The K-User Interference Channel: Strong Interference Regime

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***Abstract*—This paper presents a solution for one of the open problems in network information theory: "*What is the generalization of the strong interference regime to the K-user interference channel?*" A new approach is developed based on which one can obtain strong interference regimes not only for the multi-user interference channels but also for other interference networks with any arbitrary topology. To this development, some new lemmas are proved which have a central role in our derivations. As a result, this paper establishes the first non-trivial capacity result for the general multi-user classical interference channel (for both discrete and Gaussian channels).**

I. INTRODUCTION

One of the fundamental open problems in network information theory is to determine the capacity region of the Classical Interference Channel (CIC). The importance of this problem is by now widely acknowledged. As these channels are very useful models for wireless communication systems, in recent years they have been extensively studied. For a detailed review of the existing literatures, refer to Part I of our multi-part papers [1-7]. For the two-user CIC capacity results are known in some special cases (see [1]), however, the multi-user CICs are far less understood [8, Ch. 6, p. 157].

In 1981 [9], Sato derived a regime for the two-user Gaussian CIC wherein joint decoding both messages at both receivers is optimal and achieves the capacity. Six years later, in 1987 [10], Costa and El Gamal extended the Sato's result for the discrete channel. This regime in which the capacity region is derived by decoding both messages at both receivers is called the "strong interference regime". From that time, for about 26 years, it has been an open problem [8, Ch. 6, p. 158] that what is the generalization of the strong interference regime for the multi-user CIC? In this paper, we give a solution to this problem. Clearly, we develop a new approach based on which one can derive strong interference regime for any given interference network. To this end, we prove some new lemmas which have a central role in our derivations.

This paper addresses our main results for the classical interference channels. However, our approach can be followed to obtain strong interference regimes for arbitrary single-hop communication networks with any arbitrary topology. Please refer to Part III of our multi-part papers [3] where we have presented a general formula to derive strong interference conditions for all single-hop communication networks of arbitrary large sizes.

In this paper, we use the same notations as those given in [1], most of them are standard. Also, channel models and information theoretic concepts such as capacity region are given as usual. The details can be found in [1].

In Section II, we present our new lemmas. In Section III, we derive our main result for three-user CIC and finally in Section IV, we extend the result to the K-user CIC.

II. NEW LEMMAS

In what follows, we derive some new lemmas which are repeatedly used throughout the paper. These results are indeed critical for our future derivations.

Lemma 1) *Let $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{X}_1,\mathcal{X}_2,\dots,\mathcal{X}_{\mu_1},\mathcal{X}_{\mu_1+1},\dots,\mathcal{X}_{\mu_1+\mu_2}$ be arbitrary sets, where μ_1,μ_2 are arbitrary natural numbers. Let also $\mathbb{P}(y_1,y_2|x_1,x_2,\dots,x_{\mu_1},x_{\mu_1+1},\dots,x_{\mu_1+\mu_2})$ be a given conditional probability distribution defined on the set $\mathcal{Y}_1\times\mathcal{Y}_2\times\mathcal{X}_1\times\mathcal{X}_2\times\dots\times\mathcal{X}_{\mu_1}\times\mathcal{X}_{\mu_1+1}\times\dots\times\mathcal{X}_{\mu_1+\mu_2}$. Consider the inequality below:*

$$I(X_1,\dots,X_{\mu_1};Y_1|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2}) \leq I(X_1,\dots,X_{\mu_1};Y_2|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2}) \quad (1)$$

If the inequality (1) *holds for all PDFs $P_{X_1\dots X_{\mu_1}X_{\mu_1+1}\dots X_{\mu_1+\mu_2}}$ with the following factorization:*

$$P_{X_1\dots X_{\mu_1}X_{\mu_1+1}\dots X_{\mu_1+\mu_2}}=P_{X_1\dots X_{\mu_1}}P_{X_{\mu_1+1}}P_{X_{\mu_1+2}}\dots P_{X_{\mu_1+\mu_2}} \quad (2)$$

then, we have:

$$I(X_1,\dots,X_{\mu_1};Y_1|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},D) \leq I(X_1,\dots,X_{\mu_1};Y_2|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},D) \quad (3)$$

for all joint PDFs $P_{DX_1\dots X_{\mu_1}X_{\mu_1+1}\dots X_{\mu_1+\mu_2}}$ where $D\to X_1,\dots,X_{\mu_1},X_{\mu_1+1},\dots,X_{\mu_1+\mu_2}\to Y_1,Y_2$ forms a Markov chain.

Proof of Lemma 1) First we show that (1) implies the following inequality:

$$I(X_1,\dots,X_{\mu_1};Y_1|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},W) \leq I(X_1,\dots,X_{\mu_1};Y_2|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},W) \quad (4)$$

for all PDFs $P_{WX_1\dots X_{\mu_1}X_{\mu_1+1}\dots X_{\mu_1+\mu_2}}$ with:

$$P_{WX_1\dots X_{\mu_1}X_{\mu_1+1}\dots X_{\mu_1+\mu_2}} = P_W P_{X_1\dots X_{\mu_1}|W}P_{X_{\mu_1+1}|W}P_{X_{\mu_1+2}|W}\dots P_{X_{\mu_1+\mu_2}|W} \quad (5)$$

where $W\to X_1,\dots,X_{\mu_1},X_{\mu_1+1},\dots,X_{\mu_1+\mu_2}\to Y_1,Y_2$ forms a Markov chain. To prove this inequality, one can write:

$$\begin{aligned}
&I(X_1,\dots,X_{\mu_1};Y_1|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},W)\\
&=\textstyle\sum_w P_W(w)I(X_1,\dots,X_{\mu_1};Y_1|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},w)\\
&\overset{(a)}{\leq}\textstyle\sum_w P_W(w)I(X_1,\dots,X_{\mu_1};Y_2|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},w)\\
&=I(X_1,\dots,X_{\mu_1};Y_2|X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},W)
\end{aligned}$$

where the inequality (a) is due to (1). Now, having at hand the inequality (4), one can substitute $W\equiv(D,X_{\mu_1+1},X_{\mu_1+2},\dots,X_{\mu_1+\mu_2})$ with an arbitrary joint distribution on the set $\mathcal{D}\times\mathcal{X}_{\mu_1+1}\times\dots\times\mathcal{X}_{\mu_1+\mu_2}$. By this substitution, we derive that (3) holds for all joint PDFs $P_{DX_{\mu_1+1}\dots X_{\mu_1+\mu_2}}P_{X_1\dots X_{\mu_1}|DX_{\mu_1+1}\dots X_{\mu_1+\mu_2}}$. The proof is complete. ■

Corollary 1) *Let $\mathcal{L}$ be an arbitrary subset of $\{1,\dots,\mu_1\}$. Denote $\mathbb{X}_{\mathcal{L}}\triangleq\{X_l: l\in\mathcal{L}\}$. If the inequality* (1) *holds for all joint PDFs* (2)*, then we have:*

$$I(\{X_1,\dots,X_{\mu_1}\}-\mathbb{X}_{\mathcal{L}};Y_1|\,\mathbb{X}_{\mathcal{L}},X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},D) \leq I(\{X_1,\dots,X_{\mu_1}\}-\mathbb{X}_{\mathcal{L}};Y_2|\,\mathbb{X}_{\mathcal{L}},X_{\mu_1+1},\dots,X_{\mu_1+\mu_2},D) \quad (6)$$

for all joint PDFs $P_{DX_1\ldots X_{\mu_1}X_{\mu_1+1}\ldots X_{\mu_1+\mu_2}}$ *where* $D \to X_1,\ldots,X_{\mu_1},X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2} \to Y_1,Y_2$ *forms a Markov chain.*

Proof of Corollary 1) Replace D with $(D,\mathbb{X}_{\mathcal{L}})$ in (3). ■

Next let us consider a Gaussian transition probability function. Precisely, let the outputs Y_1 and Y_2 be given as follows:

$$\begin{cases} Y_1 \triangleq \sum_{i=1}^{\mu_1} a_i X_i + \sum_{i=\mu_1+1}^{\mu_1+\mu_2} a_i X_i + Z_1 \\ Y_2 \triangleq \sum_{i=1}^{\mu_1} b_i X_i + \sum_{i=\mu_1+1}^{\mu_1+\mu_2} b_i X_i + Z_2 \end{cases} \tag{7}$$

where Z_1 and Z_2 are zero-mean unit-variance Gaussian random variables; also, $X_1,X_2,\ldots,X_{\mu_1},X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}$ are real-valued power-constrained random variables independent of (Z_1,Z_2) and $a_1,a_2,\ldots,a_{\mu_1},a_{\mu_1+1},\ldots,a_{\mu_1+\mu_2}$ and $b_1,b_2,\ldots,b_{\mu_1},b_{\mu_1+1},\ldots,b_{\mu_1+\mu_2}$ are fixed real numbers. Our purpose is to determine sufficient conditions under which for this setup the inequality (3) holds for all joint PDFs $P_{DX_1\ldots X_{\mu_1}X_{\mu_1+1}\ldots X_{\mu_1+\mu_2}}$. The following lemma gives such conditions.

Lemma 2) *Consider the Gaussian system in* (7)*. If the following condition satisfies:*

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_{\mu_1}}{b_{\mu_1}} = \alpha, \qquad |\alpha| \le 1 \tag{8}$$

then, the inequality (3) *holds for all joint PDFs* $P_{DX_1\ldots X_{\mu_1}X_{\mu_1+1}\ldots X_{\mu_1+\mu_2}}(d,x_1,\ldots,x_{\mu_1},x_{\mu_1+1},\ldots,x_{\mu_1+\mu_2})$ *where* D *is independent of* (Z_1,Z_2).

Proof of Lemma 2) First note that if D is independent of (Z_1,Z_2), then $D \to X_1,\ldots,X_{\mu_1},X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2} \to Y_1,Y_2$ forms a Markov chain. It is sufficient to prove that (1) holds. Define:

$$\begin{aligned}\tilde{Y}_1 \triangleq{}& \alpha Y_2 + (\alpha b_{\mu_1+1} - a_{\mu_1+1})X_{\mu_1+1} + (\alpha b_{\mu_1+2} - a_{\mu_1+2})X_{\mu_1+2} \\ &+ \cdots + (\alpha b_{\mu_1+\mu_2} - a_{\mu_1+\mu_2})X_{\mu_1+\mu_2} + \sqrt{1-\alpha^2}\tilde{Z}_1\end{aligned} \tag{9}$$

where $\tilde{Z}_1$ is a Gaussian random variable with zero mean and unit variance and impendent of (Z_1,Z_2). Considering (7), it is readily derived that $\tilde{Y}_1$ is statistically equivalent to Y_1 in the sense of:

$$\begin{aligned}&\mathbb{P}(\tilde{y}_1|x_1,\ldots,x_{\mu_1},x_{\mu_1+1},\ldots,x_{\mu_1+\mu_2}) \\ &\qquad \approx \mathbb{P}(y_1|x_1,\ldots,x_{\mu_1},x_{\mu_1+1},\ldots,x_{\mu_1+\mu_2})\end{aligned}$$

Therefore, for all input distributions we have:

$$\begin{aligned}&I(X_1,\ldots,X_{\mu_1};Y_1|X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}) \\ &= I(X_1,\ldots,X_{\mu_1};\tilde{Y}_1|X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}) \\ &\le I(X_1,\ldots,X_{\mu_1};\tilde{Y}_1,Y_2|X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}) \\ &\overset{(a)}{=} I(X_1,\ldots,X_{\mu_1};Y_2|X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}) \\ &\qquad + I(X_1,\ldots,X_{\mu_1};\tilde{Y}_1|Y_2,X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}) \\ &= I(X_1,\ldots,X_{\mu_1};Y_2|X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2})\end{aligned}$$

where (a) holds because, according to (9), $X_1,\ldots,X_{\mu_1} \to Y_2,X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2} \to \tilde{Y}_1$ forms a Markov chain. The proof is thus complete. ■

Remarks:

1. The proof style of Lemma 2 indicates that under the condition (8), given $X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2}$, the signal Y_1 is a stochastically degraded version of Y_2.
2. The relation (8) is a sufficient condition under which (1) holds; however, in general the inequality (1) may not be equivalent to (8). It is also essential to note that the condition (8) is not derived by evaluating (1) for Gaussian input distributions. Only for the case of $\mu_1 = 1$, the condition (8) can be equivalently derived by evaluating (1) for Gaussian input distributions.

In the next lemma, we also provide a multi-letter extension of Lemma 1 which is necessary to identify strong interference regime for multi-user networks.

Lemma 3) *Fix the conditional PDF* $\mathbb{P}(y_1,y_2|x_1,\ldots,x_{\mu_1},x_{\mu_1+1},\ldots,x_{\mu_1+\mu_2})$*. Assume that the inequality* (1) *holds for all joint PDFs* (2)*. For a given arbitrary natural number* n*, let* $\mathbb{P}(y_1^n,y_2^n|x_1^n,x_2^n,\ldots,x_{\mu_1}^n,x_{\mu_1+1}^n,\ldots,x_{\mu_1+\mu_2}^n)$ *be a memoryless* n*-tuple extension of* $\mathbb{P}(y_1,y_2|x_1,x_2,\ldots,x_{\mu_1},x_{\mu_1+1},\ldots,x_{\mu_1+\mu_2})$*, i.e.,*

$$\begin{aligned}&\mathbb{P}(y_1^n,y_2^n|x_1^n,x_2^n,\ldots,x_{\mu_1}^n,x_{\mu_1+1}^n,\ldots,x_{\mu_1+\mu_2}^n) \\ &= \prod_{t=1}^{n} \mathbb{P}(y_{1,t},y_{2,t}|x_{1,t},x_{2,t},\ldots,x_{\mu_1,t},x_{\mu_1+1,t},\ldots,x_{\mu_1+\mu_2,t})\end{aligned} \tag{10}$$

Then, the following inequality holds:

$$\begin{aligned}&I(X_1^n,\ldots,X_{\mu_1}^n;Y_1^n|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,D) \\ &\qquad \le I(X_1^n,\ldots,X_{\mu_1}^n;Y_2^n|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,D)\end{aligned} \tag{11}$$

for all joint PDFs $P_{DX_1^n\ldots X_{\mu_1}^nX_{\mu_1+1}^n\ldots X_{\mu_1+\mu_2}^n}$ *where* $D \to X_1^n,\ldots,X_{\mu_1}^n,X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n \to Y_1^n,Y_2^n$ *forms a Markov chain.*

Proof of Lemma 3) First note that, according to Lemma 1, since (1) holds for all joint PDFs (2), the inequality (3) also holds for all joint PDFs $P_{DX_1\ldots X_{\mu_1}X_{\mu_1+1}\ldots X_{\mu_1+\mu_2}}$ where $D \to X_1,\ldots,X_{\mu_1},X_{\mu_1+1},\ldots,X_{\mu_1+\mu_2} \to Y_1,Y_2$ forms a Markov chain. Now consider the two sides of (11). For a given vector A^n, denote $A^{n\backslash t} \triangleq (A^{t-1},A_{t+1}^n)$ where $t = 1,..,n$. Define:

$$\bar{\bar{D}} \triangleq (X_{\mu_1+1}^{n\backslash t},\ldots,X_{\mu_1+\mu_2}^{n\backslash t},Y_2^{t-1},Y_{1,t+1}^n,D) \tag{12}$$

We have:

$$\begin{aligned}&I(X_1^n,\ldots,X_{\mu_1}^n;Y_2^n|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,D) \\ &\quad - I(X_1^n,\ldots,X_{\mu_1}^n;Y_1^n|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,D) \\ &\quad = \textstyle\sum_{t=1}^n I(X_1^n,\ldots,X_{\mu_1}^n;Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(X_1^n,\ldots,X_{\mu_1}^n;Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,D) \\ &\quad \overset{(a)}{=} \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,D) \\ &\quad \overset{(b)}{=} \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t},Y_{1,t+1}^n;Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t},Y_2^{t-1};Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,D) \\ &\quad = \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,Y_2^{t-1},D) \\ &\quad + \textstyle\sum_{t=1}^n I(Y_{1,t+1}^n;Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(Y_2^{t-1};Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,D) \\ &\quad \overset{(c)}{=} \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{2,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,Y_2^{t-1},D) \\ &\quad - \textstyle\sum_{t=1}^n I(X_{1,t},\ldots,X_{\mu_1,t};Y_{1,t}|X_{\mu_1+1}^n,\ldots,X_{\mu_1+\mu_2}^n,Y_{1,t+1}^n,Y_2^{t-1},D)\end{aligned}$$

$$= \sum_{t=1}^{n} \begin{pmatrix} I\left(X_{1,t},\dots,X_{\mu_1,t};Y_{2,t}\middle|X_{\mu_1+1,t},\dots,X_{\mu_1+\mu_2,t},\overline{\overline{D}}\right) \\ -I\left(X_{1,t},\dots,X_{\mu_1,t};Y_{1,t}\middle|X_{\mu_1+1,t},\dots,X_{\mu_1+\mu_2,t},\overline{\overline{D}}\right) \end{pmatrix}$$
$$\overset{(d)}{\geq} 0 \tag{13}$$

where equalities (a) and (b) hold because the memorylessness property (10) implies the following Markov relations ($t = 1,\dots,n$):

$$X_1^{n\setminus t},\dots,X_{\mu_1}^{n\setminus t},X_{\mu_1+1}^{n\setminus t},\dots,X_{\mu_1+\mu_2}^{n\setminus t},Y_2^{t-1},Y_{1,t+1}^{n},D \to$$
$$\to X_{1,t},\dots,X_{\mu_1,t},X_{\mu_1+1,t},\dots,X_{\mu_1+\mu_2,t} \to Y_{1,t},Y_{2,t}$$

Also, equality (c) is due to Csiszar-Korner identity according which the 3^{rd} and the 4^{th} expressions in the left hand side of (c) are equal; finally, (d) is due to the inequality (3) in which D is replaced by $\overline{\overline{D}}$. The proof is thus complete. ■

Let us consider the special case of $\mu_1 = \mu_2 = 2$. In [10, Appendix], it is shown that if the following condition holds:

$$I(X_1;Y_1|X_2) \leq I(X_1;Y_2|X_2) \quad \text{for all joint PDFs} \quad P_{X_1}P_{X_2} \tag{14}$$

then, we have:

$$I(X_1^n;Y_1^n|X_2^n) \leq I(X_1^n;Y_2^n|X_2^n) \quad \text{for all joint PDFs} \quad P_{X_1^n}P_{X_2^n} \tag{15}$$

The proof of [10] is based on induction which requires establishing some sophisticated Markov chains (see [10, Appendix]). Moreover, the authors of [10] are able to derive (15) only for product distributions $P_{X_1^n} \times P_{X_2^n}$. Our proof in Lemma 3 is considerably simpler since instead of sophisticated induction-based arguments, it is derived by a direct application of the Csiszar-Korner identity. Also, by using the consequence of Lemma 1, we are able to prove (15) for all arbitrary joint PDFs $P_{X_1^n X_2^n}$. As we will see throughout the paper, such extension is critical while deriving strong interference regime for multi-user networks.

By these lemmas, we are ready to develop our results in the subsequent sections.

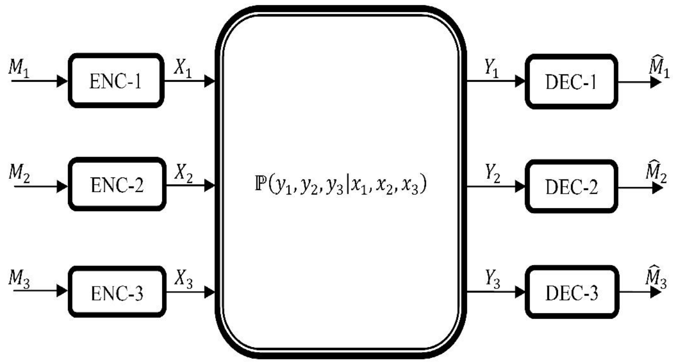

Figure 1. The three-user Classical Interference Channel (CIC).

III. The Three-User CIC

Consider the three-user CIC shown in Fig. 1. We intend to derive a strong interference regime for this network. We remark that the following theory which is presented for the three-user CIC can be developed for other interference networks with any arbitrary topology, as given in [3, Sec. V.B.3].

First note that according to the definition (see [3, Definition 4]) in the strong interference regime each receiver decodes all messages. The resulting achievable rate region by this scheme is given by (16). We need to derive conditions under which this rate region is optimal. Consider a length-n block code for the network with vanishing error probability.

$$\bigcup_{P_Q P_{X_1|Q} P_{X_2|Q} P_{X_3|Q}} \left\{ \begin{array}{l} (R_1,R_2,R_3) \in \mathbb{R}_+^3: \\ R_1 \leq \min\limits_{i=1,2,3}\{I(X_1;Y_i|X_2,X_3,Q)\} \\ R_2 \leq \min\limits_{i=1,2,3}\{I(X_2;Y_i|X_1,X_3,Q)\} \\ R_3 \leq \min\limits_{i=1,2,3}\{I(X_3;Y_i|X_1,X_2,Q)\} \\ R_1+R_2 \leq \min\limits_{i=1,2,3}\{I(X_1,X_2;Y_i|X_3,Q)\} \\ R_2+R_3 \leq \min\limits_{i=1,2,3}\{I(X_2,X_3;Y_i|X_1,Q)\} \\ R_1+R_3 \leq \min\limits_{i=1,2,3}\{I(X_1,X_3;Y_i|X_2,Q)\} \\ R_1+R_2+R_3 \leq \min\limits_{i=1,2,3}\{I(X_1,X_2,X_3;Y_1|Q)\} \end{array} \right\} \tag{16}$$

Claim: if the network transition probability function satisfies the following conditions:

$$\begin{cases} I(X_2;Y_2|X_1,X_3) \leq I(X_2;Y_3|X_1,X_3) & \text{for all joint PDFs} \quad P_{X_1}P_{X_2}P_{X_3} \\ I(X_2,X_3;Y_3|X_1) \leq I(X_2,X_3;Y_1|X_1) & \text{for all joint PDFs} \quad P_{X_1}P_{X_2X_3} \end{cases} \tag{17}$$

then, we have:

$$\begin{aligned} n(R_1+R_2+R_3) &\leq I(X_1^n,X_2^n,X_3^n;Y_1^n) + n\epsilon_n \\ &\leq \textstyle\sum_{t=1}^{n} I\left(X_{1,t},X_{2,t},X_{3,t};Y_{1,t}\right) + n\epsilon_n \end{aligned} \tag{18}$$

where $\epsilon_n \to 0$ as $n \to 0$.

Proof of Claim: Based on the Fano's inequality one can write:

$$\begin{aligned} n(R_1+R_2+R_3) &\leq I(M_2;Y_2^n) + I(M_3;Y_3^n) + I(M_1;Y_1^n) + n\epsilon_n \\ &\leq I(M_2;Y_2^n|M_1,M_3) + I(M_3;Y_3^n|M_1) + I(M_1;Y_1^n) + n\epsilon_n \\ &\overset{(a)}{=} I(X_2^n;Y_2^n|X_1^n,X_3^n,M_1,M_3) + I(X_3^n,M_3;Y_3^n|X_1^n,M_1) \\ &\qquad + I(X_1^n,M_1;Y_1^n) + n\epsilon_n \\ &\overset{(b)}{\leq} I(X_2^n;Y_3^n|X_1^n,X_3^n,M_1,M_3) + I(X_3^n,M_3;Y_3^n|X_1^n,M_1) \\ &\qquad + I(X_1^n,M_1;Y_1^n) + n\epsilon_n \\ &= I(X_2^n,X_3^n;Y_3^n|X_1^n,M_1) + I(X_1^n,M_1;Y_1^n) + n\epsilon_n \\ &\overset{(c)}{\leq} I(X_2^n,X_3^n;Y_1^n|X_1^n,M_1) + I(X_1^n,M_1;Y_1^n) + n\epsilon_n \\ &= I(X_1^n,X_2^n,X_3^n;Y_1^n) + n\epsilon_n \\ &\leq \textstyle\sum_{t=1}^{n} I\left(X_{1,t},X_{2,t},X_{3,t};Y_{1,t}\right) + n\epsilon_n \end{aligned} \tag{19}$$

where equality (a) holds because the input sequence X_i^n is given by a deterministic function of the message $M_i, i = 1,2,3$, inequality (b) is due to the first condition in (17) and its n-tuple extension in Lemma 3, and inequality (c) is due to the second condition in (17) and its n-tuple extension in Lemma 3.

Therefore, under the conditions (17), we derived one of the desired constraints on the sum-rate capacity in (16). In fact, by the conditions (17), one can achieve further results. Clearly, these conditions imply that decoding all messages at the first receiver is optimal. Let us prove this conclusion. Consider the constraints on the partial sum rates. First note that, according to Corollary 1, the second condition of (17) imply that:

$$\begin{cases} I(X_2;Y_3|X_1,X_3) \leq I(X_2;Y_1|X_1,X_3) \\ I(X_3;Y_3|X_1,X_2) \leq I(X_3;Y_1|X_1,X_2) \end{cases} \tag{20}$$

Comparing the first condition in (17) and the first condition of (20), we also obtain:

$$I(X_2;Y_2|X_1,X_3) \leq I(X_2;Y_1|X_1,X_3) \tag{21}$$

Now, we have:

$$\begin{aligned}
n(R_1+R_2) &\leq I(M_2;Y_2^n) + I(M_1;Y_1^n) + n\epsilon_n \\
&\leq I(M_2;Y_2^n|M_1,M_3) + I(M_1;Y_1^n|M_3) + n\epsilon_n \\
&= I(X_2^n;Y_2^n|X_1^n,X_3^n,M_1,M_3) \\
&\qquad + I(X_1^n,M_1;Y_1^n|X_3^n,M_3) + n\epsilon_n \\
&\overset{(a)}{\leq} I(X_2^n;Y_1^n|X_1^n,X_3^n,M_1,M_3) \\
&\qquad + I(X_1^n,M_1;Y_1^n|X_3^n,M_3) + n\epsilon_n \\
&= I(X_1^n,X_2^n;Y_1^n|X_3^n,M_3) + n\epsilon_n \\
&\leq \textstyle\sum_{t=1}^{n} I(X_{1,t},X_{2,t};Y_{1,t}|X_{3,t}) + n\epsilon_n
\end{aligned} \tag{22}$$

where inequality (a) is due to (21) and its n-tuple extension in Lemma 3. Also, by following the same lines as (19), one can derive:

$$\begin{aligned}
R_2 + R_3 &\leq I(X_2^n,X_3^n;Y_1^n|X_1^n,M_1) + n\epsilon_n \\
&\leq \textstyle\sum_{t=1}^{n} I(X_{2,t},X_{3,t};Y_{1,t}|X_{1,t}) + n\epsilon_n
\end{aligned} \tag{23}$$

Note that (23) actually is the inequality (c) of (19) in which the term $I(X_1^n,M_1;Y_1^n)$ from the right side and the corresponding rate R_1 from the left side are removed. Lastly, we have:

$$\begin{aligned}
n(R_1+R_3) &\leq I(M_3;Y_3^n) + I(M_1;Y_1^n) + n\epsilon_n \\
&\leq I(X_3^n;Y_3^n|X_1^n,X_2^n,M_1,M_2) \\
&\qquad + I(X_1^n,M_1;Y_1^n|X_2^n,M_2) + n\epsilon_n \\
&\overset{(a)}{\leq} I(X_3^n;Y_1^n|X_1^n,X_2^n,M_1,M_2) \\
&\qquad + I(X_1^n,M_1;Y_1^n|X_2^n,M_2) + n\epsilon_n \\
&= I(X_1^n,X_3^n;Y_1^n|X_2^n,M_2) + n\epsilon_n \\
&\leq \textstyle\sum_{t=1}^{n} I(X_{1,t},X_{3,t};Y_{1,t}|X_{2,t}) + n\epsilon_n
\end{aligned} \tag{24}$$

where inequality (a) is due to the second condition in (20). Finally, the desired constraints on the individual rates can be easily derived using the second condition of (20) and the condition (21). Thus, if (17) holds, then it is optimal to decode all messages at the first receiver. It is clear that we can follow the same procedure for the other receivers to derive conditions under which the strong interference criterion, i.e., the optimality of decoding all messages, is satisfied. For example, one can verify that if the following conditions hold:

$$\begin{cases} I(X_3;Y_3|X_1,X_2) \leq I(X_3;Y_1|X_1,X_2) & \text{for all joint PDFs} \quad P_{X_1}P_{X_2}P_{X_3} \\ I(X_1,X_3;Y_1|X_2) \leq I(X_1,X_3;Y_2|X_2) & \text{for all joint PDFs} \quad P_{X_1X_3}P_{X_2} \end{cases} \tag{25}$$

the second receiver, and if the following hold:

$$\begin{cases} I(X_1;Y_1|X_2,X_3) \leq I(X_1;Y_2|X_2,X_3) & \text{for all joint PDFs} \quad P_{X_1}P_{X_2}P_{X_3} \\ I(X_1,X_2;Y_2|X_3) \leq I(X_1,X_2;Y_3|X_3) & \text{for all joint PDFs} \quad P_{X_1X_2}P_{X_3} \end{cases} \tag{26}$$

the third receiver experience strong interference. Therefore, the collection of the conditions (17), (25) and (26) constitutes a strong interference regime for the three-user CIC in Fig. 1. A remarkable point is that the necessary conditions for deriving the desired constraints on the sum-rate such as (18) are indeed sufficient to prove the optimality of decoding all messages at the receivers. In other words, once we derived the desired constraints on the sum-rate capacity using certain conditions, no additional condition is required to be introduced to prove the desired constraints on the partial sum-rates.

Let us concentrate on the collection of the conditions (17), (25) and (26). According to Corollary 1, the second condition of (17) implies the first condition of (25), the second condition of (25) implies the first condition in (26) and the second condition of (26) implies the first condition of (17). Therefore, a strong interference regime for the three-user CIC is given as follows:

$$\begin{cases} I(X_2,X_3;Y_3|X_1) \leq I(X_2,X_3;Y_1|X_1) & \text{for all joint PDFs} \quad P_{X_1}P_{X_2X_3} \\ I(X_1,X_3;Y_1|X_2) \leq I(X_1,X_3;Y_2|X_2) & \text{for all joint PDFs} \quad P_{X_1X_3}P_{X_2} \\ I(X_1,X_2;Y_2|X_3) \leq I(X_1,X_2;Y_3|X_3) & \text{for all joint PDFs} \quad P_{X_1X_2}P_{X_3} \end{cases} \tag{27}$$

The conditions (27) to some extent represents a fact regarding the CICs that is the signal of each receiver is impaired by the joint effect of interference from all non-corresponding transmitters rather by each transmitter's signal separately [8, p. 157]. Note that the terms in the right side of the inequalities in (27) indeed measure the amount of interference experienced by the receivers.

It should be noted that using the conditions (27) we are able to derive all the constraints in the rate region (16); nevertheless, some of these constraints are actually redundant. In fact, if the conditions (27) hold, the rate region (16) is simplified below:

$$\bigcup_{P_QP_{X_1|Q}P_{X_2|Q}P_{X_3|Q}} \left\{ \begin{array}{l} (R_1,R_2,R_3): \\ R_1 \leq I(X_1;Y_1|X_2,X_3,Q) \\ R_2 \leq I(X_2;Y_2|X_1,X_3,Q) \\ R_3 \leq I(X_3;Y_3|X_1,X_2,Q) \\ \\ R_1+R_2 \leq \min\left\{ \begin{array}{l} I(X_1,X_2;Y_1|X_3,Q), \\ I(X_1,X_2;Y_2|X_3,Q) \end{array} \right\} \\ R_2+R_3 \leq \min\left\{ \begin{array}{l} I(X_2,X_3;Y_2|X_1,Q), \\ I(X_2,X_3;Y_3|X_1,Q) \end{array} \right\} \\ R_1+R_3 \leq \min\left\{ \begin{array}{l} I(X_1,X_3;Y_1|X_2,Q), \\ I(X_1,X_3;Y_3|X_2,Q) \end{array} \right\} \\ R_1+R_2+R_3 \leq \min\left\{ \begin{array}{l} I(X_1,X_2,X_3;Y_1|Q), \\ I(X_1,X_2,X_3;Y_2|Q), \\ I(X_1,X_2,X_3;Y_3|Q) \end{array} \right\} \end{array} \right\} \tag{28}$$

The other constraints of (16) are relaxed by the conditions in (27). Let us now consider the three-user Gaussian CIC as formulated in the following standard form:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{29}$$

where Z_1,Z_2,Z_3 are zero-mean unit-variance Gaussian noises and $\mathbb{E}[X_i^2] \leq P_i,\ i = 1,2,3$. Using Lemma 2, one can derive explicit constraints on the network gains under which the strong interference regime (27) holds, as given below:

$$\begin{cases} |a_{13}| \geq 1, \qquad |a_{21}| \geq 1, \qquad |a_{32}| \geq 1 \\ \qquad\qquad\qquad\qquad , \\ a_{12} = a_{13}a_{32}, \qquad a_{31} = a_{21}a_{32}, \qquad a_{23} = a_{21}a_{13} \end{cases} \tag{30}$$

Let examine the conditions (30). Among six parameters in the network gain matrix (29), the parameters a_{13}, a_{21} and a_{32}, which no pair of them lies in either a same row or a same column, are given by arbitrary real numbers greater than one and the other parameters are given in terms of these parameters by specific relations.

A Strong Interference Regime for the K-User Interference Channel

$$
\begin{aligned}
&I(X_2,X_3,X_4,\dots,X_K;Y_K|X_1) \le I(X_2,X_3,X_4,\dots,X_K;Y_1|X_1) && \text{for all joint PDFs} \quad P_{X_2X_3X_4\dots X_K}P_{X_1}\\
&I(X_1,X_3,X_4,\dots,X_K;Y_1|X_2) \le I(X_1,X_3,X_4,\dots,X_K;Y_2|X_2) && \text{for all joint PDFs} \quad P_{X_1X_3X_4\dots X_K}P_{X_2}\\
&I(X_1,X_2,X_4,\dots,X_K;Y_2|X_3) \le I(X_1,X_2,X_4,\dots,X_K;Y_3|X_3) && \text{for all joint PDFs} \quad P_{X_1X_2X_4\dots X_K}P_{X_3}\\
&\vdots\\
&I(X_1,X_2,\dots,X_{K-1};Y_{K-1}|X_K) \le I(X_1,X_2,\dots,X_{K-1};Y_K|X_K) && \text{for all joint PDFs} \quad P_{X_1X_2\dots X_{K-1}}P_{X_K}
\end{aligned}
\tag{31}
$$

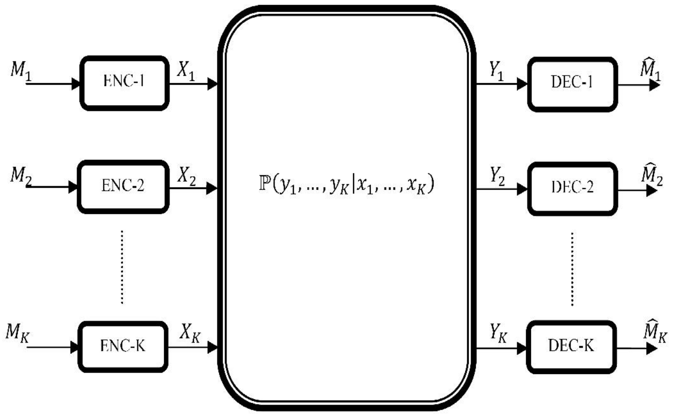

Figure 2. The K-user Classical Interference Channel (CIC).

IV. THE K-USER CIC

Now, let consider the CIC with arbitrary number of users as shown in Fig. 2. By following the same approach as three-user channel, one can indeed derive the strong interference regime given on the top of this page for the K-user CIC.

Note that the regime (31) is described by K inequalities. In fact, for the K-user CIC by following the same lines as the three-user CIC, one can derive $\left((K-1)!\right)^K$ different strong interference regimes. However, among these regimes, $(K-1)!$ ones are more significant (see [3, Sec. V.B] for more discussion) which are derived by exchanging the indices $1,2,3,\dots,K$, with $\vartheta(1),\vartheta(2),\dots,\vartheta(K)$, respectively, in (31) where $\vartheta(.)$ is a *cyclic permutation* of the elements of the set $\{1,2,3,\dots,K\}$. It is worth noting that, up to our knowledge, this is for the first time where a full characterization of the capacity region is derived for a general multi-user CIC in a non-trivial case (for both discrete and Gaussian channels).

Let us consider the Gaussian network which is formulated below:

$$
\begin{cases}
Y_1 = X_1 + a_{12}X_2 + a_{13}X_3 + \dots + a_{1K}X_K + Z_1\\
Y_2 = a_{21}X_1 + X_2 + a_{23}X_3 + \dots + a_{2K}X_K + Z_2\\
\vdots\\
Y_K = a_{K1}X_1 + a_{K2}X_2 + \dots + a_{KK-1}X_{K-1} + X_K + Z_K
\end{cases}
\tag{32}
$$

where $Z_1,\dots,Z_K$ are zero-mean unit-variance Gaussian noises and $\mathbb{E}[X_i^2] \le P_i,\ i = 1,\dots,K$. Using Lemma 2, one can show that if the following conditions are satisfied:

$$
\begin{cases}
\frac{a_{K2}}{a_{12}} = \frac{a_{K3}}{a_{13}} = \dots = \frac{a_{KK-1}}{a_{1K-1}} = \frac{1}{a_{1K}} = \alpha_1\\
\frac{1}{a_{21}} = \frac{a_{13}}{a_{23}} = \frac{a_{14}}{a_{24}} = \dots = \frac{a_{1K}}{a_{2K}} = \alpha_2\\
\frac{a_{21}}{a_{31}} = \frac{1}{a_{32}} = \frac{a_{24}}{a_{34}} = \dots = \frac{a_{2K}}{a_{3K}} = \alpha_3\\
\vdots\\
\vdots\\
\frac{a_{K-1,1}}{a_{K1}} = \frac{a_{K-1,2}}{a_{K2}} = \frac{a_{K-1,3}}{a_{K3}} = \dots = \frac{1}{a_{KK-1}} = \alpha_K
\end{cases}
, \quad |\alpha_i| \le 1
\tag{33}
$$

then, the strong interference regime (31) holds. According to the conditions (33), the parameters $a_{1K}, a_{21}, a_{32}, \dots, a_{KK-1}$, which no pair of them lies in either a same row or a same column of the gain matrix, are given by arbitrary real numbers greater than one and the other parameters are given in terms of these parameters by specific relations determined in (33). It is clear that $(K-1)!$ other strong interference regimes are derived by exchanging the indices $1,2,3,\dots,K$, with $\vartheta(1),\vartheta(2),\dots,\vartheta(K)$, respectively, in (33) where $\vartheta(.)$ is a cyclic permutation of the elements of the set $\{1,2,3,\dots,K\}$.

We finally remark that in [11] a certain condition is identified for the *symmetric Gaussian CIC* under which the interference can be perfectly canceled by all the receivers without incurring any rate penalty. That result is derived using the lattice coding where it is shown that decoding the total interference at each receiver (not the interfering messages) is optimal. Unlike to ours, the result of [11] holds only for Gaussian channels. Moreover, it is established for a very specific case, i.e., the symmetric channel where all the channel parameters and also all the powers of the transmitters are equal to each other.

CONCLUSION

This paper presents a solution for one of the open problems in network information theory that is the generalization of the strong interference regime to the K-user interference channel.

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