Reliable Uncoded Communication in the SIMO MAC via Low-Complexity Decoding

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***Abstract*—We consider a multiple access channel with a large number of transmitters sending symbols from a constellation to the receiver of a multi-antenna base station. We investigate the joint decoding of the signals from all the users using a low complexity convex relaxation of the maximum likelihood decoder (constellation search). We show that, in a rich scattering environment, and in the asymptotic limit of a large number of transmitters, reliable communication is possible even without employing coding at the transmitters.**

I. Introduction

The need for accommodating an ever increasing number of mobile subscribers within the limited available spectrum has motivated the search for techniques more efficient than orthogonalizing communication resources either in time (TDMA), space (sectorization in cellular networks) or frequency (FDMA). Although the low implementation complexity of both encoding and decoding makes orthogonalizing schemes natural candidates for practical use, schemes motivated by the information-theoretically optimal techniques of superposition coding at the encoder and joint decoding at the receiver [1] promise significantly higher capacity for multiuser networks. However such sophisticated coding schemes have numerous challenges from an implementation viewpoint. Moreover the capacity benefits from such schemes over simple time-division (or other orthogonalizing schemes) have been shown to be negligible in regimes such as the asymptotically low-power or many user regimes. ([2],[3]).

Motivated by the gains possible in the asymptotic regime of many users, in this work, we consider a multiple access setting where a large number of transmitting users communicate in a rich scattering environment with a single multi-antenna base station. We look at transmitting schemes which do not employ coding, but instead transmit symbols from the BPSK constellation and rely on the diversity inherent in a large system to achieve reliability. Such a setting closely models sensor networks or general distributed networks with energy limited transmitters and centralized receivers. A similar setting was considered in [4] where it was shown that using the optimal maximum likelihood (ML) decoder for the joint processing, the decoding can be made arbitrarily reliable for an arbitrarily small number of receiver antennas per transmitter, provided that the number of transmitters is large enough. However, due to its complexity, the ML decoder is not feasible in practice and this motivates the need for lower complexity decoder structures.

In this work, we propose a low-complexity decoder that modifies the constellation search of the ML decoder via a convex relaxation. In particular we consider the decoder obtained by restricting the search over possible symbols to intervals instead of discrete points, and then quantizing an output of the interval search to the nearest constellation point. This relaxation of the search over integer points to search over intervals allows a more efficient (polynomial time) decoding. Henceforth we will refer to this decoding technique as the interval search and quantize (ISQ) decoder. We obtain analytical bounds on the performance of the ISQ decoder for the case of a channel matrix with more receivers than transmitters. We show that reliable decoding (in a sense made precise in the later sections) is possible in the asymptotic limit of a large number of transmitters.

The rest of the paper is organized as follows. We first present the system model and describe the optimal decoder and the ISQ decoder. We then describe a bound on the error probability of this new decoder. Asymptotic analysis of this bound gives us the reliable communication result.

II. System Model

Our system model is depicted in Figure 1. We have an uplink system with n transmitters and m antenna receiver. We assume $m \geq n$. The channel matrix $\mathbf{H} \in \mathbb{R}^{m \times n}$ is chosen to model a rich scattering environment, and the entries are assumed to be drawn i.i.d. from $\mathcal{N}(0,1)$. The result also holds for arbitrary fading distributions, as will be indicated later. The k^{th} column of $\mathbf{H}$ is denoted as $\mathbf{h}_k \in \mathbb{R}^m$. Thus

$$\mathbf{H} = \begin{pmatrix} \mathbf{h_1} & \mathbf{h_2} & \dots & \mathbf{h_{n-1}} & \mathbf{h_n} \end{pmatrix}.$$

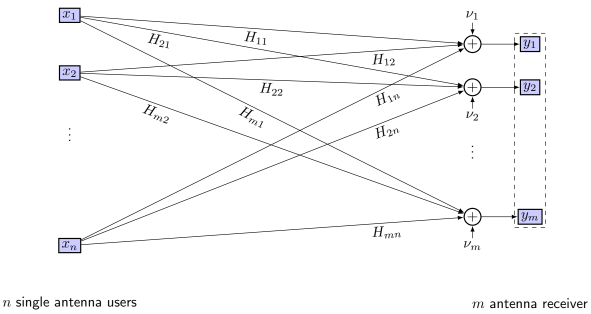

Fig. 1. System model

We also assume that the users do not cooperate with each other and they transmit symbols from the standard unit energy BPSK constellation. The components of the noise at the receiver ($\boldsymbol{\nu}$) are assumed to be i.i.d. $\mathcal{N}(0, \sigma^2)$. The received signal at the multi-antenna receiver is then

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\nu}. \tag{1}$$

The vector $\mathbf{x} \in \{-1, +1\}^n$, which consists of the transmitted symbols from the n users, is referred to as the *n-user codeword* to indicate that the receiver decodes the block of n-user constellation points simultaneously. We further assume that the receiver has perfect channel state information (CSI) and that the transmitters have no CSI.

III. Previous work and results

We now describe a few observations and results we have about the performance limits of this system. These results, derived in [4], assume that the receiver employs ML decoding, i.e. it returns

$$\hat{\mathbf{x}} = \mathrm{argmin}_{\mathbf{x} \in \{-1,+1\}^n} ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2. \tag{2}$$

With this decoder it has been shown that in the limit of a large number of transmitters the following holds:

Theorem 1. *Under ML decoding, there exists a $d > 0$ such that for all sufficiently large n, the probability of error in decoding the n-user codeword satisfies*

$$P_{\text{error}} \leq 2^{-dn}.$$

In other words the probability of decoding a particular user's transmitted symbol in error decreases exponentially with the number of users, even though the users do not employ any coding across time.

A critical component in [4] to achieve this asymptotic result was the use of ML decoding. In this work we investigate whether we can get away with similar reliability even with lower-complexity decoders. In particular, we ask whether an efficient polynomial time decoder can realize an asymptotically vanishing probability of error.

A common approach to solve hard combinatorial optimization problems (such as ML decoding) is the technique of expanding the search space from discrete points to intervals or regions [5]. Motivated by this idea, we consider a convex relaxation of the maximum likelihood decoder search as follows:

$$\hat{\mathbf{x}} = \mathrm{sgn}(\mathrm{argmin}_{\mathbf{x} \in [-1,+1]^n} ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2). \tag{3}$$

In the above $\mathrm{sgn}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$ refers to the vector obtained by the coordinatewise application of the signum function defined below for a scalar x.

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0. \\ -1 & \text{otherwise.} \end{cases}$$

The modified decoder of (3) expands the search for a valid *n-user codeword* to the interval $[-1, 1]$ per dimension and then quantizes it to integer values afterwards, hence we call it an ISQ decoder.

We show that even with the ISQ decoder, we do not lose out on the asymptotic reliability in a sense which is made precise in Theorem 2. Although we are not able to bound the probability of at least one error among n transmitter symbols (i.e. the *n-user codeword*), we show that the per-transmitter (i.e. per-user) decoding error probability vanishes. In particular, we show that the probability that a constant fraction of the transmitter symbols are decoded incorrectly can be made as small as we like for a large enough system size. We define $P_e^{k'}$ to be the probability of incorrectly decoding at least $k'n$ out of n transmitted symbols. The following states a bound on the above probability of error for the ISQ decoder.

Theorem 2. *Under ISQ decoding, for $m = \alpha n$, $\alpha \geq 1$ and any constant $k > 0$, there exists a $d > 0$ such that for all sufficiently large n,*

$$P_e^k \leq 2^{-dn \log n}.$$

Thus we see that by exploiting the *diversity* (richness in the scattering environment), one can get arbitrarily reliable communications in different asymptotic regimes (in this case for a large number of transmitters) without employing coding or ML decoders. We make this precise in the following theorem, which is our main result.

Theorem 3. *The probability of error with the ISQ decoder seen by any transmitting user, vanishes in the limit of an asymptotically large number of transmitters, with the per-transmitter number of receive antennas being at least 1.*

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In other words, not only can reliable communication be achieved with uncoded non-cooperating transmission, as was established in [4], but in fact it can be achieved reliably with practical low-complexity decoding.

The remainder of this paper discusses the proofs of Theorems 2 and 3.

IV. An Upper Bound on the Decoding Error

We describe an upper bound on the probability of decoding error. The basic idea, which is similar to what was used in [4], is an union bound consisting of the pairwise error probability of mistaking the transmitted codeword with one differing in $k'n$ positions. Since the receiver uses the decoder in (3), the probability of mistaking a codeword $\mathbf{x_0}$ for another codeword differing in i symbol positions is given by

$$P_{e,\mathbf{b_i}} \leq Q\left(\min_{\mathbf{x}:\mathrm{supp}(\mathrm{sgn}(\mathbf{x})-\mathbf{x_0})=\mathbf{b_i}} \frac{||\mathbf{H}(\mathbf{x}-\mathbf{x_0})||}{2\sigma}\right) \quad (4)$$

Here $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-x^2/2}dx$, $\mathbf{b_i}$ is a vector of size i whose entries are positions where the codewords differ (arranged in increasing order), and $\mathbf{b_i}(j)$ is the j^{th} symbol position where the codewords differ. Note that $\mathbf{b_i}$ has a one-to-one correspondence with a subset of $\{1,\ldots,n\}$ of cardinality i. $||\mathbf{x}||_0$ refers to the number of non zero entries in $\mathbf{x}$. $\mathrm{supp}(\mathbf{x})$ refers to the support (i.e. locations of the non zero entries of vector $\mathbf{x}$).

Note that the error probability above has the same statistics (with respect to the distribution of $\mathbf{H}$) regardless of which $\mathbf{x_0}$ is chosen. Hence, choosing $\mathbf{x_0} = -\mathbf{1}$, we note that the last expression can be rewritten for decoder 3 as

$$\begin{aligned} P_{e,\mathbf{b_i}} &\leq Q\left(\min_{\substack{1\leq c_j\leq 2\,\forall j\in\mathbf{b_i}\\ 0\leq c_j\leq 1\,\forall j\in\mathbf{b_i}^c}} \frac{||\sum_{j=1}^n c_j\mathbf{h_{b_i(j)}}||}{2\sigma}\right) \\ &\stackrel{(a)}{\leq} \frac{1}{2}\exp\left(-\min_{\substack{1\leq c_j\leq 2\,\forall j\in\mathbf{b_i}\\ 0\leq c_j\leq 1\,\forall j\in\mathbf{b_i}^c}} \frac{||\sum_{j=1}^n c_j\mathbf{h_{b_i(j)}}||^2}{8\sigma^2}\right) \quad (5) \end{aligned}$$

where (a) follows from the fact that $Q(x) \leq \frac{1}{2}\exp(\frac{-x^2}{2})$. We observe now that the expectation of (5) with respect to the channel fading process is independent of the particular subset of transmitted symbols that are in error and depends only on the number i of such errors. Let's call this expectation P_i. Thus

$$P_i \triangleq E_{\mathbf{H}} e^{\left(-\min_{\substack{1\leq c_j\leq 2\,\forall j\in\{1,\ldots,i\}\\ 0\leq c_j\leq 1\,\forall j\in\{i+1,\ldots,n\}}} \frac{||\sum_{j=1}^n c_j\mathbf{h_j}||^2}{8\sigma^2}\right)}.$$

If $P_e^{k'}$ is the probability of error of decoding at least $k'n$ transmitter symbols incorrectly, and S_i refers to the set of all vectors representing subsets of size i from $\{1,\ldots,n\}$, then a union bound for the error probability is

$$\begin{aligned} P_e^{k'} &\leq \sum_{k'n\leq i\leq n}\sum_{\mathbf{b}\in S_i}\frac{1}{2}P_i \quad (6)\\ &\leq \sum_{k'n\leq i\leq n}\binom{n}{i}\frac{1}{2}P_i. \quad (7) \end{aligned}$$

Note that, by the symmetry of the system, the probability of error P_e seen by each transmitting user is then upper bounded by

$$P_e \leq k' + P_e^{k'}.$$

We show that for any small k', there exists a large enough system size for which $P_e^{k'}$ becomes exponentially small, even with a convex decoder of much lower complexity.

V. Asymptotic analysis of the upper bound

We first prove bounds on the exponent appearing in the bound for $P_{e,\mathbf{b_i}}$ in (5). Specifically, we look at (ignoring constant $8\sigma^2$)

$$\min_{\substack{1\leq c_j\leq 2\,\forall j\in\{1,\ldots,i\}\\ 0\leq c_j\leq 1\,\forall j\in\{i+1,\ldots,n\}}} ||\sum_{j=1}^n c_j\mathbf{h_j}||^2.$$

Before we describe the proof, we define the ϵ-grid inside the hypercube $[-1,+1]^n$, for some $0<\epsilon<0.5$. This grid is simply the set of points

$$\mathcal{G}_{n,\epsilon} = \{\mathbf{x} : \mathbf{x}_i \mod \epsilon = 0, -1\leq \mathbf{x}_i \leq 1\ \forall i\}.$$

It may be rewritten as

$$\mathcal{G}_{n,\epsilon} = \{-1,-1+\epsilon,\ldots,1-\epsilon,1\}^n.$$

if $\frac{1}{\epsilon}\in\mathbb{N}$. Henceforth we make this assumption.

We now introduce the ϵ-ISQ decoder, so named because it replaces the interval search in the ISQ decoder by an ϵ-grid search:

$$\epsilon\text{-ISQ: } \hat{\mathbf{x}} = \mathrm{sgn}(\mathrm{argmin}_{\mathbf{x}\in\{-1,-1+\epsilon,\ldots,1-\epsilon,1\}^n}\,||\mathbf{y}-\mathbf{H}\mathbf{x}||^2).$$

The ϵ-grid error probabilities are defined similar to the definitions for the ISQ decoder in the previous section, and are indicated by an ϵ subscript. We now state some observations about the ϵ-grid error probabilities. We next use an union bounding argument for $P_{e,\mathbf{b_i},\epsilon}$.

Before that, however, we require the following lemma about the negative of the exponent in the error probability $P_{e,\mathbf{b_i},\epsilon}$:

$$\min_{\substack{c_j\in\{1,1+\epsilon,\ldots,2-\epsilon,2\}\,\forall j\in\{1,\ldots,i\}\\ c_j\in\{0,\epsilon,\ldots,1-\epsilon,1\}\,\forall j\in\{i+1,\ldots,n\}}} ||\sum_{j=1}^n c_j\mathbf{h_j}||^2.$$

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Lemma 1. *For any $i > k'n$, there exists an n_0 and $a > 0$, such that for all $n > n_0$,*

$$P(||\sum_{j=1}^{i} c_j \mathbf{h_j}||^2 < an\log n) \leq \exp(-an\log n).$$

Proof: We can show this using Markov's inequality. Let $a_1 < \frac{\alpha}{4}$. Then

$$P(||\sum_{j=1}^{n} c_j \mathbf{h_j}||^2 < a_1 n\log n) \tag{8}$$

$$= P(\exp(-t||\sum_{j=1}^{n} c_j \mathbf{h_j}||^2) > \exp(-ta_1 n\log n))$$

$$\stackrel{(a1)}{\leq} \exp(ta_1 n\log n)E(-t||\sum_{j=1}^{n} c_j \mathbf{h_j}||^2) \tag{9}$$

$$\stackrel{(a2)}{=} \exp(ta_1 n\log n)(1+2t\sum_j c_j^2)^{(-\alpha n/2)} \tag{10}$$

$$\stackrel{(b)}{\leq} \exp(ta_1 n\log n)(1+2tk'n)^{(-\alpha n/2)} \tag{11}$$

$$\stackrel{(c)}{\leq} \exp(-\tilde{a}n\log n) \quad \text{for large enough n.} \tag{12}$$

In the above $(a1)$ follows from Markov's inequality, $(a2)$ follows from the moment generating function of a chi-squared random variable, (b) follows from the fact that for at least $k'n$ errors,

$$\sum_j c_j^2 \geq k'n,$$

(c) follows by optimizing over $t > 0$, and defining $\tilde{a} = \frac{\alpha}{4}$. Defining $a = \min(a_1, \tilde{a}) = \frac{\alpha}{4}$, we get the claim in the lemma. Thus the claim is established for $\mathbf{H}$ with $\mathcal{N}(0,1)$ entries. For arbitrary fading distributions with a finite absolute third moment (i.e. $E(|x|^3) < \infty$), we can invoke the Berry-Esseen lemma ([6]) to note that

$$E(-t||\sum_{j=1}^{n} c_j \mathbf{h_j}||^2) \leq ((C+t)(k'n)^{-\frac{1}{2}})^{\alpha n}$$

in step $(a2)$, by noting again that $\sum_j c_j^2 \geq k'n$. C here is a constant dependent on the fading distribution but is independent of n. The remaining proof proceeds in the exact same fashion, except for a readjustment of the constants involved.

■

By observing that for a positive r.v., $P(x > d_0) < \exp(-d_0)$ implies

$$E(\exp(-x)) \tag{13}$$

$$\leq \exp(-d_0) + (1-\exp(-d_0))\exp(-d_0) \tag{14}$$

$$\leq 2\exp(-d_0), \tag{15}$$

we get that

$$P_{i,\epsilon} \leq E(\exp(-||\sum_{j=1}^{n} c_j \mathbf{h_j}||^2)) \tag{16}$$

$$\leq \exp(-an\log n) \quad \text{for large enough n.} \tag{17}$$

The probability of the event that there are at least $k'n$ symbols that are decoded incorrectly can then be union bounded as follows.

$$P_{e,\epsilon}^{k'} \leq \sum_{i=k'n}^{n} \binom{n}{i}\left(\frac{1}{\epsilon}\right)^n P_{i,\epsilon} \tag{18}$$

$$\stackrel{(d1)}{\leq} n2^{n(\max_{k'n\leq i\leq n} H_2(\frac{i}{n})-\log(\epsilon)-a\log n)} \quad \text{for } a>0 \text{ and large enough } n \tag{19}$$

$$\stackrel{(d2)}{\leq} 2^{-an\log n} \text{ for a large enough n.}$$

In the above,

$$H_2(x) = -x\log x - (1-x)\log(1-x).$$

$(d1)$ follows by noting that

$$\binom{n}{i} \leq 2^{H_2(i/n)}$$

and $(d2)$ follows from the observation that $H_2(\cdot)$ is bounded above by a constant. We now relate the solution from the search over the ϵ-grid $\mathcal{G}_{n,\epsilon}$ $(\hat{\mathbf{x}}_\epsilon)$ to the solution $(\hat{\mathbf{x}})$ of the ISQ decoder. We note that, since $\mathbf{H}^T\mathbf{H}$ is full rank a.s. (because $m \geq n$), $f(\mathbf{x}) = ||\mathbf{y}-\mathbf{H}\mathbf{x}||^2$ is strictly convex with probability 1. Thus the optimal solution $\hat{\mathbf{x}}$ of the ISQ decoder would satisfy (in an almost sure sense)

$$||\hat{\mathbf{x}} - \hat{\mathbf{x}}_\epsilon||_\infty \leq \frac{\epsilon}{2}.$$

Also, by following the exact union bounding technique used to bound $P_{e,\epsilon}^{k'}$, we get that, for any $k'' > 0$, the probability that $\hat{\mathbf{x}}_\epsilon$ has greater than or equal to $k''n$ entries that are close to zero, (i.e. either $-\epsilon, 0$, or ϵ) is upper bounded by $\exp(-d_1 n\log n)$, for some $d_1 > 0$. Let

$$d_2 = \min(d_1, a).$$

Thus we conclude that for a large enough n, with probability at least $1-\exp(-d_2 n\log n)$, the signs of $\hat{\mathbf{x}}_\epsilon$ will match the signs of $\mathbf{x_0}$ (i.e. the correct n-user codeword) in at least $(1-k''-k')n$ positions. By choosing k' and k'' small enough we see that the number of mismatches is sublinear in the number of transmitting users. The proof of Theorem 2 is now complete. □

To get to Theorem 3, we note that, by the symmetry of the system, the error probability P_e seen by each transmitter is the same. Thus given any target symbol error rate (SER) $\epsilon_1 > 0$, we can choose $k < \epsilon_1/2$ in Theorem 2. Then there exists an n_0 depending on k such that

$$P_e^k \leq 2^{-dn\log n} \leq \frac{\epsilon_1}{2} \;\forall n > n_0.$$

Then, assuming independent (both temporally and spatially) channel realizations, we get that the expected

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number of errors $E(N_e)$ seen by *all* transmitters in t single shot transmission slots satisfies

$$E(N_e) \leq nkt + n(1-k)P_e^k t \tag{20}$$

$$\text{or } \frac{E(N_e)}{t} \leq n\epsilon_1 \text{ for } n \text{ large enough.} \tag{21}$$

Dividing both sides by n we get that the per-transmitter error probability P_e can be made smaller than ϵ_1 for a large enough n. □

VI. Conclusion and Future Work

We have considered an uplink communication system in a rich fading environment with a large number of non-cooperating transmitters and a large number of antennas at the receiver. The transmitters send bits to the receiver *without coding*. The receiver does joint decoding of the noisy received signal with a convex relaxation of the maximum likelihood decoder. We call this new decoding technique the interval search and quantize (ISQ) decoder. Under general assumptions about the fading distribution of the channel coefficients, we have shown that with the ISQ decoder, for a large enough system size, the error probability that each user sees is below any prespecified threshold. While in this paper we have focused on the BPSK constellation for simplicity of presentation, the same techniques carry over to general constellations.

We conjecture that the bound presented in this work can be improved upon further. In particular, we conjecture that even for a sublinear number of errors in a block, the error probability can be vanishingly small in the asymptotic regime of a large number of transmitters (thereby provably improving the rate of decay of the error probability with the system size). We also conjecture that similar results continue to hold even if the per-transmitter number of receiver antennas (α) is *any* arbitrary positive constant (instead of being greater than or equal to 1). However, we do seem to lose the exponential decay rates of the error probabilities achievable using a ML decoder. Nevertheless, the question of how large the system size needs to be for the diversity-induced reliability to kick in remains a topic for future work.

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