A Lattice Singleton Bound

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***Abstract*— The binary coding theory and subspace codes for random network coding exhibit similar structures. The method used to obtain a Singleton bound for subspace codes mimic the technique used in obtaining the Singleton bound for binary codes. This motivates the question of whether there is an abstract framework that captures these similarities. As a first step towards answering this question, we use the lattice framework proposed in [1]. A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. A 'lattice scheme' is defined as a subset of a lattice. In this paper, we derive a Singleton bound for lattice schemes and obtain Singleton bounds known for binary codes and subspace codes as special cases. The lattice framework gives additional insights into the behaviour of Singleton bound for subspace codes. We also obtain a *new* upper bound on the code size for non-constant dimension codes. The plots of this bound along with plots of the code sizes of known non-constant dimension codes in the literature reveal that our bound is tight for certain parameters of the code.**

I. Introduction and Background

In the field of coding theory, calculations of good upper and lower bounds on the size of a code serves as a benchmark for code design. Techniques of constructing subspace codes for error correction in random network coding (also called 'projective space codes') are similar to the techniques used to construct binary codes. A projective space code is defined as a subset of subspaces of a vector space. The viewpoint of lattice theory for projective space codes has already been suggested [2], [3], [4]. A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. The set of all subspaces of a vector space is called a projective space and the set of all subsets of a set is called a power set. Both the power set and the projective space form lattices under inclusion order. A 'lattice scheme' is defined as a subset of a lattice. It has already been observed in [1] that binary codes and projective space codes are lattice schemes. Binary codes are power set schemes and projective space codes are projective space schemes. The lattice framework common to projective space codes and binary codes have also been observed in [3] and [4]. The authors of [4] establish that the projective space codes are analogs of classical binary codes. They also investigate the notions of linearity and complements of a code. In [1], the relationship between binary codes and projective space codes is investigated thoroughly using the framework of lattices and a generalized notion of orthogonal complement is introduced for lattices.

One of the ways to generate a bound on the code size is to capture a bare minimum structure for the bound through an appropriate mathematical abstraction and generalize the approach. For example, the introduction of 'Association Schemes' captured the notion of sphere packing and sphere covering arguments in traditional coding theory [5] and strengthened the bounds by a linear programming approach. In our paper, the notion of Singleton bound is generalized using the framework of lattices. Since error correcting codes for random network coding was proposed in [2], there has been a lot of activity in the theory of projective space codes. The authors of [2] consider projective space codes in which all the subspaces are of the same dimension. Such codes are called constant dimension codes. The Koetter Kschischang Singleton bound (KKS bound) for constant dimension codes was derived and achieved asymptotically in [2].

In the case of binary codes, Hamming distance between two binary vectors is a metric on the binary coding space. Similarly the subspace distance, introduced in [2], between two subspaces of a vector space is a metric on the projective space. In a projective space, the number of elements in a sphere of a particular radius depends on the center of that sphere [2]. However, in the case of binary codes, the number of elements in a sphere depends only on the radius of the sphere and not on the center of the sphere. So the subspace distance on the projective space behaves differently from the Hamming distance on the usual binary vectors. As noted in [6], the anticode bound introduced in [5] gives a setting in which one can get tighter sphere packing bounds for constant dimension codes. A close look at the bounds for subspace distance and non-existence of nontrivial perfect codes has been given in [6]. An appropriate generalization for Singleton bounds (analogous to association scheme for sphere packing/covering bounds) is not known yet. In this paper, we propose that the lattice framework is appropriate for generalizing Singleton bounds. We derive an upper bound on the size of a lattice scheme, which will be called Lattice Singleton Bound (LSB) for the remainder of the paper. We get the KKS bound and the classical Singleton bound as special cases of the LSB.

While most results in projective space codes assume that the dimensions of the subspaces in the code are constant, a few papers in literature have considered codes with non-constant dimension. In particular, the authors of [6] derive a Gilbert-Varshamov type bound, which we will term as 'Etzion Vardy- Gilbert Varshamov Bound' (EV-GVB), for non-constant dimension codes. We derive a *new* upper bound on the code size for non-constant dimension codes by applying the LSB to the projective lattice. To the best of our knowledge, our Singleton bound is the first upper bound for non-constant dimension codes.

The construction of constant dimension codes have been the main focus in the field of code constructions for projective spaces. The first constant dimension subspace codes introduced in [2] achieve the KKS bound asymptotically and there is no known code constructions that exactly achieve the Singleton bound proposed in [2], [6]. The question of achieving that Singleton bound is an open question. A large class of constant and non-constant dimension codes using Ferrer's diagrams and Lifted Rank Metric codes are constructed in [7], [8], [9]. The authors of [10] report marginal improvements over the size of the code compared to the codes of [7]. A plot of code sizes of the codes reported in [10], [7] compared to a plot of our upper bound shows that both the reported codes have code sizes that are close to optimal. This also establishes that our upper bound is tight for certain coding parameters.

The contributions of this paper are as follows:

1) We derive a Singleton bound in the framework of lattices. We recover the Singleton bound for binary codes and the Singleton bound for constant dimension subspace codes (KKS bound) as special cases of our LSB.
2) We obtain a new Singleton bound for non-constant dimension subspace codes. We compare our bound on the code size with the code sizes of non-constant dimension codes found in the literature. We also compare our upper bound with a Gilbert-Varshamov type lower bound. The plots reveal that our bound is tight for certain parameters of the code.

The paper is organized as follows: Section II introduces the idea of lattice schemes where it is shown that classical codes and subspace codes are both lattice schemes. Section III introduces the Singleton bound in the framework of lattice schemes and contains the main theorem of this paper. Section III also contains the main distinguishing feature of projective space codes which we believe make Singleton type bounds weak in projective spaces. In Section IV, we obtain a new upper bound for non-constant dimension codes and plot our upper bound with EV-GVB along with different codes constructed in the literature. Finally, in Section V, we conclude by summarizing our contributions and discussing the scope for future work.

Notations: A set is denoted by a capital letter and its elements by small letters (For example, $x \in X$). The set $\underline{n}$ is defined as $\underline{n} := \{i \in \mathbb{N} | 1 \leq i \leq n\}$. All the sets considered in this paper will be finite. Given a set X, $|X|$ denotes the number of elements in the set and for a subset A of X, A^c denotes the complement of the set A in X. For two sets A and B, $A \times B$ denotes the Cartesian product of the two sets, i.e. $A \times B := \{(a,b) | a \in A, b \in B\}$. $A \triangle B$ represents the symmetric difference of sets, i.e. $A \triangle B := (A^c \cap B) \cup (A \cap B^c)$. A lattice will be denoted by $(L, \vee, \wedge)$ and sometimes we will drop the join and the meet notation, simply calling it L. $\mathbb{F}_q$ denotes the finite field with q elements where q is a power of a prime number. $\mathbb{F}_q^*$ denotes all the non zero elements of $\mathbb{F}_q$. The symbol V denotes a vector space (generally over $\mathbb{F}_q$). For a subset S of V, $\langle S \rangle$ denotes the linear span of all the elements in S. Given two subspaces A and B, $A + B$ denotes the smallest subspace containing both A and B. Let V be a n dimensional space over $\mathbb{F}_q$. Then the symbol $\mathcal{G}(n,l)$ denotes the Grassmanian, i.e. the set of all l dimensional subspaces of V. The number of elements in $\mathcal{G}(n,l)$ is denoted as $\begin{bmatrix} n \\ l \end{bmatrix}_q$. $\mathbb{F}_q^n$ denotes the n dimensional vector space of n-tuples over $\mathbb{F}_q$. Given a vector $x \in \mathbb{F}_q^n$, x_i denotes the i-th co-ordinate of x. The support of $x \in \mathbb{F}_q^n$ (denoted by support$\{x\}$) is defined as the set of indices where the vector is non-zero. $(\mathbb{F}_q^n, d_H)$ denotes the vector space $\mathbb{F}_q^n$ with the Hamming metric, i.e. $d_H(a,b) = |\text{support}\{a-b\}|$. I and O represent the biggest and the smallest elements respectively, of the given lattice.

The proofs for all the claims, propositions, and theorems in the manuscript are available in [11] along with an introduction to basic notions and notations of lattice theory used in this paper.

II. Lattice Schemes

In order to develop a lattice based framework for Singleton bounds, we need a definition of a code in this framework. In this section, we define 'Lattice Schemes' which will serve as analogues of codes. We will show that Hamming space, rank metric space and the projective spaces are examples of lattice schemes. Although some of these observations have been mentioned in [1], we repeat it for the sake of continuity. Note that all the lattices are assumed to be geometric modular of finite height unless otherwise mentioned.

A lattice scheme, which is analogous to a code, is defined as follows:

Definition 1: Let L be a lattice and d_h be the metric induced by the height function h of the lattice L. A *lattice scheme* C in (L, d_h) is a subset of L and the *minimum distance of* C, denoted by d is defined as

$$d := \min_{a,b \in C, a \neq b} d_h(a,b).$$

The dimension of a lattice scheme is defined as $n := h(I)$.

A coding space (X, d_X) is a metric space where X is a set and d_X is a metric on X. A code C in a coding space (X, d_X) is a subset of X. The connection between lattice schemes and codes is made precise in the following definition.

Definition 2: Let C be a lattice scheme in (L, d_h) and $\tilde{C}$ be a code in a coding space (X, d_X). We say that the *code* $\tilde{C}$ is *equivalent to a lattice scheme* C, if there exists a function $T : X \to L$, that satisfies the following conditions:

1) $T(\tilde{C}) = C$
2) $d_h(T(a), T(b)) = d_X(a,b)$ for all $a, b \in \tilde{C}$

T is called a transform of the code $\tilde{C}$.

Remark 1: A transform T of a code is one-one. To prove this fact, assume $T(a) = T(b)$, then $d_h(T(a), T(b)) = 0$. By 2) of Definition 2, this would imply $d_X(a,b) = 0$. And since d_X is a metric, we have $a = b$. □

When a lattice scheme is equivalent to a code, we also say the code is equivalent to the scheme. A transform of a code preserves the distance between any pair of codewords.

Therefore whenever a lattice scheme is equivalent to a code, the lattice scheme will have the same minimum distance as the code. The following proposition follows from Definition 2,

Proposition 1: Let C be a lattice scheme with minimum distance d that is equivalent to code $\tilde{C}$ with minimum distance $\tilde{d}$, then

1) $d = \tilde{d}$
2) if $\tilde{A} \subseteq \tilde{C}$, then there exists $A \subseteq C$ such that the lattice scheme A is equivalent to the code $\tilde{A}$.

We use this observation to establish that every binary code is equivalent to a scheme in the power set lattice.

Example 1: Let $X = \underline{n}$, $L = (\mathit{Pow}(X), \cup, \cap)$ and $h(A) = |A|$. L is a geometric distributive lattice and $d_h(A, B) = |A \cup B| - |A \cap B| = |A \triangle B|$ as seen in the previous section. Consider codes in the coding space $(\mathbb{F}_2^n, d_H)$ where d_H is the Hamming distance between two vectors.

We claim that the entire coding space $\mathbb{F}_2^n$ is equivalent to the power set lattice L. To see this let,

$$\begin{aligned} \phi : & \mathbb{F}_2^n \rightarrow \mathit{Pow}(X) \\ & x \longmapsto \text{support}(x). \end{aligned}$$

It can be verified that $\phi(x + y) = \phi(x) \triangle \phi(y)$ (where $\triangle$ represents the symmetric difference operator) and that ϕ is onto. Further, $d_h(\phi(a), \phi(b)) = |\phi(a) \triangle \phi(b)| = |\phi(a + b)|$. The number of elements in $\phi(a+b) = \text{support}(a+b)$ will be the Hamming weight of $a+b$. Thus $|\phi(a+b)| = d_H(a+b, 0) = d_H(a, b)$. Therefore, ϕ is the power set lattice transform of the binary code.

Since the map is onto, ϕ is a bijective map. By application of the second part of Proposition 1, we see that every binary code is equivalent to a power set lattice scheme. □

Projective spaces, discussed in the previous section, also provide examples of lattice schemes. Since the projective lattice is precisely the lattice of subspaces and metric induced by the height function is the subspace distance, any subspace code is equivalent to a scheme in the projective lattice.

Example 2: Let V be a vector space over $\mathbb{F}_q, L = (\text{Sub}(V), +, \cap)$ is a projective lattice with height function $h(A) = \dim(A)$. The coding space is $(\text{Sub}(V), d_S)$ where d_S represents the subspace distance defined in [2]. For any two subspaces A and B, $d_S(A, B) := \dim(A+B) - \dim(A \cap B)$. Since $d_h(A, B) = h(A \vee B) - h(A \wedge B) = \dim(A + B) - \dim(A \cap B)$, the metric induced by the height function is the subspace distance. The identity map can be a transform in this case. And thus, subspace schemes are equivalent to subspace codes. □

Lattice schemes in the above lattice correspond to subspace codes in projective spaces [2], [6]. Certain constant dimension subspace code constructions, described in [12], make use of the rank distance Gabidulin codes [13]. The rank metric codes are actually equivalent to Gabidulin codes since they are 'lifted' from Gabidulin codes. We show that rank distance codes are equivalent to certain schemes in the projective lattice. We also show that the 'lifting' function is a transform.

The $m \times n$ matrices of the rank metric coding space can also be viewed as vectors of the space $\mathbb{F}_{q^m}^n$ by viewing the columns of the matrix as an element of the field $\mathbb{F}_{q^m}$. From this viewpoint, one can show that the rank distance between two matrices is always less than the Hamming distance of the matrices when viewed as vectors [13].

III. Our Main Result

All our main results on the bounds and their behavior are derived in this section. We introduce the notion of puncturing a scheme and investigate the effects of puncturing on the minimum distance of a lattice scheme. It will be proved that, after puncturing a scheme, the maximum drop in minimum distance will be two. However, if the lattice is known to be distributive, it is shown that the maximum drop in minimum distance is one. This observation will be applied repeatedly until the minimum distance drops to zero.

We will need the following definition of *Whitney number of the second kind*, from [14], to state the lattice Singleton bound:

Definition 3: The *Whitney numbers* $c_L(n, k)$ of a lattice L in a lattice with height h and $n = h(I)$ is defined as

$$c_L(n, k) = |\{a \in L | h(a) = k\}| .$$

The Whitney numbers of a lattice count the total number of elements in the lattice of a given height.

Definition 4: A scheme C is said to be *punctured to* C' if $C' = \{w \wedge a | a \in C\}$ for some $w \in L$. If w has a height of $h(I) - 1$, the scheme C is said to be *punctured by a dimension*.

Remark 2: For two lattice elements w and w' of equal height, we have $c_{L \wedge w}(h(w), k) = c_{L \wedge w'}(h(w'), k)$. This means that the Whitney numbers of two different punctured lattices remain the same, as long as the elements that puncture the scheme are of the same height.

We need the following lemma (called the 'distance drop lemma') to establish the proof of the main theorem later.

Lemma 1 (Distance drop lemma): Let C be a scheme in (L, d_h) with minimum distance d, and let C be punctured by a dimension to C'

1) L is distributive $\implies d_{\min}(C') \geq d - 1$
2) In general, $d_{\min}(C') \geq d - 2$.

The lemma states that in a non-distributive lattice, the drop in the minimum distance after puncturing a dimension, can be at most two units. So it would be interesting to know if it is possible that the drop of two units is exhibited by some scheme in a non-distributive lattice. The following example constructs such a scheme.

Example 3: Let V be a three dimensional space, over $\mathbb{F}_2$, spanned by $\{e_1, e_2, e_3\}$. Let $A_1 = \langle\{e_1, e_2\}\rangle$, $A_2 = \langle\{e_2, e_3 + e_1\}\rangle$ and $W = \langle\{e_2, e_3\}\rangle$. We have $d_S(A_1, A_2) = 2$ but $d_S(W \cap A_1, W \cap A_2) = 0$. Note that $A_2 \cap (A_1 + W) \neq A_2 \cap A_1 + A_2 \cap W$ (that is, the sublattice generated by A_1, A_2 and W is not distributive) as expected. If our scheme contained A_1, A_2 and W, then puncturing the scheme with W would have left W as it is and punctured only the remaining two subspaces. □

The above example clearly illustrates that the lack of distributivity in a lattice can drop the distance of a punctured scheme by two units. We will now derive a Singleton bound for lattice schemes, that establishes an upper bound on the cardinality of the scheme, for a given minimum distance for geometric modular lattices.

Theorem 1 (Lattice Singleton Bound(LSB)): If (L, d_h) is a geometric modular lattice with height h and $h(I) = n$, d_h is the metric induced by the height function, and C is a scheme of L with minimum distance d, then

$$|C| \leq \sum_{k=0}^{n-\alpha_L} c_L(n - \alpha_L, k)$$

where

1) $\alpha_L = d - 1$, when L is *distributive*
2) $\alpha_L = \lfloor \frac{d-1}{2} \rfloor$, when L is *modular*.

We remark that the bound does not equal 2, for the case $d = n$, which should yield only two elements. The bound is weaker for large minimum distance because we have no information about the range of heights of the elements of the scheme. We can tighten the bound by systematically puncturing and projecting a dimension as done in [2]. Also, as we shall see in Example 4, puncturing a dimension of a scheme does not reduce the heights of the elements of the scheme in a uniform manner. So we need the following definition,

Definition 5: Given two elements c and w in a lattice L, define $\overline{c \wedge w}$ to be equal to an element of height $h(c) - 1$ that is contained in $c \wedge w$. In particular, if w is fixed and $h(w) = h(I) - 1$, the set $\overline{C \wedge w} = \{\overline{c \wedge w} | C \in c\}$ is said to be *punctured and projected by a dimension*.

Thus when $c \leq w$, $\overline{c \wedge w}$ is arbitrarily chosen to be an element immediately below c in the Hasse diagram of the lattice. For all other cases, $\overline{c \wedge w} = c \wedge w$. The following example illustrates that puncturing a dimension reduces the heights of different elements of a scheme in an unequal manner. It also demonstrates that puncturing and projecting by a dimension equally reduces the heights of all elements of a scheme.

Example 4: Let V be a three dimensional space, over $\mathbb{F}_2$, spanned by $\{e_1, e_2, e_3\}$. Let $A_1 = \langle\{e_1, e_3\}\rangle$, $A_2 = \langle\{e_2, e_3\}\rangle$ and $W = \langle\{e_1, e_3\}\rangle$. Thus the dimension of A_1 remains the same after puncturing by W but the dimension of A_2 drops by one unit. However, after puncturing and projecting the lattice by W, both A_1 and A_2 drop by a height of one unit. To be precise, define $E_3 := W \wedge A_2 = \langle\{e_3\}\rangle$ and observe that $\overline{W \wedge A_2} = W \wedge A_2 = E_3$. But $W \wedge A_1 = W$. To define $\overline{W \wedge A_1}$ we need to choose a one dimensional subspace of A_1. For instance, we can choose $\overline{W \wedge A_1} = E_3$. □

We can now establish that puncturing and projecting a dimension of a scheme can drop the minimum distance of the scheme by at most two units. This lemma is useful since it tells us that every time we puncture and project by a dimension, all the elements in the scheme drop once in height.

Lemma 2: Let C be a scheme in a lattice (L, d_h) with minimum distance d, and let $h(I) = n$ and let $w \in L$ such that $h(w) = n - 1$. If the maximum height of C was M and the minimum height of C was m, then the maximum height of $\overline{C \wedge w}$ is $M-1$ and the minimum height is $m-1$. Additionally $d_{\min}\left(\overline{C \wedge w}\right) \geq d - 2$.

We will need Lemma 2 to get the following improvement on the code size,

Theorem 2: Let C be a scheme in a geometric modular lattice L with minimum distance d. Let m denote the smallest height of all the elements in C and M denote the largest height of all the elements in C. Then,

$$|C| \leq \sum_{k=(m-\alpha_L)}^{M-\alpha_L} c_L(n - \alpha_L, k).$$

We will now apply Theorem 1 and Theorem 2 to two important lattice schemes (namely the power set lattice and the subspace lattice) and derive the corresponding LSB. The LSB that we obtain coincides with the Singleton bound found in the literature. First, we derive the classical Singleton Bound as a corollary to the LSB theorem.

Corollary 1: Let C be a code in $(\mathbb{F}_2^n, d_H)$, with minimum distance d, then $|C| \leq 2^{n-d+1}$.

Now we derive the KKS bound reported in [2]. It follows as a direct corollary to Theorem 2 when the lattice is chosen to be the lattice of subspaces.

Corollary 2 (KKS Bound): If $m = M = l$ and $\alpha_L = \lfloor \frac{d-1}{2} \rfloor$, then

$$|C| \leq \begin{bmatrix} n - \alpha_L \\ l - \alpha_L \end{bmatrix}_q.$$

IV. A New Upper Bound for Projective space codes

It is clear that many examples fit into the framework of lattice schemes and the proposed LSB specializes to the known Singleton bounds in literature. In this section, we will show that the LSB gives a new upper bound when applied to projective spaces. A family of non-constant dimension code constructions in projective spaces have been reported by [7] and [10]. Our LSB plots show that the code sizes, of both reported codes, are near optimal for certain fixed values of the coding parameters.

Lower bounds on code sizes based on sphere covering methods can be found in [6]. Upper bounds on code sizes for projective space codes, found in the literature, are for constant dimension codes. The following bound is an upper bound on the code size of a non-constant dimension code in a projective space.

Corollary 3 (Singleton Bound for projective spaces): Let C be a code in $(\mathrm{Sub}(V), d_S)$ (where V is vector space over $\mathbb{F}_q$), with minimum distance d, then

$$|C| \leq \sum_{k=0}^{n-\lfloor \frac{d-1}{2} \rfloor} \begin{bmatrix} n - \lfloor \frac{d-1}{2} \rfloor \\ k \end{bmatrix}_q$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the Gaussian number.

The bound is weak when minimum distance d is comparable to the code dimension n. However when the minimum distance is smaller than code dimension, the tightness of the bound is

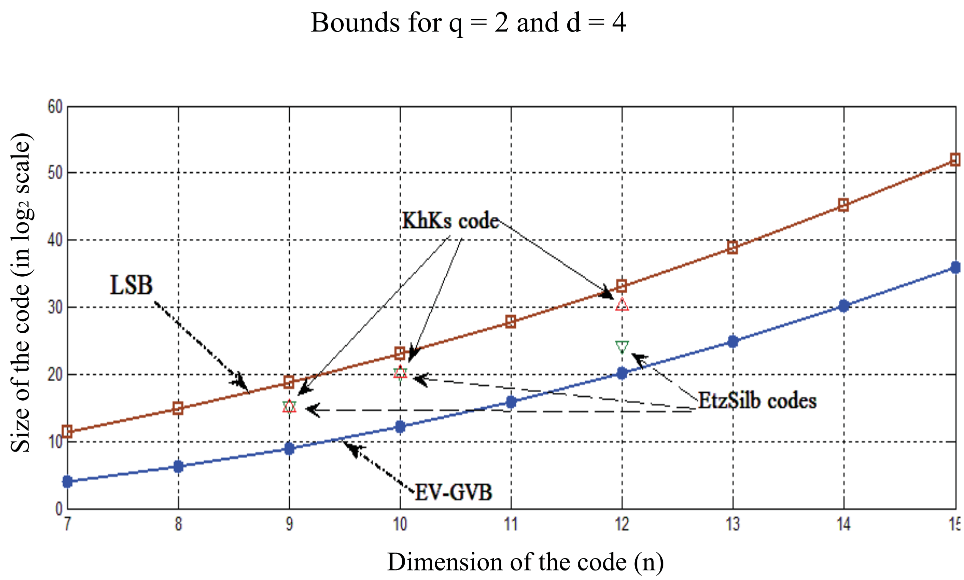

Fig. 1. A plot that compares code sizes of different non-constant dimension codes and a lower bound on the code size. Note that the KhKs code sizes are within three bits from our upper bound. The subspaces are assumed to be over $\mathbb{F}_2$ and the minimum distance is fixed at 4 for this example.

not apparent. Therefore we investigate the bounds behavior at such ranges by plotting it with other bounds in the literature. We will compare our bound (LSB) to the Gilbert Varshamov bound (EV-GVB) proposed in [6] for various values of the minimum distance. Further, we will plot points achieved by various codes in the literature.

The plot is shown in Fig. 1. The minimum distance of the projective code and the finite field size has been fixed at 4 and 2 respectively, throughout this section. The plot clearly shows our upper bound above the EV-GV lower bound. The points marked 'EtzSilb codes' and 'KhKs code' are the code parameters reported in [7] and [10] for $q = 2$ and $d = 4$ respectively.

Fig. 1 shows that the EtzSilb codes and KhKs codes are close to optimal for $q = 2$ and $d = 4$. The KhKs code sizes are roughly 3 bits away from the upper bound. It has already been observed that the distance drop in projective spaces is variable and thus we believe that our bound is weak for larger values of d.

V. Discussion

It would be interesting to investigate whether there exists lattice schemes that achieve the LSB with equality. In particular, it would be very interesting to know if there exists a Reed Solomon-like scheme which specializes to the different counterparts in binary and subspace codes. Also, whether quantum codes are lattice schemes can be investigated.

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