Integrable Communication Channels and the Nonlinear Fourier Transform

Mansoor I. Yousefi and Frank R. Kschischang
Dept. of Electrical & Computer Engineering, University of Toronto
Email: {mansoor,frank}@comm.utoronto.ca

***Abstract*—This paper considers the transmission of information over integrable channels, a class of (mainly nonlinear) channels described by a Lax operator-pair. For such channels, the nonlinear Fourier transform, a powerful tool in soliton theory and exactly solvable models, plays the same role in "diagonalizing" the channel that the ordinary Fourier transform plays for linear convolutional channels. A transmission strategy encoding information in the nonlinear Fourier spectrum, termed nonlinear frequency-division multiplexing, is proposed for integrable channels that is the nonlinear analogue of orthogonal frequency-division multiplexing commonly used in linear channels. A central and motivating example is fiber-optic data transmission, for which the proposed transmission technique deals with both dispersion and nonlinearity directly and unconditionally without the need for dispersion or nonlinearity compensation methods.**

I. Introduction

The goal of this paper is to introduce *integrable channels* and develop a method similar to the orthogonal frequency-division multiplexing (OFDM) for data transmission over these channels. These are channels generated by a pair of Lax operators (L, M). By various choices of operators L and M one can construct a variety of interesting channel models, mostly nonlinear, which go beyond the linear channel models commonly studied in data communications.

Integrable channels are typically described by an evolution equation, *i.e.*, a partial differential equation (PDE) for a complex-valued function $q(t, z)$ of the form

$$q_z = K(q), \tag{1}$$

where $K(q)$ is an expression involving q and its derivatives with respect to the components of t, and z is a scalar. The channel input and output are, respectively, $q(t, 0)$ and $q(t, \mathcal{L})$, for some $z = \mathcal{L}$. In (1) and throughout this paper, subscripts are used to denote partial derivatives with respect to the corresponding variable. In mathematical physics, the parameter z often denotes time, and t is a vector of n spatial variables, in which case one has a *temporal* evolution equation in $n + 1$ dimensions.

In the example that motivates this work, namely fiber-optic communications, t is a scalar denoting time, and z is a spatial parameter denoting distance along the fiber from the transmitter, and therefore one has a *spatial* evolution equation in $1 + 1$ dimensions. In the pertinent evolution equation—the nonlinear Schrödinger (NLS) equation—one has, in the absence of noise and after suitable normalization, $K(q) = -jq_{tt} - 2j|q|^2 q$; this evolution equation is known to be integrable [1]. Here the signal degrees-of-freedom couple together via the nonlinearity $-2j|q|^2 q$ and dispersion $-jq_{tt}$ in a complicated manner. Most current approaches to fiber-optic data transmission assume a linearly-dominated regime of operation, consider the nonlinearity as a small perturbation, or are geared towards managing and suppressing the (detrimental) effects of the nonlinear and dispersive terms. Inline dispersion management, digital backpropagation, and other forms of electronic pre-and post-compensation belong to this class of methods (see *e.g.*, [2] and references therein).

In this paper we adopt a different philosophy. Rather than treating nonlinearity and dispersion as nuisances, we seek a transmission scheme that exploits integrability and is fundamentally compatible with these effects. We develop a transmission approach for integrable channels in general that effectively "diagonalizes" such a channel with the help of the nonlinear Fourier transform (NFT), a powerful tool for solving *integrable* nonlinear dispersive PDEs [3], [4]. The NFT uncovers linear structure hidden in this class of channels and can be viewed as a generalization of the (ordinary) Fourier transform to such nonlinear systems.

The NFT represents a signal by its discrete and continuous nonlinear spectra. When a signal is transmitted over a (potentially complicated) integrable channel, the action of the channel on its spectral components is given by simple independent linear equations. Just as the (ordinary) Fourier transform converts a linear convolutional channel $y(t) = x(t) * h(t)$ into a number of parallel scalar channels, the NFT converts a nonlinear dispersive channel described by a *Lax convolution* (see Sec. II) into a number of parallel scalar channels. This suggests that information can be encoded (in analogy with orthogonal frequency-division multiplexing) in the nonlinear spectra.

The mathematical tools presented here are also described in mathematics and physics (see, *e.g.*, [4]–[6]); here we attempt to extract those aspects of the theory that are relevant for information transmission.

II. Canonical Lax Form for Integrable Models

A. Lax Pairs for Evolution Equations

We wish to consider linear differential operators defined in terms of a signal $q(t, z)$ whose spectrum are invariant even as q evolves in z (according to some evolution equation) [7]. To facilitate the discussion, it is useful to imagine a linear operator represented as a matrix.

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{D}$ be some domain that is dense in $\mathcal{H}$, and let $L(z) : \mathcal{D} \to \mathcal{H}$ be a family of bounded linear operators indexed by a parameter z. Clearly, the spectrum of $L(z)$ is in general a function of z too. However, for some operators, it might be the case that while the operator changes with z, its spectrum remains constant (independent of z). We refer to such $L(z)$ as an *isospectral* family of operators.

If diagonalizable, it follows that for each z, $L(z)$ is similar to a multiplication operator Λ, *i.e.*, $L(z) = G(z)\Lambda G^{-1}(z)$, for some operator $G(z)$. Assuming that $L(z)$ varies smoothly with z, we have

$$\begin{aligned}\frac{\mathrm{d}L(z)}{\mathrm{d}z} &= G'\Lambda G^{-1} + G\Lambda\left(-G^{-1}G'G^{-1}\right)\\ &= G'G^{-1}\left(G\Lambda G^{-1}\right) - \left(G\Lambda G^{-1}\right)G'G^{-1}\\ &= M(z)L(z) - L(z)M(z) = [M, L]\,, \end{aligned} \tag{2}$$

where $G' = \mathrm{d}G(z)/\mathrm{d}z$, $M = G'G^{-1}$, and $[M, L] \triangleq ML - LM$ is the *commutator bracket*. In other words, every diagonalizable isospectral operator $L(z)$ satisfies the differential equation (2).

The converse can be shown to hold as well, giving rise to the characterization of isospectral operators in the following lemma [7].

Lemma 1. *Let $L(z)$ be a diagonalizable operator. Then $L(z)$ is isospectral if and only if it satisfies*

$$\frac{\mathrm{d}L}{\mathrm{d}z} = [M, L], \tag{3}$$

for some operator M. If L is self-adjoint (so that L is unitarily equivalent to a multiplication operator, i.e., $L = G\Lambda G^*$), then M must be skew-Hermitian, i.e., $M^* = -M$.

Proof: The proof of the forward part was outlined above; see [8] for the converse. The skew-Hermitian property of M can be shown by differentiating $GG^* = I$. ∎

It is important to note that L and M do not have to be independent and can depend on a common parameter, *e.g.*, a function $q(t, z)$, as illustrated in Fig. 1. The isospectral property of the solution is unchanged. The commutator bracket $[M, L]$ in (3) can create nonlinear evolution equations for $q(t, z)$ in the form

$$\frac{\partial q}{\partial z} = K(q),$$

where $K(q)$ is some, in general nonlinear, function of $q(t, z)$ and its time derivatives. An example of this is the Korteweg de-Vries (KdV) equation.

Example 1 (KdV Equation). Let $D = \frac{\partial}{\partial t}$ denote the time-derivative operator, and let $q(t, z)$ be a real-valued function. Finally, let

$$L = D^2 + \frac{1}{3}q, \quad M = 4\left(D^3 + \frac{1}{4}Dq + \frac{1}{4}qD\right).$$

The Lax equation $L_z = [M, L]$ is easily simplified to

$$\frac{1}{3}q_z - \frac{1}{3}\left(q_{ttt} + qq_t\right) + (\text{some terms})D \equiv \mathbf{0},$$

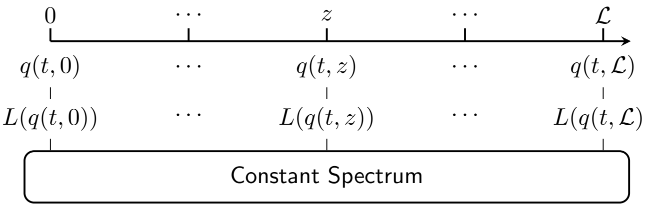


Fig. 1. An isospectral flow: the spectrum of L is held invariant even as $q(t, z)$ evolves.

where $\mathbf{0}$ is the zero operator. The zero-order term of this equation as a polynomial in D, which must be zero, produces the KdV equation $q_z = q_{ttt} + qq_t$. From Lemma 1, eigenvalues of L are preserved if q evolves according to the KdV equation. □

Definition 1 (Lax Pair)**.** A pair of operators L and M, depending on z, are called a Lax pair (L, M) if they satisfy (3). Following Lemma 1, the eigenvalues of the L operator are independent of z.

Following the work of Zakharov and Shabat on the NLS equation [1], Ablowitz et al. [3] suggested that for many equations of practical significance, the operator L can be fixed as

$$L = j\begin{pmatrix} D & -r(t, z)\\ s(t, z) & -D\end{pmatrix}, \tag{4}$$

where $r(t, z)$ and $s(t, z)$ are functions—depending on $q(t, z)$—to be determined to produce a given nonlinear evolution equation.

Example 2 (Nonlinear Schrödinger Equation). Take $r = q$, $s = -q^*$ and

$$M = \begin{pmatrix} 2j\lambda^2 - j|q(t, z)|^2 & -2\lambda q(t, z) - jq_t(t, z)\\ 2\lambda q^*(t, z) - jq_t^*(t, z) & -2j\lambda^2 + j|q(t, z)|^2\end{pmatrix}.$$

The Lax equation is simplified to the NLS equation $jq_z(t, z) = q_{tt}(t, z) + 2|q(t, z)|^2 q(t, z)$. □

B. Lax Convolution and Integrable Communication Channels

We wish to define a system in terms of a Lax pair (L, M). Here, L and M are parametrized by a waveform $q(t, z)$. Such a system accepts a waveform $x(t) = q(t, 0)$ at its input and produces a waveform $y(t) = q(t, \mathcal{L})$ at its output, according to the evolution equation induced by $L_z = [M, L]$. The time-domain input-output map is thus given by an evolution equation of the form $q_z = K(q)$, obtainable from the Lax equation (3). We refer to such a system as an *integrable system*. Note that an integrable system is completely characterized by the two operators (L, M) and the parameter $z = \mathcal{L}$, independent of the signals. We denote such a system using the triple $(L, M; \mathcal{L})$.

Definition 2 (Lax Convolution)**.** We refer to the action of an integrable system $S = (L, M; \mathcal{L})$ on the input $q(t, 0)$ as the Lax convolution of q with S. We write the system output as $q(t, \mathcal{L}) = q(t, 0) \circledast (L, M; \mathcal{L})$.

Definition 3 (Integrable Communication Channels)**.** A waveform communication channel $C: x(t) \times n(t,z) \to y(t)$ with inputs $x(t) \in L^1(\mathbb{R})$ and noise $n(t,z) \in L^2(\mathbb{R}, \mathbb{R}^+)$, and output $y(t) \in L^1(\mathbb{R})$, is said to be *integrable* if the noise-free channel is an integrable system.

The noise term $n(t,z)$ can be introduced to the system in a variety of ways, *e.g.*, additively in the form of $q_z = K(q) + n(t,z)$. In some cases channel is integrable in the presence of the noise. In other cases, noise breaks the integrability structure; here we require noise to be small so that it acts as a small perturbation to the integrability.

III. Nonlinear Fourier Transform

In this section, we assume that a function $q(t,\cdot)$ is given, and we define its nonlinear Fourier transform with respect to a Lax operator L. The variable z is irrelevant in the forward and inverse transforms and is omitted in this section. We assume that 1) $q(t) \in L^1(\mathbb{R})$, and 2) $\lim_{t\to\pm\infty} q(t) = 0$.

The nonlinear Fourier transform is defined via the spectral analysis of the L operator. The eigenproblem $Lv = \lambda v$ for the operator (4) can be rewritten as

$$v_t = P(\lambda, q)v = \begin{pmatrix} -j\lambda & q(t,z) \\ -q^*(t,z) & j\lambda \end{pmatrix} v. \tag{5}$$

A. Nonlinear Fourier Coefficients

We wish to study solutions of (5), in which vectors $v(t)$ are elements of the vector space $\mathcal{H}$. We begin by equipping the vector space $\mathcal{H}$ with a symplectic bilinear form $\mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$, which, for any *fixed* value of $t \in \mathbb{R}$, is defined as

$$\langle v(t), w(t) \rangle_s = v_1(t)w_2(t) - v_2(t)w_1(t).$$

Let us also define the adjoint of any vector v in H as $\tilde{v}(t) = [v_2^*(t), -v_1^*(t)]^T$. There are generally infinitely many solutions v of (5) for a given $\lambda \in \mathbb{C}$, parametrized by the set of all possible boundary conditions. These solutions form a subspace E_λ of continuously differentiable 2×1 vector functions (an eigenspace).

Lemma 2. *For all vectors $v(t)$ and $w(t)$ in E_λ,*

1) $\tilde{v} \in E_{\lambda*}$, *i.e.,* $\tilde{v}_t = P(\lambda^*, q)\tilde{v}$;
2) $\langle v(t), w(t) \rangle_s$ *is a constant, independent of t;*
3) *If $\langle v(t), w(t) \rangle_s \neq 0$, then v and w are linearly independent and form a basis for E_λ;*
4) $\dim(E_\lambda) = 2$.

Proof: Property 1) follows directly from (5). To see 2), note that $\frac{\mathrm{d}}{\mathrm{d}t}\langle v, w\rangle_s = \langle v_t, w\rangle_s + \langle v, w_t\rangle_s = \langle Pv, w\rangle_s + \langle v, Pw\rangle_s = \mathrm{tr}(P)\langle v, w\rangle_s = 0$ (by direct calculation). To see 3), fix t and let $u(t) \in E_\lambda$, then $u(t) = a(t)v(t) + b(t)w(t)$ for some $a(t)$ and $b(t)$. Taking the symplectic inner product of both sides with w and v, we get $a(t) = \langle u, w\rangle_s / \langle v, w\rangle_s$ and $b(t) = \langle u, v\rangle_s / \langle w, v\rangle_s$. From Property 2, $\langle u, w\rangle_s$, $\langle v, w\rangle_s$, $\langle u, v\rangle_s$, and $\langle w, v\rangle_s$ are all independent of t. It follows that a and b are also independent of t. Finally, 4) follows from 3). □

An important conclusion of Lemma 2 is that any two linearly independent solutions u and w of (5) provide a basis for the solution space. Using the assumption $q(\pm\infty) = 0$ in (5), two possible boundary conditions, bounded in the upper half complex plane $\mathbb{C}^+ = \{\lambda | \Im(\lambda) > 0\}$, are

$$v^1(t,\lambda) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{j\lambda t}, \quad t \to +\infty, \tag{6}$$

$$v^2(t,\lambda) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-j\lambda t}, \quad t \to -\infty. \tag{7}$$

We solve (5) for $\lambda \in \mathbb{C}^+$ under the boundary conditions (6)–(7), and denote the resulting eigenvectors for all $t \in \mathbb{R}$ as $v^1(t,\lambda)$ and $v^2(t,\lambda)$. We can also solve (5) for λ^* under the boundary conditions $v^1(\infty, \lambda^*)$ and $v^2(-\infty, \lambda^*)$, giving rise to two more solutions $v^1(t,\lambda^*)$ and $v^2(t,\lambda^*)$. From Lemma 2, the adjoint eigenvectors $\tilde{v}^1(t,\lambda^*)$ and $\tilde{v}^2(t,\lambda^*)$ are elements of E_λ. These four eigenvectors $v^1(t,\lambda)$, $v^2(t,\lambda)$, $\tilde{v}^1(t,\lambda^*)$, and $\tilde{v}^2(t,\lambda^*)$, all of them elements of E_λ, are called *canonical eigenvectors*.

Lemma 3. *Canonical eigenvectors satisfy:*

1) $\langle \tilde{v}^1(t,\lambda^*), v^1(t,\lambda)\rangle_s = \langle \tilde{v}^2(t,\lambda^*), v^2(t,\lambda)\rangle_s = 1$;
2) $\{v^1(t,\lambda), \tilde{v}^1(t,\lambda^*)\}$ *and* $\{v^2(t,\lambda), \tilde{v}^2(t,\lambda^*)\}$ *are independent sets in E_λ.*

Proof: 1) Since $\langle \tilde{v}^1, v^1\rangle_s$ is independent of t, using (6) and its adjoint, $\langle \tilde{v}^1(t,\lambda^*), v^1(t,\lambda)\rangle_s = \langle \tilde{v}^1(+\infty,\lambda^*), v^1(+\infty,\lambda)\rangle_s = 1$. 2) Follows from 1) and Lemma 2. □

Choosing $\tilde{v}^1(t,\lambda^*)$ and $v^1(t,\lambda)$ as a basis of E_λ, one can project $v^2(t,\lambda)$, $\tilde{v}^2(t,\lambda^*) \in E_\lambda$ on this basis to obtain

$$\begin{aligned} v^2(t,\lambda) &= a(\lambda)\tilde{v}^1(t,\lambda^*) + b(\lambda)v^1(t,\lambda), \\ \tilde{v}^2(t,\lambda^*) &= b^*(\lambda^*)\tilde{v}^1(t,\lambda^*) - a^*(\lambda^*)v^1(t,\lambda), \end{aligned} \tag{8}$$

where $a(\lambda) = \langle v^2, v^1\rangle_s$ and $b(\lambda) = \langle \tilde{v}^1, v^2\rangle_s$. A crucial property, following from Lemma 2, is that $a(\lambda)$ and $b(\lambda)$ are time-independent. The time-independent complex scalars $a(\lambda)$ and $b(\lambda)$ are called the *nonlinear Fourier coefficients* [4], [6].

B. The Nonlinear Fourier Transform

The boundary condition of eigenvectors v^1 and $\tilde{v}^1$, respectively, decay and blow up at $t = \infty$, for $\lambda \in \mathbb{C}^+$. As a result, (8) is consistent in $\mathbb{C}^+$ only if $a(\lambda) = 0$. Eigenvalues are therefore identified as the zeros of the complex function $a(\lambda)$ in $\mathbb{C}^+$. It can be shown that if $q(t) \in L_1(\mathbb{R})$, $a(\lambda)$ is an analytic function of λ on $\mathbb{C}^+$, and consequently has isolated zeros there.

It follows that the Zakharov-Shabat operator for the NLS equation has two types of spectra: a discrete (or point) spectrum, which occurs in $\mathbb{C}^+$, characterized by those $\lambda_j \in \mathbb{C}^+$ satisfying $a(\lambda_j) = 0$, $j = 1, 2, \ldots, N$, and a continuous spectrum, which in general includes the whole real line $\Im(\lambda) = 0$.

Definition 4 (Nonlinear Fourier Transform [5], [6])**.** Let $q(t)$ be a sufficiently smooth function in $L^1(\mathbb{R})$. The nonlinear Fourier transform of $q(t)$ with respect to the Lax operator L (4) consists of the continuous and discrete spectral functions $\hat{q}(\lambda): \mathbb{R} \mapsto \mathbb{C}$ and $\tilde{q}(\lambda_j): \mathbb{C}^+ \mapsto \mathbb{C}$ where

$$\hat{q}(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad \tilde{q}(\lambda_j) = \frac{b(\lambda_j)}{a'(\lambda_j)}, \quad j = 1, 2, \ldots, N,$$

in which λ_j are the zeros of $a(\lambda)$. Here, the nonlinear Fourier coefficients $a(\lambda)$ and $b(\lambda)$ are given by

$$a(\lambda) = \lim_{t\to\infty} v_1^2 e^{j\lambda t}, \quad b(\lambda) = \lim_{t\to\infty} v_2^2 e^{-j\lambda t},$$

where $v^2(t,\lambda)$ is a solution of (5) under the boundary condition (7). □

Just like the ordinary Fourier transform, the nonlinear Fourier transform can be computed analytically only in a few cases. An example is given below.

Example 3. Consider the rectangular pulse

$$q(t) = \begin{cases} A, & t \in [t_1, t_2]; \\ 0, & \text{otherwise}. \end{cases}$$

Let $T = t_2 - t_1$ and $T' = t_2 + t_1$.

In this case $P(\lambda, q)$ is time-independent when $t \in [t_1, t_2]$, and (5) under the boundary condition (7) can be easily solved in closed form. Omitting the straightforward calculations, the continuous spectrum is given by

$$\hat{q}(\lambda) = \frac{A^*}{j\lambda} e^{-2j\lambda t_2} \left(1 - \frac{\Delta}{j\lambda}\cot(\Delta T)\right)^{-1},$$

and the discrete spectrum is the set of roots of

$$j\tan(T\sqrt{|A|^2 + \lambda^2}) = \sqrt{1 + \frac{|A|^2}{\lambda^2}}, \quad \lambda \in \mathbb{C}^+.$$

Note that as $A \to 0$, $\Delta \to \lambda$, and one can see that in the limit of $AT \ll 1$ there is no discrete spectrum. Furthermore, the continuous spectrum tends to

$$\hat{q}(\lambda) = -A^* T e^{-j\lambda T'} \operatorname{sinc}(2Tf), \quad \lambda = 2\pi f,$$

which is just the ordinary Fourier transform of the $q(t)$. This property is indeed general: ordinary Fourier transform is the limit of NFT when $||q|| \ll 1$.

Fig. 2 shows the two spectra for $T = 1$ and various values of A. For small A, there is no discrete spectrum and the continuous spectrum is essentially just the ordinary Fourier transform of $q(t)$. As A is increased, the continuous spectrum deviates from the ordinary Fourier transform and one or more discrete mass points appear on the $j\omega$ axis. □

The continuous spectrum is the component of the NFT which corresponds to the ordinary Fourier transform, whereas the discrete spectrum corresponds to solitons and has no analogue in linear systems theory. The correspondence between the ordinary Fourier transform and the NFT is explained further in [8].

C. Elementary Properties of the Nonlinear Fourier Transform

Let $q(t) \leftrightarrow (\widehat{q}(\lambda), \widetilde{q}(\lambda_k))$ be a nonlinear Fourier transform pair. The following properties can be proved [8].

1) (The ordinary Fourier transform as limit of the nonlinear Fourier transform): If $||q||_{L_1} \ll 1$, there is no discrete spectrum and $\widehat{q}(\lambda) \to Q(\lambda)$, where $Q(\lambda)$ is the ordinary (linear) Fourier transform of $-q^*(t)$

$$Q(\lambda) = -\int_{-\infty}^{\infty} q^*(t) e^{-2j\lambda t} \mathrm{d}t.$$

2) (Weak nonlinearity): If $|a| \ll 1$, then $\widehat{aq}(\lambda) \approx a\widehat{q}(\lambda)$ and $\widetilde{aq}(\lambda_k) \approx a\widetilde{q}(\lambda_k)$. In general, however, $\widehat{aq}(\lambda) \neq a\widehat{q}(\lambda)$ and $\widetilde{aq}(\lambda_k) \neq a\widetilde{q}(\lambda_k)$.
3) (Constant phase change): $\widehat{e^{j\phi}q(t)}(\lambda) = e^{j\phi}\widehat{q(t)}(\lambda)$ and $\widetilde{e^{j\phi}q(t)}(\lambda_k) = e^{j\phi}\widetilde{q(t)}(\lambda_k)$.
4) (Time dilation): $\widehat{q(\frac{t}{a})} = \widehat{|a|q}(a\lambda)$ and $\widetilde{q(\frac{t}{a})} = \widetilde{|a|q}(a\lambda_k)$;
5) (Time shift): $q(t - t_0) \leftrightarrow e^{-2j\lambda t_0}\,(\widehat{q}(\lambda,)\widetilde{q}(\lambda_k))$;
6) (Frequency shift): $q(t)e^{-2j\omega t} \leftrightarrow (\widehat{q}(\lambda - \omega), \widetilde{q}(\lambda_k - \omega))$;
7) (Lax convolution): If $q_2(t) = q_1(t) * (L, M; \mathcal{L})$, then $\widehat{q_2}(\lambda) = H(\lambda, \mathcal{L})\widehat{q_1}(\lambda)$ and $\widetilde{q_2}(\lambda_k) = H(\lambda, \mathcal{L})\widetilde{q_1}(\lambda_k)$. For the NLS equation, the channel filter is $H(\lambda, \mathcal{L}) = \exp(-4j\lambda^2\mathcal{L})$.
8) (Parseval identity): $\int_{-\infty}^{\infty} \|q(t)\|^2 \, \mathrm{d}t = \hat{E} + \tilde{E}$, where

$$\hat{E} = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left(1 + |\hat{q}(\lambda)|^2\right) \mathrm{d}\lambda, \quad \tilde{E} = 4\sum_{j=1}^{N} \Im\left(\lambda_j\right).$$

The quantities $\hat{E}$ and $\tilde{E}$ represent the energy contained in the continuous and discrete spectra, respectively.

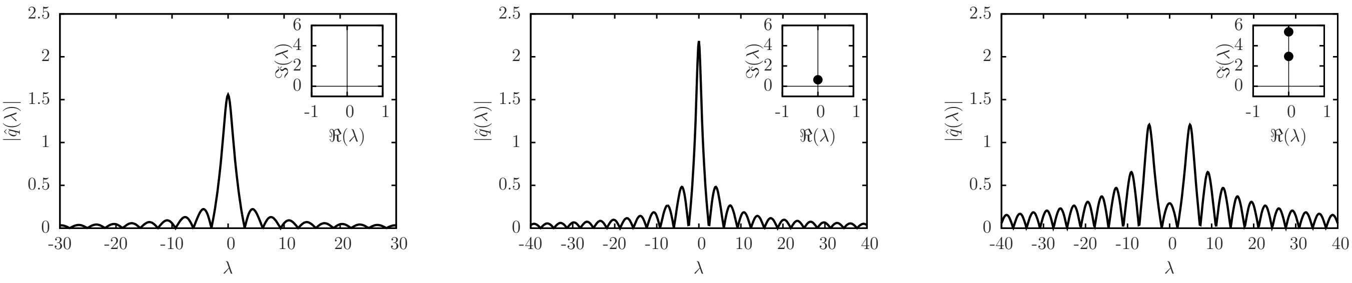
IV. Evolution of the Nonlinear Fourier Transform

It can be shown that the operation of the Lax convolution in the nonlinear Fourier domain is described by a simple multiplicative (diagonal) operator, much in the same way that the ordinary Fourier transform maps $y(t) = x(t) * h(t)$ to $Y(\omega) = X(\omega) \cdot H(\omega)$ [8]. As $q(t,z)$ propagates according to an integrable equation $q_z = K(q)$, its NFT is simply multiplied by a filter. The form of the filter depends on the channel of interest and, for instance, for the NLS equation it is given by $H(\lambda) = \exp(-4j\lambda^2 z)$ [8]. Thus in this case, NFT transforms the complicated time domain (deterministic) channel $jq_z = q_{tt} + 2|q|^2 q$ to a simple multiplication by a filter in the nonlinear frequency domain:

$$\begin{aligned} \widehat{q(t,z)}(\lambda) &= H(\lambda)\widehat{q(t,0)}(\lambda), \\ \widetilde{q(t,z)}(\lambda_j) &= H(\lambda_j)\widetilde{q(t,0)}(\lambda_j), \\ \lambda_j(z) &= \lambda_j(0), \qquad j = 1, 2, \ldots, N. \end{aligned}$$

V. An Approach to Communication over Integrable Channels

Since the nonlinear Fourier transform of a signal is essentially preserved under Lax convolution, one can immediately conceive of a nonlinear analogue of orthogonal frequency-division multiplexing for communication over integrable channels. We refer to this scheme as nonlinear frequency-division



(a) (b) (c)

Fig. 2. Discrete and continuous spectra of the rectangular pulse with $T = 1$ and (a) $A = 1$ (b) $A = 2$ (c) $A = 6$.

multiplexing (NFDM). In this scheme, the input output channel model is given by

$$\hat{Y}(\lambda) = H(\lambda)\hat{X}(\lambda) + \hat{Z}, \quad \tilde{Y}(\lambda_j) = H(\lambda_j)\tilde{X}(\lambda_j) + \tilde{Z}_j,$$

where $\hat{X}(\lambda) = \hat{q}(\lambda, 0)$ and $\tilde{X}(\lambda_j) = \tilde{q}(\lambda_j, 0)$ are spectra at the input of the channel, $\hat{Y}(\lambda) = \hat{q}(\lambda, z)$ and $\tilde{Y}(\lambda_j) = \tilde{q}(\lambda_j, z)$ are spectra at the output of the channel, and $\hat{Z}$ and $\tilde{Z}_j$ are effective noises in the spectral domain.

The proposed scheme consists of two steps.

- *The inverse nonlinear Fourier transform at the transmitter (INFT).* At the transmitter, information is encoded in the nonlinear spectra of the signal according to a suitable constellation on $\left(\hat{X}(\lambda), \tilde{X}(\lambda_j)\right)$. The time domain signal is generated by taking the inverse nonlinear Fourier transform,
$$q(t) = \mathrm{INFT}\left(\hat{X}(\lambda), \tilde{X}(\lambda)\right),$$
and is sent over the channel. (The INFT is described formally in [8].)
- *The forward nonlinear Fourier transform at the receiver (NFT).* At the receiver, the (forward) nonlinear Fourier transform of the signal,
$$\left(\hat{Y}(\lambda), \tilde{Y}(\lambda)\right) = \mathrm{NFT}(q(t, z))$$
is taken and the resulting spectra are compared against the transmitted spectra according to some metric $d\left(\hat{X}(\lambda), \tilde{X}(\lambda); \hat{Y}(\lambda), \tilde{Y}(\lambda)\right)$.

As $q(t, 0)$ propagates in the time domain based on the complicated nonlinear equation, it is significantly distorted and undergoes intersymbol interference (ISI) and interchannel interference. Despite this, in the spectral domain, in the absence of noise, all the nonlinear spectral components propagate independent of each other and the channel is decomposed into a number of linear parallel independent channels. By diagonalizing the channel in this way, the deterministic ISI and interchannel interference are removed in the spectral domain.

Application of this transmission method to the fiber-optic channel is described in [9]. Note that, in general, additive Gaussian noise in the time domain transforms to non-Gaussian correlated (signal-dependent) noise in the spectral domain; see [9] for details. Noise statistics can be approximated and exploited by the optimal receiver [9]. In fiber-optic communications, usually wavelength-division multiplexing (WDM) is used for multiuser communications. This is a linear multiplexing method that, when used in nonlinear channels, leads to interchannel interference. While WDM has been a major development in optical communication, its achievable rates are limited at high input signal powers. NFDM does not suffer from interference and has the potential to improve the spectral efficiency of optical fiber networks [8], [9].

VI. Conclusions

The nonlinear Fourier transform of a signal with respect to an operator L in a Lax pair consists of continuous and discrete spectral functions $\hat{x}(\lambda)$ and $\tilde{x}(\lambda_j)$, obtainable by solving the eigenproblem for the L operator. The NFT maps a Lax convolution to a multiplication operator in the spectral domain. Using the nonlinear Fourier transform, we propose a transmission scheme for integrable channels, termed nonlinear frequency-division multiplexing, in which the information in encoded in the nonlinear spectrum of the signal. The scheme is an extension of traditional OFDM to channels generated by Lax pairs. The class of integrable channels, though often nonlinear and complicated, are somehow "linear in disguise," and thus admit the proposed NFDM transmission scheme.

References

[1] V. E. Zakharov and A. B. Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," *Soviet Physics-JETP*, vol. 34, pp. 62–69, 1972.

[2] R. J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, "Capacity limits of optical fiber networks," *IEEE J. Lightw. Technol.*, vol. 28, no. 4, pp. 662–701, February 2010.

[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "The inverse scattering transform - Fourier analysis for nonlinear problems," *Stud. Appl. Math.*, vol. 53, pp. 249–315, 1974.

[4] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*. Springer Verlag, 2007.

[5] T. Tao and C. Thiele, "Nonlinear Fourier analysis," ArXiv:1201.5129, Jan 2012.

[6] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, 1st ed. Soc. Ind. Appl. Math. (SIAM), 2000.

[7] P. D. Lax, "Integrals of nonlinear equations of evolution and solitary waves," *Comm. Pure Appl. Math*, vol. 21, no. 5, pp. 467–490, 1968.

[8] M. I. Yousefi and F. R. Kschischang, "Information transmission using the nonlinear Fourier transform, Part I: Mathematical tools," *Submitted for publication to the IEEE Trans. Inf. Theory*, 2012.

[9] ——, "Communication over fiber-optic channels using the nonlinear Fourier transform," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT 2013)*, Istanbul, Turkey, 2013.