Rate-Distortion Function for Gamma Sources under Absolute-Log Distortion Measure

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***Abstract*—We evaluate the rate-distortion function for the i.i.d. gamma sources with respect to the absolute-log distortion measure. The logarithmic transformation reduces this rate-distortion problem to that under the absolute distortion measure. Extending the explicit evaluation of the rate-distortion function for the Gaussian sources, we obtain the parametric form of the rate-distortion function. We show that the optimal distribution of reconstruction consists of a continuous component enclosed by left and right discrete components and the left discrete component vanishes when the allowed distortion is small.**

I. Introduction

Gamma distributions are widely used for modelling positive real variables such as the inter-spike intervals in neuroscience [1], [2], the waiting times in queueing systems [3] and power spectrums in audio processing [4]. We consider the rate-distortion function of an i.i.d. gamma source under the absolute-log distortion measure. The rate-distortion function $R(D)$ shows the minimum information rate required to reconstruct the source outputs with average distortion not exceeding D. Rate-distortion functions have been explicitly evaluated for various sources and distortion measures. The Shannon lower bound (SLB) plays an important role in order to evaluate the rate-distortion functions of difference distortion measures [5], [6]. There have been only several results on the rate-distortion function when it is strictly greater than the SLB [7], [8], [9]. In particular for the absolute distortion measure, it had been considered formidable to explicitly evaluate the rate-distortion function until a method for evaluation was provided for an i.i.d. Gaussian source [8]. This analysis was extended to i.i.d. sources with constrained tail decay [8] and to those supported on a proper subset of the real line [9]. By taking the logarithmic transformation, the rate-distortion problem with respect to the absolute-log distortion measure is transformed to that of the absolute distortion measure. Under this transformation, the gamma source provides a source density having a heavier left tail than those dealt with in [8] and [9].

In this paper, we modify the analysis in [8] to deal with the heavy left tail of the gamma source and provide the explicit evaluation of its rate-distortion function. We show that as the average distortion decreases the optimal distribution of reconstruction transit from 1) the delta measure at the median to 2) a continuous distribution enclosed by two discrete components and then to 3) a continuous distribution with only the right discrete component.

II. Rate-Distortion Function

Let U and V be random variables on $\mathbf{R}$ and $d(u,v)$ be the non-negative distortion measure between u and v. The rate-distortion function $R(D)$ of the source $p(u)$ with respect to the distortion d is defined by

$$R(D) = \inf_{q(v|u):E[d(u,v)]\le D} I(q), \tag{1}$$

where

$$I(q) = \int\int q(v|u)p(u)\log\frac{q(v|u)}{\int q(v|u)p(u)du}dudv$$

is the mutual information and E denotes the expectation with respect to $q(v|u)p(u)$. $R(D)$ shows the minimum achievable rate for the i.i.d. source with the density $p(u)$ under the given distortion measure d [5].

The KKT conditions for the infinite-dimensional convex programming problem (1) identify the conditional probability distribution that achieves the minimization, which are stated in the following theorem.

Theorem 1 *[6, p. 91, Theorem 4.2.3] Let Λ_s be the set of non-negative functions $\lambda_s(u)$ satisfying*

$$c(v) = \int_{-\infty}^{\infty}\lambda_s(u)p(u)e^{sd(u,v)}du \le 1 \quad (\textit{for all } v). \tag{2}$$

Then for all $D>0$,

$$R(D) = \sup_{s\le 0,\lambda_s(u)\in\Lambda_s}\left[sD + \int p(u)\log\lambda_s(u)du\right]. \tag{3}$$

For each $s\le 0$, a necessary and sufficient condition for $\lambda_s(u)$ to realize the supremum in (3) is the existence of a probability density $q_s(v)$ such that $c(v)=1$ for almost all v for which $q_s(v)>0$ and it is related to $\lambda_s(u)$ by

$$\lambda_s(u) = \left[\int_{-\infty}^{\infty} q_s(v)e^{sd(u,v)}dv\right]^{-1} \tag{4}$$

for all $u\in\mathbf{R}$.

Using such λ_s and q_s, the rate-distortion function $R(D)$ is given parametrically in s by

$$R(D_s) = sD_s + \int_{-\infty}^{\infty} p(u) \log \lambda_s(u) du, \quad (5)$$

$$D_s = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda_s(u) q_s(v) p(u) d(u,v) \cdot \exp(sd(u,v)) du dv. \quad (6)$$

The optimal conditional density of reconstruction is then given by

$$q_s(v|u) = \lambda_s(u) q_s(v) \exp(sd(u,v)). \quad (7)$$

Note that the function $q_s(v)$ which is related to the optimal λ_s with (4) may contain Dirac delta functions. This means that the optimal distribution of reconstruction would have a discrete component.

The SLB of $R(D)$ is given by setting $\lambda_s(u)$ to make $\lambda_s(u)p(u)$ constant with respect to u [6, pp.92-94]. However, when $R(D)$ is strictly greater than its SLB, it is difficult to derive the optimal λ_s that achieves the supremum in (3). Unlike for i.i.d. finite-alphabet sources, there is no established general method for the explicit evaluation of $R(D)$ for continuous-amplitude sources when $R(D)$ is strictly greater than the SLB [10].

A systematic procedure to obtain the optimal $q_s(v)$ was proposed in [8] and applied to evaluating $R(D)$ for Gaussian sources under the absolute-magnitude distortion. In this paper, we follow this procedure, which is outlined as follows. It first chooses a set $V_s \subset \mathbf{R}$ of the support of q_s and then solves

$$\int_{-\infty}^{\infty} \frac{p(u) \exp\{sd(u,v)\}}{\int_{V_s} q_s(\tilde{v}) \exp\{sd(u,\tilde{v})\} d\tilde{v}} du = 1, \quad \forall v \in V_s \quad (8)$$

for $q_s(v)$, $v \in V_s$, the solution of which is called a tentative solution if $q_s(v) \geq 0$ for $v \in V_s$. By letting $q_s(v) = 0$ for $v \notin V_s$, it is examined if $c(v)$ defined by (2) satisfies $c(v) \leq 1$ for all $v \notin V_s$ with λ_s defined by (4). If so, Theorem 1 guarantees that $q_s(v)$ is the optimal solution and $R(D)$ is given parametrically by (5).

III. Gamma Source

In this section, we consider the i.i.d. gamma source with density for $x \geq 0$,

$$f(x) = \frac{1}{\Gamma(\alpha)} \theta^{-\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right), \quad (9)$$

where $\theta > 0$ is the scale parameter, $\alpha > 0$ is the shape parameter and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function. We focus on the rate-distortion function for the gamma source under the absolute-log distortion measure,

$$d(x,y) = |\log x - \log y|. \quad (10)$$

Since this is invariant under the scaling transformations, $x \to \theta x$ and $y \to \theta y$, we consider the case $\theta = 1$.

The properties of the rate-distortion function $R(D)$ tell that $R(D) > 0$ for $0 < D < D_{\max}$, where

$$D_{\max} = \inf_y \int f(x) d(x,y) dx, \quad (11)$$

and $R(D) = 0$ for $D \geq D_{\max}$ [6, p. 90].

Let us first define some notations.

$$\gamma(\alpha, y) = \int_0^y t^{\alpha-1} e^{-t} dt \text{ and } \Gamma(\alpha, y) = \int_y^\infty t^{\alpha-1} e^{-t} dt,$$

are the lower and upper incomplete gamma functions, $\Psi(\alpha) = (\log \Gamma(\alpha))'$ is the Psi function and we define the incomplete Psi function as follows,

$$\Psi(\alpha, y) = \frac{\int_0^y (\log t) t^{\alpha-1} e^{-t} dt}{\Gamma(\alpha)}.$$

The next lemma shows $D_{\max}$ for the gamma source (9).

Lemma 1 *For the gamma source (9) with $\theta = 1$ and the distortion measure (10),*

$$D_{\max} = \Psi(\alpha) - 2\Psi(\alpha, y^*), \quad (12)$$

where y^* is the median of $f(x)$, which satisfies

$$\frac{\gamma(\alpha, y^*)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha, y^*)}{\Gamma(\alpha)} = \frac{1}{2}. \quad (13)$$

Proof: It is not difficult to show that the minimum of (11) is achieved when y is equal to the median of $f(x)$. Since the cumulative distribution function is

$$F(x) = \int_0^x \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t} dt = \frac{\gamma(\alpha, x)}{\Gamma(\alpha)},$$

the median y^* is implicitly given by (13). Calculating $\int_0^\infty f(x) \left|\log \frac{x}{y^*}\right| dx$ yields (12). ■

A. Optimal Reconstruction

Hereafter, we represent positive real variables x and y in terms of logarithms $u = \log x$ and $v = \log y$. Then the above rate-distortion problem turns out to be that of the absolute distortion, $d(u,v) = |u - v|$ with the source,

$$p(u) = \frac{1}{\Gamma(\alpha)} \exp(u\alpha - e^u), \quad (14)$$

where we put $\theta = 1$ in (9). Since this distribution is asymmetric and has a heavier left tail than those considered in [8], we extend the analysis and assume the support of v as $V_s = [v^* - a_s, \;\; v^* + b_s]$, where $v^* = \log y^*$, $a_s > 0$, $b_s > 0$ and $a_s \to \infty$, $b_s \to \infty$ as $|s| \to \infty$. The explicit dependencies of a_s and b_s on s are to be specified later. In fact, it is proved that $a_s \to \infty$ as $|s| \to \alpha$ because of the heavy left tail.

Then (8) yields

$$A_1 e^{sv} + A_2 e^{-sv} + \int_{v^*-a_s}^{v^*+b_s} A(u) \exp\{s|u-v|\} du = 1, \quad (15)$$

for $v \in [v^* - a_s, \; v^* + b_s]$, where

$$A_1 = \frac{\gamma\left(\alpha, e^{v^*-a_s}\right)}{\Gamma(\alpha) \int_{v^*-a_s}^{v^*+b_s} q_s(t) e^{st} dt}, \quad (16)$$

$$A_2 = \frac{\Gamma\left(\alpha, e^{v^*+b_s}\right)}{\Gamma(\alpha) \int_{v^*-a_s}^{v^*+b_s} q_s(t) e^{-st} dt}, \quad (17)$$

$$\text{and } A(u) = \frac{p(u)}{\int_{v^*-a_s}^{v^*+b_s} q_s(t) \exp\{s|u-t|\} dt}, \quad (18)$$

for $u \in [v^* - a_s,\ v^* + b_s]$. Solving (15) for $A(u)$, A_1 and A_2 by using the standard Wiener-Hopf equation technique [8][11, pp.375-382], we have $A_s(u) = \frac{|s|}{2}$, $A_1 = \frac{1}{2}e^{-s(v^*-a_s)}$ and $A_2 = \frac{1}{2}e^{s(v^*+b_s)}$. Substituting these into (16), (17) and (18) yields

$$\int_{v^*-a_s}^{v^*+b_s} q_s(t)e^{s(t-v^*+a_s)}dt = 2\frac{\gamma\left(\alpha, e^{v^*-a_s}\right)}{\Gamma(\alpha)}, \quad (19)$$

$$\int_{v^*-a_s}^{v^*+b_s} q_s(t)e^{s(v^*+b_s-t)}dt = 2\frac{\Gamma\left(\alpha, e^{v^*+b_s}\right)}{\Gamma(\alpha)}, \quad (20)$$

$$\text{and} \quad \int_{v^*-a_s}^{v^*+b_s} q_s(t)\exp\{s|u-t|\}dt = \frac{2}{|s|}p(u), \quad (21)$$

for all $u \in [v^* - a_s,\ v^* + b_s]$. The boundary conditions of (21) when u approaches $v^* - a_s$ and $v^* + b_s$ are given by (19) and (20) as follows,

$$\frac{1}{|s|}p(v^* - a_s) = \frac{\gamma\left(\alpha, e^{v^*-a_s}\right)}{\Gamma(\alpha)}, \quad (22)$$

$$\text{and} \quad \frac{1}{|s|}p(v^* + b_s) = \frac{\Gamma\left(\alpha, e^{v^*+b_s}\right)}{\Gamma(\alpha)}. \quad (23)$$

These conditions are rewritten with (14) and by letting $\underline{t}_s = e^{v^*-a_s}$ and $\bar{t}_s = e^{v^*+b_s}$,

$$\frac{1}{|s|} = \frac{\gamma(\alpha, \underline{t}_s)}{\underline{t}_s^\alpha e^{-\underline{t}_s}} = \frac{\Gamma(\alpha, \bar{t}_s)}{\bar{t}_s^\alpha e^{-\bar{t}_s}}. \quad (24)$$

For $y^* = e^{v^*}$, let $s_{\max} = -2p(v^*)$. The next lemma ensures that there exist unique $a_s > 0$ and $b_s > 0$ satisfying (22) and (23) respectively such that $a_s \downarrow 0$ and $b_s \downarrow 0$ as $s \uparrow s_{\max}$, $a_s \uparrow \infty$ as $s \downarrow -\alpha$ and $b_s \uparrow \infty$ as $s \downarrow -\infty$.

Lemma 2 $g_1(t) = \frac{\gamma(\alpha,t)}{t^\alpha e^{-t}}$ *is monotonically increasing and* $g_2(t) = \frac{\Gamma(\alpha,t)}{t^\alpha e^{-t}}$ *is monotonically decreasing for* $t \geq 0$. *It holds that* $\frac{1}{\alpha} \leq g_1(t) \leq \frac{1}{|s_{\max}|}$ *for* $0 \leq t \leq y^*$, $0 < g_2(t) \leq \frac{1}{|s_{\max}|}$ *for* $t \geq y^*$ *and* $g_1(y^*) = g_2(y^*) = \frac{1}{|s_{\max}|}$.

Proof: The derivative of $g_1(t)$ is

$$g_1'(t) = \frac{1}{t^{\alpha+1}e^{-t}}\{t^\alpha e^{-t} - \gamma(\alpha,t)(\alpha - t)\}. \quad (25)$$

Let $r_1(t) = t^\alpha e^{-t} - \gamma(\alpha,t)(\alpha - t)$. Since $r_1'(t) = \gamma(\alpha,t) > 0$ and $r_1(0) = 0$, we have $r_1(t) \geq 0$ for $t \geq 0$. This implies that $g_1(t)$ is monotonically increasing for $t \geq 0$. From L'Hopital's rule,

$$\lim_{t\to 0} g_1(t) = \lim_{t\to 0}\frac{\gamma(\alpha,t)'}{(t^\alpha e^{-t})'} = \lim_{t\to 0}\frac{1}{\alpha - t} = \frac{1}{\alpha},$$

which implies $g_1(t) \geq \frac{1}{\alpha}$ for $t \geq 0$. The monotonicity of $g_2(t)$ follows from

$$g_2'(t) = \frac{1}{t^{\alpha+1}e^{-t}}\{-t^\alpha e^{-t} - \Gamma(\alpha,t)(\alpha - t)\} \leq 0, \quad (26)$$

which is proved similarly. We have $\lim_{t\to\infty} g_2(t) = 0$ from L'Hopital's rule. $g_1(y^*) = g_2(y^*) = \frac{1}{|s_{\max}|}$ follows from the definitions of y^* and $s_{\max}$. ■

Let a_s be given by (22) for $-\alpha < s < s_{\max} = -2p(v^*)$ and $a_s = \infty$ for $-\infty < s < -\alpha$ and b_s be given by (23) for $-\infty < s < s_{\max}$. We solve (21) for q_s as we solved (15) and obtain for $v \in [v^* - a_s,\ v^* + b_s]$,

$$\begin{aligned} q_s(v) \;=\; & \left\{1 - \frac{1}{s^2}\left(\alpha - e^v\right)^2 + \frac{e^v}{s^2}\right\}p(v) \\ & + \frac{|s| - \left(\alpha - e^{v^*-a_s}\right)}{s^2}p(v^* - a_s)\delta(v - v^* + a_s) \\ & + \frac{|s| + \left(\alpha - e^{v^*+b_s}\right)}{s^2}p(v^* + b_s)\delta(v - v^* - b_s), \end{aligned} \quad (27)$$

where $\delta(x)$ denotes the Dirac delta function. Note that the second term, the left discrete component, vanishes for $s \leq -\alpha$ since $a_s = \infty$ for this range of s. We confirm that $|s| - \left(\alpha - e^{v^*-a_s}\right) \geq 0$, $|s| + \left(\alpha - e^{v^*+b_s}\right) \geq 0$ and $1 - \frac{1}{s^2}\left(\alpha - e^v\right)^2 + \frac{e^v}{s^2} \geq 0$ for all v in $[v^* - a_s,\ v^* + b_s]$. Since a_s and b_s are related to s by (22) and (23), the first two inequalities are equivalent to $g_1'(t) \geq 0$ and $g_2'(t) \leq 0$ for g_1' and g_2' in (25) and (26), which we used to derive Lemma 2. The third inequality follows from the first and second ones since

$$-|s| \leq \left(\alpha - e^{v^*+b_s}\right) \leq \left(\alpha - e^v\right) \leq \left(\alpha - e^{v^*-a_s}\right) \leq |s|,$$

for $v \in [v^* - a_s,\ v^* + b_s]$. This confirms that the q_s in (27) is a tentative solution. Left panels in Figure 1 illustrate $q_s(v)$ for $\alpha = 1$ and (a) $s = s_{\max} = -\log 2$, (b) $s = -0.8$ and (c) $s = -2.0$.

Next, we prove $c(v) \leq 1$ for all $v \notin [v^* - a_s,\ v^* + b_s]$. We obtain $\lambda_s(u)$ in (4) from (27),

$$\lambda_s(u)^{-1} = \begin{cases} \frac{2}{|s|}p(v^* - a_s)e^{s(v^*-a_s-u)}, & (u < v^* - a_s), \\ \frac{2}{|s|}p(u) & (u \in [v^* - a_s,\ v^* + b_s]), \\ \frac{2}{|s|}p(v^* + b_s)e^{-s(v^*+b_s-u)}, & (u > v^* + b_s). \end{cases} \quad (28)$$

Right panels in Figure 1 illustrate $\lambda_s(u)^{-1}$ for $\alpha = 1$ and (a) $s = s_{\max} = -\log 2$, (b) $s = -0.8$ and (c) $s = -2.0$.

Thus substituting (28) into (2) shows that $c(v) \leq 1$ for $v < v^* - a_s$ is equivalent to $c_1(\tilde{v}) \leq 1$ for $\tilde{v} = v^* - a_s - v > 0$ and $c(v) \leq 1$ for $v > v^* + b_s$ is equivalent to $c_2(\tilde{v}) \leq 1$ for $\tilde{v} = v - v^* - b_s > 0$, where

$$\begin{aligned} & c_1(\tilde{v}) \\ =\; & \frac{1}{2}\exp(s\tilde{v}) + \frac{|s|\exp(-s\tilde{v})}{2p(v^* - a_s)}\int_{-\infty}^{v^*-a_s-\tilde{v}} p(u)du \\ & + \frac{|s|\exp(-s(2v^* - 2a_s - \tilde{v}))}{2p(v^* - a_s)}\int_{v^*-a_s-\tilde{v}}^{v^*-a_s} p(u)e^{2su}du \end{aligned}$$

and

$$\begin{aligned} & c_2(\tilde{v}) \\ =\; & \frac{1}{2}\exp(s\tilde{v}) + \frac{|s|\exp(-s\tilde{v})}{2p(v^* + b_s)}\int_{v^*+b_s+\tilde{v}}^{\infty} p(u)du \\ & + \frac{|s|\exp(s(2v^* + 2b_s + \tilde{v}))}{2p(v^* + b_s)}\int_{v^*+b_s}^{\tilde{v}+v^*+b_s} p(u)e^{-2su}du. \end{aligned}$$

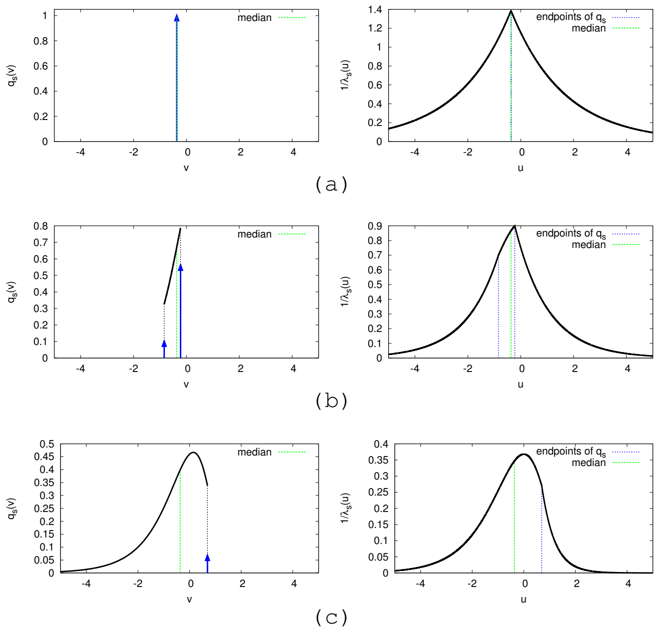

Fig. 1. Reconstruction density $q_s(v)$ (left panels) and function $\lambda_s(u)^{-1}$ (right panels) for $\alpha = 1$ and (a) $s = s_{\max} = -\log 2$, (b) $s = -0.8$ and (c) $s = -2.0$. Discrete components in $q_s(v)$ are represented by arrows whose length is equal to the coefficient in (27). Also indicated are the endpoints, $v^* - a_s$ and $v^* + b_s$, of $q_s(v)$ (in the right panels) and the median v^*.

We can prove $c_1(\tilde{v}) \le 1$ and $c_2(\tilde{v}) \le 1$ for $\tilde{v} > 0$ as in Lemma 2 of [8]. This is because we have for $i = 1, 2$,

$$c_i'(\tilde{v}) = s(c_i(\tilde{v}) - h_i(\tilde{v})), \tag{29}$$

where $h_1(\tilde{v}) = \frac{1}{e^{s\tilde{v}}} \frac{\gamma\left(\alpha, e^{v^*-a_s-\tilde{v}}\right)}{\gamma\left(\alpha, e^{v^*-a_s}\right)}$, $h_2(\tilde{v}) = \frac{1}{e^{s\tilde{v}}} \frac{\Gamma\left(\alpha, e^{v^*+a_s+\tilde{v}}\right)}{\Gamma\left(\alpha, e^{v^*+b_s}\right)}$. Then, if we assume that there is a $\tilde{v} > 0$ such that $c_i(\tilde{v}) > 1$, a contradiction is derived from (29) and the mean value theorem since $c_1(0) = c_2(0) = 1$ holds and $h_1(\tilde{v}) \le 1$ and $h_2(\tilde{v}) \le 1$ follow respectively from the monotonicity of g_1 and g_2 in Lemma 2 of this paper. This completes the proof of the optimality of q_s in (27).

B. Parametric Form of Rate-Distortion Curve

It follows from (4), (6) and (28) that

$$\begin{aligned} D_s &= \int_{-\infty}^{v^*-a_s} p(u)(v^* - a_s - u)du + \int_{v^*-a_s}^{v^*+b_s} \frac{p(u)}{|s|}du \\ &\quad + \int_{v^*+b_s}^{\infty} p(u)(u - v^* - b_s)du \qquad (30)\\ &= \frac{1}{|s|} - \frac{1}{|s|^2\Gamma(\alpha)}\left(\bar{t}_s^{\alpha} e^{-\bar{t}_s} + \underline{t}_s^{\alpha} e^{-\underline{t}_s}\right) \\ &\quad + \frac{1}{|s|\Gamma(\alpha)}\left\{(\log \underline{t}_s)\, \underline{t}_s^{\alpha} e^{-\underline{t}_s} - \left(\log \bar{t}_s\right) \bar{t}_s^{\alpha} e^{-\bar{t}_s}\right\} \\ &\quad + \Psi(\alpha) - \Psi(\alpha, \underline{t}_s) - \Psi(\alpha, \bar{t}_s), \qquad (31) \end{aligned}$$

where we put $\underline{t}_s = e^{v^*-a_s}$ and $\bar{t}_s = e^{v^*+b_s}$ and used (24).

Furthermore, the parametric form of $R(D_s)$ follows from (5), (28) and (30), which is presented in the next theorem.

Theorem 2 *Let $R(D)$ be the absolute-log criterion rate-distortion function of an i.i.d. gamma sequence (9). Then $R(D)$ for $0 < D < D_{\max}$ is given parametrically by,*

$$\begin{aligned} R(D) &= \log\frac{|s|}{2} + \alpha - 1 + \frac{1}{\Gamma(\alpha)}\left(\underline{t}_s^{\alpha} e^{-\underline{t}_s} - \bar{t}_s^{\alpha} e^{-\bar{t}_s}\right) \\ &\quad - (\alpha \log \underline{t}_s - \underline{t}_s + \alpha - 1)\frac{\underline{t}_s^{\alpha} e^{-\underline{t}_s}}{\Gamma(\alpha)|s|} \\ &\quad - \left(\alpha \log \bar{t}_s - \bar{t}_s + \alpha - 1\right)\frac{\bar{t}_s^{\alpha} e^{-\bar{t}_s}}{\Gamma(\alpha)|s|} \\ &\quad + \alpha\left\{\Psi\left(\alpha, \underline{t}_s\right) - \Psi\left(\alpha, \bar{t}_s\right)\right\} + \log\Gamma(\alpha), \qquad (32) \end{aligned}$$

where D is given parametrically by (31), $\underline{t}_s$ and $\bar{t}_s$ are identified by (24) for $-\alpha \le s \le s_{\max} = -2p(v^*)$ and by $\frac{1}{|s|} = \frac{\Gamma(\alpha, \bar{t}_s)}{\bar{t}_s^{\alpha} e^{-\bar{t}_s}}$, $\underline{t}_s = 0$, for $-\infty < s < -\alpha$.

C. Exponential Source

In the case of the exponential source distribution, that is, $\alpha = 1$, the above result is simplified as follows. (24) yields

$$\frac{1}{|s|} = \frac{e^{\underline{t}_s} - 1}{\underline{t}_s} = \frac{1}{\bar{t}_s}. \tag{33}$$

Parametric forms of $R(D)$ and D are given by

$$\begin{aligned} R(D) &= \log\frac{|s|}{2} + \underline{t}_s e^{-\underline{t}_s} - \bar{t}_s e^{-\bar{t}_s} - (\log \underline{t}_s - \underline{t}_s)\frac{\underline{t}_s e^{-\underline{t}_s}}{|s|} \\ &\quad - \left(\log \bar{t}_s - \bar{t}_s\right)\frac{\bar{t}_s e^{-\bar{t}_s}}{|s|} - \Psi\left(1, \bar{t}_s\right) + \Psi\left(1, \underline{t}_s\right), \qquad (34) \end{aligned}$$

$$\begin{aligned} D &= \frac{1}{|s|}\left(e^{-\underline{t}_s} - e^{-\bar{t}_s}\right) + (1 - e^{-\underline{t}_s})\log \underline{t}_s + e^{-\bar{t}_s}\log \bar{t}_s \\ &\quad - \gamma - \Psi\left(1, \underline{t}_s\right) - \Psi\left(1, \bar{t}_s\right), \qquad (35) \end{aligned}$$

where $\gamma = 0.577215\cdots$ is the Euler's constant. Figure 2 shows $R(D)$ for the exponential source, where we solved (33) for $\underline{t}_s$ by iterative substitution and approximated the incomplete Psi function $\Psi(1, x) = \int_0^x (\log t)e^{-t}dt$ by the trapezoidal rule.

IV. Shannon Lower Bound

Since q_s in (27) never satisfies $\int_{-\infty}^{\infty} q_s(v) g_s(u - v)dv = p(u)$, for $g_s(u) = \frac{|s|}{2} e^{s|u|}$, the SLB $R_L(D)$ is strictly smaller than $R(D)$ for all $0 < D < D_{\max}$ [6]. In this section, we compare the rate-distortion curve obtained in Section III with the SLB. To obtain the SLB, we compute the differential entropy of g_s,

$$h(g_s) = -\int_{-\infty}^{\infty} g_s(u)\log g_s(u)du = 1 - \log\frac{|s|}{2},$$

and that of $p(u)$ in (14),

$$h(p) = \alpha - \alpha\Psi(\alpha) + \log\Gamma(\alpha).$$

Then the SLB is given parametrically by [6],

$$\begin{aligned} R_L(D_s) &= h(p) - h(g_s), \\ &= \alpha - \alpha\Psi(\alpha) + \log\Gamma(\alpha) + \log\frac{|s|}{2} - 1, \end{aligned}$$

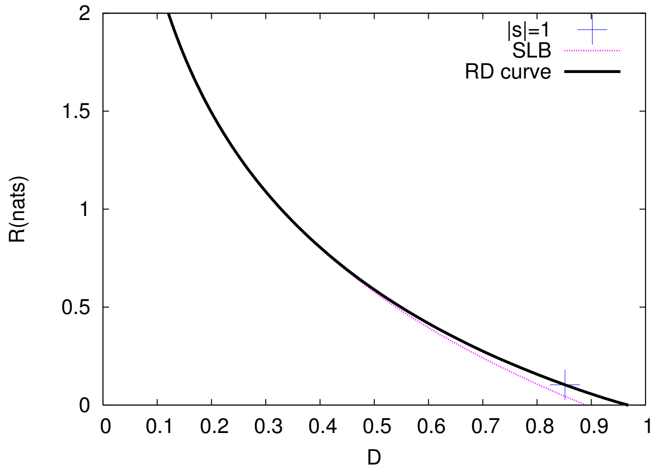

Fig. 2. Rate-distortion curve for the exponential source given parametrically by (34) and (35) (solid line) and the SLB given by (36) for $\alpha = 1$ (dotted line). The plus symbol $(+)$ indicates the point on which the left discrete component disappears from $q_s(v)$.

and the corresponding distortion is

$$D_s = \int_{-\infty}^{\infty} |u| g_s(u) du = \frac{1}{|s|}.$$

Thus the SLB is summarized as follows,

$$R_L(D) = \alpha - \alpha\Psi(\alpha) + \log\Gamma(\alpha) - \log(2D) - 1, \qquad (36)$$

for $0 < D < D^L_{\max} = \frac{1}{2}\exp\{\alpha - \alpha\Psi(\alpha) + \log\Gamma(\alpha) - 1\}$.

Figure 2 compares $R(D)$ given in Section III with $R_L(D)$ in the case of the exponential source, $\alpha = 1$. It can be seen that $R_L(D)$ approaches $R(D)$ for small enough D and hence provides a good approximation to $R(D)$ for this source.

V. Other Sources and Distortions

We have derived the rate-distortion function of the gamma sources with the absolute-log distortion measure. As in Section III, the logarithmic transformation reduces the rate-distortion problem with the absolute-log distortion to that of absolute-magnitude distortion measure.

The log-normal distribution for positive real variable corresponds to the Gaussian source in the transformed space, whose rate-distortion function is evaluated in [8]. In this case, the optimal reconstruction has the distribution with the discrete components on both endpoints of an interval around the median v^* and q_s is symmetric around v^* while for the gamma source, q_s is asymmetric and the left discrete component disappears for large $|s|$.

Similarly, the rate-distortion problem with the squared-log distortion measure,

$$d(x, y) = (\log x - \log y)^2,$$

can be casted to that with the squared distortion, $d(u, v) = (u - v)^2$ by the logarithmic transform. Under this distortion measure, the log-normal source in the original space yields the well-known rate-distortion function of Gaussian sources with respect to the squared distortion measure [6, p. 99, Theorem 4.3.2]. As for the gamma sources examined in Section III, Rose's analysis [7] suggests that the rate-distortion function with respect to the squared-log distortion measure is achieved by the discrete reconstruction distribution with distinct points of support. Analyzing this optimal discrete distribution is an important undertaking.

It is proved for the Itakura-Saito distortion measure that $R(D)$ of the gamma source coincides with the corresponding SLB [12]. This result is obtained through the Mellin transform and is analogous to that of the Gaussian source through the Fourier transform. The rate-distortion function of a particular kind of distortion measure is investigated in the case of the exponential distribution [13]. The rate-distortion function and the optimal reconstruction distribution are explicitly obtained through the Laplace transform in this case.

VI. Conclusion

In this paper, we presented the rate-distortion curve for the i.i.d. gamma source under the absolute-log distortion measure. We showed that the optimal reconstruction distribution consists of a continuous component on an interval around the median with discrete components on the endpoints of the interval and that the left discrete component disappears for low average distortion while the right one remains until the distortion approaches zero.

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