

Explicit Constructions of Quasi-Uniform Codes from Groups

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Abstract—We address the question of constructing explicitly quasi-uniform codes from groups. We determine the size of the codebook, the alphabet and the minimum distance as a function of the corresponding group, both for abelian and some nonabelian groups. Potentials applications comprise the design of almost affine codes and non-linear network codes.

I. INTRODUCTION

Let X_1, \dots, X_n be a collection of n jointly distributed discrete random variables over some alphabet of size N . We denote by \mathcal{A} a subset of indices from $\mathcal{N} = \{1, \dots, n\}$, and $X_{\mathcal{A}} = \{X_i, i \in \mathcal{A}\}$. We call the support of $X_{\mathcal{A}}$ $\lambda(X_{\mathcal{A}}) = \{x_{\mathcal{A}} : P(X_{\mathcal{A}} = x_{\mathcal{A}}) > 0\}$.

Definition 1. A probability distribution over a set of n random variables X_1, \dots, X_n is said to be quasi-uniform if for any $\mathcal{A} \subseteq \mathcal{N}$, $X_{\mathcal{A}}$ is uniformly distributed over its support $\lambda(X_{\mathcal{A}})$:

$$P(X_{\mathcal{A}} = x_{\mathcal{A}}) = \begin{cases} 1/|\lambda(X_{\mathcal{A}})| & \text{if } x_{\mathcal{A}} \in \lambda(X_{\mathcal{A}}), \\ 0 & \text{otherwise.} \end{cases}$$

The motivation for introducing quasi-uniform random variables [1] is that they possess a non-asymptotic equipartition property, by analogy to the asymptotic equipartition property, where long typical sequences have total probability close to 1, and are approximately uniformly distributed.

We call a code C of length n an arbitrary nonempty subset of $\mathcal{X}_1 \times \dots \times \mathcal{X}_n$ where \mathcal{X}_i is the alphabet for the i th codeword symbol, and each \mathcal{X}_i might be different.

We can associate to every code C a set of random variables [2] by treating each codeword $(X_1, \dots, X_n) \in C$ as a random vector with probability

$$P(X_{\mathcal{N}} = x_{\mathcal{N}}) = \begin{cases} 1/|C| & \text{if } x_{\mathcal{N}} \in C, \\ 0 & \text{otherwise.} \end{cases}$$

To the i th codeword symbol then corresponds a *codeword symbol random variable* X_i induced by C .

Definition 2. [2] A code C is said to be quasi-uniform if the induced codeword symbol random variables are quasi-uniform.

Given a code, we explained above how to associate a set of random variables, which might or not end up being quasi-uniform. Conversely, given a set of quasi-uniform random variables, a quasi-uniform code is obtained as follows. Let X_1, \dots, X_n be a set of quasi-uniform random variables with

probabilities $P(X_{\mathcal{A}} = x_{\mathcal{A}}) = 1/|\lambda(X_{\mathcal{A}})|$ for all $\mathcal{A} \subseteq \mathcal{N}$. The corresponding quasi-uniform code C of length n is given by $C = \lambda(X_{\mathcal{N}}) = \{x_{\mathcal{N}} : P(X_{\mathcal{N}} = x_{\mathcal{N}}) > 0\}$. Quasi-uniform codes were defined in [2], where some of their properties were discussed, and importantly their weight enumerator polynomial was computed.

For a linear $[n, k]$ code C of dimension k and length n (over some finite field), the weight enumerator polynomial of C is

$$W_C(x, y) = \sum_{c \in C} x^{n-wt(c)} y^{wt(c)},$$

where $wt(c)$ is the weight of c , that is the number of non-zero coefficients of c . For arbitrary codes, rather than the weight of the codewords, the distance between two codewords is of interest. Let $A_r(c) = |\{c' \in C, |\{j \in \mathcal{N}, c_j \neq c'_j\}| = r\}|$ be the distance profile of C centered at c . Note that we avoid defining A_r using $wt(c - c')$, since this already assumes that the difference of two codewords makes sense. It was shown in [2] that quasi-uniform codes are distance-invariant, meaning that the distance profile does not depend on the choice of c .

Theorem 1. [2] Let C be a quasi-uniform code of length n . Then its weight enumerator $W_C(x, y) = \sum_{j=0}^n A_j x^{n-j} y^j$ is

$$W_C(x, y) = \sum_{\mathcal{A} \subseteq \mathcal{N}} q^{H(X_{\mathcal{N}}) - H(X_{\mathcal{A}})} (x - y)^{|\mathcal{A}|} y^{n-|\mathcal{A}|}, \quad (1)$$

where $H(X_{\mathcal{A}}) = \log_q(|\lambda(X_{\mathcal{A}})|)$ is the joint entropy of the induced codeword symbol quasi-uniform random variables.

The formula for the weight enumerator shows that it only depends $H(X_{\mathcal{A}})$. In fact, [2] which introduced quasi-uniform codes focused on their information theoretic properties, rather than on their coding properties. The goal of this paper is to address the construction and understanding of such quasi-uniform codes from a constructive point of view. We will use the group theoretic approach proposed in [3], [4] for constructing quasi-uniform random variables from finite groups.

More precisely, in Section II, we recall the construction of quasi-uniform codes from groups, and compute the size of the corresponding code as a function of the group G we started with. We then consider abelian groups in Section III, and compute the alphabet as well as the minimum distance as a function of G . We next move to nonabelian groups in Section IV. The structure of group, even though it is a nonabelian one,

allows in some cases to mimic a definition for the minimum distance of the code. Potential applications to the design of almost affine codes is mentioned in Section V.

Note that codes coming from groups have been studied in the literature (see for example [6], [7], [8], [9], [10] or [11]). Looking at constructions of codes from groups is here motivated by the need to design non-linear codes for network coding (see [12] and also [13] for applications of quasi-uniform codes to network coding), apart from designing almost affine codes as mentioned above.

II. QUASI-UNIFORM CODES FROM GROUPS

Let G be a finite group of order $|G|$ with n subgroups G_1, \dots, G_n , and $G_{\mathcal{A}} = \cap_{i \in \mathcal{A}} G_i$. Given a subgroup G_i of G , the (left) coset of G_i in G is defined by $gG_i = \{gh, h \in G_i\}$. The number of (left) cosets of G_i in G is called the index of G_i in G and is denoted by $[G : G_i]$. It is known from Lagrange Theorem that $[G : G_i] = |G|/|G_i|$. If G_i is normal, the sets of cosets $G/G_i = \{gG_i, g \in G\}$ are themselves groups, called quotient groups.

Let X be a random variable uniformly distributed over G , that is $P(X = g) = 1/|G|$, for any $g \in G$. Define the new random variable $X_i = XG_i$, with support the $[G : G_i]$ cosets of G_i in G . Then $P(X_i = gG_i) = |G_i|/|G|$ and $P(X_i = gG_i, i \in \mathcal{A}) = |\cap_{i \in \mathcal{A}} G_i|/|G|$.

This shows that quasi-uniform random variables may be obtained from finite groups. More precisely:

Theorem 2. [3], [4] *For any finite group G and any subgroups G_1, \dots, G_n of G , there exist n jointly distributed quasi-uniform discrete random variables X_1, \dots, X_n such that for all non-empty subsets \mathcal{A} of \mathcal{N} , $P(X_{\mathcal{A}} = x_{\mathcal{A}}) = |G_{\mathcal{A}}|/|G|$.*

Quasi-uniform codes are obtained from these quasi-uniform distributions by taking the support $\lambda(X_{\mathcal{N}})$, as explained in the introduction. Codewords (of length n) can then be described explicitly by letting the random variable X take every possible values in the group G , and by computing the corresponding cosets as follows:

| | G_1 | \dots | G_n |
|-----------|--------------|---------|--------------|
| g_1 | g_1G_1 | | g_1G_n |
| g_2 | g_2G_1 | | g_2G_n |
| \vdots | \vdots | | \vdots |
| $g_{ G }$ | $g_{ G }G_1$ | \dots | $g_{ G }G_n$ |

Each row corresponds to one codeword of length n . The cardinality $|C|$ of the code obtained seems to be $|G|$, but in fact, it depends on the subgroups G_1, \dots, G_n . Indeed, it could be that the above table yields several copies of the same code.

Lemma 1. *Let C be a quasi-uniform code obtained from a group G and subgroups G_1, \dots, G_n . Then $|C| = |G|/|G_{\mathcal{N}}|$. In particular, if $|G_{\mathcal{N}}| = 1$, then $|C| = |G|$.*

Proof. Let $G_{\mathcal{N}} = \{h_1, \dots, h_m\}$ be the intersection of all the subgroups G_1, \dots, G_n . There are $|G|/|G_{\mathcal{N}}|$ cosets of $G_{\mathcal{N}}$ in G . Let us compute a first coset, say $g_1G_{\mathcal{N}} = \{g_1, g_1h_2, \dots, g_1h_m\}$, by assuming wlog that h_1 is the identity

element of $G_{\mathcal{N}}$. In words, we observe that every element in $g_1G_{\mathcal{N}}$ is a multiple of a non-trivial element of $G_{\mathcal{N}}$. Thus, when computing the above table, we have (by reordering the elements of G so as to list first the elements in $g_1G_{\mathcal{N}}$):

| | G_1 | \dots | G_n |
|----------|----------------------|---------|----------------------|
| g_1 | g_1G_1 | | g_1G_n |
| g_1h_2 | $g_1h_2G_1 = g_1G_1$ | | $g_1h_2G_n = g_1G_n$ |
| \vdots | \vdots | | \vdots |
| g_1h_m | $g_1h_mG_1 = g_1G_1$ | \dots | $g_1h_mG_n = g_1G_n$ |
| \vdots | \vdots | | \vdots |

where $g_1h_iG_j = g_1G_j$, for all $i, j = 1, \dots, n$ because by definition of $G_{\mathcal{N}}$, $h_i \in G_j$. Since the cosets of $G_{\mathcal{N}}$ partition G , we will get $|G|/|G_{\mathcal{N}}|$ copies of a code C . \square

One of the motivations to consider quasi-uniform codes is that they allow to go beyond abelian structures. Nevertheless, we will start by considering the case of abelian groups, which is easier to handle.

III. QUASI-UNIFORM CODES FROM ABELIAN GROUPS

Suppose that G is an abelian group, with subgroups G_1, \dots, G_n . The procedure from Section II explains how to obtain a quasi-uniform distribution of n random variables, and thus a quasi-uniform code of length n from G . To avoid getting several copies of the same code, as exposed in Lemma 1, notice that since G is abelian, all the subgroups G_1, \dots, G_n are normal, and thus so is $G_{\mathcal{N}}$. If $|G_{\mathcal{N}}| > 1$, we consider instead of G the quotient group $G/G_{\mathcal{N}}$, and we can thus assume wlog that $|G_{\mathcal{N}}| = 1$.

Lemma 2. *Let C be a quasi-uniform code obtained from a group G and subgroups G_1, \dots, G_n . Then the alphabet size of C is $\sum_{i=1}^n [G : G_i]$.*

Proof: It is enough to show that all the cosets that appear in the table are distinct. By definition, every column contains $[G : G_i]$ distinct cosets. If $|G_i| \neq |G_j|$, the respective cosets will have different sizes, so let us assume that $|G_i| = |G_j|$. If $gG_i = g'G_j$, then $g^{-1}g'G_i = G_i = g^{-1}g'G_j$ and it must be that $g^{-1}g'G_j$ is a subgroup. This implies that $g^{-1}g' \in G_j$, and thus that $g^{-1}g'G_j = G_j$. \blacksquare

The size of the alphabet can often be reduced, as explained next. Let π_i denote the canonical projection $\pi_i : G \rightarrow G/G_i$. Since G/G_i is itself an abelian group, let us denote explicitly by $\psi_i : G/G_i \rightarrow H_i$ this group isomorphism, for some abelian group H_i . Then $\pi_i(g) = gG_i \mapsto h \in H_i$ via $\psi_i(gG_i) = h$, $i = 1, \dots, n$.

Proposition 1. *Let G be an abelian group with subgroups G_1, \dots, G_n . Then its corresponding quasi-uniform code is defined over $H_1 \times \dots \times H_n$.*

Proof: Let X be again this random variable defined over G by $Pr(X = g) = 1/|G|$. Define a new random variable Z_i by $Z_i = \psi_i(\pi_i(X))$ which takes values directly in H_i . Then $Pr(Z_i = h) = Pr(\psi_i(\pi_i(X)) = h) = Pr(\pi_i(X) = gG_i) = |G_i|/|G|$. Similarly, if $Pr(Z_i = h_i, i \in \mathcal{A}) > 0$, then

$Pr(Z_i = h_i : i \in \mathcal{A}) = Pr(\psi_i(\pi_i(X)) = h_i : i \in \mathcal{A}) = Pr(\pi_i(X) = gG_i : i \in \mathcal{A}) = |\mathcal{G}_\mathcal{A}|/|G|$. ■

In other words, we get a labeling of the cosets which respects the group structures componentwise. The next result then follows naturally.

Corollary 1. *A quasi-uniform code C obtained from an abelian group is itself an abelian group.*

Proof: First notice that the zero codeword is in C , since the codeword corresponding to the identity element in G is $(\psi_1(\pi_1(G_1)), \dots, \psi_n(\pi_n(G_n))) = (0, \dots, 0)$ where each 0 corresponds to the identity element in each abelian group H_i . Let $(\psi_1(\pi_1(gG_1)), \dots, \psi_n(\pi_n(gG_n)))$ and $(\psi_1(\pi_1(g'G_1)), \dots, \psi_n(\pi_n(g'G_n)))$ be two codewords in C . Then note that the codeword in C corresponding to the element $g + g' \in G$ is $(\psi_1(\pi_1((g + g')G_1)), \dots, \psi_n(\pi_n((g + g')G_n)))$ where $\psi_i(\pi_i(g + g')) = \psi_i(\pi_i(g)) + \psi_i(\pi_i(g'))$, $i = 1, \dots, n$. Every codeword has an additive inverse for the same reason. It forms an abelian group because every group law componentwise is commutative. ■

Because every H_i is an abelian group, we can freely use 0 since it corresponds to the identity element of H_i , as well as the operation $+$ and $-$, since $+$ is the group law for H_i , and $-$ is the additive inverse. However, the alphabet H_i is possibly any abelian group, in particular, different abelian groups might be used for different components of the codewords. The classification of abelian groups tells us that each H_i can be expressed as the direct sum of cyclic subgroups of order a prime power. In the particular case where we have only one cyclic group, then (1) the group C_{p^r} is isomorphic to the integers mod p^r , and (2) the group C_p is isomorphic to the integers mod p , which in fact has a field structure, and we deal with the usual finite field \mathbb{F}_p . If all the subgroups G_1, \dots, G_n have index p , then we get an $[n, k]$ linear code over \mathbb{F}_p .

The minimum distance of an abelian quasi-uniform code is encoded in its weight enumerator, but is however not easily read from (1). We can easily express it in terms of the subgroups G_1, \dots, G_n .

Lemma 3. *The minimum distance $\min_{c \in C} wt(c)$ of a quasi-uniform code C generated by an abelian group G and its subgroups G_i , $i = 1, \dots, n$ is $n - \max_{\mathcal{A} \in \mathcal{N}, \mathcal{G}_\mathcal{A} \neq \{0\}} |\mathcal{A}|$.*

Proof: The minimum distance $\min_{c \neq c' \in C} |\{c_i \neq c'_i\}|$ can be written as $\min_{c \neq c' \in C} wt(c - c')$ since $c - c'$ makes sense. Furthermore, since $c - c' \in C$, it reduces, as for linear codes over finite fields, to $\min_{c \in C} wt(c)$, and we are left to find the weight of the codeword having maximum number of zeros. This corresponds to finding the maximum number of subgroups whose intersection contains a non-trivial element of G . ■

As an illustration, here is a simple family of abelian groups that generate $[n, k]$ linear codes over \mathbb{F}_p of length n and dimension k .

Lemma 4. *The elementary abelian group $C_p \times C_p$ generates a $[p + 1, 2]$ linear code over \mathbb{F}_p with minimum distance p .*

Proof: The group $G = C_p \times C_p$ contains $p + 1$ non-trivial subgroups, of the form $\langle (1, i) \rangle$ where $i = 0, 1, \dots, p - 1$ and $\langle (0, 1) \rangle$. They all have index p and trivial pairwise intersection. We thus get a code of length $n = p + 1$, containing p^2 codewords, which is linear over \mathbb{F}_p (by using that C_p is isomorphic to the integers mod p). Since the pairwise intersection of subgroups is trivial, the minimum distance is p . ■

We finish this section by providing a worked out example.

Example 1. Consider the elementary abelian group $G = C_3 \times C_3 \simeq \{0, 1, 2\} \times \{0, 1, 2\}$ and the four subgroups $G_1 = \langle (1, 0) \rangle = \{(0, 0), (1, 0), (2, 0)\}$, $G_2 = \langle (0, 1) \rangle = \{(0, 0), (0, 1), (0, 2)\}$, $G_3 = \langle (1, 1) \rangle = \{(0, 0), (1, 1), (2, 2)\}$, and $G_4 = \langle (1, 2) \rangle = \{(0, 0), (1, 2), (2, 1)\}$. Using the method of Section II, we obtain the following codewords (we write ij instead of (i, j) for brevity):

| | $\langle (10) \rangle$ | $\langle (01) \rangle$ | $\langle (11) \rangle$ | $\langle (12) \rangle$ |
|------|------------------------|------------------------|------------------------|------------------------|
| (00) | $\langle (10) \rangle$ | $\langle (01) \rangle$ | $\langle (11) \rangle$ | $\langle (12) \rangle$ |
| (01) | (01)(11)(21) | $\langle (01) \rangle$ | (01)(12)(20) | (01)(10)(22) |
| (02) | (02)(12)(22) | $\langle (01) \rangle$ | (10)(21)(02) | (02)(11)(20) |
| (10) | $\langle (10) \rangle$ | (10)(11)(12) | (10)(21)(02) | (01)(10)(22) |
| (11) | (01)(11)(21) | (10)(11)(12) | $\langle (11) \rangle$ | (02)(11)(20) |
| (12) | (02)(12)(22) | (10)(11)(12) | (01)(12)(20) | $\langle (12) \rangle$ |
| (20) | $\langle (10) \rangle$ | (20)(21)(22) | (01)(12)(20) | (02)(11)(20) |
| (21) | (01)(11)(21) | (20)(21)(22) | (10)(21)(02) | $\langle (12) \rangle$ |
| (22) | (02)(12)(22) | (20)(21)(22) | $\langle (11) \rangle$ | (01)(10)(22) |

Now let $H_1 = G/\langle (10) \rangle$, $H_2 = G/\langle (01) \rangle$, $H_3 = G/\langle (11) \rangle$ and $H_4 = G/\langle (12) \rangle$. Note that $H_i \simeq C_3 = \{0, 1, 2\}$ for all i . If we replace the subgroups by their quotients in the above table, we get the following code from Lemma 1:

| | $\langle (10) \rangle$ | $\langle (01) \rangle$ | $\langle (11) \rangle$ | $\langle (12) \rangle$ |
|------|------------------------|------------------------|------------------------|------------------------|
| (00) | 0 | 0 | 0 | 0 |
| (01) | 1 | 0 | 1 | 1 |
| (02) | 2 | 0 | 2 | 2 |
| (10) | 0 | 1 | 2 | 1 |
| (11) | 1 | 1 | 0 | 2 |
| (12) | 2 | 1 | 1 | 0 |
| (20) | 0 | 2 | 1 | 2 |
| (21) | 1 | 2 | 2 | 0 |
| (22) | 2 | 2 | 0 | 1 |

It is a ternary linear code of length $p + 1 = 4$ and minimum distance $p = 3$ with generator matrix $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.

Since $H(X_\mathcal{N}) = \log_q 9$, $H(X_\mathcal{A}) = \log_q 3$ when $|\mathcal{A}| = 1$ and $H(X_\mathcal{A}) = \log_q 9$ if $|\mathcal{A}| \geq 2$, we have that $q^{H(X_\mathcal{N}) - H(X_\mathcal{A})} = 9$ when \mathcal{A} is empty, $q^{H(X_\mathcal{N}) - H(X_\mathcal{A})} = q^{\log_q(9/3)} = 3$ when $|\mathcal{A}| = 1$ and 1 otherwise, so that the weight enumerator of C using Theorem 1 is $W_C(x, y) = x^4 + 8xy^3$, which can be verified using the usual methods.

IV. QUASI-UNIFORM CODES FROM NONABELIAN GROUPS

Suppose now that G is a nonabelian group. The resulting quasi-uniform codes might end up being very different depending on the nature of the subgroups considered. We next treat the different possible cases that can occur.

A. The Case of Quotient Groups

If G is a nonabelian group, but G_1, \dots, G_n are normal subgroups, then the intersection $G_\mathcal{N}$ of all subgroups G_1, \dots, G_n is a normal subgroup. Following Lemma 1, we can further consider the quotient $G/G_\mathcal{N}$. As a result, some nonabelian

groups are really reduced to abelian ones. This is the case of some dihedral groups. Let

$$D_{2m} = \langle r, s \mid r^m = s^2 = 1, rs = sr^{-1} \rangle$$

be the dihedral group of order $2m$.

Lemma 5. *Quasi-uniform codes obtained from dihedral groups of order a power of 2 and some (possibly all) of their normal subgroups are obtained from abelian groups.*

Proof: Normal subgroups H of D_{2m} are known: H is either a subgroup of $\langle r \rangle$, or $2|m$ and H is one of the two maximal subgroups of index 2 $\langle r^2, s \rangle$, $\langle r^2, rs \rangle$. Since m is a power of 2, the intersection $G_{\mathcal{N}}$ of (any choice of) these normal subgroups is necessarily a subgroup of order some power of 2, and we can take the quotient $D_{2m}/G_{\mathcal{N}}$, which is a dihedral group of smaller order also a power of 2 and the code generated by D_{2m} and $D_{2m}/G_{\mathcal{N}}$ will be the same (see Lemma 1). Since the normal subgroups of $D_{2m}/G_{\mathcal{N}}$ too are of the same form, the process can be iterated until we reach D_4 which is abelian. ■

The case when G is nonabelian and all the subgroups G_1, \dots, G_n are normal but some (possibly all) of the quotient groups G/G_i are nonabelian is very interesting. Indeed, in that case, Proposition 1 still holds (with H_i nonabelian), and in fact, the corresponding quasi-uniform code still has a group structure, but that of a nonabelian group. This gives the opportunity to go beyond abelian structures, and yet to keep a group structure. As above, we assume wlog that $|G_{\mathcal{N}}| = 1$.

Lemma 6. *A quasi-uniform code C obtained from a nonabelian group G with normal subgroups G_1, \dots, G_n where at least one quotient group G/G_i is nonabelian forms a nonabelian group.*

The proof is identical to that of Corollary 1, but one has to be cautious that the group law is not commutative. The group law $*$ is defined componentwise, and the identity element is the codeword $(1_{H_1}, \dots, 1_{H_n})$.

It is then possible to mimic the definition of minimum distance. The weight of a codeword is then the number of components which are not an identity element. The identity codeword plays the role of the whole zero codeword. The minimum distance $\min_{c \neq c' \in C} |\{c_i \neq c'_i\}|$ is then $\min_{c \neq c' \in C} wt(c * (c')^{-1})$. Indeed, $(c')^{-1} = ((c'_1)^{-1}, \dots, (c'_n)^{-1})$, and noncommutativity is not an issue, since every inverse componentwise is both a left and a right inverse. This gives a counterpart to Lemma 3.

Corollary 2. *The minimum distance $\min_{c \in C} wt(c)$ of a quasi-uniform code C generated by a nonabelian group G and its normal subgroups G_i , $i = 1, \dots, n$ is $n - \max_{A \in \mathcal{N}, G_A \neq \{0\}} |A|$, where the weight $wt(c)$ is understood as the number of components which are not an identity element in some H_i .*

Example 2. The dihedral group D_{12} has two normal subgroups $G_1 = \langle r^3 \rangle$ and $G_2 = \langle r^2, s \rangle$ with trivial intersection (see Figure 1). We can create quasi-uniform codes of length n by choosing $n - 2$ other normal subgroups. Since $D_{12}/G_1 \simeq D_6$, this gives a nonabelian quotient H_1 .

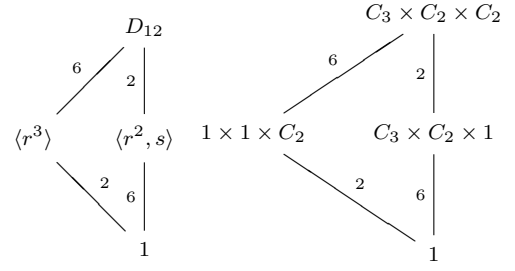


Fig. 1. On the right, the dihedral group D_{12} , and on the left, the abelian group $C_3 \times C_2 \times C_2$, both with some of their subgroups.

This example illustrates the difference between an information theoretic view of these codes, where one focuses on the joint entropy of the corresponding quasi-uniform codes, and a coding perspective, where the actual code, its alphabet, and its structure are of interest. We observe on Figure 1 that if we care about having two (normal) subgroups G_1, G_2 with respective orders 2 and 6, in a group of order 12, we could have done that with the abelian group $C_3 \times C_2 \times C_2$. The difference is in the alphabet: D_{12}/G_1 is a nonabelian group, while $C_3 \times C_2 \times C_2 / 1 \times 1 \times C_2$ is abelian. This question of distinguishing nonabelian groups whose entropic vectors can or cannot be obtained from abelian groups was addressed more generally in [14].

Definition 3. [14] *Let G be a nonabelian group and let G_1, \dots, G_n be fixed subgroups of G . Suppose there exists an abelian group A with subgroups A_1, \dots, A_n such that for every non-empty $\mathcal{A} \subseteq \mathcal{N}$, $[G : G_{\mathcal{A}}] = [A : A_{\mathcal{A}}]$. Then we say that (A, A_1, \dots, A_n) represents (G, G_1, \dots, G_n) .*

It follows immediately that if (A, A_1, \dots, A_n) represents (G, G_1, \dots, G_n) , then the quasi-uniform codes generated by both groups have the same joint entropy, and in turn the same weight enumerator, by Theorem 1. This is the case for dihedral and quasi-dihedral 2-groups as well as dicyclic 2-groups, which are abelian group representable for any number of subgroups, and all nilpotent groups for $n = 2$ [14].

If (G, G_1, \dots, G_n) cannot be abelian represented, then we can build a quasi-uniform code using exactly the same subgroups G_1, \dots, G_n and be sure that its weight enumerator cannot be obtained from an abelian group. It was also shown in [14] that dihedral groups whose order is a power of 2 are abelian group representable, however Lemma 5 is stronger, in that it shows that the code alphabet will be the same.

B. The Case of Nonnormal Subgroups

Let G be a nonabelian group, with subgroups G_1, \dots, G_n , where some (possibly all) of the subgroups are not normal. In that case, we lose the group structure on the set of cosets. To start with, Lemma 1 will still hold, since it does not depend on the normality of the subgroups, however in general, we cannot take the quotient by $G_{\mathcal{N}}$ anymore, and the copy of C we are left with does not have a group structure. This is an example of situations mentioned in the introduction, where the

minimum distance is then not of interest.

Example 3. Consider the group $S_3 = \{(), (12), (13), (23), (123), (132)\}$ of permutations on three elements described in cycle notation. The corresponding quasi-uniform code is:

| | $\langle(12)\rangle$ | $\langle(13)\rangle$ | $\langle(23)\rangle$ |
|---------|----------------------|----------------------|----------------------|
| $()$ | $\langle(12)\rangle$ | $\langle(13)\rangle$ | $\langle(23)\rangle$ |
| (12) | $\langle(12)\rangle$ | $(12), (132)$ | $(12), (123)$ |
| (13) | $(13), (123)$ | $\langle(13)\rangle$ | $(13), (132)$ |
| (23) | $(23), (132)$ | $(23), (123)$ | $\langle(23)\rangle$ |
| (123) | $(13), (123)$ | $(23), (123)$ | $(12), (123)$ |
| (132) | $(23), (132)$ | $(12), (132)$ | $(13), (132)$ |

Note that we could label the cosets using integers, however then one should keep in mind that these integers do not have any algebraic meaning.

Losing the group structure of the code however has the advantage that we have more flexibility in choosing the subgroups we deal with, and thus have more choices in terms of possible intersections that we are getting.

Proposition 2. *There exist a quasi-uniform code obtained from a non-nilpotent group which cannot be obtained by any abelian group.*

Proof: Let G be a non-nilpotent group. Then G is not abelian group representable for all n [14]. That is, there exists some subgroups G_i such that $[G : G_A] \neq [A : A_A]$ for some abelian group and subgroups (See Definition 3). The proof uses the fact that in a group which is not nilpotent, there exists a Sylow subgroup which is not normal. ■

V. ALMOST AFFINE CODES FROM GROUPS

Consider a non-empty subset \mathcal{A} of \mathcal{N} and let $C_{\mathcal{A}}$ be the projection of the code C into the coordinate space \mathcal{A} , that is, all the words of C are restricted to the positions in \mathcal{A} .

Definition 4. [5] *A q -ary code C of length n is said to be almost affine if it satisfies the condition*

$$\log_q(|C_{\mathcal{A}}|) \in \mathbb{N}, \text{ for all } \mathcal{A} \subseteq \mathcal{N}.$$

Almost affine codes were introduced in [5] as a generalization of affine codes, which are themselves generalizations of linear codes over finite fields. It was shown in [2] that almost affine codes are quasi-uniform. It is thus natural to look for such codes among codes built from groups. Because of the definition of almost affine codes, p -groups are the first candidates that come to mind.

Lemma 7. *Let G be a p -group, and let G_1, \dots, G_n be subgroups of index p . The corresponding quasi-uniform code is almost affine.*

Proof: First note that G_1, \dots, G_n are normal subgroups of G , thus so is their intersection, and wlog we may assume that $|G_{\mathcal{N}}| = 1$. Then Proposition 1 holds (though the H_i might be nonabelian), and we obtain a p -ary quasi-uniform code. Since any intersection $G_{\mathcal{A}}$ will have order a power of p , the code obtained is almost affine. ■

There are other ways to get almost affine codes from p -groups, and p -groups are not the only finite groups that can provide almost affine codes.

VI. CONCLUSION

Quasi-uniform codes were known to be constructed from groups. In this paper, we were interested in relating the properties of the obtained code as a function of the corresponding group. We determined the size of the code, its alphabet, and its minimum distance, both for abelian groups, but also for some nonabelian groups where the group structure allows to mimic the definition of minimum distance. An application to the design of almost affine codes is also given.

Current and future works involve studying further properties of these codes coming from groups, and in particular (1) codes coming from nonabelian groups which cannot be reduced to abelian groups, and (2) almost affine codes. The information theoretic point of view is also of course of interest: it is related to the understanding of entropic vectors coming from nonabelian groups. A long term goal is the design of non-linear network codes.

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REFERENCES

- [1] T. H. Chan, "A Combinatorial Approach to Information Inequalities", *Commun. Inf. Syst.*, vol. 1, 2001.
- [2] T. H. Chan, A. Grant and T. Britz, "Properties of Quasi-Uniform Codes", *2010 IEEE International Symposium on Information Theory (ISIT 2010)*, Austin, TX, 13-18 June 2010.
- [3] T. H. Chan and R. W. Young, "On a Relation Between Information Inequalities and Group Theory", *IEEE Trans. Inform. Theory*, vol. 48, pp. 1992-1995, 2002.
- [4] T. H. Chan, "Group Characterizable Entropy Functions," *2007 IEEE International Symposium on Information Theory (ISIT 2007)*, Nice, France, 24-29 June 2007.
- [5] J. Simonis and A. Ashikhmin, "Almost Affine Codes", *Designs, Codes and Cryptography*, vol. 14, no. 2, 1998.
- [6] G.D. Forney, "Geometrically Uniform Codes", *IEEE Transactions on Information Theory*, Volume 35, Issue 5, 1991.
- [7] A.A. Zain and B.S. Rajan, "Algebraic Characterization of MDS Group Codes over Cyclic groups", *IEEE Transactions on Information Theory*, Volume 41, No. 6, 1995.
- [8] H. A. Loeliger, "Signal Sets Matched to Groups", *IEEE Transactions on Information Theory*, Volume 37, Issue 6, 1991.
- [9] R. Garello and S. Benedetto, "Multilevel Construction of Block and Trellis Group Codes", *IEEE Transactions on Information Theory*, Volume 41, Issue 5, 1995.
- [10] J. Bali and B.S. Rajan, "Block-Coded PSK Modulation Using Two-Level Group Codes Over Dihedral Groups", *IEEE Transactions on Information Theory*, Volume 44, Issue 4, 1998.
- [11] E.J. Rossin, N.T. Sindhushayana and C.D. Heegard, "Trellis Group Codes for the Gaussian Channel", *IEEE Transactions on Information Theory*, Volume 41, Issue 5, 1995.
- [12] R. Dougherty, C. Freiling and K. Zeger, "Insufficiency of Linear Coding in Network Information Flow", *IEEE Transactions on Information Theory*, Volume 51, No. 8, 2005.
- [13] T. H. Chan, A. Grant, "On Capacity Regions of Non-Multicast Networks", *2010 IEEE International Symposium on Information Theory (ISIT 2010)*, Austin, USA.
- [14] E. Thomas, N. Markin and F. Oggier, "On Abelian Group Representability of Finite Groups", Preprint, 2012, available at <http://arxiv.org/abs/1212.1298>.