Optimal Multiresolution Quantization with Error Detecting Codes for Broadcast Channels

James Ho and En-hui Yang
Department of Electrical and Computer Engineering
University of Waterloo, Waterloo, Ontario, N2L 3G1
Email: james.ho@uwaterloo.ca, ehyang@uwaterloo.ca

***Abstract*—This paper investigates the design of optimal vector quantization given channel and error statistics by inclusion of cyclic redundancy checks (CRC) into the consideration. Given a broadcast system with multiresolution vector quantization (MRVQ) and error detection availability for each resolution, a closed-form formula for the weighted end-to-end distortion (EED) is first derived under a random index assignment. Based on the closed-form formula, an iterative algorithm is then proposed for designing optimal MRVQ to minimize the EED with CRC. Experiments show that for a wide range of channel error probability, the inclusion of CRC indeed reduces the EED. Finally, the best tradeoff between the number of bits for quantization and those for CRC is also investigated by experiments.**

I. Introduction

Consider the design of multiresolution vector quantizers (MRVQ) that process real-valued sources for transmission over a discrete memoryless point-to-point or broadcast channel with non-zero symbol crossover probabilities. Under joint source-channel coding (JSCC), quantization design can be tailored specifically to the given noisy channel's statistics. These types of quantizers are well-known in the literature as channel-optimized quantizers, or noisy channel quantizers.

Noisy channel quantizers for the scalar ([1], [2]) and vector ([3], [4]) cases have been previously considered, where algorithms were proposed to design optimal noisy channel quantizers for a given fixed index assignment. In all of these early works, these quantizers were experimentally demonstrated to outperform those designed via Lloyd-Max. The work in [5] further considered the joint design of noisy channel quantizers with source-optimized channel codes. For broadcast channels given fixed energy constraints, [6] jointly designed multiresolution source quantization and superposition channel coding (SPC) to minimize the end-to-end distortion (EED).

In contrast to the aforementioned works, [7] and [8] considered a JSCC system under the assumption of random index assignment (RIA). By doing so, theoretical analysis became tractable to derive a closed-form formulae of the EED, which reveal that a potentially large portion of the EED comes from a structural factor known as the scatter factor. This portion is different from and in addition to the quantization distortion. Hence, the traditional separate quantization design, where only the quantization distortion is minimized, is suboptimal. To minimize the EED, optimal noisy channel quantizer design has to balance distortion contributions from both the scatter factor and the quantization itself. It was shown in [7] that by doing so, a significant reduction in the EED in comparison with the traditional separate quantization design can be obtained.

Although optimal noisy channel quantizer design can alleviate to some degree the distortion contribution to the EED from the scatter factor at the expense of larger quantization distortion, it cannot entirely eliminate the effect of the scatter factor on the EED. An interesting question then naturally arises: is there any other way to largely reduce or even eliminate the effect of the scatter factor on the EED so as to further improve the end to end performance?

Motivated by the above question, we observe from the EED that the scatter factor's contribution is attributed to source reconstruction to incorrect symbols caused by *undetected* symbol errors. Therefore, if the decoder is somehow informed of whether or not its received symbol is correct, it can improve the EED performance by declaring an erasure state for incorrect symbols and selecting the source mean for reconstruction. This way, the decoder limits the distortion for incorrect symbols at no more than the source variance and completely gets rid of the impact from the scatter factor. Such an observation leads us to consider the inclusion of error detection via cyclic redundancy checks (CRC) into the system.

With the inclusion of error detection into the system, the above argument is certainly oversimplified. First, the inclusion of CRC would reduce the number of bits for quantization and hence, increases quantization distortion. Second, CRC is hardly perfect and as a result, the portion of the EED arising from false negative incorrect symbols is still present, even though it may now be much smaller. Hence, there is an interesting tradeoff between quantization and CRC design to minimize the EED of the CRC-coded system.

This paper targets the design of optimal MRVQ with CRC in tandem with broadcast channels under RIA to transmit real-valued sources. A closed-form expression is derived for the weighted EED of a tandem system of MRVQ, RIA, CRC, and SPC. Further, necessary conditions for minimizing the weighted EED are derived. Experimental results are conducted to demonstrate the possible reductions to EED due to CRC. The rest of the paper is organized as follows. Section II provides a brief derivation of the weighted EED formula. Section III details the two necessary optimality conditions for MRVQ design given CRC-coded channel statistics. Section IV is an analysis of optimal MRVQ/CRC design through experiments. Finally, Section V concludes the paper.

II. End-To-End Distortion

A. System and Notation

Suppose $\boldsymbol{z}$ is a k-dimensional real-valued vector source over the Euclidean space Λ with a probability density function $f(\boldsymbol{z})$, zero mean, and variance per dimension denoted as $\sigma^2 = \frac{1}{k}\int_\Lambda \|\boldsymbol{z}\|^2 f(\boldsymbol{z})d\boldsymbol{z}$. Consider transmitting $\boldsymbol{z}$ as a scalably encoded two-resolution source over the tandem source-channel coding broadcast system depicted in Fig. 1. For the lower resolution, the quantizer partitions Λ into N_1 disjoint regions denoted by $\{A_1, \cdots, A_{N_1}\}$, and represents them with respective codeword vectors $\{\boldsymbol{z}_1, \cdots, \boldsymbol{z}_{N_1}\}$. For the higher resolution, the quantizer further partitions each of the N_1 regions into N_2 subregions denoted by $\{A_{i1}, \cdots, A_{iN_2}\}$, and represents them with respective codeword vectors $\{\boldsymbol{z}_{i1}, \cdots, \boldsymbol{z}_{iN_2}\}$. Let $i = 1, \cdots, N_1$ and $j = 1, \cdots, N_2$ index the lower and higher resolution codeword vectors, respectively. The transmitted scalable coded source $\boldsymbol{z}$ is then represented by the index pair (i, j), where the first receiver attempts to reconstruct $\boldsymbol{z}$ at a higher resolution using both i and j while the second receiver only desires a lower resolution reconstruction using i.

Let $\pi_t(i, j) = (\pi_{tb}(i), \pi_{te}(j|i)) = (r, s)$ be a particular index assignment linking the multiresolution source encoder output (i, j) with the CRC-coded broadcast channel input (r, s) in a one-to-one mapping manner such that $i \in \{1, \cdots, N_1\} = \mathfrak{m}_{\rm b}$ and $j \in \{1, \cdots, N_2\} = \mathfrak{m}_2$ map to $r \in \mathfrak{m}_{\rm b}$ and $s \in \mathfrak{m}_2$, respectively. Further let $\hat{\mathfrak{m}}_{\rm b} = \mathfrak{m}_{\rm b} \cup {\rm e}_1$, and $\hat{\mathfrak{m}}_2 = \mathfrak{m}_2 \cup {\rm e}_2$, where ${\rm e}_1$ and ${\rm e}_2$ denote the erasure states for r and s.

The CRC-coded broadcast channel takes $(r, s) \in \mathfrak{m}_{\rm b} \times \mathfrak{m}_2 = \mathfrak{m}_{\rm e}$ as input and outputs $\hat{\mathfrak{m}}_e = (\hat{r}, \hat{s}) \in \hat{\mathfrak{m}}_{\rm b} \times \hat{\mathfrak{m}}_2$ to the first receiver and $\hat{\mathfrak{m}}_b = \hat{\hat{r}} \in \hat{\mathfrak{m}}_{\rm b}$ to the second receiver. The entire CRC-coded broadcast channel is hence fully characterized by a matrix of transitional probabilities

$$\{p(\hat{\mathfrak{m}}_e, \hat{\mathfrak{m}}_b|(r, s)) : (r, s) \in \mathfrak{m}_{\rm e}, \hat{\mathfrak{m}}_b \in \hat{\mathfrak{m}}_{\rm b}, \hat{\mathfrak{m}}_e \in \hat{\mathfrak{m}}_{\rm b} \times \hat{\mathfrak{m}}_2\},$$

where $p(\hat{\mathfrak{m}}_e, \hat{\mathfrak{m}}_b|(r, s))$ is the conditional probability that the CRC-coded broadcast channel outputs $\hat{\mathfrak{m}}_e$ and $\hat{\mathfrak{m}}_b$ given the input (r, s). From this matrix, transitional probability matrices

$$\{p_e(\hat{\mathfrak{m}}_e|(r, s)) : (r, s) \in \mathfrak{m}_{\rm e}, \hat{\mathfrak{m}}_e \in \hat{\mathfrak{m}}_{\rm b} \times \hat{\mathfrak{m}}_2\},$$
$$\{p_b(\hat{\mathfrak{m}}_b|(r, s)) : (r, s) \in \mathfrak{m}_{\rm e}, \hat{\mathfrak{m}}_b \in \hat{\mathfrak{m}}_{\rm b}\}$$

can be further derived to describe the channels for each of the two receivers. In presence of CRC, let $p_{bd} = \Pr\{\hat{\hat{r}} = {\rm e}_1\}$ and $p_{bu} = \Pr\{\hat{\hat{r}} \neq r, \hat{\hat{r}} \neq {\rm e}_1\}$ be the respective detected and undetected error probabilities of the second receiver. The first receiver is associated with five error probabilities based on error detection states in r and s: (i) $p_{d_1} = \Pr\{\hat{r} = {\rm e}_1\}$; (ii) $p_{d_2} = \Pr\{\hat{r} = r, \hat{s} = {\rm e}_2\}$; (iii) $p_{ud} = \Pr\{\hat{r} \neq r, \hat{r} \neq {\rm e}_1, \hat{s} = {\rm e}_2\}$; (iv) $p_{u_1} = \Pr\{\hat{r} \neq r, \hat{r} \neq {\rm e}_1, \hat{s} \neq {\rm e}_2\}$; and (v) $p_{u_2} = \Pr\{\hat{r} = r, \hat{s} \neq s, \hat{s} \neq {\rm e}_2\}$. All seven error probabilities are computed under the assumption that (r, s) is uniformly distributed over $\mathfrak{m}_{\rm e}$, where $|\mathfrak{m}_{\rm e}| = N_1 N_2$. As an example for the second receiver,

$$p_{bu} = \frac{1}{N_1}\frac{1}{N_2}\sum_{r=1}^{N_1}\sum_{s=1}^{N_2}\sum_{\substack{\hat{\hat{r}}=1,\\ \hat{\hat{r}}\neq r,{\rm e}_1}}^{N_1} p_b\left\{\hat{\hat{r}}|r, s\right\},$$

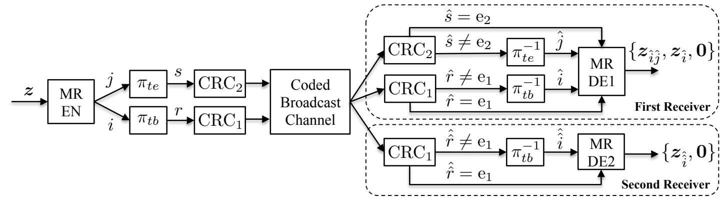

Fig. 1. A tandem source-channel CRC-coded broadcast system.

and for the first receiver,

$$p_{u_1} = \frac{1}{N_1}\frac{1}{N_2}\sum_{r=1}^{N_1}\sum_{s=1}^{N_2}\sum_{\hat{r}=1,\hat{r}\neq r,{\rm e}_1}^{N_1}\sum_{\hat{s}=1,\hat{s}\neq{\rm e}_2}^{N_2} p_e\left\{\hat{r}, \hat{s}|r, s\right\}.$$

With reference to Fig. 1, CRC introduces per-symbol erasure states for r and s at the source decoder input of both receivers. Hence, upon receiving $(\hat{r}, \hat{s})$, the first receiver has three possible outputs as follows: $\boldsymbol{z}_{\hat{i}\hat{j}}$ if $\hat{r} \neq {\rm e}_1$, $\hat{s} \neq {\rm e}_2$; $\boldsymbol{z}_{\hat{i}}$ if $\hat{r} \neq {\rm e}_1$, $\hat{s} = {\rm e}_2$; and the source mean of zero if $\hat{r} = {\rm e}_1$. Similarly, upon receiving $\hat{\hat{r}}$, the second receiver outputs $\boldsymbol{z}_{\hat{\hat{i}}}$ if $\hat{\hat{r}} \neq {\rm e}_1$ and the source mean of zero if $\hat{\hat{r}} = {\rm e}_1$. Given index assignment π_t, the crossover error probabilities from codeword vectors $\boldsymbol{z}_{ij}$ to the outputs of the two receivers are related to the channel transitional error probabilities as follows:

$$p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}\hat{j}}|\boldsymbol{z}_{ij}) = p_e\{\hat{r}, \hat{s}|r, s\}, \hat{r} \neq {\rm e}_1, \hat{s} \neq {\rm e}_2;$$
$$p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}}|\boldsymbol{z}_{ij}) = p_e\{\hat{r}, {\rm e}_2|r, s\}, \hat{r} \neq {\rm e}_1;$$
$$p_e^{\pi_t}(\boldsymbol{0}|\boldsymbol{z}_{ij}) = p_e\{\hat{r} = {\rm e}_1|r, s\};$$
$$p_b^{\pi_t}(\boldsymbol{z}_{\hat{\hat{i}}}|\boldsymbol{z}_{ij}) = p_b\{\hat{\hat{r}}|r, s\}, \hat{r} \neq {\rm e}_1;$$
$$p_b^{\pi_t}(\boldsymbol{0}|\boldsymbol{z}_{ij}) = p_b\{\hat{\hat{r}} = {\rm e}_1|r, s\}.$$

The EED is defined as the mean squared error distortion between the encoder input and the decoder output subject to each of the above codeword crossover error probabilities. Given index assignment π_t, the EED for the first receiver is defined as $D_e^{\pi_t} \triangleq D_{e_1}^{\pi_t} + D_{e_2}^{\pi_t} + D_{e_3}^{\pi_t}$, where

$$D_{e_1}^{\pi_t} \triangleq \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i},\hat{j}}\|\boldsymbol{z} - \boldsymbol{z}_{\hat{i}\hat{j}}\|^2 p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}\hat{j}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z} \quad (1)$$

$$D_{e_2}^{\pi_t} \triangleq \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i}}\|\boldsymbol{z} - \boldsymbol{z}_{\hat{i}}\|^2 p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z} \quad (2)$$

$$D_{e_3}^{\pi_t} \triangleq \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\|\boldsymbol{z}\|^2 p_e^{\pi_t}(\boldsymbol{0}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}. \quad (3)$$

Similarly, the EED for the second receiver is defined as

$$D_b^{\pi_t} \triangleq \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{\hat{i}}}\|\boldsymbol{z} - \boldsymbol{z}_{\hat{\hat{i}}}\|^2 p_b^{\pi_t}(\boldsymbol{z}_{\hat{\hat{i}}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}$$
$$+ \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\|\boldsymbol{z}\|^2 p_b^{\pi_t}(\boldsymbol{0}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}. \quad (4)$$

B. End-to-End Distortion with Imperfect Error Detection

We assume random index assignment such that one of the $(N_1!)(N_2!)^{N_1}$ possible mappings for π_t are selected randomly and uniformly. Let D_e^Π and D_b^Π respectively denote the EED for the first and second receiver averaged over all possible assignments.

Theorem 1: For any k-dimensional multiresolution quantizer in tandem with a coded broadcast channel and imperfect error detection as shown in Fig. 1,

$$D_b^{\Pi} = \left(1 - p_{bd} - \frac{N_1 p_{bu}}{N_1 - 1}\right) D_{Q_b} + \left(\frac{N_1 p_{bu}}{N_1 - 1}\right)\left(\sigma^2 + S_{Q_b}\right) + p_{bd}\sigma^2 \tag{5}$$

$$\begin{aligned} D_e^{\Pi} = &\left(1 - p_{u_1} - \frac{N_2 p_{u_2}}{N_2 - 1} - p_{d_1} - p_{d_2} - p_{ud}\right) D_{Q_e} \\ &+ \left(p_{d_2} - \frac{p_{ud}}{N_1 - 1}\right) D_{Q_b} + \frac{N_1 p_{u_1}}{N_1 - 1}\left(\sigma^2 + S_{Q_e}\right) \\ &+ \left(\frac{N_2 p_{u_2}}{N_2 - 1} - \frac{p_{u_1}}{N_1 - 1}\right)\left(\bar{\sigma}^2_{Q_e} + \bar{S}_{Q_e}\right) \\ &+ \frac{N_1 p_{ud}}{N_1 - 1}(\sigma^2 + S_{Q_b}) + p_{d_1}\sigma^2, \end{aligned} \tag{6}$$

where

$$D_{Q_e} \triangleq \frac{1}{k}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_{\boldsymbol{z}\in A_{ij}} \|\boldsymbol{z} - \boldsymbol{z}_{ij}\|^2 f(\boldsymbol{z})d\boldsymbol{z},$$

$$D_{Q_b} \triangleq \frac{1}{k}\sum_{i=1}^{N_1}\int_{\boldsymbol{z}\in A_i} \|\boldsymbol{z} - \boldsymbol{z}_i\|^2 f(\boldsymbol{z})d\boldsymbol{z},$$

$$S_{Q_e} \triangleq \frac{1}{kN_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\|\boldsymbol{z}_{ij}\|^2,$$

$$\bar{S}_{Q_e} \triangleq \sum_{i=1}^{N_1}\Pr\{\boldsymbol{z}\in A_i\}\left(\frac{1}{kN_2}\sum_{\hat{j}=1}^{N_2}\|\boldsymbol{z}_{i\hat{j}} - \boldsymbol{y}_i\|^2\right),$$

$$S_{Q_b} \triangleq \frac{1}{kN_1}\sum_{i=1}^{N_1}\|\boldsymbol{z}_i\|^2,$$

$$\bar{\sigma}^2_{Q_e} \triangleq \sum_{i=1}^{N_1}\Pr\{\boldsymbol{z}\in A_i\}\sigma_i^2,$$

$\boldsymbol{y}_i = \frac{1}{\Pr\{\boldsymbol{z}\in A_i\}}\int_{\boldsymbol{z}\in A_i}\boldsymbol{z}f(\boldsymbol{z})d\boldsymbol{z}$ is the conditional mean of $\boldsymbol{z}$ given A_i and $\sigma_i^2 = \frac{1}{k\Pr\{\boldsymbol{z}\in A_i\}}\int_{\boldsymbol{z}\in A_i}\|\boldsymbol{z} - \boldsymbol{y}_i\|^2 f(\boldsymbol{z})d\boldsymbol{z}$ is the conditional variance per dimension of $\boldsymbol{z}$ given A_i. Observe that D_{Q_b} and D_{Q_e} are the conventional quantization distortions for reconstruction at the lower and higher resolution, respectively. The scatter factors of the lower and higher resolution are respectively denoted by S_{Q_b} and S_{Q_e}, quantifying the average distance of the codeword vectors from the source mean vector $\mathbf{0}$. $\bar{S}_{Q_e}$ denotes the conditional scatter factor of the refinement coding given the set of lower resolution partitions $\{A_i\}$.

Remark 1: It is instructive to compare (5) to the scenario without error detection. From a per-symbol basis, detectable symbol errors contribute only σ^2 to the EED as opposed to $(\sigma^2 + S_{Q_b})$ for undetectable symbol errors, thus eliminating effects of the scatter factor for detectable errors. In (6), the portion of p_{d_2} symbols with a correct lower resolution index but an erasure declared for the higher resolution observe an distortion equal to the quantization distortion of the second receiver, D_{Q_b}. Hence, error detection further exploits the advantages of the coded broadcast channel by introducing an incremental lower resolution quality for the first receiver.

Proof of Theorem 1: Due to limitation of space, we only provide a sketch of the proof; for the complete proof, please refer to the full paper [9]. Take expectation of $D_e^{\pi_t}$ with respect to π_t in (1)-(3) to yield

$$\begin{aligned} D_{e_1}^{\Pi} = &\frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}} E_{\pi_t}\sum_{\hat{i},\hat{j}} p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}\hat{j}}|\boldsymbol{z}_{ij})\|\boldsymbol{z} - \boldsymbol{z}_{ij}\|^2 f(\boldsymbol{z})d\boldsymbol{z} \\ &+ \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{j}\neq j}\Big[2(\boldsymbol{z} - \boldsymbol{z}_{ij})(\boldsymbol{z}_{ij} - \boldsymbol{z}_{i\hat{j}})' \\ &\quad + \|\boldsymbol{z}_{ij} - \boldsymbol{z}_{i\hat{j}}\|^2\Big] E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{i\hat{j}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z} \\ &+ \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i}\neq i,\hat{j}}\Big[2(\boldsymbol{z} - \boldsymbol{z}_{ij})(\boldsymbol{z}_{ij} - \boldsymbol{z}_{\hat{i}\hat{j}})' \\ &\quad + \|\boldsymbol{z}_{ij} - \boldsymbol{z}_{\hat{i}\hat{j}}\|^2\Big] E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}\hat{j}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}, \end{aligned} \tag{7}$$

$$\begin{aligned} D_{e_2}^{\Pi} = &\frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\|\boldsymbol{z} - \boldsymbol{z}_i\|^2 f(\boldsymbol{z})d\boldsymbol{z}\sum_{\hat{i}} E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}}|\boldsymbol{z}_{ij}) \\ &+ \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i}\neq i}[2(\boldsymbol{z} - \boldsymbol{z}_i)(\boldsymbol{z}_i - \boldsymbol{z}_{\hat{i}})' \\ &\quad + \|\boldsymbol{z}_i - \boldsymbol{z}_{\hat{i}}\|^2\Big] E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}, \end{aligned} \tag{8}$$

$$D_{e_3}^{\Pi} = \frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\|\boldsymbol{z}\|^2 E_{\pi_t}p_e^{\pi_t}(\mathbf{0}|\boldsymbol{z}_{ij})f(\boldsymbol{z})d\boldsymbol{z}, \tag{9}$$

where the prime symbol indicates matrix transposition.

Expectation taken over π_t for each of the codeword crossover error probabilities are summarized as follows for substitution into (7)-(9). For a correct low resolution index and undetected high resolution index error such that $\hat{i} = i$ and $(\hat{j} \neq j, \hat{s} \neq \mathrm{e}_2)$, respectively,

$$E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{i\hat{j}}|\boldsymbol{z}_{ij}) = \frac{1}{N_2 - 1}p_{u_2}. \tag{10}$$

For an undetected low resolution index error such that $\hat{i} \neq i$ and $\hat{r} \neq \mathrm{e}_1$,

$$E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}\hat{j}}|\boldsymbol{z}_{ij}) = \frac{1}{N_2\left(N_1 - 1\right)}p_{u_1}. \tag{11}$$

When the lower resolution index is correct and erasure is declared for the higher resolution index such that $\hat{i} = i$ and $\hat{s} = \mathrm{e}_2$, respectively,

$$E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_i|\boldsymbol{z}_{ij}) = p_{d_2}. \tag{12}$$

When the lower resolution index is incorrect and undetected such that $\hat{i} \neq i$ and $\hat{r} \neq \mathrm{e}_1$ while an erasure is declared for the higher resolution index such that $\hat{s} = \mathrm{e}_2$,

$$E_{\pi_t}p_e^{\pi_t}(\boldsymbol{z}_{\hat{i}}|\boldsymbol{z}_{ij}) = \frac{1}{N_1 - 1}p_{ud}. \tag{13}$$

When erasure is declared for the lower resolution index,

$$E_{\pi_t}p_e^{\pi_t}(\mathbf{0}|\boldsymbol{z}_{ij}) = p_{d1}. \tag{14}$$

Substitution of (10)-(14) into (7)-(9) yields

$$\begin{aligned} D_{e_1}^{\Pi} = &\left(1 - p_{d_1} - p_{d_2} - p_{ud}\right) D_{Q_e} \\ &+ \frac{p_{u_2}}{k(N_2 - 1)}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{j}\neq j}\Big[2(\boldsymbol{z} - \boldsymbol{z}_{ij})(\boldsymbol{z}_{ij} - \boldsymbol{z}_{i\hat{j}})' \\ &\quad + \|\boldsymbol{z}_{ij} - \boldsymbol{z}_{i\hat{j}}\|^2\Big] f(\boldsymbol{z})d\boldsymbol{z} \end{aligned}$$

$$+\frac{p_{u_1}}{kN_2(N_1-1)}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i}\neq i,\hat{j}}\Big[2(\boldsymbol{z}-\boldsymbol{z}_{ij})(\boldsymbol{z}_{ij}-\boldsymbol{z}_{\hat{i}\hat{j}})' + \|\boldsymbol{z}_{ij}-\boldsymbol{z}_{\hat{i}\hat{j}}\|^2\Big]f(\boldsymbol{z})d\boldsymbol{z}, \quad (15)$$

$$D_{e_2}^{\Pi}+D_{e_3}^{\Pi}=(p_{d_2}+p_{ud})\frac{1}{k}\sum_{i,j}\int_{z\in A_{ij}}\|\boldsymbol{z}-\boldsymbol{z}_i\|^2 f(\boldsymbol{z})d\boldsymbol{z} + \frac{p_{ud}}{k(N_1-1)}\sum_{i,j}\int_{z\in A_{ij}}\sum_{\hat{i}\neq i}[2(\boldsymbol{z}-\boldsymbol{z}_i)(\boldsymbol{z}_i-\boldsymbol{z}_{\hat{i}})' + \|\boldsymbol{z}_i-\boldsymbol{z}_{\hat{i}}\|^2]\,f(\boldsymbol{z})d\boldsymbol{z}+p_{d_1}\sigma^2. \quad (16)$$

Combining (15)-(16) and simplifying leads to (6). Proof of (5) employs the same techniques and is omitted. □

III. Optimality Conditions for Multiresolution Vector Quantization Design with Error Detection

Since minimizing D_b^{Π} does not necessarily minimize D_e^{Π}, and *vise versa*, we assign certain weights to the two receiver and look to minimize a weighted EED defined as

$$\bar{D}\triangleq pD_e^{\Pi}+(1-p)D_b^{\Pi}, \quad (17)$$

where $0<p<1$ is the weight assigned to the first receiver to represent the percentage of receivers who are eligible for the higher resolution source reconstruction.

Let $(\mathfrak{A},\mathfrak{Z}_1,\mathfrak{Z}_2)$ be a triple representing a two-resolution vector quantizer employed in the system depicted in Fig. 1, where $\mathfrak{A}=\{A_{ij},1\le i\le N_1,1\le j\le N_2\}$ is the partitioning of Λ into the higher resolution regions in a way such that $\{A_i=\cup_j A_{ij}\}_{i=1}^{N_1}$ is the set of lower resolution regions, $\mathfrak{Z}_1=\{\boldsymbol{z}_i, i=1,\cdot,N_1\}$ is the set of codeword vectors respectively representing all source vectors in A_i, and $\mathfrak{Z}_2=\{\boldsymbol{z}_{ij}, i=1,\cdot,N_1, j=1,\cdots,N_2\}$ for all source vectors in A_{ij}. The design of the optimal quantizer thus has an objective function expressed as

$$\min_{\mathfrak{Z}_1,\mathfrak{Z}_2}\min_{\mathfrak{A}}\quad pD_e^{\Pi}+(1-p)D_b^{\Pi}, \quad (18)$$

where the minimization is over all possible triples of $(\mathfrak{A},\mathfrak{Z}_1,\mathfrak{Z}_2)$. The optimal solution to (18) is characterized by two necessary conditions derived from Theorem 1.

Theorem 2: Given an error detecting coded broadcast channel with transitional probabilities $p(\hat{m}_e,\hat{m}_b|(r,s))$, the optimal multiresolution vector quantizer to (18) satisfies the following two conditions:

1) Given $\{A_{ij},1\le i\le N_1,1\le j\le N_2\}$, the optimal code vectors to minimize $\bar{D}$ for the lower and higher resolution are respectively computed by

$$\boldsymbol{z}_i=\frac{\int_{\boldsymbol{z}\in A_i}\boldsymbol{z}f(\boldsymbol{z})d\boldsymbol{z}}{\frac{(1-p)p_{bu}+pp_{ud}}{(N_1-1)[(1-p)k_3+pk_4]}+\Pr\{\boldsymbol{z}\in A_i\}}, \quad i=1,\cdots,N_1, \quad (19)$$

$$\boldsymbol{z}_{ij}=\frac{k_1\int_{\boldsymbol{z}\in A_{ij}}\boldsymbol{z}f(\boldsymbol{z})d\boldsymbol{z}+k_2\int_{\boldsymbol{z}\in A_i}\boldsymbol{z}f(\boldsymbol{z})d\boldsymbol{z}}{\frac{p_{u_1}}{N_2(N_1-1)}+k_1\Pr\{\boldsymbol{z}\in A_{ij}\}+k_2\Pr\{\boldsymbol{z}\in A_i\}}, \quad i=1,\cdots,N_1, j=1,\cdots,N_2. \quad (20)$$

2) Given $\{\boldsymbol{z}_i, i=1,\cdots,N_1\}$ and $\{\boldsymbol{z}_{ij}, i=1,\cdots,N_1, j=1,\cdots,N_2\}$, the optimal higher resolution partitioning of Λ is defined as

$$A_{ij}=\{\boldsymbol{z}:2\boldsymbol{\alpha}_{ij}\boldsymbol{z}'-\boldsymbol{\beta}_{ij}\ge 2\boldsymbol{\alpha}_{i'j'}\boldsymbol{z}'-\boldsymbol{\beta}_{i'j'}\ \forall\,(i',j')\neq(i,j)\}, i=1,\cdots,N_1, j=1,\cdots,N_2 \quad (21)$$

where

$$k_1\triangleq 1-p_{u_1}-\frac{N_2p_{u_2}}{N_2-1}-p_{d_1}-p_{d_2}-p_{ud}$$

$$k_2\triangleq\frac{p_{u_2}}{N_2-1}-\frac{p_{u_1}}{N_2(N_1-1)}$$

$$k_3\triangleq 1-\frac{N_1p_{bu}}{N_1-1}-p_{bd}$$

$$k_4\triangleq p_{d_2}-\frac{p_{ud}}{N_1-1}$$

$$\boldsymbol{\alpha}_{ij}\triangleq pk_1\boldsymbol{z}_{ij}+pk_2\sum_{\hat{j}=1}^{N_2}\boldsymbol{z}_{i\hat{j}}+[(1-p)k_3+pk_4]\boldsymbol{z}_i$$

$$\boldsymbol{\beta}_{ij}\triangleq pk_1\|\boldsymbol{z}_{ij}\|^2+pk_2\sum_{\hat{j}=1}^{N_2}\|\boldsymbol{z}_{i\hat{j}}\|^2+[(1-p)k_3+pk_4]\|\boldsymbol{z}_i\|^2.$$

Remark 2: 1) and 2) suggest a gradient descent iterative algorithm. Begin with an initial multiresolution quantizer $Q^0=(\mathfrak{A}^0,\mathfrak{Z}_1^0,\mathfrak{Z}_2^0)$. Then for each iteration $n>0$, alternate between computing $\mathfrak{Z}_1^{n+1}$ and $\mathfrak{Z}_2^{n+1}$ according to (19) and (20) based on $\mathfrak{A}^n$, followed by computing $\mathfrak{A}^{n+1}$ according to (21) based on $\mathfrak{Z}_1^{n+1}$ and $\mathfrak{Z}_2^{n+1}$. Continue until the decrease in weighted EED between two consecutive iterations fall below a defined threshold. Experimental results in the next section will employ quantizers designed in this manner.

Proof of Theorem 2: The proof follows the same procedure taken in [8] and is hence omitted. □

IV. Experiments for Gaussian Broadcast Channel

Experiments were conducted to study the tradeoff between source quantization and error detection performance. We considered transmission of a one-dimensional Gaussian source with zero mean and unit variance over a coded broadcast channel using superposition coding (SPC), which can be implemented using layered modulation, a technique well-studied in the literature [10]-[12] and defined in a number of standards such as DVB-T [13] and UMB [14].

Suppose each multiresolution source symbol is transmitted over two blocks of a standard 16/64-QAM hierarchical modulation as defined in [10] for a block size of 12 bits, where $b_1=8$ and $b_2=4$ bits are respectively available for the lower and higher resolution indices. Let n_1 and l_1 respectively denote the number of data and redundancy bits for the lower resolution index, where $n_1+l_1=b_1$. Similarly, let n_2 and l_2 denote the same for the higher resolution index with $n_2+l_2=b_2$. Hence, for a particular pair of CRC polynomials, we have $N_1=2^{n_1}$ and $N_2=2^{n_2}$ available for multiresolution quantizer design, while l_1 and l_2 are selected from Table I, a list of considered polynomials of various lengths consisting of one polynomial with the best performance for each CRC size based on [15].

Suppose Karnaugh map style Gray mapping is employed to map each channel symbol to a bit stream. The matrix of

TABLE I
CONSIDERED CRC POLYNOMIALS AND LENGTHS FOR LOWER (l_1) OR HIGHER (l_2) RESOLUTION INDEX

l_1 or l_2	Nickname	Polynomial
0	-	`0x00` = 1
1	CRC-1/parity	`0x01` = $(x+1)$
3	-	`0x05` = (x^3+x+1)
4	CCITT-4	`0x09` = (x^4+x+1)
5	CRC-5/USB	`0x12` = (x^5+x^2+1)
6	CRC-6/DARC	`0x2c` = $(x+1)(x^5+x^4+x^2+x+1)$

transitional probabilities $p(\hat{m}_e, \hat{m}_b|(r,s))$ under CRC is then computable, from which p_{bd}, p_{bu}, p_{d_1}, p_{d_2}, p_{ud}, p_{u_1}, and p_{u_2} can be exactly evaluated. Given these seven error probabilities that govern the entire CRC-coded broadcast channel as well as N_1 and N_2, we can apply the iterative algorithm briefed in Remark 2 to design the multiresolution vector quantizer.

Performance gains are presented in terms of PSNR gains for $p = 0.5$ to quantify the increase in PSNR when a tradeoff is considered between data and redundancy bits instead of the allocation of all bits for data in the case of no error detection. In Table II, a number of channel SNR combinations (γ_1, γ_2) are considered for the first and second receivers experiencing AWGN power $\frac{N_0}{2}$, where γ_1 and γ_2 respectively denote the SNRs of the first and second receiver. For each (γ_1, γ_2), we consider all combinations of CRC_1 and CRC_2, which denote the CRC polynomials selected from Table I for the lower and higher resolution indices, respectively. The combination of CRC_1 and CRC_2 that achieves the largest PSNR gain, as well as the quantity of the gain itself, are listed in Table II. Note the definition of PSNR $= 10\log_{10}(\sigma^2/\bar{D})$.

Experimental results presented in Table II reveal some insight of the performance gains achievable with the reallocation of some source bits for CRC error detection. In general, inclusion of CRC error detection provides gains, which may even be significant depending on the channel conditions of the two receivers. For example, it is observed that PSNR gains are as high as 1.92 dB when $(\gamma_1, \gamma_2) = (24, 22)$.

The overall trend of the PSNR gains may actually be somewhat surprising at first glance; as the channel condition of the second receiver improves, one would expect the gains yielded from CRC to decrease. This is in contrast to the actuality summarized in Table II, where gains from employing CRC actually *increases* as γ_2 increases. This counterintuitive behaviour can be attributed to the scatter factors of MRVQ design for noisy channels with non-zero error probabilities. Consider (5) for the case of no error detection such that $p_{bd} = 0$ and $p_{bu} > 0$. As γ_2 increases, p_{bu} decreases, hence reducing the effects of S_{Q_b} on D_b^{II}. This in turn allows the optimal MRVQ to tradeoff a larger S_{Q_b} for lower D_{Q_b} in minimizing D_b^{II}. Since the gains enabled with CRC capability originate from eliminating the scatter factor's contribution to EED for detectable symbol errors, a larger S_{Q_b} naturally produces larger performance gains once CRC is implemented. With the scatter factor's effects largely eliminated, the optimal MRVQ further minimizes quantization distortion and hence produces more PSNR gains.

TABLE II
ACHIEVED PSNR GAINS [dB] WITH CRC_1 AND CRC_2 UNDER VARIOUS CHANNEL SNR FOR FIRST (γ_1) AND SECOND (γ_2) RECEIVER

	$\gamma_1 = 20$ dB			$\gamma_1 = 24$ dB		
γ_2 [dB]	Gains	CRC_1	CRC_2	Gains	CRC_1	CRC_2
8	0.3872	`0x09`	`0x05`	0.3533	`0x09`	`0x05`
10	0.5102	`0x09`	`0x05`	0.4834	`0x09`	`0x05`
12	0.6264	`0x05`	`0x05`	0.6058	`0x05`	`0x05`
14	0.7293	`0x05`	`0x05`	0.7166	`0x05`	`0x05`
16	0.8289	`0x05`	`0x05`	0.8270	`0x05`	`0x05`
18	**0.9388**	`0x05`	`0x05`	0.9576	`0x05`	`0x05`
20	-	-	-	1.2536	`0x01`	`0x05`
22	-	-	-	**1.9088**	`0x01`	`0x05`

V. CONCLUSION

This paper investigated optimal multiresolution quantization design over CRC-coded broadcast channels, based upon which a closed-form formula for end-to-end distortion was derived and applied to study the achievable gains of employing error detecting codes. Experimental results from a designed iterative algorithm demonstrated significant reductions to end-to-end distortion when striking a compromise between data and redundancy bits. Unexpectedly, larger gains are yielded from employing CRC under better channel conditions as long as the channel exhibits non-zero symbol crossover probabilities.

REFERENCES

[1] A. Kurtenbach and P. Wintz, "Quantizing for noisy channels," *IEEE Trans. Commun. Technol.*, vol. COM-17, pp. 291-302, Apr. 1969.

[2] N. Farvardin and V. Vaishampayan, "Optimal Quantizer Design for Noisy Channels: an Approach to Combined Source-Channel Coding," *IEEE Trans. Inf. Theory*, vol. IT-33, pp. 827-838, Nov. 1987.

[3] N. Farvardin and V. Vaishampayan, "On the performance and complexity of channel-optimized vector quantizers," *IEEE Trans. Inf. Theory*, vol. 37, no. 1, pp. 155-160, Jan. 1991.

[4] H. Kumazawa *et al.*, "A construction of vector quantizers for noisy channels," Electron. Eng. Japan, vol. 67-B, pp. 39-47, 1984.

[5] A. Goldsmith and M. Effros, "Joint Design of Fixed-Rate Source Codes and Multiresolution Channel Codes," *IEEE Trans. Commun.*, vol. 46, no. 10, pp. 1301-1312, Oct. 1998.

[6] I. Kozintsev and K. Ramchandran, "Robust Image Transmission Over Energy-Constrained Time-Varying Channels Using Multiresolution Joint Source-Channel Coding," *IEEE Trans. Signal Process.*, vol. 46, no. 4, pp. 1012-1026, Apr. 1998.

[7] X. Yu, H. Wang, and E.-H. Yang, "Design and analysis of optimal noisy channel quantization with random index assignment," *IEEE Trans. Inf. Theory*, vol. 56, no. 11, pp. 5796-5804, Nov. 2010.

[8] F. Teng, E.-H. Yang, and X. Yu, "Optimal Multiresolution Quantization for Broadcast Channels with Random Index Assignment," *Proc. IEEE Intern. Symp. Inf. Theory*, Austin, TX, Jun. 2010, pp. 181-185.

[9] J. Ho and E. Yang, "Designing Optimal Multiresolution Quantizers with Error Detecting Codes," submitted to *IEEE Trans. Wireless Commun.*

[10] P. Vitthaladevuni and M. Alouini, "A Recursive Algorithm for the Exact BER Computation of Generalized Hierarchical QAM Constellations," *IEEE Trans. Inf. Theory*, vol. 49, no. 1, pp. 297-307, Jan. 2003.

[11] M. Scagliola *et al.*, "A Dirty Paper Scheme for Hierarchical OFDM," *Proc. IEEE Intern. Conf. on Commun.*, Kyoto, Japan, Jun. 2011.

[12] L. Cai *et al.*, "Scalable Modulation for Video Transmission in Wireless Networks," *IEEE Trans. Vehicular Tech.*, vol. 60, no. 9, pp. 4314-4323, Nov. 2011.

[13] *DVB-T standard: ETS 300 744, Digital Broadcast Systems for Television, Sound and Data Services: Framing Structure, Channel Coding and Modulation for Digital Terrestrial Television*, ETSI Draft, vol. 1.2.1, no. EN300 744, 1999-1.

[14] *Physical Layer for Ultra Mobile Broadband (UMB) Air Interface Specification*, TSG-C, 3GPP2 C.S0084-001-0 v3.0, 2008.

[15] P. Koopman and T. Chakravarty, "Cyclic Redundancy Code (CRC) polynomial selection for embedded networks," *Proc. Intern. Conf. Dependable Systems and Networks*, Florence, Italy, Jun. 2004, pp. 145-154.