Marton-Marton Coding for A Broadcast Relay Network

Lanying Zhao
Department of Electrical Engineering
KAIST, Daejeon, South Korea
Email: lyzhao83@kaist.ac.kr

Sae-Young Chung
Department of Electrical Engineering
KAIST, Daejeon, South Korea
Email: schung@kaist.ac.kr

***Abstract*—We consider a discrete memoryless broadcast relay network that consists of one transmitter, two receivers and a relay. The transmitter sends independent messages to each receiver using Marton coding. The relay performs decode-and-forward using another Marton coding. We show the achievable rate region of the proposed scheme and also provide an outer bound for the channel.**

I. Introduction

The broadcast channel is a multi-user communication channel where a single source transmits multiple messages to multiple destinations. The source may send a common message to all users and/or separate messages to desired users. Broadcast channels were first investigated by Cover [5]. Its capacity region still remains open in general. Marton provided the best known achievable scheme to date [6]. Capacity regions for all broadcast channels with known capacity can be achieved by the Marton's coding scheme.

In wireless communication, cooperation of nodes can significantly increase the transmitting rate. Van der Meulen first introduced and studied the relay channel [2][3]. In a three-user relay channel, a relay node helps the transmitter communicate with the receiver. In the decode-and-forward scheme, the relay perfectly removes the noise and reliably recovers the message from the transmitter [4]. Then relay re-encodes the message and then sends it to the destination. In the compress-and-forward scheme the relay quantizes the received signal from the transmitter and transmits the compressed information to the destination [4].

For the broadcast relay channel shown in Fig.1, with help of one relay node, the transmitter wants to send messages to both destinations. The broadcast relay channel with common message was first studied in [7]. In [7], the relay only decodes the common message and forwards it to both destinations and the transmitter applies Marton's scheme for broadcasting. In [8], this was generalized such that the relay also decodes and forwards one of the private messages. For the combination of Marton's scheme at the source and compress-and-forward at the receiver, an achievable rate region is provided in [9].

In this work, we consider the broadcast relay channel without common message. The transmitter sends independent private messages to corresponding destinations. In our scheme, the relay helps to decode all message pairs to increase the correlation with the source. Since all message pairs are available at the relay node, to increase the broadcasting rate at the relay, we apply Marton's encoding at the relay, too. Furthermore, we also provide an outer bound for the broadcast relay channel.

The paper is organized as follows. Section II describes the system model for the broadcast relay channel. Section III presents our achievable coding scheme and the corresponding inner bound. Section IV gives an outer bound.

II. System model

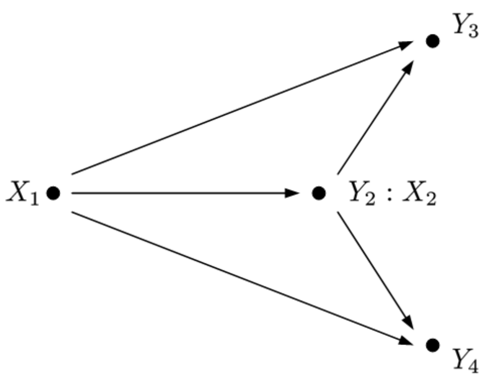

Fig. 1. Broadcast relay channel model

Consider the channel in Fig. 1. The source wishes to send the message $M_1 \in [1 : 2^{nR_1}] = \{1, 2, \dots, 2^{\lceil nR_1 \rceil}\}$ to receiver 1 and $M_2 \in [1 : 2^{nR_2}] = \{1, 2, \dots, 2^{\lceil nR_2 \rceil}\}$ to receiver 2 with the help of the relay, where $\lceil x \rceil$ is the smallest integer not smaller than x. We assume this broadcast relay channel is discrete memoryless and M_1 and M_2 are independent of each other and the message pair (M_1, M_2) is uniformly distributed over $[1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$. The source encodes a pair of messages (m_1, m_2) into a length-n codeword $x_1^n(m_1, m_2)$ and transmits it to the relay and destinations. The relay encoder assigns a symbol $x_{2i}(y_2^{i-1})$ to each past received sequence y_2^{i-1} at each time $i \in [1 : n]$. According to the received sequence y_3^n, receiver 1 estimates the message $\hat{m}_1$ or declares an error. Receiver 2 estimates $\hat{m}_2$ or declares an error from the received sequence y_4^n. Thus, the channel model $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_2, y_3, y_4|x_1, x_2), \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}_4)$ consists of five finite sets $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ and a set of conditional probability mass functions $p(y_2, y_3, y_4|x_1, x_2)$. We assume the channel is memoryless, i.e., the current channel output symbols (Y_{2i}, Y_{3i}, Y_{4i}) are conditionally independent with $(m_1, m_2, X_1^{i-1}, X_2^{i-1}, Y_2^{i-1}, Y_3^{i-1}, Y_4^{i-1})$ given the current channel input symbols (X_{1i}, X_{2i}). The average probability of error is defined as $P_e^{(n)} = P\{\hat{M}_1 \neq M_1 \text{ or } \hat{M}_2 \neq M_2\}$. A rate pair (R_1, R_2) is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $P_e^{(n)} \to 0$ as $n \to \infty$.

III. Achievable rate region

We will first provide a lemma which is slightly modified from the mutual covering lemma [1].

Lemma 1 (Modified Mutual Covering Lemma)**.** *Let $(V_1, V_2, U_1, U_2) \sim p(v_1, v_2, u_1, u_2)$ and $\epsilon' < \epsilon$. Let $(V_1^n, V_2^n) \sim p(v_1^n, v_2^n)$ be a pair of random sequences with $P\{(V_1^n, V_2^n) \in \mathcal{T}_{\epsilon'}^{(n)}\} \to 1$ as $n \to \infty$. Let $U_1^n(m_1)$, $m_1 \in [1 : 2^{nr_1}]$, be pairwise conditionally independent random sequences, each distributed according to $\prod_{i=1}^n p_{U_1|V_1}(u_{1i}|v_{1i})$. Similarly, let $U_2^n(m_2)$, $m_2 \in [1 : 2^{nr_2}]$, be pairwise conditionally independent random sequences, each distributed according to $\prod_{i=1}^n p_{U_2|V_2}(u_{2i}|v_{2i})$. Assume that $\{U_1^n(m_1) : m_1 \in [1 : 2^{nr_1}]\}$ and $\{U_2^n(m_2) : m_2 \in [1 : 2^{nr_2}]\}$ are conditionally independent given (V_1^n, V_2^n). Then, there exists $\delta(\epsilon) > 0$ that tends to zero as $\epsilon \to 0$ such that*

$$\lim_{n\to\infty} P\{(V_1^n, V_2^n, U_1^n(m_1), U_2^n(m_2)) \notin \mathcal{T}_{\epsilon}^{(n)}$$

for all $m_1 \in [1 : 2^{nr_1}], m_2 \in [1 : 2^{nr_2}]\} = 0$, *if*

$$\begin{aligned} r_1 &> I(U_1; V_2|V_1) + 3\delta(\epsilon) \\ r_2 &> I(U_2; V_1|V_2) + 3\delta(\epsilon) \\ r_1 + r_2 &> H(U_1|V_1) + H(U_2|V_2) - H(U_1, U_2|V_1, V_2) \\ &\quad +\delta(\epsilon). \end{aligned}$$

The proof of Lemma 1 is similar to that of the Mutual covering lemma in [1] and thus is omitted here.

Before beginning to state our results, We prefer mentioning that, for two receiver broadcast channel, Marton's inner bound keeps unchanged when we impose more strict decoding at receivers. In [6], both receivers decode the unique message pair $(\hat{m}_1, \hat{m}_2)$. Instead, if receivers decode unique index pair (l_1, l_2) then determine the message pair (m_1, m_2), the achievable rate region remains the same. This is the other motivation to apply Marton's coding scheme at the relay node.

The following theorem presents our achievable rate region.

Theorem 1. *A rate pair (R_1, R_2) is achievable for a DM-BRN, if*

$$\begin{aligned} R_1 &< I(U_1; U_2, Y_2|V_1, V_2), \\ R_1 &< I(U_1, V_1; Y_3) - I(U_1; V_2|V_1), \\ R_1 &< I(U_1, V_1; Y_3) + I(V_2; Y_4) - I(U_1, V_1; V_2), \quad (1) \\ R_2 &< I(U_2; U_1, Y_2|V_1, V_2), \\ R_2 &< I(U_2, V_2; Y_4) - I(U_2; V_1|V_2), \\ R_2 &< I(U_2, V_2; Y_4) + I(V_1; Y_3) - I(U_2, V_2; V_1), \quad (2) \\ R_1 + R_2 &< I(U_2; Y_2|V_1, V_2, U_1) + I(U_1; Y_2|V_1, V_2, U_2) \\ &\quad +I(U_1; U_2|V_1, V_2), \\ R_1 + R_2 &< I(U_1, U_2; Y_2|V_1, V_2), \\ R_1 + R_2 &< I(U_1, V_1; Y_3) + I(U_2; Y_2|V_1, V_2, U_1) \\ &\quad -I(U_1; V_2|V_1), \\ R_1 + R_2 &< I(U_2, V_2; Y_4) + I(U_1; Y_2|V_1, V_2, U_2) \\ &\quad -I(U_2; V_1|V_2), \\ R_1 + R_2 &< I(U_2, V_2; Y_4) + I(U_1; Y_2|V_1, V_2, U_2) \\ &\quad +I(V_1; Y_3) - I(U_2, V_2; V_1), \\ R_1 + R_2 &< I(U_1, V_1; Y_3) + I(U_2; Y_2|V_1, V_2, U_1) \\ &\quad +I(V_2; Y_4) - I(U_1, V_1; V_2), \\ R_1 + R_2 &< I(U_1, V_1; Y_3) + I(U_2, V_2; Y_4) - I(V_1; V_2) \\ &\quad -I(U_1; U_2|V_1, V_2) - I(U_1; V_2|V_1) \\ &\quad -I(U_2; V_1|V_2), \end{aligned}$$

for some pmf $p(v_1, v_2)p(u_1, u_2|v_1, v_2)$ and functions $x_2(v_1, v_2)$ and $x_1(u_1, u_2)$ such that

$$I(V_1; Y_3) + I(V_2; Y_4) > I(V_1; V_2). \quad (3)$$

Proof. Fix $p(u_1, u_2|v_1, v_2) \cdot p(v_1, v_2)$ and functions $x_1(u_1, u_2)$ and $x_2(v_1, v_2)$. We will apply Marton's scheme without rate-splitting for broadcasting both at the source node and at the relay node and decode-and-forward strategy in a block Markov manner at the relay. Thus, the achievability scheme uses b transmission blocks, each consisting of n transmissions, to send $b-1$ messages M_j, $j \in [1 : b-1]$. For each block $j \in [1 : b]$, randomly and independently generate a codebook.

Codebook generation. Fix the conditional pmf $p(u_1, u_2|v_1, v_2)p(v_1, v_2)$ and functions $x_1(u_1, u_2)$ and $x_2(v_1, v_2)$. Let $R_{21} > R_1$, $R_{22} > R_2$, $R_{11} > R_1$, and $R_{12} > R_2$. Let $\epsilon > \epsilon'' > \epsilon'$.

1) For each $j \in [1 : b]$, randomly and independently generate $2^{nR_{21}}$ sequences $v_1^n(l_{1,j-1})$, $l_{1,j-1} \in [1 : 2^{nR_{21}}]$, each according to $\prod_{i=1}^n p_{V_1}(v_{1i})$. Partition the $2^{nR_{21}}$ sequences $v_1^n(l_{1,j-1})$ into 2^{nR_1} equal size bins $B_{21}(m_{1,j-1}) = [(m_{1,j-1}-1)2^{n(R_{21}-R_1)}+1 : m_{1,j-1}2^{n(R_{21}-R_1)}]$, $m_{1,j-1} \in [1 : 2^{nR_1}]$. Similarly, randomly and independently generate $2^{nR_{22}}$ sequences $v_2^n(l_{2,j-1})$, $l_{2,j-1} \in [1 : 2^{nR_{22}}]$, each according to $\prod_{i=1}^n p_{V_2}(v_{2i})$. Partition the $2^{nR_{22}}$ sequences $v_2^n(l_{2,j-1})$ into 2^{nR_2} equal size bins $B_{22}(m_{2,j-1}) = [(m_{2,j-1}-1)2^{n(R_{22}-R_2)} + 1 : m_{2,j-1}2^{n(R_{22}-R_2)}]$, $m_{2,j-1} \in [1 : 2^{nR_2}]$.

2) For each message pair $(m_{1,j-1}, m_{2,j-1})$, find an index pair $(l_{1,j-1}, l_{2,j-1}) \in B_{21}(m_{1,j-1}) \times B_{22}(m_{2,j-1})$, such that $(v_1^n(l_{1,j-1}), v_2^n(l_{2,j-1})) \in \mathcal{T}_{\epsilon'}^{(n)}$. If there is more than one such pair, choose the smallest index $l_{1,j-1}$ first, then choose the smallest index $l_{2,j-1}$, given $l_{1,j-1}$ among all jointly typical sequences. If there is no such pair, choose $(l_{1,j-1}, l_{2,j-1}) = ((m_{1,j-1} - 1)2^{n(R_{21}-R_1)} + 1, (m_{2,j-1}-1)2^{n(R_{22}-R_2)} + 1)$.

3) Then generate $x_2^n(m_{1,j-1}, m_{2,j-1})$ as $x_{2i}(m_{1,j-1}, m_{2,j-1}) = x_2(v_{1i}(l_{1,j-1}), v_{2i}(l_{2,j-1}))$, $i \in [1:n]$.

4) For each $l_{1,j-1} \in [1 : 2^{nR_{21}}]$, randomly and conditionally independently generate $2^{nR_{11}}$ sequences $u_1^n(k_{1,j}|l_{1,j-1})$, $k_{1,j} \in [1 : 2^{nR_{11}}]$, each according to $\prod_{i=1}^n p_{U_1|V_1}(u_{1i}|v_{1i}(l_{1,j-1}))$. Partition the $2^{nR_{11}}$ sequences $u_1^n(k_{1,j}|l_{1,j-1})$ into 2^{nR_1} equal size bins $B_{11}(m_{1,j}) = [(m_{1,j}-1)2^{n(R_{11}-R_1)}+1 : m_{1,j}2^{n(R_{11}-R_1)}]$, $m_{1,j} \in [1 : 2^{nR_1}]$. Similarly, for each $l_{2,j-1} \in [1 : 2^{nR_{22}}]$, randomly and conditionally independently generate $2^{nR_{12}}$ sequences $u_2^n(k_{2,j}|l_{2,j-1})$, $k_{2,j} \in [1 : 2^{nR_{12}}]$, each according to $\prod_{i=1}^n p_{U_2|V_2}(u_{2i}|v_{2i}(l_{2,j-1}))$. Partition the $2^{nR_{12}}$ sequences $u_2^n(k_{2,j}|l_{2,j-1})$ into 2^{nR_2} equal size bins $B_{12}(m_{2,j}) = [(m_{2,j}-1)2^{n(R_{12}-R_2)}+1 : m_{2,j}2^{n(R_{12}-R_2)}]$, $m_{2,j} \in [1 : 2^{nR_2}]$.

5) Let $(l_{1,j-1}, l_{2,j-1})$ be the chosen index pair for the message pair $(m_{1,j-1}, m_{2,j-1})$ at the relay. For each message pair $(m_{1,j}, m_{2,j})$ given $(l_{1,j-1}, l_{2,j-1})$, find an index pair $(k_{1,j}, k_{2,j}) \in B_{11}(m_{1,j}) \times B_{12}(m_{2,j})$ such that $(u_1^n(k_{1,j}|l_{1,j-1}), u_2^n(k_{2,j}|l_{2,j-1})) \in \mathcal{T}_{\epsilon''}^{(n)}(U_1, U_2|v_1^n(l_{1,j-1}), v_2^n(l_{2,j-1}))$. If there is more than one such pair, choose the smallest index $k_{1,j}$ first, then choose the smallest index $k_{2,j}$ given $k_{1,j}$ among all joint typical sequences. If there is no such pair, choose $(k_{1,j}, k_{2,j}) = ((m_{1,j}-1)2^{n(R_{11}-R_1)}+1, (m_{2,j}-1)2^{n(R_{12}-R_2)}+1)$.

6) Then generate $x_1^n(m_{1,j}, m_{2,j}|m_{1,j-1}, m_{2,j-1})$ as $x_{1i}(m_{1,j}, m_{2,j}|m_{1,j-1}, m_{2,j-1}) = x_1(u_{1i}(k_{1,j}|l_{1,j-1}), u_{2i}(k_{2,j}|l_{2,j-1}))$, $i \in [1:n]$.

This defines the codebook

$$\begin{aligned}\mathcal{C}_j = \{&(x_1^n(m_{1,j}, m_{2,j}|m_{1,j-1}, m_{2,j-1}),\\ &x_2^n(m_{1,j-1}, m_{2,j-1}), u_1^n(k_{1,j}|l_{1,j-1}), u_2^n(k_{2,j}|l_{2,j-1}),\\ &v_1^n(l_{1,j-1}), v_2^n(l_{2,j-1})) : m_{1,j-1}, m_{1,j} \in [1 : 2^{nR_1}],\\ &m_{2,j-1}, m_{2,j} \in [1 : 2^{nR_2}], k_{1,j} \in [1 : 2^{nR_{11}}],\\ &k_{2,j} \in [1 : 2^{nR_{12}}], l_{1,j-1} \in [1 : 2^{nR_{21}}],\\ &l_{2,j-1} \in [1 : 2^{nR_{22}}]\}\end{aligned}$$

for $j \in [1 : b]$.

Fig. 2 and Fig. 3 show the codebook generation.

Encoding. Let $(m_{1,j-1}, m_{2,j-1})$ be the message pair transmitted in the $(j-1)^{th}$ block. In block j, to send a message pair $(m_{1,j}, m_{2,j})$, the source transmits $x_1^n(m_{1,j}, m_{2,j}|m_{1,j-1}, m_{2,j-1})$ from codebook $\mathcal{C}_j$, where $m_{1,0} = m_{1,b} = m_{2,0} = m_{2,b} = 1$ and $l_{1,0} = l_{2,0} = k_{1,b} = k_{2,b} = 1$ by convention.

Relay encoding. Let $(\tilde{l}_{1,j-1}, \tilde{l}_{1,j-1}) \in B_{21}(\tilde{m}_{1,j-1}) \times B_{22}(\tilde{m}_{1,j-1})$ be the chosen index for the previously decoded message pair $(\tilde{m}_{1,j-1}, \tilde{m}_{2,j-1})$. At the end of block j, the relay finds the unique message pair $(\tilde{m}_{1,j}, \tilde{m}_{2,j})$ such that $(u_1^n(\tilde{k}_{1,j}|\tilde{l}_{1,j-1}), u_2^n(\tilde{k}_{2,j}|\tilde{l}_{2,j-1}), v_1^n(\tilde{l}_{1,j-1}), v_2^n(\tilde{l}_{2,j-1}), y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ for some $\tilde{k}_{1,j} \in B_{11}(\tilde{m}_{1,j})$, $\tilde{k}_{2,j} \in B_{12}(\tilde{m}_{2,j})$. In block $j+1$, it transmits $x_2^n(\tilde{m}_{1,j}, \tilde{m}_{2,j})$ from codebook $\mathcal{C}_{j+1}$. And let $(\tilde{l}_{1,j}, \tilde{l}_{1,j}) \in B_{21}(\tilde{m}_{1,j}) \times B_{22}(\tilde{m}_{1,j})$ be the chosen index for decoded message pair $(\tilde{m}_{1,j}, \tilde{m}_{2,j})$.

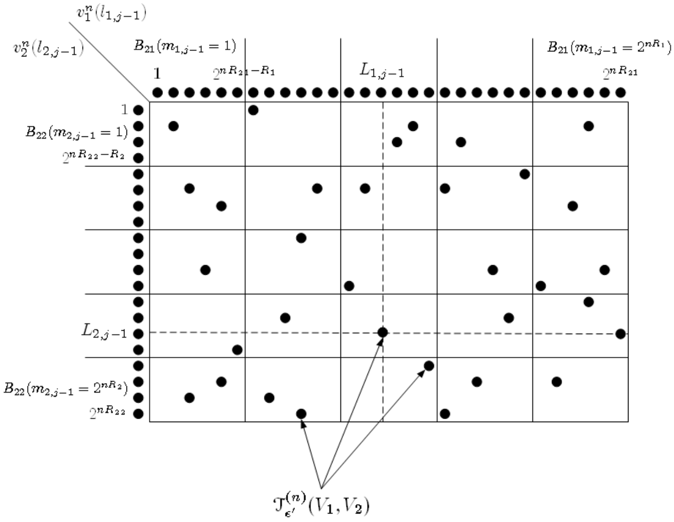

Fig. 2. Coding scheme for Marton+Marton (sliding window decoding): relay part

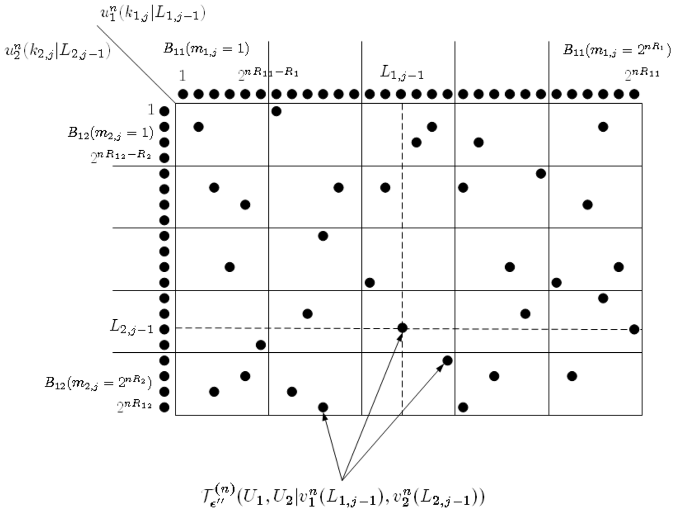

Fig. 3. Coding scheme for Marton+Marton (sliding window decoding): source part for a given $(L_{1,j-1}, L_{2,j-1})$

Decoding at receiver 1. (Unique decoding for l_1, non-unique decoding for k_1)
Let $\hat{l}_{1,j-1}$ be the uniquely decoded index at the end of the j^{th} block. At the end of block $j+1$, the receiver 1 finds the unique message $\hat{m}_{1,j}$ such that $(u_1^n(\hat{k}_{1,j}|\hat{l}_{1,j-1}), v_1^n(\hat{l}_{1,j-1}), y_3^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ and $(v_1^n(\hat{l}_{1,j}), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ for some $\hat{k}_{1,j} \in B_{11}(\hat{m}_{1,j})$, and for the unique $\hat{l}_{1,j} \in B_{21}(\hat{m}_{1,j})$ simultaneously.

Decoding at receiver 2. (Unique decoding for l_2, non-unique decoding for k_2)
Let $\hat{l}_{2,j-1}$ be the uniquely decoded index at the end of the j^{th} block. At the end of block $j+1$, the receiver 2 finds the unique message $\hat{m}_{2,j}$ such that $(u_2^n(\hat{k}_{2,j}|\hat{l}_{2,j-1}), v_2^n(\hat{l}_{2,j-1}), y_4^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ and $(v_2^n(\hat{l}_{2,j}), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ for some $\hat{k}_{2,j} \in B_{12}(\hat{m}_{2,j})$, and for the unique $\hat{l}_{2,j} \in B_{22}(\hat{m}_{2,j})$ simultaneously.

Analysis of probability of error. Assume without loss of generality $M_{1,j-1} = M_{2,j-1} = M_{1,j} = M_{2,j} = 1$. Let $(L_{1,j-1}, L_{2,j-1})$ be the chosen index pair at the relay for $(M_{1,j-1}, M_{2,j-1})$, $(L_{1,j}, L_{2,j})$ be the chosen index pair at the relay for $(M_{1,j}, M_{2,j})$ and $(K_{1,j}, K_{2,j})$ be the chosen index pair at the source for $(M_{1,j}, M_{2,j})$ given the previous message pair $(M_{1,j-1}, M_{2,j-1})$.

1) Because of the Marton's scheme at both the source and the relay, joint typical encoding is needed. This needs following rate constraints, obtained by applying the mutual covering lemma and the modified mutual covering lemma.

$$\begin{aligned}
R_{11} - R_1 + R_{12} - R_2 &> I(U_1; V_2|V_1) \\
&\quad + I(U_2; V_1|V_2) \\
&\quad + I(U_1; U_2|V_1, V_2), \\
R_{11} - R_1 &> I(U_1; V_2|V_1), \\
R_{12} - R_2 &> I(U_2; V_1|V_2), \\
R_{21} - R_1 + R_{22} - R_2 &> I(V_1; V_2).
\end{aligned}$$

2) The relay decodes the whole messages for both users from the current received sequence assuming the previously messages are decoded correctly. For this, the following rate constraints should be satisfied.

$$\begin{aligned}
R_{11} + R_{12} &< I(U_1; V_2, Y_2|V_1) \\
&\quad + I(U_1, V_1, Y_2; U_2|V_2), \\
R_{12} &< I(U_1, V_1, Y_2; U_2|V_2), \\
R_{11} - R_1 + R_{12} &< I(U_1; V_2, Y_2|V_1) \\
&\quad + I(U_1, V_1, Y_2; U_2|V_2), \\
R_{11} &< I(U_2, V_2, Y_2; U_1|V_1), \\
R_{12} - R_2 + R_{11} &< I(U_2; V_1, Y_2|V_2) \\
&\quad + I(U_2, V_2, Y_2; U_1|V_1).
\end{aligned}$$

3) Each user decodes its own private messages using sliding window joint typical decoding. For this analysis, we have the following rate constraints,

$$\begin{aligned}
R_{11} + R_{21} - R_1 &< I(U_1, V_1; Y_3), \\
R_{21} - R_1 &< I(V_1; Y_3), \\
R_{21} - R_1 + R_{11} - R_1 &< I(U_1, V_1; Y_3), \\
R_{12} + R_{22} - R_2 &< I(U_2, V_2; Y_4), \\
R_{22} - R_2 &< I(V_2; Y_4), \\
R_{22} - R_2 + R_{12} - R_2 &< I(U_2, V_2; Y_4).
\end{aligned}$$

Block	1	2	…	$b-1$	b
X_1	$x_1^n(m_{1,1}, m_{2,1}\|1,1)$ $x_1^n(u_1^n(k_{1,1}\|1), u_2^n(k_{2,1}\|1))$	$x_1^n(m_{1,2}, m_{2,2}\|m_{1,1}, m_{2,1})$ $x_1^n(u_1^n(k_{1,2}\|l_{1,1}), u_2^n(k_{2,2}\|l_{2,1}))$	… …	$x_1^n(m_{1,b-1}, m_{2,b-1}\|m_{1,b-2}, m_{2,b-2})$ $x_1^n(u_1^n(k_{1,b-1}\|l_{1,b-2}), u_2^n(k_{2,b-1}\|l_{2,b-2}))$	$x_1^n(1,1\|m_{1,b-1}, m_{2,b-1})$ $x_1^n(u_1^n(1\|l_{1,b-1}), u_2^n(1\|l_{2,b-1}))$
Y_2	$\tilde{k}_{1,1}, \tilde{k}_{2,1}, \tilde{m}_{1,1}, \tilde{m}_{2,1}, \tilde{l}_{1,1}, \tilde{l}_{2,1}$	$\tilde{k}_{1,2}, \tilde{k}_{2,2}, \tilde{m}_{1,2}, \tilde{m}_{2,2}, \tilde{l}_{1,2}, \tilde{l}_{2,2}$	…	$\tilde{k}_{1,b-1}, \tilde{k}_{2,b-1}, \tilde{m}_{1,b-1}, \tilde{m}_{2,b-1}, \tilde{l}_{1,b-1}, \tilde{l}_{2,b-1}$	1
X_2	$x_2^n(1,1)$ $x_2^n(v_1^n(1), v_2^n(1))$	$x_2^n(\tilde{m}_{1,1}, \tilde{m}_{2,1})$ $x_2^n(v_1^n(\tilde{l}_{1,1}), v_2^n(\tilde{l}_{2,1}))$	… …	$x_2^n(\tilde{m}_{1,b-2}, \tilde{m}_{2,b-2})$ $x_2^n(v_1^n(\tilde{l}_{1,b-2}), v_2^n(\tilde{l}_{2,b-2}))$	$x_2^n(\tilde{m}_{1,b-1}, \tilde{m}_{2,b-1})$ $x_2^n(v_1^n(\tilde{l}_{1,b-1}), v_2^n(\tilde{l}_{b-1}))$
Y_3	1	$\hat{l}_{1,1}, \hat{k}_{1,1}, \hat{m}_{1,1}$	…	$\hat{l}_{1,b-2}, \hat{k}_{1,b-2}, \hat{m}_{1,b-2}$	$\hat{l}_{1,b-1}, \hat{k}_{1,b-1}, \hat{m}_{1,b-1}$

TABLE I
ENCODING AND DECODING OF THE SCHEME

In analysis steps 2) and 3), since the chosen sequence indices are dependant on chosen bins, more careful analysis is needed (details omitted). Finally, by Fourier-Motzkin elimination, we can finish the proof. $\square$

Since the relay decodes both private messages, thus it can fully cooperate with the transmitter, the private message rate is influenced by the other user's channel state. That is why there is $I(V_2;Y_4)$ (or $I(V_1;Y_3)$) term in the private rate constraint (1) (or (2)). The mutual information constraint (3) needs to guarantee the unique decoding of $v_1^n(l_{1,j})$ (or $v_2^n(l_{2,j})$) at the receiver 1 (or receiver 2).

For the rate region in Theorem 1, if we set $(U_2, V_2, Y_4) =$, $U_1 = X_1$ and $V_1 = X_2$, we get

$$R_1 < \min\{I(X_1;Y_2|X_2), I(X_1,X_2;Y_3)\}$$

over the probability distribution $p(x_1,x_2)$, which is the decode-and-forward lower bound. In addition, if we set $(V_1,V_2) =$ and $Y_2 = (Y_3,Y_4)$, the region becomes

$$\begin{aligned} R_1 &< I(U_1;Y_3), \\ R_2 &< I(U_2;Y_4), \\ R_1+R_2 &< I(U_1;Y_3)+I(U_2;Y_4)-I(U_1;U_2), \end{aligned}$$

over the probability distribution $p(u_1,u_2)$, which is the Marton's inner bound without rate splitting.

IV. Outer Bound

By Fano's inequality and standard steps we have the following outer bound.

Theorem 2. *The set $\mathscr{C}_{BRN}^{OUT}$ of all rate pairs (R_1,R_2) satisfies*

$$\begin{aligned} \mathscr{C}_{BRN}^{OUT} &= \{(R_1\geq 0, R_2\geq 0): \\ R_1 &\leq I(U_1,W;Y_3), \\ R_1 &\leq I(U_1,V,W;Y_2,Y_3|X_2), \\ R_2 &\leq I(U_2,W;Y_4), \\ R_2 &\leq I(U_2,V,W;Y_2,Y_4|X_2), \\ R_1+R_2 &\leq I(W;Y_3)+I(U_2;Y_4|W)+I(U_1;Y_3|W,U_2), \\ R_1+R_2 &\leq I(W;Y_4)+I(U_1;Y_3|W)+I(U_2;Y_4|W,U_1), \\ R_1+R_2 &\leq I(V,W;Y_2,Y_3|X_2)+I(U_2;Y_2,Y_4|V,W,X_2) \\ &\quad +I(U_1;Y_2,Y_3|V,W,U_2,X_2), \\ R_1+R_2 &\leq I(V,W;Y_2,Y_4|X_2)+I(U_1;Y_2,Y_3|V,W,X_2) \\ &\quad +I(U_2;Y_2,Y_4|V,W,U_1,X_2), \\ R_1+R_2 &\leq I(W,U_1;Y_3)+I(U_2;Y_4|W,U_1), \\ R_1+R_2 &\leq I(U_1;Y_3|W,U_2)+I(W,U_2;Y_4), \\ R_1+R_2 &\leq I(V,W,U_1;Y_2,Y_3|X_2) \\ &\quad +I(U_2;Y_2,Y_4|V,W,U_1,X_2), \\ R_1+R_2 &\leq I(U_1;Y_2,Y_3|V,W,U_2,X_2) \\ &\quad +I(V,W,U_2;Y_2,Y_4|X_2)\}, \end{aligned}$$

for some pmf $p(v,w,u_1,u_2) = p(u_1)p(u_2)p(v,w|u_1,u_2)$ and functions $x_1(w,u_1,u_2)$ and $x_2(v)$.

Proof. By Fano's inequality and following standard steps, we have

$$\begin{aligned} nR_1 &\leq I(M_1;Y_3^n)+n\epsilon_n, \\ nR_2 &\leq I(M_2;Y_4^n)+n\epsilon_n, \\ nR_1 &\leq I(M_1;Y_2^n,Y_3^n)+n\epsilon_n, \\ nR_2 &\leq I(M_2;Y_2^n,Y_4^n)+n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_3^n)+I(M_2;Y_4^n)+n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_2^n,Y_3^n)+I(M_2;Y_2^n,Y_4^n) \\ &\quad +n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_3^n)+I(M_2;Y_4^n|M_1)+n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_3^n|M_2)+I(M_2;Y_4^n)+n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_2^n,Y_3^n)+I(M_2;Y_2^n,Y_4^n|M_1) \\ &\quad +n\epsilon_n, \\ n(R_1+R_2) &\leq I(M_1;Y_2^n,Y_3^n|M_2)+I(M_2;Y_2^n,Y_4^n) \\ &\quad +n\epsilon_n. \end{aligned}$$

Setting $U_{1i} \triangleq M_1$, $W_i \triangleq (Y_3^{i-1}, Y_{4,i+1}^n)$, $U_{2i} \triangleq M_2$ and $V_i \triangleq (Y_2^{i-1}, Y_{2,i+1}^n)$, with help of Csiszár-Körner identity, we can obtain the upper bounds. $\square$

Our outer bound can be shown to perform as good as that of the outer bound given in Theorem 5 in [8].

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References

[1] A. El Gamal and Y.-H. Kim, *Network information theory*. Cambridge, 2011.
[2] E. C. van der Meulen, "Transmission of information in a T-terminal discrete memoryless channel," Ph. D. dissertation, Univ. California, Berkely, CA, Jun. 1968.
[3] E. C. van der Meulen, "Three-terminal communication channels," *Adv. Appl. Prob.*, vol. 3, pp. 120-154, 1971.
[4] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 5, pp. 572-584, Sep. 1979.
[5] T. M. Cover, "broadcast channels," *IEEE Trans. Inf. Theory*, vol. 8, no. 1, pp. 2-14, Jan. 1972.
[6] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 3, pp. 306-311, May 1979.
[7] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037-3063, sep. 2005.
[8] A. Behboodi and P. Piantanida, "Cooperative strategies for simultaneous and broadcast relay channels," *arXiv:1103.5133v2[cs.IT]*, May. 2012.
[9] S. A. Yildirim and M. Yuksel, "Multiple description coding based compress-and-forward for the broadcast relay channel", in *Proc. IEEE Int. Symp. Information Theory*, Cambridge, MA, pp. 199-203, Jul. 2012.