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Optimal Codes in the Enomoto-Katona Space

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***Abstract*—Coding in a new metric space, the Enomoto-Katona space, is considered recently in connection to the study of implication structures of functional dependencies and their generalizations in relational databases. The central problem here is the determination of $C(n,k,d)$, the size of an optimal code of length n, weight k, and distance d in the Enomoto-Katona space. The value of $C(n,k,d)$ is known only for some congruence classes of n when $(k,d) \in \{(2,3),(3,5)\}$. In this paper, we obtain new infinite families of optimal codes in the Enomoto-Katona space. In particular, $C(n,k,2k-1)$ is determined for all sufficiently large n satisfying either $n \equiv 1 \bmod k$ and $n(n-1) \equiv 0 \bmod 2k^2$, or $n \equiv 0 \bmod k$.**

1. Introduction

The problem we consider is motivated by implication structures of functional dependencies in relational databases.

Let A be a set of n *attributes*. Each attribute $x \in A$ is associated a set Ω_x, called its *domain*. A *relation* is a finite set R of n-tuples (called *data items*) so that $R \subseteq \times_{x\in A}\Omega_x$. A relation R of m data items may be visualized as an $m\times n$ array (called a *table*), with columns indexed by A, such that each row corresponds to a data item. Denote this table by $R(A)$. Formally, if $R = \{(\mathsf{d}_{i,x})_{x\in A} : 1 \le i \le m\}$, then the cell in $R(A)$ with row index i and column index x has entry $\mathsf{d}_{i,x}$. A *relational database* is a set of tables, where tables may be defined over different attribute sets. Relational database, introduced by Codd [1], is the first database with a rigorous mathematical foundation, and remains the predominant choice for data storage and management today.

For a given table $R(A)$ and $X \subseteq A$, the X-*value* of a data item $\mathsf{d} = (\mathsf{d}_x)_{x\in A}$ in $R(A)$ is the $|X|$-tuple $\mathsf{d}\,|_X = (\mathsf{d}_x)_{x\in X}$. Let $X \subseteq A$ and $y \in A$ for a given table $R(A)$. We say that y *(functionally) depends*[1] on X, written $X \to y$, if no two rows of $R(A)$ agree in X but differ in y. In other words, if the X-value of a data item is known, then its $\{y\}$-value can be determined with certainty. Identifying functional dependencies is important in relational database design [2]–[5].

Demetrovics, Katona, and Sali [6] generalized functional dependencies as follows.

Definition 1.1. Let $X \subseteq A$ and $y \in A$ for a given table $R(A)$. Then for positive integers $p \le q$, we say that y (p,q)-*depends* on X, written $X \xrightarrow{(p,q)} y$, if there do not exist $q+1$ data items (rows) $\mathsf{d}_1, \mathsf{d}_2, \ldots, \mathsf{d}_{q+1}$ of $R(A)$ such that

(i) $|\{\mathsf{d}_i\,|_{\{x\}} : 1 \le i \le q+1\}| \le p$ for each $x \in X$, and
(ii) $|\{\mathsf{d}_i\,|_{\{y\}} : 1 \le i \le q+1\}| = q+1$.

Our usual concept of functional dependency is equivalent to the special case of $(1,1)$-dependency. When functional dependencies are not known, (p,q)-dependencies identified in a relational database can still be exploited for improving storage efficiency [6]–[9].

Let $p \le q$ be positive integers. For a table $R(A)$, define the operation $J^{(p,q)}_{R(A)} : 2^A \to 2^A$ so that for $X \subseteq A$, we have

$$J^{(p,q)}_{R(A)}(X) = \left\{ y \in A : X \xrightarrow{(p,q)} y \right\}.$$

We call $J^{(p,q)}_{R(A)}$ the (p,q)-*implication structure* of $R(A)$, since it specifies the subsets of attributes that are implied by some (p,q)-dependency of $R(A)$. A function $J : 2^A \to 2^A$ is said to be (p,q)-*representable* if there exists a table $R(A)$ such that $J^{(p,q)}_{R(A)} = J$.

The function $J^{(1,1)}_{R(A)}$ is a closure operator on A. Armstrong [2] showed that the converse is also true: any closure operator $J : 2^A \to 2^A$ is $(1,1)$-representable. This is, however, not true for general p and q [6]. When a function J is (p,q)-representable, there is interest in determining the table $R(A)$ with the least number of rows such that $J^{(p,q)}_{R(A)} = J$ [7]–[9]. Consideration of this problem, particularly when for fixed k, the function $J^k_n : 2^A \to 2^A$ takes the form

$$J^k_n(X) = \begin{cases} X, & \text{if } |X| < k \\ A, & \text{otherwise,} \end{cases}$$

has led to coding-theoretic problems in a new metric space, called the *Enomoto-Katona space* [10].

A. The Enomoto-Katona Space

If X is a finite set, the set of all k-subsets of X is denoted $\binom{X}{k}$. Let n and k be positive integers such that $2k \le n$ and let X be an n-set. Consider the set

$$\mathcal{E}(X,k) = \left\{ \{A,B\} \subseteq \binom{X}{k} : A \cap B = \varnothing \right\}$$

of all unordered pairs of disjoint k-subsets of X. Elements of $\mathcal{E}(X,k)$ are called *set-pairs*. The function $\mathrm{d}_{\mathcal{E}} : \mathcal{E}(X,k) \times \mathcal{E}(X,k) \to \{0,1,\ldots,2k\}$ given by

$$\mathrm{d}_{\mathcal{E}}(\{A,B\},\{S,T\}) = \min\{|A\setminus S|+|B\setminus T|, |A\setminus T|+|B\setminus S|\}$$

is a metric of $\mathcal{E}(X,k)$ and the finite metric space $(\mathcal{E}(X,k), \mathrm{d}_{\mathcal{E}})$ is called the *Enomoto-Katona space*.

An *Enomoto-Katona code* (or *EK code*, in short), is a set $\mathcal{C} \subseteq \mathcal{E}(X,k)$. More specifically, $\mathcal{C}$ is an EK code of *length* n, *weight* k, and *distance* d, or (n,k,d)-EK code, if $\mathrm{d}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) \ge d$ for all distinct $\mathsf{u},\mathsf{v} \in \mathcal{C}$.

The following example gives a construction of a table from an EK-code (see [8], [11]).

[1] By definition, if $y \in X$, then $X \to y$ trivially.

Example 1.1. Consider the following $(9,2,3)$-EK code $\mathcal{C}$, where $X = \mathbb{Z}/9\mathbb{Z}$.

$$\begin{aligned} c_1 &= \{\{0,1\},\{2,4\}\}, & c_2 &= \{\{1,2\},\{3,5\}\}, & c_3 &= \{\{2,3\},\{4,6\}\}, \\ c_4 &= \{\{3,4\},\{5,7\}\}, & c_5 &= \{\{4,5\},\{6,8\}\}, & c_6 &= \{\{5,6\},\{7,0\}\}, \\ c_7 &= \{\{6,7\},\{8,1\}\}, & c_8 &= \{\{7,8\},\{0,2\}\}, & c_9 &= \{\{8,0\},\{1,3\}\}. \end{aligned}$$

Let A be a set of nine attributes, given by $\mathcal{C}$. We construct a table $R(A)$ with nine rows indexed by X whose implication structure $J_{R(A)}^{(1,1)}$ is precisely J_9^2. Each set-pair $\{A, B\}$ constructs a column in the following manner: place 1 at rows indexed by elements of A, place 2 at rows by elements of B and place distinct elements from $\mathbb{Z}_{\geq 3}$ for the remaining rows.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
0	1	3	3	3	3	2	3	2	1
1	1	1	4	4	4	3	2	3	2
2	2	1	1	5	5	4	4	2	3
3	3	2	1	1	6	5	5	4	2
4	2	4	2	1	1	6	6	5	4
5	4	2	5	2	1	1	7	6	5
6	5	5	2	6	2	1	1	7	6
7	6	6	6	2	7	2	1	1	7
8	7	7	7	7	2	7	2	1	1

The maximum size of an (n,k,d)-EK code is denoted by $C(n,k,d)$. An (n,k,d)-EK code of size $C(n,k,d)$ is said to be *optimal*. The central problem is to determine $C(n,k,d)$.

B. Problem Status

Trivially, $C(n,k,1) = \binom{n}{k}\binom{n-k}{k}/2$, $C(n,k,2k) = \lfloor n/2k \rfloor$, so we assume $2 \leq d \leq 2k-1$ for the rest of this paper.

General upper and lower bounds on the size of codes in the Enomoto-Katona space have been obtained by Brightwell and Katona [12]. In particular, they showed for $1 \leq d \leq 2k \leq n$,

$$C(n,k,d) \leq \frac{\prod_{i=n-2k+d}^{n} i}{2\left(\prod_{i=\lceil (d+1)/2 \rceil}^{k} i\right) \cdot \left(\prod_{i=\lfloor (d+1)/2 \rfloor}^{k} i\right)}. \quad (1)$$

Brightwell and Katona [12] also showed that $C(n,k,d) = \Theta(n^{2k-d+1})$ for fixed k and d. Bollobás *et al.* [13] (see also [11]) subsequently established that the upper bound in (1) is asymptotically tight.

Theorem 1.1 (Bollobás et al. [13])**.**

$$\lim_{n\to\infty} \frac{C(n,k,d)}{n^{2k-d+1}} = \frac{1}{2 \cdot \left(\prod_{i=\lceil (d+1)/2 \rceil}^{k} i\right) \cdot \left(\prod_{i=\lfloor (d+1)/2 \rfloor}^{k} i\right)}.$$

The best known upper bound is due to Quistorff [14].

Theorem 1.2 (Quistorff Bound [14])**.** Suppose $k - d + 1 \leq e \leq \min\{k, 2k-d\}$. Then

$$C(n,k,d) \leq \left\lfloor \frac{\binom{n}{e}}{2\binom{k}{e}} \left\lfloor \frac{\binom{n-e}{2k-d-e+1}}{\binom{k}{2k-d-e+1}} \right\rfloor \right\rfloor.$$

Only the following exact values of $C(n,k,d)$ are known.

Theorem 1.3 (Bollobás et al. [13])**.**

$$C(n,2,3) = \frac{n(n-1)}{8}, \qquad \text{if } n \equiv 1 \text{ or } 9 \bmod 72,$$

$$C(n,3,5) = \frac{n(n-1)}{18}, \qquad \text{if } n \equiv 1 \text{ or } 19 \bmod 342.$$

C. Contributions

Our contributions in this paper are as follows.

Main Theorem. For any fixed $k \geq 2$, we have

$$C(n,k,2k-1) = \left\lfloor \frac{n}{2k} \left\lfloor \frac{n-1}{k} \right\rfloor \right\rfloor$$

for all sufficiently large n satisfying

(i) $n \equiv 1 \bmod k$ and $n(n-1) \equiv 0 \bmod 2k^2$, or
(ii) $n \equiv 0 \bmod k$.

Previous asymptotic results are known only when $k \in \{2,3\}$. In addition,

(i) We determine the exact value of $C(n,2,d)$ completely. Previously, the value of $C(n,2,2)$ is unknown and $C(n,2,3)$ is determined only when $n \equiv 1$ or $9 \bmod 72$.
(ii) The exact value of $C(n,3,5)$ is determined for n belonging to a set of density $4/9$. Previously, the exact value of $C(n,3,5)$ is known only for $n \equiv 1$ or $19 \bmod 342$, a set of density $1/171$.

These results are obtained by constructing EK codes (or their equivalent combinatorial objects) whose sizes meet the Quistorff bound. Owing to space constraints, we prove the Main Theorem and determine $C(n,2,2)$ in this paper, leaving the proofs for the remaining results to the full paper.

2. EK Packings and Designs

Our approach is based on combinatorial design theory. In this section, we introduce necessary concepts and establish connections to EK codes.

Throughout the rest of this paper, X denotes a set of size n. For a positive integer k, $[k]$ denotes the set of integers $\{1,2,\ldots,k\}$, while $\mathbb{Z}_{\geq k}$ denotes the set of integers at least k. The set of all (ordered) k-tuples of a finite set X with distinct components is denoted $\overline{\binom{X}{k}}$.

We use angled brackets $\langle$ and $\rangle$ for multisets. We sometimes use the exponential notation to describe multisets so that a multiset where an element g_i appears s_i times, $i \in [t]$, is denoted $g_1^{s_1} g_2^{s_2} \cdots g_t^{s_t}$.

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $\mathfrak{S}$ is the number of points in X, and the *size* of $\mathfrak{S}$ is the number of blocks in $\mathcal{A}$. Let $K \subseteq \mathbb{Z}_{\geq 0}$. The set system $(X, \mathcal{A})$ is said to be *K-uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

Let $2 \leq t < 2k$ and $0 \leq e \leq \min\{k, \lfloor t/2 \rfloor\}$. We say that the tuple $(x_1, x_2, \ldots, x_t) \in \overline{\binom{X}{t}}$ is *(e,t)-contained* in a set-pair $\{A, B\} \in \mathcal{E}(X,k)$ if either $\{x_1, x_2, \ldots, x_e\} \subseteq A$ and $\{x_{e+1}, x_{e+2}, \ldots, x_t\} \subseteq B$, or $\{x_1, x_2, \ldots, x_e\} \subseteq B$ and $\{x_{e+1}, x_{e+2}, \ldots, x_t\} \subseteq A$.

Let $\mathcal{C} \subseteq \mathcal{E}(X,k)$. Then $(X, \mathcal{C})$ is an EK packing of *strength* t, or more precisely a *t-(n,k) EK packing*[2], if for $0 \leq e \leq \lfloor t/2 \rfloor$, every t-tuple in $\overline{\binom{X}{t}}$ is (e,t)-contained in at most one set-pair in $\mathcal{C}$. A *t-(n,k) EK design* is a t-(n,k) EK packing satisfying the condition that for $e = \lfloor t/2 \rfloor$, every t-tuple in

[2]Note that $\mathcal{C} \subseteq \mathcal{E}(X,k)$, while $\mathcal{A} \subseteq 2^X$.

$\overline{\binom{X}{t}}$ is (e,t)-contained in exactly one set-pair in $\mathcal{C}$. It is easy to see that if $(X,\mathcal{C})$ is a t-(n,k) EK design, then

$$|\mathcal{C}| = \frac{\binom{n}{t}\binom{t}{\lfloor t/2\rfloor}}{2\binom{k}{\lfloor t/2\rfloor}\binom{k}{\lceil t/2\rceil}}.$$

EK packings of strength t are equivalent to EK codes of distance $2k-t+1$, while EK designs of strength t give rise to optimal EK codes of distance $2k-t+1$.

Proposition 2.1. Let $\mathcal{C} \subseteq \mathcal{E}(X,k)$. Then $(X,\mathcal{C})$ is a t-(n,k) EK packing if and only if $\mathcal{C}$ is an $(n,k,2k-t+1)$-EK code. Furthermore, if $(X,\mathcal{C})$ is a t-(n,k) EK design, then $\mathcal{C}$ is an optimal $(n,k,2k-t+1)$-EK code.

Proof: Suppose $(X,\mathcal{C})$ is a t-(n,k) EK packing and $\{A,B\}$, $\{S,T\} \in \mathcal{C}$. We claim that $\mathrm{d}_{\mathcal{E}}(\{A,B\},\{S,T\}) \geq 2k-t+1$. Suppose otherwise. Then without loss of generality, $|A\setminus S|+|B\setminus T| \leq 2k-t$ and there exists a nonnegative $e \leq \lfloor t/2\rfloor$, $I \in \binom{X}{e}$, $J \in \binom{X}{t-e}$ such that $I \subseteq A\cap S$ and $J \subseteq B\cap T$. If $I = \{x_1,x_2,\ldots,x_e\}$ and $J = \{x_{e+1},x_{e+2},\ldots,x_t\}$, we see that $(x_1,x_2,\ldots,x_t)$ is (e,t)-contained in $\{A,B\}$ and $\{S,T\}$, contradicting the fact that $(X,\mathcal{C})$ is a t-(n,k) EK packing.

Conversely, suppose $\mathcal{C}$ is an $(n,k,2k-t+1)$-EK code. If $(X,\mathcal{C})$ is not a t-(n,k) EK packing, then there exists a nonnegative $e \leq \lfloor t/2\rfloor$, $(x_1,x_2,\ldots,x_t) \in \overline{\binom{X}{t}}$, and $\{A,B\}$, $\{S,T\} \in \mathcal{C}$ such that $(x_1,x_2,\ldots,x_t)$ is (e,t)-contained in $\{A,B\}$ and $\{S,T\}$. Without loss of generality, $\{x_1,x_2,\ldots,x_e\} \subseteq A\cap S$ and $\{x_{e+1},x_{e+2},\ldots,x_t\} \subseteq B\cap T$. Hence, $|A\setminus S|+|B\setminus T| \leq 2k-(e+t-e) = 2k-t$, and consequently $\mathrm{d}_{\mathcal{E}}(\{A,B\},\{S,T\}) \leq 2k-t$, contradicting the fact that $\mathcal{C}$ is an $(n,k,2k-t+1)$-EK code.

Finally, when $(X,\mathcal{C})$ is a t-(n,k) EK design, $\mathcal{C}$ is an optimal $(n,k,2k-t+1)$-EK code, since $|\mathcal{C}|$ meets the Quistorff bound with $e = \lfloor t/2\rfloor$. ■

In view of Proposition 2.1, our strategy in constructing optimal EK codes (and hence determining $C(n,k,d)$) is to construct equivalent EK packings and designs of sizes meeting the Quistorff bound. We introduce next EK group divisible designs and their connections to EK codes and EK packings.

A. EK Group Divisible Designs

Let $\mathcal{G} = \{G_1,G_2,\ldots,G_s\}$ be a partition of an n-set X and $\mathcal{C} \subseteq \mathcal{E}(X,k)$. Then $(X,\mathcal{G},\mathcal{C})$ is an *EK group divisible design* (or EKGDD, in short) if for all $(x,y) \in \overline{\binom{X}{2}}$ such that $\{x,y\} \not\subseteq G_i$ for all $i \in [s]$, we have

(i) (x,y) is $(1,2)$-contained in exactly one set-pair $\{A,B\}$,
(ii) (x,y) is $(0,2)$-contained in at most one set-pair $\{A,B\}$.

In addition, $|G_i\cap(A\cup B)| \leq 1$ for all $i \in [s]$ and $\{A,B\} \in \mathcal{C}$. Such an EKGDD is more precisely called a (k,T)-EKGDD, where $T = \langle |G_i| : i \in [s]\rangle$.

A 2-(n,k) EK design can be regarded as a $(k,1^n)$-EKGDD, where each group contains just a single point. Furthermore, a $(k,g_1g_2\cdots g_s)$-EKGDD can be regarded as a 2-$(k,\sum_{i=1}^s g_i)$ EK packing, and hence as a $(\sum_{i=1}^s g_i,k,2k-1)$-EK code. In addition, as the following shows, certain classes of EKGDD give optimal EK codes.

Proposition 2.2. Suppose there exists a (k,k^s)-EKGDD $(X,\mathcal{G},\mathcal{C})$. Then $\mathcal{C}$ is an optimal $(ks,k,2k-1)$-EK code.

Proof: Observe $\mathcal{C}$ is a $(ks,k,2k-1)$-EK code since $(X,\mathcal{C})$ is an 2-(ks,k) EK packing. There are $(ks)\cdot(ks-k)$ ordered pairs $(x,y) \in \overline{\binom{X}{2}}$ where $\{x,y\}$ does not belong to any group. In addition, we have $2k^2$ ordered pairs in $\overline{\binom{X}{2}}$ that are $(1,2)$-contained in each set-pair. Hence, the code $\mathcal{C}$ is of size $s(s-1)/2$, which meets the Quistorff bound. ■

3. $C(n,k,2k-1)$ for Sufficiently Large n

We show that a 2-(n,k) EK design and a (k,k^n)-EKGDD exist when n belongs to certain congruence classes, provided n is sufficiently large. Our proof is an application of decompositions of edge-colored directed graphs (digraphs).

An *edge-colored directed graph* is a triple $G = (V,C,E)$, where V is a finite set of *vertices*, C is a finite set of *colors* and E is a subset of $\overline{\binom{V}{2}} \times C$. Members of E are called *edges*. The *complete edge-colored digraph* on n vertices with r colors, denoted by $K_n^{(r)}$, is the edge-colored digraph (V,C,E), where $|V| = n$, $|C| = r$, and $E = \overline{\binom{V}{2}} \times C$.

A family $\mathcal{F}$ of edge-colored subgraphs of an edge-colored digraph K is a *decomposition* of K if every edge of K belongs to exactly one member of $\mathcal{F}$. Given an edge-colored digraph G, a decomposition $\mathcal{F}$ of K is a *G-decomposition of* K if each edge-colored digraph in $\mathcal{F}$ is isomorphic to G.

Lamken and Wilson [15] studied the existence of G-decompositions of $K_n^{(r)}$ and showed that for fixed G and r, a G-decomposition exists for sufficiently large n under certain conditions. To state the theorem, we require more concepts.

Consider an edge-colored digraph $G = (V,C,E)$ with $|C| = r$. Let $((u,v),c) \in E$ denote a directed edge from u to v, colored by c. For any vertex u and color c, define the *indegree* and *outdegree* of u with respect to c as follows:

$$\mathrm{in}_c(u) = |\{v : ((v,u),c) \in E\}|,$$
$$\mathrm{out}_c(u) = |\{v : ((u,v),c) \in E\}|.$$

Then for vertex u, we define the *degree vector* of u, denoted by $\boldsymbol{\delta}(u)$, to be the vector of length $2r$. That is, $\boldsymbol{\delta}(u) = (\mathrm{in}_c(u),\mathrm{out}_c(u))_{c\in C}$. Define $\alpha(G)$ to be the least positive integer t such that $(t,t,\ldots,t)$ is an integral linear combination of the vectors in $\{\boldsymbol{\delta}(u) : u \in V\}$. The following is due to Lamken and Wilson [15].

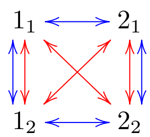
Theorem 3.1 (Lamken and Wilson [15, Theorem 1.1])**.** Let G be an edge-colored digraph with r colors and m edges of each of r different colors. There exists a constant n_0 such that there is G-decomposition of $K_n^{(r)}$ for all $n \geq n_0$ satisfying both

$$n(n-1) \equiv 0 \bmod m \text{ and } n-1 \equiv 0 \bmod \alpha(G).$$

Now for fixed $k \geq 2$ define the edge-colored digraph $G_k = (V_k, C_k, E_k)$, where

$$\begin{aligned} V_k &= \{i_j : i \in [k], j \in [2]\}, \\ C_k &= \{{\color{red}\bullet}, {\color{blue}\bullet}\}, \\ E_k &= \{((i_r, j_s), {\color{red}\bullet}) : i, j \in [k], (r, s) \in \{(1,2),(2,1)\}\} \\ &\quad \cup \{((i_r, i_s), {\color{blue}\bullet}) : i \in [k], (r, s) \in \{(1,2),(2,1)\}\} \\ &\quad \cup \left\{((i_r, j_r), {\color{blue}\bullet}) : (i, j) \in \overline{\binom{[k]}{2}}, r \in [2]\right\}. \end{aligned}$$

Example 3.1. The edge-colored graph G_2 is given by



where ${\color{blue}\longleftrightarrow}$ denotes two directed edges of color ${\color{blue}\bullet}$ (one in each direction), and ${\color{red}\longleftrightarrow}$ denotes two directed edges of color ${\color{red}\bullet}$ (one in each direction).

Proposition 3.1. If a G_k-decomposition of $K_n^{(2)}$ exists, then a 2-(n, k) EK design exists.

Proof: Let $\mathcal{F}$ be a G_k-decomposition of $K_n^{(2)}$. Then for a subgraph $G \in \mathcal{F}$, let $\phi_G : G_k \to G$ be a graph isomorphism and define

$$A_G = \{\phi_G(i_1) : i \in [k]\}, \quad B_G = \{\phi_G(i_2) : i \in [k]\}.$$

Let X be the vertex set of $K_n^{(2)}$ and

$$\mathcal{C} = \{\{A_G, B_G\} : G \in \mathcal{F}\}.$$

We claim that $(X, \mathcal{C})$ is a 2-(n, k) EK design. Since $|\mathcal{C}| = n(n-1)/(2k(k-1))$, it suffices to check that for $e \in \{0, 1\}$, each $(x, y) \in \overline{\binom{X}{2}}$ is $(e, 2)$-contained in at most one set-pair in $\mathcal{C}$.

Suppose otherwise. Then there exist $(x, y) \in \overline{\binom{X}{2}}$, $G, H \in \mathcal{F}$ and $e \in \{0, 1\}$ such that (x, y) is $(e, 2)$-contained in $\{A_G, B_G\}$ and $\{A_H, B_H\}$.

If $e = 0$, then assume that $\{x, y\} \subset A_G \cap A_H$. Hence, the edge $((x, y), {\color{red}\bullet})$ belongs to both G and H, contradicting the fact that $\mathcal{F}$ is a G_k-decomposition of $K_n^{(2)}$.

If $e = 1$, then assume that $x \in A_G \cap A_H$ and $y \in B_G \cap B_H$. Hence, the edge $((x, y), {\color{blue}\bullet})$ belongs to G and H, contradicting the fact that $\mathcal{F}$ is a G_k-decomposition of $K_n^{(2)}$. ∎

Observe there are $2k^2$ edges of each color in G_k and $\boldsymbol{\delta}(v) = (k, k, k, k)$ for all $v \in V_k$. Hence, $\alpha(G_k) = k$. The following is immediate from Propositions 2.1, 3.1, and Theorem 3.1.

Theorem 3.2. Fix $k \geq 2$. Then

$$C(n, k, 2k-1) = \frac{n(n-1)}{2k^2}$$

for all sufficiently large n satisfying $n \equiv 1 \bmod k$ and $n(n-1) \equiv 0 \bmod 2k^2$.

To determine $C(n, k, 2k-1)$ when $n \equiv 0 \bmod k$, consider the following graph. Fix $k \geq 2$ and define the edge-colored digraph $H_k = (V_k, C_k, E_k)$, where

$$\begin{aligned} V_k &= \{i_j : i \in [k], j \in [2]\}, \\ C_k &= ([k] \times [k] \times \{{\color{red}\bullet}\}) \cup \left(\overline{\binom{[k]}{2}} \times \{{\color{blue}\bullet}\}\right), \\ E_k &= \{((i_r, j_s), (i, j, {\color{red}\bullet})) : i, j \in [k], (r, s) \in \{(1,2),(2,1)\}\} \\ &\quad \cup \left\{((i_r, j_r), (i, j, {\color{blue}\bullet})) : (i, j) \in \overline{\binom{[k]}{2}}, r \in [2]\right\}. \end{aligned}$$

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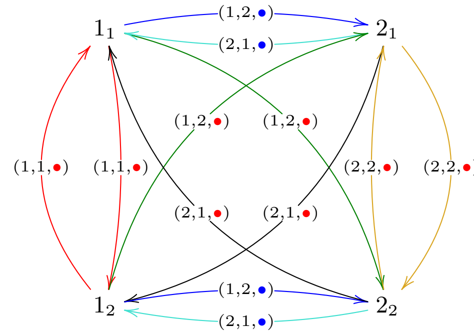

Proposition 3.2. If an H_k-decomposition of $K_n^{(2k^2-k)}$ exists, then a (k, k^n)-EKGDD exists.

Proof: Let $\mathcal{H}$ be an H_k-decomposition of $K_n^{(2k^2-k)}$. Then for a subgraph $H \in \mathcal{H}$, let $\phi_H : H_k \to H$ be a graph isomorphism and define

$$A_H = \{\phi_H(i_1)_i : i \in [k]\}, \quad B_H = \{\phi_H(i_2)_i : i \in [k]\}.$$

Let V be the vertex set of $K_n^{(2k^2-k)}$ and

$$\begin{aligned} X &= \{v_i : v \in V, i \in [k]\}, \\ \mathcal{G} &= \{\{v_i : i \in [k]\} : v \in V\}, \\ \mathcal{C} &= \{\{A_H, B_H\} : H \in \mathcal{H}\}. \end{aligned}$$

We claim that $(X, \mathcal{G}, \mathcal{C})$ is a (k, k^n)-EKGDD. Suppose otherwise. Since $|\mathcal{C}| = n(n-1)/2$, it suffices to consider the following two cases.

(i) There exist $v \in V$ and $H \in \mathcal{H}$ such that $|\{v_i : i \in [k]\} \cap (A_H \cup B_H)| \geq 2$. This contradicts the fact that H is isomorphic to H_k.

(ii) There exist $(x, y) \in \overline{\binom{X}{2}}$, $G, H \in \mathcal{H}$ and $e \in \{0, 1\}$ such that (x_i, y_j) is $(e, 2)$-contained in $\{A_G, B_G\}$ and $\{A_H, B_H\}$.
If $e = 0$, then assume that $\{x_i, y_j\} \subset A_G \cap A_H$. Hence, the edge $((x, y), (i, j, {\color{red}\bullet}))$ belongs to both G and H, contradicting the fact that $\mathcal{H}$ is an H_k-decomposition.
Similarly, if $e = 1$, then assume that $x_i \in A_G \cap A_H$ and $y_j \in B_G \cap B_H$. Hence, the edge $((x, y), (i, j, {\color{blue}\bullet}))$ belongs to both G and H, contradicting the fact that $\mathcal{H}$ is an H_k-decomposition of $K_n^{(2k^2-k)}$.

∎

Observe there are two edges of each color in H_k and $\sum_{i\in[k]} \boldsymbol{\delta}(i_1) = (1,1,\ldots,1)$. Hence, $\alpha(H_k) = 1$. From Propositions 2.2, 3.2, and Theorem 3.1, we have the following.

Theorem 3.3. Fix $k \geq 2$. Then

$$C(n,k,2k-1) = \frac{n(n-k)}{2k^2}$$

for all sufficiently large n satisfying $n \equiv 0 \bmod k$.

Theorems 3.2 and 3.3 combine to give the Main Theorem.

4. The Value of $C(n,2,2)$

In this section, we give a complete solution for $C(n,2,2)$. Our proof makes use of t-wise balanced designs.

Definition 4.1. A *t-wise balanced design*, or a t-BD(v,K), is a K-uniform set system $(X,\mathcal{A})$ of order v such that every t-subset of X is contained in exactly one block of $\mathcal{A}$.

The following existence result for 3-BDs is known.

Theorem 4.1 (Hanani [16])**.** A 3-BD$(v,\{4,6\})$ exists for all even $v \geq 4$.

The following proposition gives a recursive construction for EK designs of strength t.

Proposition 4.1 (Filling in Blocks)**.** Let $K \subseteq \mathbb{Z}_{\geq 1}$ and suppose that a t-BD(v,K) exisits. If a t-(h,k) EK design exists for all $h \in K$, then a t-(v,k) EK design exists.

Proof: Let $(X,\mathcal{A})$ be a t-BD(v,K). For each $A \in \mathcal{A}$, let $(A,\mathcal{C}_A)$ be a t-$(|A|,k)$ EK design. Then $(X, \cup_{A\in\mathcal{A}}\mathcal{C}_A)$ is a t-(v,k) EK design. ■

We first determine $C(n,2,2)$ when n is even.

Proposition 4.2. A 3-$(n,2)$ EK design exists for even $n \geq 4$.

Proof: When $n=4$, the pair $(X,\mathcal{C})$, where

$$\begin{aligned} X &= \mathbb{Z}/4\mathbb{Z}, \\ \mathcal{C} &= \{\{\{0,1\},\{2,3\}\},\{\{0,2\},\{1,3\}\},\{\{0,3\},\{1,2\}\}\}, \end{aligned}$$

is a 3-$(4,2)$ EK design.

When $n=6$, let

$$\begin{aligned} X &= \mathbb{Z}/6\mathbb{Z}, \\ \mathcal{C}_0 &= \{\{\{0,1\},\{2,4\}\},\{\{0,1\},\{3,5\}\},\{\{0,2\},\{3,1\}\}, \\ &\qquad \{\{0,3\},\{1,4\}\},\{\{0,5\},\{1,2\}\}\}, \\ \mathcal{C} &= \{\{\{a+i,b+i\},\{c+i,d+i\}\} : \\ &\qquad \{\{a,b\},\{c,d\}\} \in \mathcal{C}_0, i \in \{0,2,4\}\}. \end{aligned}$$

Then $(X,\mathcal{C})$ is a 3-$(6,2)$ EK design.

For $n \geq 8$, there exists a 3-BD$(n,\{4,6\})$ by Theorem 4.1. The result now follows from Proposition 4.1. ■

Proposition 4.3. There exists a 3-$(n,2)$ EK packing of size $n(n-1)(n-3)/8$ for all odd $n \geq 5$.

Proof: By Proposition 4.2, there exists a 3-$(n+1,2)$ EK design $(X,\mathcal{C})$. Fix any point $x \in X$ and define

$$X' = X \setminus \{x\}, \quad \mathcal{C}' = \{\{A,B\} \in \mathcal{C} : x \notin A \cup B\}.$$

Since x is contained in exactly $n(n-1)/2$ set-pairs in $\mathcal{C}$, we have $|\mathcal{C}'| = n(n+1)(n-1)/8 - n(n-1)/2 = n(n-1)(n-3)/8$. ■

Propositions 2.1, 4.2, 4.3, and Theorem 1.2 combine to give the following.

Theorem 4.2. Let $n \geq 4$. Then

$$C(n,2,2) = \begin{cases} \dfrac{n(n-1)(n-2)}{8}, & \text{if } n \text{ is even,} \\ \dfrac{n(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases}$$

5. Conclusion

New infinite families of optimal codes in the Enomoto-Katona space are obtained in this paper. In particular, we show that $C(n,k,2k-1)$ attains the Quistorff bound for infinitely many n. The value of $C(n,2,2)$ is also completely determined.

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