Interactive Interference Alignment

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***Abstract*—We study interference channels (IFC) where interaction among sources and destinations is enabled, e.g., both sources and destinations can talk to each other. The interaction can come in two ways: 1) for half-duplex radios, destinations can talk back to sources using either simultaneous out-of-band (white spaces) transmission or in-band half-duplex transmission; 2) for full-duplex radios, both sources and destinations can transmit and listen in the same channel simultaneously. The flexibility afforded by interaction among sources and destinations allows for the derivation of interference alignment (IA) strategies that have desirable "engineering properties": insensitivity to the rationality or irrationality of channel parameters, small block lengths and finite SNR operations. We show that for several classes of interference channels the interactive interference alignment scheme can achieve the optimal degrees of freedom.**

I. Introduction

Interference alignment is a novel interference management technique which makes concurrent wireless communications feasible [1], [2]. Since its introduction, substantial research has been conducted in understanding the gains of interference alignment. From a theoretical perspective, the focus has been mainly in understanding the degrees of freedom (first term in the high SNR approximation of the information theoretic capacity region). A seminal result, shown in [1], is that we can communicate at a total $\frac{K}{2}$ degrees of freedom for nearly all time-varying/frequency selective K-user interference channel (this is also an upper bound). To get to $\frac{K}{2}$ degrees of freedom, the scheme employed (vector space interference alignment) requires the channel diversity to be unbounded; in fact, one needs the channel diversity to grow like K^{2K^2} [3]. The diversity order required is huge even for modest number of users K (such as 3 and 4). This is definitely so in a practical communication system, where there is hardly enough channel diversity to make such a vector space interference alignment scheme feasible.

A practically relevant problem is to understand the fundamental degrees of freedom for a fixed deterministic channel. For fully connected channel matrix H, the total degrees of freedom are upper bounded by $\frac{K}{2}$ and [4] shows that this upper bound is achievable for almost all channel matrices H using a coding scheme based on Diophantine approximation. However, this result is limited in two ways. First, the coding scheme is very sensitive to whether entries of H are rational or irrational. Second, although it is provable $\frac{K}{2}$ degrees of freedom are achievable for almost all H, for a given H, in general it is not known what are the optimal achievable degrees of freedom. [5] shows that the $\frac{K}{2}$ result for almost all H can be derived using Rényi information dimension. Again, the result is sensitive to whether entries of H are rational or irrational, and for fixed channel matrix H, in general the optimal achievable degrees of freedom is unknown. The recent works of [6], [7] address this issue to a good extent for the case of the two-user X channel [1] and the symmetric K-user interference channel respectively , but the engineering implication of the proposed coding schemes remains unclear.

Therefore, despite significant theoretical progress on the K-user interference channel problem, it is still unclear how to make interference alignment practical. The drawbacks of existing schemes may be inherent to the channel model which assumes sources can only transmit and destinations can only listen, while in practice radios can both transmit and receive. We study new channel models where interaction among sources and destinations is enabled, e.g., both source and destination can talk to each other. The interaction can come in two ways: 1) for half-duplex radios, destinations can talk back to sources using simultaneous out-of-band transmission or in-band half-duplex transmission; 2) for full-duplex radios, both sources and destinations can transmit and listen in the same channel simultaneously. Although [10] shows that for interference channel, relays, feedback, and full-duplex operation cannot improve the degrees of freedom beyond $\frac{K}{2}$, we demonstrate that the interaction among sources and destinations enables flexibility in designing simple interference alignment scheme and in several cases achieves the optimal degrees of freedom.

Our main contribution is to propose a simple interference alignment scheme by exploiting the interactions among sources and destinations, and prove that the scheme can achieve the optimal degrees of freedom for several classes of interference channels, including half-duplex 3-user IFC, full-duplex 3-user and 4-user IFC, and full-duplex 4-user MIMO IFC. One specific aspect of our model, namely, feedback using the reciprocal interference channel, has been considered in prior work [11]. In this work, we improve on this state-of-the-art in two ways: we prove new results for this specific model and also generalize this model to exploit more general modes of interaction in order to achieve interference alignment. In addition, we do extensive numeric simulations and show the

[1]The case that the channel coefficients are complex or time-varying has been settled previously in [8], [9].

proposed interactive interactive alignment scheme also works for some other classes of IFC empirically. We use tools from algebraic geometry to show why success of numeric simulations can suggest that the scheme should work well for almost all channel parameters in a rigorous way.

The paper is organized as follows. We describe the channel model in Section II, present the interactive communication scheme for half-duplex radios and interference alignment conditions in Section III, and our main technical results from algebraic geometry on the interference alignment feasibility in Section IV. Section V and Section VI study interactive interference alignment for half-duplex and full-duplex K-user interference channels with small K, respectively. Section VII concludes this paper.

II. System Model

Consider a K-user interference channel with $2K$ radios, where radios $s_1, s_2, \ldots, s_K$ are sources and radios $t_1, t_2, \ldots, t_K$ are destinations. For each $1 \leq i \leq K$, source s_i wants to send an independent message W_i to destination t_i.

Let $H \in \mathbb{C}^{K\times K}$ denote the forward channel matrix from sources to destinations, and the input and output signals of the forward channel are related as

$$y_i[n] = \sum_{j=1}^{K} H_{ij}x_j[n] + z_i[n], \tag{1}$$

where n is the time index, $y_i[n]$ is the signal received by destination t_i at time n, $x_j[n]$ is the signal sent out by source s_j at time n, $z_i[n]$ is the channel noise, and H_{ij} is the channel coefficient from source s_j to destination t_i.

The above is a canonical channel model for K-user interference channel.

In this work, we consider two channel models where interaction among sources and destinations can be enabled.

A. Interaction from destinations to sources via reverse transmission for half-duplex radios

Since in practice radios can both actively talk (send signals) or passively listen (receive signals), we assume destinations can also actively talk back to sources either via simultaneous out-of-band (white spaces) transmission or via in-band half-duplex transmission. Let $G \in \mathbb{C}^{K\times K}$ denote the feedback channel (or reverse channel) matrix from destinations to sources. Similarly, the input-output relation of the feedback channel is

$$f_i[n] = \sum_{j=1}^{K} G_{ij}v_j[n] + \tilde{z}_i[n], \tag{2}$$

where n is the time index, $v_j[n]$ is the signal sent out by destination t_j at time n, $f_i[n]$ is the signal received by source s_i at time n, $\tilde{z}_i[n]$ is the channel noise, and G_{ij} is channel coefficient from destination t_j to source s_i.

If the destinations use in-band half-duplex transmissions, due to the reciprocity of wireless channels, we have

$$G = H^T, \tag{3}$$

where H^T denotes the transpose of H.

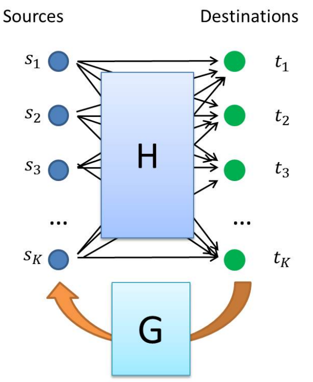

Fig. 1. System Model For Half-duplex Radios.

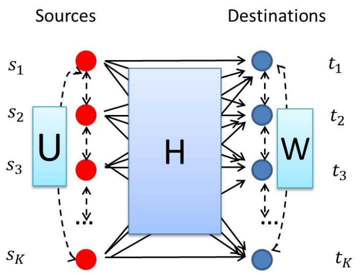

Fig. 2. System Model For Full-duplex Radios.

B. Interaction via transmitting and receiving simultaneously for full-duplex radios

Full-duplex radios can both send and receive signals using the same channel simultaneously, and this capability naturally enables the interaction among all radios in the network.

We assume that all sources and destinations have full-duplex antennas, and all nodes can transmit and receive signals using the same channel simultaneously. Let $H \in \mathbb{C}^{K\times K}$ be the channel matrix from sources to destinations, $U \in \mathbb{C}^{K\times K}$ be the channel matrix among sources, and $W \in \mathbb{C}^{K\times K}$ be the channel matrix among destinations. The input-output relation of this full-duplex communication channel is

$$y_i[n] = \sum_{j=1}^{K} H_{ij}x_j[n] + \sum_{j=1}^{K} W_{ij}v_j[n] + z_i[n], \tag{4}$$

$$f_i[n] = \sum_{j=1}^{K} H_{ji}v_j[n] + \sum_{j=1}^{K} U_{ij}x_j[n] + \tilde{z}_i[n], \tag{5}$$

where n is the time index, $x_j[n]$ and $v_j[n]$ are the signals sent out by source s_j and destination t_j at time n, respectively, $y_i[n]$ and $f_i[n]$ are the signal received by destination t_i and source s_i at time n, $z_i[n]$ and $\tilde{z}_i[n]$ are the channel noise, H_{ij} is channel coefficient from s_j to t_i and also the channel coefficient from t_i to s_j due to channel reciprocity, W_{ij} is the channel coefficient from r_j to r_i, and U_{ij} is the channel coefficient from s_j to s_i.

III. Communication Scheme and Interference Alignment Conditions

In this section we present a natural communication scheme for the half duplex models and derive the interference alignment conditions.

Consider the following natural three-phase interactive communication scheme:

- Phase 1 (forward transmission): All sources send their independent symbols simultaneously. And destinations get $y = Hx + n$, where n is the additive noise of the channel.
- Phase 2 (interaction from destinations): After receiving signals from sources in Phase 1, all destinations scale y and send back to sources using the reverse channel. sources get $f = GD_1 y + \tilde{n}$, where $\tilde{n}$ is the additive noise of the channel. Since each source and each destination only knows what signals they sent out and received, the coding matrix D_1 has to be diagonal.
- Phase 3 (forward transmission): Now each source has two sets of signals x and f, and sources can send out a linear combination of x and f to destinations via the forward channel. destinations get

$$y' = H(D_2 x + D_3 f) + \hat{n} \tag{6}$$

$$= (HD_2 + HD_3 GD_1 H)x + \text{noise}. \tag{7}$$

Again, D_2, D_3 are constrained to be diagonal.

After the three-phase communications, each destination t_i can decode x_i by taking a linear combination of y_i and y'_i to cancel the interference terms. To achieve this, the interference terms in y' should be aligned with the interference terms in y, and the signal terms are not aligned with interference terms.

More precisely, let $B \triangleq HD_2 + HD_3 GD_1 H$. Then for destination t_i, in Phase 1 it receives

$$y_i = H_{ii} x_i + \sum_{j \neq i} H_{ij} x_j + \text{noise}, \tag{8}$$

and in Phase 3 it receives

$$y'_i = B_{ii} x_i + \sum_{j \neq i} B_{ij} x_j + \text{noise}. \tag{9}$$

If for all $1 \leq i \leq K$, we have

$$\frac{B_{ij}}{H_{ij}} = \frac{B_{ij'}}{H_{ij'}} = \lambda_i, \quad \forall j, j' \neq i, \tag{10}$$

ant λ_i, then interference terms of y and y' for each destination are aligned with each other. So destination t_i can compute $y_i - \lambda_i y'_i$ and get

$$(H_{ii} - \lambda_i B_{ii}) x_i + \text{noise}. \tag{11}$$

Therefore, if $(H_{ii} - \lambda_i B_{ii}) \neq 0$, or equivalently,

$$\frac{B_{ii}}{H_{ii}} \neq \frac{B_{ij}}{H_{ij}}, \quad \forall j \neq i, \tag{12}$$

then destination t_i can get the desired signal sent by source s_i without any interference.

Therefore, to make the three-phase communication scheme work and thus achieve the optimal $\frac{K}{2}$ degrees of freedom, we need to solve $K(K-2)$ interference alignment equations with $3K$ coding variables and check that whether the solution satisfies the K inequalities.

In some cases (e.g., K = 3), we can reduce the polynomial equations corresponding to interference alignment equality constraints to linear equations, the solution of which has a closed-form expression and thus it can be easily verified whether the inequality constraints are satisfied. However, in general the system of polynomial equations is nonlinear, and it may not have closed-form solution, making it hard to check whether the inequality constraint can be satisfied. Our main approach is to convert the system of polynomial equations and polynomial inequalities to a system of polynomial equations, and then use tools from algebraic geometry to check the existence of solution for the system of polynomial equations.

It is easy to reduce the problem to checking the existence of solutions to a system of polynomial equations. Let $f_i = 0, 1 \leq i \leq N, g_j \neq 0, 1 \leq j \leq M$ be the polynomial equations and inequalities over variables $\{d_1, d_2, \ldots, d_S\}$. By introducing an auxiliary variable t, there exists $(d_1, d_2, \ldots, d_S)$ satisfying $f_i = 0, g_j \neq 0, \forall i, j$ if and only if there exists $(d_1, d_2, \ldots, d_S, t)$ satisfying $f_i = 0, \forall i$, and $t \prod_{j=1}^{M} g_j - 1 = 0$.

IV. General Solution Methodology

In this section we present our main technical results on checking the existence of solutions to the interference alignment equations for generic channel parameters H and G using tools in algebraic geometry. For more details on algebraic geometry, we refer the reader to Appendix B of [12] and the excellent textbook [13].

As discussed in Section III, our problem can be reduced to checking the existence of solutions to a system of polynomial equations. In the language of algebraic geometry, the set of solutions of polynomial equations is called affine variety, and checking whether an affine variety is empty or not is well studied for algebraically closed field. In wireless communication, the channel coefficients are represented as complex numbers $\mathbb{C}$, which is an algebraically closed field.

The standard approach to checking whether an affine variety is an empty set is to use Buchberger's algorithm to compute the Gröbner basis of the given polynomials, and from the Gröbner basis we can easily conclude whether the corresponding affine variety is empty or not. The affine variety is nonempty if and only if the Gröbner basis does not contain a nonzero constant polynomial.

One important implication of these results in algebraic geometry is

Theorem 1. *Let $f_1, \ldots, f_s$ be polynomials in $k[\xi_1, \xi_2, \ldots, \xi_n]$, where $k = \mathbb{C}$ and all coefficients of the polynomials are rational functions of variables $h_1, h_2, \ldots, h_m$. Then there exists a nontrivial polynomial equation on $h_1, h_2, \ldots, h_m$, denoted by $e(h_1, h_2, \ldots, h_m)$, such that except the set of*

$\{(h_1, h_2, \ldots, h_m) \mid e(h_1, h_2, \ldots, h_m) = 0\}$, *either for all* $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1\times m}$, $V(f_1, \ldots, f_s) \neq \emptyset$, *or for all* $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1\times m}$, $V(f_1, \ldots, f_s) = \emptyset$.

Proof: See Appendix C of [12]. ∎

For the polynomial equations describing the interference alignment problem, the coefficients of the polynomials are rational functions of channel parameters H and G in symbolic form. Therefore, in the context of the interference alignment feasibility problem, this main result can be restated as follows. *Either one* of the following two statements hold:

- For almost all[2] channel realizations of H and G, there exists solution to the system of interference alignment equations.
- Or, for almost all channel realizations of H and G, there does not exist solution to the system of interference alignment equations.

Although in theory we can compute in finite number of steps the symbolic Gröbner basis for these polynomials with symbolic coefficients to check whether for almost all H and G there exists solutions, it turns out to be computationally infeasible to run the Buchberger's algorithm for the symbolic polynomial equations for most of our interference alignment problems, due to the fact that the orders of intermediate symbolic coefficients can increase exponentially. However, it is much easier to compute a Gröbner basis numerically. Due to Theorem 1,

Corollary 1. *If we draw the channel parameters according to a continuous probability distribution, then with probability one the numeric polynomial equations have a solution if and only if for almost all channel realizations the polynomials equations have solution.*

Hence, while we may not be able to prove that certain polynomial equations have solution for almost all channel parameters because of computational difficulty, numeric simulations can let us make claims with high credibility.

V. Interactive Interference Alignment for small K with half duplex radios

In this section, we study interactive interference alignment for half-duplex K-user interference channel with small K.

A. $K = 3$

Theorem 2 **([11]).** *For 3-user interference channel with reciprocal feedback channel ($G = H^T$), and for generic channel matrix H, there exists solutions to the interference alignment equations and inequalities.*

[11] proves Theorem 2 by giving one closed-form solution to the interference alignment equations and verifying that it also satisfies the inequalities. An alternative approach is to reduce the polynomial equations to linear equations and then use dimension argument, which is given in [12]. The above result also holds for generic reverse channel.

Theorem 3. *For 3-user interference channel with out-of-band feedback channel, for generic channel matrices H and G, there exists solutions to the interference alignment equations and inequalities.*

B. $K = 4$

For 4-user interference channel, we have $K(K-2) = 8$ polynomial equation constraints and $K = 4$ inequality constraints with $3K = 12$ variables. The polynomial equations are strictly nonlinear and may not be reduced to linear equations, which makes it hard to derive closed-form solution and then verify the inequalities. Instead, we convert the system of polynomial equations and polynomial inequalities to a system of polynomial equations, and then we check whether it has solutions or note by computing the Gröbner basis of corresponding polynomials.

Due to computational difficulty, it is hard to compute the Gröbner basis for the symbolic polynomials. We do extensive numeric verifications: each time we generate random numeric channel matrices, and then apply Buchberger's algorithm to compute a Gröbner basis. In all numeric instances, the Gröbner basis does not contain a nonzero scalar, which implies that solution to (10) and (12) exists. Therefore, due to Corollary 1, although we cannot compute a symbolic Gröbner basis explicitly, from numeric verifications we believe the following claim hold.

Claim 1. *For 4-user interference channel, for generic matrix H, and generic matrix G or $G = H^T$, there exists D_1, D_2 and D_3 such that each row of $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries.*

[11] formulates the interference alignment problem as a rank constrained nonconvex optimization problem, and reports that all numeric simulations confirm the existence of solutions to the interference alignment equations and inequalities.

In some cases when H has special structures, we are able to explicitly compute the symbolic Gröbner basis of the interference alignment polynomials, and thus we can conclude whether for almost all H with such structure there exists interference alignment solutions. One positive example is that if H is a symmetric matrix and all diagonal entries are zero, then for almost all such H, the interference alignment equations are solvable. Due to space limit, we refer the readers to Theorem 4 of [12].

C. $K = 5$ *and* $K = 6$

For $K = 5$ and $K = 6$, numeric simulations suggest that for almost all H and G there is no solutions satisfying both interference alignment equations and inequalities. However, if two reverse transmissions are allowed in the second phase, numeric simulations suggest there exist interference alignment solutions. We refer the readers to Section VI.C of [12] for more details.

[2] We emphasize that in our results "almost all" means for all numeric values except a set of parameters which satisfy a nontrivial polynomial equation. Therefore, in contrast to results in [5] and [4], our results are not sensitive to whether the channel parameters are rational or irrational.

VI. Interactive Interference Alignment for small K with full duplex radios

In this section, we study how to exploit interaction in interference channel where all sources and destinations have full-duplex antennas so that all nodes can transmit and receive in the same band simultaneously. We will show how full-duplex operations can help align interferences and achieve the optimal degrees of freedom for some classes of interference channels.

A. SISO IFC with small K

Recall in the system model of full-duplex IFC, H is the channel matrix from sources to destination, U is the channel matrix among sources, and W is the channel matrix among destinations. Consider the following simple two-phase transmission scheme.

- Phase 1: All sources send out signals x simultaneously. After the transmission, sources get $f = Ux + \text{noise}$, and destinations get $y = Hx + \text{noise}$.
- Phase 2: sources send out a linear combination of x and f, and destinations send out scaled version of y. Therefore, destinations get $H(D_1 x + D_2 f) + W D_3 y + \text{noise} = (HD_1 + HD_2 U + W D_3 H)x + \text{noise}$, where D_1, D_2 and D_3 are diagonal coding matrices.

Note that in this transmission scheme no feedback channel from destinations to sources is required. Similarly, if each row of $(HD_1 + HD_2 U + W D_3 H)$ is proportional to the corresponding row of H except all diagonal entries, then interferences at all destinations are aligned and all destinations can retrieve their designed signals from sources without any interference, and thus achieve the optimal $\frac{K}{2}$ degrees of freedom. Our first result is that the above two-phase transmission scheme works for $K = 3$ and $K = 4$.

Theorem 4. *For $K = 3$ and $K = 4$, for generic channel matrices U, H and W, there exists diagonal matrices D_1, D_2 and D_3 such that each row of $(HD_1 + HD_2 U + W D_3 H)$ is proportional to the corresponding row of H except all diagonal entries.*

Since the IA equations are linear in terms of D_1, D_2 and D_3. One can use dimension argument to prove it. Furthermore, using the same proof technique, we can show that the above scheme does not work for $K = 5$ or bigger.

B. MIMO IFC with $K = 4$

For K-user MIMO interference channel where all sources and destinations are equipped with M antennas, [1] shows that for $K = 3, M > 1$, vector space interference alignment can achieve the optimal $\frac{KM}{2} = \frac{3M}{2}$ degrees of freedom and channel diversity is not required. [3] and [14] prove that in general vector space MIMO interference alignment can at most get $\frac{2KM}{K+1}$ degrees of freedom, which is strictly less than the optimal $\frac{KM}{2}$ when $K \geq 4$. We show that for four-user MIMO interference channel with full-duplex antennas, a simple two-phase transmission scheme can achieve the optimal $2M$ degrees of freedom at least for all $M \leq 15$.

Theorem 5. *For 4-user MIMO interference channel where each node have M full-duplex antennas, then for generic matrices H, U and W, the two-phase scheme can achieve the optimal $\frac{K}{2}$ degrees of freedom, at least for $M \leq 15$.*

We refer the readers to Section VI.B of [12] for the proof.

VII. Conclusion

We study new channel models where interaction among sources and destinations is enabled, and show that how interaction among sources and destinations enables flexibility in designing simple interference alignment scheme and in several cases achieves the optimal degrees of freedom.

We present a three-phase interactive communication scheme for half-duplex K-user interference channel with small K, a two-phase interference communication scheme for full-duplex K-user interference channel with small K, and show that the interactive interactive alignment scheme can achieve the optimal degrees of freedom for several classes of IFC. We use tools from algebraic geometry to show why success of numeric simulations can suggest the scheme should work well for almost all channel parameters in a rigorous way.

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