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The Diversity Multiplexing Tradeoff of the Half-Duplex Relay Network

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***Abstract*—The diversity multiplexing-gain tradeoff (DMT) of a Rayleigh faded relay network (RN) with a single source-destination pair and n half-duplex relay nodes, all with single antennas, is characterized and shown to be equal to the DMT of an $(n+1)\times 1$ MISO point-to-point channel. Further, it is shown that the *quantize map-and-forward* (QMF) scheme with fixed, channel independent scheduling for the relays can achieve the fundamental DMT of the RN. An interesting consequence is that the DMT of the RN can be achieved without any channel state information (CSI) at the relay nodes.**

***Index Terms*—Diversity-multiplexing tradeoff, Half-duplex, Relay network.**

I. Introduction

It is more the rule rather than the exception that there are multiple idle mobile phones present in the vicinity of an active mobile phone communicating with its base station. The performance of this communication link can be greatly improved if one or more of the idle mobiles are enlisted (as relays) to forward some processed copy of their received signal from the transmitter to the receiver. Such a network of communicating nodes is called the relay network (RN). While the performance improvement due to such a cooperation has been extensively analyzed and relatively well understood for the relay channel with a single relay node, the same can not be said about the channel with multiple relay nodes. In fact, even the DMT of the simplest such channel, where the communication between a single-antenna source-destination pair is helped by *two* single-antenna half-duplex[1] (HD) relays, is not known. In this paper, we characterize the DMT of an RN with n HD relays under quasi-static Rayleigh fading in which all the involved path gains or channel coefficients (see Fig. 2) are assumed to be independently and identically (Rayleigh) distributed (i.i.d.). In what follows, we refer to the relay network with n half-duplex relays simply as the n-RN.

Allowing the relays to fully cooperate with the source node cannot reduce the DMT of the channel. So, the DMT of the $(n+1)\times 1$ MISO point-to-point channel serves as an upper bound on the fundamental DMT of the n-RN, i.e.,

$$d^*(r) \leq (n+1)(1-r),\ 0 \leq r \leq 1,$$

where $d^*(r)$ denotes the highest diversity order achievable on the n-RN at multiplexing gain r.

The n-RN has been studied before but under an idealized assumption. In particular, it was shown in [1] that under the assumption that the relay nodes are mutually *non-interfering*, the previously stated full cooperation upper bound can be achieved. The more realistic scenario is one where each relay can also hear the transmissions of all other relays besides that of the source node. It is not clear whether this inter-relay chatter at each relay fundamentally degrades the DMT performance of the n-RN. We show herein that it does not.

The best known DMT performance on the n-RN, where the relays are not assumed to be non-interfering, is achieved by the *dynamic decode-and-forward* (DDF) [2] and Amplify-and-Forward (AF) protocol [3] for general n and for $n=2$, respectively. The performance of the DDF protocol on a 4-RN is depicted in Fig. 1 with the black dashed line. The red solid line in the figure represents the DMT of the *Dynamic-Compress and Forward* (DCF) protocol on a relay channel with a single 4-antenna relay node. Evidently, the DMT of the DDF protocol is far from the MISO bound. Comparing the two DMTs, it is easy to see that the difference between the two also increases with the number of relay nodes.

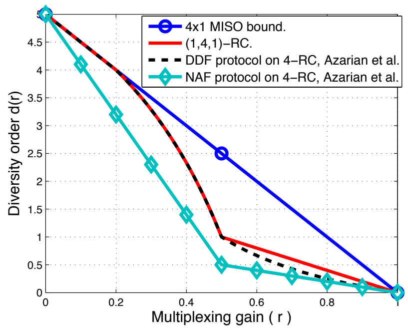

Fig. 1: Comparison of the performance of different coding schemes with the MISO bound.

In this paper, we show that the fundamental DMT of the n-RN is identical to that of the MISO channel, i.e., allowing the relays to interfere with each other does not hamper its DMT performance. Moreover, no channel state information (CSI) is needed at the relays to achieve this optimal tradeoff. The rest of the paper is organized as follows: in Section II, we provide a description of the system model and some preliminaries including an expression for the cut-set bound for the n-RN. In Section III, we obtain the DMT of the n-RN. Section IV concludes the paper.

II. System Model and Preliminaries

We consider a Rayleigh faded SISO HD-RN with n HD relays, denoted by $r_1, \cdots r_n$, which help the source node,

[1]A relay node is called a half-duplex relay if it can only either transmit or receive at a given time but not both.

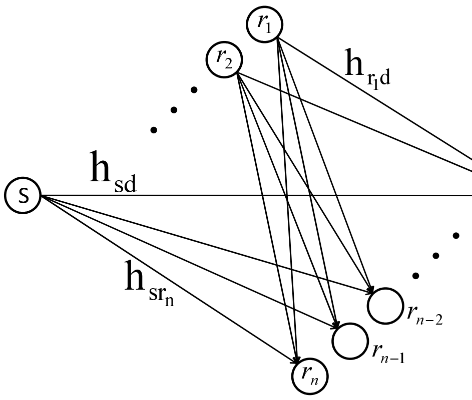

Fig. 2: System model of the RN with n relay nodes.

s to communicate with the destination node, d. We denote the collection of all the nodes in the network by $\mathcal{V}$ and the collection of relays by $\mathcal{V}_r$, i.e., $\mathcal{V}_r = \mathcal{V} \setminus \{s, d\}$. Due to the HD constraint, a relay can either transmit or receive but not both simultaneously.[2] Since a relay can operate in one of these 2 modes and there are n relays, there are $2^n = M$ different combinations in which the relays can operate. Each of these combinations is called a state of the network and will be denoted by $m \in \{1, \cdots, 2^n\}$. For instance, Fig. 2 depicts a state of the network where only the relays above the horizontal line are in the transmitting mode and the relays below are in the receiving mode. Note that in this figure, we do not show any *active*[3] link ending on the relay nodes that are transmitting because the relays are assumed to be HD. Let us denote by t_m the fraction of time for which the network remains in state m, where $0 \le t_m \le 1$, then $\sum_{m=1}^{M} t_m = 1$. The sequence of these network state durations, i.e., $\{t_m\}_{m=1}^{M}$ will also be referred to as the "transmit-receive schedule" of the relays.

The channel coefficient between transmitter i and receiver j is denoted by h_{ij}. The different channel coefficients of the network, $h_{r_1d}, \cdots h_{sd}, \cdots, h_{sr_n}$ are assumed to be independent and identically distributed (i.i.d.) as $\mathcal{CN}(0, 1)$. For economy of notation, all these channel coefficients will collectively denoted by $\mathcal{H}$. The relay nodes are assumed to know only their incoming channel coefficients, i.e., r_i knows only h_{sr_i} for $1 \le i \le n$, and the destination knows all the channel coefficients of the network, i.e., the *global* CSI.

Let us denote by $x_a[t]$ the signal transmitted by node $a \in \{s, r_1, \cdots, r_n\}$ and by $X_V^{(m)}$ a vector whose components are the $x_a[t]$'s with $a \in V \subseteq \mathcal{V} \setminus \{d\}$ and are in the transmit mode in state m. Similarly, we denote by $Y_W^{(m)}$ a vector whose components are the $y_b[t]$'s with $b \in W \subseteq \mathcal{V} \setminus \{s\}$ and are in the receive mode in state m. With this notation, the received signal at any node $b \in \mathcal{V} \setminus \{s\}$, in network state m can be written as

$$y_b^{(m)} = \sum_{a \in \mathcal{V} \setminus \{d\}} x_a^{(m)} h_{ab} + z_b,$$

where $x_a^{(m)} = x_a$, if a is in the transmit mode in state m and $x_a^{(m)} = 0$ otherwise and similarly, $y_b^{(m)} = y_b$, if b is in the receive mode in state m and $y_b^{(m)} = 0$, otherwise. The additive noises $\{z_b\}_b$ are i.i.d. zero-mean, unit variance, complex Gaussian random variables (denoted as $\mathcal{CN}(0, 1)$). Assuming that the length of the codeword transmitted by the source is L_c, the input at node a satisfies the following average power constraint:

$$\frac{1}{L_c} \sum_{m=1}^{M} \sum_{t = L_c \sum_{l=1}^{(m-1)} t_l + 1}^{L_c \sum_{l=1}^{m} t_l} \mathbb{E} \left| x_a^{(m)}[t] \right|^2 \le \rho. \tag{1}$$

A. The cut-set upper bound

The n-RN considered in this paper, having finite number of states, is a special case of the general relay network considered in [4]. From Theorem 2 of [4] it follows that any rate, R, at which information can be transferred from the source to the destination with vanishing probability of error is upper bounded as follows

$$\begin{aligned} R \le & \max_{\left\{ \begin{array}{c} p\left(\{x_a^{(m)}\}_{a \in \mathcal{V},\ m \in \{1, \cdots, M\}}\right) \\ t_m :\ 0 \le t_m \le 1,\ \sum_{m=1}^{M} t_m = 1 \end{array} \right\}} \\ & \min_{\Omega \in \Lambda_s} \sum_{m=1}^{M} t_m I\left(X_{\Omega}^{(m)}; Y_{\Omega^c}^{(m)} | X_{\Omega^c}^{(m)} \right), \\ \triangleq & \max_{\left\{ t_m :\ 0 \le t_m \le 1,\ \sum_{m=1}^{M} t_m = 1 \right\}} \bar{C}\left(\mathcal{H}, \{t_m\}\right), \end{aligned} \tag{2}$$

where $X_{\mathcal{S}}^{(m)} = \{x_a^{(m)} : a \in \mathcal{S}\}$, $Y_{\mathcal{S}}^{(m)} = \{y_b^{(m)} : b \in \mathcal{S}\}$ and Λ_s is the set of all possible subsets of $\mathcal{V}_r \cup \{s\}$, or equivalently, represents the set of all possible cuts that separate the source from the destination. Note that in (2), $\bar{C}\left(\mathcal{H}, \{t_m\}\right)$ represents the cut-set upper bound on the rate of reliable information transfer from the source to the destination, for the given sequence of network state's durations, $\{t_m\}$ which are known at the different relay nodes.

B. Diversity order and multiplexing gain

Let $\{\mathcal{C}(\rho)\}$ be a sequence of codebooks, where for each ρ the corresponding codebook $\mathcal{C}(\rho)$ consists of $2^{L_c R(\rho)}$ codewords, each of which is a $1 \times L_c$ vector satisfying the input average power constraint, (1). The sequence of codebooks is said to have a multiplexing gain of r if

$$\lim_{\rho \to \infty} \frac{R(\rho)}{\log(\rho)} = r.$$

Further, suppose for such a coding scheme $\mathcal{C}(\rho)$, $P_{e,\mathcal{C}}(r, \rho)$ represents the average probability of decoding error at the destination node (averaged over the Gaussian noise, channel realizations and the different codewords of the codebook) at

[2] However, the source node is always in transmit mode and the destination node in receive mode.

[3] A communication link emerging from a transmitting node and ending onto a listening node is called an active link.

an SNR of ρ, then the optimal diversity order (over all coding schemes at multiplexing gain of r) is defined as

$$d^*_{\mathcal{C}}(r) = \lim_{\rho\to\infty} \frac{-\log\left(P_{e,\mathcal{C}}(r,\rho)\right)}{\log(\rho)}. \quad (3)$$

III. DMT OF THE n-RN

Theorem 1: The DMT of the n-RN (shown in Fig. 2) is the same as that of an $(n+1)\times 1$ MISO point-to-point channel. Furthermore, it is achievable by the QMF scheme with a uniform,[4] channel-independent scheduling of the relay nodes where the network stays in each of the possible states the same amount of time.

Proof: As stated previously, it is well-known that the optimal diversity order, $d^*(r)$ of the n-RN at multiplexing gain of r admits the MISO point-to-point channel upper bounded as follows:

$$d^*(r) \le (n+1)(1-r), \text{ for } 0 \le r \le 1. \quad (4)$$

The proof of achievability is given in Sections III-A and III-B. ■

Remark 1: We know from [5] that when the QMF protocol operates with a fixed channel independent scheduling of the relays, the relay nodes do not require any CSI. Therefore, the DMT of the channel can be achieved without any CSI at the relay nodes.

Remark 2: In Theorem 4.2 of [1], the DMT of the n-RN with non-interfering relays was obtained and shown to be equal to that of an $(n+1)\times 1$ MISO point-to-point channel. Theorem 1 asserts that the DMT of the channel is unchanged even if the relays interfere with each other.

Remark 3: As a corollary of the theorem, it can be concluded that any coding scheme that can achieve the cut-set bound of the channel within a bounded gap for a fixed scheduling can also achieve the MISO bound and hence the DMT of the n-RN, as does the compress-and-forward scheme of [6]. However, the compress-and-forward scheme scheme in [6] requires *global* channel state information (CSI), namely, all the channel coefficients of the network, at each of the relays to compute the rate of compression at each relay.

A. Proof of achievability: a lower bound on the DMT

Since the fundamental DMT is the supremum of the achievable DMTs of all possible coding schemes, the achievable DMT of a particular coding scheme yields a lower bound to it. In this section, we derive such a lower bound for the n-RN by computing the achievable DMT of the QMF protocol [5]. But first we start with a brief review of the QMF scheme.

1) The coding and decoding strategies of the QMF scheme: The encoding method at the source node of the QMF scheme is a two step procedure involving two different codes. The inner codebook, denoted by $\mathcal{T}_{\bar{x}_S}$ has 2^{RT} mutually independent codewords, i.e.,

$$\mathcal{T}_{\bar{x}_S} = \{\bar{x}_S^{(w)} : w = 1,\cdots,2^{RT}\},$$

where $\bar{x}_S^{(w)}$ for each $w \in \{1,\cdots,2^{RT}\}$ is a random T length vector with i.i.d. $\mathcal{CN}(0,1)$ components. Each codeword of the inner codebook is treated as a symbol of the outer code. To transmit a message $\mathcal{U} \in \{1,\cdots,2^{NRT}\}$, the source node first maps it onto a N length sequence of symbols of the outer code, i.e., $w_1, w_2, \cdots, w_N$. Each of these symbols are then encoded into a codeword in $\mathcal{T}_{\bar{x}_S}$, i.e., $w_k \to \bar{x}_S^{(w_k)}$ for $1 \le k \le N$. Finally, the message $\mathcal{U}$ is transmitted over NT channel uses and hence at a rate of R bits per channel use.

The network passes through all the possible 2^n states during each of the inner codeword and stays in the m-th state for Tt_m channel uses, for $m = 1,\cdots,2^n$. Suppose, the network is in the m-th state during the transmission of the k-th codeword, $\bar{x}_S^{(w_k)}$. A relay node, say the i-th relay, can be either in the transmit or the listening mode in this state. If it is in the listening mode, then it receives the signal $y_{r_i}^{(k,m)} \in \mathbb{C}^{1\times Tt_m}$. Otherwise, if it is in the transmit phase, it quantizes all the received signals during the previous inner codeword (i.e., $y_{r_i}^{(k-1,m)}$, $m = 1,\cdots,2^n$) to $\hat{y}_{r_i}^{(k,m)} \in \mathbb{C}^{1\times Tt_m}$, randomly maps into a Gaussian codeword $x_{r_i}^{(k,m)} \in \mathbb{C}^{1\times Tt_m}$ using a random mapping function $f_{r_i}(\hat{y}_{r_i}^{(k,m)})$ and sends it during the Tt_m channel uses. The same procedure is repeated by the relay for all N inner codewords that make up the source codeword.[5] Given the knowledge of the relay mappings, $f_{r_i}(.)$, for $1 \le i \le n$, the global CSI, and the received sequence $y_d^{(w_k)} \in \mathbb{C}^{1\times T}$ for $k = 1,\cdots,N$, the destination node decodes the message.

2) Achievable DMT of the QMF protocol: By Theorem 8.3 of [5], for any fixed transmit-receive schedule for the relays, i.e., given the sequence $\{t_m\}_{m=1}^M$ for the duration of different states of the network, the QMF protocol can achieve a rate $R_q(\mathcal{H}, \{t_m\}_{m=1}^M)$, such that

$$R_q(\mathcal{H}, \{t_m\}_{m=1}^M) \ge \bar{C}(\mathcal{H}, \{t_m\}_{m=1}^M) - \tau, \quad (5)$$

where τ is a constant independent of the channel parameters and the SNR, ρ. Let us assume a uniform transmit-receive scheduling of the relays, i.e., $t_m = \frac{1}{2^n}$ for all $m \in \{1,\cdots,2^n\}$ and denote the corresponding achievable rate of the QMF protocol by $R_q(\mathcal{H}, \text{uniform})$ and the corresponding cut-set bound by $\bar{C}(\mathcal{H}, \text{uniform})$. Then, from Theorem 8.3 of [5] or (5) again, we have

$$R_q(\mathcal{H}, \text{uniform}) \ge \bar{C}(\mathcal{H}, \text{uniform}) - \tau. \quad (6)$$

Since we are considering a fading channel, for any given target data rate, $R = r\log(\rho)$, there exists channel coefficients for which R can not be achieved by the QMF protocol. Such an event is called an outage event, denoted by $\mathcal{O}_q(r)$ and formally defined as

$$\mathcal{O}_q(r) = \{\mathcal{H} : R_q(\mathcal{H}, \text{uniform}) < r\log(\rho)\}. \quad (7)$$

Let the average probability of error (averaged over the random channel coefficients) of the QMF scheme at the target

[4] A "uniform" scheduling refers to the scheduling, where $t_m = \frac{1}{M}$ for all $m \in \{1,\cdots,M\}$.

[5] During the transmission of $\bar{x}_S^{(w_1)}$ none of the relay nodes can transmit, however as $N\to\infty$ the loss due to this vanishes.

rate of $R = r\log(\rho)$ and SNR of ρ be denoted by $P_{e,q}(r,\rho)$ then we have

$$\begin{aligned} P_{e,q}(r,\rho) =& \Pr(\mathcal{E}|\mathcal{O}_q(r))\Pr(\mathcal{O}_q(r)) + \Pr(\mathcal{E}|\mathcal{O}_q^c(r))\Pr(\mathcal{O}_q^c(r)) \\ \le& \Pr(\mathcal{O}_q(r)) + \Pr(\mathcal{E}|\mathcal{O}_q^c(r)) \\ \le& \Pr(\mathcal{O}_q(r)), \end{aligned} \tag{8}$$

where $\Pr(\mathcal{E}|\mathcal{O}_q(r))$ represents the probability of error given the channel is in outage and (8) follows from the fact that, at any given ρ if the channel is not in outage, the destination can decode the message with probability of error going to zero (which follows from (6) and (7)), i.e., by letting the block length to be sufficiently large $\Pr(\mathcal{E}|\mathcal{O}_q^c(r))$ can be made to decay much faster than $\Pr(\mathcal{O}_q(r))$.

Now, substituting the expression for the outage event from (7) in (8), and using (6), we get

$$P_{e,q}(r,\rho) \le \Pr\left(\bar{C}(\mathcal{H},\text{uniform}) - \tau < r\log(\rho)\right), \tag{9}$$

where the last step follows from (5). Next, substituting (9) into (3), we get

$$d_{\text{QMF}}(r) \ge \lim_{\rho\to\infty} -\frac{\log\left(\Pr\left\{\bar{C}(\mathcal{H},\text{uniform}) < r\log(\rho)\right\}\right)}{\log(\rho)}, \tag{10}$$

because τ is a constant independent of ρ.

Lemma 1: i. For the uniform transmit-receive schedule of the relays the cut-set upper bound of the channel is lower bounded as

$$\begin{aligned} \bar{C}(\mathcal{H},\text{uniform}) \ge& \min_{\Omega\in\Lambda_s} \underbrace{\frac{1}{2^n}\sum_{m=1}^{M} I\left(\widetilde{X}_\Omega^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right)}_{\bar{C}_\Omega^G} \\ \triangleq& \min_{\Omega\in\Lambda_s} \bar{C}_\Omega^G, \end{aligned} \tag{11}$$

where $I\left(\widetilde{X}_\Omega^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right)$ is $I\left(X_\Omega^{(m)};Y_{\Omega^c}^{(m)}|X_{\Omega^c}^{(m)}\right)$ but with the inputs being i.i.d. as $\mathcal{CN}(0,\rho)$.

ii. Further for all $\Omega \in \Lambda_s$,

$$\Pr\left\{\bar{C}_\Omega^G \le r\log(\rho)\right\} \le K_\Omega \rho^{-(n+1)(1-r)^+}, \tag{12}$$

where $K_\Omega = o(\log(\rho))$.[6]

Proof: The proof is given in Section III-B. ■

Putting the lower bound from (11) in (10) and using the union bound we get

$$d_{\text{QMF}}(r) \ge \lim_{\rho\to\infty} -\frac{\log\left(\sum_{\Omega\in\Lambda_s}\Pr\left\{\bar{C}_\Omega^G < r\log(\rho)\right\}\right)}{\log(\rho)}. \tag{13}$$

Substituting (12) from the second part of Lemma 1 into (13), we get

$$\begin{aligned} d_{\text{QMF}}(r) \ge& (n+1)(1-r) - \lim_{\rho\to\infty}\frac{\log\left(\sum_{\Omega\in\Lambda_s} K_\Omega\right)}{\log(\rho)} \\ =& (n+1)(1-r). \end{aligned} \tag{14}$$

[6]For a real number $u \in \mathbb{R}$, we say $u = o(\log(\rho))$ if $\lim_{\rho\to\infty}\frac{\log(u)}{\log(\rho)} = 0$.

B. Proof of Lemma 1

1) Proof of Part (i): Starting with expression for the cut-set bound from (2), and substituting $t_m = \frac{1}{2^n}$ for all m, we have

$$\begin{aligned} \bar{C}(\mathcal{H},\text{uniform}) =& \max_{\{p(\{x_a\}_{a\in\mathcal{V}})\}} \min_{\Omega\in\Lambda_s} \frac{1}{2^n}\sum_{m=1}^{M} I\left(X_\Omega^{(m)};Y_{\Omega^c}^{(m)}|X_{\Omega^c}^{(m)}\right), \\ \ge& \min_{\Omega\in\Lambda_s} \frac{1}{2^n}\sum_{m=1}^{M} I\left(\widetilde{X}_\Omega^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right) \end{aligned} \tag{15}$$

where (15) follows from the fact that instead of maximizing the quantity on the right hand side among all possible input distributions at different nodes the maximization is restricted to one particular distribution.

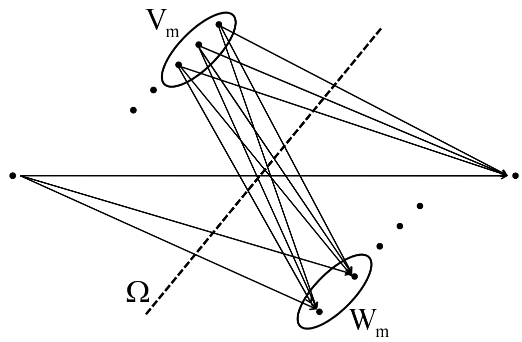

Fig. 3: An arbitrary cut Ω, in the m-th state of the network.

2) Proof of Part (ii): The proof will be divided into two parts: in part (ii.a), we find a lower bound on $\bar{C}_\Omega^G$ in terms of the different channel coefficients as follows:

$$\begin{aligned} \bar{C}_\Omega^G \ge& \frac{1}{(n+1)}\Bigg(\log(1+\rho|h_{sd}|^2) + \sum_{a\in\Omega\setminus\{s\}}\log(1+\rho|h_{ad}|^2) \\ &+ \sum_{b\in\Omega^c\setminus\{d\}}\log(1+\rho|h_{sb}|^2)\Bigg), \end{aligned} \tag{16}$$

and then in part (ii.b), we use this lower bound on $\bar{C}_\Omega^G$ to derive an upper bound to $\Pr(\bar{C}_\Omega^G \le r\log(\rho))$.

We prove part (ii.a) first. Consider an arbitrary cut, Ω, as shown in Fig. 3, where the network operates in state m. We denote the set of transmitting relays in Ω by V_m and the receiving relays in Ω^c by W_m. By this assumption, the relays in $\Omega\cap V_m^c$ do not transmit and therefore, $I\left(\widetilde{X}_\Omega^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right)$ represents the mutual information of a $(|V_m|+1)\times(|W_m|+1)$ MIMO point-to-point channel. Let us denote the channel matrix from the set of transmitters in $V_m\cup\{s\}$ to the set of receivers in $W_m\cup\{d\}$ by H_Ω. Recall that the components of $\widetilde{X}_\Omega^{(m)}$ are i.i.d. as $\mathcal{CN}(0,\rho)$, which in turn by the above assumptions imply that $\widetilde{X}_\Omega^{(m)} \sim \mathcal{CN}(0,\rho I_{(|V_m|+1)})$. Using this fact and elementary results in algebra it can be shown that

$$I\left(\widetilde{X}_{\Omega}^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right) \geq \max_{a\in V_m\cup\{s\},\ b\in W_m\cup\{d\}} \log\left(1+\rho|h_{ab}|^2\right) \triangleq f(V_m,W_m). \quad (17)$$

To obtain the desired lower bound we need to use a result presented in Lemma B.2 of [1] which we state below:

Lemma 2 (Lemma B.2 of [1]): Consider a set of numbers $s_0, s_1, \cdots s_n$. Assume function $\hat{f}(.)$ is such that for any set $V \subseteq \{1,\cdots,n\}$ we have

$$\hat{f}(V) \geq \max(s_0, s_V), \text{ where } s_V = \{s_i | i \in V\}. \quad (18)$$

Then

$$\frac{1}{2^n}\sum_{V\subseteq\{1,\cdots,n\}} \hat{f}(V) \geq \frac{s_0+\sum_{i=1}^n s_i}{(n+1)}. \quad (19)$$

Before using the above lemma, let us verify that the function in (17) satisfies the condition (18). For any $V_m \subseteq \Omega\setminus\{s\}$ and $W_m \subseteq \Omega^c\setminus\{d\}$, we have

$$\begin{aligned} f(V_m,W_m) =& \max_{\{a\in V_m\cup\{s\},\ b\in W_m\cup\{d\}\}} \log\left(1+\rho|h_{ab}|^2\right) \\ \geq& \max\{g_{sd}, \mathcal{N}_{V_m,d}, \mathcal{N}_{s,W_m}\}, \end{aligned}$$

where $g_{ab} = \log(1+\rho|h_{ab}|^2)$, $\mathcal{N}_{V_m,d} = \{g_{ad}|a \in V_m\}$ and $\mathcal{N}_{s,W_m} = \{g_{sb}|b \in W_m\}$.

Now, from (11) we have

$$\begin{aligned} \bar{C}_{\Omega}^{G} =& \frac{1}{2^n}\sum_{m=1}^{M} I\left(\widetilde{X}_{\Omega}^{(m)};\widetilde{Y}_{\Omega^c}^{(m)}|\widetilde{X}_{\Omega^c}^{(m)}\right) \\ \geq& \frac{1}{2^n}\sum_{V_m\subseteq\Omega\setminus\{s\}}\ \sum_{W_m\subseteq\Omega^c\setminus\{d\}} f(V_m,W_m) \quad (20) \\ \geq& \frac{1}{(n+1)}\left(g_{sd}+\sum_{a\in\Omega\setminus\{s\}} g_{ad}+\sum_{b\in\Omega^c\setminus\{d\}} g_{sb}\right), \quad (21) \end{aligned}$$

where (20) follows from (17) and the fact that while the network visits its different states V_m and W_m visit all the subsets of $\Omega\setminus\{s\}$ and $\Omega^c\setminus\{d\}$, respectively, and the last step follows from Lemma 2. This completes the proof of part (ii.a).

We next prove part (ii.b). Let us define the following set

$$\begin{aligned} \mathcal{O}_{\Omega}(r) =& \Bigg\{ \left(g_{sd}, \{g_{ad}\}_{a\in\Omega\setminus\{s\}}, \{g_{sb}\}_{b\in\Omega^c\setminus\{d\}}\right) : \\ & \frac{1}{(n+1)}\left(g_{sd}+\sum_{a\in\Omega\setminus\{s\}} g_{ad}+\sum_{b\in\Omega^c\setminus\{d\}} g_{sb}\right) \leq r\log(\rho) \Bigg\} \\ =& \left\{\bar{\alpha} : \alpha_{sd}+\sum_{a\in\Omega\setminus\{s\}}\alpha_{ad}+\sum_{b\in\Omega^c\setminus\{d\}}\alpha_{sb} \leq (n+1)r\right\} \quad (22) \end{aligned}$$

where $\bar{\alpha} = \left(\alpha_{sd}, \{\alpha_{ad}\}_{a\in\Omega\setminus\{s\}}, \{\alpha_{sb}\}_{b\in\Omega^c\setminus\{d\}}\right)$ and for $a,b \in \mathcal{V}$

$$\alpha_{ab} = \lim_{\rho\to\infty}\frac{\log(1+\rho|h_{ab}|^2)}{\log(\rho)}. \quad (23)$$

It can be easily shown that

$$f_{\alpha_{ab}}(\alpha_{ab}) = K_{ab}\rho^{-(1-\alpha_{ab})}, \quad 0 \leq \alpha_{ab} \leq 1 \quad (24)$$

where $K_{ab} = o(\log(\rho))$. Since h_{sd}, $\{h_{ad}\}_{a\in\Omega\setminus\{s\}}$ and $\{h_{sb}\}_{b\in\Omega^c\setminus\{d\}}$ are mutually independent, so are α_{sd}, $\{\alpha_{ad}\}_{a\in\Omega\setminus\{s\}}$ and $\{\alpha_{sb}\}_{b\in\Omega^c\setminus\{d\}}$, which in turn implies that

$$f_{\bar{\alpha}}(\bar{\alpha}) = K'_{\Omega}\rho^{-\left\{(n+1)-\left(\alpha_{sd}+\sum_{a\in\Omega\setminus\{s\}}\alpha_{ad}+\sum_{b\in\Omega^c\setminus\{d\}}\alpha_{sb}\right)\right\}} \quad (25)$$

with $0 \leq \alpha_{ab} \leq 1$, and where $K'_{\Omega} = \Pi K_{ab} = o(\log(\rho))$.

Combining (21) and (22) we have

$$\begin{aligned} \Pr(\bar{C}_{\Omega}^{G} \leq r\log(\rho)) \leq& \Pr\{\mathcal{O}_{\Omega}(r)\} = \int_{\bar{\alpha}\in\mathcal{O}_{\Omega}(r)} f_{\bar{\alpha}}(\bar{\alpha})d\bar{\alpha} \\ =& \int_{\substack{\bar{\alpha}\in\mathcal{O}_{\Omega}(r)\\ 0\leq\bar{\alpha}\leq 1}} K_{\Omega}\cdot \\ & \rho^{-\left\{(n+1)-\left(\alpha_{sd}+\sum_{a\in\Omega\setminus\{s\}}\alpha_{ad}+\sum_{b\in\Omega^c\setminus\{d\}}\alpha_{sb}\right)\right\}}d\bar{\alpha} \\ =& K_{\Omega}\rho^{-d_{\Omega}(r)}, \end{aligned} \quad (26)$$

where $K_{\Omega} = o(\log(\rho))$ and

$$\begin{aligned} d_{\Omega}(r) =& \inf_{\substack{\bar{\alpha}\in\mathcal{O}_{\Omega}(r)\\ 0\leq\bar{\alpha}\leq 1}} \Bigg\{(n+1)- \\ & \left(\alpha_{sd}+\sum_{a\in\Omega\setminus\{s\}}\alpha_{ad}+\sum_{b\in\Omega^c\setminus\{d\}}\alpha_{sb}\right)\Bigg\} \\ =& (n+1)(1-r). \end{aligned} \quad (27)$$

The last step in the above set of equations follows from (22).

IV. Conclusion

A fundamental characterization in terms of the diversity multiplexing tradeoff is obtained for the relay network with a single source-destination pair and with n half-duplex relays under quasi-static Rayleigh fading. It is shown that the QMF protocol under the uniform transmit-receive schedule of the n relays (and hence with no CSI at the relays) achieves the DMT of the $(n+1\times 1)$ MISO point-to-point channel. The DMT of previously proposed protocols for the half duplex relay network such as the DDF and non-orthogonal amplify-forward (NAF) protocols fall well short of being DMT optimal.

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