Ergodic Sum-Rate of Proportional Fair Scheduling with Multiple Antennas

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***Abstract*—This paper deals with the ergodic sum-rate of a proportional fair scheduler in wireless systems where both the base stations and the user terminals are equipped with multiantenna transceivers. The scheduling process is allowed to operate jointly over multiple parallel subchannels, e.g., those created by frequency-division multiaccess/multiplexing. Exact expressions are derived for arbitrary numbers of users and antennas, and arbitrary fading distributions. These results are then specialized to Rayleigh fading and, further, to the low- and high-power regimes. Informative expressions are also derived for the regime of large numbers of users.**

I. Introduction

Closed-form expressions have been derived for the ergodic capacity of both single-antenna and (spatially uncorrelated) multiantenna channels subject to the most common fading laws, chiefly Rayleigh fading [1]. The Rayleigh law is a well-tested and thoroughly accepted model for the distribution of multipath fading over a given channel resource, understood as a combination of time slot and frequency band.

In modern systems, however, the fading distribution is modified by a key engine of such systems: the scheduler. This engine determines which of the many users within a given sector or cell is allowed access to a given resource at each point in time. Allocating each resource to the user with the best fading thereon is the aim of the scheduler, albeit the underlying objective of maximizing the aggregate bit rate must be suitably tempered by the need to ensure a certain degree of fairness [2]. Among the many policies that can be formulated, proportional fair scheduling (PFS) is widely accepted as offering a satisfying compromise between sum-rate and fairness. PFS selects for each resource the user with the highest preference metric, with such metric defined as the ratio of instantaneous-to-average rate [3].

As a result of the scheduling process, the fading experienced by a given user no longer conforms to its original law, e.g., Rayleigh, but rather to a more favorable law that further depends on the scheduling policy and the number of users. The advantage gained through this process is referred to as *multiuser diversity*. Obtaining expressions for the corresponding ergodic rate region, and specifically the ergodic sum-rate, is a very relevant problem and some expressions therefore have been derived asymptotically in the number of users [3–5]. An approximate integral form was also derived in [6], under the assumption that the rates of the individual links are normally distributed; this implicitly makes the solution also asymptotic, as it is only asymptotically that such normality holds. As often happens, the asymptotic expressions are simple and insightful, but in this particular problem they suffer from two drawbacks:

- The multiuser diversity gains arise largely for small numbers of users, quickly diminishing thereafter. These gains cannot be quantified through asymptotic expressions.
- The asymptotic expressions are suspect because, when the number of users is large, artifact results take hold on account of the unbounded support of the Rayleigh law assumed for each individual link. Put differently, the Rayleigh law is a valid model for multipath fading except in its upper tail, which is precisely what dominates the asymptotic behavior of schedulers such as PFS.

These considerations motivate the interest in deriving nonasymptotic expressions for the ergodic sum-rate with scheduling, and chiefly with PFS, and extensive work has been done in recent years to advance this problem for single-antenna communication [5–9]. The analysis in these works runs into obstacles associated with the preference metric and, to circumvent them, alternative versions of the preference metric are typically considered. An SNR-based preference metric, in particular, facilitates the analysis while yielding similar ergodic sum-rate performance [9]. A number of references have considered this alternative metric [5, 8, 9]. Specifically, the exact ergodic sum-rate is given in [8, 9] in terms of the incomplete gamma function. More general multiantenna scenarios are considered in [10, 11]. However, all of these analyses considered scheduling processes operating over a single channel rather than over the various parallel subchannels created by multiaccess/multiplexing schemes.

This paper presents an exact analysis of the ergodic sum-rate of PFS with multiantenna diversity at both the base stations (BSs) and the user terminals. The scheduling process is allowed to operate jointly over multiple parallel subchannels such as those that arise with frequency-division or orthogonal code-division multiaccess/multiplexing. We derive expressions that apply with arbitrary numbers of users and antennas, and arbitrary fading distributions. These results are then specialized to Rayleigh fading per link and, further, to the low- and high-SNR regimes. Informative expressions are also derived for the regime of large numbers of users.

II. System Model

We consider an uplink where K_{t} users with N_{t} transmit antennas are connected to a BS with N_{r} receive antennas.

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Multiantenna diversity is fully exploited at both the users and the BS ends: either transmit antenna selection (TAS) or space-time block codes (STBC) is implemented at the user terminals, whereas either selection combining (SC) or maximal ratio combining (MRC) is applied at the BS. As will be emphasized, any combination of these transmit and receive diversity schemes results in an output SNR distribution that can be put in a unified form; this motivates us to keep the analysis general. By means of orthogonal multiaccess/multiplexing, $K_\mathrm{a} \leq K_\mathrm{t}$ users are served in parallel, i.e., K_a subchannels are allocated at each transmission block (time slot) to a subset of $K_\mathrm{a} \leq K_\mathrm{t}$ active users. Although we focus on the uplink, the analysis applies also to the dual downlink.

Let us consider a frequency-flat channel divided into K_a subchannels. That can be seen as the coherence bandwidth of a wider frequency-selective channel divided into K_a subchannels. The K_a codewords transmitted from the different users are affected by frequency-flat fading and corrupted by additive white Gaussian noise (AWGN) with zero mean and unit variance. The fading channel for user k is modeled by the $N_\mathrm{r} \times N_\mathrm{t}$ matrix $\mathbf{H}_k = \{h_{r,t}^{(k)}\}$ with independent and identically distributed (i.i.d.) entries denoting the channel gain between the tth user antenna and the rth BS antenna. The matrices $\mathbf{H}_k$ are unknown at the transmitters, but perfectly known at the receiver.

The transmitted symbols x_k, $k = 1, ..., K_\mathrm{a}$, are drawn from the capacity-achieving complex Gaussian distribution satisfying $E\left[x_k x_k^*\right] = 1$; there is no loss of generality because any scaling factor can be absorbed into $\mathbf{H}_k$. After multiantenna diversity processing, the received signal from user k is

$$y_k = g(\mathbf{H}_k)\, x_k + n_k,$$

where n_k is the received noise and $g(\mathbf{H}_k)$ is the complex-valued effective channel for the kth user.

The instantaneous SNR for the kth user, $k = 1, ..., K_\mathrm{t}$, is given by $\left|g(\mathbf{H}_k)\right|^2 = \tilde{\rho}_k \rho_k$, where ρ_k denotes the average SNR for the kth user and $\tilde{\rho}_k$ its (unit variance) normalized SNR. The normalized SNRs for the K_t users are i.i.d. random variables (RVs) with unit variance and marginal CDF $F_{\tilde{\rho}}(\cdot)$. We denote by $\tilde{\rho}_{K_\mathrm{t},r}$ with CDF $F_{\tilde{\rho}_{K_\mathrm{t},r}}(\cdot)$ the order statistics of $\tilde{\rho}_k$, e.g., $\tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}}$ is the highest normalized SNR among the K_t users.

The PFS algorithm selects the K_a (out of K_t) users with highest preference metric within each transmission block. We consider the SNR-based PFS algorithm with the normalized SNR $\tilde{\rho}_k$ as preference metric. The user s_i having the $(i+1)$th highest normalized SNR, i.e., $\tilde{\rho}_{s_i} = \tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}$, is selected for transmission on subchannel $i = 0, ..., K_\mathrm{a} - 1$.

III. Ergodic Sum-Rate Analysis

A. General Analysis

The ergodic sum-rate is given by

$$C = \frac{1}{K_\mathrm{a}} \sum_{i=0}^{K_\mathrm{a}-1} E\left[\mathcal{I}_i\right] \tag{1}$$

with $\mathcal{I}_i$ the mutual information of subchannel $i = 0, ...K_\mathrm{a} - 1$,

$$\mathcal{I}_i = \log_2\left(1 + \tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}\, \rho_{s_i}\right), \tag{2}$$

where s_i is the selected user for subchannel i, ρ_{s_i} indicates its average SNR, and $\tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}$ gives its normalized SNR, which is the $(i+1)$th highest value among the K_t users.

Proportional fairness is ensured by utilizing the normalized SNR as the scheduling metric and, since the normalized SNRs are i.i.d., each user is scheduled with probability $1/K_\mathrm{t}$. Then, the CDF of $\mathcal{I}_i$ can be written as

$$\begin{aligned} F_{\mathcal{I}_i}(\xi) = \frac{1}{K_\mathrm{t}} (&\Pr\{\mathcal{I}_i \leq \xi \,|s_i = 1\} + \cdots \\ &+ \Pr\{\mathcal{I}_i \leq \xi \,|s_i = K_\mathrm{t}\}). \end{aligned} \tag{3}$$

The probabilities on the right-hand side of (3) are simply the CDF of $\mathcal{I}_i$ conditioned on a certain user being selected. Defining $\mathcal{I}_{i,k}$ as the mutual information conditioned on the selection of user k, we can rewrite (3) as

$$F_{\mathcal{I}_i}(\xi) = \frac{1}{K_\mathrm{t}} \left(F_{\mathcal{I}_{i,1}}(\xi) + ... + F_{\mathcal{I}_{i,K_\mathrm{t}}}(\xi)\right) \tag{4}$$

where the mutual information for subchannel i, given that user k is selected, is

$$\mathcal{I}_{i,k} = \log_2\left(1 + \tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}\rho_k\right) \tag{5}$$

with CDF

$$F_{\mathcal{I}_{i,k}}(\xi) = F_{\tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}}\left(\frac{2^\xi - 1}{\rho_k}\right). \tag{6}$$

In light of (4),

$$E\left[\mathcal{I}_i\right] = \frac{1}{K_\mathrm{t}} \sum_{k=1}^{K_\mathrm{t}} E\left[\mathcal{I}_{i,k}\right] \tag{7}$$

and, therefore, the ergodic sum-rate can be obtained as

$$C = \frac{1}{K_\mathrm{a} K_\mathrm{t}} \sum_{i=0}^{K_\mathrm{a}-1} \sum_{k=1}^{K_\mathrm{t}} E\left[\mathcal{I}_{i,k}\right]. \tag{8}$$

At this point, we merely need to obtain the expectation (over the normalized SNR) of $\mathcal{I}_{i,k}$. On the one hand, from (6),

$$E\left[\mathcal{I}_{i,k}\right] = \int_0^\infty \left(1 - F_{\tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}}\left(\frac{2^\xi - 1}{\rho_k}\right)\right) d\xi. \tag{9}$$

On the other hand, from order statistics [12] it is known that

$$F_{\tilde{\rho}_{K_\mathrm{t},K_\mathrm{t}-i}}\left(\frac{2^\xi - 1}{\rho_k}\right) = I_{F_{\tilde{\rho}}\left(\frac{2^\xi-1}{\rho_k}\right)}\left(K_\mathrm{t} - i, i + 1\right) \tag{10}$$

where $I_\xi(\cdot, \cdot)$ is the regularized incomplete beta function [13, 8.392]. Plugging (10) into (9) and performing some algebraic manipulations based on [13, 8.391] and on the series expansion of the hypergeometric function ${}_2F_1(\cdot, \cdot; \cdot, \cdot)$, we obtain

$$E\left[\mathcal{I}_{i,k}\right] = \int_0^\infty \left(1 - \sum_{r=0}^{i} \alpha_{i,r}\, F_{\tilde{\rho}}^{K_\mathrm{t}-i+r}\left(\frac{2^\xi - 1}{\rho_k}\right)\right) d\xi \tag{11}$$

with coefficients

$$\alpha_{i,r} = \frac{(-1)^r K_\mathrm{t}!\,(K_\mathrm{t} - i)_r\,(i)_r}{i!r!\,(K_\mathrm{t} - i)!\,(K_\mathrm{t} - i + 1)_r} \tag{12}$$

TABLE I
PARAMETERIZATION OF MULTIANTENNA SCHEMES

Scheme	TAS-MRC[2]	TAS-SC	STBC-SC	STBC-MRC
$(\delta,\, \nu)$	$(N_{\mathrm{t}},\, N_{\mathrm{r}})$	$(N_{\mathrm{r}}N_{\mathrm{t}},\, 1)$	$(N_{\mathrm{r}},\, N_{\mathrm{t}})$	$(1,\, N_{\mathrm{r}}N_{\mathrm{t}})$

and with $(m)_r = \frac{(m+r-1)!}{(m-1)!}$ denoting the Pochhammer symbol.

Altogether, we have expressed $E\left[\mathcal{I}_{i,k}\right]$, and therefore the ergodic sum-rate in (8), as an integral involving the marginal CDF of the normalized SNR. This result is valid for any statistical distribution of the SNR, thereby encompassing different multiantenna configurations and fading models. In the next section, we particularize the analysis to Rayleigh fading and derive an explicit expression.

B. Rayleigh Fading

The kth user output SNR for any of the considered multiantenna schemes is distributed, in general, as the maximum of the squared row (or column) norms of $\mathbf{H}_k$. Under Rayleigh fading, the normalized SNRs are therefore distributed as the maximum of δ chi-squared RVs with 2ν degrees of freedom, i.e., with CDF[1]

$$F_{\tilde{\rho}}\left(\xi\right) = \left(\frac{1}{\Gamma\left(\nu\right)}\gamma\left(\nu,\xi\right)\right)^{\delta} \tag{13}$$

where $\gamma\left(\cdot,\cdot\right)$ is the lower incomplete gamma function, and the parameters δ and ν depend on the multiantenna scheme, as specified in Table I.

Lemma 1: Let the normalized SNRs $(\tilde{\rho}_1,\ldots,\tilde{\rho}_{K_{\mathrm{t}}})$ be i.i.d. RVs with CDF given by (13). Then, the ergodic rate for subchannel i if user k is selected for transmission is

$$E\left[\mathcal{I}_{i,k}\right] = -\log_2 e \sum_{r=0}^{i} \alpha_{i,r} \sum_{n=1}^{\delta(K_{\mathrm{t}}-i+r)} \binom{\delta(K_{\mathrm{t}}-i+r)}{n} \cdot \left(-1\right)^n e^{n/\rho_k} \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{b!}{\rho_k^b}\,\Gamma\left(-b,\frac{n}{\rho_k}\right) \tag{14}$$

where $\Gamma\left(\cdot,\cdot\right)$ is the upper incomplete gamma function, $\alpha_{i,r}$ are the combinatorial coefficients defined in (12), and

$$\beta_{n,b}^{(1)} = 1, \qquad \beta_{n,b}^{(2)} = \frac{n!}{b!\,(n-b)!},$$

$$\beta_{n,b}^{(\nu)} = \sum_{i_\nu=0}^{\left\lfloor\frac{b}{\nu-1}\right\rfloor}\sum_{i_{\nu-1}=0}^{\left\lfloor\frac{b-(\nu-1)i_\nu}{\nu-2}\right\rfloor}\cdots\sum_{i_3=0}^{\left\lfloor\frac{b-3i_4-\ldots-(\nu-1)i_\nu}{2}\right\rfloor} \cdot \frac{n!}{\left(b-\sum_{r=3}^{\nu}(r-1)i_r\right)!\,\left(n-b+\sum_{r=3}^{\nu}(r-2)i_r\right)!} \cdot \frac{1}{\prod_{r=3}^{\nu}(r-1)!^{i_r} i_r!}, \qquad \nu > 2. \tag{15}$$

Proof: See [14]. ∎

The final expression for the ergodic sum-rate in Rayleigh fading is obtained after substituting (14) in (8) and rearranging the terms in the sums, yielding

$$C = \frac{-\log_2 e}{K_{\mathrm{a}}K_{\mathrm{t}}} \sum_{k=1}^{K_{\mathrm{t}}} \sum_{i=0}^{K_{\mathrm{a}}-1} \alpha_i \sum_{n=1}^{\delta(K_{\mathrm{t}}-i)} \binom{\delta(K_{\mathrm{t}}-i)}{n} (-1)^n \cdot e^{n/\rho_k} \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{b!}{\rho_k^b}\,\Gamma\left(-b,\frac{n}{\rho_k}\right) \tag{16}$$

with the coefficients $\beta_{n,b}^{(\nu)}$ as defined in (15) and with

$$\alpha_i = \sum_{r=i}^{K_{\mathrm{a}}-1} \frac{(-1)^{r-i} K_{\mathrm{t}}!\,(K_{\mathrm{t}}-r)_{r-i}\,(r)_{r-i}}{r!(r-i)!\,(K_{\mathrm{t}}-r)!\,(K_{\mathrm{t}}-r+1)_{r-i}}. \tag{17}$$

It can be shown that $\sum_{i=0}^{K_{\mathrm{a}}-1}\alpha_i = K_{\mathrm{a}}$ and, thus, the sum-rate in (16) with K_{a} subchannels and K_{t} users is a weighted average of the sum-rates with a single subchannel and $K_{\mathrm{t}}-i$ users, $i=0,...,K_{\mathrm{a}}-1$. It follows that the sum-rate is maximized with $K_{\mathrm{a}}=1$, i.e., when there is a single subchannel and only the user with highest normalized SNR is scheduled. Having multiple parallel subchannels, however, reduces the user waiting time and hence the transmission delay. The ensuing tradeoff between sum-rate and delay can be evaluated with the aid of (16), which quantifies the rate penalty associated with multiple parallel subchannels ($K_{\mathrm{a}}>1$).

C. Special Cases

A number of particular cases can be further analyzed from (16), leading to simplified expressions.

1) Same average SNR for all users: If $\rho_k=\rho$ for $k=1,...,K_{\mathrm{t}}$, (16) reduces to

$$C = \frac{-\log_2 e}{K_{\mathrm{a}}} \sum_{i=0}^{K_{\mathrm{a}}-1} \alpha_i \sum_{n=1}^{\delta(K_{\mathrm{t}}-i)} \binom{\delta(K_{\mathrm{t}}-i)}{n} (-1)^n e^{n/\rho} \cdot \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{b!}{\rho^b}\,\Gamma\left(-b,\frac{n}{\rho}\right). \tag{18}$$

In this case, PFS is equivalent to the opportunistic (best-channel) scheduling algorithm. Since all the users have the same average SNR, the rate per user is simply C/K_{t}.

In light of (18), the sum-rate for K_{a} subchannels, $C^{(K_{\mathrm{a}})}$, approaches $C^{(1)}$ asymptotically as $K_{\mathrm{t}}\to\infty$, i.e., the sum-rate loss for $K_{\mathrm{a}}>1$ is asymptotically erased as the number of users grows. This is illustrated in Fig. 1, which depicts $C^{(K_{\mathrm{a}})}/C^{(1)}$ for TAS-MRC with $N_{\mathrm{t}}=1$, $N_{\mathrm{r}}=2$, and $\rho=10$ dB, parameterized by K_{a}. For instance, the sum-rate loss for $K_{\mathrm{a}}=6$ reduces from 25% to 12% with only 20 users.

2) TAS-SC: The normalized SNR is distributed as the maximum of $\delta=N_{\mathrm{r}}N_{\mathrm{t}}$ exponential RVs ($\nu=1$). Then,

$$C = \frac{-\log_2 e}{K_{\mathrm{a}}K_{\mathrm{t}}} \sum_{k=1}^{K_{\mathrm{t}}} \sum_{i=0}^{K_{\mathrm{a}}-1} \alpha_i \cdot \sum_{n=1}^{\delta(K_{\mathrm{t}}-i)} \binom{\delta(K_{\mathrm{t}}-i)}{n} (-1)^n e^{n/\rho_k}\,\Gamma\left(0,\frac{n}{\rho_k}\right). \tag{19}$$

[1]The same CDF applies to Nakagami-m fading with $\nu' = m\nu$.

[2]The dual scheme, MRT-SC with $\mathbf{H}_k$ known at the transmitters, can be analyzed by setting $\delta = N_{\mathrm{r}}$ and $\nu = N_{\mathrm{t}}$.

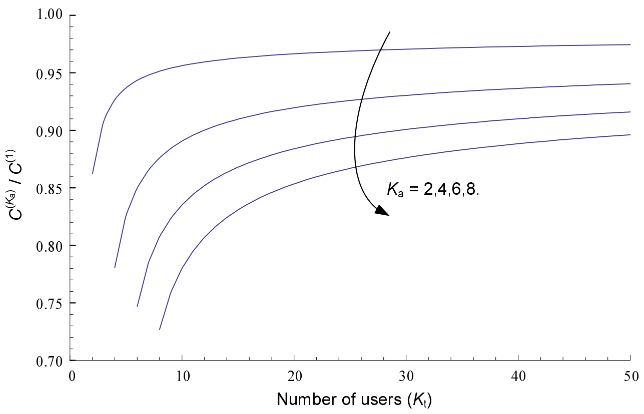

Fig. 1. Ergodic sum-rate as function of the number of users, for TAS-MRC with $N_\mathrm{t} = 1$, $N_\mathrm{r} = 2$, and $\rho = 10$ dB, parameterized by K_a

3) TAS-SC with a single subchannel: Plugging $K_\mathrm{a} = 1$ in (19), we obtain

$$C = \frac{-\log_2 e}{K_\mathrm{t}} \sum_{k=1}^{K_\mathrm{t}} \sum_{n=1}^{\delta K_\mathrm{t}} \binom{\delta K_\mathrm{t}}{n} (-1)^n \, e^{n/\rho_k} \Gamma\left(0, \frac{n}{\rho_k}\right)$$

with $\delta = N_\mathrm{r} N_\mathrm{t}$, which is equivalent, for the single-antenna case ($\delta = 1$), to the result reported in [8].

IV. Asymptotic Analysis

In this section, the ergodic sum-rate is analyzed asymptotically in different scenarios of interest.

A. Large number of users (K_t)

The conditioned mutual information in (5) is just a transformation of the order statistics $\tilde{\rho}_{K_\mathrm{t}, K_\mathrm{t}-i}$, whose expected value can be approximated by the inverse function of the marginal CDF [12]. Since $\tilde{\rho}_1, \ldots, \tilde{\rho}_{K_\mathrm{t}}$ are i.i.d., the expected value of the conditioned mutual information for subchannel i can be approximated (for large K_t) by

$$E\left[\mathcal{I}_{i,k}\right] \approx G^{-1}\left(\frac{K_\mathrm{t} - i}{K_\mathrm{t} + 1}\right) \tag{20}$$

where $G^{-1}(\cdot)$ is the inverse function of $G(\xi) = F_{\tilde{\rho}}\left(\frac{2^\xi - 1}{\rho_k}\right)$, with $F_{\tilde{\rho}}(\cdot)$ given in (13). Plugging the solution to (20) in (8),

$$C \approx \frac{1}{K_\mathrm{a} K_\mathrm{t}} \sum_{k=1}^{K_\mathrm{t}} \sum_{i=0}^{K_\mathrm{a}-1} \log_2\left(1 + \rho_k \gamma^{-1}\left(\nu, \Gamma(\nu)\left(\tfrac{K_\mathrm{t}-i}{K_\mathrm{t}+1}\right)^{\frac{1}{\delta}}\right)\right). \tag{21}$$

For TAS-SC, $\tilde{\rho}$ is exponentially distributed ($\nu = 1$) and

$$C \approx \frac{1}{K_\mathrm{a} K_\mathrm{t}} \sum_{k=1}^{K_\mathrm{t}} \sum_{i=0}^{K_\mathrm{a}-1} \log_2\left(1 - \rho_k \log\left(1 - \left(\tfrac{K_\mathrm{t}-i}{K_\mathrm{t}+1}\right)^{\frac{1}{\delta}}\right)\right). \tag{22}$$

From (22), we can recover the well-known behavior [3]

$$C = \log_2 \log(K_\mathrm{t}) + \mathcal{O}(1). \tag{23}$$

However, (22) is more informative than (23) and returns tight approximations for intermediate and even small numbers of users ($K_\mathrm{t} > 5$), bridging the gap between the exact expression in (16) and the asymptote in (23).

B. High SNR

For a high-SNR analysis, we consider that all users have the same average SNR, $\rho_k = \rho$ for $k = 1, \ldots, K_\mathrm{t}$, with $\rho \to \infty$. In view of (18), all the terms in the sum exhibit the product of an exponential and the incomplete gamma function; this product can be expanded for $\rho \to \infty$. To that end, we distinguish between two cases depending on the argument of the incomplete gamma function. Invoking [13, 8.352.8],

$$e^{n/\rho}\,\Gamma\left(0, \frac{n}{\rho}\right) = \log_e \rho - \mathcal{C} - \log_e n + \mathcal{O}\left(\frac{1}{\rho}\right) \tag{24}$$

$$\frac{e^{n/\rho}}{\rho^b}\,\Gamma\left(-b, \frac{n}{\rho}\right) = \frac{(b-1)!}{b!}\left(\frac{1}{n}\right)^b + \mathcal{O}\left(\frac{1}{\rho}\right) \tag{25}$$

where $\mathcal{C}$ is Euler's constant and $b > 0$. Plugging (24) and (25) into (18) and after some algebra, we obtain an affine expansion of the ergodic sum-rate. Precisely,

$$C = \log_2 \rho + \mathcal{L}_\infty + \mathcal{O}\left(\frac{1}{\rho}\right) \tag{26}$$

where the zero-order term $\mathcal{L}_\infty$, often referred to as *power offset* [15], is readily identified as

$$\mathcal{L}_\infty = \frac{\log_2 e}{K_\mathrm{a}} \sum_{i=0}^{K_\mathrm{a}-1} \alpha_i \sum_{n=1}^{\delta(K_\mathrm{t}-i)} \binom{\delta(K_\mathrm{t}-i)}{n} (-1)^n \cdot \left(\mathcal{C} + \log_e n - \sum_{b=1}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{(b-1)!}{n^b}\right). \tag{27}$$

From (26) and (27), the same particular cases characterized in the previous section may be analyzed in the high SNR regime. For instance, with TAS-SC ($\nu = 1$) and a single subchannel ($K_\mathrm{a} = 1$) we obtain

$$\mathcal{L}_\infty = \log_2 e \sum_{n=1}^{\delta K_\mathrm{t}} \binom{\delta K_\mathrm{t}}{n} (-1)^n \left(\mathcal{C} + \log_e n\right). \tag{28}$$

For small K_t, and consequently small n, $\log n \approx 2\frac{n-1}{n+1}$. Then, (28) can be simplified into

$$\mathcal{L}_\infty \approx \log_2 e \left(\frac{2\delta K_\mathrm{t} - 2}{\delta K_\mathrm{t} + 1} - \mathcal{C}\right). \tag{29}$$

Note that this approximation holds only for relatively small K_t and thus it does not contradict the findings of Section IV-A.

C. Low SNR

Consider again that $\rho_k = \rho$, for $k = 1, \ldots, K_\mathrm{t}$, but with $\rho \to 0$. In this regime, the capacity is often expressed as a function of the energy per bit, E_b/N_0, rather than the SNR [16]. The key quantities are $\frac{E_\mathrm{b}}{N_0}_{\min}$ (the minimum energy per information bit required for a positive rate), and the slope therein, S_0. The sum-rate as function of E_b/N_0 expands as

$$\mathsf{C}\left(\frac{E_\mathrm{b}}{N_0}\right) = S_0\left(\log_2 \frac{E_\mathrm{b}}{N_0} - \log_2 \frac{E_\mathrm{b}}{N_0}_{\min}\right) + \epsilon \tag{30}$$

where ϵ is a lower-order term and

$$\frac{E_\mathrm{b}}{N_0}_{\min} = \frac{1}{\dot{C}\,(0)} \qquad\qquad S_0 = \frac{-2\,[\dot{C}(0)]^2}{\ddot{C}\,(0)\log_2 e} \tag{31}$$

with $\dot{C}\,(0)$ and $\ddot{C}\,(0)$ being, respectively, the first and second derivatives with respect to the SNR at $\rho = 0$.

Lemma 2: The ergodic sum-rate expands as $C = \dot{C}\,(0)\,\rho + \frac{\ddot{C}(0)}{2}\rho^2 + o(\rho^2)$ with

$$\dot{C}\,(0) = \frac{-\log_2 e}{K_\mathrm{a}}\mathcal{F}_0 \tag{32}$$

$$\ddot{C}\,(0) = \frac{2\log_2 e}{K_\mathrm{a}}\mathcal{F}_1 \tag{33}$$

and

$$\mathcal{F}_j = \sum_{i=0}^{K_\mathrm{a}-1}\alpha_i \sum_{n=1}^{\delta(K_\mathrm{t}-i)} \binom{\delta(K_\mathrm{t}-i)}{n} (-1)^n \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{(b+j)!}{n^{b+1+j}} \tag{34}$$

Proof: See [14]. ∎

From Lemma 2, $\frac{E_\mathrm{b}}{N_0}_{\min}$ and S_0 can be readily obtained via (31). Again, particular cases lead to more compact expressions as shown next.

For TAS-SC ($\nu = 1$), (32) and (33) simplify to

$$\dot{C}\,(0) = \frac{\log_2 e}{K_\mathrm{a}} \sum_{i=0}^{K_\mathrm{a}-1} \alpha_i\, H_{\delta(K_\mathrm{t}-i)} \tag{35}$$

$$\ddot{C}\,(0) = \frac{-\log_2 e}{K_\mathrm{a}} \sum_{i=0}^{K_\mathrm{a}-1} \alpha_i \left(H^2_{\delta(K_\mathrm{t}-i)} + H_{\delta(K_\mathrm{t}-i),2}\right) \tag{36}$$

where $\delta = N_\mathrm{r} N_\mathrm{t}$, $H_{k,r} = \sum_{n=1}^{k} 1/n^r$ is the kth harmonic number of r and $H_{k,1} = H_k$.

In turn, for single-antenna single-subchannel systems, $\dot{C}\,(0) = H_{K_\mathrm{t}} \log_2 e$ and $\ddot{C}\,(0) = -\left(H^2_{K_\mathrm{t}} + H_{K_\mathrm{t},2}\right)\log_2 e$ and, subsequently,

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{H_{K_\mathrm{t}}} \qquad\qquad S_0 = \frac{2\,H^2_{K_\mathrm{t}}}{H^2_{K_\mathrm{t}} + H_{K_\mathrm{t},2}}. \tag{37}$$

Fig. 2 depicts both $\frac{E_\mathrm{b}}{N_0}_{\min}$ and S_0 as a function of K_t, for $K_\mathrm{a} = \nu = \delta = 1$. For $K_\mathrm{t} = 1$ we recover the well-known values obtained without scheduling, namely $\frac{E_\mathrm{b}}{N_0}_{\min} = \log_e 2 = -1.59$ dB and $S_0 = 1$. Then, as the number of users available for scheduling increases, two trends are observed:

- The value of $\frac{E_\mathrm{b}}{N_0}_{\min}$ diminishes steadily indicating how the transmit power required for reliable communication abates. For large K_t, this reduction can be assessed from the asymptotic behavior $H_{K_\mathrm{t}} = \log_e K_\mathrm{t} + \mathcal{C} + o(1)$. However, this asymptotic behavior does not capture what happens for small K_t, which is where the most substantial reduction takes place. Again, our exact analysis provides a complete characterization, shedding light on the improvement at every K_t. For instance, we can affirm from (37) and Fig. 2 that with 5 users to select from, $\frac{E_\mathrm{b}}{N_0}_{\min}$ shrinks by 3.58 dB.
- The slope S_0 improves rapidly and approaches $S_0 = 2$ for $K_\mathrm{t} \to \infty$. This value is precisely the low-SNR slope of an unfaded channel [16], hence offering the pleasing interpretation of scheduling ultimately erasing the impact of fading as far as S_0 is concerned.

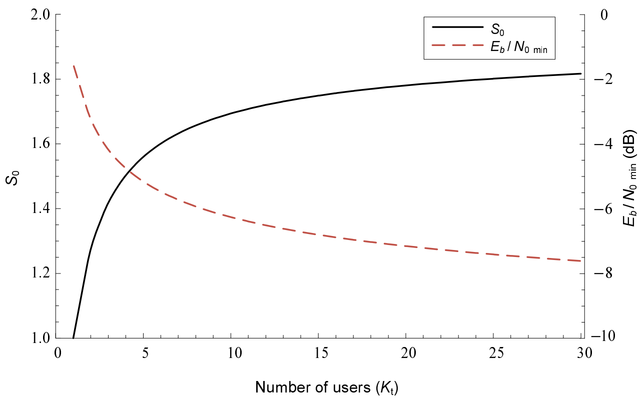

Fig. 2. $\frac{E_b}{N_0}_{\min}$ and S_0 for $K_\mathrm{a} = \delta = \nu = 1$ as a function of K_t.

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