Towards a strong converse for the quantum capacity (of degradable channels)

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***Abstract*—We exhibit a possible road towards a strong converse for the quantum capacity of degradable channels. In particular, we show that all degradable channels obey what we call a "pretty strong" converse: When the code rate increases above the quantum capacity, the fidelity makes a discontinuous jump from 1 to at most $\frac{1}{\sqrt{2}}$, asymptotically. A similar result can be shown for the private (classical) capacity.**

Furthermore, we can show that if the strong converse holds for symmetric channels (which have quantum capacity zero), then degradable channels obey the strong converse: The above-mentioned asymptotic jump of the fidelity at the quantum capacity is then from 1 down to 0.

I. Introduction

Communication via noisy channels is one of the information tasks by which, following the fundamental work of Shannon [19], we have learned to quantify information and noise. One of the most important models considered from these early days of information theory is that of a discrete memoryless channel, for which Shannon gave his famous single-letter formula for the capacity (i.e. the maximum communication rate achievable by asymptotic error-free block coding).

The analogous model in quantum Shannon theory is the memoryless quantum channel $\mathcal{N}^{\otimes n}$ (for asymptotically large integer n), given by a completely positive and trace preserving (cptp) map $\mathcal{N}:\mathcal{L}(A')\to\mathcal{L}(B)$, with Hilbert spaces A' and B that we assume to be finite dimensional throughout this paper.

The quantum capacity $Q(\mathcal{N})$ of $\mathcal{N}$ is informally defined as the maximum rate at which quantum information can be transmitted asymptotically faithfully over that channel, when using it $n\to\infty$ times. As for all channel capacity theorems, the quantum capacity theorem consists of a direct part and a converse. The direct part states that for rates below a certain threshold there exist codes with decoding error (quantified as a certain distance from noiseless transmission) tending to 0 in the number of channel uses. The converse states that if the rate lies above this threshold then the error does not go to 0 for any sequence of codes. To be precise, this is known as a *weak converse* and the threshold rate sometimes called *weak capacity*. A *strong converse* is the statement that for rates above the capacity the error converges to its maximum 1 as $n\to\infty$.

Strong converse theorems have been shown to hold for other types of information sent over memoryless quantum channels, including classical information encoded into product states [15], [28] and for general input states (i.e. allowing the possibility of entangled input signal states) over certain classes of quantum channels [11]. The strong converse also holds for entanglement-assisted classical communication over memoryless quantum channels, by the Quantum Reverse Shannon Theorem [1], [2]. Strong converses do not hold by default; certain quantum channels with memory have a weak capacity but fail the strong converse [6], [9].

The paper is structured as follows: In section II we recall the definition of codes, error criteria and the quantum capacity. Then, in section III we discuss the weak converse for the quantum capacity and the possibility of strong converses. In section IV, we review the concept of degradable channels and some of the analysis of Devetak and Shor [8] of their quantum capacity. Then in section V, we state our main result (Theorem 2) strongly bounding the rate of channels with sufficiently small error. Subsequently, we state an analogous rate bound for the private classical capacity (Theorem 4), and then show that a strong converse for all symmetric channels implies the strong converse for all degradable channels (Theorem 8). The proofs for all theorems and lemmas stated in this sections are proved in the longer version of this paper [14]. We conclude in section VI with a brief discussion of what was achieved and highlight open problems.

On notation: In this paper, $\log$ is always the binary logarithm, and $\exp$ its inverse, the exponential function to base 2. The natural logarithm is denoted $\ln x$, the natural exponential function e^x.

II. Quantum channel capacity

For a given channel $\mathcal{N}:\mathcal{L}(A')\to\mathcal{L}(B)$, we consider encoding and decoding of quantum information, given by cptp maps

$$\mathcal{E}:\mathcal{L}(C)\to\mathcal{L}(A'),$$
$$\mathcal{D}:\mathcal{L}(B)\to\mathcal{L}(C),$$

which together form a *quantum code*. The idea is that the information to be sent is subjected to the overall effective channel $\mathcal{D}\circ\mathcal{N}\circ\mathcal{E}:\mathcal{L}(C)\to\mathcal{L}(C)$. For a Hilbert space

$\mathcal{H}$, we denote the set of states by

$$\mathcal{S}(\mathcal{H}) = \{\rho \geq 0 \text{ s.t. } \operatorname{Tr}\rho = 1\}.$$

There are many ways of defining mathematically the notion that the output is a good approximation of the input, and we refer the reader to the comprehensive treatment of Kretschmann and Werner [12] for a discussion of all the concomitant ways of defining the capacity and the proof that asymptotically and for vanishing error they are the same. In the present paper we will measure the degree of approximation between states by the fidelity, given as

$$F(\rho,\sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 = \max|\langle\varphi|\psi\rangle|,$$

where the maximization is over all purifications $|\varphi\rangle$, $|\psi\rangle$ of ρ and σ, respectively [27], [10].

The maximum dimension $|C|$ of C such that there exists a quantum code for $\mathcal{N}^{\otimes n}$ with error ϵ, is denoted $N(n,\epsilon)$, or more precisely $N(n,\epsilon|\mathcal{N})$ if we want to refer explicitly to the channel.

If we have a code with error $\leq \epsilon$, this means that we can use it with the maximally entangled state $|\Phi\rangle^{CC'}$ at the input, to get an output state

$$\sigma^{CC'} = (\mathrm{id}\otimes\mathcal{D}\circ\mathcal{N}\circ\mathcal{E})\Phi,$$

which is ϵ-close to being maximally entangled, using the fidelity as a distance measure. This motivates the definition of an *entanglement-generating code with error* ϵ. The maximum dimension $|C|$ of C such that there exists an entanglement-generating code for $\mathcal{N}^{\otimes n}$ with error ϵ, is denoted $N_E(n,\epsilon)$, or more explicitly, $N_E(n,\epsilon|\mathcal{N})$. Clearly, $N(n,\epsilon) \leq N_E(n,\epsilon)$.

The quantum capacity is now defined as

$$Q(\mathcal{N}) = \inf_{\epsilon>0}\liminf_{n\to\infty}\frac{1}{n}\log N(n,\epsilon).$$

One obtains the same capacity when using $\limsup$ and N_E, see [12] for a proof of this and the equivalence of other variations of the definition.

A Shannon-style formula for the quantum capacity was first stated by Lloyd [13] and proved rigorously by Shor [21] and Devetak [7]: more precisely, in these papers the direct (achievability) part was proven; earlier, Schumacher and Nielsen [18] had shown that the same quantity is an upper bound, i.e. the weak converse.

The formula is given in terms of the *coherent information*

$$I(A\rangle B)_\rho = -S(A|B)_\rho = S(\rho^B) - S(\rho^{AB}),$$

where $S(\rho) = -\operatorname{Tr}\rho\log\rho$ is the von Neumann entropy, of a state $\rho^{AB} = (\mathrm{id}\otimes\mathcal{N})\phi^{AA'}$ with a "test state" ϕ on AA'. Namely,

$$Q(\mathcal{N}) = \lim_{n\to\infty}\frac{1}{n}Q^{(1)}(\mathcal{N}^{\otimes n}),$$

with the single-letter expression

$$Q^{(1)}(\mathcal{N}) = \max_{\phi\in\mathcal{S}(AA')}\{I(A\rangle B)_\rho : \rho = (\mathrm{id}\otimes\mathcal{N})\phi\}.$$

Remark The quantum capacity is known to be non-additive [26]. So is the single-letter quantity $Q^{(1)}(\mathcal{N})$ [23], meaning that the regularization above is necessary, at least as long as we base our capacity formula on the coherent information. It is not known whether there is a single-letter formula for $Q(\mathcal{N})$, or even an efficient approximation scheme [22]. As a matter of fact, we do not even know how to characterize the quantum capacity of the qubit depolarizing channel as a function of the noise. ■

III. WEAK AND STRONG CONVERSE

The fact that the coherent information gives an upper bound on the quantum capacity of general channels has been known since Schumacher and Nielsen [18]. To be precise, they showed that for any entanglement generating code with code space C, for a channel $\mathcal{N}:\mathcal{L}(A')\to\mathcal{L}(B)$ with error ϵ, there exists an input test state $\phi^{AA'}$ such that with $\rho^{AB} = (\mathrm{id}\otimes\mathcal{N})\phi$,

$$(1-2\epsilon)\log|C| \leq I(A\rangle B)_\rho + 1.$$

Applying this to a maximal code for $\mathcal{N}^{\otimes n}$ yields, for $\epsilon < \frac{1}{2}$,

$$\frac{1}{n}\log N_E(n,\epsilon) \leq \frac{1}{1-2\epsilon}\frac{1}{n}Q^{(1)}(\mathcal{N}^{\otimes n}) + \frac{1}{(1-2\epsilon)n}, \quad (1)$$

hence the result that for $n\to\infty$ and $\epsilon\to 0$, the optimal rate cannot exceed $\lim_n \frac{1}{n}Q^{(1)}(\mathcal{N}^{\otimes n})$, which we know is also asymptotically achievable, thanks to Lloyd-Shor-Devetak.

However, for any non-zero $\epsilon > 0$, the upper bound in Eq. (1) is a constant factor away from the capacity, which is the hallmark of a weak converse; it leaves room for a trade-off between communication rate and error, asymptotically.

If the quantum capacity $Q(\mathcal{N})$ is zero, Eq. (1) says something a bit stronger, namely that $N_E(n,\epsilon) \leq O(1)$, at least when $\epsilon < \frac{1}{2}$. In this article we call such a statement *pretty strong converse*, i.e. a proof amounting to

$$\limsup_{n\to\infty}\frac{1}{n}\log N_E(n,\epsilon) \leq Q(\mathcal{N}),$$

at least for error ϵ below some threshold ϵ_0. A strong converse would require the above for all $\epsilon < 1$.

Here are two simple examples of channels for which the strong converse holds.

Example *(PPT entanglement binding channels).* If $\mathcal{N}$ is such that all $\rho = (\mathrm{id}\otimes\mathcal{N})\phi$ have positive partial transpose (PPT), then any entanglement generating code for a maximally entangled state of Schmidt rank d, Φ_d, using any number n of channel uses and even arbitrary classical communication on the side, is still a PPT state. Twirling by the symmetries $U\otimes\overline{U}$ of the maximally entangled state does not change the fidelity between the resulting state and the maximally entangled state. But the resulting isotropic state

$$\rho = p\Phi_d + (1-p)\frac{1}{d^2-1}(\mathbb{1}-\Phi_d)$$

is still PPT, and it is well-known that this can only hold for $p \leq \frac{1}{d}$ [16]. I.e., the error is at least $\sqrt{1-\frac{1}{d}}$, which in the setting of n channel uses ($\mathcal{N}^{\otimes n}$) goes to 1 exponentially fast for positive rates (meaning $d = 2^{nR}$ for $R > 0$). ■

Example *(Ideal channel).* Consider the identity $\mathrm{id}_2 : \mathcal{L}(\mathbb{C}^2) \rightarrow \mathcal{L}(\mathbb{C}^2)$ on a qubit. Consider an entanglement-generating code for n uses of it, $\mathrm{id}_2^{\otimes n}$ for a maximally entangled state of rank d. It is evident that the state shared between sender and receiver after the transmission is of Schmidt rank $\leq 2^n$, and so is any state obtained by the receiver's decoding. Hence the fidelity of the code is upper bounded by

$$\max\left\{|\langle\Phi_d|\psi\rangle| \,:\, \text{Schmidt rank of } |\psi\rangle \text{ at most } 2^n\right\} = \sqrt{\frac{2^n}{d}}.$$

Consequently, as soon as the rate is above the capacity $Q(\mathrm{id}_2) = 1$, i.e. $d = 2^{nR}$ for $R > 1$, the error goes to 1 exponentially fast. ■

IV. DEGRADABLE AND ANTI-DEGRADABLE CHANNELS

By the Stinespring dilation theorem, any channel can be defined by an isometric embedding $U : A' \longrightarrow B \otimes E$ followed by a partial trace over the environment system E, such that $\mathcal{N}(\rho) = \mathrm{Tr}\,_E U\rho U^\dagger$. Tracing over B rather than E we obtain the corresponding complementary channel, $\mathcal{N}^c(\rho) := \mathrm{Tr}\,_B U\rho U^\dagger$.

As we are interested in the channel's behaviour, we will without loss of generality assume from now on that E is chosen of minimal dimension (which makes U unique up to isometries on E). Furthermore, since $\mathcal{N}$ is the complementary channel of $\mathcal{N}^c$, we may equally reduce the dimension of B if necessary; this can equivalently be described as finding the subspace $\widehat{B} \subset B$ that contains all supports of all $\mathcal{N}(\rho)$ for states ρ on A', which is in fact the supporting subspace of $\mathcal{N}(\mathbb{1})$, and viewing $\mathcal{N}$ as a mapping into $\mathcal{L}(\widehat{B})$.

A channel $\mathcal{N}$ is called *degradable* if it can be degraded to its complementary channel, i.e. if there exists a cptp map $\mathcal{M}$ such that $\mathcal{N}^c = \mathcal{M} \circ \mathcal{N}$. Introducing the Stinespring dilation of $\mathcal{M}$ by an isometry $V : B \longrightarrow F \otimes E'$, the channel output system B can be mapped to the composite system $E' \otimes F$ such that the channel taking A' to E is the same as the channel taking A' to E' (with an isomorphism between E and E' fixed once and for all). We may also assume F to be minimal. The above information process is illustrated in Fig. 1.

If the complementary channel is degradable, i.e. if $\mathcal{N} = \mathcal{M} \circ \mathcal{N}^c$ for some cptp map, we call $\mathcal{N}$ *anti-degradable*. A channel that is both degradable and anti-degradable is called *symmetric* [24].

The identity between the channels $\mathcal{L}(A') \rightarrow \mathcal{L}(E)$ and $\mathcal{L}(A') \rightarrow \mathcal{L}(E')$ (defined by conjugating by VU and tracing over $E'F$ and EF, respectivey) is expressed by the equation

$$\psi^{AE} = \psi^{AE'}, \tag{2}$$

modulo the implicit isomorphism between E and E'. This was enough for Devetak and Shor [8] to prove that for degradable channels the coherent information is additive; see also [5, Sec. A.2]. The crucial point in their argument is that the coherent information can be rewritten as a conditional entropy,

$$I(A\rangle B)_\varphi = S(F|E')_\psi.$$

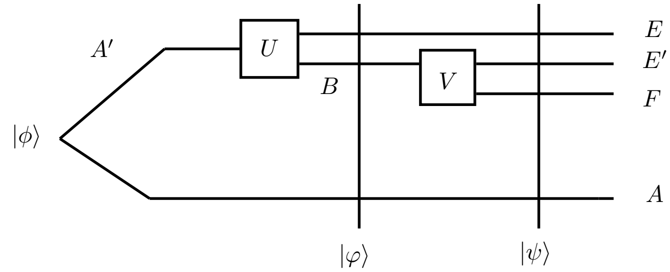

Fig. 1. Schematic of a degradable quantum channel, with the input state ϕ between A' and the reference A, the channel output and environment state φ and the state ψ shared between A, F and the two copies of the original environment, E and E'.

Then, based on the observation that the state $\psi^{FE'}$ on the r.h.s. is a linear function of the reference state $\rho^{A'} = \mathrm{Tr}\,_A\phi$, and using strong subadditivity, one gets subadditivity of the coherent information of a product channel, hence additivity of $Q^{(1)}$.

Denoting $\mathrm{SWAP}_{EE'}$ the swap unitary between systems E and E', i.e. $\mathrm{SWAP}|u\rangle|v\rangle = |v\rangle|u\rangle$ (always modulo the implicit identification of E with E'), we have the following statement strengthening Eq. (2):

Lemma 1 *Consider degradable channel $\mathcal{N}$ with Stinespring dilation $U : A' \hookrightarrow B \otimes E$. Then there exist a degrading map $\mathcal{M}$ with Stinespring dilation $V : B \hookrightarrow F \otimes E'$ (not necessarily with minimal dimension $|F|$) and a unitary X on F, which may be chosen as an involution (i.e. $X^2 = \mathbb{1}$), such that*

$$(X_F \otimes \mathrm{SWAP}_{EE'})VU = VU.$$

In particular, for arbitrary state vector $|\phi\rangle^{AA'}$ and $|\psi\rangle^{AFEE'} := (\mathbb{1} \otimes VU)|\phi\rangle$,

$$(\mathbb{1}_A \otimes X_F \otimes \mathrm{SWAP}_{EE'})|\psi\rangle^{AFEE'} = |\psi\rangle^{AFEE'}.$$

■

V. MAIN RESULTS

Our main results are stated the the following subsections.

A. Pretty strong converse

Theorem 2 *Let $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ be a degradable channel with finite quantum systems A and B. Then, for error $\epsilon < \frac{1}{\sqrt{2}}$ and every integer n,*

$$\log N(n, \epsilon) \leq \log N_E(n, \epsilon) \leq nQ^{(1)}(\mathcal{N}) + O\left(\sqrt{n \log n}\right).$$

■

The *proof* of this theorem and the others below is quite long and complex, and we refer to the full paper [14]. Very broadly, the strategy is to use the calculus of min-entropies to upper bound the rate (for fixed error) by a concave quantity; here the degradability is used. This in turn allows us to formulate an upper bound in terms of smooth min-entropies first of symmetric and then, by invocation of a de Finetti theorem, of i.i.d. states. The last step invokes the well-established asymptotic equipartition property for the min-entropy.

Together with the direct part (achievability proved in [13], [7], [21]) we thus get:

Corollary 3 *For a degradable channel $\mathcal{N}$, the quantum capacity is given by*

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \log N(n,\epsilon) = \lim_{n\to\infty} \frac{1}{n} \log N_E(n,\epsilon),$$

for any $0 < \epsilon < \frac{1}{\sqrt{2}}$. Compared to the original definition this is simpler as we do not need to vary ϵ, and there is convergence rather than reference to $\liminf$ *or* $\limsup$. ■

Remark The error $\frac{1}{\sqrt{2}}$ is precisely that achieved asymptotically by a single 50%-50% erasure channel acting on the code space, and of other suitable symmetric (i.e., degradable and anti-degradable) channels. ■

B. Pretty strong converse for the private capacity

We start by reviewing the basic definitions, which we adapt from Renes and Renner [17]: A *private classical code* for a channel $\mathcal{N} : \mathcal{L}(A') \to \mathcal{L}(B)$ consists of a family of signal states $\rho_x \in \mathcal{S}(A')$ ($x = 1,\ldots,M$), and a decoding measurement (POVM) $(D_x)_{x=1}^M$, i.e. $D_x \geq 0$, $\sum_x D_x = \mathbb{1}_B$. The latter can also be viewed as a cptp map $\mathcal{D} : \mathcal{L}(B) \to \widehat{X}$.

For a given channel $\mathcal{N}$, we denote the largest M such that there exists a private classical code with error ϵ and privacy δ (which is itself defined in terms of the fidelity of the complementary channel $\mathcal{N}^c$) by $M(n,\epsilon,\delta)$. The *(weak) private capacity* of $\mathcal{N}$ is then defined as

$$P(\mathcal{N}) = \inf_{\epsilon,\delta>0} \liminf_{n\to\infty} \frac{1}{n} \log M(n,\epsilon,\delta).$$

The above quantity was determined in [7], [4], and like Q it is only known as a regularized characterization in general. By the monogamy of entanglement, we know that $P(\mathcal{N}) \geq Q(\mathcal{N})$, but in general this inequality is strict.

However for degradable channels, it was proved by Smith [25] that the private capacity $P(\mathcal{N})$ equals the quantum capacity $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$, and is hence given by a simple single-letter formula.

Next we state a theorem for a pretty strong converse for the private capacity.

Theorem 4 *Let $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ be a degradable channel with finite quantum systems A and B. Then, for error ϵ and privacy δ such that $\epsilon + 2\delta < \frac{1}{\sqrt{2}}$ (e.g. $\epsilon = \delta < \frac{1}{3\sqrt{2}} \approx .2357$), and every integer n,*

$$\log M(n,\epsilon,\delta) \leq nQ^{(1)}(\mathcal{N}) + O\left(\sqrt{n\log n}\right).$$

■

Together with the direct part (achievability proved in [7], [4]) we thus get:

Corollary 5 *For a degradable channel $\mathcal{N}$, the private capacity is given by*

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \log M(n,\epsilon,\delta),$$

for any $\epsilon,\delta > 0$ such that $\epsilon + 2\delta < \frac{1}{\sqrt{2}}$. ■

C. Strong converse for symmetric channels implies it for degradable channels

The main result of this subsection, Theorem 8, is valid for degradable channels satisfying the following technical condition.

Definition 6 *We say that a degradable channel $\mathcal{N}$ is of* type I *(for* invariance*) if one can choose a Stinespring dilation U of it, and a Stinespring dilation V of a degrading channel $\mathcal{M}$, such that the unitary X_F in Lemma 1 is a global phase (hence ± 1). I.e.,*

$$(\mathbb{1}_F \otimes \mathrm{SWAP}_{EE'})UV = \pm UV.$$

The following lemma relates degradable channels of type I to symmetric channels.

Lemma 7 *Let $\mathcal{N}$ be a degradable channel of type I, and choose a Stinespring dilation U as well as a dilation V of a degrading map, according to Lemma 1, s.t. $X_F = \pm\mathbb{1}$.*

For any test state $|\phi_0\rangle \in AA'$ of maximal Schmidt rank, let $|\psi_0\rangle^{AFEE'} = VU|\phi_0\rangle^{AA'}$ and denote the supporting subspace of ψ_0^{AF} by G.

Then there is a symmetric channel $\mathcal{M}$ with Stinespring isometry $W : G' \hookrightarrow E \otimes E'$ (i.e. $\mathrm{SWAP}_{EE'}W = \pm W$) such that every state $|\psi\rangle^{AFEE'} = VU|\phi\rangle$, for $|\phi\rangle \in AA'$ can be written as $W|\xi\rangle \in GEE'$ for a suitable test state $|\xi\rangle \in GG'$, up to a (state-dependent) isometry $\widehat{W} : G \hookrightarrow AF$:

$$|\psi\rangle^{AFEE'} = (\widehat{W} \otimes W)|\xi\rangle^{GG'}.$$

■

Finally we state our theorem that if a strong converse holds for symmetric channels then it holds for degradable channels.

Theorem 8 *Let $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ be a degradable channel, which w.l.o.g. we assume to be of type I (by Lemma 17 in [14]). Denote its environment by E and the associated symmetric channel by $\mathcal{M}$, with Stinespring dilation $W : G \hookrightarrow E \otimes E'$ from Lemma 7. Then $\mathcal{N}$ obeys the strong converse for its quantum capacity, if $\mathcal{M}$ does (note that by the no-cloning argument, $Q(\mathcal{M}) = 0$). More precisely, there exists a constant μ such that*

$$\log N_E(n,\epsilon|\mathcal{N}) \leq nQ^{(1)}(\mathcal{N}) + \mu\sqrt{n\ln\frac{64n^{|A|^2}}{\lambda^2}} + 8\log\frac{1}{\lambda} + O(\log n) + \log N_E(n,1-\lambda|\mathcal{M}),$$

with $\lambda = \frac{1-\epsilon}{5}$. ■

VI. Conclusion

For degradable quantum channels, whose quantum and private capacities are given by the single-letter maximization of the coherent information (which is then also additive on the class of all degradable channels), we have shown bounds on the optimal quantum and private classical rate, for every finite blocklength n. These bounds improve on the well-known weak converse in that they give asymptotically the capacity as soon as the error (parametrized by the purified distance) is small enough: for Q this was $\frac{1}{\sqrt{2}}$, the error of a 50%-50% erasure channel, for P we could get $\frac{1}{3\sqrt{2}}$. Since this says equivalently that the minimum attainable error jumps from 0 to at least some threshold as the coding rate increases above the capacity, we speak of a "pretty strong" converse (halfway between a weak and a proper strong converse).

We have shown furthermore that it is enough to prove a strong converse for certain universal symmetric (degradable and anti-degradable) channels, namely those whose Stinespring dilation is the embedding of $\mathrm{Sym}^2(E)$ into $E \otimes E'$ as a subspace; then the strong converse would follow for all degradable channels.

To close this discussion, we note that most channels are of course not degradable (or anti-degradable). For practically all these others we do not have any approach to obtain a strong or even a pretty strong converse. One might speculate that other channels with additive coherent information, hence with a single-letter capacity formula, are amenable, too, to our method. But already the very attractive-looking class of *conjugate degradable* [3] channels poses new difficulties.

Note on related work. Sharma and Wasri [20] posted a preprint where they show that one may formulate upper bounds on the fidelity of codes in terms of the rate and so-called generalized divergences. Their approach doesn't appear to be related to ours, but it is conceivable that it may lead to proofs of strong converses for certain channels' quantum capacity. This however seems to presuppose that channel parameters derived from these divergences have strong additivity properties, which can only hold for channels with additive coherent information.

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