

Lattice Coding for the Two-way Two-relay Channel

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Abstract—We develop a novel lattice coding scheme for the Two-way Two-relay Channel: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, where Node 1 and 4 communicate with each other through two relay nodes 2 and 3. Each node only communicates with its neighboring nodes. The key technical contribution is the lattice-based achievability strategy, where each relay is able to remove the noise while decoding the sum of several signals in a Block Markov strategy and then re-encode the signal into another lattice codeword using the so-called “Re-distribution Transform”. This allows nodes further down the line to again decode sums of lattice codewords. The symmetric rate achieved by the proposed lattice coding scheme is within $\frac{1}{2} \log 3$ bit/Hz/s of the symmetric rate capacity.

I. INTRODUCTION

Lattice codes may be viewed as linear codes in Euclidean space: the sum of two lattice codewords is again a codeword. This group property is exploited in additive white Gaussian noise (AWGN) relay networks, such as [1], [2], [3], [4], where it has been shown that lattice codes may sometimes outperform i.i.d. random codes, particularly when interested in decoding a linear combination of the received codewords. One such example is the AWGN Two-way Relay channel, where two users communicate with each other through a relay node [3], [4]. If the two users employ lattice codewords, the relay node may decode the sum of the codewords from both users directly at higher rates than decoding them individually. It is then sufficient for the relay to broadcast this sum of codewords to both users since each user, knowing the sum and its own message, may determine the other desired message.

Past work. Beyond their use in the Two-way Relay Channel, nested lattice codes have been shown to be capacity achieving in the point-to-point Gaussian channel [5], the Gaussian Multiple-access Channel [2], Broadcast Channel [6], and to achieve the same rates as those achieved by i.i.d. Gaussian codes in the Decode-and-Forward rate and Compress-and-Forward rates [1] of the Relay Channel [7]. Lattice codes have also been shown to be useful in the Compute-and-Forward framework for decoding linear equations of codewords of [2].

The Two-way Two-relay Channel: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ where Nodes 1 and 4 exchange information through the relay nodes 2 and 3 is related to [8], which considers the throughput of i.i.d. random code-based Amplify-and-Forward and Decode-and-Forward approaches for this channel model, or the i.i.d. random coding based schemes of [9] where there are also links between all nodes. This model is also different from that in

[10] where a two-way relay channel with two *parallel* (rather than sequential as in this work) relays are considered.

Contributions. The proposed scheme for the Two-way Two-relay Channel $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ may be seen as a generalization of the lattice based scheme of [3], [4] for the Two-way Relay Channel $1 \leftrightarrow 2 \leftrightarrow 3$. However, this generalization is not straightforward as the multiple relays need to repeatedly be able to decode the sum of codewords. One may enable this by having the relays use lattice codewords as well – something not required in the Two-way Relay Channel. The scheme includes multiple Block Markov phases where the end users send new messages encoded by lattice codewords and the relays decode a combination of lattice codewords. The relays then perform a “Re-distribution Transform” on the decoded lattice codeword combinations, and broadcast the resulting lattice codewords. The novelty of our scheme lies in this “Re-distribution Transform” which enables both messages to fully utilize the relays’ power. Furthermore, all decoders are lattice decoders (more computationally efficient than joint typicality decoders) and only a single nested lattice pair is needed.

II. PRELIMINARIES ON LATTICE CODES AND NOTATION

Our notation for (nested) lattice codes for transmission over AWGN channels follows that of [6], [11]. An n -dimensional lattice Λ is a discrete subgroup of Euclidean space \mathbb{R}^n with Euclidean norm $\|\cdot\|$ under vector addition. We use bold \mathbf{x} to denote column vectors, \mathbf{x}^T to denote the transpose of \mathbf{x} , and $\mathbf{0}$ denote the all zeros vector. All vectors lie in \mathbb{R}^n unless otherwise stated, all logarithms are base 2, and $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian random variable (or vector) of mean μ and variance σ^2 . Further define or note that

- The *nearest neighbor lattice quantizer* of Λ as $Q(\mathbf{x}) = \arg \min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|$;
- The *mod Λ operation* as $\mathbf{x} \bmod \Lambda := \mathbf{x} - Q(\mathbf{x})$;
- The *Voronoi region* of Λ as $\mathcal{V} := \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\}$, which is of volume $V := \text{Vol}(\mathcal{V})$
- The *second moment per dimension of a uniform distribution over \mathcal{V}* as $\sigma^2(\Lambda) := \frac{1}{V} \cdot \frac{1}{n} \int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}$;
- For any $\mathbf{s} \in \mathbb{R}^n$,

$$(\alpha(\mathbf{s} \bmod \Lambda)) \bmod \Lambda = (\alpha\mathbf{s}) \bmod \Lambda, \quad \alpha \in \mathbb{Z}. \quad (1)$$

$$\beta(\mathbf{s} \bmod \Lambda) = (\beta\mathbf{s}) \bmod \beta\Lambda, \quad \beta \in \mathbb{R}. \quad (2)$$

- The definitions of Rogers and Poltyrev good lattices are in [1]; we will not need these definitions explicitly. Rather, we will use the results derived from lattices with these properties.

Portions of this work were performed at Samsung Electronics, US R&D Center during Yiwei Song’s summer internship in 2012.

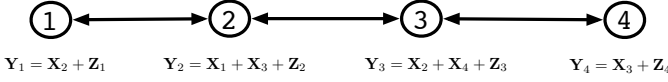


Fig. 1. The Gaussian Two-way Two-relay Channel Model.

A. Nested lattice codes

Consider two lattices Λ and Λ_c such that $\Lambda \subseteq \Lambda_c$ with fundamental regions $\mathcal{V}, \mathcal{V}_c$ of volumes V, V_c respectively; (Λ, Λ_c) is termed a *nested lattice pair*. Here Λ is termed the *coarse* lattice which is a sublattice of Λ_c , the *fine* lattice, and hence $V \geq V_c$. When transmitting over the AWGN channel, using the set $\mathcal{C}_{\Lambda_c, \mathcal{V}} = \{\Lambda_c \cap \mathcal{V}(\Lambda)\}$ as codebook, the coding rate R is

$$R = \frac{1}{n} \log |\mathcal{C}_{\Lambda_c, \mathcal{V}}| = \frac{1}{n} \log \frac{V}{V_c}.$$

Nested lattice pairs satisfying certain properties were shown to be capacity achieving for the AWGN channel [5].

In this work, we only need one “good” nested lattice pair $\Lambda \subseteq \Lambda_c$, in which Λ is both Rogers good and Poltyrev good and Λ_c is Poltyrev good (see definitions in [1]). The existence of such a pair may be guaranteed by [5]; and may be generated by Construction A [5], [2], which maps the codebook of a linear block code over a finite field into real lattice points. Then, as described in [12], one may construct a one-to-one mapping (1:1) denoted by $\phi(\cdot)$ which maps an element in the finite field $w \in \mathbb{F}_{P_{\text{prime}}} = \{0, 1, \dots, P_{\text{prime}} - 1\}$ to a point in n -dimension real space $\mathbf{t} \in \mathcal{C}_{\Lambda_c, \mathcal{V}}$: $\mathbf{t} = \phi(w)$ and $w = \phi^{-1}(\mathbf{t})$.

B. Technical lemmas

The following lemmas, proven in [12], are needed in the proposed two-way lattice based scheme. Let \mathbf{t}_{ai} and $\mathbf{t}_{bi} \in \mathcal{C}_{\Lambda_c, \mathcal{V}}$ be generated from w_{ai} and $w_{bi} \in \mathbb{F}_{P_{\text{prime}}}$ as $\mathbf{t}_{ai} = \phi(w_{ai})$, $\mathbf{t}_{bi} = \phi(w_{bi})$. Furthermore, let $\alpha, \alpha_i, \beta_i \in \mathbb{Z}$ such that $\frac{\alpha}{P_{\text{prime}}}, \frac{\alpha_i}{P_{\text{prime}}}, \frac{\beta_i}{P_{\text{prime}}} \notin \mathbb{Z}$ and $\theta \in \mathbb{R}$. We use \oplus, \otimes and \ominus to denote modulo P_{prime} addition, multiplication, and subtraction over the finite field $\mathbb{F}_{P_{\text{prime}}}$.

Lemma 1. *There exists a 1:1 mapping between $\mathbf{v} = (\sum_i \alpha_i \theta \mathbf{t}_{ai} + \sum_i \beta_i \theta \mathbf{t}_{bi}) \bmod \theta \Lambda$ and $\mathbf{u} = \bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi}$.*

Lemma 2. *There exists a 1:1 mapping between $\alpha \otimes w$ and w .*

Lemma 3. *If w_{ai} and w_{bi} are uniformly distributed over $\mathbb{F}_{P_{\text{prime}}}$, then $(\sum_i \alpha_i \theta \mathbf{t}_{ai} + \sum_i \beta_i \theta \mathbf{t}_{bi}) \bmod \theta \Lambda$ is uniformly distributed over $\{\theta \Lambda_c \cap \mathcal{V}(\theta \Lambda)\}$.*

III. CHANNEL MODEL

In the Gaussian Two-way Two-relay Channel, Node 1 and 4 exchange messages w_a, w_b of respective rates R_a, R_b through multiple full-duplex relays (Node 2 and 3) and multiple hops as shown in Figure 1. Each node can only communicate with its neighboring nodes. The channel model may be expressed as (all bold symbols are n dimensional)

$$\begin{aligned} \mathbf{Y}_1 &= \mathbf{X}_2 + \mathbf{Z}_1, & \mathbf{Y}_2 &= \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2 \\ \mathbf{Y}_3 &= \mathbf{X}_2 + \mathbf{X}_4 + \mathbf{Z}_3, & \mathbf{Y}_4 &= \mathbf{X}_3 + \mathbf{Z}_4 \end{aligned}$$

where \mathbf{Z}_i ($i \in \{1, 2, 3, 4\}$) is an i.i.d. Gaussian noise vector with variance N_i : $\mathbf{Z}_i \sim \mathcal{N}(\mathbf{0}, N_i \mathbf{I})$, and the input \mathbf{X}_i is subject to the transmit power constraint P_i : $\frac{1}{n} E(\mathbf{X}_i^T \mathbf{X}_i) \leq P_i$. Standard definitions of achievable rate regions for the pairs (R_a, R_b) are omitted due to space constraints; see [12]. We first need the following tangential result, which forms the basis for our Two-way Two-Relay Channel achievability scheme.

IV. LATTICE CODES IN THE BC PHASE OF THE TWO-WAY RELAY CHANNEL

The work [3], [4] introduces a two-phase lattice scheme for the Gaussian Two-way Relay Channel $1 \leftrightarrow 2 \leftrightarrow 3$, where nodes 1 and 3 exchange information through node 2: the Multiple-access Channel (MAC) phase and the Broadcast Channel (BC) phase. In the MAC phase, if the codewords are from nested lattice codebooks, the relay may decode the sum of the two codewords directly without decoding them individually. This is sufficient, as then, in the BC phase, the relay may broadcast the sum of the codewords to both users who may determine the other message using knowledge of their own transmitted message. In the scheme of [3], The relay re-encodes the decoded sum into a codeword from an i.i.d. random codebook in [3], and a lattice codebook in [4].

In extending the schemes of [3], [4] to multiple relays we would want to use lattice codebooks in the BC phase, as in [4]. This would, for example, allow the signal sent by Node 2 to be aligned with Node 4’s transmitted signal (aligned is used to mean that the two codebooks are nested) in the Two-way Two-relay Channel: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ and hence enable the decoding of the sum of codewords again at Node 3. However, the scheme of [4] is only applicable to channels in which the SNR from the users to the relay are symmetric, i.e. $\frac{P_1}{N_2} = \frac{P_3}{N_2}$. In this case the relay can simply broadcast the decoded (and possibly scaled) sum of codewords sum without re-encoding it. Thus, before tackling the Two-way Two-relay channel, we first devise a lattice-coding scheme for the BC phase in the Two-way Relay Channel with *arbitrary* uplink SNRs $\frac{P_1}{N_2} \neq \frac{P_3}{N_2}$.

In the Two-way Relay Channel [3] Nodes 1 and 3 exchange messages through the relay Node 2, with channel model:

$$\mathbf{Y}_1 = \mathbf{X}_2 + \mathbf{Z}_1, \quad \mathbf{Y}_2 = \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2, \quad \mathbf{Y}_3 = \mathbf{X}_2 + \mathbf{Z}_3$$

where \mathbf{Z}_i ($i \in \{1, 2, 3\}$) is an i.i.d. Gaussian noise vector with variance N_i : $\mathbf{Z}_i \sim \mathcal{N}(\mathbf{0}, N_i \mathbf{I})$, and the input \mathbf{X}_i is subject to the transmit power constraint P_i : $\frac{1}{n} E(\mathbf{X}_i^T \mathbf{X}_i) \leq P_i$. Definitions of achievability are as in [12].

We devise an achievability scheme which uses lattice codes in both the MAC phase and BC phase. For simplicity, to demonstrate the central idea of a lattice-based BC phase which is going to be used in the Two-way Two-relay Channel, we do not use dithers nor MMSE scaling as in [5], [3], [4].

We assume that $P_1 = N^2 p^2$ and $P_3 = p^2$ where $p \in \mathbb{R}$ and $N \in \mathbb{Z}$. In the next section, we incorporate arbitrary power

Dithers and MMSE scaling allows one to go from achieving rates proportional to $\log(\text{SNR})$ to $\log(1 + \text{SNR})$. However, we initially forgo the “1+” term for simplicity and so as not to clutter the main idea with additional dithers and MMSE scaling.

constraints by first truncating the powers to have the desired form; we show that even with this sub-optimal truncation, constant gap-to-capacity results are possible. We focus on the symmetric rate, i.e. when the rates of the two messages are identical.

Codebook generation: Consider the messages $w_a, w_b \in \mathbb{F}_{P_{\text{prime}}}$, where $P_{\text{prime}} = [2^{nR_{\text{sym}}}]$, where R_{sym} is the symmetric coding rate and $[\cdot]$ denotes rounding to the nearest prime. Nodes 1 and 2 send the codewords $\mathbf{X}_1 = Np\mathbf{t}_a = Np\phi(w_a)$ and $\mathbf{X}_2 = p\mathbf{t}_b = p\phi(w_b)$ where $\phi(\cdot)$ is defined in Section II-A with the nested lattices $\Lambda \subseteq \Lambda_c$. Notice that their codebooks are scaled versions of the codebook $\mathcal{C}_{\Lambda_c, \mathcal{V}}$. The symmetric coding rate is then $R_{\text{sym}} := \frac{1}{n} \log \frac{V}{V_c}$.

In the MAC phase, the relay receives $\mathbf{Y}_2 = \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2$ and decodes $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ with arbitrarily low probability of error as $n \rightarrow \infty$ with rate constraints

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_1}{N_2} \right) \right]^+, \quad R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_3}{N_2} \right) \right]^+$$

according to [12, Lemma 4]. We note that the first rate constant is redundant as $P_1 \geq P_3$; we are including it in the theorem statement for intuition, so as to make it easier to understand the proofs of the main theorems in Section V.

In the BC phase, if, mimicking the steps of [4] the relay simply broadcasts the scaled version of $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$:

$$\frac{\sqrt{P_2}}{Np} ((Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda) = \left(\sqrt{P_2}\mathbf{t}_a + \frac{\sqrt{P_2}}{N}\mathbf{t}_b \right) \bmod \sqrt{P_2}\Lambda, R_a, R_b < \min \left(\left[\frac{1}{2} \log \left(\frac{P_1}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_2}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_3}{N_4} \right) \right]^+, \right. \\ \left. \left[\frac{1}{2} \log \left(\frac{P_4}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_3}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_2}{N_1} \right) \right]^+ \right) \quad (3)$$

we would achieve the rate $R_{\text{sym}} < [\frac{1}{2} \log \frac{P_2}{N_3}]^+$ for the direction $2 \rightarrow 3$ and the rate $R_{\text{sym}} < [\frac{1}{2} \log \frac{P_2}{NN_1}]^+$ for the $1 \leftarrow 2$ direction. While the rate constraint for the direction $2 \rightarrow 3$ is as large as expected, the rate constraint for the direction $1 \leftarrow 2$ does not fully utilize the power at the relay, i.e. the codeword \mathbf{t}_b appears to use only the power P_2/N rather than the full power P_2 . One would thus want to somehow transform the decoded sum $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ such that both \mathbf{t}_a and \mathbf{t}_b of the transformed signal would be uniformly distributed over $\mathcal{V}(\sqrt{P_2}\Lambda)$. Notice that the relay can only operate on $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ rather than $Np\mathbf{t}_a$ and $p\mathbf{t}_b$ individually.

Re-distribution Transform: To alleviate this problem we propose the following ‘‘Re-distribution Transform’’ operation which consists of three steps:

- 1) multiply the decoded signal by N to obtain $N((Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda)$,
- 2) then perform $\bmod \Lambda$ to obtain $N((Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda) \bmod \Lambda = (N^2p\mathbf{t}_a + Np\mathbf{t}_b) \bmod Np\Lambda$ according to the operation rule in (1), and finally
- 3) re-scale the signal to be of second moment P_2 as $\frac{\sqrt{P_2}}{Np} ((N^2p\mathbf{t}_a + Np\mathbf{t}_b) \bmod Np\Lambda) = (N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ according to (2). Notice that $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ is uniformly distributed over $\{\sqrt{P_2}\Lambda_c \cap \mathcal{V}(\sqrt{P_2}\Lambda)\}$ by Lemma 3.

The relay broadcasts

$$\mathbf{X}_2 = (N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda.$$

Notice that $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ is uniformly distributed over $\{\sqrt{P_2}\Lambda_c \cap \mathcal{V}(\sqrt{P_2}\Lambda)\}$, and so its coding rate is R_{sym} . Node 1 and Node 3 receive $\mathbf{Y}_1 = \mathbf{X}_2 + \mathbf{Z}_1$ and $\mathbf{Y}_3 = \mathbf{X}_2 + \mathbf{Z}_3$ respectively and, according to [12, Lemma 5], may decode $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ at rate

$$R_{\text{sym}} < \left[\frac{1}{2} \log \frac{P_2}{N_1} \right]^+, \quad R_{\text{sym}} < \left[\frac{1}{2} \log \frac{P_2}{N_3} \right]^+.$$

Nodes 1 and 2 then map the decoded $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod p\Lambda$ to $Nw_a \oplus w_b$ by Lemma 1. With side information w_a , Node 1 may then determine w_b ; likewise with side information w_b , Node 2 obtains Nw_a and w_a by Lemma 2.

We note that for the single relay Two-way Relay Channel, we achieve lower rates than those in [3], [4]. We are describing it here to explain the intuition behind the scheme which will be used for multiple relays (the goal of this work) Section V.

V. TWO-WAY TWO-RELAY CHANNEL

We first consider the full-duplex Two-way Two-relay channel where every node transmits and receives at the same time.

Theorem 4. *For the channel model described in Section III, if $P_1 = p^2$, $P_2 = M^2q^2$, $P_3 = N^2p^2$ and $P_4 = q^2$, where $p, q \in \mathbb{R}^+$ and $M, N \in \mathbb{Z}^+$ the following rate region*

is achievable using lattice codes.

We again note that some terms are redundant, but are included to allow for a simple, easily generalizable expression in accordance with Theorem 6.

Proof: **Codebook generation:** We consider the good nested lattice pair $\Lambda \subseteq \Lambda_c$ with corresponding codebook $\mathcal{C}_{\Lambda_c, \mathcal{V}} = \{\Lambda_c \cap \mathcal{V}\}$, and two messages $w_a, w_b \in \mathbb{F}_{P_{\text{prime}}}$ in which $P_{\text{prime}} = [2^{nR_{\text{sym}}}]$ (R_{sym} is the coding rate). The codewords associated with the messages w_a and w_b are $\mathbf{t}_a = \phi(w_a)$ and $\mathbf{t}_b = \phi(w_b)$, where the mapping $\phi(\cdot)$ from $\mathbb{F}_{P_{\text{prime}}}$ to $\mathcal{C}_{\Lambda_c, \mathcal{V}} \in \mathbb{R}^n$ is defined in Section II-A.

Encoding and decoding steps: We use a Block Markov Encoding/Decoding scheme where Node 1 and 4 transmit a new message w_{ai} and w_{bi} , respectively, at the beginning of block i . To satisfy the transmit power constraints, Node 1 and 4 send the scaled codewords $\mathbf{X}_{1i} = p\mathbf{t}_{ai} = p\phi(w_{ai}) \in \{p\Lambda_c \cap \mathcal{V}(p\Lambda)\}$ and $\mathbf{X}_{4i} = q\mathbf{t}_{bi} = q\phi(w_{bi}) \in \{q\Lambda_c \cap \mathcal{V}(q\Lambda)\}$ respectively in block i . Node 2 and 3 send \mathbf{X}_{2i} and \mathbf{X}_{3i} , and Node j ($j \in \{1, 2, 3, 4\}$) receives \mathbf{Y}_{ji} in block i . The procedure of the first few blocks (the initialization steps) are described and then a generalization is made. We note that in general the coding rates R_a for w_a and R_b for w_b may be different, as long as $R_{\text{sym}} = \max(R_a, R_b)$, since we may always send dummy messages to make the two coding rates equal.

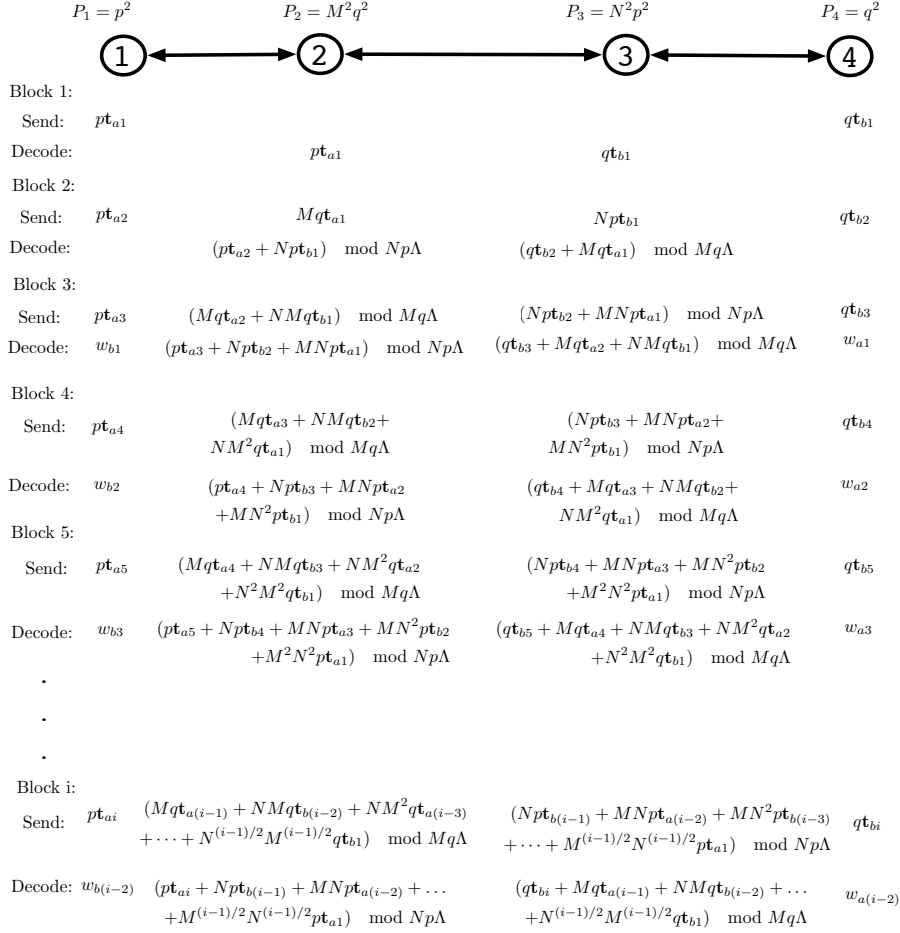


Fig. 2. Multi-phase Block Markov achievability strategy for Theorem 4.

Block 1: Codewords $\mathbf{X}_{11} = p\mathbf{t}_{a1}$ and $\mathbf{X}_{41} = q\mathbf{t}_{b1}$ sent from Nodes 1 and 4 to Nodes 2 and 3, may be decoded if, resp.

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_1}{N_2} \right) \right]^+, \quad (4)$$

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_4}{N_3} \right) \right]^+ \quad (5)$$

according to [12, Lemma 5].

Block 2: Node 1 and 4 send new codewords $\mathbf{X}_{12} = p\mathbf{t}_{a2}$ and $\mathbf{X}_{42} = q\mathbf{t}_{b2}$, while Node 2 and 3 broadcast $\mathbf{X}_{22} = Mq\mathbf{t}_{a1}$ and $\mathbf{X}_{32} = Np\mathbf{t}_{b1}$ received in the last block. Note they are scaled to fully utilize the transmit power $P_2 = M^2q^2$ and $P_3 = N^2p^2$. Node 2 receives $\mathbf{Y}_{22} = \mathbf{X}_{12} + \mathbf{X}_{32} + \mathbf{Z}_{22}$ and decodes $(p\mathbf{t}_{a2} + Np\mathbf{t}_{b1}) \bmod Np\Lambda$ if (4) and

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_3}{N_2} \right) \right]^+. \quad (6)$$

Node 3 decodes $(q\mathbf{t}_{b2} + Mq\mathbf{t}_{a1}) \bmod Mq\Lambda$ if (5) and

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_2}{N_3} \right) \right]^+. \quad (7)$$

Block 3: • *Encoding*: Node 1 and 4 send new codewords as in the previous blocks. Node 2 further processes its decoded

codewords combination according to the three steps of the Re-distribution Transform from previous block as

$$\begin{aligned} & (N((p\mathbf{t}_{a2} + Np\mathbf{t}_{b1}) \bmod Np\Lambda)) \bmod Np\Lambda \\ &= (Np\mathbf{t}_{a2} + N^2p\mathbf{t}_{b1}) \bmod Np\Lambda \end{aligned}$$

and scales this to utilize the full transmit power $P_2 = M^2q^2$ as $\frac{Mq}{Np}(Np\mathbf{t}_{a2} + N^2p\mathbf{t}_{b1}) \bmod Np\Lambda = (Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$. It then broadcasts $\mathbf{X}_{23} = (Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$. Notice that since

$$(Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda \in \{Mq\Lambda_c \cap \mathcal{V}(Mq\Lambda)\}$$

according to Lemma 3, its coding rate is R_{sym} . Similarly, Node 3 broadcasts $\mathbf{X}_{33} = (Np\mathbf{t}_{b2} + MNp\mathbf{t}_{a1}) \bmod Np\Lambda$ again at coding rate R_{sym} .

• *Decoding*: At the end of this block, Node 2 is able to decode $(p\mathbf{t}_{a3} + Np\mathbf{t}_{b2} + MNp\mathbf{t}_{a1}) \bmod Np\Lambda$ with rate constraints (4) and (6) according to [12, Lemma 4], and Node 3 decodes $(q\mathbf{t}_{b3} + Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$ if (7) and (5). Node 1 decodes $(Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$ sent by Node 2 as in the point-to-point channel with rate constraint

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_2}{N_1} \right) \right]^+ \quad (8)$$

according to Lemma [12, Lemma 5]. From the decoded $(Mq\mathbf{t}_{a2} + Nm\mathbf{q}\mathbf{t}_{b1}) \bmod Mq\Lambda$, it obtains $w_{a2} \oplus Nw_{b1}$ (Lemma 1). With its own information w_{a2} , Node 1 can then obtain $N \otimes w_{b1} = w_{a2} \oplus Nw_{b1} \ominus w_{a2}$, which may be mapped to w_{b1} since P_{prime} is a prime number (Lemma 2). Notice $P_{\text{prime}} = \lfloor 2^{nR_{\text{sym}}} \rfloor \rightarrow \infty$ as $n \rightarrow \infty$, so $N \ll P_{\text{prime}} = \lfloor 2^{nR} \rfloor$ and $\frac{N}{P_{\text{prime}}} \notin \mathbb{Z}$. Similarly, Node 4 can decode w_{a1} with rate constraint

$$R_{\text{sym}} < \left[\frac{1}{2} \log \left(\frac{P_2}{N_3} \right) \right]^+. \quad (9)$$

Block 4 and 5 proceed similarly, as shown in Figure 2.

Block i : To generalize, in Block i (assume i is odd),

• **Encoding:** Node 1 and 4 send new messages $\mathbf{X}_{1i} = p\mathbf{t}_{ai}$ and $\mathbf{X}_{4i} = q\mathbf{t}_{bi}$, resp. Node 2 and 3 broadcast

$$\begin{aligned} \mathbf{X}_{2i} &= (Mq\mathbf{t}_{a(i-1)} + Nm\mathbf{q}\mathbf{t}_{b(i-2)} + NM^2\mathbf{q}\mathbf{t}_{a(i-3)} + \\ &\quad N^2M^2\mathbf{q}\mathbf{t}_{b(i-4)} + \dots + N^{(i-1)/2}M^{(i-1)/2}\mathbf{q}\mathbf{t}_{b1}) \bmod Mq\Lambda \\ \mathbf{X}_{3i} &= (Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + MN^2p\mathbf{t}_{b(i-3)} + \\ &\quad M^2N^2p\mathbf{t}_{a(i-4)} + \dots + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda. \end{aligned}$$

• **Decoding:** Node 1 decodes the codeword from Node 2 with rate constraint (8) ([12, Lemma 5]) and maps it to $w_{a(i-1)} \oplus Nw_{b(i-2)} \oplus NMw_{a(i-3)} \oplus N^2Mw_{b(i-4)} \oplus \dots \oplus N^{(i-1)/2}M^{(i-1)/2}w_{b1}$ (Lemma 1). With its own messages w_{ai} ($\forall i$) and the messages it decoded previously $\{w_{b1}, w_{b2}, \dots, w_{b(i-3)}\}$, Node 1 can obtain $N \otimes w_{b(i-2)}$ and determine $w_{b(i-2)}$ accordingly (Lemma 2). Similarly, Node 4 can decode $w_{a(i-2)}$ subject to rate constraint (9).

• **Re-distribution Transform:** In block i , Node 2 decodes $(p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + MN^2p\mathbf{t}_{b(i-3)} + M^2N^2p\mathbf{t}_{a(i-4)} + \dots + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda$ from the received $\mathbf{Y}_{2i} = \mathbf{X}_{1i} + \mathbf{X}_{3i} + \mathbf{Z}_{2i}$ subject to (4) and (6) ([12, Lemma 4]). It then uses the Re-distribution Transform to obtain $(N(p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + \dots + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda) \bmod Np\Lambda = Np\mathbf{t}_{ai} + N^2p\mathbf{t}_{b(i-1)} + MN^2p\mathbf{t}_{a(i-2)} + \dots + M^{(i-1)/2}N^{(i-1)/2+1}p\mathbf{t}_{a1} \bmod Np\Lambda$ and scales it to utilize the full transmit power: $\frac{Mq}{Np}(Np\mathbf{t}_{ai} + N^2p\mathbf{t}_{b(i-1)} + MN^2p\mathbf{t}_{a(i-2)} + \dots + M^{(i-1)/2}N^{(i-1)/2+1}p\mathbf{t}_{a1}) \bmod Np\Lambda = Mq\mathbf{t}_{ai} + NMq\mathbf{t}_{b(i-1)} + NM^2\mathbf{q}\mathbf{t}_{a(i-2)} + \dots + N^{(i-1)/2}M^{(i-1)/2+1}\mathbf{q}\mathbf{t}_{a1} \bmod Mq\Lambda$.

This signal will be transmitted in the next block $i+1$. Node 3 performs similar operations, decoding $q\mathbf{t}_{bi} + Mq\mathbf{t}_{a(i-1)} + Nm\mathbf{q}\mathbf{t}_{b(i-2)} + \dots + N^{(i-1)/2}M^{(i-1)/2}\mathbf{q}\mathbf{t}_{b1} \bmod Mq\Lambda$ subject to constraints (7) and (5), and transforms it into $Np\mathbf{t}_{bi} + MNp\mathbf{t}_{a(i-1)} + MN^2p\mathbf{t}_{b(i-2)} + \dots + M^{(i-1)/2}N^{(i-1)/2+1}p\mathbf{t}_{b1} \bmod Mq\Lambda$, which is transmitted in the next block.

Combining all rate constraints, we obtain

$$R_{\text{sym}} < \min \left(\left[\frac{1}{2} \log \left(\frac{P_1}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_2}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_3}{N_4} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_4}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_3}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P_2}{N_1} \right) \right]^+ \right)$$

For i even we have analogous steps with slightly different indices as may be extrapolated from the difference between Block 4 and 5 in Fig. 2.

Assuming there are I blocks in total, the final achievable rate is $\frac{I-2}{I}R_{\text{sym}}$, which, as $I \rightarrow \infty$, approaches R_{sym} . ■

We may achieve the same region for the permuted powers:

Lemma 5. *The rates of Theorem 4 may also be achieved when $P_1 = N^2p^2$, $P_3 = p^2$ and/or $P_2 = q^2$, $P_4 = M^2q^2$.*

Proof: The proof is shown in [12]. ■

Theorem 4 and Lemma 5 both hold for powers for which P_1/P_3 and/or P_2/P_4 are either the squares of integers or the reciprocal of the squares of integers. However, these scenarios do not cover general power constraints with arbitrary ratios. We next present an achievable rate region for arbitrary powers:

Theorem 6. *For the Two-way Two-relay Channel with arbitrary transmit power constraints, any rates satisfying*

$$R_a, R_b < \max_{\substack{P'_i \leq P_i, \\ P'_3 = N^2 \text{ or } \frac{1}{N^2}, \\ P'_4 = M^2 \text{ or } \frac{1}{M^2}}} \min \left(\left[\frac{1}{2} \log \left(\frac{P'_1}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P'_2}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P'_3}{N_4} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P'_4}{N_3} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P'_3}{N_2} \right) \right]^+, \left[\frac{1}{2} \log \left(\frac{P'_2}{N_1} \right) \right]^+ \right)$$

for some $N, M \in \mathbb{Z}^+$ and $i \in \{1, 2, 3, 4\}$, are achievable. This rate region for $R_a = R_b$ is within $\frac{1}{2} \log 3$ bit/Hz/s per user from the symmetric rate capacity.

Proof: The proof is shown in [12]. ■

The half-duplex case and extensions to more than two relays are also discussed in [12].

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