On the Capacity Region of Gaussian Interference Channels with State

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***Abstract*—The Gaussian interference channel with additive state at two receivers is investigated, in which the state information is noncausally known at both transmitters but not known at either receiver. For the very strong Gaussian interference channel with state, the capacity region is obtained under certain conditions on channel parameters. For the strong (but not very strong) Gaussian interference channel with state, points on the boundary of the capacity region are characterized under corresponding conditions on channel parameters. Finally, for the weak Gaussian interference channel with state, the sum capacity is obtained for certain channel parameters. All the above capacity-achieving rate points achieve the capacity for the corresponding channel without state.**

I. Introduction

The interference channel with state has recently caught a lot of attention, in which the state information is noncausally known at the transmitters, but not at either receiver. The scenario with the same state at two receivers has been studied in [1], [2]. Various achievable schemes have been designed and the corresponding achievable regions are compared in [1], and the gap between inner and outer bounds on the capacity region have been characterized within certain finite bits in [2]. The scenario with different states at two receivers has been studied in [3], in which various achievable schemes have been studied, and the capacity region has been characterized for an asymptotic case when the power of the state sequence becomes large. We also note that two cognitive interference channel models with state has been studied in [4] and [5], [6], in which inner and outer bounds on the capacity region for these models have been developed and have been shown to be tight for some special cases.

In this paper, we study the Gaussian interference channel with the same state, but scaled differently at two receivers. The state sequence is noncausally known at both transmitters, but not known at either receiver. Differently from the previous studies of this model in [1], [2], our focus here is to characterize the capacity region, or points on the boundary of the capacity region. We note that the capacity region/the sum capacity has been characterized for the Gaussian interference channel *without* state in the following three regimes: (1) very strong interference channels [7]; (2) strong interference channels [8]; and (3) a certain weak interference channel [9]–[11] (based on the technique developed in [12]). We study in this paper whether or not the capacity region/the sum capacity in these regimes are achievable when the two receivers' outputs are also corrupted by differently scaled state, and if so, what transmission schemes are capacity achieving.

The solution to such a problem has been provided for the point-to-point Gaussian channel with state in [13], and it has been shown that dirty paper coding achieves the capacity of the Gaussian channel without state, and hence is capacity achieving for the channel with state. However, such a problem for the interference channel can be challenging, because the input from each transmitter appears in two outputs, in which the channel state S is scaled differently. It is then challenging to design dirty paper coding for the input to take care of such differently scaled states in two receivers' outputs. Such difficulty has also been observed for the compound channel with states, e.g. in [14]. In this paper, we explore the properties of the interference channel in three regimes and characterize conditions on the channel parameters under which the capacity region/the sum capacity for the channel without state can be achieved, and hence the capacity region/the sum capacity for the channel with state is characterized.

In particular, for the very strong and strong interference channels with state, we first design an achievable scheme based on message splitting, superposition, and sequential decoding, which aims at achieving the boundary points of the capacity region for the Gaussian multiple access channel (MAC), as the capacity of the strong interference channel without state is the same as the compound MAC. Based on such a scheme, we design dirty paper coding for the Gaussian channel. For the very strong Gaussian interference channel with state, we obtain the capacity region under certain conditions on channel parameters. For the strong (but not very strong) Gaussian interference channel with state, we characterize points on the boundary of the capacity region under corresponding conditions on channel parameters. Finally, for the weak Gaussian interference channel with state, we obtain the sum capacity for certain channel parameters. All the above capacity-achieving rate points achieve the capacity for the corresponding channel without state.

The rest of the paper is organized as follows. In Section II, we describe the channel model and explain the notation used in this paper. In Section III, we derive an achievable scheme that is useful for studying very strong and strong interference channels with state. In Sections IV, V, and VI, we present our results for the very strong, strong, and weak Gaussian interference channels with state, correspondingly. Finally, in Section VII, we conclude with a few remarks.

II. Channel Model

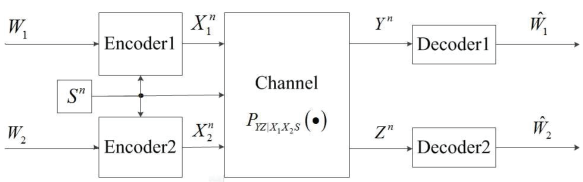

Fig. 1. A model of the interference channel with state

We consider the interference channel with state (as shown in Fig. 1), in which transmitter 1 sends a message W_1 to receiver 1, and transmitter 2 sends a message W_2 to receiver 2. The channel is corrupted by an independent and identically distributed (i.i.d). state sequence S^n, which is assumed to be known *noncausally* at both transmitters. Transmitter 1 maps a message $w_1 \in \{1, \ldots, 2^{nR_1}\}$ and a state sequence s^n to an input x_1^n, and transmitter 2 maps a message $w_2 \in \{1, \ldots, 2^{nR_2}\}$ and a state sequence s^n to an input x_2^n. These two inputs are then sent over the memoryless interference channel characterized by $P_{YZ|X_1X_2S}$. Receiver 1 is required to decode W_1 and receiver 2 is required to decode W_2, with the probability of error approaching zero as the codeword length n goes to infinity. The capacity region is defined to be the closure of the set of all achievable rate pairs (R_1, R_2).

In this paper, we study the Gaussian channel with the outputs at receivers 1 and 2 for one channel use given by

$$\begin{aligned} Y &= X_1 + aX_2 + S + N_1 \\ Z &= bX_1 + X_2 + cS + N_2 \end{aligned} \tag{1}$$

where a, b and c are constants, the noise variables N_1, $N_2 \sim \mathcal{N}(0, 1)$, and $S \sim \mathcal{N}(0, Q)$. Both the noise variables and the state variable are i.i.d. over channel uses. The channel inputs X_1 and X_2 are subject to the average power constraints P_1 and P_2.

The goal of this paper is to characterize the capacity region, points on the boundary of the capacity region, or sum capacity for very strong, strong, and weak Gaussian interference channels with state under certain conditions on the channel parameters (a, b, c, P_1, P_2, Q).

III. A Useful Achievable Scheme

In this section, we derive an achievable rate region for the discrete memoryless interference channel with state based on Gel'fand-Pinsker scheme and sequential decoding. This region is useful to obtain achievable regions for the Gaussian channel with state and to further show that such regions match the capacity region or partially match the boundary of the capacity region of the corresponding Gaussian channel without state. The sequential coding is motivated by the fact that the capacity of the strong Gaussian interference channel without state is the same as that of the compound Gaussian MAC, and the capacity of the Gaussian MAC can be achieved by sequential decoding.

The idea of the achievable scheme is to split the message W_1 into two parts W_{11} and W_{12}, and split the message W_2 into two parts W_{21} and W_{22}. Both receivers decode both messages with the decoding order $W_{11}, W_{21}, W_{22}, W_{12}$ at receiver 1 and the decoding order $W_{21}, W_{11}, W_{12}, W_{22}$ at receiver 2. The achievable rate region based on this scheme is given in the following Lemma. The details of the proof is omitted due to the space limitations.

Lemma 1. *The following region is achievable for the interference channel with state noncausally known at both transmitters, which consists of rate pair (R_1, R_2) satisfying:*

$$\begin{aligned} R_1 \leqslant\ & \min\{I(U_1; Y), I(U_1; V_1 Z)\} \\ & + \min\{I(U_2; V_1 V_2 Y|U_1), I(U_2; V_1 Z|U_1)\} - I(U_1 U_2; S) \\ R_2 \leqslant\ & \min\{I(V_1; Z), I(V_1; U_1 Y)\} \\ & + \min\{I(V_2; U_1 U_2 Z|V_1), I(V_2; U_1 Y|V_1)\} - I(V_1 V_2; S) \end{aligned} \tag{2}$$

for some distribution

$$\begin{aligned} P_{SU_1U_2V_1V_2X_1X_2YZ} = & P_S P_{U_1|S} \\ & \cdot P_{U_2|SU_1} P_{X_1|U_1U_2S} P_{V_1|S} P_{V_2|SV_1} P_{X_2|V_1V_2S} P_{YZ|SX_1X_2} \end{aligned}$$

where U_1, U_2, V_1, and V_2 are auxiliary random variables.

We note that although the above scheme may not perform sufficiently well for general interference channels with state, it is shown in Sections IV and V that such a scheme achieves the capacity region or partial boundary of the capacity region for these channels with state under certain conditions on channel parameters.

IV. Very Strong Gaussian Interference Channel With State

In this section, we study the very strong Gaussian interference channel with state. The channel parameters are assumed to satisfy $P_1 + a^2 P_2 + 1 \geq (1 + P_1)(1 + P_2)$ and $b^2 P_1 + P_2 + 1 \geq (1 + P_1)(1 + P_2)$. For such interference channel without state (i.e. the very strong Gaussian interference channel), the capacity region contains rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leqslant \frac{1}{2} \log (1 + P_1) \\ R_2 &\leqslant \frac{1}{2} \log (1 + P_2). \end{aligned} \tag{3}$$

The following theorem characterizes the conditions on the channel parameters, under which the above capacity region of the channel without state can be achieved for the corresponding interference channel with state. We thus characterized the capacity region for these channels with state.

Theorem 1. *For the Gaussian interference channel with state noncausally known at both transmitters, if its channel parameters* (a, b, c, P_1, P_2, Q) *satisfy the following conditions:*

$$\frac{(b^2P_1+P_2+c^2Q+1)}{(1+P_2)(1+\frac{(1+P_2)(c+cP_1-bP_1)^2Q+QP_1(1+P_2-acP_2)^2}{((1+P_1)(1+P_2)-abP_1P_2)^2})} \geqslant 1+P_1 \quad (4)$$

$$\frac{(P_1+a^2P_2+Q+1)}{(1+P_1)(1+\frac{P_2(c+cP_1-bP_1)^2Q+Q(1+P_1)(1+P_2-acP_2)^2}{((1+P_1)(1+P_2)-abP_1P_2)^2})} \geqslant 1+P_2, \quad (5)$$

then the capacity region consists of rate pairs (R_1, R_2) *satisfying*

$$R_1 \leqslant \frac{1}{2}\log(1+P_1)$$
$$R_2 \leqslant \frac{1}{2}\log(1+P_2). \quad (6)$$

Remark 1. *Conditions* (4) *and* (5) *reduces to the conditions that define the very strong Gaussian interference channel if* $Q = 0$*, i.e., the state power is zero.*

Proof: (Outline) For the achievable region given in (2), we set $U_1 = \phi$, $V_1 = \phi$, $U_2 = U$ and $V_2 = V$, and obtain the following achievable region containing rate pairs (R_1, R_2) satisfying

$$R_1 \leqslant I(U;VY) - I(U;S) \quad (7)$$
$$R_1 \leqslant I(U;Z) - I(U;S) \quad (8)$$
$$R_2 \leqslant I(V;Y) - I(V;S) \quad (9)$$
$$R_2 \leqslant I(V;UZ) - I(V;S). \quad (10)$$

For the Gaussian channel, we set U and V as $U = X_1 + \alpha S$, $V = X_2 + \beta S$, where X_1, X_2 and S are independent Gaussian variables with mean zero and variances P_1, P_2 and S, correspondingly, and α and β are given by

$$\alpha = \frac{P_1(1+P_2-acP_2)}{(1+P_1)(1+P_2)-abP_1P_2}$$
$$\beta = \frac{cP_2(C(1+P_1)-bP_1)}{(1+P_1)(1+P_2)-abP_1P_2}.$$

Then the bounds (7) and (10) becomes

$$R_1 \leqslant \frac{1}{2}\log(1+P_1)$$
$$R_2 \leqslant \frac{1}{2}\log(1+P_2). \quad (11)$$

The above bounds characterize the achievable region if

$$\frac{1}{2}\log(1+P_1) \leqslant I(U;Z) - I(U;S)$$
$$\frac{1}{2}\log(1+P_2) \leqslant I(V;Y) - I(V;S). \quad (12)$$

By computing the mutual information terms in the above equations based on the chosen distributions for U and V, we obtain the conditions given in the theorem.

Since such an achievable region is the same as the corresponding very strong Gaussian interference channel without state, it can be shown that it is the capacity region for the channel with state. ■

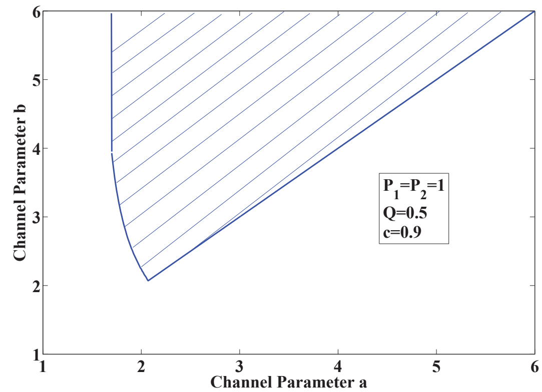

Fig. 2. Conditions on channel parameters (a, b) under which capacity is obtained for the very strong Gaussian interference channel with state

In Fig. 2, we set $c = 0.9$, $P_1 = P_2 = 1$, $Q = 0.5$, and plot the range of parameter pairs (a, b), for which the capacity region of the very strong Gaussian interference channel with state is characterized in Theorem 1, and achieves the capacity region of the corresponding interference channel without state.

Following Theorem 1, we also obtain the following result for the symmetric Gaussian interference channel as a special case.

Corollary 1. *For the symmetric Gaussian interference channel with state noncausally known at the transmitters, i.e.,* $a = b$*,* $c = 1$*, and* $P_1 = P_2$*, the capacity region contains rate pairs* (R_1, R_2) *satisfying*

$$R_1 \leqslant \frac{1}{2}\log(1+P)$$
$$R_2 \leqslant \frac{1}{2}\log(1+P), \quad (13)$$

if $a \geq a_{th}$*, where* a_{th} *solves the following equation*

$$\frac{(P+a^2P+Q+1)(1+P+aP)^2}{(1+P)[(1+P+aP)^2+Q(1+2P)]} = 1+P. \quad (14)$$

Proof: If $a = b$ and $c = 1$, then the conditions (5) and (4) reduce to the following single condition:

$$\frac{(P+a^2P+Q+1)(1+P+aP)^2}{(1+P)[(1+P+aP)^2+Q(1+2P)]} \geqslant 1+P. \quad (15)$$

Such a condition is equivalent to the one given in the corollary. ■

V. Strong Gaussian Interference channel with State

Since the very strong interference channel is studied separately in Section IV, in this section, we focus on strong interference channels which are not very strong interference channels. Hence, we require the channel parameters to satisfy $a \geq 1$, $b \geq 1$, $\min\{P_1 + a^2P_2 + 1, b^2P_1 + P_2 + 1\} \leqslant (1+P_1)(1+P_2)$. Without loss of generality, we assume that $P_1 + a^2P_2 + 1 \leqslant b^2P_1 + P_2 + 1$ due to symmetry of two users. For such interference channels *without* state (i.e., strong but not very strong), the capacity region contains rate pairs (R_1, R_2) satisfying

$$R_1 \leqslant \frac{1}{2}\log(1+P_1), R_2 \leqslant \frac{1}{2}\log(1+P_2)$$
$$R_1 + R_2 \leqslant \frac{1}{2}\log(1+P_1+a^2P_2). \tag{16}$$

We illustrate such a capacity region in Fig. 3, which is the pentagon O-A-B-D-E.

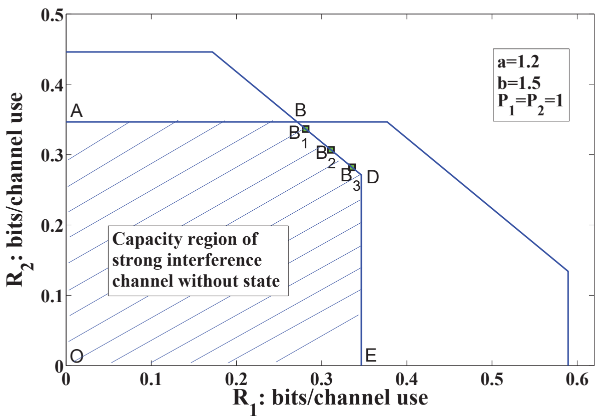

Fig. 3. Capacity region of the strong interference channel without state

Our goal here is to study whether a point on the boundary of such pentagon (i.e., the boundary of the capacity region of the interference channel without state) can be achieved for the corresponding interference channel with state. In particular, we focus on the sum rate boundary of the pentagon (i.e., the line B-D in Fig. 3), as the points on the line A-B and the line D-E are achievable if the two corner points B and D on the sum rate boundary are achievable.

We note that any rate points on the line B-D can be characterized by

$$R_1 = \frac{1}{2}\log\left(1+\frac{P_1'}{P_1''+a^2P_2+1}\right) + \frac{1}{2}\log(1+P_1'')$$
$$R_2 = \frac{1}{2}\log\left(1+\frac{a^2P_2}{P_1''+1}\right) \tag{17}$$

for some $P_1', P_1'' \geqslant 0$ and $P_1' + P_1'' \leqslant P_1$.

For any such point, we characterize the conditions on the channel parameters under which this point is on the boundary of the capacity region of the corresponding interference channel with state. We state our result in the following theorem.

Theorem 2. *Any rate point given in* (17) *is on the boundary of the capacity region of the corresponding interference channel with state noncausally known at the transmitters if the channel parameters satisfy the following conditions*

$$\frac{1}{2}\log\left(1+\frac{P_1'}{P_1''+a^2P_2+1}\right) \leqslant I(U_1;V_1Z)$$
$$\frac{1}{2}\log(1+P_1'') \leqslant I(U_2;V_1Z|U_1)$$
$$\frac{1}{2}\log\left(1+\frac{a^2P_2'}{P_1''+a^2P_2''+1}\right) \leqslant I(V_1;Z)$$
$$\frac{1}{2}\log\left(1+\frac{a^2P_2''}{P_1''+1}\right) \leqslant I(V_2;U_1U_2Z|V_1) \tag{18}$$

for some $P_2', P_2'' \geqslant 0$, such that $P_2' + P_2'' \leqslant P_2$.

To compute the mutual terms in the above conditions, the auxiliary random variables U_1, U_2, V_1 and V_2 are chosen to be

$$U_1 = X_1' + \alpha_1 S, \quad U_2 = X_1'' + \alpha_2 S$$
$$V_1 = X_2' + \beta_1 S, \quad V_2 = X_2'' + \beta_2 S \tag{19}$$

where X_1', X_1'', X_2', X_2'' are independent Gaussian variables with mean zero and variances P_1', P_1'', P_2' and P_2'', correspondingly, $X_1 = X_1' + X_1''$, $X_2 = X_2' + X_2''$, and $\alpha_1, \alpha_2, \beta_1$ and β_2 are given by

$$\alpha_1 = \frac{P_1'}{P_1+a^2P_2+1}, \alpha_2 = \frac{P_1''}{P_1+a^2P_2+1}$$
$$\beta_1 = \frac{a^2P_2'}{P_1+a^2P_2+1}, \beta_2 = \frac{a^2P_2''}{P_1+a^2P_2+1}.$$

Proof: (Outline) The achievability follows from Lemma 1 by choosing the auxiliary random variables U_1, U_2, V_1, and V_2 as in (19) based on the dirty paper coding for removing the state effect at the received signal Y so that the rate point given in (17) is achievable at receiver 1. For this rate point to be achievable also at receiver 2, following Lemma 1, the following conditions should be satisfied

$$I(U_1;Y) \leqslant I(U_1;V_1Z)$$
$$I(U_2;V_1V_2Y|U_1) \leqslant I(U_2;V_1Z|U_1)$$
$$I(V_1;U_1Y) \leqslant I(V_1;Z)$$
$$I(V_2;U_1Y|V_1) \leqslant I(V_2;U_1U_2Z|V_1). \tag{20}$$

By substituting the auxiliary random variables defined in (19) into (20), we obtain the conditions (18) on the channel parameters, under which the given boundary point is achievable over the interference channel with state.

It can also be shown that the capacity region of the strong Gaussian interference channel without state is an outer bound on the capacity of the corresponding channel with state. Hence, if a rate point is on the boundary of the capacity region for the channel without state, can be achieved by the channel with state, it is on the boundary of the capacity region for the channel with state. ■

Theorem 2 provides a computable way for checking whether any point on the sum rate boundary of the capacity of the interference channel without state is also on the capacity boundary for the corresponding channel with state under certain channel parameters. In Fig. 4, we plot the range of such channel parameters. In particular, we set $a = 1.2$, $P_1 = P_2 = 1$, and $a \leqslant b$. For this set of parameters, the capacity region of the interference channel without state is plotted Fig. 3. In Fig. 4, we plot the ranges of (b, c) under which the rate points

B_1, B_2 and B_3 are on the boundary of the capacity region of the interference channel with state. It can be seen that as the rate point gets closer to the corner point D, the range of the parameters becomes larger indicating that for a larger set of channel parameters, such point is on the capacity boundary for the channel with sate.

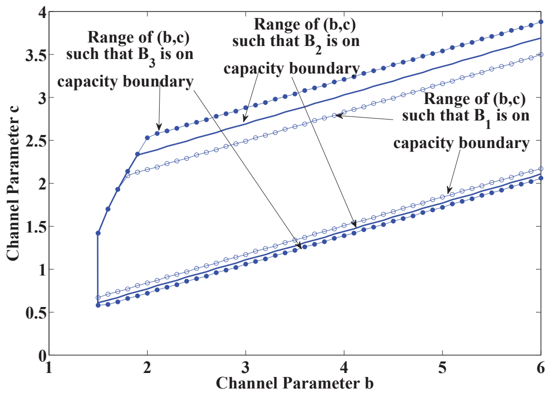

Fig. 4. Ranges of (b, c) under which certain points are on the boundary of the capacity region for the strong interference channel with state

Remark 2. *Following Theorem 2, we can also characterize conditions on the channel parameters, under which a portion of sum rate boundary of the capacity region of the channel without state (say, the line B_1-D) is on the capacity boundary of the channel with state, by taking intersections of the conditions for the point B_1 and the point D.*

Our numerical results also indicate that the range of parameters that guarantee the points B_1, B_2, B_3 to be on the capacity boundary of the channel with state also guarantee the lines B_1-D-E, B_2-D-E and B_3-D-E, correspondingly, are on the capacity boundary.

VI. Weak Gaussian Interference Channel with State

As having been shown in [9], [10] and [11], the sum capacity for the weak Gaussian interference channel can be achieved by treating interference as noise at each receiver, if the channel satisfies the condition $|a(1 + b^2P_1)| + |b(1 + a^2P_2)| \leqslant 1$. For the corresponding interference channel with state, the sum capacity remains the same.

Theorem 3. *For the weak interference channel with state noncausally known at both transmitters, if $|a(1 + b^2P_1)| + |b(1 + a^2P_2)| \leqslant 1$, the sum capacity is given by*

$$R_1 + R_2 \leqslant \frac{1}{2}\log\left(1 + \frac{P_1}{a^2P_2 + 1}\right) + \frac{1}{2}\log\left(1 + \frac{P_2}{b^2P_1 + 1}\right) \tag{21}$$

The idea to achieve the sum capacity is to apply dirty paper coding for X_1 treating $aX_2 + N_1$ as noise and dirty paper coding for X_2 treating $bX_1 + N_1$ as noise. Hence, the above sum capacity is achievable. Since it achieves the sum capacity for the interference channel without state, it can be shown that it is also optimal for the channel with state. The details of the proof is omitted due to the space limitations.

VII. Conclusion

In this paper, we have studied the two-user Gaussian interference channel with state noncausally known at both transmitters and not known at either receiver. The conditions on channel parameters are characterized, under which the capacity region, points on the boundary of the capacity region, or sum capacity are obtained. Future work includes generalizing the current study to none i.i.d. state sequences, studying the interference channel with causal state information, and applying the techniques developed in this paper to nonGaussian channels with state.

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