Full-Rate, Full-Diversity, Finite Feedback Space-Time Schemes with Minimum Feedback and Transmission Duration

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***Abstract*—In this paper a MIMO quasi static block fading channel with finite N-ary delay-free, noise-free feedback is considered. The transmitter uses a set of N Space-Time Block Codes (STBCs), one corresponding to each of the N possible feedback values, to encode and transmit information. The feedback function used at the receiver and the N component STBCs used at the transmitter together constitute a *Finite Feedback Scheme (FFS)*. If each of the component codes encodes K independent complex symbols and is of transmission duration T, the rate of the FFS is $\frac{K}{T}$ complex symbols per channel use. Although a number of FFSs are available in the literature that provably achieve full-diversity, there is no known universal criterion to determine whether a given arbitrary FFS achieves full-diversity or not. Further, all known full-diversity FFSs for $T < N_t$ where N_t is the number of transmit antennas, have rate at the most 1. In this paper a universal necessary condition for any FFS to achieve full-diversity is given, using which the notion of *Feedback-Transmission duration optimal (FT-optimal)* FFSs–schemes that use minimum amount of feedback N given the transmission duration T, and minimum transmission duration given the amount of feedback to achieve full-diversity–is introduced. When there is no feedback ($N = 1$) an FT-optimal scheme consists of a single STBC with $T = N_t$, and the proposed necessary condition reduces to the well known necessary and sufficient condition for an STBC to achieve full-diversity, viz. every non-zero codeword difference matrix of the STBC must be of rank N_t. Also, a sufficient condition for full-diversity is given for those FFSs in which the component STBC yielding the largest minimum Euclidean distance is chosen. Using this sufficient condition, full-rate (rate N_t) full-diversity FT-optimal schemes are constructed for all (N_t, T, N) with $NT = N_t$. These are the first full-rate full-diversity FFSs reported in the literature for $T < N_t$. Simulation results show that the new schemes have the best error performance among all known FFSs.**

I. Introduction

We consider quasi-static block fading multiple-input multiple-output (MIMO) wireless channel with Rayleigh flat fading. We assume that the receiver has full-channel state information, and the transmitter has only a partial knowledge of the channel obtained through a delay-free noise-free N-ary feedback index conveyed by the receiver. The transmitter is equipped with N Space-Time Block Codes (STBCs), one corresponding to each of the N different values of the feedback index, and based on the received feedback value, it uses the corresponding STBC to encode and transmit information bits. The receiver, knowing the feedback index that it has sent to the transmitter and hence the STBC used for encoding, performs maximum-likelihood (ML) decoding of transmitted codeword to estimate the information bits. The feedback function used by the receiver to generate the N-ary feedback index, and the N component STBCs used by the transmitter determine the communication protocol implemented on the MIMO channel with feedback. Throughout this paper we will refer to the combination of the particular feedback function used at the receiver with the N component STBCs used at the transmitter as a *Finite Feedback Scheme (FFS)*. If each of the component STBCs encodes K independent complex symbols and has transmission duration T, the FFS has rate $R = \frac{K}{T}$ complex symbols per channel use. This definition of FFS is universal and subsumes all schemes available in the literature for delay-free noise-free finite feedback channels with quasi-static block fading, such as transmit antenna selection [1], precoding for spatial multiplexing systems [2], beamforming [3], [4], combining space-time codes with beamforming [5], [6], extending orthogonal STBCs [7], switching between orthogonal STBC and spatial multiplexing [8], and code diversity [9] (see Section II-A for formal definition of an FFS, and Table I for a summary of some of the FFSs available in the literature).

A number of FFSs are available in the literature that provably achieve full-diversity such as transmit antenna selection [1] and the schemes in [3]–[9]. However, there is no known universal criterion (applicable to any finite feedback scheme, including those in [1]–[9] as special cases) to determine whether a given arbitrary FFS achieves full-diversity or not. Further, all known full-diversity FFSs for $T < N_t$, where N_t is the number of transmit antennas, have rate at the most 1. In this context, the contributions (and organization) of this paper are as follows.

- We first give a universal necessary condition for any FFS to achieve full-diversity (Theorem 1, Section II-B), and then introduce the notion of *Feedback-Transmission duration optimal (FT-optimal)* FFSs–schemes that use minimum amount of feedback given the transmission duration and minimum transmission duration given the amount of feedback to achieve full-diversity. When there is no feedback ($N = 1$) an FT-optimal scheme consists of a single STBC with $T = N_t$, and the universal necessary condition reduces to the well known necessary and sufficient condition for an STBC to achieve full-diversity, viz. every non-zero codeword difference matrix of the STBC must be of rank N_t.

- For FFSs which use the feedback function that chooses the component STBC yielding the largest minimum Euclidean distance, we give a sufficient condition for full-diversity (Theorem 2, Section II-C).
- Using the sufficient criterion and tools from algebraic number theory we construct full-rate (rate $R = N_t$) full-diversity FT-optimal schemes for all triples (N_t, T, N) with $NT = N_t$ (Section III). These are the first full-rate full-diversity FFSs reported in the literature for $T < N_t$.
- Simulation results show that the new FFSs have the best error performance among all known schemes (Section IV).

The proofs for all the theorems, propositions and other claims in this paper have been omitted due to space considerations, but have been made available in [10].

Notation: Throughout the paper, matrices (column vectors) are denoted by bold, uppercase (lowercase) letters. For a complex matrix $\mathbf{A}$, the transpose, the conjugate-transpose and the Frobenius norm are denoted by $\mathbf{A}^T$, $\mathbf{A}^H$ and $\|\mathbf{A}\|_F$ respectively. For a square matrix $\mathbf{A}$, $\det(\mathbf{A})$ is the determinant of $\mathbf{A}$. For a positive integer n, $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{0}$ is the all zero matrix of appropriate dimension. Unless used as a subscript i denotes $\sqrt{-1}$. For any vector $\mathbf{u}$, its ℓ^{th} component is denoted by $\mathbf{u}(\ell)$.

II. Full-Diversity Criteria

A. System Model

We consider an $N_t \times N_r$ quasi-static Rayleigh flat fading MIMO channel $\mathbf{Y} = \sqrt{E}\mathbf{X}\mathbf{H} + \mathbf{W}$, where $\mathbf{Y}$ is the $T \times N_r$ received matrix, $\mathbf{X}$ is the $T \times N_t$ transmit matrix, $\mathbf{H}$ is the $N_t \times N_r$ channel matrix, $\mathbf{W}$ is the $T \times N_r$ matrix representing the additive noise at the receiver and E is the average transmit power. The entries of $\mathbf{H}$ and $\mathbf{W}$ are independent, zero mean, circularly symmetric complex Gaussian random variables, with the variance of each entry of $\mathbf{H}$ being 1, and the variance of each entry of $\mathbf{W}$ being N_0. The receiver uses a feedback function $\mathsf{f} : \mathbb{C}^{N_t \times N_r} \to \{1, \dots, N\}$ to send the feedback index $\mathsf{f}(\mathbf{H})$ to the transmitter through a delay-free, noise-free feedback channel. A *Space-Time Block Code (STBC)* $\mathcal{C}$ is a finite set of $T \times N_t$ complex matrices. The transmitter is equipped with N STBCs $\mathcal{C}_1, \dots, \mathcal{C}_N$, with $|\mathcal{C}_1| = \cdots = |\mathcal{C}_N|$. When $\mathsf{f}(\mathbf{H}) = n$, the transmitter uses $\mathcal{C}_n$ to encode the information bits. Upon receiving $\mathbf{Y}$, knowing the feedback index, and hence knowing the codebook used for transmission, the receiver performs ML decoding of the codeword $\hat{\mathbf{X}} = \arg\min_{\mathbf{X} \in \mathcal{C}_{\mathsf{f}(\mathbf{H})}} \|\mathbf{Y} - \sqrt{E}\mathbf{X}\mathbf{H}\|_F^2$.

Definition 1: A *Finite Feedback Scheme (FFS)* for an $N_t \times N_r$ MIMO channel with N-ary noise-free, delay-free feedback and transmission duration T is a tuple $(\mathsf{f}, \mathcal{C}_1, \dots, \mathcal{C}_N)$, where $\mathsf{f} : \mathbb{C}^{N_t \times N_r} \to \{1, \dots, N\}$ is the feedback function, and $\mathcal{C}_1, \dots, \mathcal{C}_N$ are $T \times N_t$ STBCs.

If an STBC $\mathcal{C} \subset \mathbb{C}^{T \times N_t}$ encodes K independent complex symbols its rate is $\frac{K}{T}$. An FFS $(\mathsf{f}, \mathcal{C}_1, \dots, \mathcal{C}_N)$ is of rate R if each of the N STBCs $\mathcal{C}_1, \dots, \mathcal{C}_N$ is of rate R, and the FFS is of *full-rate* if $R = N_t$. Table I summarizes some of the FFSs available in the literature. The scheme from [8] uses two codes of different rates (the Alamouti code with rate 1 and spatial multiplexing with rate 2), hence the rate of this FFS is not defined.

B. A Universal Necessary Condition

For any STBC $\mathcal{C}$, let $\Delta\mathcal{C}$ denote the set of non-zero codeword difference matrices, i.e., $\Delta\mathcal{C} = \{\mathbf{X}_1 - \mathbf{X}_2 \mid \mathbf{X}_1, \mathbf{X}_2 \in \mathcal{C}, \mathbf{X}_1 \neq \mathbf{X}_2\}$. For a given FFS $\mathcal{S} = (\mathsf{f}, \mathcal{C}_1, \dots, \mathcal{C}_N)$ define the set $\Delta\mathcal{S}$ as

$$\Delta\mathcal{S} = \left\{ \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_N \end{bmatrix} \;\middle|\; \mathbf{X}_1 \in \Delta\mathcal{C}_1, \dots, \mathbf{X}_N \in \Delta\mathcal{C}_N \right\}.$$

Further, let $\mathsf{r}(\Delta\mathcal{S}) = \min\{\mathsf{rank}(\mathbf{X}) | \mathbf{X} \in \Delta\mathcal{S}\}$. Since the matrices in $\Delta\mathcal{S}$ are of dimension $NT \times N_t$, $\mathsf{r}(\Delta\mathcal{S}) \leq N_t$.

Theorem 1: If an FFS $\mathcal{S}$ achieves full-diversity, then $\mathsf{r}(\Delta\mathcal{S}) = N_t$ and $NT \geq N_t$.

Definition 2: A full-diversity FFS is said to be *Feedback-Transmission duration optimal (FT-optimal)* if $NT = N_t$.

From Theorem 1, an FT-optimal scheme uses the least N for a given T, and the least T for a given N to attain full-diversity. When there is no feedback, i.e., when $N = 1$ (last row of Table I), an FT-optimal scheme consists of a single STBC with $T = N_t$, and the necessary condition of Theorem 1 reduces to the well known necessary and sufficient condition of [11] for an STBC to achieve full-diversity, viz. every non-zero codeword difference matrix of the STBC must be of rank N_t.

C. A Sufficient Condition

Let $\mathsf{f}_d(\mathbf{H}) = \arg\max_{n \in \{1,\dots,N\}} \left\{ \min_{\mathbf{X} \in \Delta\mathcal{C}_n} \|\mathbf{X}\mathbf{H}\|_F^2 \right\}$, i.e., the function that returns the index of the codebook with largest minimum Euclidean distance for the given channel $\mathbf{H}$.

Theorem 2: The FFS $\mathcal{S} = (\mathsf{f}_\mathsf{d}, \mathcal{C}_1, \dots, \mathcal{C}_N)$ achieves full-diversity if $\mathsf{r}(\Delta\mathcal{S}) = N_t$.

As an example for the application of Theorem 2, we now construct a new $N = 2$, $T = 1$ FT-optimal, full-rate, full-diversity FFS for $N_t = 2$ antennas using the same field extensions as the Golden code [12]. Let x_1, x_2 be complex symbols encoded using a QAM constellation $\mathcal{A} \subset \mathbb{Z}[i]$. Let $\mathbb{Q}(i, \sqrt{5})$ be the field obtained from $\mathbb{Q}$ by the adjunction of elements $i = \sqrt{-1}$ and $\sqrt{5}$, and σ be the automorphism on $\mathbb{Q}(i, \sqrt{5})$ that fixes $\mathbb{Q}(i)$ and maps $\sqrt{5}$ to $-\sqrt{5}$. Define $\mathcal{C}_1 = \left\{ \begin{bmatrix} \alpha(x_1 + x_2\theta) & \sigma\left(\alpha(x_1 + x_2\theta)\right) \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathcal{A} \right\}$ and $\mathcal{C}_2 = \left\{ \begin{bmatrix} \alpha(x_1 + x_2\theta) & i\sigma\left(\alpha(x_1 + x_2\theta)\right) \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathcal{A} \right\}$, where $\theta = \frac{1+\sqrt{5}}{2}$ and $\alpha = 1 + i - i\theta$. In order to show that $\mathcal{S} = (\mathsf{f}_\mathsf{d}, \mathcal{C}_1, \mathcal{C}_2)$ achieves full-diversity, we need to prove that every $\mathbf{X} \in \Delta\mathcal{S}$ has full rank. Since

TABLE I
EXAMPLES OF FINITE FEEDBACK SCHEMES AVAILABLE IN THE LITERATURE

<table>
<tr><th>Scheme</th><th>Setting</th><th>Component Code
$\mathcal{C}_n$</th><th>Feedback function
$\mathsf{f}(\mathbf{H})$</th><th>Rate
R</th></tr>
<tr><td rowspan="3">Precoded Spatial-Multiplexing [2]</td><td rowspan="3">$N_t, N > 1$, $T = 1$, $M < N_t$
$\mathbf{F}_1, \ldots, \mathbf{F}_N \in \mathbb{C}^{M \times N_t}$</td><td rowspan="3">$\{\mathbf{s}^T \mathbf{F}_n \vert \mathbf{s} \in \mathcal{A}^M\}$</td><td>$\mathsf{f}_{\mathsf{d}}(\mathbf{H})$</td><td rowspan="3">$M$</td></tr>
<tr><td>$\arg\max_{n \in \{1,\ldots,N\}}$
$\lambda_{\min}(\mathbf{F}_n \mathbf{H})$</td></tr>
<tr><td>$\arg\max_{n \in \{1,\ldots,N\}}$
$\det(\mathbf{I}_M + \frac{E}{N_0} \mathbf{F}_n \mathbf{H}\mathbf{H}^H \mathbf{F}_n^H)$</td></tr>
<tr><td>Grassmannian Beamforming [4]</td><td>$N_t, N > 1$, $T = 1$,
$\mathbf{u}_1, \ldots, \mathbf{u}_N \in \mathbb{C}^{N_t \times 1}$
have unit norm</td><td>$\{s\mathbf{u}_n^T \vert s \in \mathcal{A}\}$</td><td>$\arg\max_{n \in \{1,\ldots,N\}}$
$\Vert\mathbf{u}_n^T \mathbf{H}\Vert_F^2$</td><td>1</td></tr>
<tr><td>Precoded Orthogonal STBCs [5]</td><td>$N_t, N > 1$, $M < N_t$, $\mathcal{C}$ is a
$T \times M$ rate R orthogonal STBC,
$\mathbf{F}_1, \ldots, \mathbf{F}_N \in \mathbb{C}^{M \times N_t}$</td><td>$\{\mathbf{X}\mathbf{F}_n \vert \mathbf{X} \in \mathcal{C}\}$</td><td>$\arg\max_{n \in \{1,\ldots,N\}}$
$\Vert\mathbf{F}_n \mathbf{H}\Vert_F^2$</td><td>$\leq 1$</td></tr>
<tr><td>Heath, Jr. & Paulraj [8]</td><td>$N_t = N = T = 2$,
$\vert\mathcal{A}\vert = \vert\mathcal{A}'\vert^2$</td><td>$\mathcal{C}_1$ is Alamouti code using $\mathcal{A}$
$\mathcal{C}_2 = \left\{ \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \middle\vert s_i \in \mathcal{A}' \right\}$</td><td>$\mathsf{f}_{\mathsf{d}}(\mathbf{H})$</td><td>NA</td></tr>
<tr><td>No feedback [11]</td><td>$N = 1$, $N_t, T \geq 1$</td><td>$\mathcal{C}_1 \subset \mathbb{C}^{T \times N_t}$</td><td>1</td><td>$\leq N_t$</td></tr>
</table>

Notation: (i) $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}$ are complex constellations such as QAM, HEX or PSK.
(ii) $\mathsf{f}_{\mathsf{d}}(\mathbf{H}) = \arg\max_{n \in \{1,\ldots,N\}} \left\{\min_{\mathbf{X} \in \Delta\mathcal{C}_n} \Vert\mathbf{X}\mathbf{H}\Vert_F^2\right\}$, where $\Delta\mathcal{C}_n = \{\mathbf{X}_1 - \mathbf{X}_2 \mid \mathbf{X}_1, \mathbf{X}_2 \in \mathcal{C}_n, \mathbf{X}_1 \neq \mathbf{X}_2\}$.
(iii) $\lambda_{\min}(\mathbf{A})$ is the smallest singular value of $\mathbf{A}$.

both $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear, for any given $\mathbf{X} \in \Delta\mathcal{S}$ there exist $[u_1\ u_2]^T, [v_1\ v_2]^T \in \mathbb{Z}[i]^2 \setminus \{\mathbf{0}\}$, such that $\mathbf{X} = \begin{bmatrix} \alpha(u_1 + u_2\theta) & \sigma\left(\alpha(u_1 + u_2\theta)\right) \\ \alpha(v_1 + v_2\theta) & i\sigma\left(\alpha(v_1 + v_2\theta)\right) \end{bmatrix}$. Since $u_1, u_2 \in \mathbb{Q}(i)$ and $\{1, \theta\}$ is a basis of $\mathbb{Q}(i, \sqrt{5})$ as a vector space over $\mathbb{Q}(i)$, we have that $u = \alpha(u_1 + u_2\theta) \neq 0$, and similarly $v = \alpha(v_1 + v_2\theta) \neq 0$. Since $\det(\mathbf{X}) = iu\sigma(v) - v\sigma(u)$ and σ^2 is the identity map on $\mathbb{Q}(i, \sqrt{5})$, we have $\det(\mathbf{X}) = iz - \sigma(z)$, where $z = u\sigma(v) \in \mathbb{Q}(i, \sqrt{5}) \setminus \{0\}$. If $\mathbf{X}$ is not of full rank, $\det(\mathbf{X}) = 0$, i.e., $i = \frac{\sigma(z)}{z}$ for some $z \in \mathbb{Q}(i, \sqrt{5})$. This would imply that $i = \sigma(i) = \sigma\left(\frac{\sigma(z)}{z}\right) = \frac{z}{\sigma(z)} = \left(\frac{\sigma(z)}{z}\right)^{-1} = -i$, a contradiction. Hence $\mathbf{X}$ is of full rank.

III. New Full-Rate Full-Diversity FT-Optimal Finite Feedback Schemes

A. Review of some results from Algebraic Number Theory

For any two fields $\mathbb{K}$ and $\mathbb{F}$, if $\mathbb{F} \subseteq \mathbb{K}$ then $\mathbb{F}$ is a *subfield* of $\mathbb{K}$. For any $\alpha \in \mathbb{K}$, $\mathbb{F}(\alpha)$ denotes the smallest subfield of $\mathbb{K}$ that contains both $\mathbb{F}$ and α. An element $\alpha \in \mathbb{C}$ is said to be an *algebraic number*, or simply *algebraic*, if there exists a non-zero polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. The sum, difference, product and quotient of algebraic numbers are themselves algebraic numbers [13]. For example, for any $a \in \mathbb{Q}$ and positive integer n, $\sqrt[n]{a}$ is algebraic, since it satisfies the equation $x^n - a = 0$. Hence $\sqrt{2}, \sqrt{3}, i = \sqrt{-1}$ are all algebraic. Also, $\frac{1+\sqrt{5}}{2}$ is algebraic since it is a root of the equation $x^2 - x - 1 = 0$. If α is algebraic, the field $\mathbb{Q}(\alpha)$ is said to be an *algebraic number field*.

The following result from [14] gives a procedure to construct a set of algebraic numbers, of any desired finite cardinality, that is linearly independent over $\mathbb{Q}$. Let $n_1, \ldots, n_m$ be positive integers, $p_1, \ldots, p_m$ be distinct primes, and $b_1, \ldots, b_m$ be positive integers not divisible by any of these primes. For $k = 1, \ldots, m$, let α_k be the real algebraic number $(b_k p_k)^{\frac{1}{n_k}}$. Then the set $\left\{\alpha_1^{\ell_1}\alpha_2^{\ell_2} \cdots \alpha_m^{\ell_m} \mid 0 \leq \ell_k < n_k,\ k = 1, \ldots, m\right\}$, with cardinality $\prod_{k=1}^m n_k$, is linearly independent over $\mathbb{Q}$ [14]. For example, for $m = 2$, let $p_1 = 2$ and $p_2 = 3$ be the two distinct primes, $b_1 = b_2 = 1$, $n_1 = n_2 = 2$. Then we have $\alpha_1 = \sqrt{2}$ and $\alpha_2 = \sqrt{3}$, and $\left\{\alpha_1^{\ell_1}\alpha_2^{\ell_2} \mid 0 \leq \ell_1, \ell_2 < 2\right\} = \left\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\right\}$ is linearly independent over $\mathbb{Q}$. On multiplying each of the elements of this set by i, we see that $\{i, i\sqrt{2}, i\sqrt{3}, i\sqrt{6}\}$ is linearly independent over $\mathbb{Q}$.

In [15] rotation matrices $\mathbf{U} \in \mathbb{C}^{m \times m}$ were constructed for all $m > 1$ with the property that for any $\mathbf{a} \in \mathbb{Z}[i]^m \setminus \{\mathbf{0}\}$ and $\mathbf{s} = \mathbf{U}\mathbf{a}$, $\prod_{\ell=1}^m |\mathsf{s}(\ell)| > 0$, where $\mathsf{s}(\ell)$ denotes the ℓ^{th} component of $\mathbf{s}$. Further, these matrices were constructed over algebraic number fields, i.e., each element of the matrix $\mathbf{U}$ is an algebraic number. These matrices are known as *full-diversity algebraic rotations*, and a table of such known rotations is available in [16].

B. New Finite Feedback Schemes with $T = 1$

Let $\mathbf{U} \in \mathbb{C}^{N_t \times N_t}$ be any full-diversity algebraic rotation, $\alpha \in \mathbb{C}$ be any non-zero algebraic number, and $\gamma = e^{\alpha}$. The proposed FFS uses the following $N = N_t$ STBCs

$$\begin{aligned}
\mathcal{C}_1 &= \left\{ \begin{bmatrix} \gamma\mathsf{s}(1) & \mathsf{s}(2) & \cdots & \mathsf{s}(N_t) \end{bmatrix} \,\middle|\, \mathbf{s} = \mathbf{U}\mathbf{a},\ \mathbf{a} \in \mathcal{A}^{N_t} \right\}, \\
\mathcal{C}_2 &= \left\{ \begin{bmatrix} \mathsf{s}(1) & \gamma\mathsf{s}(2) & \cdots & \mathsf{s}(N_t) \end{bmatrix} \,\middle|\, \mathbf{s} = \mathbf{U}\mathbf{a},\ \mathbf{a} \in \mathcal{A}^{N_t} \right\}, \\
&\ \ \vdots \\
\mathcal{C}_{N_t} &= \left\{ \begin{bmatrix} \mathsf{s}(1) & \mathsf{s}(2) & \cdots & \gamma\mathsf{s}(N_t) \end{bmatrix} \,\middle|\, \mathbf{s} = \mathbf{U}\mathbf{a},\ \mathbf{a} \in \mathcal{A}^{N_t} \right\}.
\end{aligned}$$

Each of the above STBCs is obtained from $\mathbf{s}^T$ by multiplying one of its components with γ.

Lemma 1: If $\mathbf{U}$ is a full-diversity algebraic rotation and α is a non-zero algebraic number, the FFS $\mathcal{S} = (\mathsf{f}_{\mathsf{d}}, \mathcal{C}_1, \ldots, \mathcal{C}_{N_t})$

achieves full-diversity.

C. New Finite Feedback Schemes for $T > 1$

1) Some notations: The structure of the component codes of the new FFSs for $T > 1$ is similar to the threaded space-time architecture proposed in [17]. For any $T > 1$ denote addition modulo T by $\oplus_T$. For a set of T vectors $\mathbf{s}_1, \dots, \mathbf{s}_T \in \mathbb{C}^{T\times 1}$, we define a $T \times T$ matrix $\mathcal{T}(\mathbf{s}_1, \dots, \mathbf{s}_T) = [t_{i,j}]$ as follows. The entries of $\mathcal{T} = [t_{i,j}]$ are partitioned into T *threads*, one corresponding to each of the vectors $\mathbf{s}_1, \dots, \mathbf{s}_T$. The first thread of $\mathcal{T}$ originates at $t_{1,1}$ and occupies the main diagonal $\{t_{i,i} | i = 1, \dots, T\}$. These entries are populated by the components of the first vector $\mathbf{s}_1$. For $\ell = 2, \dots, T$, the ℓ^{th} thread consists of those entries of $\mathcal{T}$ that are $(\ell - 1)$ places to the right (in cyclic sense) of the entries of the main diagonal. These entries of $\mathcal{T}$ are occupied by the components of the vector $\mathbf{s}_\ell$. Hence, for $1 \le \ell, i \le T$ we have $t_{i,1+\left((i-1)\oplus_T(\ell-1)\right)} = \mathbf{s}_\ell(i)$. For example, for $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \mathbb{C}^{3\times 1}$, we have $\mathcal{T}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \begin{bmatrix} \mathbf{s}_1(1) & \mathbf{s}_2(1) & \mathbf{s}_3(1) \\ \mathbf{s}_3(2) & \mathbf{s}_1(2) & \mathbf{s}_2(2) \\ \mathbf{s}_2(3) & \mathbf{s}_3(3) & \mathbf{s}_1(3) \end{bmatrix}$.

For any $\mathbf{s} \in \mathbb{C}^{T\times 1}$ and $1 \le m \le n \le T$ we denote by $\mathbf{s}(m:n)$ the length $(n-m+1)$ vector $[\mathbf{s}(m)\ \mathbf{s}(m+1)\ \cdots\ \mathbf{s}(n)]^T$. If $\mathcal{T}_1, \dots, \mathcal{T}_N$ are $T \times T$ complex matrices, we define $\pi\left([\mathcal{T}_1\ \ \mathcal{T}_2\ \ \cdots\ \ \mathcal{T}_{N-1}\ \ \mathcal{T}_N]\right) = [\mathcal{T}_N\ \ \mathcal{T}_1\ \ \mathcal{T}_2\ \ \cdots\ \ \mathcal{T}_{N-1}]$, which is a cyclic shift of the $T \times T$ blocks one place to the right. For any $\mathcal{C} \subset \mathbb{C}^{T\times NT}$, let $\pi(\mathcal{C}) = \{\pi(\mathbf{X}) | \mathbf{X} \in \mathcal{C}\}$.

2) New FFSs for $T > 1$: Let $NT = N_t$, $\mathbf{U}$ be an $N_t \times N_t$ full-diversity algebraic rotation, $\mathcal{A} \subset \mathbb{Z}[i]$ be a QAM constellation, $\mathbf{U}\mathcal{A}^{N_t} = \{\mathbf{U}\mathbf{a} | \mathbf{a} \in \mathcal{A}^{N_t}\}$, $\mathbf{s}_1, \dots, \mathbf{s}_T \in \mathbf{U}\mathcal{A}^{N_t}$, $\beta_1, \dots, \beta_T$ be algebraic numbers that are linearly independent over $\mathbb{Q}$, and $\gamma_\ell = e^{\beta_\ell}$ for $\ell = 1, \dots, T$. The scalars $\beta_1, \dots, \beta_T$ can be obtained using the procedure explained in Section III-A. Now for each $\ell = 1, \dots, T$, partition the N_t-length vector $\mathbf{s}_\ell$ into N vectors $\mathbf{s}_\ell^{(1)}, \mathbf{s}_\ell^{(2)}, \dots, \mathbf{s}_\ell^{(N)}$ of length T each such that $\mathbf{s}_\ell{}^T = [\mathbf{s}_\ell^{(1)T}\ \ \mathbf{s}_\ell^{(2)T}\ \ \cdots\ \ \mathbf{s}_\ell^{(N)T}]$, i.e., $\mathbf{s}_\ell^{(1)} = \mathbf{s}_\ell(1:T)$, $\mathbf{s}_\ell^{(2)} = \mathbf{s}_\ell(T+1:2T), \dots, \mathbf{s}_\ell^{(N)} = \mathbf{s}_\ell(N_t - T + 1 : N_t)$. We now construct N matrices $\mathcal{T}_1, \dots, \mathcal{T}_N$ as

$$\mathcal{T}_1 = \mathcal{T}\left(\gamma_1 \mathbf{s}_1^{(1)}, \gamma_2 \mathbf{s}_2^{(1)}, \dots, \gamma_T \mathbf{s}_T^{(1)}\right), \text{ and}$$
$$\mathcal{T}_n = \mathcal{T}\left(\mathbf{s}_1^{(n)}, \mathbf{s}_2^{(n)}, \dots, \mathbf{s}_T^{(n)}\right) \text{ for } n = 2, \dots, N.$$

Finally, the N codebooks are

$$\mathcal{C}_1 = \left\{ [\mathcal{T}_1\ \ \mathcal{T}_2\ \ \cdots\ \ \mathcal{T}_N] \ \middle|\ \mathbf{s}_1, \dots, \mathbf{s}_T \in \mathbf{U}\mathcal{A}^{N_t} \right\}, \quad (1)$$
$$\mathcal{C}_n = \pi(\mathcal{C}_{n-1}),\ n = 2, \dots, N. \quad (2)$$

Example 1: To construct the FFS for $N_t = 4$, $N = T = 2$, consider the 4×4 rotation $\mathbf{U}$ from [16], $\gamma_1 = e^{i\sqrt{2}}$ and $\gamma_2 = e^{i\sqrt{3}}$. Note that in Section III-A we showed that $\{\beta_1, \beta_2\} = \{i\sqrt{2}, i\sqrt{3}\}$ is linearly independent over $\mathbb{Q}$. The two component STBCs of the proposed FFS are $\mathcal{C}_1 = \left\{ \begin{bmatrix} \gamma_1\mathbf{s}_1(1) & \gamma_2\mathbf{s}_2(1) & \mathbf{s}_1(3) & \mathbf{s}_2(3) \\ \gamma_2\mathbf{s}_2(2) & \gamma_1\mathbf{s}_1(2) & \mathbf{s}_2(4) & \mathbf{s}_1(4) \end{bmatrix} \ \middle|\ \mathbf{s}_1, \mathbf{s}_2 \in \mathbf{U}\mathcal{A}^4 \right\}$,

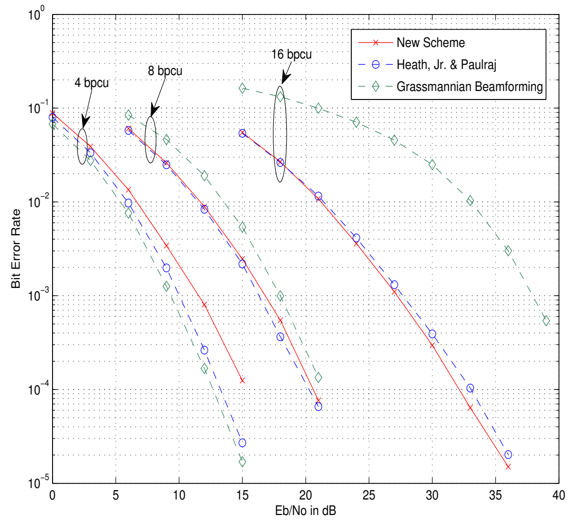

Fig. 1. FFSs for 2×2 MIMO with $N = 2$.

$\mathcal{C}_2 = \left\{ \begin{bmatrix} \mathbf{s}_1(3) & \mathbf{s}_2(3) & \gamma_1\mathbf{s}_1(1) & \gamma_2\mathbf{s}_2(1) \\ \mathbf{s}_2(4) & \mathbf{s}_1(4) & \gamma_2\mathbf{s}_2(2) & \gamma_1\mathbf{s}_1(2) \end{bmatrix} \ \middle|\ \mathbf{s}_1, \mathbf{s}_2 \in \mathbf{U}\mathcal{A}^4 \right\}$. With $\mathcal{T}_1 = \mathcal{T}\left(\gamma_1\mathbf{s}_1(1:2), \gamma_2\mathbf{s}_2(1:2)\right)$ and $\mathcal{T}_2 = \mathcal{T}\left(\mathbf{s}_1(3:4), \mathbf{s}_2(3:4)\right)$, each codeword of $\mathcal{C}_1$ is of the form $[\mathcal{T}_1\ \ \mathcal{T}_2]$, and the two STBCs are related as $\mathcal{C}_2 = \pi(\mathcal{C}_1)$.

Theorem 3: If $\mathbf{U}$ is a full-diversity algebraic rotation and $\beta_1, \dots, \beta_T$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, the FFS $\mathcal{S} = (\mathsf{f_d}, \mathcal{C}_1, \dots, \mathcal{C}_N)$ achieves full-diversity, where $\mathcal{C}_1, \dots, \mathcal{C}_N$ are given by (1) and (2)

Every component STBC of each of the proposed FFSs has the property that each entry of the codeword matrix is a linear combination of QAM information symbols. Hence, sphere-decoding [18] can be used to implement both the ML decoder and the feedback function $\mathsf{f_d}$ for all new FFSs.

IV. Simulation Results

All the codes discussed in this section use square QAM constellations, and Gray encoding to map information bits into QAM symbols. In all the simulations, the new FFSs have the best performance while utilizing the least amount of feedback and transmission duration.

Fig. 1 shows the comparison of the performance of the new scheme constructed in Section II-C with Grassmannian Beamforming [4] and the scheme of Heath, Jr. & Paulraj [8] for bitrates 4, 8 and 16 bpcu and $N_t = N_r = N = 2$. While the new FFS does not fare well for 4 bpcu, its relative performance improves as the bitrate increases, and for 16 bpcu it has the lowest BER among the three schemes. Fig. 2 shows that the new FFS with $T = 1, N = 3$ constructed using the procedure of Section III-B with the 3×3 rotation from [16] and $\gamma = e^{i\left(\frac{1+\sqrt{5}}{2}\right)}$ outperforms Grassmannian Beamforming [4] ($N = 3$ and $N = 16$) and the FFS of Wu & Calderbank [9] ($N = 3$) for 6 bpcu and $N_t = N_r = 3$.

We now consider schemes for $N_t = N_r = 4$ that use $N \ge 4$. Fig. 3 shows that the new FFS with $N = 4, T = 1$ constructed using the procedure in Section III-B using $\gamma = e^{i\left(\frac{1+\sqrt{5}}{2}\right)}$ and

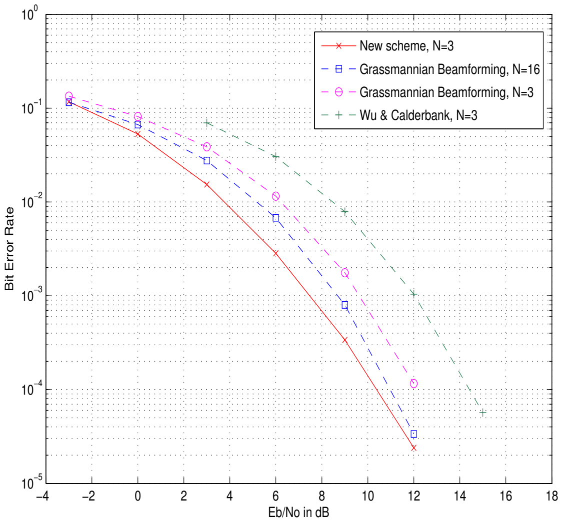

Fig. 2. FFSs for 3×3 MIMO with 6 bpcu.

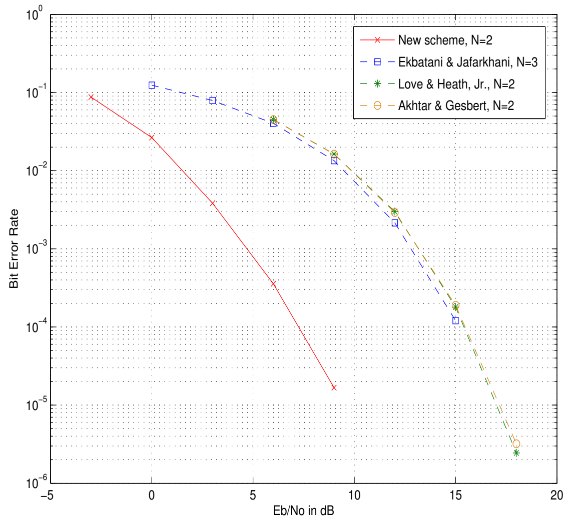

Fig. 4. FFSs for 4×4 MIMO with 8 bpcu and $N < 4$.

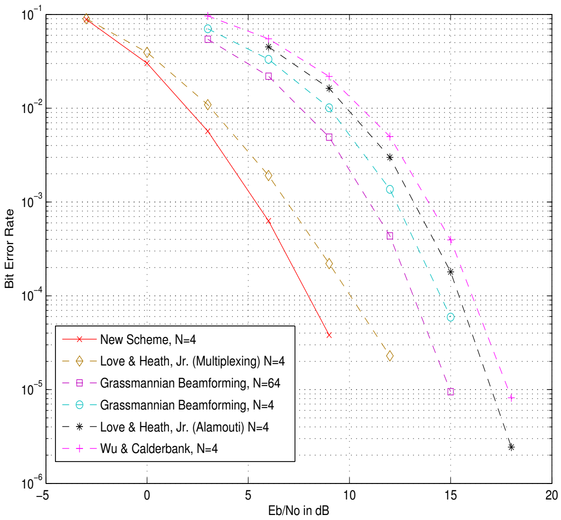

Fig. 3. FFSs for 4×4 MIMO with 8 bpcu and $N \geq 4$.

the 4×4 rotation from [16] has better performance when compared with the following schemes for 8 bpcu: *(i)* the $N = 4$ scheme of Love & Heath, Jr. [2] that precodes a rate 2 spatial multiplexing input and uses $\mathsf{f} = \mathsf{f}_\mathsf{d}$, *(ii)* Grassmannian Beamforming [4] with $N = 64$, *(iii)* Grassmannian Beamforming with $N = 4$, *(iv)* the $N = 4$ scheme of Love & Heath, Jr. [5] that precodes the Alamouti code, *(v)* the $N = T = 4$ FFS of Wu & Calderbank [9]. Finally, the performance of schemes for $N_t = N_r = 4$ that use $N < 4$ is shown in Fig. 4 for the bitrate of 8 bpcu. The new scheme of Example 1 is compared with: *(i)* $N = 3$ scheme of Ekbatani & Jafarkhani [6], *(ii)* $N = 2$ scheme of Love & Heath, Jr. [5], and *(iii)* $N = 2$ scheme of Akhtar & Gesbert [7].

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