{P_{1i}}{N})$$

$$\overset{(a)}{\leq} \frac{d}{n}\log(1 + \frac{\sum_{i=1}^{d} P_{1i}}{d \times N})$$

$$\overset{(b)}{\leq} \frac{d}{n}\log(1 + \frac{n}{d}\frac{P_1}{N}) \triangleq \epsilon_n^{(1)}(d) \quad (12)$$

where (a) follows from concavity of the $\log$ function, and (b) follows from (2). Similarly, for the second term in the right hand side of (11), we have

$$\frac{1}{n}\sum_{i=n+1}^{n+d} I(X_{2(i-d)}; Y_i) \leq \frac{d}{n}\log(1 + \frac{n}{d}\frac{P_2}{N}) = \epsilon_n^{(2)}(d). \quad (13)$$

We now upper bound the third term in the right hand side of (11) as follows

$$\frac{1}{n}\sum_{i=1}^{d} I(X_{1i}, X_{2(n-d+i)}; \tilde{Y}_i) = \frac{1}{n}\sum_{i=1}^{d} h(\tilde{Y}_i) - \frac{1}{n}\sum_{i=1}^{d} h(Z_i)$$

$$= \frac{1}{n}\sum_{i=1}^{d} h(X_{1i} + X_{2(n-d+i)} + Z_i) - \frac{1}{n}\sum_{i=1}^{d} h(Z_i)$$

$$\leq \frac{1}{n}\sum_{i=1}^{d} \log(1 + \frac{P_{1i} + P_{2(n+i-d)} + 2\sqrt{P_{1i}P_{2(n+i-d)}}}{N})$$

$$\overset{(a)}{\leq} \frac{1}{n}\sum_{i=1}^{d} \log(1 + \frac{2(P_{1i} + P_{2(n+i-d)})}{N}) \quad (14)$$

$$\overset{(b)}{\leq} \frac{d}{n}\log\left[1 + \frac{2\sum_{i=1}^{d}(P_{1i} + P_{2(n+i-d)})}{d \times N}\right]$$

$$\overset{(c)}{\leq} \frac{d}{n}\log\left[1 + \frac{n}{d}\frac{2(P_1 + P_2)}{N}\right] \triangleq \epsilon_n^{(3)}(d), \quad (15)$$

where (a) follows by the geometric inequality: $2\sqrt{ab} \leq a + b$, for $a, b \geq 0$, (b) follows by concavity of the $\log$ function, and (c) follows by the power constraint (2).

Based on (12), (13), and (15), we can now bound the absolute difference between the mutual informations in (6) by $\epsilon_n^{(1)}(d) + \epsilon_n^{(2)}(d) + \epsilon_n^{(3)}(d)$. By defining $\epsilon_n^{(4)}(d) \triangleq \epsilon_n^{(1)}(d) + \epsilon_n^{(2)}(d) + \epsilon_n^{(2)}(d)$, one can see that $\max_{1 \leq d \leq \mathsf{d}_{\max}(n)} \epsilon_n^{(4)}(d) \to 0$ as $n \to \infty$ since for any $a > 0$, $\sup_{0 < x \leq \mathsf{d}_{\max}(n)/n} x\log(1 + a/x) \to 0$ as $\mathsf{d}_{\max}(n)/n \to 0$, and the lemma is proved. ■

III. CONVERSE

In this section, we derive an outer bound on the capacity region and prove a necessary condition for reliable communication of (U_1, U_2) over the Gaussian TA-MAC.

Lemma 5: *Reliable Communication over a TA-MAC*: Consider a Gaussian TA-MAC with power constraints P_1, P_2 on the transmitters, and an unknown offset $0 \leq d \leq \mathsf{d}_{\max}(n)$ between the encoders. Moreover, assume the offset d is known to the receiver, $\mathsf{d}_{\max}(n) \to \infty$, and $\mathsf{d}_{\max}(n)/n \to 0$ as $n \to \infty$. Then, a necessary condition for reliably communicating a source pair $(U_1, U_2) \sim \prod_i p(u_{1i}, u_{2i})$, over such Gaussian TA-MAC, in the sense of Definition 2, is given by

$$H(U_1|U_2) \leq \log(1 + P_1/N), \quad (16)$$

$$H(U_2|U_1) \leq \log(1 + P_2/N), \quad (17)$$

$$H(U_1, U_2) \leq \log(1 + (P_1 + P_2)/N). \quad (18)$$

proof: First, fix a TA-MAC with given offset d, a codebook $\mathcal{C}$, and induced *empirical* distribution $p(u_1^n, u_2^n, x_1^n, x_2^n, y^{n+d})$. Since for this fixed choice of d, $P_e^n(d) \to 0$, from Fano's inequality, we have

$$\frac{1}{n}H(U_1^n, U_2^n|Y^{n+d}, d) \leq \frac{1}{n}P_e^n(d)\log\|\mathcal{U}_1^n \times \mathcal{U}_2^n\| + \frac{1}{n} \triangleq \delta_n,$$

and $\delta_n \to 0$, where convergence is uniform in d by (3).

We first bound $H(U_1|U_2)$ as follows. The second bound of the lemma on $H(U_2|U_1)$ can be derived similarly.

$$H(U_1|U_2) = \frac{1}{n}H(U_1^n|U_2^n)$$

$$= \frac{1}{n}I(U_1^n; Y^{n+d}|U_2^n, X_2^n) + \frac{1}{n}H(U_1^n|Y^{n+d}, U_2^n)$$

$$\leq \frac{1}{n}I(X_1^n; Y^{n+d}|U_2^n, X_2^n) + \delta_n$$

$$\leq \frac{1}{n}I(X_1^n; Y^{n+d}|X_2^n) + \delta_n$$

$$= \frac{1}{n}[h(Y^{n+d}|X_2^n) - h(Y^{n+d}|X_1^n, X_2^n)] + \delta_n$$
$$\leq \frac{1}{n}\sum_{i=1}^{n} h(X_{1i}+Z_i) + \frac{1}{n}\sum_{i=n+1}^{n+d} h(Z_i) - \frac{1}{n}\sum_{i=1}^{n+d} h(Z_i) + \delta_n$$
$$= \frac{1}{n}\sum_{i=1}^{n}[h(X_{1i}+Z_i) - h(Z_i)] + \delta_n$$
$$\leq \frac{1}{n}\sum_{i=1}^{n} \log(1+\frac{P_{1i}}{N}) \leq \log(1+\frac{P_1}{N}) + \delta_n. \quad (19)$$

where the last step follows by the concavity of the log function and power constraints (2). Hence (16), and similarly (17) are established. We now bound $H(U_1, U_2)$ to derive the third lemma's inequality (18):

$$H(U_1,U_2) = \frac{1}{n}H(U_1^n, U_2^n)$$
$$= \frac{1}{n}I(U_1^n, U_2^n; Y^{n+d}|d) + \frac{1}{n}H(U_1^n, U_2^n|Y^{n+d}, d)$$
$$\leq \frac{1}{n}I(U_1^n, U_2^n; Y^{n+d}|d) + \delta_n$$
$$\leq \frac{1}{n}I(X_1^n, X_2^n; Y^{n+d}|d) + \delta_n$$
$$\leq \frac{1}{n}I(X_1^n, X_2^n; \tilde{Y}^n|d) + \epsilon_n + \delta_n, \quad (20)$$

where the last step follows from Lemma 4.

Now, let D be a random variable uniformly distributed on the set $\{0, 1, \cdots, \mathsf{d}_{\max}(n)\}$. Since (20) is true for every choice of $d \in \{0, 1, \cdots, \mathsf{d}_{\max}(n)\}$, $H(U_1, U_2)$ can also be upper bounded by the average over d of $\frac{1}{n}I(X_1^n, X_2^n; \tilde{Y}^n|d) + \epsilon_n + \delta_n$. Hence,

$$H(U_1,U_2) \leq \frac{1}{n}I(X_1^n, X_2^n; \tilde{Y}^n|D) + \epsilon_n + \delta_n$$
$$\stackrel{(a)}{=} \frac{1}{n}I(X_1^n, X_2^n; \hat{\tilde{Y}}^n|D) + \epsilon_n + \delta_n, \quad (21)$$

where $\hat{\tilde{Y}}^n = \mathrm{DFT}(\tilde{Y}^n)$, and (a) follows from the fact that the DFT is a bijection. Expanding $I(X_1^n, X_2^n; \hat{\tilde{Y}}^n|D)$ in the right hand side of (21),

$$H(U_1,U_2) \leq \frac{1}{n}[h(\hat{\tilde{Y}}^n|D) - h(\hat{\tilde{Y}}^n|X_1^n, X_2^n, D)] + \epsilon_n + \delta_n$$
$$\leq \frac{1}{n}[h(\hat{\tilde{Y}}^n) - h(\hat{Z}^n)] + \epsilon_n + \delta_n,$$

where $\hat{Z}^n$ has i.i.d entries with $\hat{Z}_i \sim \mathcal{CN}(0, N)$. Recall $\hat{X}_1^n = \mathrm{DFT}(X_1^n), \hat{X}_2^n = \mathrm{DFT}(X_2^n)$. Then,

$$h(\hat{\tilde{Y}}^n) = h(\hat{X}_1^n + e^{-j\boldsymbol{\theta}(D)} \odot \hat{X}_2^n + \hat{Z}^n)$$
$$\leq \sum_{i=1}^{n} h(\hat{X}_{1i} + e^{\frac{-j2\pi(i-1)D}{n}}\hat{X}_{2i} + \hat{Z}_i),$$

where $e^{-j\boldsymbol{\theta}(D)} = (e^{\frac{-j2\pi(i-1)D}{n}})_{i=1}^n$, and $\odot$ denotes element-wise vector multiplication. Let $\hat{V}_{2i} = e^{\frac{-j2\pi(i-1)D}{n}}\hat{X}_{2i}$. Thus,

$$H(U_1,U_2) \leq \frac{1}{n}\sum_{i=1}^{n}[h(\hat{X}_{1i}+\hat{V}_{2i}+\hat{Z}_i) - h(\hat{Z}_i)] + \epsilon_n + \delta_n$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\log(1+\frac{\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2}{N}) + \epsilon_n + \delta_n. \quad (22)$$

Then define a function $\alpha(n): \mathbb{N} \to \mathbb{N}$ such that

$$\frac{\alpha(n)}{n} \to 0, \quad \frac{\alpha(n)\mathsf{d}_{\max}(n)}{n} \to \infty. \quad (23)$$

An example of such an $\alpha(n)$ is the function $\alpha(n) = \lceil \frac{n}{\mathsf{d}_{\max}(n)} \log \mathsf{d}_{\max}(n) \rceil$.

Now, breaking up the sum from $i = 1$ to n into three terms: from 1 to $\alpha(n)$, from $\alpha(n)+1$ to $n-\alpha(n)$ and from $\alpha(n)+1$ to n, we upper bound each of the three terms in the sequel. For $1 \leq i \leq \alpha(n)$, we have

$$\frac{1}{n}\sum_{i=1}^{\alpha(n)}\log(1+\frac{\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2}{N})$$
$$\stackrel{(a)}{\leq} \frac{1}{n}\sum_{i=1}^{\alpha(n)}\log(1+\frac{2(\hat{P}_{1i}+\hat{P}_{2i})}{N})$$
$$\stackrel{(b)}{\leq} \frac{\alpha(n)}{n}\log(1+\frac{2n(P_1+P_2)}{\alpha(n)N}) \triangleq \lambda(n), \quad (24)$$

where (a) follows by the fact that $\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2 \leq 2\mathbb{E}|\hat{X}_{1i}|^2 + 2\mathbb{E}|\hat{V}_{2i}|^2$ (which is in turn a direct result of the geometric inequality $2\sqrt{ab} \leq a+b$, for $a, b \geq 0$), and (b) follows exactly the same way as the derivation of (15) from (14). Furthermore, $\lambda(n) \to 0$ since $\alpha(n)/n \to 0$ asymptotically by assumption (23).

A similar upper bound can be derived on the sum term for $n-\alpha(n)+1 \leq i \leq n$ by symmetry of the problem:

$$\frac{1}{n}\sum_{i=n-\alpha(n)+1}^{n}\log(1+\frac{\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2}{N}) \leq \lambda(n). \quad (25)$$

Now, in order to upper bound the third sum for $\alpha(n)+1 \leq i \leq n-\alpha(n)$, we use the inequality

$$\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2 \leq (\hat{P}_{1i}+\hat{P}_{2i})(1+\mu_n(i)), \quad (26)$$

where we define $\mu_n(i) \triangleq \left[\mathsf{d}_{\max}(n) \times |\sin(\frac{\pi(i-1)}{n})|\right]^{-1}$, and which is proved in Appendix A.

Thus, for the sum over $\alpha(n)+1 \leq i \leq n-\alpha(n)$, we have

$$\frac{1}{n}\sum_{i=\alpha(n)+1}^{n-\alpha(n)}\log(1+\frac{\mathbb{E}|\hat{X}_{1i}+\hat{V}_{2i}|^2}{N})$$
$$\stackrel{(a)}{\leq} \frac{1}{n}\sum_{i=\alpha(n)+1}^{n-\alpha(n)}\log\left[1+\left(\frac{\hat{P}_{1i}+\hat{P}_{2i}}{N}\right)(1+\mu_n(\alpha(n)))\right],$$
$$\stackrel{(b)}{\leq} \frac{n-2\alpha(n)}{n}\log\left[1+\frac{n(P_1+P_2)(1+\mu_n(\alpha(n)))}{[n-2\alpha(n)]N}\right], \quad (27)$$

where (a) follows by (26) and the fact that $\mu_n(i)$ for $\alpha(n)+1 \leq i \leq n-\alpha(n)$ is bounded above by $\mu_n(\alpha(n))$, and (b) follows by the concavity of the log function, and the power constraints (2). By combining the three upper bounds (24), (25), and (27) on the three sums, and noting that due to the choice of $\alpha(n)$ in (23),

$$\mu_n(\alpha(n)) = \frac{1}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi[\alpha(n)-1]}{n})|} \to \frac{n}{\pi\mathsf{d}_{\max}(n)\alpha(n)} \to 0$$

as $n \to \infty$, and $2\lambda(n)$, ϵ_n, $\delta_n \to 0$, as $n \to \infty$. Thus (18) is established and the proof of the lemma is complete.

IV. Achievability

We now demonstrate the achievability of the region that was proved to be an outer bound on the capacity region in Lemma 4 and thus conclude that the region is actually the capacity region. To establish the achievability argument, we follow a *tandem* (separate) source-channel coding scheme. Thus, the communication process will be divided into two parts:

Source Coding: From Slepian-Wolf coding [17], for the correlated source (U_1^n, U_2^n), if we have two n-length sequences of source codes with rates (R_1, R_2), for asymptotically lossless representation of the source, we have

$$H(U_1|U_2) < R_1, \tag{28}$$

$$H(U_2|U_1) < R_2, \tag{29}$$

$$H(U_1, U_2) < R_1 + R_2. \tag{30}$$

Channel Coding: Next, for fixed source codes with rates (R_1, R_2), we make channel codes for the TA-MAC separately such that the channel codes can be reliably decoded at the receiver side. Then using a time-sharing strategy as in [1], and also due to the assumption that $\mathsf{d}_{\mathsf{max}}(n)/n \to 0$, we derive the following sufficient conditions on R_1, R_2 [1]:

$$R_1 < \log(1 + P_1/N), \tag{31}$$

$$R_2 < \log(1 + P_2/N), \tag{32}$$

$$R_1 + R_2 < \log(1 + (P_1 + P_2)/N). \tag{33}$$

Lemma 6: (16)-(18) give a sufficient condition for reliable communication of the source (U_1, U_2) over the TA-MAC defined in (1), with $\leq$ being replaced by $<$.

Proof: From (16)-(18), it can be easily seen that there exist choices of R_1, R_2 such that the Slepian-Wolf conditions (28)-(30) and the channel coding conditions (31)-(33) are simultaneously satisfied. ■

V. Conclusion

The problem of sending arbitrarily correlated sources over a time asynchronous multiple-access channel with maximum offset between encoders $\mathsf{d}_{\mathsf{max}}(n) \to \infty$, as $n \to \infty$, is considered. Necessary and sufficient conditions for reliable communication are presented under the *weak* asynchronism assumption of $\mathsf{d}_{\mathsf{max}}(n)/n \to 0$. Namely, an outer bound on the capacity region is derived and is shown to match the separate source-channel coding achievable region. Therefore, separation is shown to be optimal and as a result, joint source-channel coding is not necessary under time asynchronism.

Appendix A
Proof of (26)

First, note that $\mathbb{E}|\hat{X}_{1i}|^2 = \hat{P}_{1i}$, and $\mathbb{E}|\hat{V}_{2i}|^2 = \mathbb{E}|e^{\frac{j2\pi(i-1)D}{n}}\hat{X}_{2i}|^2 = \mathbb{E}|\hat{X}_{2i}|^2 = \hat{P}_{2i}$. Thus, $\mathbb{E}|\hat{X}_{1i} + \hat{V}_{2i}|^2 = \hat{P}_{1i} + \hat{P}_{2i} + 2\Re[\mathbb{E}(X_{1i}V_{2i}^*)]$. Let $\rho_i = \frac{\mathbb{E}[X_{1i}X_{2i}^*]}{\sqrt{\hat{P}_{1i}\hat{P}_{2i}}}$, with $|\rho_i| \leq 1$. Then, letting $k \triangleq i - 1$, we have

$$\begin{aligned}
\Re[\mathbb{E}(X_{1i}V_{2i}^*)] &\leq |\mathbb{E}(X_{1i}V_{2i}^*)| = |\mathbb{E}_D\mathbb{E}(X_{1i}V_{2i}^*|D)| \\
&= \left|\frac{1}{\mathsf{d}_{\mathsf{max}}(n)+1}\sum_{d=0}^{\mathsf{d}_{\mathsf{max}}(n)}\mathbb{E}(X_{1i}X_{2i}^* e^{\frac{j2\pi kd}{n}})\right| \\
&= \left|\frac{1}{\mathsf{d}_{\mathsf{max}}(n)+1}\rho_i\sqrt{\hat{P}_{1i}\hat{P}_{2i}}\sum_{d=0}^{\mathsf{d}_{\mathsf{max}}(n)} e^{\frac{j2\pi kd}{n}}\right| \\
&\leq \frac{1}{\mathsf{d}_{\mathsf{max}}(n)}\sqrt{\hat{P}_{1i}\hat{P}_{2i}}\left|\frac{1-\exp(\frac{j2\pi k(\mathsf{d}_{\mathsf{max}}+1)}{n})}{1-\exp(\frac{j2\pi k}{n})}\right| \\
&\leq \frac{1}{\mathsf{d}_{\mathsf{max}}(n)}\frac{2\sqrt{\hat{P}_{1i}\hat{P}_{2i}}}{\left|1-\exp(\frac{j2\pi k}{n})\right|} \\
&= \frac{\sqrt{\hat{P}_{1i}\hat{P}_{2i}}}{\mathsf{d}_{\mathsf{max}}(n)|\sin(\frac{\pi k}{n})|} \overset{(a)}{\leq} \frac{\hat{P}_{1i}+\hat{P}_{2i}}{2\mathsf{d}_{\mathsf{max}}(n)|\sin(\frac{\pi k}{n})|},
\end{aligned}$$

where (a) follows by the geometric inequality ($2\sqrt{ab} \leq a + b$) and therefore (26) is directly established.

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