Precoding Based Network Alignment and the Capacity of a Finite Field X Channel

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***Abstract*—Precoding based network alignment (PBNA) is a network coding paradigm inspired by wireless networks where all the intelligence resides at the sources and destination nodes whereas intermediate relay nodes only perform arbitrary linear network coding operations, creating an effective one-hop finite field linear network between sources and destinations. The main question explored in this work is how degrees of freedom (DoF) results from wireless networks can be translated into capacity results for their finite field counterparts. A finite field X channel, i.e., a multiple unicast network comprised of 2 source nodes, 2 destination nodes and 4 independent messages (one for each source-destination pair) is considered in this work, where the channel outputs are arbitrary linear combinations of channel inputs over a finite field $\mathbb{F}_{p^n}$. Like its wireless counterpart, which has 4/3 sum DoF, this channel is shown to have a sum capacity of 4/3 symbols per channel use for most channel realizations. The main insight is that, with a few exceptions that are pointed out, scalar (SISO) finite field channels over $\mathbb{F}_{p^n}$ are analogous to $n \times n$ complex vector (MIMO) channels in the wireless setting, so the DoF optimal precoding solutions for wireless networks can be translated into capacity optimal solutions for their finite field counterparts.**

I. Introduction

Precoding based network alignment is a network communication paradigm inspired by linear network coding and interference alignment principles [4], [5]. While intermediate nodes only perform arbitrary linear network coding operations which transform the network into a one-hop linear finite field network, all the intelligence resides at the source and destination nodes where information theoretically optimal encoding (precoding) and decoding is performed to achieve the capacity of the resulting linear network. The two restricting assumptions — restricting the intelligence to the source and destination nodes, and restricting to linear operations at intermediate nodes — are motivated by the reduced complexity of network optimization and also by the potential to apply the insights and techniques developed for one-hop wireless networks. Indeed, the PBNA paradigm gives rise to settings that are analogous to 1-hop wireless networks, albeit over finite fields. To highlight this distinction, we simply refer to these networks as finite field networks. There is a finite field counterpart to every 1-hop wireless network and vice versa. A number of interesting interference alignment techniques have been developed for 1-hop wireless networks and shown to be optimal from a degrees of freedom (DoF) perspective. Translating the DoF optimal schemes for wireless networks into capacity optimal schemes for finite field networks is therefore a promising research avenue. For example, the CJ scheme originally conceived in [7] for the K user time-varying wireless interference channel is applied to the 3 unicast problem by Das et al. in [4], [5]. The strongly asymptotic character of the CJ scheme, and the requirement of infinite diversity (channel variations) along with perfect channel knowledge makes it rather impractical. However, there are many instances of wireless networks where simpler and interference alignment schemes, that are not asymptotic and do not require any channel variations, are known to be DoF optimal. In this work, we study the simplest such setting, the X channel.

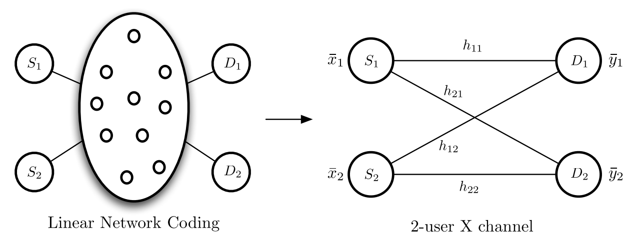

Fig. 1. Wired network modeled as 2-user X channel

An X network is an all-unicast setting, i.e., there is an independent message from each source node to each destination node. In this work we study an X network with 2 source nodes, 2 destination nodes, and 4 independent messages, also known simply as the X channel. Specifically we explore the capacity of a finite field constant X channel.

Note that if intermediate nodes were intelligent, i.e., operations at intermediate nodes could be optimized, then the sum-capacity of an all-unicast network, i.e., an X network, is known to be achievable by routing [6]. However, due to practical limitations, optimization of intermediate nodes may not be possible. While the overhead and complexity of learning and optimizing individual coding coefficients at all intermediate nodes may be excessive, it is much easier to learn only the end-to-end channel coefficients, e.g., through network tomography, with no knowledge of the internal structure of the network or the individual coding coefficients at the intermediate nodes. This is the setting that we explore in this work.

II. Channel Model: Finite Field X Channel

Consider the finite field X channel

$$\begin{aligned}\bar{y}_1(t) &= h_{11}\bar{x}_1(t) + h_{12}\bar{x}_2(t)\\ \bar{y}_2(t) &= h_{22}\bar{x}_2(t) + h_{21}\bar{x}_1(t)\end{aligned}$$

where, over the t^{th} channel use, $\bar{x}_i(t)$ is the symbol sent by source i, h_{ji} represents channel coefficient between source i and destination j and $\bar{y}_j$ represents the received symbol at destination j. All symbols $\bar{x}_i(t), h_{ji}, \bar{y}_j(t)$ and addition and multiplication operations are in a finite field $\mathbb{F}_{p^n}$. The channel coefficients h_{ji} are constant across t and assumed to be perfectly known at all sources and destinations. There are four independent messages, with W_{ji} denoting the message that originates at source i and is intended for destination j.

A coding scheme over T channel uses, that assigns to each message W_{ji} a rate R_{ji}, measured in units of $\mathbb{F}_{p^n}$ symbols per channel use, corresponds to a encoding function at each source i that maps the messages originating at that source into a sequence of T transmitted symbols, and a decoding function at each destination that maps the sequence of T received symbols into decoded messages $\hat{W}_{ji}$.

$$\begin{aligned}
&\text{Encoder 1:} && (W_{11}, W_{21}) \rightarrow \bar{x}_1(1)\bar{x}_1(2)\cdots\bar{x}_1(T) && (1)\\
&\text{Encoder 2:} && (W_{12}, W_{22}) \rightarrow \bar{x}_2(1)\bar{x}_2(2)\cdots\bar{x}_2(T) && (2)\\
&\text{Decoder 1:} && \bar{y}_1(1)\bar{y}_1(2)\cdots\bar{y}_1(T) \rightarrow (\hat{W}_{11}, \hat{W}_{12}) && (3)\\
&\text{Decoder 2:} && \bar{y}_2(1)\bar{y}_2(2)\cdots\bar{y}_2(T) \rightarrow (\hat{W}_{21}, \hat{W}_{22}) && (4)
\end{aligned}$$

Each message W_{ji} is uniformly distributed over $\{1, 2, \cdots, \lceil p^{nTR_{ji}} \rceil\}$, $\forall i, j \in \{1, 2\}$. An error occurs if $(\hat{W}_{11}, \hat{W}_{12}, \hat{W}_{21}, \hat{W}_{22}) \neq (W_{11}, W_{12}, W_{21}, W_{22})$. A rate tuple $(R_{11}, R_{12}, R_{21}, R_{22})$ is said to be achievable if there exist encoders and decoders such that the probability of error can be made arbitrarily small by choosing a sufficiently large T. The closure of all achievable rate pairs is the capacity region and the maximum value of $R_{11} + R_{12} + R_{21} + R_{22}$ across all rate tuples that belong to the capacity region, is the sum-capacity, that we will refer to as simply the capacity, for brevity. Note that the capacity depends only on the values of the given channel coefficients h_{ji}.

III. Zero Channels

First, let us deal with trivial cases where some of the channel coefficients are zero.

Theorem 1: If one or more of the channel coefficients h_{ji} is equal to zero, the capacity is given by:

1) If $h_{12} = h_{21} = 0$ and $h_{11}, h_{22} \neq 0$, then $C = 2$.
2) If $h_{11} = h_{22} = 0$ and $h_{12}, h_{21} \neq 0$, then $C = 2$.
3) If $h_{11} = h_{12} = h_{21} = h_{22} = 0$, then $C = 0$.
4) In all other cases where at least one channel coefficient is zero, $C = 1$.

Proof: Cases 1, 2, 3 are trivial. The resulting channel for Case 4 is a MAC, BC or Z channel. MAC and BC have capacity 1 by min-cut max-flow theorem, and the proof for the Z channel follows from the corresponding DoF result presented in [1] for the wireless setting. ∎

IV. Channel Normalization

Based on Theorem 1, henceforth we will assume that all channel coefficients are non-zero. Without loss of generality, let us normalize the channel coefficients by invertible operations at the sources and destinations shown in Fig. 2. Since these are invertible operations, they do not affect the channel capacity:

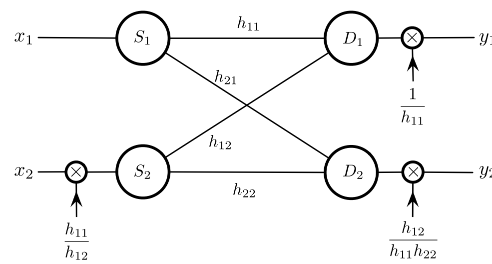

Fig. 2. Normalization in X channel

Destination 1 normalizes symbols by h_{11} : $y_1 = \frac{\bar{y}_1}{h_{11}}$
Destination 2 normalizes symbols by $\frac{h_{11}h_{22}}{h_{12}}$: $y_2 = \frac{\bar{y}_2 h_{12}}{h_{11}h_{22}}$
Source 2 normalizes symbols by $\frac{h_{11}}{h_{12}}$: $x_2 = \frac{\bar{x}_2 h_{12}}{h_{11}}$
Source 1 performs no normalization : $x_1 = \bar{x}_1$
The normalized X channel is represented as

$$\begin{aligned} y_1 &= x_1 + x_2\\ y_2 &= hx_1 + x_2\end{aligned}$$

wherein we have reduced channel parameters to single channel coefficient h, defined as

$$h = \frac{h_{12}h_{21}}{h_{11}h_{22}} \tag{5}$$

All symbols are still over $\mathbb{F}_{p^n}$.

V. Main Insight: $n \times n$ MIMO and $\mathbb{F}_{p^n}$

It is well known that the multiple input multiple output (MIMO) wireless X channel where each node is equipped with n antennas has $\frac{4n}{3}$ DoF [1], [2]. For almost all channel realizations in the wireless setting, the DoF are achieved through a linear vector space interference alignment scheme. If n is a multiple of 3, no symbol extensions are needed and spatial beamforming is sufficient. For example, if each node is equipped with 3 antennas, then it suffices to send 1 symbol per message, each along its assigned 3×1 signal vector. The vectors are chosen such that the two undesired symbols at each destination align in the same dimension leaving the remaining 2 dimensions free to resolve the desired signals. If n is not a multiple of 3 then 3 symbol extensions (i.e., coding over 3 channel uses) are needed to create a vector space within which a third of the dimensions are assigned to each message. When translating these insights into the finite field X channel with only scalar inputs and scalar outputs (SISO) the key insight from this work is that a SISO network over $\mathbb{F}_{p^n}$ is analogous to a $n \times n$ MIMO network, albeit with some structure imposed on the channel matrix. Note that structured channels are also encountered in the wireless setting — channels obtained by symbol extensions have a block diagonal structure [1], asymmetric complex signaling based schemes used for the SISO X channel have a unitary matrix structure [2]. Exploring the structural implications of vector space interpretations of finite fields $\mathbb{F}_{p^n}$ is a key aspect of this work. We start with a brief review of the fundamentals.

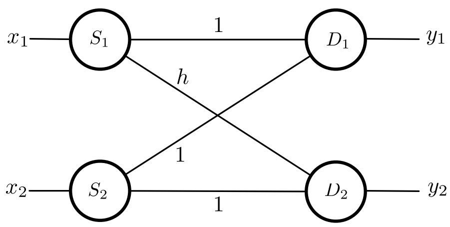

Fig. 3. Normalized X channel with Non-Zero Coefficients

The finite field $\mathbb{F}_{p^n}$ can be used to generate a n-dimensional vector space as follows. Each element of $\mathbb{F}_{p^n}$ can be represented in the form

$$z = x_{n-1}s^{n-1} + x_{n-2}s^{n-2} + \ldots + x_1 s^1 + x_0 \tag{6}$$

wherein $z \in \mathbb{F}_{p^n}$, $x_i \in \mathbb{F}_p$.

As an example consider $\mathbb{F}_{3^3}$ which contains 27 elements $\{0, 1, \ldots, 26\}$ and each element $a \in \mathbb{F}_{3^3}$ is of the form 3^2a_2+$3a_1$+a_0, wherein $a_2, a_1, a_0 \in \mathbb{F}_3$ with values from {0,1,2}. Hence every element can be written in a vector notation with coefficients $[a_2; a_1; a_0]$, e.g., $a = 22$ can be written as $[2\ ;\ 1\ ;\ 1]$.

Next, let us see how multiplication with the channel coefficient $h \in \mathbb{F}_{p^3}$ is represented as a multiplication with a 3×3 matrix with elements in $\mathbb{F}_3$. Consider the monic irreducible cubic polynomial $s^3 + 2s + 1$ which is treated as zero in the field. The field itself consists of all polynomials with coefficients in $\mathbb{F}_3$, modulo $s^3 + 2s + 1$. Since $s^3 + 2s + 1 = 0$ in $\mathbb{F}_{3^3}$, it follows that

$$s^3 = -2s - 1 = (3-2)s + (3-1) = s + 2 \tag{7}$$

$$s^4 = s(s^3) = s(s+2) = s^2 + 2s \tag{8}$$

Since $h, x \in \mathbb{F}_{3^3}$ they can be represented as $h = h_2s^2 + h_1s + h_0,\ x = x_2s^2 + x_1s + x_0$ where $h_i, x_i \in \mathbb{F}_3$. The product $y = hx \in \mathbb{F}_{3^3}$ can be written as

$$\begin{aligned} y = hx \equiv (h_2s^2 + h_1s + h_0)(x_2s^2 + x_1s + x_0) = \\ s^4(h_2x_2) + s^3(h_2x_1 + h_1x_2) + s^2(h_2x_0 + h_0x_2 + h_1x_1) + \\ s(h_1x_0 + h_0x_1) + (h_0x_0) \end{aligned}$$

Equivalently,

$$\mathbf{y} = \mathbf{H}\mathbf{x} = \begin{bmatrix} h_2 + h_0 & h_1 & h_2 \\ 2h_2 + h_1 & h_2 + h_0 & h_1 \\ 2h_1 & 2h_2 & h_0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}$$

wherein $\mathbf{x}, \mathbf{y}$ are 3×1 vector with entries from $\mathbb{F}_3$ and $\mathbf{H}$ is a 3×3 matrix with its 9 entries from $\mathbb{F}_3$. Here the equivalence of SISO channel over $\mathbb{F}_{3^3}$ and MIMO channel over $\mathbb{F}_3$ is established through the 3×3 linear transformation, $\mathbf{H}$. Note also the structure inherent in the matrix representation $\mathbf{H}$. While there are 3^9 possible 3×3 matrices over $\mathbb{F}_3$, there are only 27 valid $\mathbf{H}$ matrices, because $\mathbb{F}_{3^3}$ has only 27 elements.

Next we formalize the insights identified above by characterizing the capacity of finite field X channel in field $\mathbb{F}_{p^n}$.

VI. X Channel over $\mathbb{F}_{p^3}$

Theorem 2: The X channel over $\mathbb{F}_{p^3}$ has capacity $\frac{4}{3}$ if

$$h = \frac{h_{12}h_{21}}{h_{11}h_{22}} \notin \mathbb{F}_p \tag{9}$$

Proof: We will omit the details of the information theoretic $\frac{4}{3}$ outer bound except to mention that it is a straightforward extension of the DoF outer bound for the wireless setting presented in [1], a combination of the Z channel bounds, with minor adjustments to account for finite field channels.

For achievability, consider the normalized X channel which can be characterized by single channel coefficient $h = \frac{h_{12}h_{21}}{h_{11}h_{22}}$ from $\mathbb{F}_{p^3}$. We consider superposition coding at the sources, wherein messages from source 1 (W_{j1}) are independently encoded into symbols x_{j1} and added to obtain the transmitted symbol $x_1 = x_{11} + x_{21}$ and messages from source 2 (W_{j2}) are similarly encoded as $x_2 = x_{21} + x_{22}$. We choose input symbols $x_{ji} \in \mathbb{F}_p$. Received symbols can be expressed as

$$\begin{aligned} y_1 &= x_{11} + x_{12} + x_{22} + x_{21} \\ y_2 &= hx_{21} + hx_{11} + x_{22} + x_{12} \end{aligned}$$

wherein $h, y_j \in \mathbb{F}_{p^3}$ and $x_{ij} \in \mathbb{F}_p$.

As described earlier, $\mathbb{F}_{p^3}$ can be split into a 3-dimensional space over subfield $\mathbb{F}_p$ so that the output has 3 dimensions (each of $\mathbb{F}_p$) within which 2 desired symbols and 2 interference symbols are present at each destination. To achieve capacity, interference symbols should be aligned at each destination such that they occupy only one dimension at that destination while remaining distinguishable at the other destination where they are desired. To this end, we will assign a precoding vector v_{ji} to each symbol x_{ji}.

$$\begin{aligned} y_1 &= v_{11}x_{11} + v_{12}x_{12} + v_{22}x_{22} + v_{21}x_{21} \\ y_2 &= v_{22}x_{22} + hv_{21}x_{21} + hv_{11}x_{11} + v_{12}x_{12} \end{aligned}$$

wherein $h, y_j, v_{ji} \in \mathbb{F}_{p^3}$ and $x_{ji} \in \mathbb{F}_p$. Equivalently, using vector notation,

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{v}_{11}x_{11} + \mathbf{v}_{12}x_{12} + \mathbf{v}_{22}x_{22} + \mathbf{v}_{21}x_{21} \\ \mathbf{y}_2 &= \mathbf{v}_{22}x_{22} + \mathbf{H}\mathbf{v}_{21}x_{21} + \mathbf{H}\mathbf{v}_{11}x_{11} + \mathbf{v}_{12}x_{12} \end{aligned}$$

wherein $\mathbf{y}_j, \mathbf{v}_{ji} \in \mathbb{F}_p^{3\times1}$ and $\mathbf{H} \in \mathbb{F}_p^{3\times3}$ representing $h \in \mathbb{F}_{p^3}$.

Interference alignment conditions can be expressed as

$$\text{span}(\mathbf{v}_{22}) = \text{span}(\mathbf{v}_{21})\ \&\ \text{span}(\mathbf{v}_{12}) = \text{span}(\mathbf{H}\mathbf{v}_{11}) \tag{10}$$

For each choice of v_{21}, v_{11} (one can try all possible choices in the worst case), set the other two vectors as

$$v_{22} = v_{21} \qquad \& \qquad v_{12} = hv_{11} \tag{11}$$

then interference is aligned at each destination along one dimension of order p. At the destinations, the spaces occupied by the two desired symbols and the aligned interference symbol are represented using matrices S_1 and S_2.

$$S_1 = [v_{11} \quad v_{12} \quad v_{21}] = [v_{11} \quad hv_{11} \quad v_{21}] \tag{12}$$

$$S_2 = [v_{22} \quad hv_{21} \quad v_{12}] = [v_{21} \quad hv_{21} \quad hv_{11}] \tag{13}$$

When $h \notin \mathbb{F}_p$, we will now show that we can choose v_{11} and v_{21} such that elements of S_1 and S_2 are linearly independent over $\mathbb{F}_p$. Set $v_{21} = 1$. Then S_1 and S_2 can be written as

$$S_1 = [v_{11} \quad hv_{11} \quad 1] \qquad \& \qquad S_2 = [1 \quad h \quad hv_{11}] \tag{14}$$

Consider S_1. Note that v_{11} and hv_{11} are linearly independent over $\mathbb{F}_p$ since $h \notin \mathbb{F}_p$. Hence elements of S_1 are linearly independent if $\frac{1}{v_{11}}$ is not a linear combination (with coefficients from $\mathbb{F}_p$) of 1 and h. This is guaranteed if

$$v_{11} \notin A \triangleq \left\{ \frac{1}{\alpha + \beta h} : \alpha, \beta \in \mathbb{F}_p, (\alpha, \beta) \neq (0,0) \right\} \cup \{0\} \tag{15}$$

Similarly, consider S_2. Note that 1 and h are linearly independent over $\mathbb{F}_p$, since $h \notin \mathbb{F}_p$. Hence, elements of S_2 are linearly independent if v_{11} is not a linear combination of $\frac{1}{h}$ and 1 over $\mathbb{F}_p$. This is guaranteed if

$$v_{11} \notin B \triangleq \left\{ \alpha + \frac{\beta}{h} : \alpha, \beta \in \mathbb{F}_p, (\alpha, \beta) \neq (0,0) \right\} \cup \{0\} \tag{16}$$

Since $|A| \leq p^2$ and $|B| \leq p^2$, and all p constant polynomials are contained in both A and B, we must have

$$|A \cup B| \leq 2p^2 - p \tag{17}$$

Unless $A \cup B$ contains all p^3 elements of $\mathbb{F}_{p^3}$ there is at least one choice of v_{11} that satisfies both (15) and (16). In other words, the scheme works if $p^3 > 2p^2 - p$ which is true for all $p \geq 2$. Thus, we have proved the achievability of rate $\frac{1}{3}$ per message, and a sum-rate of $\frac{4}{3}$ for all p, which matches the capacity outer bound. Note that a $\mathbb{F}_{p^3}$ symbol represents $\frac{1}{3}$ of an $\mathbb{F}_p$ symbol and the capacity is measured in $\mathbb{F}_{p^3}$ units because the original channel alphabet is from $\mathbb{F}_{p^3}$. ■

Example: An instance of the problem and its solution are illustrated in Fig. 4 using scalar notation and again in Fig. 5 using vector notation.

Remark 1: Using the same reasons described in the proof, it is possible to show that no vector linear scheme for the X channel over $\mathbb{F}_{p^n}$ can achieve rate more than 1 when $h \in \mathbb{F}_p$, regardless of the values of p, n. For a definition of vector linear schemes, see [9]. Signal level alignment schemes, such as the recent work in [10] are an interesting research avenue for these channel instances, although we expect that there might be a capacity loss in these cases relative to the $\frac{4}{3}$ value.

Remark 2: There are $p^3 - 1$ possible non-zero values for h, out of which the scheme presented, fails for $p - 1$ values. If this behavior is representative for all larger p, the fraction of channel instances for which the scheme would fail is

$$\frac{(p-1)}{(p^3-1)} = \frac{1}{p^2+p+1} \to 0 \quad \text{for large } p.$$

which is consistent with the intuition from wireless setting that the linear precoding based interference alignment scheme achieves $4/3$ DoF for almost all channel realizations.

Similar to splitting a field $\mathbb{F}_{p^3}$ to form a 3-dimensional space in field of order p, other fields of order p^n can be split to a n-dimensional field of order p. However, in order to achieve the optimal capacity of $\frac{4}{3}$, symbol extensions would be required when n is not a multiple of 3. To illustrate this, we next consider $\mathbb{F}_{p^2}$ in next section, wherein we show that capacity is $\frac{4}{3}$ over 3 symbol extensions of the channel. In fact, the scheme discussed is equivalent to asymmetric complex signaling scheme used in wireless networks [2] for most p.

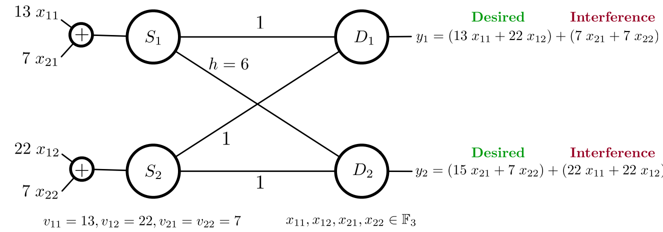

Fig. 4. An instance of the X channel over $\mathbb{F}_{3^3}$ and its capacity optimal solution represented in scalar notation.

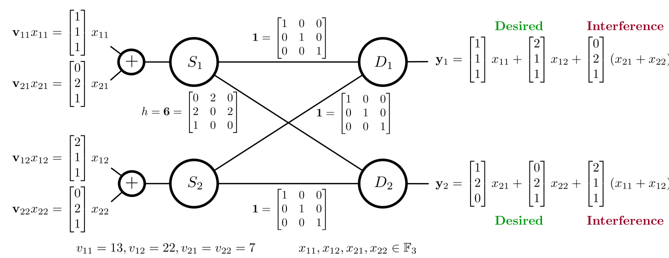

Fig. 5. The same example and solution as Fig. 4, illustrated in vector notation.

VII. X Channel over $\mathbb{F}_{p^2}$

$\mathbb{F}_{p^2}$ can be viewed as a 2-dimensional vector space over subfield $\mathbb{F}_p$, much like the field of complex numbers can be viewed as a 2-dimensional vector space over reals ($\mathbb{R}$), which is also the essential idea behind the asymmetric complex signaling scheme used in [2] to achieve $4/3$ DoF for the constant SISO wireless X channel with complex coefficients. Specifically, we can represent each element of $\mathbb{F}_{p^2}$ as

$$z = x + y\sqrt{c} \quad \text{or} \quad x + ys \tag{18}$$

wherein $z \in \mathbb{F}_{p^2}$, $x, y \in \mathbb{F}_p$ and c is a quadratic non-residue (an element of $\mathbb{F}_p$ that does not have a square root in $\mathbb{F}_p$) similar to -1 (which does not have a square root over reals) in the field of complex numbers. ($s = \sqrt{c} \equiv j$).

For example, consider $\mathbb{F}_{3^2}$ defined with irreducible polynomial s^2+1 which has $c = -1(\text{mod } 3) = 2$ as the quadratic non-residue, since $\sqrt{2} \notin \mathbb{F}_3$. Field $\mathbb{F}_{3^2}$ contains 9 elements and every element $a_1 s + a_0$ can be written in a vector notation with coefficients $[a_1; a_0]$ wherein $a_1, a_0 \in \mathbb{F}_3$ $= \{0, 1, 2\}$ and assigned a scalar integer label $\{0, 1, \ldots, 8\}$ as $3a_1 + a_o$. For example, the field element labeled $a = 7$ can be represented as $[2\ ;\ 1]$ in vector notation, as $2s{+}1$ in polynomial notation, or as $2\sqrt{2}+1$ in the quadratic non-residue notation. Here, h can be represented as a 2×2 MIMO channel. Let $h = h_1 s + h_0, \quad x = x_1 s + x_0$ wherein $h_i, x_i \in \mathbb{F}_3$. Then the product $y = hx \in \mathbb{F}_{3^2}$ can be written as

$$\begin{aligned} y = hx = (h_1 s + h_0)(x_1 s + x_0) = \\ s^2(h_1 x_1) + s(h_1 x_0 + h_0 x_1) + (h_0 x_0) \end{aligned} \tag{19}$$

and in vector notation as

$$\mathbf{y} = \mathbf{H}\mathbf{x} = \begin{bmatrix} h_0 & 2h_1 \\ h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

wherein $\mathbf{x} \in \mathbb{F}_3^{2\times 1}$ and $\mathbf{H} \in \mathbb{F}_3^{2\times 2}$. It can be noted that above 2×2 linear transformation is equivalent to complex multiplication and stacking the resulting real and imaginary parts in a 2×1 vector.

Theorem 3: The X channel over $\mathbb{F}_{p^2}$ has capacity $\frac{4}{3}$ if

$$\mathbf{h} = \frac{h_{12}h_{21}}{h_{11}h_{22}} \notin \mathbb{F}_p \tag{20}$$

Proof: Since the outerbound is $\frac{4}{3}$ as before, we focus on the achievability. We consider the X channel with 3 symbol extensions, wherein we can represent the channel between source i and destination j as $H_{ji} = h_{ji} I_3$ where I_3 is the 3×3 identity matrix and h_{ji} is the scalar channel coefficient from $\mathbb{F}_{p^2}$. The inputs are represented as $x_{ji} \in \mathbb{F}_p$ and outputs as $y_j \in \mathbb{F}_{p^2}$, and three channel uses over $\mathbb{F}_{p^2}$ can be seen as a 6 dimensional vector space over $\mathbb{F}_p$ within which 4 desired symbols and 4 interference symbols are present at each destination. In order to achieve capacity, interference should be aligned within 2 dimensions at each destination. To this end, we will construct beamforming vectors at each source such that interference is aligned. Received symbols at the destinations, in vector notation, are given by

$$\begin{aligned} \mathbf{Y}_1 &= \mathbf{V}_{11}X_{11} + \mathbf{V}_{12}X_{12} + \mathbf{V}_{22}X_{22} + \mathbf{V}_{21}X_{21} \\ \mathbf{Y}_2 &= \mathbf{V}_{22}X_{22} + \bar{\mathbf{H}}\mathbf{V}_{21}X_{21} + \bar{\mathbf{H}}\mathbf{V}_{11}X_{11} + \mathbf{V}_{12}X_{12} \end{aligned}$$

Here $\mathbf{Y}_j \in \mathbb{F}_p^{6\times 1}$, $\mathbf{V}_{ji} \in \mathbb{F}_p^{6\times 2}$ is the beamforming matrix chosen so as to satisfy alignment constraints, $X_{ji} \in \mathbb{F}_p^{2\times 1}$ represents the input symbol vector. $\bar{\mathbf{H}} \in \mathbb{F}_p^{6\times 6}$ is the linear transformation that is equivalent to multiplication by hI_3 wherein $h \in \mathbb{F}_{p^2}$. Over 3 symbol extensions of the channel, linear transformation $\bar{\mathbf{H}}$ for $p = 3$, is given by $[h_0 I_3 \ \ 2h_1 I_3; \ \ h_1 I_3 \ \ h_0 I_3]$ which is the symbol extended version of the linear transformation $H = [h_0 \ \ 2h_1; h_1 \ \ h_0]$, that was discussed earlier. In order to achieve capacity, interference should be aligned at both destinations:

$$\text{span}(\mathbf{V}_{22}) \equiv \text{span}(\mathbf{V}_{21}) \quad \& \quad \text{span}(\mathbf{V}_{12}) \equiv \text{span}(\bar{\mathbf{H}}\mathbf{V}_{11}) \tag{21}$$

For every choice of $\mathbf{V}_{21}, \mathbf{V}_{11}$, we set

$$\mathbf{V}_{22} = \mathbf{V}_{21} \qquad \& \qquad \mathbf{V}_{12} = \bar{\mathbf{H}}\mathbf{V}_{11} \tag{22}$$

to ensure interference alignment. Choices for $\mathbf{V}_{11}, \mathbf{V}_{21}$ and linear independence proof are discussed in full version of the paper - [12], that establishes the capacity result. ■

For all other field extensions $\mathbb{F}_{p^n}$ $(n > 3)$, full version of this work - [12] provides linear independence proofs, establishing that the finite field X channel has capacity of 4/3 when $h \notin \mathbb{F}_p$. In the full paper, interesting parallels between p and SNR & n and channel diversity, are discussed.

VIII. Conclusions

We explored capacity results for the finite field X channel, translating precoding based interference alignment schemes from corresponding DoF results for the wireless setting. The main insight is that the finite field $\mathbb{F}_{p^n}$ can be viewed as analogous to the $n \times n$ MIMO wireless setting, albeit with some structure imposed on the channels. We also described the conditions on channel coefficients under which the linear beamforming scheme would fail.

IX. Acknowledgements

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