

Balanced Sparsest Generator Matrices for MDS Codes

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Abstract—We show that given n and k , for q sufficiently large, there always exists an $[n, k]_q$ MDS code that has a generator matrix G satisfying the following two conditions:

- (C1) **Sparsest:** each row of G has Hamming weight $n - k + 1$;
- (C2) **Balanced:** Hamming weights of the columns of G differ from each other by at most one.

I. INTRODUCTION

We study the existence and provide a construction of a *sparsest* and *balanced* generator matrix of Maximum Distance Separable (MDS) codes. A generator matrix is the sparsest if it contains the least number of nonzero entries among all generator matrices of the same MDS code. A generator matrix is balanced if every column contains approximately the same number of nonzero entries. More specifically, we require that the number of nonzero entries in each column differs from each other by at most one.

Apart from being of theoretical interest, our study on balanced sparsest generator matrices for MDS codes was motivated by its application in error correction for sensor networks (a closely related work can be found in [1]). Suppose n sensors, S_1, \dots, S_n , collectively measure k conditions x_1, \dots, x_k , such as temperature, pressure, light intensity, etc. Let $\mathbf{x} = (x_1, \dots, x_k)$, where $x_i \in \mathbb{F}_q$, a finite field of q elements. These sensors transmit the information they collected to a base station, which is a data collector. Furthermore, each sensor performs some encoding on the information it has, before transmitting the information back to the base station in the following way. Let G be an $k \times n$ generator matrix of an $[n, k, d]_q$ error-correcting code. Sensor S_i transmits the scalar product of \mathbf{x} and column i of G to the base station. It is well known in classical coding theory that this coding scheme allows the base station to retrieve \mathbf{x} when at most $\lfloor \frac{d-1}{2} \rfloor$ sensors transmit wrong information. Moreover, the base station can also identify the malfunctioned sensors. For each sensor S_i , only those conditions corresponding to nonzero entries of column i of G are involved into encoding. Hence S_i only needs to measure such conditions. Thus, if G is sparse then in average, each sensor only needs to measure a few among k conditions to achieve the desired error correction capability. On top of that, if columns of G have approximately the same number of nonzero entries then the sensors are required to measure approximately the same number of conditions. This balance guarantees an even distribution of workload among sensors, which is an important criterion for sensor networks where energy saving is a critical issue.

In fact, any error-correcting code can be used in the aforementioned scheme for sensor networks. We choose to study MDS codes first because their structure, especially their weight distribution, is well studied (see, for instance [2, Ch. 11]). Moreover, they have optimal error-correcting capability, given the length and the dimension. We prove that over a sufficiently large field, there always exists an MDS code that has a balanced and sparsest generator matrix, which is ideally suitable for the above encoding scheme for sensor networks.

Necessary notations and definitions are provided in Section II. We state and prove our main result in Section III.

II. PRELIMINARIES

We denote by \mathbb{F}_q the finite field with q elements. Let $[n]$ denote the set $\{1, 2, \dots, n\}$. The *support* of a vector $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{F}_q^n$ is defined by $\text{supp}(\mathbf{u}) = \{i \in [n] : u_i \neq 0\}$. The (Hamming) *weight* of \mathbf{u} is $|\text{supp}(\mathbf{u})|$. We can also define weight and support of a row or a column of a matrix over some finite field, by regarding them as vectors over that field. Apart from Hamming weight, we also use other standard notions from coding theory such as minimum distance, linear $[n, k]_q$ and $[n, k, d]_q$ codes, MDS codes, and generator matrices (for instance, see [2]).

For a matrix $G = (g_{i,j}) \in \mathbb{F}_q^{k \times n}$, the *support matrix* of G , denoted $\text{supp}(G)$, is a $k \times n$ binary matrix $M = (m_{i,j})$ where $m_{i,j} = 0$ if $g_{i,j} = 0$ and $m_{i,j} = 1$ if $g_{i,j} \neq 0$. Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. We denote by $\text{var}(M) = (v_{i,j})$ the matrix obtained from M by replacing every nonzero entry $m_{i,j} = 1$ by $\xi_{i,j}$, where $\xi_{i,j}$'s are indeterminates. More formally, $v_{i,j} = 0$ if $m_{i,j} = 0$ and $v_{i,j} = \xi_{i,j}$ if $m_{i,j} = 1$. We also denote by $\text{gr}(M)$ the bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined as follows. The vertex set \mathcal{V} can be partitioned into two parts, namely, the left part $L = \{\ell_1, \dots, \ell_k\}$, and the right part $R = \{r_1, \dots, r_n\}$. The edge set is

$$\mathcal{E} = \{(\ell_i, r_j) : i \in [k], j \in [n], m_{i,j} \neq 0\}.$$

For any $k \times n$ matrix N , we define $f(N) = \prod_P \det(P)$, where the product is taken over all $\binom{n}{k}$ submatrices P of order k of N .

III. MAIN RESULT

A sparsest generator matrix of an $[n, k]_q$ MDS code would have precisely $n - k + 1$ nonzero entries in every row. Moreover, if it is balanced, then each column contains either $\lfloor \frac{k(n-k+1)}{n} \rfloor$ or $\lceil \frac{k(n-k+1)}{n} \rceil$ nonzero entries. Hereafter, we often use R_i ,

$i \in [k]$, and C_j , $j \in [n]$, to denote the supports of row i and column j , respectively, of a $k \times n$ binary matrix M . Note that $R_i \subseteq [n]$ and $C_j \subseteq [k]$.

Lemma 1. *Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. Suppose that each row of M has weight $n - k + 1$. Then M is the support matrix of a generator matrix of some $[n, k]_q$ MDS code over a sufficiently large field \mathbb{F}_q ($q > \binom{n-1}{k-1}$) if and only if $f(\text{var}(M)) \neq 0$.*

Proof: Suppose $M = \text{supp}(G)$, where $G = (g_{i,j})$ is a generator matrix of some $[n, k]_q$ MDS code. Due to a well-known property of MDS codes (see [2, p. 319]), every submatrix of order k of G has nonzero determinant. Therefore, $f(G) \neq 0$. Note that $f(\text{var}(M))$ can be regarded as a multivariable polynomial in $\mathbb{F}_q[\dots, \xi_{i,j}, \dots]$. Moreover, since $M = \text{supp}(G)$, we deduce that $f(G)$ can be obtained from $f(\text{var}(M))$ by substituting $\xi_{i,j}$ by $g_{i,j}$ for all i, j where $g_{i,j} \neq 0$. As $f(G) \neq 0$, we conclude that $f(\text{var}(M)) \neq 0$.

Now suppose that $f(\text{var}(M)) \neq 0$. Note that each column of $\text{var}(M)$ belongs to precisely $\binom{n-1}{k-1}$ submatrices of order k of $\text{var}(M)$. Hence the exponent of each $\xi_{i,j}$ in $f(\text{var}(M))$ is at most $\binom{n-1}{k-1}$. Since $f(\text{var}(M)) \neq 0$, by [3, Lemma 4], if $q > \binom{n-1}{k-1}$ then there exist $g_{i,j} \in \mathbb{F}_q$ (for i, j where $m_{i,j} = 1$) so that $f(\text{var}(M))(\dots, g_{i,j}, \dots) \neq 0$. Let $G = (g_{i,j})$ (for i, j where $m_{i,j} = 0$ we set $g_{i,j} = 0$). Since $f(G) = f(\text{var}(M))(\dots, g_{i,j}, \dots) \neq 0$, again by [2, p. 319], we deduce that G is a generator matrix of an $[n, k]_q$ MDS code. Therefore, each row of G has weight at least $n - k + 1$, due to the Singleton Bound (see [2, p. 33]). Since each row of M also has weight $n - k + 1$, we deduce that $g_{i,j} \neq 0$ whenever $m_{i,j} = 1$. Therefore, $M = \text{supp}(G)$. ■

Lemma 2. *Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. Then $f(\text{var}(M)) \neq 0$ if and only if every bipartite subgraph induced by the k left-vertices and some k right-vertices in $\text{gr}(M)$ has a perfect matching.*

Proof: Let $\mathcal{G} = \text{gr}(M)$. Each submatrix P of order k of $\text{var}(M)$ corresponds to a bipartite subgraph \mathcal{H}_P induced by the k left-vertices and some k right-vertices in \mathcal{G} . In the literature, P is usually referred to as the *Edmonds matrix* of \mathcal{H}_P . It is well known (see [4, p. 167]) that a bipartite graph has a perfect matching if and only if the determinant of its Edmonds matrix is not identically zero. Hence the proof follows. ■

Lemma 3. *Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. Then every bipartite subgraph induced by the k left-vertices and some k right-vertices in $\text{gr}(M)$ has a perfect matching if and only if*

$$|\cup_{j \in J} C_j| \geq |J|, \text{ for every subset } J \subseteq [n], |J| \leq k. \quad (1)$$

Proof: Let $\mathcal{G} = \text{gr}(M)$. Each submatrix P of order k of $\text{var}(M)$ corresponds to a bipartite subgraph \mathcal{H}_P induced by the k left-vertices and some k right-vertices in \mathcal{G} . The lemma follows by applying Hall's marriage theorem to each of such subgraphs of $\text{gr}(M)$. ■

Lemma 4. *Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. The condition (1) is equivalent to*

$$|\cup_{i \in I} R_i| \geq n - k + |I|, \text{ for every subset } \emptyset \neq I \subseteq [k]. \quad (2)$$

Proof: Suppose that (1) holds and that there exists a nonempty set $I \subseteq [k]$ satisfying

$$|\cup_{i \in I} R_i| \leq n - k + |I| - 1. \quad (3)$$

We aim to obtain a contradiction. The condition (3) is equivalent to

$$|\cap_{i \in I} \overline{R_i}| \geq k - |I| + 1. \quad (4)$$

Hence there exists a set J of $k - |I| + 1$ columns of M that satisfies $|\cap_{j \in J} \overline{C_j}| \geq |I|$. Equivalently we have

$$|\cup_{j \in J} C_j| \leq k - |I| < k - |I| + 1 = |J|. \quad (5)$$

We obtain a contradiction between (1) and (5). The “only if” direction can be proved in a similar manner. ■

Lemma 5. *Let $M = (m_{i,j})$ be a $k \times n$ binary matrix. Suppose that each row of M has weight $n - k + 1$. Then M is the support matrix of a generator matrix of some $[n, k]_q$ MDS code over a sufficiently large field \mathbb{F}_q ($q > \binom{n-1}{k-1}$) if and only if (2) holds.*

Proof: The proof follows from Lemma 1-4. ■

We present below our main result.

Theorem 6 (Main Theorem). *Suppose $1 \leq k \leq n$ and $q > \binom{n-1}{k-1}$. Then there always exists an $[n, k]_q$ MDS code that has a generator matrix G satisfying the following two conditions.*

- (C1) *Sparsest: each row of G has weight $n - k + 1$.*
- (C2) *Balanced: column weights of G differ from each other by at most one.*

By Lemma 5, to prove Theorem 6, we need to show that there always exists a $k \times n$ binary matrix M satisfying the following properties

- (P1) each row of M has weight $n - k + 1$,
- (P2) column weights of M differ from each other by at most one,
- (P3) $|\cup_{i \in I} R_i| \geq n - k + |I|$, for every subset $\emptyset \neq I \subseteq [k]$, where R_i denotes the support of row i of M .

We prove the existence of such a binary matrix by designing an algorithm (Algorithm 1) that starts from an initial binary matrix which satisfies (P1) and (P3). In each iteration, the matrix at hand is slightly modified so that it still satisfies (P1) and (P3) and its column weights become more balanced. When the algorithm terminates, it produces a matrix that satisfies (P1), (P2), and (P3).

Observe that it is fairly easy to construct a binary matrix that satisfies (P1) and (P2), using the Gale-Ryser Theorem (see Manfred [5]). However, (P1) and (P2) do not automatically guarantee (P3). Indeed, the matrix P given below satisfies both (P1) and (P2). However, (P3) is violated if we choose $I = \{1, 2, 3\}$.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let \widetilde{M} be any $k \times n$ binary matrix that satisfies both (P1) and (P3). For instance, we can shift the vector $(\underbrace{1 \ 1 \ \dots \ 1}_{n-k+1} \ 0 \ 0 \ \dots \ 0)$ k times cyclically to produce k rows of such a matrix as below.

$$\widetilde{M} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

The Algorithm 1 takes \widetilde{M} as an input parameter.

Algorithm 1

Input: n, k, \widetilde{M} ;

Initialization: $M := \widetilde{M}$;

- 1: **repeat**
- 2: Let max and min be the maximum and minimum weights of columns of M ;
- 3: **if** $\max - \min \leq 1$ **then**
- 4: Return M ;
- 5: **end if**
- 6: Find two columns j_{\max} and j_{\min} that have weights max and min, respectively;
- 7: Find a row i_s satisfying $m_{i_s, j_{\max}} = 1$ and $m_{i_s, j_{\min}} = 0$ and moreover, if we set $m_{i_s, j_{\max}} := 0$ and $m_{i_s, j_{\min}} := 1$ then M still satisfies (P1) and (P3);
- 8: Swapping: set $m_{i_s, j_{\max}} = 0$ and $m_{i_s, j_{\min}} := 1$;
- 9: **until** $\max - \min \leq 1$;

Due to space constraint, we have prepared a separate note at [6] with an example to demonstrate the algorithm.

Lemma 7. *Suppose in every iteration, Algorithm 1 can always find a legitimate row described in Step 7. Then the algorithm terminates after finitely many iterations and returns a matrix satisfying (P1), (P2), and (P3).*

Proof: At a certain iteration, let $\Delta = \max - \min$. After swapping the two entries $m_{i_s, j_{\max}}$ and $m_{i_s, j_{\min}}$, the weight of column j_{\max} is decreased by one whereas the weight of column j_{\min} is increased by one. Therefore, after at most $\lfloor n/2 \rfloor$ iterations, Δ is decreased by at least one. Hence, the algorithm must terminate after finitely many iterations. The output matrix obviously satisfies (P1), (P2), and (P3). ■

Lemma 8. *In every iteration of Algorithm 1, a row i_s as described in Step 7 of the algorithm can always be found.*

Since column j_{\max} has a larger weight than column j_{\min} , there always exists at least one row i_s where $m_{i_s, j_{\max}} = 1$ and $m_{i_s, j_{\min}} = 0$. Obviously, swapping $m_{i_s, j_{\max}}$ and $m_{i_s, j_{\min}}$ does not make M violate (P1). The stricter criterion is that M must still satisfy (P3) after the swap. We need a few more auxiliary results before we can prove Lemma 8.

Suppose at a certain iteration, we choose some columns j_{\max} and j_{\min} that have maximum and minimum weights,

respectively. Without loss of generality, we assume that the first t rows are all the rows of M satisfying the property that each of them has a one at column j_{\max} and a zero at column j_{\min} . In other words, assume that

$$\{i \in [k] : m_{i, j_{\max}} = 1 \text{ and } m_{i, j_{\min}} = 0\} = [t].$$

Since $\max - \min \geq 2$, we have $t \geq 2$.

Suppose, for contradiction, that none of these t rows satisfy the condition in Step 7 of Algorithm 1. Let $M^{(i)}$, $i \in [t]$, be the matrix obtained from M after swapping the two entries $m_{i, j_{\max}}$ and $m_{i, j_{\min}}$. Then $M^{(i)}$, $i \in [t]$, does not satisfy (P3). Since M satisfies (P3) and the only difference between $M^{(i)}$ and M is the row i , the set of rows of $M^{(i)}$ that violates the condition (P3) must contain row i . Therefore, for each $i \in [t]$, there exists a set $I_i \subset [k]$, $i \notin I_i$, such that $\{i\} \cup I_i$ is a set of rows that violates (P3) in $M^{(i)}$. For our purpose, for each $i \in [t]$, we choose I_i to be of minimum size among those sets that satisfied the aforementioned requirement. Since for each $i \in [t]$, $|R_i^{(i)}| = |R_i| = n - k + 1$, we deduce that $I_i \neq \emptyset$.

Let $R_r^{(i)}$ denote the support of row r of $M^{(i)}$, $i \in [t]$, $r \in [k]$. Note that R_r denotes the support of row r of M , $r \in [k]$. For simplicity, we use $R_I^{(i)}$ to denote the union $\cup_{r \in I} R_r^{(i)}$ for any subset $I \subseteq [k]$. Since $\{i\} \cup I_i$ is the set of rows of $M^{(i)}$ that violates (P3), for every $i \in [t]$ we have

$$|R_{\{i\} \cup I_i}^{(i)}| \leq n - k + |\{i\} \cup I_i| - 1 = n - k + |I_i|. \quad (6)$$

Lemma 9. *For all $i, i' \in [t]$, the following statements hold*

- a) $R_{i'}^{(i)} = \begin{cases} R_{i'}, & \text{if } i' \neq i, \\ (R_{i'} \setminus \{j_{\max}\}) \cup \{j_{\min}\}, & \text{if } i' = i, \end{cases}$
- b) $R_{I_i}^{(i)} = R_{I_i}$, c) $j_{\max} \notin R_{I_i}$, d) $j_{\min} \in R_{I_i}$,
- e) $i \notin I_{i'}$, f) $|R_{\{i\} \cup I_i}^{(i)}| = n - k + |I_i|$.

Proof: Proof of a). Note that all the rows of $M^{(i)}$ except for the row i are the same as that of M . Therefore, $R_{i'}^{(i)} = R_{i'}$ if $i' \neq i$. As row i of $M^{(i)}$ is obtained from row i of M by swapping $m_{i, j_{\max}} = 1$ and $m_{i, j_{\min}} = 0$, we deduce that

$$R_i^{(i)} = (R_i \setminus \{j_{\max}\}) \cup \{j_{\min}\}.$$

Proof of b). By definition of I_i , $i \notin I_i$. Therefore, using Part a), we conclude that $R_{I_i}^{(i)} = R_{I_i}$.

Proof of c). Suppose, for contradiction, that $j_{\max} \in R_{I_i}$. Due to Part a) and b), we have

$$\begin{aligned} R_{\{i\} \cup I_i}^{(i)} &= R_i^{(i)} \cup R_{I_i}^{(i)} \\ &= ((R_i \setminus \{j_{\max}\}) \cup \{j_{\min}\}) \cup R_{I_i} \\ &= ((R_i \setminus \{j_{\max}\}) \cup R_{I_i}) \cup \{j_{\min}\} \\ &= (R_i \cup R_{I_i}) \cup \{j_{\min}\} \supseteq R_{\{i\} \cup I_i}. \end{aligned}$$

As M satisfies (P3), we have

$$|R_{\{i\} \cup I_i}^{(i)}| \geq |R_{\{i\} \cup I_i}| \geq n - k + |\{i\} \cup I_i| = n - k + |I_i| + 1.$$

This inequality contradicts (6).

Proof of d). Suppose, for contradiction, that $j_{\min} \notin R_{I_i}$. Then by Part a) and b) we have

$$R_{\{i\} \cup I_i}^{(i)} = ((R_i \setminus \{j_{\max}\}) \cup \{j_{\min}\}) \cup R_{I_i} \supseteq \{j_{\min}\} \cup R_{I_i}.$$

Therefore, using the fact that M satisfies (P3), we deduce that

$$|R_{\{i\} \cup I_i}^{(i)}| \geq |\{j_{\min}\} \cup R_{I_i}| = 1 + |R_{I_i}| \geq 1 + n - k + |I_i|.$$

This inequality contradicts (6).

Proof of e). Note that $j_{\max} \in R_i$. However, by Part c), $j_{\max} \notin R_{I_{i'}}$. Hence, $i \notin I_{i'}$.

Proof of f). Using Part a) we have

$$|R_{\{i\} \cup I_i}^{(i)}| = |R_i^{(i)} \cup R_{I_i}^{(i)}| = |R_i^{(i)} \cup R_{I_i}| \geq |R_{I_i}| \geq n - k + |I_i|, \quad (7)$$

where the last inequality comes from the fact that M satisfies (P3). Combining (6) and (7), the proof of f) follows. ■

Lemma 10. For all $i, i' \in [t]$, $i \neq i'$, it holds that $I_i \cap I_{i'} = \emptyset$.

Proof: Without loss of generality, we prove that $I_1 \cap I_2 = \emptyset$. Suppose, for contradiction, that there exists $\ell \in I_1 \cap I_2$. We first present three claims, which are used later in this proof.

Claim 1: For $i = 1, 2$ we have

$$|R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| = n - k + |I_i|, \quad (8)$$

and

$$R_\ell = R_\ell^{(i)} \subseteq R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}. \quad (9)$$

Proof of Claim 1: Indeed, because of the minimality of I_i , the set $\{i\} \cup (I_i \setminus \ell)$ does not violate (P3) in $M^{(i)}$. Therefore,

$$|R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| \geq n - k + |\{i\} \cup (I_i \setminus \ell)| = n - k + |I_i|.$$

On the other hand, $R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)} \subseteq R_{\{i\} \cup I_i}^{(i)}$, which also has cardinality $n - k + |I_i|$, due to Lemma 9 f). Therefore,

$$R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)} = R_{\{i\} \cup I_i}^{(i)},$$

and

$$|R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| = |R_{\{i\} \cup I_i}^{(i)}| = n - k + |I_i|.$$

We also deduce that $R_\ell^{(i)} \subseteq R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}$. By Lemma 9 a), we have $R_\ell = R_\ell^{(i)}$. Thus we complete the proof of Claim 1. ■

Claim 2: Let $K = (I_1 \setminus \{\ell\}) \cap (I_2 \setminus \{\ell\})$. Then for $i = 1, 2$, the following holds

$$|R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K} \setminus \{j_{\max}\}| \leq |R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| - |R_{\{\ell\} \cup K}|. \quad (10)$$

Proof of Claim 2: Using Lemma 9 a) and b), we have

$$\begin{aligned} R_{\{i\} \cup I_i \setminus \{\ell\}} &= R_i \cup R_{I_i \setminus \{\ell\}} \\ &= R_i \cup R_{I_i \setminus \{\ell\}}^{(i)} \\ &= ((R_i^{(i)} \cup \{j_{\max}\}) \setminus \{j_{\min}\}) \cup R_{I_i \setminus \{\ell\}}^{(i)} \\ &\subseteq \{j_{\max}\} \cup (R_i^{(i)} \cup R_{I_i \setminus \{\ell\}}^{(i)}) \\ &= \{j_{\max}\} \cup R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}. \end{aligned}$$

Therefore, $R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus \{j_{\max}\} \subseteq R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}$. Hence

$$\begin{aligned} |(R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \setminus \{j_{\max}\}| &= |(R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus \{j_{\max}\}) \setminus R_{\{\ell\} \cup K}| \\ &\leq |R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)} \setminus R_{\{\ell\} \cup K}| \\ &= |R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| - |R_{\{\ell\} \cup K}|, \end{aligned}$$

where the last equality can be explained by the fact that $R_{\{\ell\} \cup K} \subseteq R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}$. Indeed, we have

$$R_\ell \subseteq R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}$$

due to (9). Moreover, $K \subseteq I_i \setminus \{\ell\}$. Hence the aforementioned inclusion holds. We complete the proof of Claim 2. ■

Claim 3: If $I_1 \cap I_2 = \{\ell\}$ then for $i = 1, 2$, we have

$$|R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus R_\ell \setminus \{j_{\max}\}| \leq |I_i| - 1. \quad (11)$$

Proof of Claim 3: Applying (10) with $K = \emptyset$, we obtain

$$\begin{aligned} |R_{\{i\} \cup I_i \setminus \{\ell\}} \setminus R_\ell \setminus \{j_{\max}\}| &\leq |R_{\{i\} \cup I_i \setminus \{\ell\}}^{(i)}| - |R_\ell| \\ &\stackrel{(8)}{=} (n - k + |I_i|) - (n - k + 1) \\ &= |I_i| - 1. \end{aligned}$$

We complete the proof of Claim 3. ■

The remaining of the proof of Lemma 10 is divided into two cases. Our goal is to obtain contradictions in both cases.

Case 1: $I_1 \cap I_2 = \{\ell\}$.

We aim to show that

$$|R_{\{1,2\} \cup I_1 \cup I_2}| < n - k + |\{1, 2\} \cup I_1 \cup I_2|. \quad (12)$$

This is a contradiction of our assumption that M satisfies (P3). Firstly, since $I_1 \cap I_2 = \{\ell\}$, we have

$$n - k + |\{1, 2\} \cup I_1 \cup I_2| = n - k + |I_1| + |I_2| + 1. \quad (13)$$

Secondly, we consider

$$\begin{aligned} R_{\{1,2\} \cup I_1 \cup I_2} &= R_\ell \cup (R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_\ell) \cup (R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_\ell) \\ &= R_\ell \cup \{j_{\max}\} \cup ((R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_\ell) \setminus \{j_{\max}\}) \\ &\quad \cup ((R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_\ell) \setminus \{j_{\max}\}). \end{aligned}$$

Therefore,

$$\begin{aligned} |R_{\{1,2\} \cup I_1 \cup I_2}| &\leq |R_\ell| + 1 + |(R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_\ell) \setminus \{j_{\max}\}| \\ &\quad + |(R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_\ell) \setminus \{j_{\max}\}| \\ &\stackrel{(11)}{\leq} (n - k + 1) + 1 + (|I_1| - 1) + (|I_2| - 1) \\ &= n - k + |I_1| + |I_2|. \end{aligned} \quad (14)$$

Combining (13) and (14), we obtain (12). We complete the analysis of Case 1.

Case 2: $(I_1 \setminus \{\ell\}) \cap (I_2 \setminus \{\ell\}) = K \neq \emptyset$.

We aim to prove that

$$R_\ell \subseteq R_K, \quad (15)$$

and

$$|R_K| = n - k + |K|. \quad (16)$$

If both (15) and (16) hold then

$$|R_{\{\ell\} \cup K}| = |R_K| = n - k + |K| < n - k + |\{\ell\} \cup K|,$$

which contradicts our assumption that M satisfies (P3).

Let $\delta = |R_\ell \setminus R_K| \geq 0$. As M satisfies (P3), let

$$|R_K| = n - k + |K| + \varepsilon,$$

where $\varepsilon \geq 0$. Then

$$|R_{\{\ell\} \cup K}| = |R_K| + |R_\ell \setminus R_K| = n - k + |K| + \varepsilon + \delta. \quad (17)$$

We have

$$\begin{aligned}
R_{\{1,2\} \cup I_1 \cup I_2} &= R_{\{\ell\} \cup K} \cup (R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \\
&\quad \cup (R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \\
&= R_{\{\ell\} \cup K} \cup \{j_{\max}\} \\
&\quad \cup ((R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \setminus \{j_{\max}\}) \\
&\quad \cup ((R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \setminus \{j_{\max}\}).
\end{aligned}$$

Therefore

$$\begin{aligned}
|R_{\{1,2\} \cup I_1 \cup I_2}| &\leq |R_{\{\ell\} \cup K} \cup \{j_{\max}\}| \\
&\quad + |(R_{\{1\} \cup I_1 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \setminus \{j_{\max}\}| \\
&\quad + |(R_{\{2\} \cup I_2 \setminus \{\ell\}} \setminus R_{\{\ell\} \cup K}) \setminus \{j_{\max}\}| \\
&\stackrel{(10)}{\leq} |R_{\{\ell\} \cup K}| + 1 \\
&\quad + |R_{\{1\} \cup I_1 \setminus \{\ell\}}^{(1)}| - |R_{\{\ell\} \cup K}| \\
&\quad + |R_{\{2\} \cup I_2 \setminus \{\ell\}}^{(2)}| - |R_{\{\ell\} \cup K}| \\
&= |R_{\{1\} \cup I_1 \setminus \{\ell\}}^{(1)}| + |R_{\{2\} \cup I_2 \setminus \{\ell\}}^{(2)}| \\
&\quad - |R_{\{\ell\} \cup K}| + 1 \\
&\stackrel{(8)(17)}{=} (n - k + |I_1|) + (n - k + |I_2|) \\
&\quad - (n - k + |K| + \varepsilon + \delta) + 1 \\
&\leq n - k + |I_1| + |I_2| - |K| + 1.
\end{aligned} \tag{18}$$

Moreover, as $I_1 \cap I_2 = \{\ell\} \cup K$, we have

$$\begin{aligned}
|\{1, 2\} \cup I_1 \cup I_2| &= 2 + |I_1| + |I_2| - |\{\ell\} \cup K| \\
&= |I_1| + |I_2| - |K| + 1.
\end{aligned} \tag{19}$$

As M satisfies (P3), from (18) and (19), we conclude that

$$|R_{\{1,2\} \cup I_1 \cup I_2}| = n - k + |I_1| + |I_2| - |K| + 1.$$

Therefore, all of the inequalities in (18) must be equalities. In particular, the last equality forces $\varepsilon = 0$ and $\delta = 0$. As $\delta = 0$ implies that (15) holds and $\varepsilon = 0$ implies that (16) holds, we complete the analysis of Case 2.

In any cases, we always derive a contradiction. Therefore, our assumption that there exists some $\ell \in I_1 \cap I_2$ is wrong. Hence $I_1 \cap I_2 = \emptyset$. It follows immediately that $I_i \cap I_{i'} = \emptyset$ for every $i, i' \in [t]$, $i \neq i'$. ■

We are now in position to prove Lemma 8, which in turn implies Theorem 6.

Proof of Lemma 8: Recall that we assume that

$$\{i \in [k] : m_{i,j_{\max}} = 1 \text{ and } m_{i,j_{\min}} = 0\} = [t]. \tag{20}$$

Moreover, we suppose, for contradiction, that none of these t rows satisfy the second condition in Step 7 of Algorithm 1. As shown by Lemma 9 c), d), e), and Lemma 10, we can associate to each $i \in [t]$ a subset $I_i \subset [k]$ satisfying the following

- (S1) $i \notin I_{i'}$, for all $i, i' \in [t]$,
- (S2) $j_{\max} \notin R_{I_i}$, for all $i \in [t]$,
- (S3) $j_{\min} \in R_{I_i}$, for all $i \in [t]$,
- (S4) $I_i \cap I_{i'} = \emptyset$, for all $i, i' \in [t]$, $i \neq i'$.

Due to (S2) and (S3), for each $i \in [t]$, there exists a row $r(i) \in I_i$ that has a zero at column j_{\max} and a one at column j_{\min} .

By (S1) and (S4), $r(i) \neq i'$ for all $i, i' \in [t]$ and $r(i) \neq r(i')$ whenever $i \neq i'$.

$$\begin{array}{cccc}
\text{row 1} & \text{---} & 1 & \text{---} & 0 & \text{---} \\
\text{row 2} & \text{---} & 1 & \text{---} & 0 & \text{---} \\
& & \vdots & & \vdots & \\
\text{row } t & \text{---} & 1 & \text{---} & 0 & \text{---} \\
\hline
\text{row } r(1) & \text{---} & 0 & \text{---} & 1 & \text{---} \\
\text{row } r(2) & \text{---} & 0 & \text{---} & 1 & \text{---} \\
& & \vdots & & \vdots & \\
\text{row } r(t) & \text{---} & 0 & \text{---} & 1 & \text{---}
\end{array}
\begin{array}{l}
\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} t \text{ rows} \\
\\
\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} t \text{ rows}
\end{array}$$

Along the rows in the set $[t] \cup \{r(i) : i \in [t]\}$, the weights of the two columns j_{\max} and j_{\min} are the same (equal to t). The other rows of M , because of (20), must contribute at least as much to the weight of column j_{\min} as to the weight of column j_{\max} . Therefore, in total, the weight of column j_{\max} is not larger than the weight of column j_{\min} of M . This conclusion contradicts the fact that $\max \geq \min + 2$. ■

We now discuss the complexity of Algorithm 1. In the initial matrix \widetilde{M} , the difference between the maximum and the minimum column weights is at most $k - 1$. Therefore, according to the proof of Lemma 7, the repeat loop finishes after at most $(k - 1) \lfloor \frac{n}{2} \rfloor$ iterations. It is obvious that all steps in each iteration can be done in polynomial time in n and k , except for Step 7. It is not straightforward that the verification of (P3) for a given $k \times n$ matrix can be done in polynomial time. However, it can be shown that by considering a special one-source k -sink network (of linear size in k and n) associated with each matrix, (P3) is equivalent to the condition that in this network, the minimum capacity of a cut between the source and any sink is at least n . On any network, this condition can be verified in polynomial time using the famous network flow algorithm (see, for instance [7]). Therefore, Algorithm 1 runs in polynomial time in k and n . We omit the proof due to lack of space. Interested reader can find the proof online at [6].

IV. ACKNOWLEDGMENT

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