The Symmetric Ergodic Capacity of Phase-Fading Interference Channels to within a Constant Gap: 3 Users in the Strong and Very Strong Regimes

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***Abstract*—Consider a three-user time-varying Gaussian interference channel under uniform phase fading. Prior work on ergodic interference alignment has shown that each user can simultaneously achieve half its interference-free rate. In this paper, we combine ideas from ergodic alignment and compute-and-forward to characterize the symmetric ergodic capacity of this channel in the strong and very strong regimes to within a constant gap.**

I. Introduction

The capacity region of the K-user Gaussian interference channel is a long-standing open problem. While for $K = 2$ users the capacity region is now known to within one bit [1], the $K > 2$ case has proven considerably more challenging, due to the possibility of interference alignment [2], [3]. Prior work has characterized the approximate capacity for symmetric channel gains [4], [5] and many-to-one channels [6]. Part of the difficulty arises from the fact that, for static channels, the capacity fluctuates rapidly with respect to the channel gains, ultimately leading to discontinuities in the degrees-of-freedom [7], [8]. On the other hand, for time-varying channels, interference alignment admits a much simpler form. For instance, ergodic interference alignment permits each user to achieve half its interference-free rate at any signal-to-noise ratio (SNR) [9]. In this paper, we make progress towards characterizing the symmetric capacity of three-user time-varying interference channels to within a constant gap.

As shown in [10], ergodic alignment achieves the capacity region of phase-fading interference channels if the channel is in a bottleneck state, i.e., each receiver sees an interferer with equal strength to its desired signal. Here, we relax this constraint and examine the setting where each receiver sees interference with strength greater than or equal to its desired signal, corresponding to the strong and very strong regimes (whose capacity regions are known in the two-user case [11], [12], [13]). In this setting, ergodic alignment alone does not suffice. For instance, if all interferers are much stronger than desired signals then each receiver can easily decode and remove the interference prior to recovering its desired message at the full interference-free rate.

Our scheme combines ideas from ergodic alignment and compute-and-forward in a manner similar to the computation alignment scheme proposed in [14] for multi-hop networks. The goal of the alignment phase is to create subchannels that are well-suited to lattice interference alignment. Over these subchannels, we transmit lattice codewords which are decoded at the receiver using compute-and-forward techniques similar to those proposed for symmetric interference channels in [5]. Note that, unlike [5], our capacity approximation is up to a universal constant gap, owing to the fact that we average out rate fluctuations across different realizations of the phases across time.

II. Problem Setup

Consider the 3-user time-varying Gaussian interference channel. Each transmitter (indexed by $k = 1, 2, 3$) has a message ω_k that is generated independently and uniformly over $\{1, 2, \ldots, 2^{TR_{\text{SYM}}}\}$. Each transmitter has an encoding function $\mathcal{E}_k : \{1, 2, \ldots, 2^{TR_{\text{SYM}}}\} \to \mathbb{C}^T$ that maps its message ω_k into a sequence of T channel inputs $x_k[1], \ldots, x_k[T]$ satisfying an average power constraint $\frac{1}{T}\sum_{t=1}^{T} |x_k[t]|^2 \le P$.

Each receiver (indexed by $m = 1, 2, 3$) aims to recover its message ω_m. The channel output $y_m[t]$ at receiver m is

$$y_m[t] = \sum_{k=1}^{3} h_{m,k}[t] x_k[t] + z_m[t] \quad \text{for } m = 1, 2, 3 \tag{1}$$

where $h_{m,k}[t]$ denotes the channel gain from transmitter k to receiver m at time t. Let $\mathbf{H}[t] = \{h_{m,k}[t]\}$ denote the channel matrix at time t. We will focus on the important special case where the magnitudes of the channel gains are constant across time, i.e., $|h_{m,k}[t]| = |h_{m,k}[\tilde{t}]|$ for all t and $\tilde{t}$. Furthermore, we assume that the phase of each channel gain is i.i.d. across time according to a uniform distribution. The noise $z_m[t]$ is i.i.d. across time according to a circularly symmetric complex Gaussian with mean zero and unit variance, $z_m[t] \sim \mathcal{CN}(0, 1)$, and is independent of all channel inputs.

Each receiver makes an estimate $\hat{\omega}_m$ of its desired message ω_m using a decoding function $\mathcal{D}_m : \mathbb{C}^T \to \{1, 2, \ldots, 2^{TR_{\text{SYM}}}\}$. A symmetric rate R_{SYM} is achievable if, for any $\epsilon > 0$ and T large enough, there exist encoders and decoders that can attain probability of error at most ϵ,

$$\mathbb{P}\left(\{\hat{\omega}_1 \neq \omega_1\} \cup \{\hat{\omega}_2 \neq \omega_2\} \cup \{\hat{\omega}_3 \neq \omega_3\}\right) < \epsilon \ .$$

The *symmetric capacity* C_{SYM} is the supremum of all achievable symmetric rates.

III. Main Results

Our main result shows that, up to a constant gap, we can communicate over the 3-user phase-fading interference channel as if each receiver observed only a single interferer. Since we have chosen to focus on the symmetric ergodic capacity, the performance will be dictated by the strength of the weakest interferer. More formally, define

$$\mathsf{SNR}_m = |h_{m,m}|^2 P \qquad \forall m \tag{2}$$
$$\mathsf{INR}_{m,k} = |h_{m,k}|^2 P \qquad \forall k \neq m \tag{3}$$
$$\mathsf{INR}_m = \min_{k \neq m} \mathsf{INR}_{m,k} \tag{4}$$
$$\alpha_m = \frac{\log(\mathsf{INR}_{m,k})}{\log(\mathsf{SNR}_m)} \tag{5}$$
$$R_{\text{lower},m} = \begin{cases} \log(\mathsf{SNR}_m) - 3 & \alpha_m \in [2,\infty) \\ \frac{1}{2}\log(\mathsf{INR}_m) - 10 & \alpha_m \in [1,2) \end{cases} \tag{6}$$

$$R_{\text{upper},m} = \begin{cases} \log(\mathsf{SNR}_m) + 1 & \alpha_m \in [2,\infty) \\ \frac{1}{2}\log(\mathsf{INR}_m) + 1 & \alpha_m \in [1,2) \end{cases} \tag{7}$$

The parameter α_m captures the relative strength of the weakest interferer for receiver m and is reminiscent of the parameter used in [1] to characterize the generalized degrees-of-freedom of the two-user Gaussian interference channel.

Theorem 1: Given that $\mathsf{SNR} \geq 1$, the symmetric capacity per user C_{SYM} is lower and upper bounded as

$$\min_m R_{\text{lower},m} \leq C_{\text{SYM}} \leq \min_m R_{\text{upper},m} \tag{8}$$

The upper bound follows directly from the two-user bounds proposed in [11] for the very strong regime and [12], [13] for the strong regime. The lower bound combines ideas from compute-and-forward [15] and ergodic alignment [9] (in a fashion similar to computation alignment [14]) as well as earlier work on the symmetric capacity of the (static) symmetric K-user Gaussian interference channel [5]. Due to space limitations, we are not able present all of the necessary proofs in full detail. In particular, for ergodic alignment, each channel realization is paired with several others to create aligned subchannels. We will assume that this matching is performed exactly and refer interested readers to [9], [14] for a detailed analysis of the (negligible) rate loss associated with approximate channel matching. The rest of the paper is organized as follows. In Section III-A, we describe the core ideas underlying our alignment scheme through a motivating example over two matched time slots. In Section III-B, we describe our lattice coding techniques in Section III-B from the perspective of a single aligned subchannel at a single receiver. Finally, in Section IV, we describe our alignment scheme for L time slots.

A. Motivating Example

Each transmitter aims to send 2 symbols which are denoted by $s_{k,\upsilon}$ for $\upsilon = 1,2$. Let $\mathbf{s}_k = [s_{k,1} \ s_{k,2}]^T$. Consider a time slot t_1 and the channel gains $h_{m,k}[t_1]$. For simplicity, we will drop the time index and use $h_{m,k}$ to denote $h_{m,k}[t_1]$. Hence, we have

$$\mathbf{H}[t_1] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix} \tag{9}$$

We will code over two time slots, t_1 and t_2, where t_2 is chosen so that, with careful power allocation, the interference at each receiver is aligned. Assume that we can find t_2 such that

$$\mathbf{H}[t_2] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & -h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix}$$

In [9], it was shown that, for any $\delta > 0$, almost all time slots can be matched up within tolerance δ for T large enough if the phases of the channel gains are independent and i.i.d. uniform. Each transmitter will first perform power allocation and then send two orthogonal linear combinations of symbols. Let the channel input of transmitter k be $\mathbf{x}_k = \left[x_k[t_1] \ x_k[t_2]\right]^T$ where

$$\mathbf{x}_1 = \begin{bmatrix} \rho_1 s_{1,1} + \rho_2 s_{1,2} \\ \rho_1 s_{1,1} - \rho_2 s_{1,2} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \eta_1 s_{2,1} + \eta_2 s_{2,2} \\ \eta_1 s_{2,1} - \eta_2 s_{2,2} \end{bmatrix},$$
$$\mathbf{x}_3 = \begin{bmatrix} \gamma_1 s_{3,1} \\ \gamma_1 s_{3,1} \end{bmatrix}$$

and $\rho_\upsilon, \eta_\upsilon, \gamma_\upsilon \in \mathbb{C}$ are the power allocation parameters. Our goal is to choose these parameters to have magnitudes in the interval $(\frac{1}{2}, 1]$ in order to ensure that

a) The power constraint is never violated.
b) In the worst case, the signal power is reduced to one fourth. This ensures that the power allocation process will not reduce the achievable rate by more than 2 bits.

Now, let the channel output at receiver m be $\mathbf{y}_m = \left[y_m[t_1] \ y_m[t_2]\right]^T$ for $m = 1,2,3$.

To extract its message, each receiver first inverts the linear operations performed by its transmitter. Let $\tilde{\mathbf{y}}_m = \left[\tilde{y}_{m,1} \ \tilde{y}_{m,2}\right]^T$ be the resulting effective channel outputs where

$$\tilde{y}_{m,1} = \frac{1}{2}\left(y_m[t_1] + y_m[t_2]\right) \qquad \tilde{y}_{m,2} = \frac{1}{2}\left(y_m[t_1] - y_m[t_2]\right) \ .$$

Thus, for $m = 1,2,3$, $\tilde{\mathbf{y}}_m$ is given by

$$\tilde{\mathbf{y}}_1 = \begin{bmatrix} h_{1,1}\rho_1 s_{1,1} + h_{1,2}\eta_1 s_{2,1} + h_{1,3}\gamma_1 s_{3,1} \\ h_{1,1}\rho_2 s_{1,2} + h_{1,2}\eta_2 s_{2,2} \end{bmatrix} + \tilde{\mathbf{z}}_1 \tag{10}$$
$$\tilde{\mathbf{y}}_2 = \begin{bmatrix} h_{2,1}\rho_1 s_{1,1} + h_{2,2}\eta_1 s_{2,1} \\ h_{2,1}\rho_2 s_{1,2} + h_{2,2}\eta_2 s_{2,2} + h_{2,3}\gamma_1 s_{3,1} \end{bmatrix} + \tilde{\mathbf{z}}_2 \tag{11}$$
$$\tilde{\mathbf{y}}_3 = \begin{bmatrix} h_{3,1}\rho_1 s_{1,1} + h_{3,2}\eta_1 s_{2,1} + h_{3,3}\gamma_1 s_{3,1} \\ h_{3,1}\rho_2 s_{1,2} + h_{3,2}\eta_2 s_{2,2} \end{bmatrix} + \tilde{\mathbf{z}}_3 \tag{12}$$

where $\tilde{\mathbf{z}}_m = [\tilde{z}_{m,1} \ \tilde{z}_{m,2}]^T$ and

$$\tilde{z}_{m,1} = \frac{1}{2}\left(z_m[t_1] + z_m[t_2]\right) \qquad \tilde{z}_{m,2} = \frac{1}{2}\left(z_m[t_1] - z_m[t_2]\right) \ .$$

We will start by fixing $\rho_2 = 1$ and aligning interference for $\tilde{y}_{2,2}$. Beginning with the case $|h_{2,3}| \geq |h_{2,1}|$, we should choose γ_1 so that $\frac{h_{2,3}}{h_{2,1}}\gamma_1$ is equal to an integer $b_2 \in \mathbb{Z}$. To ensure that $|\gamma_1| \in (\frac{1}{2}, 1]$ we set $b_2 = \left\lfloor \frac{|h_{2,3}|}{|h_{2,1}|} \right\rfloor$.

Plugging into (11), we get

$$\tilde{y}_{2,2} = h_{2,2}\eta_2 s_{2,2} + h_{2,1}\left(s_{1,2} + b_2 s_{3,1}\right) + \tilde{z}_{2,2} \quad (13)$$

Now, consider the case where $|h_{2,3}| < |h_{2,1}|$ and, following the same argument as above, set $b_2 = \left\lceil \frac{|h_{2,1}|}{|h_{2,3}|} \right\rceil$ and

$$\gamma_1 = \frac{h_{2,1}}{h_{2,3}b_2} \quad \Rightarrow \quad \frac{1}{2} < |\gamma_1| \leq 1 \ .$$

Thus, the effective channel described in (11) becomes

$$\tilde{y}_{2,2} = h_{2,2}\eta_2 s_{2,2} + h_{2,3}\gamma_1\left(s_{3,1} + b_2 s_{1,2}\right) + \tilde{z}_{2,2} \quad (14)$$

Hence, we are able to achieve alignment in either case.

Now, we determine η_1 as a function of γ_1. This can be done by considering $\tilde{y}_{1,1}$ and following the same steps as before. Again, we carefully distinguish two cases:

- If $\frac{|h_{1,2}|}{|h_{1,3}|} \geq |\gamma_1|$

$$a_1 \triangleq \left\lfloor \frac{|h_{1,2}|}{|h_{1,3}||\gamma_1|} \right\rfloor \quad \Rightarrow \quad \eta_1 = \frac{\left\lfloor \frac{|h_{1,2}|}{|h_{1,3}||\gamma_1|} \right\rfloor}{\frac{h_{1,2}}{h_{1,3}\gamma_1}}$$

$$\tilde{y}_{1,1} = h_{1,1}\rho_1 s_{1,1} + h_{1,3}\gamma_1(s_{3,1} + a_1 s_{2,1}) + \tilde{z}_{1,1}$$

- But if $\frac{|h_{1,2}|}{|h_{1,3}|} < |\gamma_1|$

$$a_1 \triangleq \left\lceil \frac{|h_{1,3}||\gamma_1|}{|h_{1,2}|} \right\rceil \Rightarrow \eta_1 = \frac{\frac{h_{1,3}\gamma_1}{h_{1,2}}}{\left\lceil \frac{|h_{1,3}||\gamma_1|}{|h_{1,2}|} \right\rceil}$$

$$\tilde{y}_{1,1} = h_{1,1}\rho_1 s_{1,1} + h_{1,2}\eta_1(s_{2,1} + a_1 s_{3,1}) + \tilde{z}_{1,1}$$

Note that $|\eta_1| \in (\frac{1}{2}, 1]$ as desired. After fixing η_1, we are ready to fix ρ_1 by considering $\tilde{y}_{3,1}$ and performing a similar analysis as above. This leads to the following choices:

- If $\frac{|h_{3,1}|}{|h_{3,2}|} \geq |\eta_1|$

$$c_1 \triangleq \left\lfloor \frac{|h_{3,1}|}{|h_{3,2}||\eta_1|} \right\rfloor \quad \Rightarrow \quad \rho_1 = \frac{\left\lfloor \frac{|h_{3,1}|}{|h_{3,2}||\eta_1|} \right\rfloor}{\frac{h_{3,1}}{h_{3,2}\eta_1}}$$

$$\tilde{y}_{3,1} = h_{3,3}\gamma_1 s_{3,k} + h_{3,2}\eta_1\left(s_{2,1} + c_1 s_{1,1}\right) + \tilde{z}_{3,1}$$

- But if $\frac{|h_{3,1}|}{|h_{3,2}|} < |\eta_1|$

$$c_1 \triangleq \left\lceil \frac{|h_{3,2}||\eta_1|}{|h_{3,1}|} \right\rceil \quad \Rightarrow \quad \rho_1 = \frac{\frac{h_{3,2}\eta_1}{h_{3,1}}}{\left\lceil \frac{|h_{3,2}||\eta_k|}{|h_{3,1}|} \right\rceil}$$

$$\tilde{y}_{3,1} = h_{3,3}\gamma_1 s_{3,k} + h_{3,1}\rho_1\left(s_{1,1} + c_1 s_{2,1}\right) + \tilde{z}_{3,1}$$

Now, all interferers are aligned at all effective channels. Note that we did not need to use η_2 as $s_{2,2}$ is the only interferer at $\tilde{y}_{1,2}$. We can thus send a total of 5 symbols over 2 channel uses corresponding to a rate loss factor of $\frac{5}{6}$. Below, we argue that these effective channels are well-suited for lattice coding. In Section IV, we show how to code across L symbols to improve the rate loss factor to $\frac{L-2}{L}$.

B. Lattice Decoding

Assume that over T time slots, we can obtain n effective channels of the form shown in Section III-A. For instance, assume that receiver 2 has access to

$$\tilde{y}_{2,2}[i] = h_{2,2}\beta_2 s_{2,2}[i] + h_{2,3}\gamma_1\left(s_{3,1}[i] + b_2 s_{1,2}[i]\right) + \tilde{z}_{2,2}[i]$$

for $i = 1, \ldots, n$. If the codewords corresponding to $s_{2,2}[i], s_{3,1}[i], s_{1,2}[i]$ are drawn from the same lattice code, then the receiver observes a multiple-access with two effective users $s_{2,2}[i]$ and $s_{3,1}[i] + b_2 s_{1,2}[i]$. Below, we give conditions under which receiver 2 can decode $s_{2,2}[i]$, which can be generalized to all receivers in a straightforward fashion.

The lemma below shows that, in the very strong interference regime ($\alpha \geq 2$), the receiver can decode its own message as if there were no interference (up to a constant gap). Note that, without any alignment, we might require α to be as large as 3 to achieve similar rates.

Lemma 1: Given $\alpha \geq 2$, receiver 2 can decode $s_{2,2}[i]$ if

$$R_{\text{SYM}} < \log\left(\mathsf{SNR}_2\right) - 3 \ .$$

Proof: Using the compute-and-forward framework [15], we can first decode $(s_{3,1}[i] + b s_{1,2}[i])$ by treating $s_{2,2}[i]$ as noise if the rate is less than

$$\log\left(\frac{|h_{2,3}\gamma_1|^2 P}{1 + |h_{2,2}\beta_2|^2 P}\right) \geq \log\left(\frac{|h_{2,3}|^2 P}{1 + |h_{2,2}|^2 P}\right) - 2$$
$$\geq \log\left(\mathsf{SNR}_2\right) - 3$$

where the last inequality holds because $\alpha \geq 2$ and $\mathsf{SNR} \geq 1$. Afterwards, the receiver can cancel $(s_{3,1}[i] + b s_{1,2})$ and decode $s_{2,2}[i]$ from the resulting interference-free channel. ■

The next lemma generalizes the "outage set" result from [5, Theorem 1] to complex-valued channels. Note that here the outage set is defined with respect to the channel phases, whereas in [5] it is defined with respect to the channel strengths.

Lemma 2: There exist measurable disjoint subsets $\mathcal{S}_2(i), i = 1, 2, \ldots$ with measure $\mu\left(\mathcal{S}_2(i)\right) = 2^{-i}$ that partition the phases, $\bigcup_{i=1}^{\infty} \mathcal{S}_2(i) = [0, 2\pi)^3$ such that if $(\theta_1, \theta_2, \theta_3) \in \mathcal{S}_2(i)$, the rate $R_{\text{SYM}} = \frac{1}{2}\log(\mathsf{INR}_2) - 9 - \frac{i}{2}$ is achievable at receiver 2.

Due to space limitations, we omit this proof. Note that this lemma can be applied simultaneously at all receivers since the weakest $\mathsf{INR}_{m,k}$ dominates.

Finally, the lemma below bounds the rate achieved via Lemma 2 if we average across the time-varying phases.

Lemma 3: Given $\alpha_2 \in [1, 2)$, receiver 2 can decode if

$$R_{\text{SYM}} < \frac{1}{2}\log\left(\mathsf{INR}_2\right) - 10 \ . \quad (15)$$

Proof: We assume that coding occurs across many realizations of the subchannel (13), meaning that we have access to many realizations of the underlying phases. The expected rate can be written as

$$\mathbb{E}_\Theta\left[R(\Theta)\right] = \int\limits_{[0,2\pi)^3} R(\boldsymbol{\theta}) f_\Theta(\boldsymbol{\theta}) d\boldsymbol{\theta} = \sum_{i=1}^{\infty} \int_{\mathcal{S}_2(i)} R(\boldsymbol{\theta}) f_\Theta(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

where the disjoint subsets $\mathcal{S}_2(i)$ are taken from Lemma 2. From Lemma 2, we have that

$$\int_{\mathcal{S}_2(i)} R(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \;\geq\; \left(\frac{1}{2}\log(\mathsf{INR}_2) - 9 - \frac{i}{2}\right) 2^{-i}$$

Therefore,

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\Theta}}\left[R(\boldsymbol{\Theta})\right] &\geq \sum_{i=1}^{\infty}\left(\frac{1}{2}\log(\mathsf{INR}_2) - 9 - \frac{i}{2}\right) 2^{-i} \\ &= \frac{1}{2}\log(\mathsf{INR}_2) - 10\end{aligned}$$

■

IV. Alignment over L Time Slots

The motivating example from Section III-A suffers from a rate loss factor of $5/6$. To approach the highest rates, we need to remove this penalty by coding across more subchannels generated using a larger number of symbol extensions. In this section, we will generalize the alignment process from the motivating example to $L-2$ channel uses. Let $\mathbf{H}[t_1]$ be given by (9) and

$$\mathbf{H}[t_\ell] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} e^{\frac{j2\pi(\ell-1)}{L}} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix} \text{ for } \ell = 2, \ldots, L .$$

As before, we can turn to the channel matching arguments of [9] to argue that nearly all channel matrices can be matched successfully.

Transmitter k wants to transmit L symbols $\mathbf{s}_k = [s_{k,1} \ \cdots \ s_{k,L}]^T$. We will carefully perform power allocation, by multiplying each symbol by a complex constant with magnitude less than unity. We will then send L orthogonal linear combinations of the symbols, $\mathbf{x}_k$, over the L matched time slots, that is

$$\begin{aligned}\mathbf{x}_k &\triangleq [x_k[t_1] \ \ldots \ x_k[t_L]]^T = \mathsf{DFT}(L)\,\mathrm{diag}(\boldsymbol{\beta}_k)\mathbf{s}_k, \\ \boldsymbol{\beta}_k &= [\beta_1 \ \ldots \ \beta_L]^T \quad \text{ for } \ k = 1,2,3\end{aligned}$$

$$\mathsf{DFT}(L) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{L}} & \cdots & e^{-j\frac{2\pi}{L}(L-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{L}(L-1)} & \cdots & e^{-j\frac{2\pi}{L}(L-1)^2} \end{bmatrix}$$

Clearly, the larger the magnitude of the parameters $\{\beta_{k,\upsilon}\}$, the less rate we lose, and hence our task is to find $\boldsymbol{\beta}_k$ such that alignment is achieved at the receiver side while ensuring that the magnitudes of the entries of $\boldsymbol{\beta}_k$ are bounded below by $\frac{1}{2}$ (with the exception of $\beta_{3,L}$ which is set to zero).

At the receiver side, and for a given time slot t_ℓ, the received symbol at receiver m denoted by $y_m[t_\ell]$ is given by (1). Define $\mathbf{y}_m \triangleq [y_m[t_1] \ \ldots \ y_m[t_L]]^T$ for $m = 1,2,3$. The receiver will invert the linear operations performed at the transmitter side,

$$\tilde{\mathbf{y}}_m = [\tilde{y}_{m,1} \ \ldots \ \tilde{y}_{m,L}] = \mathsf{DFT}^{-1}(L)\ \mathbf{y}_m$$

where

$$\mathsf{DFT}^{-1}(L) = \frac{1}{L}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{L}} & \cdots & e^{j\frac{2\pi}{L}(L-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2\pi}{L}(L-1)} & \cdots & e^{j\frac{2\pi}{L}(L-1)^2} \end{bmatrix}$$

It can be shown that the above effective channels can be written as (16), (17), and (18) for $\upsilon = 1,2,\ldots,L-2$,

$$\tilde{y}_{1,\upsilon} = \sum_{k=1}^{3} h_{1,k}\beta_{k,\upsilon}s_{k,\upsilon} + \tilde{z}_{1,\upsilon} \tag{16}$$

$$\tilde{y}_{2,\upsilon} = \sum_{k=1}^{2} h_{2,k}\beta_{k,\upsilon}s_{k,\upsilon} + h_{2,3}\beta_{3,\upsilon+1}s_{3,\upsilon+1} + \tilde{z}_{2,\upsilon} \tag{17}$$

$$\tilde{y}_{3,\upsilon} = \sum_{k=1}^{3} h_{3,k}\beta_{k,\upsilon}s_{k,\upsilon} + \tilde{z}_{3,\upsilon} \tag{18}$$

where

$$\begin{aligned}\tilde{z}_{m,\upsilon} &= \frac{1}{L}\sum_{\ell=1}^{L} e^{j\frac{2\pi}{L}(\upsilon-1)(\ell-1)} z_m[t_\ell] \\ \mathbb{E}\left[|\tilde{z}_{m,\upsilon}|^2\right] &= \frac{1}{L^2}\sum_{\ell=1}^{L}|z_m[t_\ell]|^2 = \frac{1}{L}\end{aligned}$$

For simplicity, we set $s_{k,\upsilon} = 0$ for all k and $\upsilon = L-1, L$. The above set of equations (16) - (18) makes it clear how we can pick the power allocation parameters $\{\beta_{k,\upsilon}\}$ in the same way as in the example. Starting with (16) we fix $\beta_{3,1}$ to be 1. We first distinguish between the two cases:

Case I: $\frac{|h_{1,2}|}{|h_{1,3}|} > |\beta_{3,1}|$

We set

$$\begin{aligned} c_{1,1} &\triangleq \left\lfloor \frac{|h_{1,2}|}{|h_{1,3}||\beta_{3,1}|} \right\rfloor \\ \beta_{2,1} &= c_{1,1}\frac{h_{1,3}\beta_{3,1}}{h_{1,2}} \qquad \Rightarrow \quad |\beta_{2,1}| \in (0.5, 1]\end{aligned}$$

By substituting for $\beta_{3,1}$ in (16) we get

$$\tilde{y}_{1,1} = h_{1,1}\beta_{1,1}s_{1,1} + h_{1,3}\beta_{3,1}\left(s_{3,1} + c_{1,1}s_{2,1}\right) + \tilde{z}_{1,L} \tag{19}$$

Case II: $\frac{|h_{1,2}|}{|h_{1,3}|} < |\beta_{3,1}|$

In this case, we set

$$\begin{aligned} c_{1,1} &\triangleq \left\lceil \frac{|h_{1,3}||\beta_{3,1}|}{|h_{1,2}|} \right\rceil \\ \beta_{2,1} &= \frac{1}{c_{1,1}}\frac{h_{1,3}\beta_{3,1}}{h_{1,2}} \qquad \Rightarrow \quad |\beta_{2,1}| \in (0.5, 1]\end{aligned}$$

By substituting for $\beta_{3,1}$ in (16) we get

$$\tilde{y}_{1,1} = \quad h_{1,1}\beta_{1,1}s_{1,1} + h_{1,2}\beta_{2,1}\left(s_{2,1} + c_{1,1}s_{3,1}\right) + \tilde{z}_{1,L} \tag{20}$$

Now, we are done with the alignment at $\tilde{y}_{1,1}$ since both interferers are in integer relation in either cases. We can use (18) to fix $\beta_{1,1}$ in the same way, then we can use (17) to fix $\beta_{3,2}$ which leads to fixing $\beta_{2,2}$ using (16) and so on. Generally, we fix the parameters as follows.

Receiver 1:

Case I: $\frac{|h_{1,2}|}{|h_{1,3}|} > |\beta_{3,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-2$, we set $c_{1,\upsilon} \triangleq \left\lfloor \frac{|h_{1,2}|}{|h_{1,3}||\beta_{3,\upsilon}|} \right\rfloor$ and

$$\beta_{2,\upsilon} = c_{1,\upsilon} \frac{h_{1,3}\beta_{3,\upsilon}}{h_{1,2}} \quad \Rightarrow \quad |\beta_{2,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{2,\upsilon}$ in (16) we get

$$\tilde{y}_{1,\upsilon} = h_{1,1}\beta_{1,\upsilon}s_{1,\upsilon} + h_{1,3}\beta_{3,\upsilon}\left(s_{3,\upsilon} + c_{1,\upsilon}s_{2,\upsilon}\right) + \tilde{z}_{1,\upsilon}$$

Case II: $\frac{|h_{1,2}|}{|h_{1,3}|} < |\beta_{3,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-2$, we set $c_{1,\upsilon} \triangleq \left\lceil \frac{|h_{1,3}||\beta_{3,\upsilon}|}{|h_{1,2}|} \right\rceil$ and

$$\beta_{2,\upsilon} = \frac{1}{c_{1,\upsilon}} \frac{h_{1,3}\beta_{3,\upsilon}}{h_{1,2}} \quad \Rightarrow \quad |\beta_{2,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{2,\upsilon}$ in (16) we get

$$\tilde{y}_{1,\upsilon} = h_{1,1}\beta_{1,\upsilon}s_{1,\upsilon} + h_{1,2}\beta_{2,\upsilon}\left(s_{2,\upsilon} + c_{1,\upsilon}s_{3,\upsilon}\right) + \tilde{z}_{1,\upsilon}$$

Receiver 3:

Case I: $\frac{|h_{3,1}|}{|h_{3,2}|} > |\beta_{2,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-2$, we set $c_{3,\upsilon} \triangleq \left\lfloor \frac{|h_{3,2}|}{|h_{3,2}||\beta_{2,\upsilon}|} \right\rfloor$ and

$$\beta_{1,\upsilon} = c_{3,\upsilon} \frac{h_{3,2}\beta_{2,\upsilon}}{h_{3,1}} \quad \Rightarrow \quad |\beta_{1,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{1,\upsilon}$ in (18) we get

$$\tilde{y}_{3,\upsilon} = h_{3,3}\beta_{3,\upsilon}s_{3,\upsilon} + h_{3,2}\beta_{2,\upsilon}\left(s_{2,\upsilon} + c_{3,\upsilon}s_{1,\upsilon}\right) + \tilde{z}_{3,\upsilon}$$

Case II: $\frac{|h_{3,1}|}{|h_{3,2}|} < |\beta_{2,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-2$, we set $c_{3,\upsilon} \triangleq \left\lceil \frac{|h_{3,2}||\beta_{2,\upsilon}|}{|h_{3,1}|} \right\rceil$ and

$$\beta_{1,\upsilon} = \frac{1}{c_{3,\upsilon}} \frac{h_{3,2}\beta_{2,\upsilon}}{h_{3,1}} \quad \Rightarrow \quad |\beta_{1,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{1,\upsilon}$ in (18) we get

$$\tilde{y}_{3,\upsilon} = h_{3,3}\beta_{3,\upsilon}s_{3,\upsilon} + h_{3,1}\beta_{1,\upsilon}\left(s_{1,\upsilon} + c_{3,\upsilon}s_{2,\upsilon}\right) + \tilde{z}_{3,\upsilon}$$

Receiver 2:

Case I: $\frac{|h_{2,3}|}{|h_{2,1}|} > |\beta_{1,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-3$, we set $c_{2,\upsilon} \triangleq \left\lfloor \frac{|h_{2,3}|}{|h_{2,1}||\beta_{1,\upsilon}|} \right\rfloor$ and

$$\beta_{3,\upsilon+1} = c_{2,\upsilon} \frac{h_{2,1}\beta_{1,\upsilon}}{h_{2,3}} \quad \Rightarrow \quad |\beta_{3,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{3,\upsilon+1}$ in (17) we get

$$\tilde{y}_{2,\upsilon} = h_{2,2}\beta_{2,\upsilon}s_{2,\upsilon} + h_{2,1}\beta_{1,\upsilon}\left(s_{1,\upsilon} + c_{2,\upsilon}s_{3,\upsilon+1}\right) + \tilde{z}_{2,\upsilon}$$

Case II: $\frac{|h_{2,3}|}{|h_{2,1}|} < |\beta_{1,\upsilon}|$

For $\upsilon = 1, 2, \ldots, L-3$, we set $c_{2,\upsilon} \triangleq \left\lceil \frac{|h_{2,1}||\beta_{1,\upsilon}|}{|h_{2,3}|} \right\rceil$ and

$$\beta_{3,\upsilon+1} = \frac{1}{c_{2,\upsilon}} \frac{h_{2,1}\beta_{1,\upsilon}}{h_{2,3}} \quad \Rightarrow \quad |\beta_{2,\upsilon}| \in (0.5, 1]$$

By substituting for $\beta_{3,\upsilon+1}$ in (17) we get

$$\tilde{y}_{2,\upsilon} = h_{2,2}\beta_{2,\upsilon}s_{2,\upsilon} + h_{2,3}\beta_{3,\upsilon+1}\left(s_{3,\upsilon+1} + c_{2,\upsilon}s_{1,\upsilon}\right) + \tilde{z}_{2,\upsilon}$$

In general, $L-2$ symbols are transmitted over L effective channel in which both interferers are aligned, resulting in a rate loss factor of $\frac{L-2}{L}$. By taking L large enough, we can eliminate this penalty.

For any given receiver m in the very strong regime ($\alpha_m \geq 2$), we can apply Lemma 1 to show that the following rate is achievable at the given receiver

$$R_{\text{SYM}} = \frac{L-2}{L} \log\left(\mathsf{SNR}_m\right) - 3 \tag{21}$$

In the strong regime ($\alpha_m \in [1, 2)$), we can apply Lemma 3 to show that the following rate is achievable at receiver m

$$R_{\text{SYM}} = \frac{L-2}{2L} \log\left(\mathsf{INR}_m\right) - 10 \tag{22}$$

For L large enough, both (21) and (22) show that (8) is achievable.

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