# Lattice codes for many-to-one cognitive interference networks

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***Abstract*—In this work we consider the cognitive many-to-one interference network. We first extend existing coding schemes from the two-user case to this network scenario. Then we present a novel coding scheme using compute-and-forward and show it can enlarge the achievable rate region considerably for a wide range of parameters. Numerical evaluations are given to compare the performance of different schemes. Specializing the results to symmetric settings, for a range of parameters, our achievable rate region is shown to be within a constant gap from capacity, regardless of the number of cognitive users.**

## I. Introduction

Recently, with growing requests on high data rate and increasing numbers of intelligent communication devices, the concept of *cognitive radio* has been intensively studied to boost spectral efficiency. As one of its information-theoretic abstractions, a model of the cognitive radio channel of two users was proposed and analyzed in [1]. In this model, the cognitive user is assumed to know the message of the primary user non-causally before transmissions take place. When the channel model is Gaussian, the capacity region of this network is known for most of the parameter region, see [2].

In this work we extend this cognitive radio channel model to include many cognitive users. We consider a simple many-to-one interference scenario illustrated in Figure 1. In this network, interference management becomes an important issue. We extend existing coding schemes to this network and show that they have significant shortcomings in a wide range of interference parameters. To address this, we develop a novel coding strategy based on compute-and-forward [3] thus dealing with interference in a beneficial fashion.

Our scheme is based on rate-splitting and compute-and-forward. Instead of decoding its message directly, the primary decoder first recovers several linear combinations of messages and then extracts its intended message. In order to do so, appropriate nested lattice codes should be carefully constructed for every user according to inherent requirements of the scheme and the choice of linear combinations should also be addressed. We show our scheme outperforms conventional coding schemes for a wide range of parameters. The advantages are most notable in the case of strong interference from the cognitive users to the primary receiver. Finally, for a certain range of parameters, we can also show that the novel coding strategy is near-optimal (in a constant-gap sense) regardless of the number of cognitive users.

We also note that the standard, non-cognitive many-to-one interference channel was studied in [4], where a similar idea of aligning interference was used, with a different technique. Specializing our results to this non-cognitive case, our analysis provides refined results for a range of interference parameters.

We use the notation $[a:b]$ to denote a set of increasing integers $\{a, a+1, \ldots, b\}$, $\log$ to denote $\log_2$ and $\log^+(x)$, $[x]^+$ to denote the function $\max\{\log(x), 0\}, \max\{x, 0\}$, respectively. We also use $\bar{x}$ for $1-x$ to lighten the notation at some places.

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## II. System Model

We consider a multi-user network consisting of one transmitter-receiver pair called primary user and $K$ other pairs ($K \geq 1$) called cognitive users, as in Figure 1. We denote user 0 as the primary user and user $k \in [1:K]$ as the cognitive users. Each user has some message $W_k$ from the set $\mathcal{W}_k$ to send to its corresponding receiver. We assume that all the cognitive users also have access to the primary user's message $W_0$ hence can help the communication of the primary user.

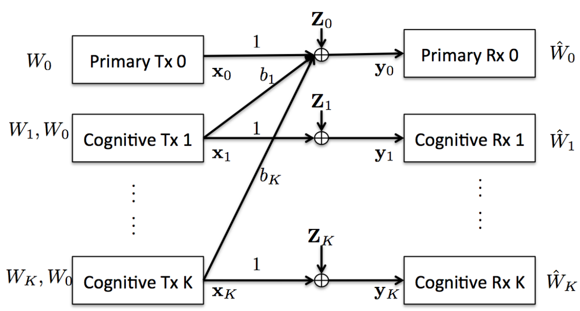


Fig. 1. A many-to-one cognitive network.

The real-valued channel is modeled as

$$\mathbf{y}_0 = \mathbf{x}_0 + \sum_{k=1}^{K} b_k \mathbf{x}_k + \mathbf{z}_0,$$
$$\mathbf{y}_k = \mathbf{x}_k + \mathbf{z}_k, \quad k \in [1:K],$$

where $\mathbf{x}_k$, $\mathbf{y}_k$ denotes the channel input and output of user $k$, respectively. The noise $\mathbf{z}_k$ is assumed to be Gaussian with zero mean and unit variance. Let $b_k \in \mathbb{R}$ denote the channel gain from cognitive transmitter $k$ to the primary receiver. The motivation for considering the many-to-one network because as it is already shown in the two-user case ($K = 1$), the interference from primary to cognitive user is easier to combat because of the cognitive user's knowledge of $W_0$ and techniques such as dirty paper coding can be used to remove the interference from the primary transmitter. This simplified model captures the main characteristics of the problem while keeping the results relatively clean.

Each transmitter has an encoder $\mathcal{E}_k : \mathcal{W}_k \to \mathbb{R}^n$ which maps the message to its channel input as $\mathbf{x}_0 = \mathcal{E}_k(W_0)$ and $\mathbf{x}_k = \mathcal{E}_k(W_k, W_0)$ for $k \in [1 : K]$. The channel input is subject to the power constraint $||\mathbf{x}_k|| \le \sqrt{nP}$ for all $k$. Each receiver has a decoder $\mathcal{D}_k : \mathbb{R}^n \to \mathcal{W}_k$ which estimates message $\hat{W}_k$ from $\mathbf{y}_k$ as $\hat{W}_k = \mathcal{D}_k(\mathbf{y}_k)$ for all $k$. The rate of each user is $R_k = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{W}_k|$ under the average error probability requirement $\Pr\left(\bigcup_{k=0}^{K} \{\hat{W}_k \neq W_k\}\right) \to \epsilon$ for any $\epsilon > 0$.

## III. Conventional coding schemes

In this section we revisit some existing coding schemes for the two-user cognitive interference channel and extend them to our cognitive network. The extensions are straightforward from the two-user case whose details can be found, for example, in [2], and we only state the results here. Throughout the paper the notation $\lambda_0^K$ defined below is used to denote the set of power-splitting parameters. Without loss of generality we can order them as

$$\begin{aligned}\lambda_0^K := \{\lambda_0 = 0, \quad & \lambda_0 \le \lambda_1 \le \ldots \le \lambda_K, \\ & \lambda_k \in [0, 1] \text{ for } k \in [1 : K]\}. \qquad (1)\end{aligned}$$

The first coding scheme applies rate-splitting and dirty-paper coding. The cognitive users split the power and use part of it to transmit the message of the primary user. Luckily this part of the signal will not cause interference to the cognitive receiver since it can be completely canceled out using dirty-paper coding. We briefly describe this coding scheme:

- **Primary encoder.** User 0 generates the codewords $\mathbf{x}_0$ i.i.d. according to distribution $\mathcal{N}(0, P)$ for message $W_0$.
- **Cognitive encoders.** User $k$ generates $\hat{\mathbf{x}}_k$ i.i.d. according to distribution $\mathcal{N}(0, \bar{\lambda}_k P)$ for any given $\lambda_0^K$ and let $\mathbf{u}_k = \hat{\mathbf{x}}_k + \beta\sqrt{\lambda_k}\mathbf{x}_1$ with $\beta = \bar{\lambda}_k P/(1 + \bar{\lambda}_k P)$, $k \ge 1$. The channel input is given by

$$\mathbf{x}_k = \sqrt{\lambda_k}\mathbf{x}_0 + \hat{\mathbf{x}}_k, \quad k \in [1 : K].$$

- **Primary decoder.** Decoder 0 decodes $\mathbf{x}_0$ by treating other signal as noise from $\mathbf{y}_0$.
- **Cognitive decoders.** Decoder $k$ decodes $\hat{\mathbf{u}}_k$ from $\mathbf{y}_k$ for $k \ge 1$.

This coding scheme gives the following achievable rate region:

*Proposition 1 (DPC):* The above dirty paper coding scheme achieves the rate region:

$$\begin{aligned} R_0 &\le \frac{1}{2}\log\left(1 + \frac{(\sqrt{P} + \sum_{k\ge1} b_k\sqrt{\lambda_k P})^2}{\sum_{k\ge1} b_k^2 \bar{\lambda}_k P + 1}\right), \\ R_k &\le \frac{1}{2}\log\left(1 + \bar{\lambda}_k P\right), \quad k \in [1 : K], \end{aligned}$$

for any power-splitting parameter $\lambda_0^K$ defined in (1).

It is worth noting that this scheme achieves the capacity in the two-user case when $|b_1| \le 1$, $K = 1$, see [2] for example.

Another coding scheme which performs well in the two-user case when $|b_1| > 1$, is to let the primary user decode the cognitive message as well [2]. We extend this by performing simultaneous nonunique decoding (SND) [5, Ch. 6] at the primary decoder. SND improves the cognitive rates over uniquely decoding $\hat{\mathbf{x}}_k, k \ge 1$ at primary decoder. The coding scheme is:

- **Primary encoder.** User 0 generates the codewords $\mathbf{x}_0$ i.i.d. according to distribution $\mathcal{N}(0, P)$ for message $W_0$.
- **Cognitive encoders.** User $k$ generates $\hat{\mathbf{x}}_k$ i.i.d. according to distribution $\mathcal{N}(0, \bar{\lambda}_k P)$ for message $W_k, k \ge 1$. The channel input is given by

$$\mathbf{x}_k = \sqrt{\lambda_k}\mathbf{x}_0 + \hat{\mathbf{x}}_k.$$

- **Primary decoder.** Decoder 0 simultaneously decodes $\mathbf{x}_0, \hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_K$ from $\mathbf{y}_1$ for some $\hat{\mathbf{x}}_k, k \ge 1$ .
- **Cognitive decoders.** Decoder $k$ decodes $\hat{\mathbf{x}}_k$ from $\mathbf{y}_k$ for $k \ge 1$.

*Proposition 2 (SND at Rx 0):* The above coding scheme achieves the rate region

$$\begin{aligned} R_0 &\le \frac{1}{2}\log(1 + (\sqrt{P} + \sum_{k\ge1} b_k\sqrt{\lambda_k P})^2), \\ R_0 + \sum_{k\in\mathcal{J}} R_k &\le \frac{1}{2}\log(1 + \sum_{k\in\mathcal{J}} b_k^2\bar{\lambda}_k P \\ &\qquad + (\sqrt{P} + \sum_{k\ge1} b_k\sqrt{\lambda_k P})^2), \\ R_k &\le \frac{1}{2}\log\left(1 + \frac{\bar{\lambda}_k P_k}{1 + \lambda_k P_k}\right), \end{aligned}$$

for any power-splitting parameter $\lambda_0^K$ defined in (1) and every $\mathcal{J} \subseteq [1 : K]$.

The first two inequalities give the achievable rate region at Rx 0 and the last inequality is due to the rate constraint of cognitive decoder. This scheme achieves capacity for certain high interference regime for $K = 1$, see [2] for example.

We can further extend the above coding schemes by combining both dirty paper coding and SND. This will result in a much more complicated rate expression but only negligible improvements in our scenario. To the best knowledge of the authors, the above two schemes are the best known strategies for the two-user ($K = 1$) cognitive Z-interference channel.

## IV. THE COMPUTE-AND-FORWARD SCHEME

In this section we apply the compute-and-forward scheme [3] to cognitive networks. The key idea of this approach is that instead of decoding the desired codeword directly at the primary receiver, it is sometimes more beneficial to first recover several linear combinations involving this codeword and solve for the codeword. We first briefly introduce the nested lattice codes used for this coding scheme and then describe how to adapt compute-and-forward to our network.

### A. Nested lattice codes

A lattice $\Lambda$ is a discrete subgroup of $\mathbb{R}^n$ with the property that if $\mathbf{t}_1, \mathbf{t}_2 \in \Lambda$, then $\mathbf{t}_1 + \mathbf{t}_2 \in \Lambda$. The details about lattice and lattice codes can be found, for example, in [6] and we only cite the basic concepts here. Define the lattice quantizer $Q_\Lambda : \mathbb{R}^n \to \Lambda$ as:

$$Q_\Lambda(\mathbf{x}) = \text{argmin}_{\mathbf{t}\in\Lambda} ||\mathbf{t} - \mathbf{x}|| ,$$

and define the fundamental Voronoi region of the lattice to be $\mathcal{V} := \{\mathbf{x} \in \mathbb{R}^n : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$. The modulo operation gives the quantization error with respect to the lattice:

$$[\mathbf{x}]\text{mod } \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}).$$

Two lattices $\Lambda$ and $\Lambda'$ are said to be nested if $\Lambda' \subseteq \Lambda$. A nested lattice code $\mathcal{C}$ can be constructed using the coarse $\Lambda'$ for *shaping* and fine lattice $\Lambda$ for codewords:

$$\mathcal{C} := \{\mathbf{t} \in \mathbb{R}^n : \mathbf{t} \in \Lambda \cap \mathcal{V}'\}, \tag{2}$$

where $\mathcal{V}'$ is the Voronoi region of $\Lambda'$. The second moment of the lattice $\Lambda'$ per dimension is defined to be

$$\sigma^2(\Lambda') = \frac{1}{n\text{Vol }(\mathcal{V}')} \int_{\mathcal{V}'} ||\mathbf{x}||^2 \, d\mathbf{x},$$

which is also the average power of code $\mathcal{C}$ defined in (2) if the codewords $\mathbf{t}$ are uniformly distributed in $\mathcal{V}'$. Erez and Zamir [6] have shown that there exist nested lattice codes which are *simultaneously good* for shaping and channel coding.

Now we construct the nested lattice codes for our problem. As shown in [7], we can find $K+1$ simultaneously good nested lattices such that $\Lambda_0 \subseteq \ldots \subseteq \Lambda_K \subseteq \Lambda$ with second moments $\sigma_k^2 = (1-\lambda_k)P$ for all $k \geq 0$ with given $\lambda_0^K$ defined in (1). The codebooks are given by $\mathcal{C}_k := \{\mathbf{t}_k \in \mathbb{R}^n : \mathbf{t}_k \in \Lambda \cap \mathcal{V}_k\}$ for $k \geq 0$. Notice that the nesting property of lattices implies $\mathcal{C}_k \subseteq \mathcal{C}_0$ for $k \geq 1$, which is an indispensable property for the decoding process at the primary receiver. It is also shown in [7] that such nested lattice codes have the following relationship on the *coding rate*

$$\begin{aligned} R_0 &:= \frac{1}{n} \log |\mathcal{C}_0| \\ &= R_k + \frac{1}{2} \log \frac{1}{1-\lambda_k} + o_n(1), \quad k \in [1:K], \end{aligned} \tag{3}$$

which means once $R_0$ is determined, so are all other coding rates $R_k$. It is not always desirable to fix the relationship of all rates. Hence whenever we need a rate $R_k$ less than $R_0 - \frac{1}{2}\log\frac{1}{1-\lambda_k}$, we will replace $\Lambda$ with some coarser $\Lambda' \subseteq \Lambda$ ($\Lambda'$ could be different for different $k$) in the construction of $\mathcal{C}_k$, to get a lower rate $R_k$ while always ensuring $\mathcal{C}_k \subseteq \mathcal{C}_0$ for $k \geq 1$.

### B. Compute-and-forward for cognitive network

Equipped with the nested lattice codes constructed above, we are ready to specify the coding scheme. Each cognitive user splits its power and uses one part to help the primary receiver. As mentioned earlier, the primary decoder will decode appropriately chosen linear combinations of the desired codeword and interfering codewords hence effectively reduce the amount of interference it experiences.

Messages $W_k \in \mathcal{W}_k$ are represented as elements in the finite field $\mathbb{Z}/p\mathbb{Z}$ of dimension $m_k$ denoted by $\mathbf{w}_k \in \mathbb{F}_p^{m_k}$ with $p$ prime. The vectors $\mathbf{w}_k$ are mapped surjectively to lattice points $\mathbf{t}_k \in \mathcal{C}_k$ for all $k$. Given the cognitive messages $W_k$ we form

$$\hat{\mathbf{x}}_k := [\mathbf{t}_k + \mathbf{d}_k]\text{mod } \Lambda_k, \quad k \in [1:K],$$

where $\mathbf{d}_k \in \mathcal{V}_k$ (called *dither*) is a random vector independent of $\mathbf{t}_k$ and uniformly distributed in $\mathcal{V}_k$. It follows that $\hat{\mathbf{x}}_k$ is also uniformly distributed in $\mathcal{V}_k$ [6]. The channel input for each transmitter is given by

$$\begin{aligned} \mathbf{x}_0 &= [\mathbf{t}_0 + \mathbf{d}_0]\text{mod } \Lambda_0, \\ \mathbf{x}_k &= \sqrt{\lambda_k}\mathbf{x}_0 + \hat{\mathbf{x}}_k, \quad k \in [1:K]. \end{aligned}$$

One can easily check that all power constraints are satisfied.

The received signal $\mathbf{y}_0$ at the primary decoder is

$$\mathbf{y}_0 = h_0\mathbf{x}_0 + \sum_{k\geq 1} b_k\hat{\mathbf{x}}_k + \mathbf{z}_0,$$

where we define

$$h_0 := 1 + \sum_{k\geq 1} b_k\sqrt{\lambda_k}. \tag{4}$$

Given a set of integers $a_0^K := \{a_k \in \mathbb{Z}, k \in [0:K]\}$ and some scalar $\alpha \in \mathbb{R}$ we can form the following:

$$\begin{aligned} \tilde{\mathbf{y}}_0 &:= [\alpha\mathbf{y}_0 - \sum_{k\geq 0} a_k\mathbf{d}_k]\text{mod } \Lambda_0 \\ &= [\tilde{\mathbf{z}}_0(\alpha) + a_0\mathbf{t}_0 + \sum_{k\geq 1} a_k\tilde{\mathbf{t}}_k]\text{mod } \Lambda_0, \end{aligned}$$

where $\tilde{\mathbf{z}}_0(\alpha)$ is the equivalent noise at the receiver given by

$$\tilde{\mathbf{z}}(\alpha)_0 := \alpha\mathbf{z}_0 + (\alpha h_0 - a_0)\mathbf{x}_0 + \sum_{k\geq 1}(\alpha b_k - a_k)\hat{\mathbf{x}}_k,$$

and $\tilde{\mathbf{t}}_k$ is the equivalent interference from user $k$

$$\tilde{\mathbf{t}}_k := \mathbf{t}_k - Q_{\Lambda_k}(\mathbf{t}_k + \mathbf{d}_k).$$

Following an argument similar to [3, Section V], from $\tilde{\mathbf{y}}_0$ we can decode the modulo linear combination:

$$\mathbf{v}(a_0^K) := [a_0\mathbf{t}_0 + \sum_{k\geq 1} a_k\tilde{\mathbf{t}}_k]\text{mod } \Lambda_0, \tag{5}$$

at any rate less or equal to the *computation rate*

$$\begin{aligned} R_{comp}(\mathbf{h}(\lambda_0^K), \mathbf{a}(\lambda_0^K, a_0^K)) &:= \max_{\alpha\in\mathbb{R}} \frac{1}{2}\log^+\left(\frac{P}{\mathbb{E}\,||\tilde{\mathbf{z}}_0(\alpha)||^2}\right) \\ &= \frac{1}{2}\log^+\left(||\mathbf{a}||^2 - \frac{P|\mathbf{h}^T\mathbf{a}|^2}{1+P\,||\mathbf{h}||^2}\right)^{-1}, \end{aligned} \tag{6}$$

with

$$\mathbf{h}(\lambda_0^K) := [h_0, b_1\sqrt{\bar{\lambda}_1}, \ldots, b_K\sqrt{\bar{\lambda}_K}], \tag{7a}$$

$$\mathbf{a}(\lambda_0^K, a_0^K) := [a_0, a_1\sqrt{\bar{\lambda}_1}, \ldots, a_K\sqrt{\bar{\lambda}_K}], \tag{7b}$$

and $h_0$ defined in (4). From the modulo linear combination (5) we are able to recover the corresponding modulo sum of messages in the finite field [3, Lemma 6]:

$$\mathbf{u}(q_0^K) = \bigoplus_{k=0}^{K} q_k \mathbf{w}_k,$$

at the same rate

$$R_{comp}(\lambda_0^K, q_0^K) := R_{comp}(\mathbf{h}(\lambda_0^K), \mathbf{a}(\lambda_0^K, a_0^K)), \tag{8}$$

with $q_k := [a_k] \bmod p$ and the RHS is defined in (6).

We still need to select the coefficients $q_0^K$ in the modulo sum $\mathbf{u}(q_0^K)$ to find $\mathbf{w}_0$. If we set $q_0 \neq 0$ and $q_k = 0$ for all $k \geq 1$, we recover $q_0\mathbf{w}_0$ hence the desired codeword $\mathbf{w}_0$. However, we can do better by decoding multiple modulo sums of $\mathbf{w}_0, \ldots, \mathbf{w}_K$ with judiciously chosen coefficients $q_0^K$ and recover $\mathbf{w}_0$ from these modulo sums. Yet another observation (also made in [8]) is that, once we have decoded linear combinations of the form (5), it is possible to recover the real sum $a_0\mathbf{x}_0 + \sum_{k\geq 1} a_l\mathbf{x}_k$ and use it together with the output $\mathbf{y}_0$ to decode a new modulo sum. Specifically, assume we have decoded $l-1$ modulo linear combinations with coefficients $a_0^K(j), j = 1, \ldots, l-1$, and now want to decode another modulo linear combination with coefficients $a_0^K(l)$, then in the same way as (8), we can show the achievable rate using *successive compute-and-forward* is

$$R_{comp}(\lambda_0^K, q_0^K(l)|q_0^K(1), \ldots, q_0^K(l-1)) := \max_{\alpha_1^l \in \mathbb{R}^l} \frac{1}{2}\log^+\left(\frac{P}{\mathbb{E}\left|\left|\tilde{\mathbf{z}}_0(\alpha_1^l)\right|\right|^2}\right) \tag{9}$$

now with the modified equivalent noise

$$\tilde{\mathbf{z}}_0(\alpha_1^l) = \alpha_l\mathbf{z}_0 + (\alpha_l h_0 - a_0(l) - \sum_{j=1}^{l-1}\alpha_j a_0(j))\mathbf{x}_0 + \sum_{k\geq 1}(\alpha_l b_k - a_k(l) - \sum_{j=1}^{l-1}\alpha_j a_k(j))\hat{\mathbf{x}}_k$$

where $q_k(j) := [a_k(j)] \bmod p$ for $j = 1, \ldots, l$.

In general, let $L$ be the number of modulo sums we want to decode from $\hat{\mathbf{y}}_0$ and write the $L$ sets of coefficients in matrix form where the $l$-th row $\mathbf{Q}_l$ represents the coefficients for the $l$-th modulo sum:

$$\mathbf{Q} = \begin{pmatrix} q_0(1) & q_1(1) & q_2(1) & \ldots & q_K(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_0(L) & q_1(L) & q_2(L) & \ldots & q_K(L) \end{pmatrix}, \tag{10}$$

with $1 \leq L \leq K+1$. Let $\mathbf{Q}' \in \mathbb{F}_p^{L\times K}$ denote the same matrix $\mathbf{Q}$ without the first column. We define a set of matrices as

$$\mathcal{Q}(L) := \{\mathbf{Q} \in \mathbb{F}_p^{L\times(K+1)} : \text{rank}(\mathbf{Q}) = m+1, \text{rank}(\mathbf{Q}') = m, \text{ for some integer } m, 0 \leq m \leq \min\{L, K\}\}. \tag{11}$$

*Theorem 1:* For any set of power-splitting parameter $\lambda_0^K$ defined in (1), the following rate is achievable for the many-to-one cognitive interference networks

$$R_0 \leq \max_{\substack{L\in[1:K+1]\\ \mathbf{Q}\in\mathcal{Q}(L)}} \min_{l\in[1:L]} R_{comp}(\lambda_0^K, \mathbf{Q}_l|\mathbf{Q}_1, \ldots, \mathbf{Q}_{l-1}) \tag{12a}$$

$$R_k \leq \min\left\{\frac{1}{2}\log\left(1 + \frac{\bar{\lambda}_k P}{1+\lambda_k P}\right), \left[R_0 - \frac{1}{2}\log\frac{1}{1-\lambda_k}\right]^+\right\} \quad \text{for } k \in [1:K], \tag{12b}$$

where $R_{comp}(\lambda_0^K, \mathbf{Q}_l|\mathbf{Q}_1, \ldots, \mathbf{Q}_{l-1})$ and $\mathcal{Q}(L)$ are defined in (9), (11) respectively.

We provide a proof sketch here. First it can be checked easily that one can solve for $\mathbf{w}_0$ using $L$ modulo sums with coefficients $(q_0^K(1), \ldots, q_0^K(L))$ if its matrix representation $\mathbf{Q}$ as defined in (10) is in the set $\mathcal{Q}(L)$ in (11). Furthermore when we decode the $l$-th modulo sum, we will utilize the previously decoded $l-1$ sums. The actual transmission rate of $\mathbf{w}_0$ equals the computation rate under which *all* $L$ modulo sums can be decoded reliably. The search for the best $L$ modulo sums is encoded in the maximization operation in (12a) and the minimization operation there reflects the fact the transmission rate is determined by the lowest computation rate of the $L$ modulo sums.

Now consider the decoding procedure of the cognitive users. After receiving $\mathbf{y}_k$, the cognitive user $k$ processes it to get

$$\tilde{\mathbf{y}}_\mathbf{k} := [\alpha(\mathbf{z}_k + \sqrt{\lambda_k}\mathbf{x}_1) + (\alpha-1)\hat{\mathbf{x}}_k + \mathbf{t}_k] \bmod \Lambda_k.$$

The same as in [6] we can show that the codeword $\mathbf{t}_k$ can be decoded reliably with an appropriate $\mathcal{C}_k$ for any rate less than the first term in (12b). But recall the relationship of nested lattice codes in (3), if we fix $R_0$ to be the achievable rate of $\mathbf{w}_0$ in (12a), the coding rate of $\mathcal{C}_k$ is also fixed to be the second term in (12b). If it exceeds this value, we need to lower the coding rate of $\mathcal{C}_k$ as in section IV-A such that this rate can be supported by the direct channel of this cognitive user.

## V. Symmetric Cognitive interference channel

In this section we consider a symmetric system with $b_k = b$ for all $k$ and consider the case when all cognitive users have the same rate. By symmetry all the power-splitting parameters $\lambda_k$ should be the same. We state the following result on the optimal coefficients $(q_0^K(1), \ldots, q_0^K(L))$ without the proof:

*Lemma 1:* For the symmetric many-to-one cognitive interference channel, we need to decode at most two modulo sums: $L \leq 2$. For the case $L = 2$, we can set $q_1(l) = \ldots = q_K(l)$ for $l = 1, 2$ without loss of optimality.

Figure 2 shows the achievable rate region for a symmetric network. The dot-dash and dashed lines are achievable with DPC in Proposition 1 and SND at Rx 0 in Proposition 2, respectively. Achievable points for the compute-and-forward based scheme are marked with dots for different power splitting parameter $\lambda_k \in [0, 1]$. Finally the solid line depicts the rate region using time sharing between compute-and-forward scheme and two other points where the primary and cognitive rate are maximized, respectively. Notice the conventional schemes are not much better than the trivial time sharing scheme in the multi-user scenario, especially for strong interference, due to their inherent inefficiencies on interference suppression.

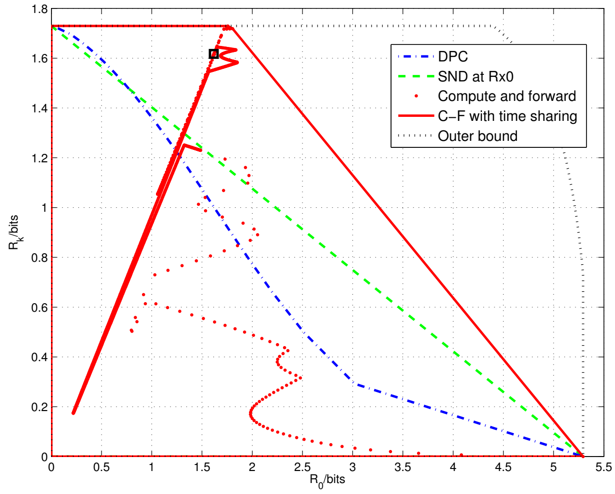


Fig. 2. Achievable rate region for a many-to-one symmetric cognitive interference network with power $P = 10$, channel gain $b = 3.8$ and $K = 3$ cognitive users. The $X$ and $Y$ axis represents the primary rate $R_0$ and cognitive rate $R_k, k \geq 1$, respectively. In this case we only need two points to obtain the whole convex region of the compute-and-forward scheme. The points correspond to $\lambda = 10^{-4}$ with two best linear combinations of coefficients $(q_0, q_k) = (1, 3), (2, 7)$ giving $(R_0 = 1.757, R_k = 1.729)$ and $\lambda = 16 \cdot 10^{-4}$ with two best linear combinations of coefficients $(q_0, q_k) = (1, 2), (8, 15)$ giving $(R_0 = 1.806, R_k = 1.717)$. We note the important role of cognition here that this small $\lambda$ will make a difference compared to by setting $\lambda = 0$ (marked by the black square). Indeed, with (4), we see even a very small $\lambda$ can change the effective channel $h_0$ of the primary user considerably with not too small $K$ and $b$. The outer bound uses a sum rate bound by simply considering the whole network as a large MIMO system.

We see the compute-and-forward scheme is particularly good in the regime where we want the cognitive users to maintain a relatively high rate while still to help the primary user as much as they can. Figure 3 makes this point clearer. Now we require that the cognitive rate $R_k$ is no less than 90% of its maximum rate, in this case $R_{k,l} := 0.9 \times 1/2 \log(1 + P) = 1.56$ bits. We depict the maximum primary rate $R_0$ for different coding schemes versus the channel gain $b$ ranging in $[0, 15]$. When $b$ is small the compute-and-forward scheme performs worse than the other schemes. The reason is that the primary rate $R_0$ is low for small $b$ and $R_k$ will be determined by the second term in (12b) due to the nested construction.

We can specialize our scheme to the non-cognitive many-to-one interference channel by letting $\lambda_k = 0$ for all $k$:

*Proposition 3:* Consider a symmetric (non-cognitive) many-to-one interference channel with integer cross gain . We have the following lower bound on the achievable rate:

$$R_0, R_k \geq \begin{cases} \frac{1}{2}\log\left(\frac{KPb^2+P+1}{1+Kb^2+K(1+P-2b)}\right) & \text{if } 2b < 1+P, \\ \frac{1}{2}\log\left(\frac{1}{1+Kb^2}+P\right) & \text{if } 2b \geq 1+P, \end{cases}$$

We omit the formal proof here. With $\lambda_k = 0$, the above lower bound is obtained by choosing $q_0 = 1$ and optimizing over

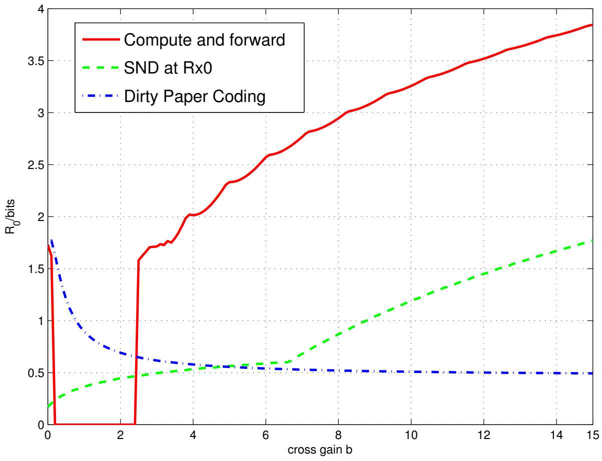


Fig. 3. A many-to-one cognitive interference network with $P = 10$ and $K = 3$ cognitive users. The maximum achievable rate of $R_0$ versus cross channel gain $b$ when cognitive rate $R_k$ is required to be no less than 90% of its maximum rate. In the range $b \in [0.2, 2.6]$, the compute-and-forward scheme cannot provide a cognitive rate $R_k$ above 90% of its maximum.

the coefficients $q_k, k \geq 1$. Notice this choice of coefficients is in general sub-optimal.

*Corollary 1:* Consider a symmetric (non-cognitive) many-to-one interference channel with integer cross gain $b$. If $2b \geq 1 + P$, all users are 0.5 bit from the capacity for any $K \geq 1$.

To see this, notice the rate of each user is upper bounded by $\frac{1}{2}\log(1+P)$ in our setting. When $2b \geq 1+P$, by Proposition 3, the difference between achievable rate and upper bound is further upper bounded by $\frac{1}{2}\log(1 + P) - \frac{1}{2}\log(P)$ which is less than 0.5 bit for $P \geq 1$. If $P < 1$, the capacity of each user is less than 0.5 bit. Hence the claim follows.

*Remark 1:* For this channel, [4] established an approximate capacity result within a gap of $(3K+3)(1+\log(K+1))$ bits per user for *any* channel gain. Our scheme gives a constant gap result independent of $K$ for a certain range of parameters. We expect that the constant gap result would hold for a larger range of parameters (also for non-integer $b$) if we optimally choose coefficients $q_0^K$.

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