On Frames from Abelian Group Codes

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***Abstract*—Designing low coherence matrices and low-correlation frames is a point of interest in many fields including compressed sensing, MIMO communications and quantum measurements. The challenge is that one must control the $\binom{n}{2}$ pairwise inner products between the frame elements. In this paper, we exploit the group code approach of David Slepian [1], which constructs frames using unitary group representations and which in general reduces the number of distinct inner products to $n-1$. We demonstrate how to efficiently find optimal representations of cyclic groups, and we show how basic abelian groups can be used to construct tight frames that have the same dimensions and inner products as those arising from certain more complex nonabelian groups. We support our work with theoretical bounds and simulations.**

***Index Terms*—Coherence, tight frame, group code, abelian group, dihedral group, unitary system.**

I. Introduction and Previous Work

Let $\mathbf{M} \in \mathbb{C}^{m\times n}$ be a complex matrix with columns $\{f_i\}_{i=1}^n$ which form a frame. The frame is called *tight* if $\mathbf{MM}^*$ is a scalar multiple of the identity $\mathbf{I_m}$, and *unit norm* if $||f_k||_2 = 1, \forall k$. We define the *coherence* μ of $\mathbf{M}$ to be the maximum correlation between any two distinct columns:

$$\mu = \max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2}. \tag{1}$$

Designing matrices and frames with low coherence is a problem that has applications in a wide range of fields, including compressive sensing [3]–[8], spherical codes [9], [12], MIMO communications [10], [11], quantum measurements [13], [14], etc.

A frame is called *equiangular* if the magnitude of the inner product between any two distinct frame elements is constant: $|\langle f_i, f_j\rangle| = \alpha$ for some α and all $i \neq j$. If a frame is both tight and equiangular, then it achieves the following lower bound on coherence, known as the *Welch bound* [12]:

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Theorem 1: Let $\mathbb{E}$ be a field, and $\{f_k\}_{k=1}^n$ be a frame for $\mathbb{E}^m$. Then

$$\max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2} \geq \sqrt{\frac{n-m}{m(n-1)}}, \tag{2}$$

with equality if and only if $\{f_k\}_{k=1}^n$ is both tight and equiangular.

Frames that are both tight and equiangular do not exist for all values of m and n, but if there are few distinct inner product magnitudes between the elements of a tight frame, then it can be shown that it will tend to have low coherence. Thus, it is of interest to construct tight frames with few mutual inner products between the elements.

It should be mentioned that the study of frames is interesting in its own right and has received substantial attention in both engineering and applied math communities (see [16]–[18]). A great deal of research has been done in structured frames, including some which are tight and/or equiangular [12], [19], [20]. Several of these have employed group theoretic methods [1], [15], [21], some of which we will describe.

The challenge in designing a low-coherence frame is that we need to control $\binom{n}{2}$ inner products. Slepian [1] proposed the following construction which reduces this number to $n-1$, and it has since been generalized [15]. Let $\mathcal{U} = \{\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n\}$ be a (multiplicative) group of unitary matrices such that for each i, we have $\mathbf{U}_i \in \mathbb{C}^{m\times m}$. Let $\mathbf{v} = [v_1, ..., v_m]^T \in \mathbb{C}^{m\times 1}$ be any vector, and let $\mathbf{M}$ be the matrix whose i^{th} column is $\mathbf{U}_i\mathbf{v}$:

$$\mathbf{M} = \begin{bmatrix}\mathbf{U}_1\mathbf{v} & \mathbf{U}_2\mathbf{v} & \dots & \mathbf{U}_n\mathbf{v}\end{bmatrix}.$$

Since $\mathcal{U}$ is a unitary group, we have $\mathbf{U}_i^*\mathbf{U}_j = \mathbf{U}_i^{-1}\mathbf{U}_j = \mathbf{U}_k$, for some $k \in [n]$. So the inner product between columns i and j of $\mathbf{M}$ is

$$\langle \mathbf{U}_i\mathbf{v}, \mathbf{U}_j\mathbf{v}\rangle = \mathbf{v}^*\mathbf{U}_i^*\mathbf{U}_j\mathbf{v} = \mathbf{v}^*\mathbf{U}_k\mathbf{v}. \tag{3}$$

Thus, the number of inner products has dropped from $\binom{n}{2}$ to only $n-1$, where each distinct inner product corresponds to one of the $\mathbf{U}_k$ (ignoring the inner product corresponding to the identity element). Furthermore, it is not too difficult to see that each of these inner products occurs with the same multiplicity.

In [22], we consider the case where $\mathcal{U}$ is a cyclic unitary group, which we represent as the powers of a single matrix $\mathbf{U} = \operatorname{diag}(\omega^{k_1}, \omega^{k_2}, ..., \omega^{k_m})$ of order n, where $\omega = e^{\frac{2\pi i}{n}}$ and each k_i is an integer modulo n: $\mathcal{U} = \{\mathbf{U}, \mathbf{U}^2, ..., \mathbf{U}^{n-1}, \mathbf{U}^n = \mathbf{I_m}\}$. We choose n to be an odd prime, and m to be a divisor of $n-1$. Then we take the set $K := \{k_1, ..., k_m\}$ to be the unique subgroup of $G := (\mathbb{Z}/n\mathbb{Z})^\times$ of order m (where $(\mathbb{Z}/n\mathbb{Z})^\times$ is the multiplicative group of nonzero integers modulo n, which is cyclic for n prime). If we set $\mathbf{v} = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T$, then the inner products take the form

$$\frac{|\mathbf{v}^*\mathbf{U}^\ell \mathbf{v}|}{||\mathbf{v}||_2^2} = \frac{1}{m}\left|\sum_{i=1}^{m} \omega^{\ell \cdot k_i}\right|. \quad (4)$$

Here we can see that there is a distinct inner product for each *coset* of K in G. A coset is a set in the form $\ell K = \{\ell k_1, ..., \ell k_m\}$, and the number of cosets of K in G is simply $r := \frac{|G|}{|K|} = \frac{n-1}{m}$.

With $\mathbf{U}$ and $\mathbf{v}$ chosen as above, the rows of $\mathbf{M} = \begin{bmatrix}\mathbf{v} & \mathbf{U}\mathbf{v} & \mathbf{U}^2\mathbf{v} & \dots & \mathbf{U}^{n-1}\mathbf{v}\end{bmatrix}$ will be distinct scaled rows of the $n \times n$ Fourier matrix, making this a so-called *harmonic* frame, and thus our frame will be tight. In our previous work [24], we give the following upper bounds on the coherence of our frames constructed from cyclic groups:

Theorem 2: In our frames constructed above, the coherence is upper-bounded by

$$\mu \leq \frac{1}{r}\left((r-1)\sqrt{\frac{1}{m}\left(r - \frac{1}{m}\right)} + \frac{1}{m}\right). \quad (5)$$

Theorem 3: If m is odd, then the coherence of our frames is upper-bounded by

$$\mu \leq \frac{1}{r}\sqrt{\left(\frac{1}{m} + \left(\frac{r}{2} - 1\right)\beta\right)^2 + \left(\frac{r}{2}\right)^2 \beta^2}, \quad (6)$$

where $\beta = \sqrt{\frac{1}{m}\left(r + \frac{1}{m}\right)}$.

The proofs of these theorems rely on rather involved extensions of a connection between harmonic frames and difference sets posed by Xia, Zhou and Giannakis [2].

II. Optimizing Coherence Over Cosets

While the preceding results give us a deterministic way to construct very low coherence matrices, we can hope to generalize our construction to yield an entire class of group-theoretically based matrices over which we can optimize to find even lower coherences.

As before, let us take n to be a prime p, and m a divisor of $n-1$, and let $G = (\mathbb{Z}/n\mathbb{Z})^\times$. As we remarked, there is a unique subgroup K of G of any order m dividing $n-1$, and it is cyclic. But suppose m' is a divisor of m, and let $K' = \{k'_1, ..., k'_{m'}\}$ be the unique subgroup of G of order m'. For convenience, let $d = \frac{m}{m'}$. If K is the unique subgroup of G of order m, then K' is a subgroup of K. Taking g to be a generator for G, then $g^{\frac{n-1}{m}}$ is a generator for K and $g^{d\frac{n-1}{m}}$ is a generator for K'.

Now, the set of cosets of K' in G form the group G/K'. This is a cyclic group of size $\frac{n-1}{m'} = d\frac{n-1}{m}$, generated by the coset gK'. We will construct our unitary group $\mathcal{U}$ as follows: take a set of d cosets of K' in G, $\{\ell_1 K', ..., \ell_d K'\}$. Then, for ω a primitive n^{th} root of unity, we let $\mathcal{U}$ be the cyclic group generated by the matrix $\mathbf{U}$, defined as follows: For any $\ell \in G$, let $\mathbf{D}_\ell = \text{diag}\left(\omega^{\ell k'_1}, ..., \omega^{\ell k'_{m'}}\right)$, an $m' \times m'$ diagonal matrix with the elements of $\omega^{\ell k'}, k' \in K'$ along the diagonal. Then define $\mathbf{U}$ to be the (block) diagonal matrix $\mathbf{U} := \text{diag}(\mathbf{D}_{\ell_1}, \mathbf{D}_{\ell_2}, ..., \mathbf{D}_{\ell_d})$.

Now, since each coset of K' in G consists only of elements relatively prime to n, then we see that this matrix will indeed maintain the property of having multiplicative order n, as in our original framework. In fact, if we choose $\ell_a K' = g^{a\frac{n-1}{m}} K'$, for each $a = 1, ..., d$, then since $g^{\frac{n-1}{m}}$ is a generator for K, the unique subgroup of G of order m, we find that the cosets $\{\ell_1 K', ..., \ell_d K'\}$ are precisely the cosets of K' as a subgroup of K. These cosets partition the elements of K, so we retrieve the matrix obtained from our original construction, with $\mathbf{U} = \text{diag}\left(\omega^{k_1}, ..., \omega^{k_m}\right)$ up to a permutation of the elements of K (which will not affect the values of the inner products). Thus, this new construction is a direct generalization of our original work. Another special case is when $K' = 1$, the trivial subgroup. In this case, selecting cosets for K' is nothing more than selecting specific rows of the $n \times n$ Fourier matrix for $\mathbf{M}$, with the exception of the row of all 1's.

As we cycle through the powers of $\mathbf{U}$, each $\mathbf{D}_{\ell_i}$ cycles through the different cosets of K' in some order. Since some powers of $\mathbf{U}$ may give rise to permutations of the same cosets, and hence lead to the same corresponding inner product from Equation 4, it can take some care to determine precisely how many distinct inner products we have in our constructed matrix. We know that it can be as few as $\frac{n-1}{m}$, as is the case when the chosen cosets of K' partition K. In general, we have the following theorem:

Theorem 4: The matrix $\mathbf{M}$ produced by the coset construction has at most $\frac{d\cdot(n-1)}{m}$ distinct inner products between its normalized columns.

Proof: We know that the distinct inner products between the normalized columns of $\mathbf{M}$ will correspond to the powers of $\mathbf{U}$. The b^{th} power of $\mathbf{U}$ can be written as $\mathbf{U}^b = \text{diag}\left(\mathbf{D}_{\ell_1}^b, ..., \mathbf{D}_{\ell_d}^b\right)$. Thus, the inner product corresponding to this power of $\mathbf{U}$ is

$$\frac{1}{m}\left|\sum_{k'\in K'} \omega^{\ell_1(bk')} + \sum_{k'\in K'} \omega^{\ell_2(bk')} + ... + \sum_{k'\in K'} \omega^{\ell_d(bk')}\right|. \quad (7)$$

Thus, there can only be as many such sums as there are cosets bK'. Since there are $\frac{n-1}{m'} = \frac{d\cdot(n-1)}{m}$ cosets of K' in G, we have our result. ■

This coset construction offers us a tradeoff. By using the smaller group K' (of size $\frac{m}{d}$) to construct our matrix as opposed to K (of size m), we gain the possibility of having nice cancelation properties among the sums

$\sum_{k' \in K'} \omega^{\ell_i(bk')}$ in (7) at the cost of having more inner products to control. But since the number of distinct inner products can increase only by a factor of d at most, this can turn out to be a worthwhile tradeoff, and indeed, we have examples where we can strictly decrease the coherence of $\mathbf{M}$ by using this construction. (See Fig. 1).

We can now can formulate the problem of constructing low-coherence matrices as an optimization problem, where we can optimize over both the choice of K' and the set of cosets $\{\ell_1 K', ..., \ell_d K'\}$. For fixed m and n, where n is a prime p and m a divisor of $p-1$, we must solve the following:

$$\min_{m'|m,\ \underline{\ell} \in G^{\times \frac{m}{m'}}} \left(\max_{b \in G} \frac{1}{m} \left| \sum_{i=1}^{m/m'} \left(\sum_{k \in K_{m'}} \omega^{\ell_i(bk)} \right) \right| \right) \tag{8}$$

where $G = (\mathbb{Z}/n\mathbb{Z})^{\times}$, $\underline{\ell} = (\ell_1, ..., \ell_d)$, $G^{\times \frac{m}{m'}}$ denotes the Cartesian product of G with itself $\frac{m}{m'}$ times, and $K_{m'}$ denotes the unique subgroup of G of size m'.

In practice, it is typically not feasible to perform this exact optimization since it requires a search over the lattice $G^{\times \frac{m}{m'}}$ for every m' dividing m. One simple way to deal with this problem is to fix m' and randomly sample $\underline{\ell} \in G^{\times \frac{m}{m'}}$ to search for the smallest value of the objective function. Note that if $m_2'|m_1'$ (or equivalently, $K_{m_2'} \leq K_{m_1'}$), then searching over cosets of $K_{m_2'}$ *encompasses* the search over cosets of $K_{m_1'}$, since we can express $K_{m_1'}$ as a union of cosets of $K_{m_2'}$. One might therefore be tempted to argue that it is unnecessary to search over cosets of $K_{m_1'}$ at all. There is, however, value in searching over these cosets, since this search will converge to its optimal value much faster than the search over cosets of the smaller group. See Figure 1.

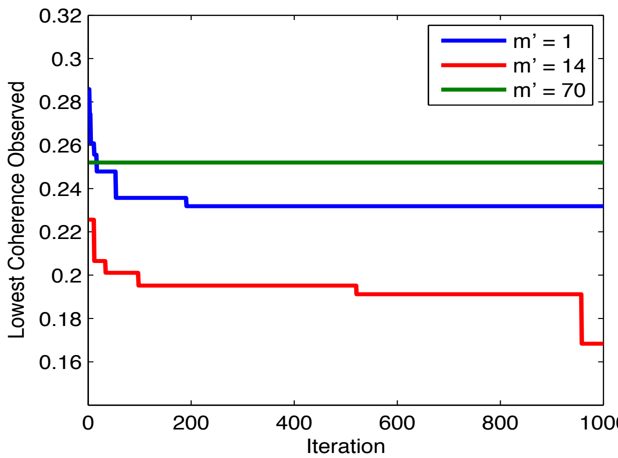

Fig. 1. Randomly sampling $\underline{\ell}$ to search for the optimal coherence over cosets of subgroups of size m' for various values of m' (= 1, 14, 70). (Plot shows the lowest coherence found up to a given iteration). Here, $n = p = 491$, and $m = 70$. The figure shows that $m' = 14$ quickly achieves the lowest values of coherence.

Of course, we can still bound the coherence of the frames that can arise from this construction using that of our previous frames. Given a set of integers $S = \{s_1, ..., s_m\}$, we will use ω^S to denote the ordered set $\{\omega^{s_1}, ..., \omega^{s_m}\}$.

Theorem 5: Let n be a prime, m a divisor of $n-1$, m' a divisor of m, and $d = \frac{m}{m'}$. Let K' be the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ of order m', and $\mu_{K'}$ the coherence of our original m'-dimensional frame constructed from the cyclic group generated by the powers of $\mathrm{diag}(\omega^{K'})$ and the m'-dimensional vector $\mathbf{v}$ of all 1s. Then, if we let $\mu_{\underline{\ell}}$ be the coherence of the m-dimensional frame constructed from the cyclic group generated by $\mathrm{diag}(\omega^{\ell_1 K'}, \omega^{\ell_2 K'}, ..., \omega^{\ell_d K'})$ and the m-dimensional vector of all 1s, we have $\mu_{\underline{\ell}} \leq \mu_{K'}$.

Proof: This comes from a simple application of the triangle inequality: from equation (7), we see that

$$\mu_{\underline{\ell}} = \max_b \frac{1}{m} \left| \sum \omega^{\ell_1(bK')} + ... + \sum \omega^{\ell_d(bK')} \right| \tag{9}$$

$$\leq \frac{1}{dm'} \left(\max_{b_1} \left| \sum \omega^{\ell_1(b_1 K')} \right| + ... + \max_{b_d} \left| \sum \omega^{\ell_d(b_d K')} \right| \right) \tag{10}$$

$$= \frac{1}{d} (d \mu_{K'}) = \mu_{K'}. \tag{11}$$

■

Theorem 5 naturally allows us to use the bounds from Theorems 2 and 3 to explicitly bound the coherence from our coset optimization in terms of r, m and d, though it is worth noting that in practice we achieve coherence significantly lower than these bounds.

III. More General Abelian Groups

We now begin to examine what we can gain by using representations of more general abelian groups in our constructions. Any finite abelian group can be realized as the direct product of cyclic groups, and since the irreducible representations of any abelian group are all 1-dimensional, we can without loss of generality assume that all the matrices in our unitary group $\mathcal{U}$ remain diagonal, with roots of unity as the diagonal elements. Thus, it again makes sense for us to choose our rotated vector $\mathbf{v}$ to be the (normalized) vector of all 1s.

A general abelian group G can be represented as follows: First express G as a direct product of, say, L cyclic groups of orders $n_1, ..., n_L$, so that $G \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \times ... \times \frac{\mathbb{Z}}{n_L\mathbb{Z}}$. Then let $\omega_1, ..., \omega_L$ be the corresponding primitive roots of unity: $\omega_j = e^{2\pi i/n_j}$. Then we set $\mathbf{U}_j = \mathrm{diag}(\omega_j^{k_{1j}}, ..., \omega_j^{k_{mj}})$, where we will assume that the k_{ij} are distinct integers modulo n_j. The abelian group generated by the diagonal matrices $\{\mathbf{U}_1, ..., \mathbf{U}_L\}$ is isomorphic to G, and an arbitrary element will take the form $\mathbf{U}_1^{a_1}\mathbf{U}_2^{a_2} \dots \mathbf{U}_L^{a_L}$, where $a_j \in \{0, ..., n_j - 1\}$. Our frame matrix $\mathbf{M}$ will then take the form $\mathbf{M} = [\dots (\mathbf{U}_1^{a_1}\mathbf{U}_2^{a_2} \dots \mathbf{U}_L^{a_L}\mathbf{v}) \dots]_{0 \leq a_j \leq n_j - 1}$.

As we commented in the case where $\mathcal{U}$ was cyclic (that is, $L = 1$), the resulting frame yielded the columns of a matrix $\mathbf{M}$ whose rows were a subset of the rows of the discrete $n \times n$ Fourier matrix, hence they were orthogonal and the frame was tight. It turns out that the frame remains tight for our more general abelian groups.

Theorem 6: Let $\mathcal{U} = \langle \mathbf{U}_1, ..., \mathbf{U}_L \rangle$ be the group generated by the diagonal matrices $\mathbf{U}_j = \text{diag}(\omega_j^{k_{1j}}, ..., \omega_j^{k_{mj}})$ where ω_j is a primitive n_j-th root of unity and $\{k_{1j}, ..., k_{mj}\}$ is a set of distinct integers modulo n_j. Let $\mathbf{v} = \frac{1}{\sqrt{m}}\begin{bmatrix}1 & ... & 1\end{bmatrix}^T$, the normalized m-dimensional vector of all 1s. Then the columns of the matrix $M = [\ldots(\mathbf{U}_1^{a_1}\mathbf{U}_2^{a_2}\ldots\mathbf{U}_L^{a_L}\mathbf{v})\ldots]_{0\le a_j\le n_j-1}$ form a tight frame.

Proof: We have already established that for any j, the columns of the matrix $\mathbf{A}_j = \begin{bmatrix}\mathbf{v}, \mathbf{U}_j\mathbf{v}, ..., \mathbf{U}_j^{n_j-1}\mathbf{v}\end{bmatrix}$ form a tight frame. Furthermore, it is not too difficult to see that the rows of $\mathbf{M}$ are a subset of the rows of the Kronecker Product $\mathbf{A}_{\text{Kron}} = \mathbf{A}_1 \otimes ... \otimes \mathbf{A}_L$. Thus, it suffices to show that the rows of this Kronecker Product are orthogonal. But by the properties of the Kronecker product,

$$\mathbf{A}_{\text{Kron}}\mathbf{A}_{\text{Kron}}^* = (\mathbf{A}_1 \otimes ... \otimes \mathbf{A}_L)(\mathbf{A}_1 \otimes ... \otimes \mathbf{A}_L)^* \quad (12)$$

$$= (\mathbf{A}_1 \otimes ... \otimes \mathbf{A}_L)(\mathbf{A}_1^* \otimes ... \otimes \mathbf{A}_L^*) \quad (13)$$

$$= \mathbf{A}_1\mathbf{A}_1^* \otimes ... \otimes \mathbf{A}_L\mathbf{A}_L^*, \quad (14)$$

and since each $\mathbf{A}_j\mathbf{A}_j^*$ is a multiple of the identity matrix, so is their Kronecker product. It follows that the columns of $\mathbf{M}$ are indeed a tight frame. ∎

IV. Matching Generalized Dihedral Groups with Abelian Groups

In [24], we began experimenting with representations of nonabelian groups to see if the resulting frames might have benefits over those of our original cyclic constructions. In particular, we examined the following class of nonabelian groups, which arises in [23]:

$$G_{n,r} = \langle \sigma, \tau \mid \sigma^n = 1, \tau^D = 1, \tau\sigma\tau^{-1} = \sigma^r \rangle. \quad (15)$$

Here, D is the order of r modulo n, and $r-1$ is chosen to be relatively prime to n. This last condition is automatically satisfied when we take n to be an odd prime as in the cyclic case. This is precisely a semidirect product in the form $\frac{\mathbb{Z}}{n\mathbb{Z}} \rtimes \frac{\mathbb{Z}}{D\mathbb{Z}}$, and if we take $D = 2$ and $r = -1$, we see that we obtain the familiar dihedral group D_{2n}. We showed that we can use these groups to extend our results from the cyclic case to obtain frames with new dimensions and non-prime numbers of elements. Furthermore, we proved that this could be done without drastically affecting the original coherence of our cyclic frames. We now show that using just abelian groups, we can construct frames that obtain not only the same coherence, but the same exact inner products as in the generalized dihedral case.

In [24], we exploited the following irreducible representation of $G_{n,r}$:

$$\sigma \mapsto \mathbf{S} := \text{diag}(\omega, \omega^r, ..., \omega^{r^{D-1}}), \quad (16)$$

$$\tau \mapsto \mathbf{T} := \begin{bmatrix} & \mathbf{I_{D-1}} \\ 1 & \end{bmatrix}, \quad (17)$$

where $\omega = e^{\frac{2\pi i}{n}}$ and $\mathbf{I_{D-1}}$ is the $(D-1)\times(D-1)$ identity matrix (see again [23]). Similarly to our cyclic constructions, we chose a representation in the form

$$\sigma \mapsto [\sigma] := \text{diag}(\mathbf{S}^{k_1}, ..., \mathbf{S}^{k_m}), \quad (18)$$

$$\tau \mapsto [\tau] := \text{diag}(\mathbf{T}, ..., \mathbf{T}) \quad (19)$$

where $K = \{k_1, ..., k_m\}$ was again the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ of size m. Thus we created a Dm-dimensional representation of $G_{n,r}$, so our resulting frame matrices had dimensions $Dm \times Dn$.

Now, in order to maintain low coherence, we were forced to deviate from our original construction by choosing our rotated vector to be $\mathbf{v} = \begin{bmatrix}\mathbf{w}^T & \mathbf{w}^T & ... & \mathbf{w}^T\end{bmatrix}^T$, where $\mathbf{w}$ is a D-dimensional vector $\mathbf{w} = [w_1, ..., w_D]^T$ whose entries are chosen to be a Zadoff-Chu (ZC)sequence,

$$w_d = e^{\frac{i\pi d^2}{D}} \text{ if } D \text{ is even}$$

$$w_d = e^{\frac{i\pi d(d+1)}{D}} \text{ if } D \text{ is odd.}$$

Since the elements of our group could be parametrized as $[\sigma]^a[\tau]^b$, where $0 \le a \le n-1$ and $0 \le b \le D-1$, we saw that there were at most $D \cdot \frac{n-1}{m}$ distinct inner products between the frame elements, each taking the form

$$\frac{\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = \frac{1}{m\cdot D}\sum_d w_d^* w_{d+b}\sum_{k\in K}\omega^{kar^{d-1}} \quad (20)$$

$$= \frac{1}{m\cdot D}\sum_d w_d^* w_{d+b}\sum_{k\in K}\omega^{ka'}, \quad (21)$$

where $a' = ar^{d-1}$. Furthermore, we proved that the resulting frames were tight.

Now consider the following alternative construction: Let $\gamma = e^{\frac{2\pi i}{D}}$ be a primitive D^{th} root of unity. We again define $\mathbf{S} := \text{diag}(\omega, \omega^r, ..., \omega^{r^{D-1}})$ and $[\sigma] := \text{diag}(\mathbf{S}^{k_1}, ..., \mathbf{S}^{k_m})$, exactly as in the generalized dihedral case, but we now alter $\mathbf{T}$ and $[\tau]$ to be

$$\mathbf{T}' := \text{diag}(1, \gamma, \gamma^2, ..., \gamma^{D-1}) \quad (22)$$

$$[\tau'] := \text{diag}(\mathbf{T}', ..., \mathbf{T}'), \quad (23)$$

where we have altered $\mathbf{T}$ to be a diagonal matrix, $\mathbf{T}'$, but maintain the property that $[\tau']$ is a block diagonal matrix with m copies of $\mathbf{T}'$ on the diagonal.

Now, since both $[\sigma]$ and $[\tau']$ are diagonal matrices, they will generate group $\mathcal{U}$ which is abelian, in this case isomorphic to the direct product $\frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{D\mathbb{Z}}$. The group elements again take the form $[\sigma]^a[\tau']^b$, where $0 \le a \le n-1$ and $0 \le b \le D-1$. As in our original constructions, we choose $\mathbf{v}' = \frac{1}{\sqrt{Dm}}\begin{bmatrix}1 & ... & 1\end{bmatrix}$ to be the normalized vector of all 1s.

Thus we construct the frame

$$\mathbf{M} = \begin{bmatrix}\ldots & [\sigma]^a[\tau']^b\mathbf{v}' & \ldots\end{bmatrix}.$$

Theorem 6 shows that this will be a tight frame, and it is not too difficult to see that its inner products will be

$$\frac{\mathbf{v}'^*[\sigma]^a[\tau']^b\mathbf{v}'}{||\mathbf{v}'||_2^2} = \frac{1}{m\cdot D}\sum_d \gamma^{d\cdot b}\sum_{k\in K}\omega^{kar^{d-1}}. \quad (24)$$

We would like to compare (24) to (20). On this note, a quick calculation shows that when $\mathbf{w}$ is chosen to be a ZC sequence as described above, we have

$$w_d^* w_{d+b} = e^{\frac{i\pi}{D}(2db+b^2)} = \gamma^{db} e^{\frac{i\pi b^2}{D}} \qquad \text{if } D \text{ is even,} \tag{25}$$

$$w_d^* w_{d+b} = e^{\frac{i\pi}{D}(2db+b^2+b)} = \gamma^{db} e^{\frac{i\pi(b^2+b)}{D}} \qquad \text{if } D \text{ is odd.} \tag{26}$$

We can now see by comparing (24) to (20) that

$$\frac{\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = e^{\frac{i\pi b^2}{D}} \frac{\mathbf{v}'^*[\sigma]^a[\tau']^b\mathbf{v}'}{||\mathbf{v}'||_2^2} \qquad \text{if } D \text{ is even,} \tag{27}$$

$$\frac{\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = e^{\frac{i\pi(b^2+b)}{D}} \frac{\mathbf{v}'^*[\sigma]^a[\tau']^b\mathbf{v}'}{||\mathbf{v}'||_2^2} \qquad \text{if } D \text{ is odd,} \tag{28}$$

which proves our main result of this section:

Theorem 7: Both the frames constructed from the generalized dihedral group $\langle[\sigma],[\tau]\rangle$ (with ZC-vector $\mathbf{v}$) and the abelian group $\langle[\sigma],[\tau']\rangle$ (with all-1-vector $\mathbf{v}'$) have corresponding inner products with the same absolute value:

$$\frac{\left|\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}\right|}{||\mathbf{v}||_2^2} = \frac{\left|\mathbf{v}'^*[\sigma]^a[\tau']^b\mathbf{v}'\right|}{||\mathbf{v}'||_2^2}. \tag{29}$$

Thus, abelian groups can not only match the coherence of the significantly more complicated generalized dihedral groups, but in essence form *equivalent* frames which have the same dimensions and number of elements as well as matching inner product norms between the frame elements.

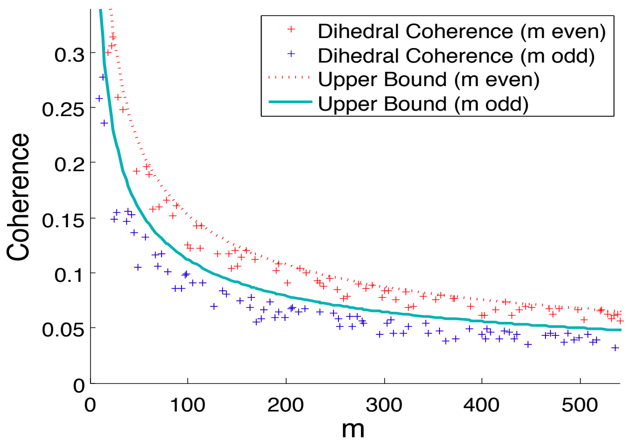

Fig. 2. Coherences arising dihedral representations for $r = 4$, which we show can also be realized by abelian representations. We also plot upper bounds derived from Theorems 2 and 3.

V. Further Directions

We have only begun to compare the different types of frames that we can achieve from abelian and nonabelian groups. While it is reasonable to expect that more complicated nonabelian groups can achieve a richer set of frames with low coherence, the results of this paper show that frames arising from abelian group constructions should not be overlooked. It would be interesting to see exactly how much overlap there is between the sets of frames that can arise from each set of groups. Furthermore we have yet to elucidate the connection between the unitary group $\mathcal{U}$ and the optimal choice of vector $\mathbf{v}$ to be rotated.

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