# On Orthogonal Signalling in Gaussian Multiple Access Channel With Peak Constraints

Kamyar Moshksar
Dep. of Electrical & Computer Eng.
University of Waterloo
Waterloo, ON, Canada
Email: kmoshksa@uwaterloo.ca

Babak Mamandipoor
Dep. of Electrical & Computer Eng.
University of Waterloo
Waterloo, ON, Canada
Email: bmamandi@uwaterloo.ca

Amir K. Khandani
Dep. of Electrical & Computer Eng.
University of Waterloo
Waterloo, ON, Canada
Email: khandani@uwaterloo.ca

***Abstract*—This paper is a follow up to [1] on the two-user Gaussian Multiple Access Channel (MAC) with peak constraints at the transmitters. It is shown that there exist an infinite number of *sum-rate-optimal* points on the boundary of the capacity region. In contrast to the Gaussian MAC with power constraints, we verify that Time Division (TD) can not achieve any of the sum-rate-optimal points in the Gaussian MAC with peak constraints. Using the so-called I-MMSE identity of Guo *et.al*, the largest achievable sum-rate by Orthogonal Code Division (OCD) is characterized where it is shown that Walsh-Hadamard spreading codes of length 2 are optimal. In the symmetric case where the peak constraints at both transmitters are similar, we verify that OCD can achieve a sum-rate that is strictly larger than the highest sum-rate achieved by TD. Finally, it is demonstrated that there are values for the maximum peak at the transmitters such that OCD can not achieve any of the sum-rate-optimal points on the boundary of the capacity region.**

## I. Introduction

### A. Summary of Prior Work

Extending the lines of proof applied by Smith [2] to study a point to point channel with peak constraint at the transmitter, [1] proves the following:

*Theorem 1:* Let $\boldsymbol{u}$ be a random variable with support $[-A, A]$ for some $A > 0$ and $\boldsymbol{z} \sim \mathrm{N}(0,1)$. For any $B > 0$, a unique and discrete random variable $\boldsymbol{x}$ with a finite number of mass points in $[-B, B]$ is the answer to the optimization problem $\sup_{\boldsymbol{x}:|\boldsymbol{x}|\leq B} I(\boldsymbol{x};\boldsymbol{x}+\boldsymbol{u}+\boldsymbol{z})$.

Using the result of Theorem 1, the authors in [1] study the largest achievable sum-rate in a two-user Gaussian MAC with peak constraints at the transmitters. The received vector at the common receiver is given by

$$\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{z}, \tag{1}$$

where $\boldsymbol{x}_i$ is the signal transmitted by user $i$ and is subject to the peak constraint $|\boldsymbol{x}_i| \leq A_i$ for $i = 1, 2$. Also, $\boldsymbol{z} \sim \mathrm{N}(0, 1)$ represents the ambient additive noise. Note that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{z}$ are independent random variables. Let us define $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ as optimal choices for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ such that the sum-rate in the network is maximized, i.e.,

$$\boldsymbol{x}_1^*, \boldsymbol{x}_2^* = \arg \sup_{\boldsymbol{x}_i:|\boldsymbol{x}_i|\leq A_i,\, i=1,2} I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}). \tag{2}$$

Note that there might be more than one choice of $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ that satisfy (2). Since $I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}) = h(\boldsymbol{y}) - h(\boldsymbol{z})$, one can alternatively write (2) as

$$\boldsymbol{x}_1^*, \boldsymbol{x}_2^* = \arg \sup_{\boldsymbol{x}_i:|\boldsymbol{x}_i|\leq A_i,\, i=1,2} h(\boldsymbol{y}). \tag{3}$$

Let us fix $\boldsymbol{x}_1 = \boldsymbol{x}_1^*$. Note that the distribution of $\boldsymbol{x}_1^*$ is unknown at this point. Define $\widetilde{\boldsymbol{x}}_2$ by

$$\widetilde{\boldsymbol{x}}_2 \triangleq \arg \sup_{\boldsymbol{x}_2:|\boldsymbol{x}_2|\leq A_2} h(\boldsymbol{x}_1^* + \boldsymbol{x}_2 + \boldsymbol{z}). \tag{4}$$

Therefore,

$$h(\boldsymbol{x}_1^* + \boldsymbol{x}_2^* + \boldsymbol{z}) \leq h(\boldsymbol{x}_1^* + \widetilde{\boldsymbol{x}}_2 + \boldsymbol{z}). \tag{5}$$

According to (3),

$$h(\boldsymbol{x}_1^* + \boldsymbol{x}_2^* + \boldsymbol{z}) \geq h(\boldsymbol{x}_1^* + \widetilde{\boldsymbol{x}}_2 + \boldsymbol{z}). \tag{6}$$

Comparing (5) and (6), $h(\boldsymbol{x}_1^* + \boldsymbol{x}_2^* + \boldsymbol{z}) = h(\boldsymbol{x}_1^* + \widetilde{\boldsymbol{x}}_2 + \boldsymbol{z})$. By Theorem 1, the answer to (4) is a *unique* discrete random variable $\widetilde{\boldsymbol{x}}_2$ with a finite number of mass points. Hence, $\boldsymbol{x}_2^* = \widetilde{\boldsymbol{x}}_2$. This shows any $\boldsymbol{x}_2^*$ satisfying (3) must be discrete with a finite number of mass points. A similar argument can be applied to verify the same property for $\boldsymbol{x}_1^*$.

Although no claim is made on uniqueness, it is shown that any answer to (3) must be discrete with a finite number of mass points. This brings us to the contributions made in this paper where in particular it is shown that indeed the answer to (3) is not unique.

### B. Contributions

The contributions made in this paper are twofold:

*1) On the number of sum-rate-optimal points on the boundary of the capacity region:* It is interesting to see if there are more that a single point on the boundary of the capacity region that achieve the largest sum-rate in (2). In fact, it is shown in section II that this statement is correct, i.e, there exists a line segment on the boundary such that any point on this segment achieves the largest sum-rate as shown in Fig. 1. We refer to this segment as the set of *sum-rate-optimal* points on the boundary of the capacity region. Note that this observation shows in particular that the answer to (3) is not unique.

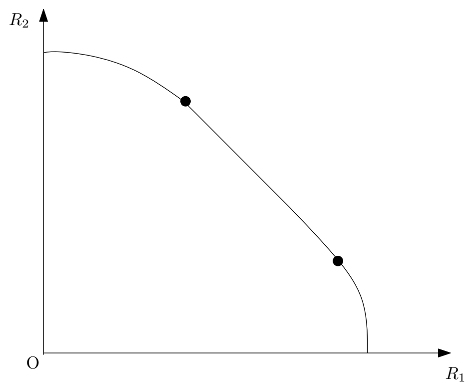


Fig. 1. Capacity region of a Gaussian MAC with peak constraints

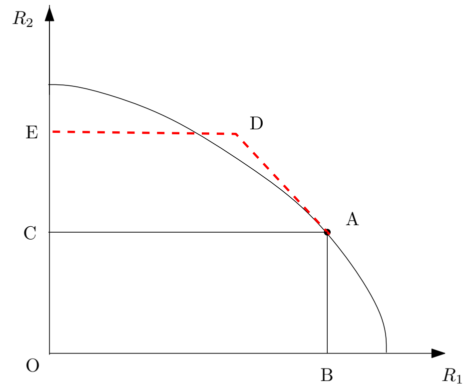


Fig. 2. Capacity region of a Gaussian MAC with peak constraints as if there was only a unique sum-rate-optimal point on the boundary

*2) Achievable sum-rate by orthogonal schemes:* It is well-known [7] that in a Gaussian MAC with only power constraints at the transmitters, any orthogonal scheme meets the boundary of the capacity region at a point where the sum-rate is at its largest value. In section III, we pose the question if orthogonal schemes can achieve the largest sum-rate in a Gaussian MAC with peak constraints at the transmitters. It is verified that Time Division (TD) is suboptimal in the sense that it can not achieve any of the sum-rate-optimal points on the boundary of the capacity region where the sum-rate is maximized. Invoking the so-called I-MMSE identity of Guo *et.al* [3], we characterize the largest achievable sum-rate using Orthogonal Code Division (OCD) where it is shown that Walsh-Hadamard spreading codes of length 2 are optimal. It is also verified that OCD achieves a sum-rate that is strictly larger than the largest sum-rate achieved by TD in the symmetric case of similar peak constraints at both transmitters. Finally, based on a result of Raginsky [6], we show that there exist values for the peak constraints at the transmitters such that the largest sum-rate attained by OCD is strictly less than the largest achievable sum-rate in the network.

*Notation:* The set of real numbers is shown by $\mathbb{R}$. The Euclidean space of real vectors of dimension $n$ is shown by $\mathbb{R}^n$. For any event $\mathcal{E}$, its probability is shown by $\mathbb{P}(\mathcal{E})$. The conditional probability of an event $\mathcal{E}$ given event $\mathcal{F}$ is denoted by $\mathbb{P}(\mathcal{E}|\mathcal{F})$. Random variables are shown in bold such as $\boldsymbol{x}$ with realization $x$. Vectors are shown by an arrow on top such as the random vector $\vec{\boldsymbol{x}}$ with realization $\vec{x}$. The transpose of $\vec{x}$ is denoted by $\vec{x}^{\,\mathrm{t}}$. For any $\vec{x}\in\mathbb{R}^n$, we define $\|\vec{x}\|_2\triangleq\sqrt{\vec{x}^{\,\mathrm{t}}\vec{x}}$ and $\|\vec{x}\|_\infty\triangleq\max_{1\leq i\leq n}|x_i|$ where $x_i$ is the $i^{th}$ coordinate of $\vec{x}$. The Probability Density Function (PDF) of a continuous random variable $\boldsymbol{x}$ is shown by $p_{\boldsymbol{x}}(\cdot)$. The conditional PDF of a continuous random variable given an event $\mathcal{E}$ is shown by $p_{\boldsymbol{x}}(\cdot|\mathcal{E})$. The expectation of a random variable $\boldsymbol{x}$ is denoted by $\mathbb{E}[\boldsymbol{x}]$ and the conditional expectation of $\boldsymbol{x}$ given $\boldsymbol{y}$ is denoted by $\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}]$. The differential entropy of a continuous random variable $\boldsymbol{x}$ is shown by $h(\boldsymbol{x})$, the entropy of a discrete random variable $\boldsymbol{x}$ is shown by $H(\boldsymbol{x})$ and $I(\boldsymbol{x};\boldsymbol{y})$ denotes the mutual information between random variables $\boldsymbol{x}$ and $\boldsymbol{y}$. A normal random vector with mean $\vec{m}$ and covariance matrix $C$ is denoted by $\mathrm{N}(\vec{m},C)$. An $n\times 1$ vector whose all elements are 0 is denoted by $0_{n\times1}$.

## II. On the number of sum-rate-optimal points on the boundary of the capacity region

In this section, we show that there are at least two sum-rate-optimal points on the boundary of the capacity region of the Gaussian MAC with peak constraints. By time-sharing arguments, it follows that there exists a line segment on the boundary such that each point on this segment is sum-rate-optimal as shown in Fig. 1. The proof is by contradiction. Assume on the contrary that there is a unique point A on the boundary of the capacity region that is sum-rate optimal. This situation is shown in Fig. 2. As such, A must be a corner point of the region

$$\begin{cases} R_1\leq I(\boldsymbol{x}_1^*;\boldsymbol{y}^*|\boldsymbol{x}_2^*)\\ R_2\leq I(\boldsymbol{x}_2^*;\boldsymbol{y}^*|\boldsymbol{x}_1^*)\\ R_1+R_2\leq I(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*;\boldsymbol{y}^*)\end{cases}, \tag{7}$$

where $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ are discrete random variables given by (2) and

$$\boldsymbol{y}^*\triangleq\boldsymbol{x}_1^*+\boldsymbol{x}_2^*+\boldsymbol{z}. \tag{8}$$

However, if the region in (7) represents a pentagon such as OBADE in Fig. 2, then there exists an achievable point, say point $D$, that lies outside the capacity region. Therefore, (7) can not be a pentagon and hence, it must represent the rectangle OBAC in Fig. 2. This yields

$$I(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*;\boldsymbol{y}^*)=I(\boldsymbol{x}_1^*;\boldsymbol{y}^*|\boldsymbol{x}_2^*)+I(\boldsymbol{x}_2^*;\boldsymbol{y}^*|\boldsymbol{x}_1^*). \tag{9}$$

However,

$$I(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*;\boldsymbol{y}^*)=I(\boldsymbol{x}_1^*;\boldsymbol{y}^*)+I(\boldsymbol{x}_2^*;\boldsymbol{y}^*|\boldsymbol{x}_1^*). \tag{10}$$

By (9) and (10),

$$I(\boldsymbol{x}_1^*;\boldsymbol{y}^*)=I(\boldsymbol{x}_1^*;\boldsymbol{y}^*|\boldsymbol{x}_2^*). \tag{11}$$

On the other hand,

$$\begin{aligned} I(\boldsymbol{x}_1^*;\boldsymbol{y}^*|\boldsymbol{x}_2^*) &= H(\boldsymbol{x}_1^*|\boldsymbol{x}_2^*) - H(\boldsymbol{x}_1^*|\boldsymbol{x}_2^*,\boldsymbol{y}^*) \\ &= H(\boldsymbol{x}_1^*) - H(\boldsymbol{x}_1^*|\boldsymbol{x}_2^*,\boldsymbol{y}^*) \\ &= I(\boldsymbol{x}_1^*;\boldsymbol{x}_2^*,\boldsymbol{y}^*) \\ &= I(\boldsymbol{x}_1^*;\boldsymbol{y}^*) + I(\boldsymbol{x}_1^*;\boldsymbol{x}_2^*|\boldsymbol{y}^*), \end{aligned} \tag{12}$$

where the second step is due to independence of $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$. Combining (11) with (12), we get

$$I(\boldsymbol{x}_1^*;\boldsymbol{x}_2^*|\boldsymbol{y}^*) = 0, \tag{13}$$

i.e., $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ are conditionally independent given $\boldsymbol{y}^*$, or equivalently, $\boldsymbol{x}_1^* \to \boldsymbol{y}^* \to \boldsymbol{x}_2^*$ is a Markov chain. Let $\mathbb{P}(\boldsymbol{x}_1^* = a_i) = p_i$ for $i = 1,\cdots,m$ and $\mathbb{P}(\boldsymbol{x}_2^* = b_j) = q_j$ for $j = 1,\cdots,n$ where $m,n \geq 2$, $a_1,\cdots,a_m \in [-A_1,A_1]$, $b_1,\cdots,b_n \in [-A_2,A_2]$ and $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j = 1$. The condition $\boldsymbol{x}_1 \to \boldsymbol{y} \to \boldsymbol{x}_2$ holds if and only if

$$\mathbb{P}(\boldsymbol{x}_2^* = b|\boldsymbol{y}^* = y, \boldsymbol{x}_1^* = a) = \mathbb{P}(\boldsymbol{x}_2^* = b|\boldsymbol{y}^* = y), \tag{14}$$

for any $a \in \{a_1,\cdots,a_m\}$, $b \in \{b_1,\cdots,b_m\}$ and $y \in \mathbb{R}$. One can alternatively write (14) as

$$\begin{aligned} &\frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a, \boldsymbol{x}_2^* = b)\mathbb{P}(\boldsymbol{x}_1^* = a\}\mathbb{P}(\boldsymbol{x}_2^* = b)}{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a)\mathbb{P}(\boldsymbol{x}_1^* = a)} \\ &= \frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_2^* = b)\mathbb{P}(\boldsymbol{x}_2^* = b)}{p_{\boldsymbol{y}^*}(y)}, \end{aligned} \tag{15}$$

or equivalently,

$$p_{\boldsymbol{y}^*}(y) = \frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a)p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_2^* = b)}{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a, \boldsymbol{x}_2^* = b)}. \tag{16}$$

This implies that $\frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^*=a)p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_2^*=b)}{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^*=a,\boldsymbol{x}_2^*=b)}$ must be a PDF. But,

$$\begin{aligned} &\frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a)p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_2^* = b)}{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^* = a, \boldsymbol{x}_2^* = b)} \\ &= \frac{\frac{1}{2\pi}\sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{-\frac{1}{2}\left((y-a-b_j)^2+(y-b-a_i)^2\right)}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-a-b)^2}} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{-\frac{1}{2}\left((y-a-b_j)^2+(y-b-a_i)^2-(y-a-b)^2\right)} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^m\sum_{j=1}^n p_i q_j \\ &\qquad \times e^{-\frac{1}{2}\left(y^2-2y(a_i+b_j)+(a+b_j)^2+(b+a_i)^2-(a+b)^2\right)} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{-\frac{1}{2}(y-a_i-b_j)^2} \\ &\qquad \times e^{-\frac{1}{2}\left((a+b_j)^2+(b+a_i)^2-(a+b)^2-(a_i+b_j)^2\right)} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{-\frac{1}{2}(y-a_i-b_j)^2} e^{(a_i-a)(b_j-b)}. \end{aligned} \tag{17}$$

Integrating both side of (17) over $y \in \mathbb{R}$ and recalling that $\frac{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^*=a)p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_2^*=b)}{p_{\boldsymbol{y}^*}(y|\boldsymbol{x}_1^*=a,\boldsymbol{x}_2^*=b)}$ must be a PDF, we get

$$1 = \sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{(a_i-a)(b_j-b)}, \tag{18}$$

where we have used $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-a_i-b_j)^2}\mathrm{d}y = 1$. Equation (18) must hold for any choice of $a \in \{a_1,\cdots,a_m\}$ and $b \in \{b_1,\cdots,b_n\}$. In particular, let us take $a = \min_{1\leq i\leq m} a_i$ and $b = \min_{1\leq j\leq n} b_j$. Then $e^{(a_i-a)(b_j-b)} \geq 1$ for any $i$ and $j$ and there is $i_0 \in \{1,\cdots,m\}$ and $j_0 \in \{1,\cdots,n\}$ such that $(a_{i_0} - a)(b_{j_0} - b) > 0$ or equivalently, $e^{(a_{i_0}-a)(b_{j_0}-b)} > 1$. Therefore, $\sum_{i=1}^m\sum_{j=1}^n p_i q_j e^{(a_i-a)(b_j-b)} > \sum_{i=1}^m\sum_{j=1}^n p_i q_j = 1$. This is a contradiction to (18). Hence, there must exist at least two points on the boundary of the capacity region that are sum-rate-optimal as shown in Fig. 1.

## III. Achievable sum-rate by orthogonal schemes

Throughout this section, we need the following Lemma which is the main result of [3], [4]:

*Lemma 1:* For any random variable $\boldsymbol{x}$ and Gaussian $\boldsymbol{z} \sim \mathrm{N}(0,1)$ such that $I(\boldsymbol{x};\sqrt{\mathsf{snr}}\,\boldsymbol{x}+\boldsymbol{z})$ is finite, we have

$$\frac{\mathrm{d}}{\mathrm{d}\mathsf{snr}} I(\boldsymbol{x};\sqrt{\mathsf{snr}}\,\boldsymbol{x}+\boldsymbol{z}) = \frac{1}{2}\,\mathrm{mmse}(\boldsymbol{x},\mathsf{snr}), \tag{19}$$

where

$$\mathsf{mmse}(\boldsymbol{x},\mathsf{snr}) = \mathbb{E}\left[\left(\boldsymbol{x} - \mathbb{E}\left[\boldsymbol{x}|\sqrt{\mathsf{snr}}\,\boldsymbol{x}+\boldsymbol{z}\right]\right)^2\right] \tag{20}$$

is the minimum mean-squre error (MMSE) in estimating $\boldsymbol{x}$ from $\sqrt{\mathsf{snr}}\,\boldsymbol{x}+\boldsymbol{z}$.
We refer to the identity in (19) as the I-MMSE identity.

### A. Time Division Multiplexing

For any $A > 0$, let $\boldsymbol{x}_A^{\mathrm{su}}$ be the optimum input distribution for a single user scenario under a peak constraint $A$ on the input[1]. More precisely,

$$\boldsymbol{x}_A^{\mathrm{su}} = \arg\sup_{\boldsymbol{x}:|\boldsymbol{x}|\leq A} I(\boldsymbol{x};\boldsymbol{x}+\boldsymbol{z}),\ \boldsymbol{z}\sim\mathrm{N}(0,1). \tag{21}$$

By [2], $\boldsymbol{x}_A^{\mathrm{su}}$ is a unique discrete random variable with a finite number of mass points in $[-A,A]$. Under TD, user 1 is active over a fraction $\lambda$ of time, while user 2 is silent. Also, user 2 transmits over a fraction $(1-\lambda)$ of time, while user 1 is silent. By definition of $\boldsymbol{x}_A^{\mathrm{su}}$, the highest achievable sum-rate under TD for the Gaussian MAC with peak constraints is given by

$$R_1^{(\mathrm{TD})} + R_2^{(\mathrm{TD})} = \lambda I(\boldsymbol{x}_{A_1}^{\mathrm{su}};\boldsymbol{x}_{A_1}^{\mathrm{su}}+\boldsymbol{z}) + (1-\lambda)I(\boldsymbol{x}_{A_2}^{\mathrm{su}};\boldsymbol{x}_{A_2}^{\mathrm{su}}+\boldsymbol{z}). \tag{22}$$

Without loss of generality, let $A_1 \leq A_2$. Then

$$\begin{aligned} I(\boldsymbol{x}_{A_1}^{\mathrm{su}};\boldsymbol{x}_{A_1}^{\mathrm{su}}+\boldsymbol{z}) &\overset{(a)}{\leq} I\Big(\boldsymbol{x}_{A_1}^{\mathrm{su}};\frac{A_2}{A_1}\boldsymbol{x}_{A_1}^{\mathrm{su}}+\boldsymbol{z}\Big) \\ &\overset{(b)}{=} I\Big(\frac{A_2}{A_1}\boldsymbol{x}_{A_1}^{\mathrm{su}};\frac{A_2}{A_1}\boldsymbol{x}_{A_1}^{\mathrm{su}}+\boldsymbol{z}\Big) \\ &\overset{(c)}{\leq} I(\boldsymbol{x}_{A_2}^{\mathrm{su}};\boldsymbol{x}_{A_2}^{\mathrm{su}}+\boldsymbol{z}), \end{aligned} \tag{23}$$

where $(a)$ is by I-MMSE identity, $(b)$ is due to the fact that $I(\boldsymbol{u};\boldsymbol{v}) = I(t\boldsymbol{u};\boldsymbol{v})$ for any random variables $\boldsymbol{u},\boldsymbol{v}$ and nonzero $t \in \mathbb{R}$ and $(c)$ is by definition of $\boldsymbol{x}_{A_2}^{\mathrm{su}}$ and the fact that

[1]The superscript "su" stands for "single user".

$|\frac{A_2}{A_1}\boldsymbol{x}^{\rm su}_{A_1}| \leq A_2$, thus complies with the peak power constraint. By (22) and (23),

$$R_1^{\rm (TD)} + R_2^{\rm (TD)} \leq I(\boldsymbol{x}^{\rm su}_{A_2}; \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}). \quad (24)$$

Recalling the definition of $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ in (2),

$$\begin{aligned} I(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*; \boldsymbol{x}_1^* + \boldsymbol{x}_2^* + \boldsymbol{z}) &\geq I(\boldsymbol{x}^{\rm su}_{A_1}, \boldsymbol{x}^{\rm su}_{A_2}; \boldsymbol{x}^{\rm su}_{A_1} + \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}) \\ &= I(\boldsymbol{x}^{\rm su}_{A_1}; \boldsymbol{x}^{\rm su}_{A_1} + \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}) \\ &\quad + I(\boldsymbol{x}^{\rm su}_{A_2}; \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}) \\ &> I(\boldsymbol{x}^{\rm su}_{A_2}; \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}), \end{aligned} \quad (25)$$

where in the last step we have used the fact that $I(\boldsymbol{x}^{\rm su}_{A_1}; \boldsymbol{x}^{\rm su}_{A_1} + \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}) > 0$. By (24) and (25),

$$R_1^{\rm (TD)} + R_2^{\rm (TD)} < I(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*; \boldsymbol{x}_1^* + \boldsymbol{x}_2^* + \boldsymbol{z}). \quad (26)$$

Note that (26) holds regardless of the value of $\lambda \in [0, 1]$. Hence, the largest sum-rate achieved by TD is strictly less than the largest achievable sum-rate in a Gaussian MAC with peak constraints.

### *B. Orthogonal Code Division Multiplexing*

Under OCD, user 1 and user 2 transmit $\boldsymbol{x}_1\vec{s}_1$ and $\boldsymbol{x}_2\vec{s}_2$ along $N \geq 2$ consecutive transmission slots indexed by $n = 1, \cdots, N$, respectively, where $\vec{s}_1$ and $\vec{s}_2$ are two orthogonal $N \times 1$ spreading vectors, i.e., $\vec{s}_1^{\rm t}\vec{s}_2 = 0$. Under a peak constraint at both transmitters, we require

$$\|\boldsymbol{x}_i\vec{s}_i\|_\infty \leq A_i, \;\; i = 1, 2. \quad (27)$$

Let us denote the received $N \times 1$ vector at the common receiver by

$$\vec{\boldsymbol{y}} = \boldsymbol{x}_1\vec{s}_1 + \boldsymbol{x}_2\vec{s}_2 + \vec{\boldsymbol{w}}, \quad (28)$$

where $\vec{\boldsymbol{w}} \sim \mathrm{N}(0_{N\times 1}, I_N)$ is the vector of ambient additive noise samples. Assuming $G$ is an $N \times (N-2)$ matrix whose columns together with $\vec{s}_1$ and $\vec{s}_2$ constitute an orthogonal basis for $\mathbb{R}^N$, we have

$$\begin{aligned} & I(\boldsymbol{x}_1, \boldsymbol{x}_2; \vec{\boldsymbol{y}}) \overset{(a)}{=} I(\boldsymbol{x}_1, \boldsymbol{x}_2; G^{\rm t}\vec{\boldsymbol{y}}, \vec{s}_1^{\rm t}\vec{\boldsymbol{y}}, \vec{s}_2^{\rm t}\vec{\boldsymbol{y}}) \\ =\;& I\Big(\boldsymbol{x}_1, \boldsymbol{x}_2; G^{\rm t}\vec{\boldsymbol{w}}, \|\vec{s}_1\|_2^2\boldsymbol{x}_1 + \vec{s}_1^{\rm t}\vec{\boldsymbol{w}}, \|\vec{s}_2\|_2^2\boldsymbol{x}_2 + \vec{s}_2^{\rm t}\vec{\boldsymbol{w}}\Big) \\ \overset{(b)}{=}\;& I\Big(\boldsymbol{x}_1, \boldsymbol{x}_2; \|\vec{s}_1\|_2^2\boldsymbol{x}_1 + \vec{s}_1^{\rm t}\vec{\boldsymbol{w}}, \|\vec{s}_2\|_2^2\boldsymbol{x}_2 + \vec{s}_2^{\rm t}\vec{\boldsymbol{w}}\Big) \\ \overset{(c)}{=}\;& I\Big(\boldsymbol{x}_1; \|\vec{s}_1\|_2^2\boldsymbol{x}_1 + \vec{s}_1^{\rm t}\vec{\boldsymbol{w}}\Big) + I\Big(\boldsymbol{x}_2; \|\vec{s}_2\|_2^2\boldsymbol{x}_2 + \vec{s}_2^{\rm t}\vec{\boldsymbol{w}}\Big) \\ \overset{(d)}{=}\;& I\Big(\boldsymbol{x}_1; \|\vec{s}_1\|_2^2\boldsymbol{x}_1 + \|\vec{s}_1\|_2\boldsymbol{z}\Big) \\ & + I\Big(\boldsymbol{x}_2; \|\vec{s}_2\|_2^2\boldsymbol{x}_2 + \|\vec{s}_2\|_2\boldsymbol{z}\Big) \\ =\;& I\Big(\boldsymbol{x}_1; \|\vec{s}_1\|_2\boldsymbol{x}_1 + \boldsymbol{z}\Big) + I\Big(\boldsymbol{x}_2; \|\vec{s}_2\|_2\boldsymbol{x}_2 + \boldsymbol{z}\Big), \end{aligned} \quad (29)$$

where $(a)$ is by the fact that having $\vec{\boldsymbol{y}}$ is equivalent to having $G^{\rm t}\vec{\boldsymbol{y}}, \vec{s}_1^{\rm t}\vec{\boldsymbol{y}}$ and $\vec{s}_2^{\rm t}\vec{\boldsymbol{y}}$, $(b)$ follows by independence of $G^{\rm t}\vec{\boldsymbol{w}}$ from $\vec{s}_1^{\rm t}\vec{\boldsymbol{w}}, \vec{s}_2^{\rm t}\vec{\boldsymbol{w}}, \boldsymbol{x}_1$ and $\boldsymbol{x}_2$, $(c)$ is due to independence of $\vec{s}_1^{\rm t}\vec{\boldsymbol{w}}, \vec{s}_2^{\rm t}\vec{\boldsymbol{w}}$, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and finally, $\boldsymbol{z} \sim \mathrm{N}(0, 1)$ in $(d)$. Therefore, the largest achievable sum-rate for fixed $\vec{s}_i$, $i = 1, 2$, is given by

$$\begin{aligned} & \sup_{\boldsymbol{x}_i : \|\boldsymbol{x}_i\vec{s}_i\|_\infty \leq A_i} \frac{1}{N} I\Big(\boldsymbol{x}_1; \|\vec{s}_1\|_2\boldsymbol{x}_1 + \boldsymbol{z}\Big) + \frac{1}{N} I\Big(\boldsymbol{x}_2; \|\vec{s}_2\|_2\boldsymbol{x}_2 + \boldsymbol{z}\Big) \\ =\;& \frac{1}{N} \sup_{\boldsymbol{x}_1 : \|\boldsymbol{x}_1\vec{s}_1\|_\infty \leq A_1} I\Big(\boldsymbol{x}_1; \|\vec{s}_1\|_2\boldsymbol{x}_1 + \boldsymbol{z}\Big) \\ & + \frac{1}{N} \sup_{\boldsymbol{x}_2 : \|\boldsymbol{x}_2\vec{s}_2\|_\infty \leq A_2} I\Big(\boldsymbol{x}_2; \|\vec{s}_2\|_2\boldsymbol{x}_2 + \boldsymbol{z}\Big). \end{aligned} \quad (30)$$

For any $i = 1, 2$, one can write

$$\begin{aligned} & \sup_{\boldsymbol{x}_i : \|\boldsymbol{x}_i\vec{s}_i\|_\infty \leq A_i} I\Big(\boldsymbol{x}_i; \|\vec{s}_i\|_2\boldsymbol{x}_i + \boldsymbol{z}\Big) \\ =\;& \sup_{\boldsymbol{x}_i : \|\boldsymbol{x}_i\vec{s}_i\|_\infty \leq A_i} I\Big(\boldsymbol{x}_i; \Big\|\frac{\vec{s}_i}{\|\vec{s}_i\|_\infty}\Big\|_2 \|\vec{s}_i\|_\infty\boldsymbol{x}_i + \boldsymbol{z}\Big) \\ \overset{(a)}{\leq}\;& \sup_{\boldsymbol{x}_i : \|\boldsymbol{x}_i\vec{s}_i\|_\infty \leq A_i} I\Big(\boldsymbol{x}_i; \sqrt{N}\|\vec{s}_i\|_\infty\boldsymbol{x}_i + \boldsymbol{z}\Big) \\ \overset{(b)}{\leq}\;& \sup_{\boldsymbol{x}_i : |\boldsymbol{x}_i| \leq A_i} I\Big(\boldsymbol{x}_i; \sqrt{N}\boldsymbol{x}_i + \boldsymbol{z}\Big), \end{aligned} \quad (31)$$

where $(a)$ is due to $\Big\|\frac{\vec{s}_i}{\|\vec{s}_i\|_\infty}\Big\|_2 \leq \sqrt{N}$ for any nonzero vector $\vec{s}_i$ and the IMMSE identity which guarantees $I\Big(\boldsymbol{x}_i; \Big\|\frac{\vec{s}_i}{\|\vec{s}_i\|_\infty}\Big\|_2 \|\vec{s}_i\|_\infty\boldsymbol{x}_i + \boldsymbol{z}\Big) \leq I\Big(\boldsymbol{x}_i; \sqrt{N}\|\vec{s}_i\|_\infty\boldsymbol{x}_i + \boldsymbol{z}\Big)$ for any random variable $\boldsymbol{x}_i$ and $(b)$ follows by the simple observation that $\|\boldsymbol{x}_i\vec{s}_i\|_\infty = |\boldsymbol{x}_i|\|\vec{s}_i\|_\infty = |\|\vec{s}_i\|_\infty\boldsymbol{x}_i|$. Note that the inequality in $(a)$ is tight if all elements of $\vec{s}_i$ have identical absolute values and the inequality in $(b)$ is tight if $\|\vec{s}_i\|_\infty = 1$. Therefore, the so-called Walsh-Hadamard (WH) spreading vectors achieve the largest sum-rate in the OCD scenario for any fixed value of $N$. As such, we assume $\vec{s}_i$, $i = 1, 2$ are two columns of a Hadamard matrix of size $N$. It is shown in appendix A that for any random variable $\boldsymbol{x}$, $\alpha I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big)$ is a nondecreasing function of $\alpha$ for $\alpha > 0$. Hence, setting $N = 2$ results in the largest achievable sum-rate by OCD. Denoting the achievable rate of user $i$ by $R_i^{\rm (OCD)}$ in this case,

$$\begin{aligned} R_1^{\rm (OCD)} + R_2^{\rm (OCD)} &= \frac{1}{2} \sup_{\boldsymbol{x}_1 : |\boldsymbol{x}_1| \leq A_1} I\Big(\boldsymbol{x}_1; \sqrt{2}\boldsymbol{x}_1 + \boldsymbol{z}\Big) \\ & \quad + \frac{1}{2} \sup_{\boldsymbol{x}_2 : |\boldsymbol{x}_2| \leq A_2} I\Big(\boldsymbol{x}_2; \sqrt{2}\boldsymbol{x}_2 + \boldsymbol{z}\Big) \\ & \overset{(a)}{=} \frac{1}{2} \sup_{\boldsymbol{x}_1 : |\boldsymbol{x}_1| \leq \sqrt{2}A_1} I\Big(\boldsymbol{x}_1; \boldsymbol{x}_1 + \boldsymbol{z}\Big) \\ & \quad + \frac{1}{2} \sup_{\boldsymbol{x}_2 : |\boldsymbol{x}_2| \leq \sqrt{2}A_2} I\Big(\boldsymbol{x}_2; \boldsymbol{x}_2 + \boldsymbol{z}\Big) \\ & \overset{(b)}{=} \frac{1}{2} I\Big(\boldsymbol{x}^{\rm su}_{\sqrt{2}A_1}; \boldsymbol{x}^{\rm su}_{\sqrt{2}A_1} + \boldsymbol{z}\Big) \\ & \quad + \frac{1}{2} I\Big(\boldsymbol{x}^{\rm su}_{\sqrt{2}A_2}; \boldsymbol{x}^{\rm su}_{\sqrt{2}A_2} + \boldsymbol{z}\Big), \end{aligned} \quad (32)$$

where $(a)$ is due to the fact that the collection of random variables $\{\sqrt{2}\boldsymbol{x}_i : |\boldsymbol{x}_i| \leq A_i\}$ is the same as the collection $\{\boldsymbol{x}_i : |\boldsymbol{x}_i| \leq \sqrt{2}A_i\}$ for $i = 1, 2$ and $(b)$ is by definition of $\boldsymbol{x}^{\rm su}_A$ in (21) for any $A > 0$. Next, we consider two scenarios where $R_1^{\rm (OCD)} + R_2^{\rm (OCD)}$ is strictly larger than $R_1^{\rm (TD)} + R_2^{\rm (TD)}$ as follows:

- Fixing $\lambda = \frac{1}{2}$ in (22), the achievable sum-rate by TD is $\frac{1}{2} I\Big(\boldsymbol{x}^{\rm su}_{A_1}; \boldsymbol{x}^{\rm su}_{A_1} + \boldsymbol{z}\Big) + \frac{1}{2} I\Big(\boldsymbol{x}^{\rm su}_{A_2}; \boldsymbol{x}^{\rm su}_{A_2} + \boldsymbol{z}\Big)$. Comparing this with the right side of (32) and noting that by I-MMSE identity, $I\Big(\boldsymbol{x}_{A_i}^{\rm su}; \sqrt{2}\boldsymbol{x}_{A_i}^{\rm su} + \boldsymbol{z}\Big) > I\Big(\boldsymbol{x}_{A_i}^{\rm su}; \boldsymbol{x}_{A_i}^{\rm su} + \boldsymbol{z}\Big)$ for $i = 1, 2$, we conclude that the achievable sum-rate by TD for $\lambda = \frac{1}{2}$ is strictly smaller than $R_1^{(\rm OCD)} + R_2^{(\rm OCD)}$.

- Let $A_1 = A_2 = A$, i.e., the peak constraints are identical at both transmitters. By (22), the largest sum-rate achieved by TD is $I(\boldsymbol{x}_A^{\rm su}; \boldsymbol{x}_A^{\rm su} + \boldsymbol{z})$ regardless of the value of $\lambda \in [0, 1]$. By (32), the largest sum-rate achieved by OCD is $I(\boldsymbol{x}_A^{\rm su}; \boldsymbol{x}_{\sqrt{2}A}^{\rm su} + \boldsymbol{z})$. Noting that $I\Big(\boldsymbol{x}_A^{\rm su}; \sqrt{2}\boldsymbol{x}_A^{\rm su} + \boldsymbol{z}\Big) > I\Big(\boldsymbol{x}_A^{\rm su}; \boldsymbol{x}_A^{\rm su} + \boldsymbol{z}\Big)$ by I-MMSE identity, OCD achieves a sum-rate that is strictly larger than the highest achievable sum-rate by TD.

We end the paper by making the observation that the largest achievable sum-rate by OCD can be strictly smaller than the largest achievable sum-rate in the network given in (2). We need the following Lemma which is an equivalent statement of Theorem 3.1 in [6]:

*Lemma 2:* If $A \leq 1.1025$, then

$$\mathbb{P}(\boldsymbol{x}_A^{\rm su} = A) = \mathbb{P}(\boldsymbol{x}_A^{\rm su} = -A) = \frac{1}{2}. \tag{33}$$

Let us focus on the symmetric case where $A_1 = A_2 = A$. By (32),

$$R_1^{(\rm OCD)} + R_2^{(\rm OCD)} = I\Big(\boldsymbol{x}_{\sqrt{2}A}^{\rm su}; \boldsymbol{x}_{\sqrt{2}A}^{\rm su} + \boldsymbol{z}\Big). \tag{34}$$

Lemma 2 enables us to (numerically) calculate the exact value of $I\Big(\boldsymbol{x}_{\sqrt{2}A}^{\rm su}; \boldsymbol{x}_{\sqrt{2}A}^{\rm su} + \boldsymbol{z}\Big)$ for $A \leq \frac{1.1025}{\sqrt{2}} \approx 0.7796$. Also, let us consider a scenario where both users in the Gaussian MAC with similar peak constraints $A_1 = A_2 = A$ select $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as independent random variables taking on $-A$ and $A$ with equal probabilities. Plots of $I\Big(\boldsymbol{x}_{\sqrt{2}A}^{\rm su}; \boldsymbol{x}_{\sqrt{2}A}^{\rm su} + \boldsymbol{z}\Big)$ and $I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{z})$ are presented in Fig. 3. It is seen that for any $A \in (0.61, 0.7796)$, $I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{z}) > I\Big(\boldsymbol{x}_{\sqrt{2}A}^{\rm su}; \boldsymbol{x}_{\sqrt{2}A}^{\rm su} + \boldsymbol{z}\Big)$. This confirms that OCD is in general not capable of achieving the sum-rate-optimal points on the boundary of the capacity region.

## Appendix

In this appendix, we show that $f(\alpha) := \alpha I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big)$ is a nondecreasing function of $\alpha$ for any $\alpha > 0$ and any random variable $\boldsymbol{x}$ where $\boldsymbol{z} \sim \mathrm{N}(0, 1)$. We have

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\alpha}(\alpha) &= I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big) + \alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big) \\ &= I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big) - \frac{1}{2\alpha}\mathsf{mmse}\left(\boldsymbol{x}, \frac{1}{\alpha}\right), \end{aligned} \tag{35}$$

where the last step is by I-MMSE identity. Hence, $\frac{\mathrm{d}f}{\mathrm{d}\alpha}(\alpha) \geq 0$ holds for any $\alpha > 0$ if and only if

$$I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big) \geq \frac{1}{2\alpha}\mathsf{mmse}\left(\boldsymbol{x}, \frac{1}{\alpha}\right), \ \alpha > 0. \tag{36}$$

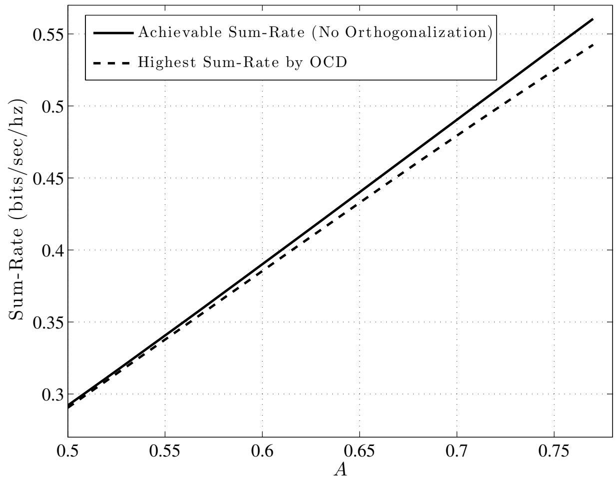


Fig. 3. Plots of $\max_{\boldsymbol{x}:|\boldsymbol{x}| \leq \sqrt{2}A} I(\boldsymbol{x}; \boldsymbol{x} + \boldsymbol{z})$ and $I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{z})$ where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independent signals taking on $-A$ and $A$ with equal probabilities.

To prove (36), we use the I-MMSE identity again. Note that

$$\begin{aligned} I\Big(\boldsymbol{x}; \frac{1}{\sqrt{\alpha}}\boldsymbol{x} + \boldsymbol{z}\Big) &= \frac{1}{2}\int_0^{\frac{1}{\alpha}} \mathsf{mmse}\,(\boldsymbol{x}, \gamma)\,\mathrm{d}\gamma \\ &\overset{(a)}{\geq} \frac{1}{2}\int_0^{\frac{1}{\alpha}} \mathsf{mmse}\left(\boldsymbol{x}, \frac{1}{\alpha}\right)\mathrm{d}\gamma \\ &= \frac{1}{2\alpha}\mathsf{mmse}\left(\boldsymbol{x}, \frac{1}{\alpha}\right), \end{aligned} \tag{37}$$

where $(a)$ is due to the fact that $\mathsf{mmse}(\boldsymbol{x}, \gamma)$ is a nonincreasing function of $\gamma > 0$ which in turn follows from Proposition 9 in [5].

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