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# On Efficient Min-Cut Approximations in Half-Duplex Relay Networks

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***Abstract*—Computing the cut-set bound in half-duplex (HD) relay networks is a challenging optimization problem that involves an exponential number of variables and constraints (exponential in the number of nodes in the network). We present a general technique for efficiently computing the HD schedule that maximizes the cut-set bound (with i.i.d. input distribution) in layered Gaussian relay networks. We use simulations to show running time improvements over alternative methods and compare the performance of various HD scheduling approaches in different SNR regimes.**

## I. Introduction

One of the apparent discrepancies between the current theory and practice of wireless communications is whether relay nodes are *full-duplex* (FD); i.e., whether they can transmit and receive simultaneously in the same frequency band. Even though most of the theoretical studies assume all network nodes to be FD, the practical design of RF radios with this capability is very sophisticated and expensive. Therefore, it is more practical to consider *half-duplex* (HD) relay nodes; i.e., nodes that can operate either in transmit or receive mode.

The main reason for the limited adoption of the HD constraint in theoretical works is the inherent combinatorial aspect related to scheduling the transmit and receive modes of the HD nodes. Therefore, in order to find capacity-achieving schemes, one must inevitably solve an optimization problem that finds the optimal transmit/receive schedule.

Two main challenges lie in the way of solving this problem. The first one is that the number of variables in the schedule optimization problem is $2^{|V|}$, where $V$ is the set of nodes in the network, and, as we consider larger networks, the problem quickly becomes intractable. The second challenge is that for most networks, even for a fixed HD schedule, the capacity is unknown. Thus we cannot even characterize how good a schedule is, let alone compute the optimal one.

To handle the latter challenge, we consider a performance metric that is easier to characterize than the capacity. One intuitive approach is to look for the schedule that maximizes the *cut-set bound*. Besides being the only known general outer bound for the capacity of relay networks, recent results indicate that it is in fact a good outer bound in many cases. As shown in [1], for FD linear deterministic wireless relay networks, the capacity is exactly given by the cut-set bound, and, for FD Gaussian relay networks, the cut-set bound is a constant-gap capacity approximation. Moreover, [1] shows that the HD schedule that maximizes the cut-set bound supports rates within a constant gap of the HD capacity.

Motivated by the approximate tightness of the cut-set bound and by recent results which show that, in the FD case, an approximation to the cut-set bound of Gaussian relay networks can be efficiently computed [2], we study the problem of efficiently finding the HD schedule that maximizes the cut-set bound of Gaussian relay networks with i.i.d. input distribution.

This problem can be formulated as a linear optimization whose variables describe the HD schedule. A direct solution to this linear program requires a time complexity that is exponential on the number of nodes due to the exponential number of variables and cut constraints in the program. The time complexity can be reduced to some extent by using the techniques introduced in [2], which exploit a submodularity property of mutual information to avoid the need to evaluate the capacity of each of the exponentially many cuts in the network. However, we are still left with an exponential number of variables, reflecting the fact that, in a HD network, the number of ways to define which nodes are talking and which nodes are listening is exponential in the number of nodes.

Motivated by the fact that special network topologies may simplify the computation of half-duplex schedules (e.g., [3]), our main result is a technique that exploits specific properties of the network topology to reduce the number of variables in the linear program. This is done by showing that, if the node set can be divided into smaller parts satisfying certain properties, the probability distribution describing the HD schedule can be broken into factors involving the nodes in these parts. This results in a drastic complexity reduction.

In this paper we present our results for layered Gaussian relay networks. In [4] we show that the same approach can be applied to other non-layered network classes and we consider a more general optimization objective, which allows, for example, to minimize the energy consumption of relays.

## II. Problem Setting

We will focus on wireless relay networks under the Gaussian model, although most of the results we present can be extended to other models (e.g., the models from [1] and [5]).

A Gaussian relay network is defined by a directed graph $G = (V, E)$ with a source node $S \in V$, a destination node $D \in V$, and a channel gain $h_e \in \mathbb{R}$ for each edge $e \in E$. The nodes in the network are all *half-duplex*, meaning that they cannot simultaneously transmit and receive. Therefore, a coding scheme for a network with HD nodes must specify, besides the usual encoding, decoding and relaying functions, which nodes transmit and which nodes receive at any given time. In other words, a coding scheme with blocklength $n$ must define, for each time $t = 1, 2, ..., n$, a partition of the node set $V$ into a set of transmitter nodes $Tx[t]$ and a set of receiver nodes $Rx[t]$. At time $t = 1, 2, ..., n$, each node $v \in Tx[t]$ may transmit any real-valued signal $X_v[t]$, whereas a node $v \in Rx[t]$, must transmit $X_v[t] = 0$. Then, at time $t = 1, 2, ..., n$, the signal received by a node $v \in Rx[t]$ is given by $Y_v[t] = \sum_{u:(u,v)\in E} h_{u,v} X_u[t] + Z_v[t]$, where $Z_v[t]$ is an i.i.d. $\mathcal{N}(0, 1)$ random process. If, instead, $v \in Tx[t]$, its received signal is $Y_v[t] = 0$. We adopt a fixed-schedule model, where the transmit/receive schedule of a given node is independent of its received signals.

Communication takes place over $n$ discrete time steps and follows a coding scheme which specifies the encoding function for $S$, decoding function for $D$, and causal relaying functions for all remaining nodes $v \in V \setminus \{S, D\}$. Without loss of generality, we assume that the encoding and relaying functions are such that the average power consumed by each node over the $n$ time steps does not exceed 1. Decoding errors, achievable rates, and capacity $C$ are defined in the usual way.

To establish a constant-gap HD capacity approximation, [1] uses the important concept of a *mode configuration*. Since at each time $t$, each node $v \in V$ must be either in transmit or receive mode, the modes of all nodes in the network at a given time $t$ can be described by a binary $|V|$-dimensional vector $\mathbf{m}$, the mode configuration vector. For a given coding scheme we can let $q(\mathbf{m})$ be the fraction of the $n$ time steps where the network uses the mode configuration described by $\mathbf{m}$. Thus $q$ is essentially a probability mass function over the $2^{|V|}$ mode configuration vectors. For a mode configuration $\mathbf{m}$, we will let $\mathcal{T}(\mathbf{m})$ be its transmitter nodes and $\mathcal{R}(\mathbf{m})$ be its receiver nodes. We formally define $\mathcal{T}(\mathbf{m}) = \{v \in V : m_v = 0\}$ and $\mathcal{R}(\mathbf{m}) = \{v \in V : m_v = 1\}$. For Gaussian relay networks with HD nodes, [1] establishes that a constant-gap capacity approximation is given by $\overline{C}_{iid} =$

$$\max_q \min_\Omega \sum_{\mathbf{m}} q(\mathbf{m}) I\left(X_{\Omega\cap\mathcal{T}(\mathbf{m})}; Y_{\Omega^c\cap\mathcal{R}(\mathbf{m})} | X_{(\Omega\cap\mathcal{T}(\mathbf{m}))^c}\right),$$

where $X_i$, $i \in V$, are independent and distributed as $\mathcal{N}(0, 1)$. The technique we introduce for finding a HD schedule can be described as finding the distribution $q$ that maximizes $\overline{C}_{\text{i.i.d.}}$. We note that, for any fixed mode configuration distribution $q$,

$$\min_\Omega \sum_{\mathbf{m}} q(\mathbf{m}) I\left(X_{\Omega\cap\mathcal{T}(\mathbf{m})}; Y_{\Omega^c\cap\mathcal{R}(\mathbf{m})} | X_{(\Omega\cap\mathcal{T}(\mathbf{m}))^c}\right) \quad (1)$$

can be computed efficiently using the results in [2]. In effect, the authors show that, when the the transmit signals $X_i$, $i \in V$, are i.i.d. Gaussian random variables, the conditional mutual information $I(X_\Omega; Y_{\Omega^c} | X_{\Omega^c})$ is a submodular function of $\Omega \in 2^V$. Thus, the minimization in (1) can be solved efficiently without evaluating the capacity of every cut in the network.

Nonetheless, when we consider the computation of $\overline{C}_{iid}$, the maximization over mode configuration distributions $q(\mathbf{m})$ adds an extra layer of difficulty to the problem. This is easily seen by noticing that $q(\mathbf{m})$ is a probability distribution on $2^{|V|}$ elements, rendering the brute force approach computationally inefficient. This approach for computing $\overline{C}_{iid}$ can be formally described through the following linear program.

**Problem 1:** $\underset{R, q(\mathbf{m})}{\text{maximize}}\ R$ subject to: (2)

$$R < \sum_{\mathbf{m}\in\{0,1\}^{|V|}} q(\mathbf{m})\, I\left(X_{\Omega\cap\mathcal{T}(\mathbf{m})}; Y_{\Omega^c\cap\mathcal{R}(\mathbf{m})} | X_{(\Omega\cap\mathcal{T}(\mathbf{m}))^c}\right),$$

$$\forall \Omega \in 2^V : S \in \Omega, D \notin \Omega \quad (3)$$

$$0 \leqslant q(\mathbf{m}), \forall \mathbf{m} \in \{0,1\}^{|V|}; \sum_{\mathbf{m}\in\{0,1\}^{|V|}} q(\mathbf{m}) = 1 \quad (4)$$

$X_i$, $i \in V$, are independent $\mathcal{N}(0, 1)$ random variables

We point out that in Problem 1 there is no optimization of the power allocation of the nodes over the various modes. While power optimization may lead to a smaller gap to the cut-set bound it would also destroy the convex structure of the problem. In addition, the analysis of channels subject to the HD constraint is more meaningful in the high SNR regime, where the channel capacity is degrees-of-freedom (DoF) limited. In this regime, optimization over the power allocations would not provide significant performance improvements.

Notice that, although Problem 1 is defined as a linear program, it involves $\Omega(2^{|V|})$ constraints and variables and, therefore, cannot be solved efficiently. Our main goal will be to find equivalent formulations of Problem 1 which can be efficiently computed for some network topologies.

## III. A SIMPLE EXAMPLE

To illustrate the main ideas presented in this paper consider the example Gaussian relay network in Fig. 1 with node set $V = \{S, 1, 2, 3, 4, 5, D\}$. As explained in Section II, the HD capacity of this network can be characterized within a constant gap by computing the cut-set bound with independent Gaussian encoding at the various nodes. Therefore, the schedule $q$ computed via Problem 1 is approximately optimal.

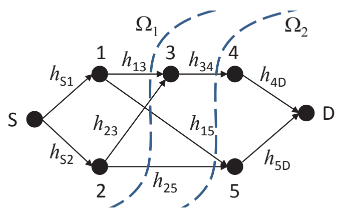


Fig. 1. Gaussian relay network with 5 relays. Two different cuts $\Omega_1 = \{S, 1, 2\}$ and $\Omega_2 = \{S, 1, 2, 3\}$ are shown.

The computational complexity of solving Problem 1 for the network in Fig. 1 arises from the $2^5 = 32$ inequalities of the form (3) involving all the cuts in the network, and the $2^5 = 32$ scalar variables required to represent the distribution $q(\mathbf{m})$ of the HD schedules. We would like to find a more efficient approach for solving this optimization. As shown in [2], it is possible to exploit a submodularity property of mutual information to simplify the computation of the min-cut, and as a result, it is not necessary to compute (3) for each of the 32 cuts. To further simplify the computation we will show that it is possible to reduce the number of optimization variables represented by $q(\mathbf{m})$. Consider the cut $\Omega_1 = \{S,1,2\}$ (shown in Fig. 1), and the corresponding inequality of the form (3)

$$R < \sum_{\mathbf{m}\in\{0,1\}^5} q(\mathbf{m}) \cdot I\left(X_{\{S,1,2\}\cap\mathcal{T}(\mathbf{m})}; Y_{\{3,4,5,D\}\cap\mathcal{R}(\mathbf{m})} | X_{(\{S,1,2\}\cap\mathcal{T}(\mathbf{m}))^c}\right).$$

Due to the network connectivity, this cut capacity only depends on the variables (input, output, and transmit/receive mode) of nodes 1, 2, 3 and 5. As a result, the inequality becomes

$$\begin{aligned} R &< \sum_{\mathbf{m}\in\{0,1\}^5} q(\mathbf{m}) I\left(X_{\{1,2\}\cap\mathcal{T}_{12}}; Y_{\{3,5\}\cap\mathcal{R}_{35}} | X_{\{1,2\}\cap\mathcal{R}_{12})}\right) \\ &= \sum_{(m_1,m_2,m_3,m_5)\in\{0,1\}^4} \left[\sum_{m_4\in\{0,1\}} q(m_1,\dots,m_5)\right] \cdot \\ &\quad I\left(X_{\{1,2\}\cap\mathcal{T}_{12}(m_1,m_2)}; Y_{\{3,5\}\cap\mathcal{R}_{35}} | X_{\{1,2\}\cap\mathcal{R}_{12}}\right) \\ &= \sum_{(m_1,m_2,m_3,m_5)\in\{0,1\}^4} q_{1235}(m_1,m_2,m_3,m_5) \cdot \\ &\quad I\left(X_{\{1,2\}\cap\mathcal{T}_{12}}; Y_{\{3,5\}\cap\mathcal{R}_{35}} | X_{\{1,2\}\cap\mathcal{R}_{12}}\right) \end{aligned}$$

where we let $\mathcal{T}_{i_1,\dots,i_k}(m_{i_1},\dots,m_{i_k}) = \mathcal{T}(\mathbf{m}) \cap \{i_1,\dots,i_k\}$, $\mathcal{R}_{i_1,\dots,i_k}(m_{i_1},\dots,m_{i_k}) = \mathcal{R}(\mathbf{m}) \cap \{i_1,\dots,i_k\}$ and define $q_{1235}(m_1,m_2,m_3,m_5) = \sum_{m_4\in\{0,1\}} q(m_1,\dots,m_5)$.

Similarly, inequality (3) for cut $\Omega_2 = \{S,1,2,3\}$ in Fig. 1 can be rewritten as

$$\begin{aligned} R < &\sum_{(m_1,m_2,m_5)\in\{0,1\}^3} q_{125}(m_1,m_2,m_5) \\ &\cdot I\left(X_{\{1,2\}\cap\mathcal{T}_{12}}; Y_{\{5\}\cap\mathcal{R}_5} | X_{\{1,2\}\cap\mathcal{R}_{12}}\right) \\ &+ \sum_{(m_3,m_4)\in\{0,1\}^2} q_{34}(m_3,m_4) I\left(X_{\{3\}\cap\mathcal{T}_3}; Y_{\{4\}\cap\mathcal{R}_4}\right) \end{aligned}$$

where $q_{125}$ and $q_{34}$ are the corresponding marginals of $q$.

By considering each of the 32 cuts, we observe that all the cut capacities can be decomposed into terms that depend only on the marginals $q_{1235}$ and $q_{34}$. Since $q_{1235}$ and $q_{34}$ must come from some joint distribution $q$, they must satisfy

$$\sum_{\substack{(m_1,m_2,m_5)\\ \in\{0,1\}^3}} q_{1235}(m_1,m_2,m_3,m_5) = \sum_{m_4\in\{0,1\}} q_{34}(m_3,m_4).$$

Assuming that this consistency requirement is satisfied, we can find a joint distribution $\tilde{q}$ with these marginals:

$$\tilde{q}(\mathbf{m}) = q_{1235}(m_1,m_2,m_3,m_5) \cdot q_{34}(m_3,m_4)/q_3(m_3)$$

where $q_3(m_3) = \sum_{m_4\in\{0,1\}} q_{34}(m_3,m_4)$. While $\tilde{q}$ may be different from $q$, they both achieve the same rate in (2), since they result in the same cut capacities. Thus, we can simplify Problem 1 obtaining a new problem with variables $q_{1235}$ and $q_{34}$ instead of $q$. The simplification arises from the smaller number of variables in $q_{1235}$ and $q_{34}$, with a total of $2^4+2^2 = 20$ vs. $2^5 = 32$ appearing in $q$. While in this simple network the computational savings are small, this approach can lead to substantial simplification of the problem in large networks.

## IV. Main Results

In this section, we state and sketch the proof of our main results. First, based on the intuition provided in Section III, we show that, for node subsets $V_1,...,V_k \subseteq V$ satisfying certain properties, Problem 1 can be rewritten as another optimization problem with a smaller number of variables (Problem 2). We then show how to obtain $V_1,...,V_k$ in layered Gaussian networks. Extensions to other network classes are explored in [4] together with a general approach for finding $V_1,...,V_k$.

### A. Finding an Equivalent Optimization Problem

In order to generalize the ideas shown in the example in Section III to other networks, we identify the main elements that led to the simplification of the optimization problem.

- The cut capacities for all cuts depend on marginals $q_{V_1},\dots,q_{V_k}$ of $q$ of (significantly) smaller dimension, i.e. $\max_i |V_i| \ll |V|$, where $V$ is the set of nodes and $V_i$s are the subsets of nodes corresponding to the marginals.
- Given an arbitrary set of marginals $\{q_{V_1},\dots,q_{V_k}\}$ satisfying some consistency constraints, there exists a joint distribution with these marginals.

To formalize the first requirement we define, for $\Omega \subseteq 2^V$, the cut graph $G_\Omega = (V,E_\Omega)$ with $E_\Omega = \{(i,j)\in E : i \in \Omega, j\in\Omega^c\}$. We let $G_{\Omega,i} = (V_{\Omega,i},E_{\Omega,i})$ be the $i$th connected component of $G_\Omega$, for $i = 1,\dots,N(\Omega)$, where $N(\Omega)$ is the number of such connected components.

The consistency constraints in the second bullet can be formalized by requiring that, for two sets $V_i$ and $V_j$ and respective marginals $q_i$ and $q_j$, their sub-marginals corresponding to the node set $V_i \cap V_j$ are the same. Thus, we are interested in node subsets $V_1,V_2,\dots,V_k \subseteq V$ satisfying

P1. $V_{\Omega,i} \subseteq V_j$ for some $j\in\{1,\dots,k\}$, for all connected components $V_{\Omega,i}$, for all cuts $\Omega$.

P2. Given a set of consistent marginals $\{q_1,\dots,q_k\}$, there exists a joint distribution $\tilde{q}$ with these marginals.

Given node subsets $V_1,...,V_k$ satisfying properties P1 and P2, we can define the following optimization problem,

**Problem 2:** $\underset{R,\{q_i\}_{i=1}^k}{\text{maximize}}$ $R$ subject to: (5)

$$R < \sum_{i=1}^{N(\Omega)} \sum_{\mathbf{m}\in\{0,1\}^{|V_{r(\Omega,i)}|}} q_{r(\Omega,i)}(\mathbf{m}) \cdot I\left(X_{\Omega\cap\mathcal{T}(\Omega,i,\mathbf{m})}; Y_{\Omega^c\cap\mathcal{R}(\Omega,i,\mathbf{m})} | X_{(\Omega\cap\mathcal{T}(\Omega,i,\mathbf{m}))^c}\right),$$

$$\forall\,\Omega\in 2^V : S\in\Omega, D\notin\Omega, \tag{6}$$

$$0 \leqslant q_i(\mathbf{m}), \mathbf{m}\in\{0,1\}^{|V_i|}, \quad \text{for } i = 1,2\dots,k \tag{7}$$

$$\sum_{\mathbf{m}\in\{0,1\}^{|V_i|}} q_i(\mathbf{m}) = 1, \quad \text{for } i = 1,2,\dots,k \tag{8}$$

$$\sum_{\substack{\mathbf{m}\in\{0,1\}^{|V_i|}:\\ (m_j)_{j\in V_i\cap V_l}=\mathbf{m}_1}} q_i(\mathbf{m}) = \sum_{\substack{\mathbf{m}\in\{0,1\}^{|V_l|}:\\ (m_j)_{j\in V_i\cap V_l}=\mathbf{m}_1}} q_l(\mathbf{m})$$

$$\forall \mathbf{m}_1\in\{0,1\}^{|V_i\cap V_l|}, i\neq l \text{ and } i,l\in\{1,2,\dots,k\}, \tag{9}$$

$X_i$, $i\in V$, are independent $\mathcal{N}(0,1)$ random variables where we let $\mathcal{T}(\Omega, i, \mathbf{m}) = \{v \in V_{\Omega,i} : m_v = 1\}$, $\mathcal{R}(\Omega, i, \mathbf{m}) = \{v \in V_{\Omega,i} : m_v = 0\}$, and $r(\Omega, i)$ is a function that returns the index $j$ of a set $V_j$ containing $V_{\Omega,i}$.

The following theorem establishes that Problem 2 can be seen as a simplification of Problem 1.

**Theorem 1.** *For a Gaussian relay network given by graph $G = (V, E)$ with node subsets $V_1, ..., V_k$ satisfying P1 and P2, the optimal solutions of Problems 1 and 2 are the same. Moreover, if $k \leqslant |V|$, Problem 2 has $O(|V|2^{\max_i |V_i|})$ variables (as opposed to the $O(2^{|V|})$ variables in Problem 1), and the optimization can be solved in a time complexity that is polynomial in $|V|2^{\max_i |V_i|}$ and $\log((|V|^2 + \sum_{i,j} |h_{i,j}| + \varepsilon^2)/\varepsilon)$ where $\varepsilon$ is the desired accuracy of the solution.*

*Proof Sketch:* (See [4] for a complete proof) As illustrated in Section III, the sum in (3) can be decomposed into terms involving mutual information expressions containing random variables associated to each of the connected components of $G_\Omega$. This, together with P1, allows us to replace $q(\mathbf{m})$ with some marginal $q_i$ in each of the terms. In addition, the consistency requirements (7)-(9) together with P2 guarantee that every feasible solution in Problem 2 is feasible in Problem 1. Furthermore, every feasible solution in Problem 1 can be used to construct a set of marginals $\{q_1, \ldots, q_k\}$ which are feasible in Problem 2 and lead to the same value of $R$. As a result, Problems 1 and 2 have the same optimal value.

The computability of the solution in polynomial time in $|V|2^{\max_i |V_i|}$ and $\log((|V|^2 + \sum_{i,j} |h_{i,j}| + \varepsilon^2)/\varepsilon)$ can be shown by exploiting the submodularity of mutual information with independent encoding at the nodes [2]. The main steps are showing that the program is convex, the objective and its subgradient can be computed in polynomial time, and that for a properly defined infeasibility measure Infeas$(\cdot)$, it can be determined in polynomial time if Infeas$(\mathbf{x}) \leqslant \varepsilon$, and when Infeas$(\mathbf{x}) > \varepsilon$, it is possible to find in polynomial time a separating hyperplane between $\mathbf{x}$ and the feasible set [6]. ∎

### B. Finding Node Subsets $\{V_i\}_i$ in Gaussian Layered Networks

Consider a layered network consisting of a set of nodes $V$, for $n = |V|$, partitioned into $L$ layers denoted by $\mathcal{V}_\ell$ for $\ell = 1, 2, \ldots, L$. We assume that $\mathcal{V}_1 = \{S\}$ and $\mathcal{V}_L = \{D\}$.

**Theorem 2.** *For a HD Gaussian layered relay network, Problem 2 can be used to find a constant-gap optimal transmit/receive schedule in polynomial time, provided that the $|\mathcal{V}_i|$s grow at most logarithmically with $|V|$.*

*Proof:* Let $k = L - 1$ and set $V_i = \mathcal{V}_i \cup \mathcal{V}_{i+1}$, for $i = 1, ..., k$. For any cut $\Omega$, it is easy to see that $G_\Omega$ has connected components $G_{\Omega,i} = (V_{\Omega,i}, E_{\Omega,i})$, where $V_{\Omega,i} = (\mathcal{V}_i \cap \Omega) \cup (\mathcal{V}_{i+1} \cap \Omega^c)$ and $E_{\Omega,i} = \{(i,j) \in E_\Omega : i, j \in V_{\Omega,i}\}$, for $i = 1, ..., k$. Hence, $V_{\Omega,i} \in V_i$, and P1 is satisfied.

To verify P2, we let $\{q_1, \ldots, q_k\}$ be a set of consistent marginals defined for the sets $V_1, \ldots, V_k$, and let

$$\tilde{q}(m_1, \ldots, m_n) \stackrel{\text{def}}{=} \frac{\prod_{i=1}^{k} q_i(\{m_j : j \in V_i\})}{\prod_{i=2}^{k} \left( \sum_{m_j : j \in \mathcal{V}_{i+1}} q_i(\{m_s : s \in V_i\}) \right)}.$$

We can verify that

$$\sum_{\{m_i : i \in \mathcal{V}_L\}} \tilde{q}(\mathbf{m}) = \frac{\prod_{i=1}^{k-1} q_i(\{m_j : j \in V_i\})}{\prod_{i=2}^{k-1} \left( \sum_{m_j : j \in \mathcal{V}_{i+1}} q_i(\{m_s : s \in V_i\}) \right)}$$

and by induction $\sum_{\{m_i : i \in \mathcal{V}_3 \cup \ldots \cup \mathcal{V}_L\}} \tilde{q}(\mathbf{m}) = q_1(\{m_j : j \in V_1\})$. Therefore $\sum_{\{m_i : i \in \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_L\}} \tilde{q}(\mathbf{m}) = 1$. By definition, $\tilde{q}(\mathbf{m}) \geqslant 0$, so it follows that $\tilde{q}$ is a probability distribution.

Similarly, by induction, we can show that for $j = 1, \ldots, k$,

$$\sum_{\{m_i : i \in \mathcal{V}_1 \cup \ldots \mathcal{V}_{j-1} \cup \mathcal{V}_{j+2} \cup \ldots \mathcal{V}_L\}} \tilde{q}(\mathbf{m}) = q_j(\{m_s : s \in V_j\})$$

and conclude that P2 is satisfied.

Finally, by Theorem 1, Problem 2 can be solved in time that is polynomial in $|V|2^{\max_i |\mathcal{V}_i|}$. ∎

## V. Numerical Results

In this section we show that with our proposed approach we can evaluate the HD cut-set bound and determine the associated HD schedules for networks of larger size than alternative algorithms. In addition, we show that by optimizing the HD schedules we can get close to the performance of FD networks. We focus on a Gaussian layered relay network, where the channel gains are randomly chosen with i.i.d. $\mathcal{CN}(0, P)$ distribution. We use Theorem 2 to solve Problem 2 for various network sizes as we vary the number of layers $L$ keeping the number of nodes in each layer constant. While in previous sections we assumed a real Gaussian channel model, all the results can be easily extended to the complex case considered in this section, which is of practical relevance for baseband models of wireless systems.

We set a baseline for comparison by solving Problem 1 using Matlab's linear optimization function `linprog`. Then, we solve Problem 2 efficiently using Theorem 2 and exploiting submodularity with the method proposed in [7] and the Matlab package SFO [8]. In addition, to evaluate the improvements from the reduction in the number of variables in going from Problem 1 to Problem 2, we also solve Problem 1 exploiting submodularity with the SFO package.

Fig. 2 shows the average running times of the various algorithms as the number of layers $L$ increases for fixed number of relays $|\mathcal{V}_l| = 2$ in each layer. The average running time is computed over 10 random realizations of the channel gains drawn i.i.d. with distribution $\mathcal{CN}(0, 1)$. We first note that the brute force approach of solving Problem 1 with `linprog` leads to smaller running times than the approach that uses SFO. This may be explained by noting that Problem 1 has exponential complexity on the number of nodes, whether the optimization is solved by brute force or by submodular optimization methods. While SFO may simplify the computation of the cut-set constraints, it introduces overheads which grow with the number of variables involved. Fig. 2 shows that these overheads dominate the running times for the values of $L$ in the plot, but due to the smaller slope of the curve, the solution with submodular optimization should be faster for large enough $L$. The overheads of SFO also explain why the

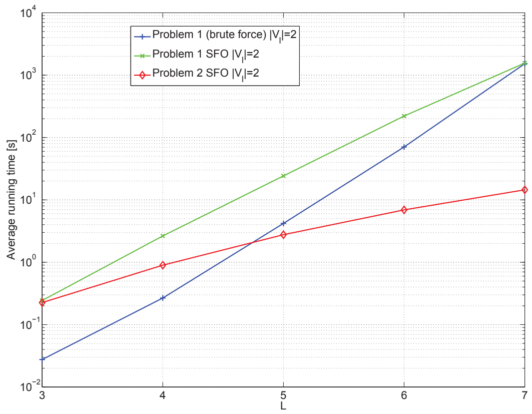


Fig. 2. Average running times of the algorithms that solve Problem 1 by brute force and by using submodular optimization (SFO) compared to that of the algorithm that solves Problem 2 using Theorem 2 as a function of the number of layers $L$ with $|\mathcal{V}_l| = 2$ nodes per layer.

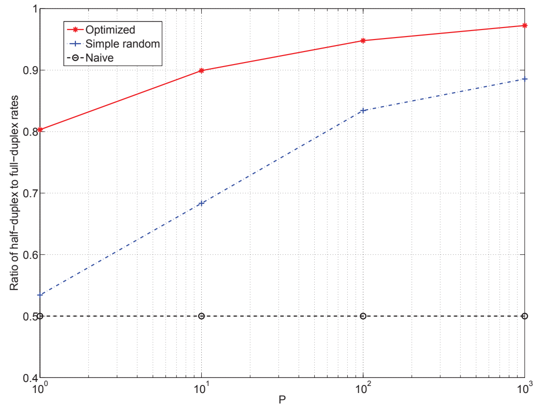


Fig. 3. Ratio of half-duplex to full-duplex rate for various scheduling approaches as a function of the signal-to-noise ratio $P$.

algorithm that solves Problem 1 by brute force via `linprog` exhibits smaller running times than the algorithm that solves Problem 2 for networks with 6 or fewer nodes. However, as the number of nodes in the network grow the running times of the algorithm implementing Problem 2 become significantly smaller ($\approx 1/100$ for $L = 7$) than those of the other algorithms. In addition to having smaller running times, the algorithm implementing Problem 2 allows to solve problems for networks of large size for which the other algorithms become impractical due to memory and time requirements.

We next evaluate the benefits of using optimal HD schedules (from solving Problem 2) compared to simpler scheduling approaches for networks with $L = 4$ layers, $|\mathcal{V}_l| = 4$ nodes per layer, and i.i.d. $\mathcal{CN}(0, P)$ channel gains. We compare the ratio between the HD and FD cut-set bound (averaged over 10 different random network realizations) for three HD scheduling techniques:

- Optimized: the HD schedule that results from solving Problem 2.
- Naïve: time is divided in timeslots of equal length. The nodes in even layers transmit in even timeslots and receive in odd timeslots, while the nodes in odd layers transmit in odd timeslots and receive in even timeslots.
- Simple random: the network is divided into 2 subnetworks, with half of the nodes in each layer randomly assigned to subnetwork 1, while the other half is assigned to subnetwork 2. The source and destination nodes are part of both subnetworks. Each subnetwork is operated using the naïve HD schedule with phase offset of 1 timeslot. That is, half of the nodes in each relay layer transmit at a given time while the other half receive at that time. The source is always in transmit mode and the destination is always in receive mode.

Fig. 3 shows that the optimized HD schedule allows to obtain a large fraction of the FD performance, which increases with the SNR (represented by $P$). The simple random schedule outperforms the naïve schedule but does not match the performance of the optimized schedule for small and moderate SNR. At large SNR, the DoF limits the rates, and the simple random schedule (which is DoF optimal) approaches the performance of the optimized schedule. We note that as SNR $\to \infty$, the ratio of HD to FD capacity approaches 1 due to the fact that the DoF of both networks (HD and FD) is 1 (with probability 1) whenever $|\mathcal{V}_l| \geqslant 2$.

## VI. Conclusions

We showed that for layered Gaussian relay networks an approximation to the HD cut-set bound can be efficiently computed. This computation not only gives a capacity characterization within a constant gap, but also provides "good" HD schedules which can be used in communication schemes. Whenever the complexity of implementing HD schedules is a concern, as is the case in fading channels, it may be of interest to implement simple HD schedules involving few timeslots. The performance of these simple schedules can be evaluated using the cut-set bound as a metric, and compared to what is achievable under optimal scheduling using the techniques that we presented.

## References


[1] A. S. Avestimehr, S. Diggavi, and D. Tse, "Wireless network information flow: a deterministic approach," *IEEE Transactions on Information Theory*, vol. 57, no. 4, April 2011.

[2] F. Parvaresh and R. Etkin, "Efficient capacity computation and power optimization for relay networks," *submitted to IEEE Transactions on Information Theory*, 2011.

[3] M. A. Khojastepour, A. Sabharwal, and B. Aazhang, "On the capacity of 'cheap' relay networks," *Proceedings of the Conference on Information Sciences and Systems*, March 12 - 14 2003.

[4] R. Etkin, F. Parvaresh, I. Shomorony, and S. Avestimehr, "On min-cut algorithms for half-duplex relay networks," *preprint, arXiv:1305.2548 [cs.IT]*, 2013.

[5] A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 789–804, March 2006.

[6] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization. Analysis, Algorithms, and Engineering Applications.* SIAM, Philadelphia, PA, 2001.

[7] S. Fujishige, T. Hayashi, and S. Isotani, *The Minimum-Norm-Point Algorithm Applied to Submodular Function Minimization and Linear Programming*. Kyoto University, Kyoto, Japan, 2006.

[8] R. A. Krause, "Matlab toolbox for submodular function optimization. [http://www.cs.caltech.edu/ krausea/sfo/]."