Local Graph Coloring and Index Coding

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***Abstract*—We present a novel upper bound for the optimal index coding rate. Our bound uses a graph theoretic quantity called the local chromatic number. We show how a good local coloring can be used to create a good index code. The local coloring is used as an alignment guide to assign index coding vectors from a general position MDS code. We further show that a natural LP relaxation yields an even stronger index code. Our bounds provably outperform the state of the art on index coding but at most by a constant factor.**

I. Introduction

Index coding is a multiuser communication problem that is recently receiving significant attention. It is perhaps the simplest possible network problem (since it can mapped on a communication network with only one finite capacity link) and yet remarkable connections to many information theory problems have been recently discovered. The problem was introduced by Birk and Kol [1] and has received significant attention, see *e.g.* [2]–[5]. It was very recently shown that any (even nonlinear) network coding problem on an arbitrary graph can be reduced to an equivalent index coding instance [6]. Further, intriguing connections between index coding and the concept of interference alignment appear in [5]. Our work addresses the construction of index codes. Conceptually, we view index coding as an interference alignment problem, and show how index codes can be created by exploiting a special type of graph coloring.

In 1986, Erdős *et al.* [7] defined the *local chromatic number* of a graph. Given an undirected graph G, a coloring of the vertices is called proper if no two adjacent vertices receive identical colors. The *chromatic number* $\chi(G)$ is the minimum number of colors needed for such a proper coloring. The local chromatic number $\chi_\ell(G)$ is defined as the maximum number of different colors that appear in the closed neighborhood of any vertex, minimized over all proper colorings. Here, a closed neighborhood of vertex v includes v and all its neighbored vertices. For example, the five cycle C_5 has local chromatic number of 3 obtained by *any* valid coloring (as the closed neighborhood of any vertex in C_5 is of size 3) and it also has a chromatic number of 3. In general, it is clear that $\chi_\ell(G) \leq \chi(G)$. Erdős *et al.* [7] showed the non-trivial fact that the local chromatic number can indeed be arbitrarily smaller than $\chi(G)$.

In this work, we consider the index coding problem where user requests are distinct. For this case, the index coding problem is defined on a side-information graph G. Significant recent attention [2], [4], [5] has been focused on obtaining bounds for the optimal communication rate $\beta(G)$. An achievable scheme for index coding on undirected graphs is the number of cliques required to cover G, which is equal to the chromatic number of the complement graph $\chi(\bar{G})$. This corresponds to the well-known bound $\beta(G) \leq \chi(\bar{G})$.

The natural LP relaxation of this quantity is called the *fractional chromatic number* $\chi_f(G)$ which also corresponds to an achievable (vector-linear) index code (as shown in [4]). Therefore, it is known that $\beta(G) \leq \chi_f(\bar{G}) \leq \chi(\bar{G})$ and both inequalities can be strict for certain graphs. The fractional chromatic number is the best known general bound for index coding [4].

A. Our contribution

In this paper, we show that the *local chromatic number* provides an achievable index coding bound, i.e. $\beta(G) \leq \chi_\ell(\bar{G})$.

For directed graphs G_d, a natural generalization of the local chromatic number was defined by Körner *et al.* [8]. This was introduced to bound the Sperner capacity of a graph (the natural directed generalization of the Shannon graph capacity [9]).

The local chromatic number of a directed graph G_d, denoted $\chi_\ell(G_d)$, is the number of colors in the most colorful closed out-neighborhood of a vertex (defined formally later). We show that for any directed side information graph G_d: $\beta(G_d) \leq \chi_\ell(\bar{G}_d)$ where $\bar{G}_d$ is the directed complement of G_d. We also show that the natural LP relaxation of the local chromatic number, called the *fractional local chromatic number* $\chi_{f\ell}$ is a stronger bound on index coding, i.e. $\beta(G_d) \leq \chi_{f\ell}(\bar{G}_d)$. Note that there exist (directed) graphs where the fractional local chromatic number is strictly smaller than the fractional chromatic number, i.e. $\chi_{f\ell}(G_d) < \chi_f(G_d)$.

B. Comparison with previous results

We investigate the relation of our bounds to previously known results. For undirected graphs G (equivalently, bidirected digraphs), previous graph theoretic work [8] established that $\chi_{f\ell}(G) = \chi_f(G)$. Therefore for undirected graphs we obtain no new interesting bound.

For directed graphs, however, we show that there can be a linear additive gap between the local chromatic number and the fractional chromatic number. We explicitly construct a directed graph where $\chi_f = n$ and $\chi_\ell \leq \frac{n}{2} + 1$.

In terms of multiplicative gaps, we explicitly present a directed graph $\bar{G}_d$ for which $\chi_f(\bar{G}_d) > (2.5244)\, \chi_\ell(\bar{G}_d)$.

It was recently communicated to us [10] that this multiplicative gap cannot exceed the constant e for any directed graph $\bar{G}_d$, i.e. $\chi_f(\bar{G}_d) \leq \mathrm{e}\, \chi_{f\ell}(\bar{G}_d)$. In this work, we present a proof that the ratio $\chi_f(\bar{G}_d)/\chi_{f\ell}\left(\bar{G}_d\right)$ is at most a constant, obtained in parallel to our communication [10].

The remainder of this paper is structured as follows. We start with precise definitions of the index coding and graph coloring concepts we need. We then state our results that show how index codes can be created with the appropriate lengths related to various coloring numbers. We subsequently discuss the multiplicative and additive gaps between different coloring based bounds and conclude.

II. Definitions

In an index coding problem, there are n users each requesting a *distinct* packet x_i, $\forall i \in \{1, 2, 3 \ldots n\}$ from a common broadcasting agent who needs to deliver these packets with a minimum number of bits over a public broadcast channel. In addition, each user has some side information packets denoted by $S(i)$, which is a subset of packets that other users want. Let $[n]$ denote the set $\{1, 2 \ldots n\}$. Here, $S(i) \subseteq [n] - x_i$. This index coding problem with distinct user requests and individual user side information can be represented as a *directed side information graph* $G_d(V, E_d)$ where $(i, j) \in E_d$, i.e. there is a directed edge from node i to node j if user i has packet x_j as side information. Each node corresponds to a user in this digraph. Note that a more general version of index coding allows multiple users to request the same packet. This corresponds to hypergraphs and we do not consider it in this work. In what follows, we define the minimum broadcast rate for the index coding problem. To be consistent, we follow the definitions of Blasiak *et al.* [4] very closely.

Definition 1: (*Valid index code*) Let $x_i \in \Sigma$, where $|\Sigma| = 2^t$ for some integral t. Here, x_i is the packet desired by user i. A *valid index code* over the alphabet Σ is a set $(\phi, \{\gamma_i\})$ consisting of:

1) An encoding function $\phi : \Sigma^n \rightarrow \{0, 1\}^p$ which maps the n messages to a transmitted message of length p bits for some integral p.
2) n decoding functions γ_i such that for every user i, $\gamma_i(\phi\,(x_1, x_2 \ldots x_n)\,, \{x_j\}_{j \in S(i)}) = x_i$. In other words, every user is able to decode its desired message from the transmitted message and the side information available at user i.

$\Diamond$

The *broadcast rate* $\beta_\Sigma(G_d, \phi, \{\gamma_i\})$ of the $(\phi, \{\gamma_i\})$ index code is the number of transmitted bits per received message bit at every user, i.e. $\beta_\Sigma(G_d, \phi, \{\gamma_i\}) = \frac{p}{\log_2 |\Sigma|} = \frac{p}{t}$.

Definition 2: (*Minimum broadcast rate:*) The minimum broadcast rate $\beta(G_d)$ of the given problem $G_d(V, E_d)$ is the minimum possible broadcast rate of all valid index codes over all alphabets Σ, i.e. $\beta(G_d) = \inf\limits_{\Sigma} \inf\limits_{\phi, \{\gamma_i\}} \beta_\Sigma(G_d, \phi, \{\gamma_i\})$. $\Diamond$

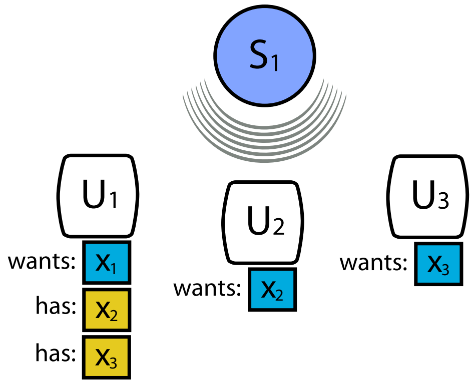

Fig. 1. Index coding example: We have three users U_1, U_2, U_3 and a broadcasting base station S_1. Each user has some side information packets and requests a distinct packet from S_1. The base station S_1 knows everything and can simultaneously broadcast to all three users noiselessly. User U_i requests packet x_i. User U_1 has packets x_2 and x_3 as side information while users U_2 and U_3 have no side information. In this example three transmissions are required, so $\beta = 3$.

If $t = 1$, and the encoding function $\phi : \{0, 1\}^n \rightarrow \{0, 1\}^p$ is a concatenation of separable linear encoding functions, i.e. $\phi = (\phi_1, \phi_2 \ldots \phi_p)$ such that $\phi_j : \{0, 1\}^n \rightarrow \{0, 1\}$ is linear over the input bits x_i, then the index code is called a valid *binary linear scalar* index code. The minimum broadcast rate over all binary linear scalar index codes for a given problem represented by digraph G_d is denoted $\beta_2(G_d)$. $\beta_2(G_d)$ [2] was shown to be equal to the graph parameter $\mathrm{minrank}_2(G_d)$. It is clear from the definitions that $\beta(G_d) \leq \beta_2(G_d)$.

We recall few bounds from [4] that are relevant to this work. The first one is the *fractional weak hyperclique cover*, denoted by ψ_f. The second is the fractional strong hyperclique cover, denoted by $\bar{\chi}_f$. We note that last two bounds are identical for the case when user requests are distinct or equivalently when the problem can be represented as a directed side information graph G_d. Hence, $\bar{\chi}_f(G_d) = \psi_f(G_d)$. We refer the reader to [4] for the exact definitions of hyperclique covers for the general problem. Here, we define $\bar{\chi}_f(G_d)$ only for a directed graph G_d. For this, we need the following definitions.

Definition 3: (*Interference graph*) The interference graph, denoted by $\bar{G}_d(V, \bar{E}_d)$ of an index coding problem is a *directed complement* of the directed side information graph G_d. For every vertex i, $(i, j) \in \bar{E}_d$ iff $(i, j) \notin E_d$. $\Diamond$

Definition 4: (*Underlying undirected side information graph*) Consider a directed side information graph $G_d(V, E_d)$. The *underlying undirected side information graph*, denoted by $G_u(V, E_u)$, is the graph obtained by deleting uni-directed edges (i.e. $(i, j) \in E_d$ but $(j, i) \notin E_d$) and all remaining bi-directed edges are replaced by an undirected edge, denoted by $\{i, j\}$. $\Diamond$

We observe that the complement of the underlying undirected side information graph G_u, denoted by $\bar{G}_u(V, \bar{E}_u)$, can alternatively be obtained by ignoring the orientation of the edges in the interference graph $\bar{G}_d$ (and, bi-directed edges in $\bar{G}_d$ can be replaced by a single undirected edge in $\bar{G}_u$). The

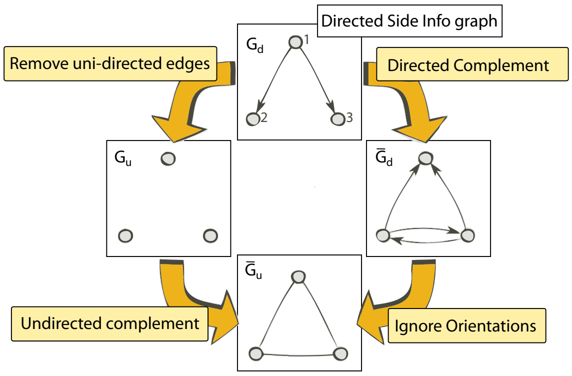

Fig. 2. Index coding representation using the directed side information graph G_d. There are two alternate ways to reach $\bar{G}_u$. One through G_u, the underlying undirected side information graph. The other way is through $\bar{G}_d$, the interference graph. Clearly $\chi(\bar{G}_u) = 3$.

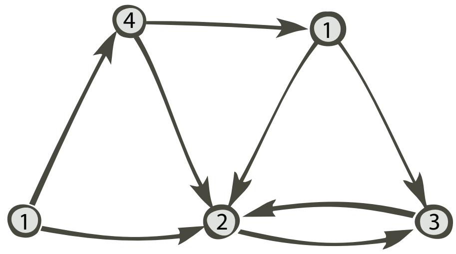

Fig. 3. Example of the local chromatic number. The vertices of this directed graph have been assigned the colors $\{1, 2, 3, 4\}$. This is a proper coloring of the underlying undirected graph. This coloring assignment corresponds to a local chromatic value of 3, since the most colorful closed out-neighborhood has 3 colors. For example, consider the unique vertex colored with color 4: its out-neighborhood has the colors $\{1, 2\}$ plus the color 4 for the vertex itself. The closed out-neighborhood of the vertex colored with color 3 has only two colors. The local chromatic number is computed by taking the minimum over proper colorings of the maximum number of colors in a closed out-neighborhood of a vertex. Note that in this graph there is a proper coloring with only 3 colors total. It is non-trivial to create graphs where the local chromatic number is strictly smaller than the chromatic number.

various graphs associated with the index coding problem and relationships between them are illustrated in Fig. 2. The graphs in Fig. 2 correspond to the index coding problem defined in Fig. 1

Definition 5: The fractional chromatic number of an undirected graph $G(V, E)$, denoted by $\chi_f(G)$, is given by the LP: $\min \sum_{I \in \mathcal{I}} x_I$ s.t. $\sum_{I : v \in I} x_I \geq 1 \ \forall v \in V, \ x_I \in \mathbb{R}^+ \ \forall I \in \mathcal{I}$ where $\mathcal{I}$ is the set of all independent sets in G and $\mathbb{R}^+$ is the set of non negative real numbers. $\Diamond$

Definition 6: $\bar{\chi}_f(G_d)$ is defined to be the fractional chromatic number, $\chi_f(\bar{G}_u)$, of the complement of the underlying undirected side information graph, denoted by $\bar{G}_u$. $\Diamond$

We note that the definition of $\bar{\chi}_f$ given here for directed side information graphs is equivalent to the definition of strong and weak hyper clique covers of [4] restricted to directed graphs.

It was shown [4] that $\beta(G_d) \leq \bar{\chi}_f(G_d)$ and $\bar{\chi}_f(G_d)$ corresponds to an achievable binary vector coding solution to the problem. In general, $\mathrm{minrank}_2(G_d)$ and $\bar{\chi}_f(G_d)$ are incomparable for digraphs. In this sense, the fractional chromatic number is one of the best known bounds for the index coding problem.

We consider two other graph parameters previously studied in [8] [11], namely, the local chromatic number $\chi_\ell(\bar{G}_d)$ and the fractional local chromatic number $\chi_{f\ell}(\bar{G}_d)$ of the interference graph $\bar{G}_d$, which we show have corresponding achievable index coding schemes and hence are related to the index coding problem.

Definition 7: (*Local chromatic number*) Denote the closed out-neighborhood of a given vertex i in a directed graph by $N^+(i)$, i.e. $j \in N^+(i)$ iff (i, j) is a directed edge or $j = i$. The local chromatic number of a directed graph $\bar{G}_d$ is the maximum number of colors in any out-neighborhood minimized over all proper colorings of the undirected graph obtained from $\bar{G}_u$ by ignoring the orientation of edges in $\bar{G}_d$. Let $c : V \to [k]$ be any proper coloring for $\bar{G}_u$ for some integer k. Let $|c(N^+(i))|$ be the number of colors in the closed out neighborhood of the directed graph taking the orientation into account. Then, $\chi_\ell(G) = \min_c \max_{i \in V} |c(N^+(i))|$ $\Diamond$

An example of local chromatic number is shown in Fig. 3.

Definition 8: The fractional local chromatic number of a directed graph $\bar{G}_d$, denoted by $\chi_{f\ell}\left(\bar{G}_d\right)$, is given by the LP:

$$\min t \tag{1}$$

$$\text{s.t.} \sum_{I : v \in I} x_I \geq 1, \ \forall v \in V$$

$$\sum_{I \cap N^+(v) \neq \phi} x_I \leq t \ \forall v \in V, \ x_I \in \mathbb{R}^+ \ \forall I \in \mathcal{I} \tag{2}$$

where $\mathcal{I}$ is the set of all independent sets in the undirected graph $\bar{G}_u$ obtained by ignoring the orientation of edges in $\bar{G}_d$. $\mathbb{R}^+$ is the set of non negative real numbers. $\Diamond$

The fractional chromatic number is a relaxation of the local chromatic number. The local chromatic number was initially defined in [7]. More recently, both the fractional local chromatic number and the local chromatic number have been studied in [8] in the context of bounds for Shannon and Sperner graph capacities.

Throughout this work, we will use the notion of a (p, q) Maximum-Distance Separable (MDS) code defined as a set of p vectors of length q that are in general position, *i.e.* any q of the p are linearly independent.

III. Achievability for Local Chromatic Numbers

In this section, we show that $\chi_\ell(\bar{G}_d)$ and $\chi_{f\ell}(\bar{G}_d)$ correspond to linear and vector linear achievable schemes over higher fields. We have the following achievability results.

Theorem 1: Given a directed side information graph G_d,

$$\beta(G_d) \leq \chi_\ell(\bar{G}_d).$$

Proof: Let $\mathcal{I}$ denote the family of independent sets in the undirected graph $\bar{G}_u$ or the graph obtained from $\bar{G}_d$ by ignoring the orientation of the edges. Coloring this graph, involves assigning 0's and 1's to the independent sets in the graph. Let $\mathcal{J} \subseteq \mathcal{I}$ be the set of color classes in the optimal local coloring. Let $\chi_\ell(\bar{G}_d)$ be the local coloring number. Let $J : V \rightarrow \mathcal{J}$ be the coloring function. To each color class (independent set assigned 1), we assign a column vector from $\mathbb{F}_q^m$ of a suitable length m and over a suitable field $\mathbb{F}_q$ by a map $\mathbf{b} : \mathcal{J} \rightarrow \mathbb{F}_q^m$. If the message desired by each user is from the finite field $\mathbb{F}_q$, i.e. $x_i \in \mathbb{F}_q,\ \forall i \in V$, then we transmit the vector

$$[\mathbf{b}(J(1)), \mathbf{b}(J(2)) \ldots \mathbf{b}(J(n))]\,[x_1, x_2 \ldots x_n]^T.$$

Clearly the length of the code is m field symbols. If the index code is valid, then the broadcast rate is m.

We now exhibit a mapping $\mathbf{b}$ with $m = \chi_\ell(\bar{G}_d)$ and $q \geq |\mathcal{J}|$. Let the colors classes in $\mathcal{J}$ be ordered in some way. Consider the generator matrix $\mathbf{G}$ of a $(|\mathcal{J}|, \chi_\ell(\bar{G}_d))$ MDS code over a suitable field $\mathbb{F}_q$ where $q \geq |\mathcal{J}|$. For instance, Reed Solomon code constructions could be used. Assign the different columns of $\mathbf{G}$ to each color class, i.e. $\mathbf{b}(j) = \mathbf{G}_j,\ \forall j \in \mathcal{J}$ where $\mathbf{G}_j$ is the j-th column. Under this mapping $\mathbf{b}$ and the previous description of the index code, it remains to be shown that this is a valid code. For any vertex i, the closed out-neighborhood $N^+(i)$ contains $|J(N^+(i))|$ colors. Because, the coloring J corresponds to the optimal local coloring, there are at most m colors in any closed out neighborhood. Therefore, $|J(N^+(i))| \leq \chi_\ell(\bar{G}_d) = m$.

Every vertex (user) i must be able to decode its own packet x_i. User i possesses packets x_k as side information when k is not in the closed out-neighborhood $N^+(i)$ in the interference graph $\bar{G}_d$. Hence, $\mathbf{b}(J(k))x_k$ can be canceled for all $k \notin N^+(i)$. The only interfering messages for user i are $\{\mathbf{b}(J(k))x_k\}_{k \in N^+(i)-\{i\}}$. If we show that $b(J(i))$ is linearly independent from all $\{\mathbf{b}(J(k))\}_{k \in N^+(i)-\{i\}}$, then user i would be able to decode the message x_i from its *interferers* in $N^+(i) - \{i\}$.

Since $\mathcal{J}$ represents a proper coloring over $\bar{G}_u$, $\mathbf{b}(J(i))$ is different from $\mathbf{b}(J(k))$ for any $k \in N^+(i) - \{i\}$. Also, any m distinct vectors are linearly independent by the MDS property of the generator $\mathbf{G}$. Since, $|J(N^+(i))| \leq \chi_\ell(\bar{G}_d) = m$, i.e. the number of colors in any closed out-neighborhood is at most χ_ℓ, the distinct vectors in any closed-out neighborhood are linearly independent . This implies that $b(J(i))$ is linearly independent from $\{\mathbf{b}(J(k))\}_{k \in N^+(i)-\{i\}}$. Hence, every user i would be able to decode the message it desires. Hence it is a valid index code and the broadcast rate is $\chi_\ell(\bar{G}_d)$. ∎

We show that adding a logarithmic additive overhead in the code length suffices to make the field size binary in the above scheme. This leads to the following corollary:

Corollary 1: Given a directed side information graph G_d, $\beta_2(G_d) \leq \chi_\ell(\bar{G}_d) + 2\log n$

Proof: Proof has been omitted due to lack of space and can be found in [12] ∎

The above corollary shows that, if field size is binary, with some extra length of $2 \log n$ over and above $\chi_\ell(\bar{G}_d)$ in Theorem 1, we can find a good index code.

Theorem 2: Given a directed side information graph G_d,

$$\beta(G_d) \leq \chi_{f\ell}(\bar{G}_d).$$

Proof: We combine the arguments of [4] for the achievability result for the fractional chromatic number and the above arguments of using MDS codes for the local chromatic number to produce an achievable index coding scheme for the fractional local chromatic number. For the interference graph, $\bar{G}_d$, let $\chi_{f\ell}(\bar{G}_d) = t$ in the program (1). This implies existence of the optimal solution $x_I \geq 0,\ x_I \in \mathbb{R}^+\ \forall I \in \mathcal{I}$ such that:

$$\sum_{I \cap N^+(v) \neq \phi} x_I \leq t\ \forall v \in V \text{ and } \sum_{I: v \in I} x_I \geq 1,\ \forall v \in V. \quad (3)$$

Here, we recall that $\mathcal{I}$ is a collection of all independent sets in the undirected graph, denoted $\bar{G}_u$, obtained from $\bar{G}_d$ ignoring the orientations of the edges. Since the program (1) has only integer coefficients, all x_I can be assumed to be rational. Let r be the least common multiple of all the denominators of the rationals x_I. Then we can redefine: $y_I = r x_I$. The equations become:

$$\sum_{I \cap N^+(v) \neq \phi} y_I \leq rt = s,\ \forall v \in V \text{ and } \sum_{I: v \in I} y_I \geq r,\ \forall v \in V. \quad (4)$$

Let $p = \sum_I y_I$. As y_I is integral, we will call y_I the number of 'colors' assigned to I. Since a subset of an independent set is independent, one can make sure all the inequalities $\sum_{I: v \in I} y_I \geq r$ are equalities by carrying out the following operation, repeatedly for any vertex v for which $\sum_{I: v \in I} y_I > r$, by choosing a suitable I: Choose the subset $J = I - \{v\}$ and increase y_J by 1 and decrease y_I by 1 if $y_I > 0$. This operation would not affect any of the inequalities including the locality constraints. For the locality constraints, the colors in any closed out-neighborhood will only reduce or remain the same due to this operation as colors for a bigger independent set are assigned to a smaller independent set contained in it.

After these operations, if $y_I > 1$, one may consider a *collection* of independent sets $\mathcal{I}$ where independent sets are repeated but each $y_I = 1$. Up until now, the arguments remain similar to that in [4] with the exception of taking note of the newly added locality constraints.

We have a sequence of p independent sets (possibly repeated) and every vertex is in r of them and the out-neighborhood sees at most s of them. Clearly, $s \geq r$. Like in the proof of Theorem 1, we generate a (p, s) MDS code over a field of size greater than p. To every independent set in the possibly repeated sequence of p sets, we assign one column of the generator matrix. Let $\mathcal{I}(v)$ denote the independent sets (possible repeated) that contain the vertex v. Under the assignment of columns to the independent sets (possibly repeated), all columns assigned to independent sets in $\mathcal{I}(v)$ are collected in the matrix $\mathbf{B}(v)$. Since every vertex is a part of exactly r independent sets (possibly repeated), $\mathbf{B}(v)$ is an $s \times r$ matrix. Since $r \leq s$, the columns of $\mathbf{B}(v)$ are linearly independent by the MDS property. Since the number of independent sets intersecting $N^+(v)$ is at most s, for any vertex v, columns of $\mathbf{B}(v)$ are linearly independent from the columns assigned to neighborhood vertices, given in a concatenated form as $[\mathbf{B}(u_1)\ \mathbf{B}(u_2)\ldots\mathbf{B}(u_k)]$ where $\{u_1\ u_2\ldots u_k\} = N^+(v) - \{v\}$.

Let each message $\mathbf{x}_i \in \mathbb{F}_q^{r\times 1}$ be a vector of r field symbols where $q \geq p$. The index code is given by: $[\mathbf{B}(1)\ \mathbf{B}(2)\ldots\mathbf{B}(n)]\left[(\mathbf{x}_1)^T\ \ (\mathbf{x}_2)^T\ldots(\mathbf{x}_n)^T\right]^T$. By the MDS property of the code used and by previous arguments, this code is a valid index code. The broadcast rate is $s/r = t = \chi_{f\ell}(\bar{G}_d)$. ■

IV. Multiplicative Gap between fractional and local chromatic variants

From the definition of the fractional local chromatic number, due to additional locality constraints, it is clear that for any directed graph G_d, $\chi_{f\ell}(\bar{G}_d) \leq \chi_f(\bar{G}_u) = \bar{\chi}_f(G_d)$. In other words, the fractional chromatic number uses just the information contained in the bi-directed edges of the side information graph. Through locality, the fractional local chromatic number exploits the directionality of certain uni-directed edges which are neglected by the fractional chromatic number. If the problem is over a bi-directed side information graph G_d (i has x_j iff j has packet x_i) or an undirected side information graph G, then, from results in [8], we have the following lemma:

Lemma 1: [8] $\chi_{f\ell}(\bar{G}) = \chi_f(\bar{G}) \leq \chi_\ell(\bar{G}) \leq \chi(\bar{G})$

To establish gaps, we consider digraphs which have some uni-directed edges. We give an example of a digraph (which is not bi-directed) over n edges where the difference between fractional and local chromatic number is $\Theta(n)$. Consider a complete undirected graph on n vertices. Number the vertices from 1 to n. Further, label the vertices 'odd' or 'even' depending on the number assigned to the vertices. Now, we construct a directed graph from the complete graph by orienting every edge in exactly one direction or the other. Consider two vertices numbered odd, i.e. $2p+1$ and $2q+1$. If $p < q$, there is an edge directed from vertex numbered $2p+1$ to the vertex numbered $2q+1$. Consider two vertices numbered $2p$ and $2q+1$. If $2p > 2q+1$, then there is a direced edge from vertex numbered $2p$ to the vertex numbered $2q+1$. If $2p < 2q+1$, then there is a direced edge from vertex numbered $2q+1$ to the vertex numbered $2p$. Consider two vertices numbered even, i.e. $2p$ and $2q$. If $p < q$, there is a directed edge from vertex numbered $2p$ to the vertex numbered $2q$. Consider any vertex numbered odd. It has edges directed outwards towards all odd vertices bigger than itself and all even vertices smaller than itself in the numbering. Similarly, consider any vertex numbered even. It has edges directed outwards towards all odd vertices smaller than itself and all even vertices bigger than itself. Therefore, the out-degree of every vertex is bounded by $\frac{n}{2}+1$. Hence, the local chromatic number of the directed version is at most $\frac{n}{2}+1$ and the fractional chromatic number is n.

We consider the question of a multiplicative gap. We came to know of a parallel work in progress [10] that has established a tighter upper bound of e for the ratio $\chi_f(\bar{G}_d)/\chi_{f\ell}(\bar{G}_d)$. In this work , using results regarding graph homomorphisms, we prove that the ratios $\chi_f(\bar{G}_d)/\chi_\ell(\bar{G}_d)$ and $\chi_f(\bar{G}_d)/\chi_{f\ell}(\bar{G}_d)$ are upper bounded by a constant. These results were obtained in parallel to [10].

Theorem 3: $\chi_f(\bar{G}_d)/\chi_\ell(\bar{G}_d) \leq \frac{5}{4}\mathrm{e}^2$

Proof: The proof is omitted and can be found in [12]. ■

Theorem 4: $\chi_f(\bar{G}_d)/\chi_{f\ell}(\bar{G}_d) \leq \frac{5}{4}\mathrm{e}^2$

Proof: The proof is omitted and can be found in [12]. ■

V. Conclusion

We presented novel index coding upper bounds based on local and fractional local chromatic numbers for the case when user requests are distinct. We presented a problem instance where the additive gap between the new bounds and the bound based on fractional chromatic number is arbitrarily large. We also proved that the multiplicative gap between the new bounds and the bound based on the fractional chromatic number is at most a constant. Studying local coloring concepts in the context of the general index coding problem with overlapping user requests is an interesting direction for future study.

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