Combinatorial Optimization for Improving QC LDPC Codes Performance

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***Abstract*—Techniques for searching for good quasi-cyclic (QC) LDPC block codes of short and moderate lengths which are suitable for practical purposes are studied. To facilitate implementations only codes whose parity-check matrices having bi-diagonal structure of their submatrices and consequently having low encoding complexity are considered. The problem of finding QC LDPC codes with the near-optimum frame or bit error rate performance is split into two independent steps: searching for the near-optimum column degree distribution of the parity-check matrix together with the best base matrix for this degree distribution and searching for the near-optimum labeling of the chosen base matrix. Sets of parameters and criteria for both steps are introduced and discussed. They allow further reduction of the search complexity without significant loss of the search optimality. New QC LDPC block codes of various code rates are obtained and their BER and FER performances are compared with those of the LDPC block codes as well as the turbo codes defined in the IEEE 802.16 WiMAX standard.**

I. Introduction

Starting with their rediscovery in 1995 [1] low-density parity-check (LDPC) codes have got a lot of attention due to their excellent performance near the Shannon limit. Quasi-cyclic (QC) LDPC codes with low complexity encoding are typically considered as practical candidates for current and future communication standards [2], [3], [4]. Although nowadays codes from this class are already efficiently used in many applications there exists a need of better codes in terms of error-correcting capability and implementation complexity. For wireless applications, we note that LDPC codes are still weaker than turbo-codes for the low rate region and for short code blocks (see, for example, [5]). The goal of this paper is to show that combinatorial optimization of the parity-check matrices of QC LDPC codes leads to significant improvement in code performance compared to known codes with the same delay and complexity. Moreover, such optimized QC LDPC codes have the same or even slightly superior performance than their turbo competitors currently used in the communication standards.

Let n and R denote block length and rate, respectively, of a QC LDPC code. Assuming fixed n and R we search for QC LDPC codes minimizing the error probability of belief propagation (BP) decoding. Note that the frame error rate (FER) is typically used for selection of short and moderate length QC LDPC codes for wireless communications (due to retransmission of codewords in ARQ applications). For this reason FER has been chosen as the search criterion, that is, for binary PSK-modulated signals transmitted over the AWGN channel with signal-to-noise ratio SNR per bit we search for the parity-check matrix H^* which is a solution of

$$H^* = \arg\min_H \{\mathrm{FER}(H,\mathrm{SNR})\} \tag{1}$$

under encoding and decoding complexity restrictions which will be discussed below.

Since the search space (of all possible H) is finite, the solution of (1) exists and can be found by exhaustive search. For each candidate H we have to compute the function $\mathrm{FER}(H,\mathrm{SNR})$. The only available approach for computing (estimating) $\mathrm{FER}(H,\mathrm{SNR})$ is the Monte-Carlo method. In this paper we try to reduce the search complexity measured as the number of candidate H (number computed estimates for $\mathrm{FER}(H,\mathrm{SNR})$) preserving near-optimality of the solution and taking into account the low-complexity requirement.

Depending on the application area, different ranges of SNRs and different FER requirements should be considered. For wireless channels, due to feedback, FER of the order 10^{-3} is considered as good enough. Thus, in our search we assume that FER is in the range from 10^{-2} to 10^{-4}.

In order to solve (1) numerically we select a set of P structural properties of H such as base matrix, determining the locations of nonzero elements, Tanner graph, etc. For each of these structural properties we introduce such a finite set of s_{P} numerical parameters $\boldsymbol{\theta} = \boldsymbol{\theta}(H)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_{s_{\mathrm{P}}})$ for which we assume that FER is a monotonic function of each entry θ_i of $\boldsymbol{\theta}$. Then we find an optimum achievable $\boldsymbol{\theta}^*$ and search for the best candidate among those H which satisfy $\boldsymbol{\theta}(H) = \boldsymbol{\theta}^*$. Moreover, to further reduce the search complexity we separate our search over vectors $\boldsymbol{\theta}$ into s_{P} candidate set searches over each component θ_i, $i = 1, 2, \ldots, s_{\mathrm{P}}$. In other words, our optimization is based on constructing candidate sets of parity-check matrices satisfying certain constraints.

Applying these arguments to general linear codes under maximum-likelihood decoding, $\boldsymbol{\theta}$ would probably include both the minimum distance and spectrum of the code. However, as soon as complexity requirements are taken into account, as for turbo or LDPC codes, the problem of determining a proper set $\boldsymbol{\theta}$ becomes difficult.

II. Definitions and Problem Formulation

It is convenient to consider a QC LDPC code as a tailbitten (TB) parent convolutional code determined by a polynomial parity-check matrix whose entries are polynomials with very few (typically only one) nonzero coefficients.

A rate $R = b/c$ parent convolutional LDPC code can be determined by its polynomial parity-check matrix $H(D)$ of syndrome memory m

$$H(D) = \begin{pmatrix} h_{11}(D) & h_{12}(D) & \dots & h_{1c}(D) \\ h_{21}(D) & h_{22}(D) & \dots & h_{2c}(D) \\ \vdots & \vdots & \ddots & \vdots \\ h_{(c-b)1}(D) & h_{(c-b)2}(D) & \dots & h_{(c-b)c}(D) \end{pmatrix} \quad (2)$$

where $h_{ij}(D)$ is either zero or a monomial entry, that is, $h_{ij}(D) = D^{w_{ij}}$ with w_{ij} being a nonnegative integer, $w_{ij} \le m$. Its *degree matrix* $W = \{w_{ij}\}, i = 1, ..., c-b, j = 1, ..., c$, follows as the $(c-b)\times c$ matrix with entries w_{ij} at the positions of the monomials $D^{w_{ij}}$ and $w_{ij} = -1$ at the zero positions. If each column of $H(D)$ contains J nonzero elements and each row contains K nonzero elements the QC LDPC convolutional code is (J, K)-regular; and irregular otherwise.

By tailbiting the parent convolutional code to length $M > m$ we obtain the parity-check matrix

$$H_{\mathrm{TB}}^{\mathrm{T}} = \begin{pmatrix} H_0^{\mathrm{T}} & H_1^{\mathrm{T}} & \cdots & H_m^{\mathrm{T}} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \cdots & \dots & \dots & \cdots & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & H_0^{\mathrm{T}} & H_1^{\mathrm{T}} & \dots & H_m^{\mathrm{T}} \\ H_m^{\mathrm{T}} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & H_0^{\mathrm{T}} & \dots & H_{m-1}^{\mathrm{T}} \\ \vdots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\ H_1^{\mathrm{T}} & \dots & H_m^{\mathrm{T}} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & H_0^{\mathrm{T}} \end{pmatrix} \quad (3)$$

of an (Mc, Mb) QC LDPC block code, where

$$H(D) = H_0 + H_1 D + \dots + H_m D^m$$

and H_i, $i = 0, 1, \dots, m$, are binary $(c-b)\times c$ matrices.

The polynomial parity-check matrix $H(D)$ given in (2) can be interpreted as a base matrix B labeled by monomials, where B is the $(c-b)\times c$ binary matrix with ones at the positions of the nonzero entries of $H(D)$, satisfying

$$B = H(D)|_{D=1}$$

Interpreting both B and H_{TB} as *biadjacency matrices* [6] yields their corresponding Tanner graphs $\mathcal{G}_{\mathrm{B}}$ and $\mathcal{G}$, respectively. Moreover, the parity-check matrix $H(D)$ of the parent convolutional code corresponds to an infinite Tanner graph obtained by unwrapping the Tanner graph $\mathcal{G}$ of the tailbiting block code and extending it to infinity in the time domain. Hence, the problem of finding new QC LDPC codes can be reduced to finding suitable base matrices B and proper labelings for the base Tanner graphs $\mathcal{G}_{\mathrm{B}}$ determined by B. The girth g of graph $\mathcal{G}$ is often used as a target when searching for good QC LDPC codes (see, for example, [7], [8], [9], [10]).

For irregular QC LDPC codes, let $J_{\min}$ and $J_{\max}$ denote the minimum and maximum numbers of the nonzero entries in any column of $H(D)$, respectively. Then the *column degree distribution* (CDD) of the parity-check matrix $H(D)$ and its base matrix B is given by the vector

$$\boldsymbol{d}^{\mathrm{p}} = (n_{J_{\min}}\, n_{J_{\min}+1}\, \dots\, n_{J_{\max}})$$

where n_i denotes the number of columns having i nonzero elements. Since the degree distribution can contain zeros, it can be efficiently represented as a set

$$\lambda_s = \left\{ d^{(1)}(n_1), d^{(2)}(n_2), \dots, d^{(l)}(n_l) \right\}$$

where n_i is the number of columns with $d^{(i)}$ nonzero elements, $d^{(1)} = J_{\min}$, $d^{(l)} = J_{\max}$, and $\sum_{i=1}^{l} n_i = c$. Notice that the degree distribution is invariant with respect to column permutations.

Denote as H_{TB}^{i}, $i = 1, 2, \dots, Mc$, a submatrix of H_{TB} formed by its i columns with the smallest column weight and let g_i be the girth of the corresponding Tanner graph with biadjacency matrix H_{TB}^{i}. Then we call the Mc-tuple

$$\boldsymbol{g}^{\mathrm{p}} = (g_1\, g_2\, \dots\, g_{Mc} = g)$$

where $g_i > g_{i+1}$, $i = 1, 2, ..., Mc$, the *girth profile* of the QC LDPC block code determined by H_{TB}. It can be efficiently represented as the ordered set

$$\mathcal{S}_g = \left\{ g^{(1)}(n_1), g^{(2)}(n_2), \dots, g^{(l)}(n_l) \right\} \quad (4)$$

where $n_j < n_{j+1}$, $n_l = Mc$, and $g_i = g^{(j)}$ for $n_{j-1} < i \le n_j$ with $n_0 = 0$. A girth profile $\boldsymbol{g}^{\mathrm{p}}$ is said to be superior to $(>)$ another profile $\boldsymbol{g}^{\mathrm{p}^*}$ if there exists a positive integer p such that

$$g_j = \begin{cases} = g_j^* & j = c, c-1, \dots, p+1 \\ > g_j^* & j = p \end{cases}$$

Let w denote the length of a sliding window containing w consecutive columns of the parity-check matrix ordered by their weights. Consider a sequence $\mathcal{T}$ of Tanner graphs determined by the corresponding sliding-window submatrices of the parity-check matrix. In [11] a *sliding-window girth* g_w was introduced. It is equal to the minimum over the sequence of girth values of the Tanner graphs in $\mathcal{T}$.

For simplicity, in the sequel we restrict ourselves to girth profiles for which n_i in (4) are multiples of M and represent the corresponding girth profiles as vectors of length c

$$\widetilde{\boldsymbol{g}}^{\mathrm{p}} = (\widetilde{g}_1\, \widetilde{g}_2\, \dots\, \widetilde{g}_c = g)$$

Hence, the corresponding set $\mathcal{S}_g$ in (4) contains at most c elements.

Using the introduced notations we can formulate the problem of optimizing QC LDPC codes of given length and rate as follows. We choose as the two main structural properties of parity-check matrices the corresponding base matrix B and the degree matrix W. The base matrix B of a given size we characterize by the column degree-distribution λ_s and the numbers of common nonzero positions in its columns and rows, C_{c} and C_{r}, respectively. Since the degree matrix W determines the corresponding Tanner graph $\mathcal{G}$ we characterize it by the girth profile $\mathcal{S}_g$, sliding-window size w, and sliding-window girth g_w of $\mathcal{G}$.

Now the code search problem may be formulated as searching over all enumerated parameters and over all base matrices B and degree matrices W with given parameters. Although the sets of base and degree matrices are finite, the problem is still intractable. Thus, we simplify the search by restricting the size of each set. Thus, our suboptimum search is performed as

$$\mathrm{FER}_{\mathrm{subopt}}(SNR) = \min_{B\in\mathcal{B}}\left\{\min_{W\in\mathcal{W}}\{\mathrm{FER}(\mathrm{SNR}, B, W)\}\right\}$$

The next sections are devoted to selecting such sets $\mathcal{B} = \{B\}$ and $\mathcal{W} = \{W\}$ which on one hand, are small enough to make an exhaustive search over the product of these sets feasible, and, on the other hand, do not lead to an essential loss of optimality compared to an unrestricted search over all QC LDPC codes with given parameters b, c, and M.

III. Base Matrix Optimization: Degree Distribution, Row and Column Correlation

In this section we focus on base matrix optimization, that is on selecting the set $\mathcal{B}$.

Commonly, the degree distribution is selected similar to that which maximizes the asymptotic performance [12]. Since such a choice can be far from optimum for finite lengths, we do not follow this way.

A restriction on the shape of the base matrix that seems to be important for practical purposes, that is, for yielding low encoding complexity, is the following. The matrix B should contain a set of columns either in lower triangular form or being easily reducible to this form. Similarly to [4] we consider polynomial parity-check matrix of the parent convolutional code in the form

$$H(D) = \begin{pmatrix} H_{\mathrm{bd}}(D) & \boldsymbol{h}_0(D) & H_{(c-b)\times b}(D)\end{pmatrix} \tag{5}$$

where the first part $H_{\mathrm{bd}}(D)$ is a bi-diagonal matrix of size $(c-b)\times(c-b-1)$, $\boldsymbol{h}_0(D)$ is a column with exactly 3 nonzero elements, two of which are identical monomials, and $H_{(c-b)\times b}(D)$ can be any monomial submatrix of proper size.

If we preselect the shape of the parity-check matrix as shown in (5), then the component n_2 of the degree distribution vector $\boldsymbol{d}^{\mathrm{p}}$ is fixed (or at least limited from below), $n_2 \geq c-b-1$, moreover $n_3 \geq 1$.

Notice that the complexity of an exhaustive search over possible column degree distributions is not that high as it would seem. Since we choose base matrices from the limited set determined by (5), then the weights of the $(c-b)$ columns of the parity part of B are fixed. We should distribute the weights from the set $\{3, 4, \ldots, c-b\}$ between the b columns of the information part of B. Let $l \leq c-b-2$ denote the number of different column weights. Then it is easy to compute that for $b \geq l$ the total number of candidate degree distributions is equal to the number of compositions of the number b into l parts, that is, equal to $\binom{b+l-1}{l-1}$ [13, Ch.5].

For example, for rate $R = 1/2$, $b = 12$, $c = 24$, $l = 10$ there exist $\binom{21}{9}$=293930 different degree distributions. This value is rather large but it was found that choosing $l > 4$ does not improve code performance significantly. Then it is enough to check not more than $\binom{15}{3}$=455 candidates. Moreover, many of the bad degree distributions (*e.g.*, too dense and too sparse) can be rejected very quickly.

Consider the problem of selecting a base matrix for a given degree distribution, that is, we construct a base matrix $B = \{\boldsymbol{b}_t\}$, $t = 1, 2, \ldots, c$, where $\boldsymbol{b}_t$ denotes the tth column of B, randomly column-by-column in order of increasing column weight, using the following criteria, listed in the order of their usage:

- The number of common positions in the columns $C_1(t) = \sum_{j=1}^{t-1} w(\boldsymbol{b}_t \wedge \boldsymbol{b}_j)$, where $w(\boldsymbol{x})$ is weight of the vector $\boldsymbol{x}$ (here $\wedge$ stands for positionwise logical "and") .
- The number of common triples in the columns intersecting in more than three positions $C_2(t) = \sum_{j<t, w(\boldsymbol{b}_t\wedge\boldsymbol{b}_j)\geq 3}\binom{w(\boldsymbol{b}_t\wedge\boldsymbol{b}_j)}{3}$.
- The girth g_{B}^t of the Tanner graph of the current base matrix B_t of size $(c-b)\times t$, $t = 1, 2, \ldots, c$, $C_3(t) = g_{\mathrm{B}}^t$.
- The sum of weights of the columns participating in the shortest cycle (approximate cycle extrinsic message degrees [14] (ACE)-like criterion) of B_t $C_4(t) = \sum_{\boldsymbol{b}\in\mathcal{D}} w(\boldsymbol{b})$.
- The multiplicity of cycles with minimum ACE-like value $C_5(t)$.

Let $\mathcal{S}$ denote the set of candidate column degree distributions (CDD), N_{B} the number of candidate base matrices for each CDD, N_{b} the number of column candidates, N_{W} the number of candidate labeling matrices, and N_{C} the number of criteria. For selecting base matrix B we apply the algorithm shown in Fig. 1.

IV. Degree Matrix Optimization: Girth, Sliding Window Girth, Girth Profile

Now we are searching for the best degree matrix W, that is, the best labeling for a given base matrix B.

Since we restricted ourselves by the class of bi-diagonal matrices (5), the first $c-b-1$ columns of W are fixed. Since dividing a column of $H(D)$ by any monomial D^a we obtain the parity-check matrix of an equivalent code, we always choose the first nonzero element of each column equal to 1. For the same reason two identical elements of $\boldsymbol{h}_0$ in (5) are also chosen equal to 1.

We select all other elements of W at random from the set $\{1, 2, ..., M\}$, column by column, for $t = 1, 2, ..., c$ taking into account the restrictions formulated above. In order to select the best tth column-candidate, we use the following criteria:

- The girth g of the Tanner graph determined by the current matrix W_t has to be greater than or equal to a predetermined value g^t, *i.e.*, $C_1(t) = g^t$.
- The "sliding window" girth g_w has to be greater than or equal to g_{w_t}, where w_t and g_{w_t} are predetermined integers, *i.e.,* $C_2 = g_{w_t}$. Notice that since the columns of the base matrix are sorted according to increasing weights, then g_{w_t} is the girth of the Tanner graph determined by the last w_t columns of $H(D)$.

Initialization
Chose any binary $(c-b)\times c$ matrix of form (5) as $B_{\rm best}$
Loop over CDD
for *CDD* $\in \mathcal{S}$ **do**
  Loop over the candidate base matrices
  for $I_{\rm B}=1$ **to** $N_{\rm B}$ **do**
    Set $t=c-b-1$, and let $B_{t,I_{\rm B}}$ be a bidiagonal matrix consisting of the first t columns of $B_{I_{\rm B}}$
    Loop over the columns
    for $t=c-b$ **to** c **do**
      Choose any column satisfying CDD as $\boldsymbol{b}_{\rm best}$
      for $I_{\boldsymbol{b}}=1$ **to** $N_{\boldsymbol{b}}=1$ **do**
        1. Generate randomly the tth column $\boldsymbol{b}$ of weight corresponding to the chosen CDD
        2. Loop over search criteria
        for $I_{\rm C}=1$ **to** $N_{\rm C}$ **do**
          Compute the criterion $C_{I_{\rm C}}$ value for extension $[B_{t-1,I_{\rm B}}\boldsymbol{b}]$
          if $C_{I_{\rm C}}([B_{t-1,I_{\rm B}}\boldsymbol{b}])\geq C_{I_{\rm C}}([B_{t-1,I_{\rm B}}\boldsymbol{b}_{\rm best}])$ **then** $\boldsymbol{b}_{\rm best}\leftarrow \boldsymbol{b}$
        end
      end
      Update: $B_{t,I_{\rm B}}=[B_{t-1,I_{\rm B}}\boldsymbol{b}_{\rm best}]$
    end
    Loop over the labeling matrices
    for $I_{\rm W}=1$ **to** $N_{\rm W}$ **do**
      Generate the random labeling $W_{I_{\rm W}}$
      Estimate the FER $(B_{I_{\rm B}},W_{I_{\rm W}})$
      if *FER*$(B_{I_{\rm B}},W_{I_{\rm W}})\leq$ *FER*$(B_{\rm best},W_{I_{\rm W}})$ **then** $B_{\rm best}\leftarrow B_{I_{\rm B}}$
    end
  end
end

Fig. 1. Optimization of base matrix and column degree distribution

We run the procedure for $i=(c-b)-1,...,c$. If for a given set of restrictions $\{g_t,w_t,g_{w_t}\},t=(c-b)-1,...,c$, the algorithm failed to find any matrix W or found too few matrices, we relax some restrictions and start the algorithm again. Finally, for all candidate matrices W (typically about one thousand) we estimate FER for the target SNR and select the final winner.

V. Search Results and Discussion

In this section we present a detailed performance comparison of the new QC LDPC codes with both QC LDPC codes and Turbo codes from the WIMAX standard [4]. We also compare performances of the new codes with QC LDPC codes from [5] since they outperform the QC LDPC codes from the WIMAX standard.

We consider codes of rate $R=1/2$ and $R=3/4$, in order to cover both low-rate and high-rate requirements. We have chosen $c=24$ as in [4] and two extreme delay values, covering both low delay (M=24, n=576) and relatively large delay (M=96, n=2304).

In the standard it is accepted that for a given rate R the same degree matrix W is used for all delay values and for all target FER requirements. We use the same restriction here.

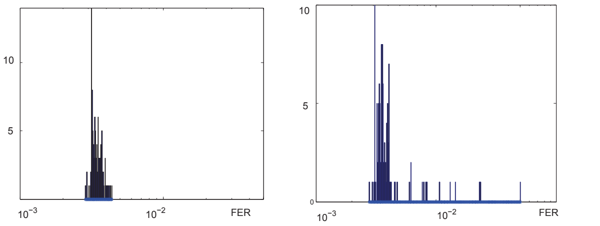

Fig. 2. Histograms of FER-values for 105 versions of the degree matrix W for (2304,1152)-QC LDPC code with a) optimized CDD and restrictions on the girth profile and the sliding window girth; and b) random degree matrix with girth 6

In order to verify the efficiency of our combinatorial optimization of the degree matrix W we have compared the FER performance of $R=1/2$ QC LDPC code of block length 2304 simulated for candidate W matrices obtained by random labeling with the target girth $g=6$ and candidate W matrices with optimized labeling. In Fig. 2 the FER performance histograms for the optimized (left) and random (right) labelings are shown. It follows from the histograms that random labelings provide rather diffuse distributions of FER values while the optimized labelings correspond to approximately the same FER value for all candidate W matrices. It supports our conjecture that the chosen set of combinatorial parameters of QC LDPC codes works well: all codes with a given girth profile have almost the same performance.

Below we present simulation results performed with 50 iterations of BP decoding and accumulating not less than 100 error events for each point. Figs. 3, 4 show the FER and BER performances of the constructed QC LDPC codes of block length $n=576$ and rates $R=1/2$ and $R=3/4$, respectively, and their counterparts from [4]. It is easy to see that the new codes provide significant improvements in both FER and BER performance compared to the QC LDPC codes from the WIMAX standard. Moreover, this improvement grows with SNR. Fig. 5 illustrates the FER and BER performances of the new QC LDPC codes of rate $R=1/2$ and block length $n=2304$ in comparison with their counterparts from the standard. It is easy to see that the optimization gain is larger for shorter lengths and lower rates.

In [5] performances of QC LDPC codes from [4] were improved by using $c=48$ instead of $c=24$ while keeping the structure of the base matrix allowing fast encoding. In Fig. 6 the FER and BER performances of the new QC LDPC codes with base matrices of size 12×24 and 16×32 of rate $R=1/2$ and block length $n=2304$ are compared with the FER and BER performances of QC LDPC codes of the same rate and length with base matrix of size 24×48 from [5] and with turbo codes with the same parameters. The new QC LDPC code with base matrix of size 16×32 provides significant gain compared to the QC LDPC code from [5] and slightly outperforms the turbo code. The new QC LDPC code with base matrix of size 12×24 loses compared with QC LDPC code from [5] for low SNRs and wins at high SNRs.

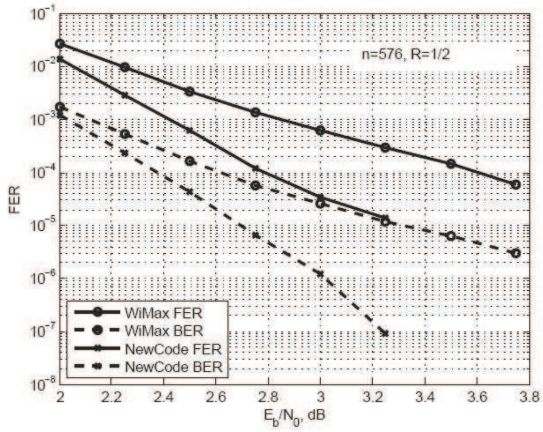

Fig. 3. FER/BER plots for the code from the WiMax standard and the new code, $n = 576$, $R = 1/2$

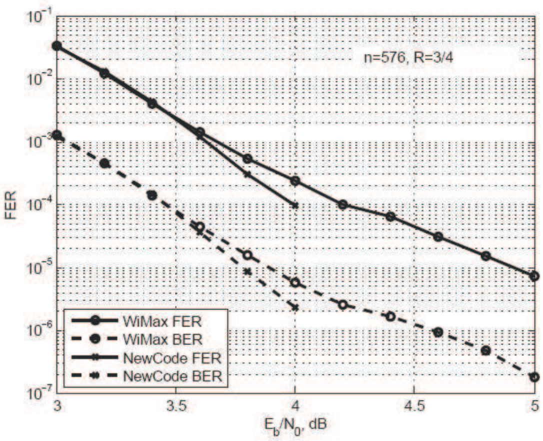

Fig. 4. FER/BER plots for the code from the WiMax standard and the new code, $n = 576$, $R = 3/4$

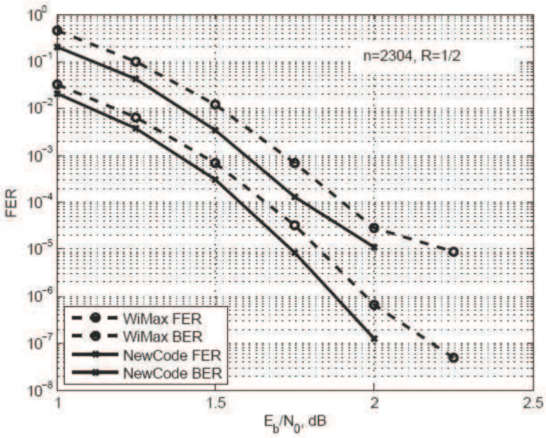

Fig. 5. FER/BER plots for the QC LDPC code from the WiMax standard and the new code, $n = 2304$, $R = 1/2$

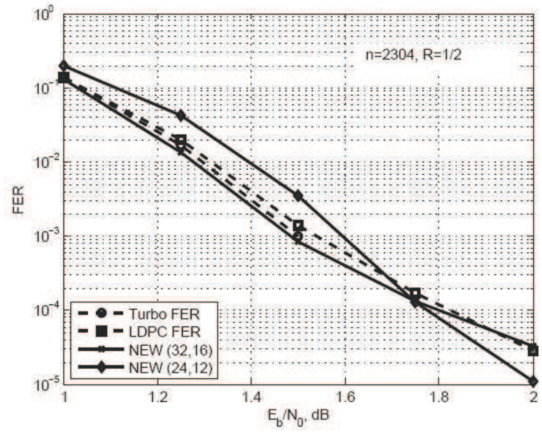

Fig. 6. FER plots for the turbo code, the QC LDPC code from [5] and the new codes with base matrices of size (24,12) and (32,16), $n = 2304$, $R = 1/2$

ACKNOWLEDGEMENTS

This research was supported in part by the Swedish Research Council under Grant 621-2010-5847.

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