Non-linear smoothers for discrete-time, finite-state Markov chains

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***Abstract*—The problem of enhancing the quality of system state estimates is considered for a special class of dynamical systems. Specifically, a system characterized by a discrete-time, finite-state Markov chain state and observed via conditionally Gaussian measurements is assumed. The associated mean vectors and covariance matrices are tightly intertwined with the system state and a control input selected by a controller. Exploiting an innovations approach, finite-dimensional, non-linear approximate MMSE smoothing estimators are derived for the Markov chain system state. The resulting smoothers are driven by a control policy determined by a stochastic dynamic programming algorithm, which minimizes the MSE filtering error, and was proposed in our earlier work. An application of the smoothers derived in this paper is presented for the problem of physical activity detection in wireless body sensing networks, which illustrates the performance enhancement due to smoothing.**

I. Introduction

Recent advances in low-power integrated circuits, signal processing and wireless communications have enabled the development of heterogeneous and multi-modal wireless sensor networks (WSNs) for addressing several, real-world problems. A common goal of such WSNs is to accurately track a time-varying process by adaptively exploiting heterogeneous sources such as different sensor types or actuation. This problem is one of *controlled sensing*, *i.e.* observations are actively selected by a controller, and is relevant in applications like sensor management for object classification and tracking [1], localization in robotics [2], health care [3] and adaptive estimation of sparse signals [4].

We previously considered in [5] the problem of tracking a discrete-time, finite-state Markov chain observed via conditionally Gaussian measurement vectors, which are shaped by the chain and an exogenous control input. Following an innovations approach, we derived a non-linear approximate Minimum Mean-Squared Error (MMSE) estimator for the Markov chain system state, which proved to be *formally* similar to the classical Kalman filter (KF) [6]. To obtain a control strategy, we proposed a partially observable Markov decision process (POMDP) formulation, where the filter's mean-squared error (MSE) served as optimization criterion.

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In the current work, we enhance system state estimates by exploiting both past and future observations and control inputs. Building upon our prior joint framework of tracking and control, we derive non-linear approximate MMSE smoothing estimators (fixed-point, fixed-interval, fixed-lag) to obtain improved state estimates of the Markov chain system state. We also present numerical results showing that smoothing can increase performance if processing delays are tolerable.

In POMDPs, the *belief state* [7], which corresponds to the conditional probability distribution associated with the chain states, constitutes a good estimate of the hidden state. As we will see in Section III, the MMSE Markov chain state estimate coincides with the belief state. Thus, focusing on MMSE estimation enables us to determine good belief state estimates, which in turn satisfies our goal of high detection accuracy. Furthermore, in contrast to our prior work [3], which necessitated the usage of upper bounds for the detection error probability due to the specific signal model, MMSE estimation gives rise to closed-form expressions for the MSE perfomance and allows us to focus on the true tracking performance.

The flexibility of how future measurements are processed and the need for numerically stable implementations has given rise to fixed–interval [8], [9], fixed–point [10], [11] and fixed–lag smoothers [10], [12], [13] for the basic, discrete-time linear, Gauss-Markov state-variable model. As with KFs, smoothers have been developed for more general, nonlinear and (non)-Gaussian systems with or without correlated and colored noises, *e.g.* the Extended Kalman smoother [14] and the Unscented Kalman smoother [15], [16]. Exploiting an innovations approach, Kalman-type smoothers for discrete-time, finite-state Markov chains observed via discrete-observations were derived in [17], [18] but without exerting control. The same problem was addressed in [19] via a change of measure with [20] extending it to the case of white Gaussian measurement noise. For the same model as in [20], fixed–lag and sawtooth lag smoothers were derived in [21]. In contrast to all these works, our proposed smoothers exploit the innovations approach [17], [18], [22] for discrete-time, finite-state Markov chains observed via controlled conditionally Gaussian measurements.

Our contributions are as follows. We derive recursive formulae for the three fundamental types of smoothing (fixed–point, fixed–interval, fixed-lag) for discrete-time, finite-state Markov chains observed via controlled conditionally Gaussian measurements. Our focus is on active sensing applications, *i.e.* the control input controls the measurements, not the state evolution. Thus, the proposed smoothers along with the Kalman-like filter and the control policy from our prior work [5] constitute an important toolbox for addressing several controlled sensing applications, *e.g.* sequential multiple hypothesis testing [4], [23], [24] (in fact, we allow the underlying hypothesis to change with time). Finally, we illustrate the performance of the proposed framework using real data from a practical application in body sensing networks.

II. Problem Description

A. System Model

We consider a special class of dynamical systems known as Partially Observable Markov Decision Processes (POMDPs) [7], where time is divided in discrete time slots denoted by $k \in \{0, 1, \dots\}$. The system state at time step k corresponds to a finite state, first-order Markov chain with n states, *i.e.* $\mathcal{X} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ denotes the unit vector with 1 in the i-th position and zeros everywhere else. We characterize the Markov chain by a transition probability matrix $\mathbf{P}$ with components $P_{j|i} = P(\mathbf{x}_{k+1} = \mathbf{e}_j | \mathbf{x}_k = \mathbf{e}_i)$ for $\mathbf{e}_i, \mathbf{e}_j \in \mathcal{X}$. We also assume that the state transition probabilities do not change with time, *i.e.* the Markov chain is stationary.

At each time step, a set of observations is generated. Namely, the system state sequence $\{\mathbf{x}_k\}$ is hidden and observed through a sequence of measurement vectors $\{\mathbf{y}_k\}$. Conditioned on the underlying system state $\mathbf{x}_k$ and an exogenous control input $\mathbf{u}_{k-1}$ selected by a controller at time step $k-1$, each such measurement vector $\mathbf{y}_k$ follows a multivariate Gaussian model of the form

$$\mathbf{y}_k | \mathbf{e}_i, \mathbf{u}_{k-1} \sim f(\mathbf{y}_k | \mathbf{e}_i, \mathbf{u}_{k-1}) = \mathcal{N}(\mathbf{m}_i^{\mathbf{u}_{k-1}}, \mathbf{Q}_i^{\mathbf{u}_{k-1}}) \quad (1)$$

for all $\mathbf{e}_i \in \mathcal{X}$. We denote the mean vector and covariance matrix of the measurement vector for system state $\mathbf{e}_i$ and control input $\mathbf{u}_{k-1}$ as $\mathbf{m}_i^{\mathbf{u}_{k-1}}$ and $\mathbf{Q}_i^{\mathbf{u}_{k-1}}$, respectively. The control input $\mathbf{u}_{k-1}$ can be defined to affect the size of the measurement vector $\mathbf{y}_k$, its form or both and is selected by the controller based on the available information, *i.e.* history of previous control inputs and measurement vectors. Finally, we assume that there is a finite number of control inputs supported by the system, *i.e.* $\mathbf{u}_k \in \mathcal{U} = \{\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^\alpha\}$, where α indicates the total number of available control inputs.

B. Innovations Representation

We begin by introducing the source sequence of true states $X^k = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k\}$, the control input sequence $U^k = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ and the observations sequence $Y^k = \{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k\}$. We then define the *global history* $\mathcal{B}_k = \sigma\{X^k, Y^k, U^k\}$, the *history* $\mathcal{B}_k^- = \sigma\{X^k, Y^{k-1}, U^{k-1}\}$ and the *observation-control history* $\mathcal{F}_k = \sigma\{Y^k, U^{k-1}\}$, where $\sigma\{z\}$ denotes the σ-algebra generated by z. Each control input $\mathbf{u}_k$ is determined based on the observation-control history $\mathcal{F}_k$, *i.e.* $\mathbf{u}_k = \eta_k(\mathcal{F}_k)$.

The *innovations sequence* $\{\mathbf{w}_k\}$ related to $\{\mathbf{x}_k\}$ with respect to $\mathcal{B}_k$ is defined as

$$\mathbf{w}_{k+1} \doteq \mathbf{x}_{k+1} - \mathbb{E}\{\mathbf{x}_{k+1} | \mathcal{B}_k\}, \quad (2)$$

and exploiting the Markov property results in

$$\mathbf{w}_{k+1} = \mathbf{x}_{k+1} - \mathbf{P}\mathbf{x}_k. \quad (3)$$

Next, the *innovations sequence* $\{\mathbf{v}_k\}$ for the process $\{\mathbf{y}_k\}$ with respect to $\mathcal{B}_k^-$ [22] is defined as

$$\mathbf{v}_k \doteq \mathbf{y}_k - \mathbb{E}\{\mathbf{y}_k | \mathcal{B}_k^-\} = \mathbf{y}_k - \mathcal{M}(\mathbf{u}_{k-1})\mathbf{x}_k, \quad (4)$$

where $\mathcal{M}(\mathbf{u}_{k-1}) = [\mathbf{m}_1^{\mathbf{u}_{k-1}}, \dots, \mathbf{m}_n^{\mathbf{u}_{k-1}}]$. Note that both innovations sequences $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are Martingale Difference (MD) sequences [22] with respect to $\{\mathcal{B}\}$ and $\{\mathcal{B}^-\}$, respectively, which implies that 1) $\mathbb{E}\{\mathbf{w}_{k+1} | \mathcal{B}_k\} = 0$ and $\mathbf{w}_{k+1} \in \sigma\{X^{k+1}, Y^k, U^k\}$, and 2) $\mathbb{E}\{\mathbf{v}_k | \mathcal{B}_k^-\} = 0$ and $\mathbf{v}_k \in \mathcal{B}_k$. Finally, the *Doob–Meyer decompositions* [25] of $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ with respect to $\mathcal{B}_k$ and $\mathcal{B}_k^-$, respectively, take on the forms

$$\mathbf{x}_{k+1} = \mathbf{P}\mathbf{x}_k + \mathbf{w}_{k+1}, \; k \geqslant 0, \quad (5)$$

$$\mathbf{y}_k = \mathcal{M}(\mathbf{u}_{k-1})\mathbf{x}_k + \mathbf{v}_k, \; k \geqslant 1. \quad (6)$$

III. Kalman-like filter and Control Policy Design

We next summarize our prior work on developing a Kalman-like estimator of the Markov chain system state and designing an appropriate control strategy [5].

We define the probability distribution of $\mathbf{x}_k$ conditioned on $\mathcal{F}_k$, known as the *belief state* [7], as $\mathbf{p}_{k|k} \doteq \left[p_{k|k}^1, \dots, p_{k|k}^n\right]^T$ with $p_{k|k}^i = P(\mathbf{x}_k = \mathbf{e}_i | \mathcal{F}_k), \forall \mathbf{e}_i \in \mathcal{X}$. This coincides with the expected value of $\mathbf{x}_k$ conditioned on the observation-control history $\mathcal{F}_k$, denoted by $\mathbf{x}_{k|k}$ [5]. Theorem 1 states the recursive formulae for the proposed approximate MMSE estimator.

Theorem 1 ([5])**.** *The Markov chain system estimate at time step k is recursively defined as*

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{G}_k[\mathbf{y}_k - \mathbf{y}_{k|k-1}], \; k \geqslant 0 \quad (7)$$

with

$$\mathbf{x}_{k|k-1} = \mathbf{P}\mathbf{x}_{k-1|k-1}, \quad (8)$$

$$\mathbf{y}_{k|k-1} = \mathcal{M}(\mathbf{u}_{k-1})\mathbf{x}_{k|k-1}, \quad (9)$$

$$\mathbf{G}_k = \mathbf{\Sigma}_{k|k-1}\mathcal{M}^T(\mathbf{u}_{k-1}) \times (\mathcal{M}(\mathbf{u}_{k-1}) \times \mathbf{\Sigma}_{k|k-1}\mathcal{M}^T(\mathbf{u}_{k-1}) + \mathbf{J}_k)^{-1}, \quad (10)$$

where $\mathbf{x}_{0|-1} = \pi$, and π is the initial distribution over the system states, $\mathbf{\Sigma}_{k|k-1}$ is the conditional covariance matrix of the prediction error and $\mathbf{J}_k = \sum_{i=1}^n x_{k|k-1}^i \mathbf{Q}_i^{\mathbf{u}_{k-1}}$.

Remark 1. The proposed estimator is a non-linear approximate MMSE estimator with gain $\mathbf{G}_k$ *depending on the observations.*

The MSE performance of the filter and the predictor is characterized by the *conditional filtering error covariance* *matrix* and the *conditional prediction error covariance matrix*, respectively, which are defined as follows

$$\mathbf{\Sigma}_{k|k} \doteq \mathbb{E}\{(\mathbf{x}_k - \mathbf{x}_{k|k})(\mathbf{x}_k - \mathbf{x}_{k|k})^T|\mathcal{F}_k\}, \tag{11}$$

$$\mathbf{\Sigma}_{k|k-1} \doteq \mathbb{E}\{(\mathbf{x}_k - \mathbf{x}_{k|k-1})(\mathbf{x}_k - \mathbf{x}_{k|k-1})^T|\mathcal{F}_{k-1}\}. \tag{12}$$

Finally, to determine an admissible control strategy, we addressed in [5] the following *finite horizon, partially observable stochastic control problem*

$$\min_{\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{L-1}} \mathop{\mathbb{E}}_{\mathbf{y}_0,\mathbf{y}_1,\ldots,\mathbf{y}_L} \left\{ \sum_{k=1}^{L} \mathrm{tr}\left(\mathbf{\Sigma}_{k|k}(\mathbf{y}_k, \mathbf{u}_{k-1})\right)\right\}, \tag{13}$$

where $\mathrm{tr}(\cdot)$ is the trace operator and we minimize the MSE over an horizon, $L < \infty$. In particular, we derived a dynamic programming (DP) algorithm [7] for determining the optimal control policy.

IV. Smoothing Estimators

In this section, we develop suboptimal MMSE smoothers for estimating the discrete-time, finite-state Markov chain system state at each time step based on the theory introduced in [22], thus extending [5] by exploiting both past and future observations and controls. We seek recursive formulae for $\mathbf{x}_{k|s}, s > k$.

We start by defining the *estimate innovations sequence* $\{\gamma_s\}$ and the *observation innovations sequence* $\{\zeta_s\}$ as [17], [18]

$$\gamma_s \doteq \mathbf{x}_{k|s} - \mathbf{x}_{k|s-1} = \mathbb{E}\{\mathbf{x}_k|\mathcal{F}_s\} - \mathbb{E}\{\mathbf{x}_k|\mathcal{F}_{s-1}\}, \tag{14}$$

$$\zeta_s \doteq \mathbf{y}_s - \mathbf{y}_{s|s-1} = \mathbf{y}_s - \mathbb{E}\{\mathbf{y}_s|\mathcal{F}_{s-1}\}, \tag{15}$$

where $\mathbf{x}_{k|s}$ and $\mathbf{x}_{k|s-1}$ denote smoothed system state estimates at time step k and $\mathbf{y}_{s|s-1}$ denotes the predicted measurement estimate at time step s. We can prove that the sequences $\{\gamma_s\}$ and $\{\zeta_s\}$ are $\{\mathcal{F}\}$–MD sequences via their definitions in (14) and (15), respectively. The MD representation theorem [17], [22] relates the sequences $\{\gamma_s\}$ and $\{\zeta_s\}$ via $\gamma_s = \mathbf{C}_s\zeta_s$ given that we can determine an $\{\mathcal{F}\}$–predictable sequence $\{\mathbf{C}_s\}$, which in that case is given by

$$\mathbf{C}_s = \mathbb{E}\{\gamma_s\zeta_s^T|\mathcal{F}_{s-1}\}\left[\mathbb{E}\{\zeta_s\zeta_s^T|\mathcal{F}_{s-1}\}\right]^{-1}. \tag{16}$$

Since we wish to determine a recursive solution, we impose recursivity as a design constraint and we use (16) as an approximation.

Theorem 2 gives the general, finite-dimensional expression for the proposed suboptimal MMSE smoother for the Markov chain system state.

Theorem 2. *The R–stage, smoothed estimator of* $\mathbf{x}_k$*, denoted by* $\mathbf{x}_{k|R}$ *with* $R \geqslant k+1, k \geqslant 0$*, is given by the expression*

$$\mathbf{x}_{k|R} = \mathbf{x}_{k|k} + \sum_{s=k+1}^{R} \mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1}) \tag{17}$$

with

$$\mathbf{C}_s = (\mathbf{\Theta}_{k,s} - \mathbf{x}_{k|s-1}\mathbf{x}_{s|s-1}^T)\mathcal{M}^T(\mathbf{u}_{s-1})\times (\mathbf{J}_s + \mathcal{M}(\mathbf{u}_{s-1})\mathbf{\Sigma}_{s|s-1}\mathcal{M}^T(\mathbf{u}_{s-1}))^{-1}, \tag{18}$$

where

$$\mathbf{\Theta}_{k,s} = \mathbb{E}\{\mathbf{x}_k\mathbf{x}_{s-1}^T|\mathcal{F}_{s-1}\}\mathbf{P}^T, \tag{19}$$

$$\mathbb{E}\{\mathbf{x}_k\mathbf{x}_{s-1}^T|\mathcal{F}_{s-1}\} = \frac{\mathbf{\Theta}_{k,s-1}\mathbf{r}(\mathbf{y}_{s-1},\mathbf{u}_{s-2})}{\mathbf{1}_n^T[\mathbf{\Theta}_{k,s-1}\mathbf{r}(\mathbf{y}_{s-1},\mathbf{u}_{s-2})]\mathbf{1}_n} \tag{20}$$

with $\mathbf{1}_n$ *denoting the* $n \times 1$ *vector of ones,* $\mathbf{r}(\mathbf{y}_k, \mathbf{u}_{k-1}) = \mathrm{diag}\left(f(\mathbf{y}_k|\mathbf{e}_1, \mathbf{u}_{k-1}), \ldots, f(\mathbf{y}_k|\mathbf{e}_n, \mathbf{u}_{k-1})\right)$ *the* $n \times n$ *diagonal matrix of pdfs,* $\mathbb{E}\{\mathbf{x}_0\mathbf{x}_0^T|\mathcal{F}_0\} = \mathrm{diag}(\mathbf{p}_{0|0})$ *and* $\mathbf{J}_s = \sum_{i=1}^n x_{s|s-1}^i\mathbf{Q}_i^{\mathbf{u}_{s-1}}$.

Proof: To determine the exact form of the R–stage smoother, we exploit the Doob-Meyer decompositions (5)–(6), the sequences defined in (14)–(15) as well as the relation $\gamma_s = \mathbf{C}_s\zeta_s$. Specifically, summing the latter relation from $s = k$ to $s = R$ gives us

$$\sum_{s=k}^{R}\gamma_s = \sum_{s=k}^{R}(\mathbf{x}_{k|s} - \mathbf{x}_{k|s-1}) = \mathbf{x}_{k|R} - \mathbf{x}_{k|k-1}, \tag{21}$$

and exploiting the MD representation theorem, we get

$$\mathbf{x}_{k|R} = \mathbf{x}_{k|k} + \sum_{s=k+1}^{R}\mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1}). \tag{22}$$

At this point, we need to compute $\mathbf{C}_k$ and as seen by (16), this necessitates evaluating $\mathbb{E}\{\gamma_s\zeta_s^T|\mathcal{F}_{s-1}\}$ and $\mathbb{E}\{\zeta_s\zeta_s^T|\mathcal{F}_{s-1}\}$. For the first term, we have

$$\begin{aligned}\mathbb{E}\{\gamma_s\zeta_s^T|\mathcal{F}_{s-1}\} &= \mathbb{E}\{\mathbb{E}\{\mathbf{x}_k|\mathcal{F}_s\}\zeta_s^T|\mathcal{F}_{s-1}\} \\ &\quad -\mathbf{x}_{k|s-1}\mathbb{E}\{\zeta_s^T|\mathcal{F}_{s-1}\} \\ &\overset{(a)}{=} \mathbb{E}\{\mathbb{E}\{\mathbf{x}_k\zeta_s^T|\mathcal{F}_s\}|\mathcal{F}_{s-1}\} \\ &\overset{(b)}{=} \mathbb{E}\{\mathbf{x}_k\mathbf{x}_s^T|\mathcal{F}_{s-1}\}\mathcal{M}^T(\mathbf{u}_{s-1}) \\ &\quad +\mathbb{E}\{\mathbf{x}_k\mathbf{v}_s^T|\mathcal{F}_{s-1}\} - \mathbf{x}_{k|s-1}\mathbf{y}_{s|s-1}^T,\end{aligned} \tag{23}$$

where we have exploited that (a) ζ_s is a $\{\mathcal{F}\}$–MD sequence and (b) $\mathbf{u}_{s-1} = \eta_{s-1}(\mathcal{F}_{s-1})$. After some manipulations, the term $\mathbf{\Theta}_{k,s} = \mathbb{E}\{\mathbf{x}_k\mathbf{x}_s^T|\mathcal{F}_{s-1}\}$ is recursively calculated as

$$\mathbf{\Theta}_{k,s} = \mathbb{E}\{\mathbf{x}_k\mathbf{x}_{s-1}^T|\mathcal{F}_{s-1}\}\mathbf{P}^T \tag{24}$$

$$\mathbb{E}\{\mathbf{x}_k\mathbf{x}_{s-1}^T|\mathcal{F}_{s-1}\} = \frac{\mathbf{\Theta}_{k,s-1}\mathbf{r}(\mathbf{y}_{s-1},\mathbf{u}_{s-2})}{\mathbf{1}_n^T[\mathbf{\Theta}_{k,s-1}\mathbf{r}(\mathbf{y}_{s-1},\mathbf{u}_{s-2})]\mathbf{1}_n} \tag{25}$$

with $\mathbb{E}\{\mathbf{x}_0\mathbf{x}_0^T|\mathcal{F}_0\} = \mathrm{diag}(\mathbf{p}_{0|0})$ and $\mathbf{r}(\mathbf{y}_k, \mathbf{u}_{k-1}) = \mathrm{diag}\left(f(\mathbf{y}_k|\mathbf{e}_1, \mathbf{u}_{k-1}), \ldots, f(\mathbf{y}_k|\mathbf{e}_n, \mathbf{u}_{k-1})\right)$. Also, we have

$$\begin{aligned}\mathbb{E}\{\mathbf{x}_k\mathbf{v}_s^T|\mathcal{F}_{s-1}\} &= \mathbb{E}\{\mathbb{E}\{\mathbf{x}_k\mathbf{v}_s^T|\mathcal{B}_s^-\}|\mathcal{F}_{s-1}\} \\ &= \mathbb{E}\{\mathbf{x}_k\mathbb{E}\{\mathbf{v}_s^T|\mathcal{B}_s^-\}|\mathcal{F}_{s-1}\} = 0,\end{aligned} \tag{26}$$

where we have exploited the MD property of $\mathbf{v}_s$ and the fact that $\mathbf{x}_k \in \mathcal{B}_s^-, \forall s > k$. The above results allows to rewrite (23) as

$$\mathbb{E}\{\gamma_s\zeta_s^T|\mathcal{F}_{s-1}\} = (\mathbf{\Theta}_{k,s} - \mathbf{x}_{k|s-1}\mathbf{x}_{s|s-1}^T)\mathcal{M}^T(\mathbf{u}_{s-1}). \tag{27}$$

For the second term, we have from the definition of ζ_s in (15)

$$\mathbb{E}\{\zeta_s\zeta_s^T|\mathcal{F}_{s-1}\} = \mathbb{E}\{\mathbf{y}_s\mathbf{y}_s^T|\mathcal{F}_{s-1}\} - \mathbf{y}_{s|s-1}\mathbf{y}_{s|s-1}^T, \tag{28}$$

and to determine the exact form of $\mathbb{E}\{\mathbf{y}_s\mathbf{y}_s^T|\mathcal{F}_{s-1}\}$, we first determine $\mathbf{p}(\mathbf{y}_s|\mathcal{F}_{s-1})$ as follows

$$\begin{aligned}\mathbf{p}(\mathbf{y}_s|\mathcal{F}_{s-1}) &= p(\mathbf{y}_s|\mathbf{y}_0,\dots,\mathbf{y}_{s-1},\mathbf{u}_0,\dots,\mathbf{u}_{s-2})\\ &= p(\mathbf{y}_s|\mathbf{y}_0,\dots,\mathbf{y}_{s-1},\mathbf{u}_0,\dots,\mathbf{u}_{s-1})\\ &= \sum_{i=1}^{n} p^i_{s|s-1} f(\mathbf{y}_s|\mathbf{e}_i,\mathbf{u}_{s-1}),\end{aligned} \tag{29}$$

where we have exploited the fact that $\mathbf{u}_{s-1} = \eta_{s-1}(\mathcal{F}_{s-1})$. Then, from the definition of the conditional correlation matrix, the term $\mathbb{E}\{\mathbf{y}_s\mathbf{y}_s^T|\mathcal{F}_{s-1}\}$ proves to be

$$\mathbb{E}\{\mathbf{y}_s\mathbf{y}_s^T|\mathcal{F}_{s-1}\} = \sum_{i=1}^{n} p^i_{s|s-1}\big[\mathbf{Q}_i^{\mathbf{u}_{s-1}} + \mathbf{m}_i^{\mathbf{u}_{s-1}}(\mathbf{m}_i^{\mathbf{u}_{s-1}})^T\big] \tag{30}$$

and substituting back to (28), we get

$$\begin{aligned}\mathbb{E}\{\zeta_s\zeta_s^T|\mathcal{F}_{s-1}\} = &\sum_{i=1}^{n} p^i_{s|s-1}\big[\mathbf{Q}_s^{\mathbf{u}_{s-1}} + \mathbf{m}_i^{\mathbf{u}_{s-1}}(\mathbf{m}_i^{\mathbf{u}_{s-1}})^T\big]\\ &-\mathbf{y}_{s|s-1}\mathbf{y}^T_{s|s-1}.\end{aligned} \tag{31}$$

Exploiting the definition of the conditional prediction error covariance matrix and after some manipulations, we have

$$\mathbb{E}\{\zeta_s\zeta_s^T|\mathcal{F}_{s-1}\} = \mathbf{J}_s + \mathcal{M}(\mathbf{u}_{s-1})\mathbf{\Sigma}_{s|s-1}\mathcal{M}^T(\mathbf{u}_{s-1}) \tag{32}$$

with $\mathbf{J}_s = \sum_{i=1}^n x^i_{s|s-1}\mathbf{Q}_i^{\mathbf{u}_{s-1}}$. Substituting (27) and (32) back to (16) completes the proof. ∎

We underline that, as in the case of our Kalman-like filter [5], the gain matrix $\mathbf{C}_s$ depends *non-linearly* on the observations. Furthermore, since we did not impose any constraints during the derivation of the smoother, we cannot guarantee that it will correspond to a valid distribution. Following the approach in [18], we can apply a suitable memoryless (linear or non-linear) transformation to the smoothed estimate to obtain a valid probability mass function (pmf). The MSE performance of the smoother in (17) is given by the *conditional smoothing error covariance matrix* $\mathbf{\Sigma}_{k|R} \doteq \mathbb{E}\{(\mathbf{x}_k - \mathbf{x}_{k|R})(\mathbf{x}_k - \mathbf{x}_{k|R})^T|\mathcal{F}_R\}$. Note that there are three smoothers depending on how observations are processed: *fixed–point*, *fixed–interval*, and *fixed–lag* [11]. Before proceeding to derive these smoothers for the discrete state case, we make the following remark.

Remark 2. The smoothers in this paper constitute non-linear approximate MMSE estimators since the gain $\mathbf{C}_s$ depends non-linearly on the observations.

A. Fixed–point smoother

The fixed–point smoother is defined as $\mathbf{x}_{k|R} \doteq \mathbb{E}\{\mathbf{x}_k|\mathcal{F}_R\}, R = k+1, k+2, \dots$, where k is a fixed positive integer [11], and the system state estimate is improved using future observations and control inputs. The fixed–point smoothed estimator of $\mathbf{x}_k$ is provided by Theorem 2.

B. Fixed–interval smoother

The fixed–interval smoother is defined as $\mathbf{x}_{k|L} \doteq \mathbb{E}\{\mathbf{x}_k|\mathcal{F}_L\}, k = 0, 1, \dots, L-1$, where L is a fixed positive integer [11], and the system state is estimated using all observations and control inputs. Proposition 1 gives the expressions for the fixed–interval smoother.

Proposition 1. *The fixed–interval smoothed estimator of* $\mathbf{x}_k, \mathbf{x}_{k|L}, k = 1, 2, \dots, L-1$, *is given by the expression*

$$\mathbf{x}_{k|L} = \mathbf{P}\mathbf{x}_{k-1|L} + (\mathbf{I}_n - \mathbf{P})\sum_{s=k}^{L}\mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1}), \tag{33}$$

where $\mathbf{I}_n$ *is the* $n\times n$ *identity matrix, and is initialized by* $\mathbf{x}_{0|L} = \mathbf{x}_{0|0} + \sum_{s=1}^{L}\mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1})$, *which is obtained from the fixed–point smoothed estimator by setting* $k = 0$.

C. Fixed–lag smoother

The fixed–lag smoother is defined as $\mathbf{x}_{k|k+\Delta} \doteq \mathbb{E}\{\mathbf{x}_k|\mathcal{F}_{k+\Delta}\}, k = 0, 1, \dots,$ where Δ is a fixed positive integer [11], that is a fixed–look ahead interval.

Proposition 2. *The fixed–lag smoothed estimator of* $\mathbf{x}_k, \mathbf{x}_{k|k+\Delta}, k = 0, 1, \dots,$ *is given by the expression*

$$\begin{aligned}\mathbf{x}_{k|k+\Delta} = &\mathbf{P}\mathbf{x}_{k-1|k+\Delta-1} + \Gamma(k,\Delta)\\ &+(\mathbf{I}_n - \mathbf{P})\sum_{s=k+1}^{k+\Delta-1}\mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1}),\end{aligned} \tag{34}$$

where $\mathbf{I}_n$ *is the* $n\times n$ *identity matrix,* $\Gamma(k,\Delta) \doteq \mathbf{C}_{k+\Delta}(\mathbf{y}_{k+\Delta} - \mathbf{y}_{k+\Delta|k+\Delta-1}) - \mathbf{x}_{k+1|k} - \mathbf{x}_{k|k-1} + \mathbf{P}\mathbf{x}_{k|k-1}$, *and the smoother is initialized by* $\mathbf{x}_{0|\Delta} = \mathbf{x}_{0|0} + \sum_{s=1}^{\Delta}\mathbf{C}_s(\mathbf{y}_s - \mathbf{y}_{s|s-1})$, *which is obtained from the fixed–point smoothed estimator by setting* $k = 0$.

V. Numerical Example

In this section, we provide a short numerical example to illustrate the performance of smoothing in a body sensing application [3], [5]. The goal is to detect an individual's time-evolving physical activity state using information from three biometric sensors. We assume four physical activity states (Sit, Stand, Run, Walk) with transition probability matrix $\mathbf{P} = [0.6\ 0\ 0.2\ 0.4, 0.1\ 0.4\ 0.1\ 0, 0\ 0.1\ 0.3\ 0.3,\ 0.3\ 0.3\ 0.6\ 0.3]$. The control input indicates the requested number of samples from each sensor at each time step. The optimal control policy was numerically determined by solving the DP recursion for total requested number of samples $N \leqslant 2$. Table I shows the signal model distributions for different control inputs. For control inputs $\mathbf{u}^r, r\in\{1,2,3\}$, the variance is determined by $g(\sigma_i^2) = 1.0667\sigma_i^2 + \sigma_z^2$, where $\sigma_z^2 = 2$. For control inputs $\mathbf{u}^r, r \in \{4,\dots,9\}$ and a specific activity state, the mean vectors are defined by $\gamma_{i,j}^{\text{state}} = [m_i^{\text{state}}, m_j^{\text{state}}]^T$ and the covariance matrices by $\mathbf{\Delta}_{i,j}^{\text{state}} = [\Delta(\sigma^2_{i,\text{state}})\ \mathbf{O}_2; \Delta(\sigma^2_{j,\text{state}}))\ \mathbf{O}_2]$, $\Delta(\sigma^2_{i,\text{state}}) = 1.0667\sigma^2_{i,\text{state}}[1\ 0.25; 0.25\ 1] + \sigma_z^2\mathbf{I}_2$, where $\mathbf{O}_2$ is the 2×2 zero matrix and $\mathbf{I}_2$ the 2×2 identity matrix, for control superscript pairs $(i,j)\in\{(i,j)|i\leqslant j$ and $i,j\in\{1,2,3\}\}$.

TABLE I

CONTROLLED STATE DISTRIBUTIONS.

	$\mathbf{u}^1$	$\mathbf{u}^2$	$\mathbf{u}^3$	$\mathbf{u}^r, r \in \{4, \dots, 9\}$
Sit	$\mathcal{N}(0.56, g(2))$	$\mathcal{N}(0.18, g(2.01))$	$\mathcal{N}(247.35, g(2.08))$	$\mathcal{N}(\gamma_{i,j}^{\text{Sit}}, \mathbf{\Delta}_{i,j}^{\text{Sit}})$
Stand	$\mathcal{N}(0.55, g(2))$	$\mathcal{N}(0.39, g(2.05))$	$\mathcal{N}(265.50, g(2.18))$	$\mathcal{N}(\gamma_{i,j}^{\text{Stand}}, \mathbf{\Delta}_{i,j}^{\text{Stand}})$
Run	$\mathcal{N}(0.34, g(2))$	$\mathcal{N}(33.8, g(2.81))$	$\mathcal{N}(256.27, g(2.12))$	$\mathcal{N}(\gamma_{i,j}^{\text{Run}}, \mathbf{\Delta}_{i,j}^{\text{Run}})$
Walk	$\mathcal{N}(0.43, g(2))$	$\mathcal{N}(27.15, g(4.07))$	$\mathcal{N}(258.55, g(2.12))$	$\mathcal{N}(\gamma_{i,j}^{\text{Walk}}, \mathbf{\Delta}_{i,j}^{\text{Walk}})$

TABLE II

FILTERING AND SMOOTHING DETECTION ACCURACY.

Filtering	85%
Smoothing, $R = k + 1$	87%
Smoothing, $R = k + 2$	88%
Smoothing, $R = k + 3$	88.2%
Smoothing, $R = k + 4$	88.4%

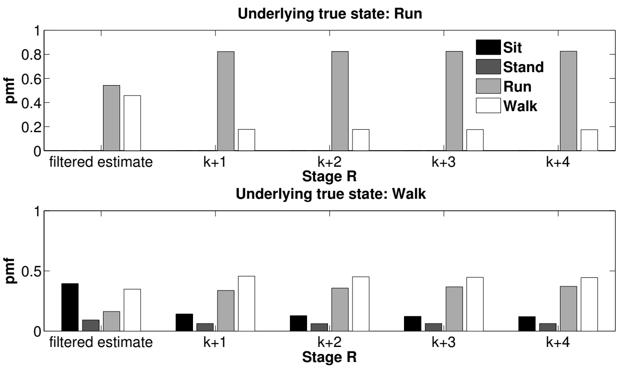

Fig. 1. Exemplary effect of stage R on the smoothed state estimates / probability mass functions (pmfs). The initial filtered estimate is also given for comparison.

Table II summarizes the detection accuracy of filtering and smoothing operations. We observe that as expected, smoothing enhances detection accuracy. However, also expected, the smoothing performance saturates as the stage R increases. We underscore that different Markov chains and/or signal model statistics would result in different smoothing performance improvements. In Fig. 1, we present an example of the effect of increasing the smoother's stage on the pmf over the underlying state. We observe that future information can enhance or overturn our belief with respect to the true system state unveiling its true value. As R increases, our belief stabilizes, which is also is supported by the results in Table II.

VI. CONCLUSIONS

In this work, we addressed the problem of smoothing for discrete-time, finite-state Markov chains observed via controlled conditionally Gaussian measurements. Our results differ from prior work in that we jointly consider time-varying systems, discrete states and active control over measurements. We derived three types of smoothers (fixed–point, fixed–lag, fixed–interval) and illustrated the smoothing procedure applied to a body sensing application. Our results indicate the feasibility of enhancing performance via smoothing in controlled sensing applications.

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