Ergodic Interference Alignment with Noisy Channel State Information

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***Abstract*—We investigate the time-varying Gaussian interference channel (IC) in which each source desires to communicate to an intended destination. For the ergodic time-varying IC with global perfect CSI at all terminals, it has been known that with an interference alignment technique each source-destination pair can communicate at half of the interference-free achievable rate. In practice, the channel gains are estimated by transmitting known pilot symbols from the sources, and the channel estimation procedure is hence prone to errors. In this paper, we model the channel estimation error at the destinations by an independent additive Gaussian noise and study the behavior of the ergodic interference alignment scheme with the global noisy CSI at all terminals. Toward this end, we present a closed-form inner bound on the achievable rate region by which we conclude that the achievable degrees of freedom with global perfect CSI can be preserved, if the variance of channel estimation error is proportional to the inverse of the transmitted power.**

I. Introduction

Characterizing the capacity region of interference channels (ICs) has attracted much interest for decades, e.g. the two-user IC has been the subject of extensive research. Although certain achievable rate regions and outer bounds on the capacity region of the two-user IC have been proposed [1], [2], the exact capacity region is still unknown except for certain special cases (e.g. when interference is either very weak or strong [3], [4]). Furthermore, extension of the results on the two-user ICs to general K-user ICs is not straightforward. Recently, via a novel interference management technique referred to as *interference alignment* [5], [6], it has been shown that ICs may *not* be interference limited in high signal-to-noise ratio (SNR) regime. Through properly designing the transmitted signals, the received interference signals at each destination can be aligned such that they occupy only a sub-space of the received signal space. Consequently, a K-user time-varying (or frequency-selective) IC can achieve the sum-rate of $\frac{K}{2}\log(\mathrm{SNR}) + o(\log(\mathrm{SNR}))$, where $\lim_{\mathrm{SNR}\to\infty} o(\log(\mathrm{SNR}))/\log(\mathrm{SNR}) = 0$ [6]. This achievable sum-rate linearly scales with the number of users at high SNR and is substantially higher than that of the time-division-multiple-access (TDMA) scheme, which is only $\log(\mathrm{SNR}) + o(\log(\mathrm{SNR}))$. Furthermore, when the channel gains are ergodic time-varying and symmetrically distributed (e.g. Rayleigh fading channels), the *ergodic interference alignment* scheme has been developed in [7] which achieves the sum-rate of $\frac{K}{2}\mathbb{E}[\log(1 + 2|h|^2\mathrm{SNR})]$. This sum-rate is achieved by exploiting the time variations of the channel and retransmission of the same symbols over properly chosen time slots. This implies that IC under time-varying channel environments is *not* interference limited at any SNR.

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To achieve the performance promised by the aforementioned schemes, however, global channel state information (CSI) is assumed to be perfectly known at all the sources and destinations. Since acquiring such perfect CSI is a challenging problem, references [7]–[11] have investigated the cases in which each destination perfectly knows CSI of its incoming channel gains. Thus, it provides either the quantized version or the un-coded version of the channel gains to the other terminals through digital feedback signals or analog feedback signals, respectively. It has been shown that if the number of feedback bits of the digital feedback signals [7]–[10] or the powers of the analog feedback signals [11] properly scale with transmit power, the outstanding performance of interference alignment is still achievable. Furthermore, it has been shown in reference [10] that even with a limited number of feedback bits, the throughput of applying interference alignment can still be larger than that achieved by TDMA.

Nevertheless, the channel estimation at each destination in general is not perfect. Thus, the available CSI at each terminal is subject to some estimation errors. Such errors can potentially degrade the performance of the network. To the best of our knowledge, it has been unknown how accurate the channel estimation is required to be to attain similar performance as the one which is achievable by applying

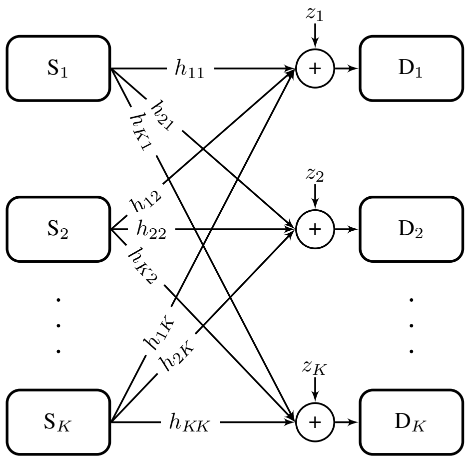

Fig. 1: K-user SISO interference channel

interference alignment based on perfect CSI. Therefore, we investigate an achievable rate region of IC in this case, when the ergodic interference alignment scheme is applied. Our results reveal that, when the variance of channel estimation error is fixed, the IC is interference limited at high SNR, i.e. simply increasing SNR would not improve the achievable sum-rate. However, if the variance of the channel estimation error is proportional to the inverse of transmit power, then the achievable degrees of freedom region is the same as the one with perfect CSI.

This paper is organized as follows. Section II describes the system model and the transmission scheme. An achievable rate region with noisy CSI is derived in Section III. Numerical evaluations are presented in Section V. Finally, Section VI concludes the paper.

II. Time-Varying IC with Noisy CSI

This paper considers a time-varying wireless IC composed of K sources and K destinations, as illustrated in Fig. 1. The sources and destinations are denoted by S_k and D_k ($k \in \{1, 2, ..., K\}$), respectively. The source S_k intends to communicate to D_k and chooses message m_k independently with a uniform distribution from the set $\mathcal{M} = \{1, 2, ..., 2^{n\tilde{R}_k}\}$, where $\tilde{R}_k > 0$ is the code rate. It encodes its message m_k to a length n codeword $\{X_k^t\}_{t=1}^n$ which satisfies the power constraint

$$\frac{1}{n}\sum_{t=1}^{n}\left|X_k^t\right|^2 \leq P. \tag{1}$$

Since all sources share the wireless transmission medium, at time slot t, the destination D_k receives its desired message from the corresponding source S_k over the *direct link* with corresponding channel gain h_{kk}^t, as well as interference signals transmitted from all other sources S_l ($l \in \{1, 2, ..., K\}$, $l \neq k$) over *interference links* with channel gains h_{kl}^t. Therefore, the channel output at D_k is

$$Y_k^t = h_{kk}^t X_k^t + \sum_{l=1, l\neq k}^{K} h_{kl}^t X_l^t + Z_k^t, \tag{2}$$

where Z_k^t denotes a zero-mean additive white Gaussian noise (AWGN), i.e. $Z_k^t \sim \mathcal{CN}(0, 1)$. The channel gains are ergodic time-varying and have independent and identical distribution across different time slots. At time slot t, the channel gains are independently drawn from a complex Gaussian distribution, i.e. $h_{kl}^t \sim \mathcal{CN}(0, 1)$.

We assume that only imperfect estimation of global CSI is available at each terminal. To obtain this, at the beginning of each time slot, each destination estimates its incoming channels. This estimation is subject to some errors in general. Next, all destinations broadcast their estimations to all the other terminals through orthogonal feedback channels. For instance, each destination sends out the quantized versions of the estimated channel gains with sufficiently high quantization resolution. The feedback channels are assumed to be error-free.

The channel estimation is modeled according to

$$h_{kl}^t = \tilde{h}_{kl}^t + \varepsilon_{kl}^t, \ \forall l, k \in \{1, 2, ..., K\}, \tag{3}$$

where $\tilde{h}_{kl}^t$ and ε_{kl}^t denote estimated channel gain and estimation error, respectively. We denote the estimated channel matrix of the considered network at time slot t as $\widetilde{\mathbf{H}}^t$. All terminals only know $\widetilde{\mathbf{H}}^t$, thus, the estimation error can be interpreted as *noise* which degrades available CSI at terminals. The estimation error is independent of the estimated channel gain and we have $\varepsilon_{kk}^t \sim \mathcal{CN}(0, \sigma_{\varepsilon,\mathrm{I}}^2)$ and $\varepsilon_{kl}^t \sim \mathcal{CN}(0, \sigma_{\varepsilon,\mathrm{II}}^2)$ ($\forall l \neq k \in \{1, 2, ..., K\}$). Also, $\tilde{h}_{kk}^t \sim \mathcal{CN}(0, 1 - \sigma_{\varepsilon,\mathrm{I}}^2)$ and $\tilde{h}_{kl}^t \sim \mathcal{CN}(0, 1 - \sigma_{\varepsilon,\mathrm{II}}^2)$ ($\forall l \neq k \in \{1, 2, ..., K\}$). The parameters $\sigma_{\varepsilon,\mathrm{I}}^2$ and $\sigma_{\varepsilon,\mathrm{II}}^2$ indicate the variance of error for direct link estimation and interference links estimations, respectively. If $\sigma_{\varepsilon,\mathrm{I}}^2, \sigma_{\varepsilon,\mathrm{II}}^2 \to 0$, then the model reduces to the case that perfect CSI is available at terminals, and when $\sigma_{\varepsilon,\mathrm{I}}^2, \sigma_{\varepsilon,\mathrm{II}}^2 \to 1$ it represents that no CSI is available terminals. The parameters $\sigma_{\varepsilon,\mathrm{I}}^2$ and $\sigma_{\varepsilon,\mathrm{II}}^2$ can potentially have different values corresponding to different accuracy of channel estimation of the direct links and that of the interference links.

We apply the ergodic interference alignment transmission scheme proposed in [7], but assume that only the estimated channel gains are available at terminals. Thus, if estimated channel gains at time slots t and t_p ($t_p > t$) satisfy $\tilde{h}_{kk}^{t_p} = \tilde{h}_{kk}^t$ and $\tilde{h}_{kl}^{t_p} = -\tilde{h}_{kl}^t$ ($\forall k, l \in \{1, 2, ..., K\}$, $k \neq l$), then S_k at time t_p retransmits the codeword which was transmitted at time t, i.e. $X_k^{t_p} = X_k^t$. To avoid measure zero events, this channel pairing can be performed based on quantized version of the estimated channel gain with sufficiently fine quantizer [7]. Therefore, D_k receives the following signals at the corresponding time slots

$$Y_k^t = h_{kk}^t X_k^t + \sum_{l=1, l\neq k}^{K} h_{kl}^t X_l^t + Z_k^t \tag{4}$$

$$Y_k^{t_p} = h_{kk}^{t_p} X_k^t + \sum_{l=1, l\neq k}^{K} h_{kl}^{t_p} X_l^t + Z_k^{t_p}. \tag{5}$$

The destination D_k combines these received signals to obtain the following signal

$$\overline{Y}_k^t = Y_k^t + Y_k^{t_p} = \left(2\tilde{h}_{kk}^t + \left(\varepsilon_{kk}^t + \varepsilon_{kk}^{t_p}\right)\right) X_k^t + \sum_{l=1, l\neq k}^{K} \left(\varepsilon_{kl}^t + \varepsilon_{kl}^{t_p}\right) X_l^t + \left(Z_k^t + Z_k^{t_p}\right). \quad (6)$$

Next, it decodes the observed channel output $\{\overline{Y}_k^t\}_{t=1}^{n/2}$ to an estimate $\hat{m}_k$ of the transmitted message.

Definition 1: A rate tuple $(R_1, R_2, ..., R_K)$ is *achievable* if for all $\epsilon > 0$ and sufficiently large code length n, channel encoding and decoding functions exist such that

$$\tilde{R}_k > R_k - \epsilon,\ k \in \{1, 2, ..., K\}$$
$$\Pr\left(\bigcup_{k=1}^{K} \{\hat{m}_k \neq m_k\}\right) < \epsilon. \quad (7)$$

III. Achievable Rate Region with Noisy CSI

Proposition 1: In the considered K-user time-varying IC, a rate tuple $(R_1, R_2, ..., R_K)$ is achievable, where

$$R_k = \frac{1}{2}\mathbb{E}\left[I\left(X_k; \overline{Y}_k \mid \widetilde{\mathbf{H}}\right)\right],\ \forall k \in \{1, 2, ..., K\} \quad (8)$$

and $\overline{Y}_k$ is given in (6).

Proof: The proof follows that of [7, *Theorem* 2]. The difference is that in [7, *Theorem* 2] the ergodic interference alignment scheme is applied based on the assumption that each destination has perfect knowledge of its incoming channel gains, but here only imperfect estimations of the channel gains are available at destinations and sources. ■

We next present a closed-form inner bound on the achievable rate region in (8).

Proposition 2: An inner bound on the achievable rate region in (8) is $R_k \geq R_k^{\mathcal{L}}$ $(\forall k \in \{1, 2, ..., K\})$, where

$$R_k^{\mathcal{L}} = \frac{1}{2}\mathbb{E}\left[\log\left(1 + \frac{2\left|\tilde{h}_{kk}\right|^2 P}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2\right) P}\right)\right]. \quad (9)$$

Proof: The term $I\left(X_k; \overline{Y}_k \mid \widetilde{\mathbf{H}}\right)$ in (8) can be lower bounded as

$$\begin{aligned} I\left(X_k; \overline{Y}_k \mid \widetilde{\mathbf{H}}\right) &\overset{(a)}{=} h\left(X_k \mid \widetilde{\mathbf{H}}\right) - h\left(X_k \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right) \\ &\overset{(b)}{=} h(X_k) - h\left(X_k \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right) \\ &\overset{(c)}{=} h(X_k) - h\left(X_k - \widehat{X}_k \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right) \\ &\overset{(d)}{=} \log 2\pi e P - h\left(X_k - \widehat{X}_k \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right) \\ &\overset{(e)}{\geq} \log 2\pi e P - \log 2\pi e \sigma^2 \end{aligned} \quad (10)$$

where σ^2 is the conditional variance of $\left(X_k - \widehat{X}_k\right)$. In this equation (a) follows the definition of the conditional mutual information; (b) holds since the transmitted codeword is chosen independent of the noisy CSI; (c) follows the fact that $\widehat{X}_k$ is a function of $\widetilde{\mathbf{H}}$ and $\overline{Y}_k$ which will be specified in the below; (d) follows the assumption that X_k is a complex Gaussian random variable; (e) follows [12, *Theorem* 8.6.5] that shows the entropy of a random variable with given bounded variance is upper bounded by that of a random variable with Gaussian distribution. To obtain a tight lower bound on the achievable rate in (10), we choose $\widehat{X}_k$ to be a minimum mean square error (MMSE) estimate of X_k; that is

$$\begin{aligned} \widehat{X}_k &= \frac{\mathbb{E}\left[X_k \left(\overline{Y}_k\right)^* \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right]}{\mathbb{E}\left[\overline{Y}_k \left(\overline{Y}_k\right)^* \mid \widetilde{\mathbf{H}}, \overline{Y}_k\right]} \overline{Y}_k \\ &= \frac{\left(\widetilde{h}_{kk}\right)^* P}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2 + 2\left|\widetilde{h}_{kk}\right|^2\right) P} \overline{Y}_k \end{aligned} \quad (11)$$

which yields

$$\sigma^2 = \frac{P}{1 + \frac{2\left|\tilde{h}_{kk}\right|^2 P}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2\right) P}}. \quad (12)$$

The details of the derivation of σ^2 are given in Appendix A. The proof is completed by substituting (12) in (10). ■

We next characterize achievable degrees of freedom region with noisy CSI and present a sufficient condition on channel estimation error that preserves the achievable degrees of freedom region of interference alignment with perfect CSI.

IV. Acheivable Degree of Freedom Region with Noisy CSI

The following corollary charactrizes the acheivable rate region at asymptotically high SNR region.

Corollary 1: When $\sigma_{\varepsilon,\mathrm{I}}^2$ and $\sigma_{\varepsilon,\mathrm{II}}^2$ are fixed, if $P \to \infty$, then an inner bound on the achievable rate region is

$$R_k^{\mathcal{L}} = \frac{1}{2}\mathbb{E}\left[\log\left(1 + \frac{2|\tilde{h}_{kk}|^2}{\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2}\right)\right],\ \forall k \in \{1, ..., K\}. \quad (13)$$

Proof: The proof is completed by taking the limit of the lower bound on the achievable rates in (9) and using the monotone convergence theorem [13]. ■

This result implies that for fixed variances of channel estimation errors, the channel is basically interference limited, i.e. increasing SNR does not improve the achievable rates at high SNR. The following corollary, however, reveals that the achievable degrees of freedom with perfect CSI can be preserved, if the variance of channel estimation error properly decays as transmit power increases. First, we define achievable degree of freedom region.

Definition 2: A tuple $(d_1, d_2, ..., d_K)$ denotes achievable degrees of freedom in which $d_k = \lim_{P\to\infty} \frac{R_k}{\log P}$, where R_k is an achievable rate.

Proposition 3: Assuming that only noisy CSI is available at all terminals, if the variance of channel estimation error is proportional to $P^{-\alpha}$ $(\alpha \in \mathbb{R})$, then the degrees of freedom region $(d_1, d_2, ..., d_K)$ is achievable, where

$$d_k = \begin{cases} 0 & \alpha \leq 0 \\ \alpha/2 & 0 < \alpha < 1 \\ 1/2 & \alpha \geq 1 \end{cases},\ k \in \{1, 2, ..., K\}. \quad (14)$$

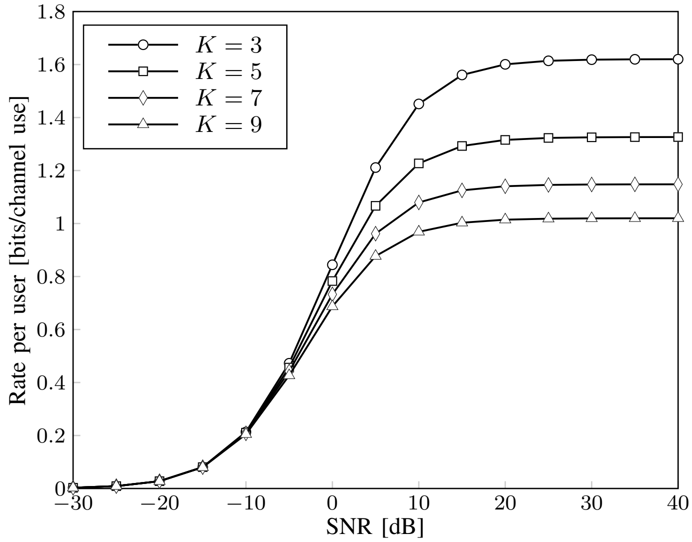

Fig. 2: The lower bound on the achievable rate per user in a K-user IC with noisy CSI, $\sigma^2_{\varepsilon,\mathrm{I}} = \sigma^2_{\varepsilon,\mathrm{II}} = 0.1$.

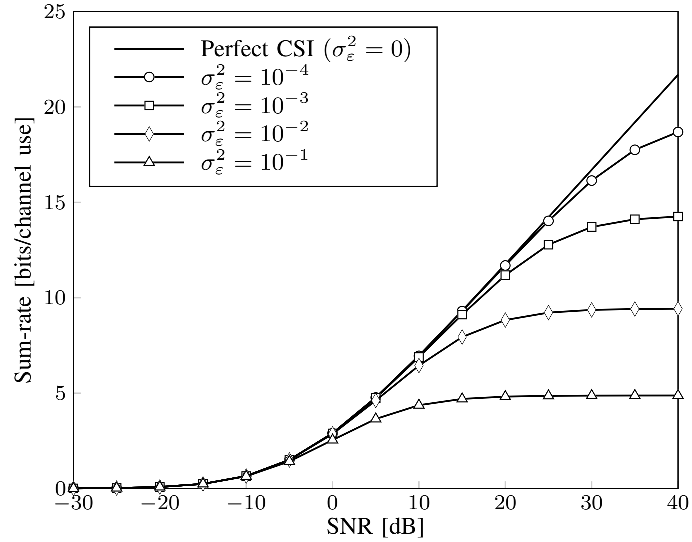

Fig. 3: The lower bound on the achievable sum-rate in a three-user IC with noisy CSI, $\sigma^2_{\varepsilon,\mathrm{I}} = \sigma^2_{\varepsilon,\mathrm{II}} = \sigma^2_\varepsilon$.

Proof: Assume that $\sigma^2_{\varepsilon,\mathrm{I}} \propto P^{-\alpha}$ and $\sigma^2_{\varepsilon,\mathrm{II}} \propto P^{-\alpha}$. This is corresponding to $\sigma^2_{\varepsilon,\mathrm{I}} = aP^{-\alpha}$ and $\sigma^2_{\varepsilon,\mathrm{II}} = bP^{-\alpha}$, where a and b are constant values. If $0 < \alpha < 1$, then we have

$$\begin{aligned} d_k &= \lim_{P\to\infty} \frac{\frac{1}{2}\mathbb{E}\left[\log\left(1 + \frac{2|\tilde{h}_{kk}|^2 P}{1+(a+(K-1)b)P^{1-\alpha}}\right)\right]}{\log P} \\ &= \lim_{P\to\infty} \frac{\frac{1}{2}\log P^{\alpha} + \frac{1}{2}\mathbb{E}\left[\log\left(P^{-\alpha} + \frac{2|\tilde{h}_{kk}|^2 P^{1-\alpha}}{1+(a+(K-1)b)P^{1-\alpha}}\right)\right]}{\log P} \\ &\overset{(a)}{=} \frac{\alpha}{2} + \frac{1}{2}\mathbb{E}\left[\lim_{P\to\infty} \frac{\log\left(P^{-\alpha} + \frac{2|\tilde{h}_{kk}|^2 P^{1-\alpha}}{1+(a+(K-1)b)P^{1-\alpha}}\right)}{\log P}\right] \\ &= \frac{\alpha}{2} \end{aligned} \tag{15}$$

where (a) follows the dominated convergence theorem [13]. We can similarly prove the achievable degrees of freedom for $\alpha \geq 1$ and $\alpha \leq 0$. ∎

This result shows that the same degrees of freedom as the one with perfect CSI is achievable if the variance of channel estimation error is proportional to the inverse of transmitted power at high SNR. This can be achieved, for instance, by splitting the transmitted power P to two parts βP and $(1-\beta)P$ ($\beta \in (0,1)$) and allocate them for training and data transmission, respectively. Applying an MMSE based estimation of the channel gains, if the transmit power is large, the variance of channel estimation error is proportional to $1/P$ and the mentioned result can be achieved.

V. Numerical Evaluation

This section presents numerical evaluations of the lower bound given in *Proposition* 2 on the achievable rate of the considered network. Fig. 2 shows the lower bound on the achievable rate per user of K-user ICs versus SNR. We observe that when the variance of channel estimation error is fixed, the achievable rate monotonically increases and at high SNR saturates. Also, we can see that the achievable rate monotonically decreases as the number of the users increases. These observations confirm that the IC in this case is interference limited which coincides with *Corollary* 1.

Fig. 3 illustrates the lower bound on the achievable sum-rate of a three-user IC for different variances of channel estimation error. It can be observed that the sum-rate increases as the variance of channel estimation error decreases. Indeed, the achievable sum-rate with noisy CSI approaches the one with perfect CSI, if the variance of channel estimation error becomes sufficiently small. This can be exploited to allocate minimum required resources for channel estimation to attain a desired transmission rate.

Fig. 4 shows the lower bound on the achievable sum-rate of a three-user IC when the variance of channel estimation error is equal to $P^{-\alpha}$, where P is the transmit power and $\alpha > 0$. We can see that, at high SNR, the sum-rate linearly scales with the power. This observation coincides with *Proposition* 3. The sum-rate has different behavior for $0 < \alpha < 1$ and $1 \leq \alpha$ at high SNR; when $0 < \alpha < 1$, the slope of the sum-rate versus SNR curve increases linearly with α, however, when $1 \leq \alpha$ the curves have similar slopes which are the same as the one with perfect CSI. Furthermore, when $1 \leq \alpha$ we can see a gap between the achievable sum-rate with noisy CSI and the one with perfect CSI at high SNR which decays as α increases.

VI. Conclusion

We have investigated the achievable rate region of the Gaussian time-varying IC when only noisy estimations of the channel gains are available at all terminals. We have shown that when the variance of channel estimation error is fixed, IC is basically interference limited, i.e. increasing SNR would

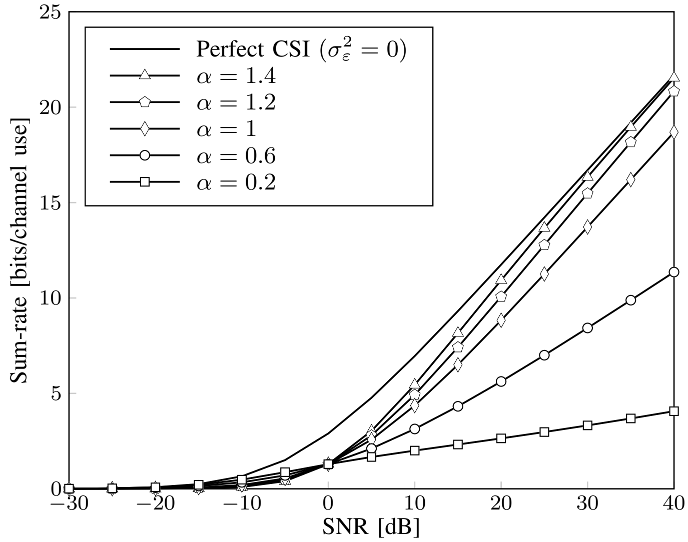

Fig. 4: The lower bound on the achievable sum-rate in a three-user IC with noisy CSI, $\sigma_{\varepsilon,\mathrm{I}}^2 = \sigma_{\varepsilon,\mathrm{II}}^2 = P^{-\alpha}$.

not improve the achievable rates at high SNR. However, if the variance of channel estimation error is proportional to $P^{-\alpha}$ ($\alpha \geq 0$), each user can achieve the degrees of freedom of $\min\{\alpha/2, 1/2\}$; indeed, if $\alpha = 1$, then the achievable degrees of freedom are the same as those when global perfect CSI is available at all terminals. Therefore, channel estimation with certain accuracy is sufficient to attain the outstanding performance of the ergodic interference alignment scheme.

Appendix A
The Proof of Proposition 2

The variance in (12) can be derived as follows

$$\begin{aligned}
\sigma^2 &= \mathbb{E}\Big[\Big(X_k - \widehat{X}_k\Big)\Big(X_k - \widehat{X}_k\Big)^* \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big] \\
&\quad - \Big|\mathbb{E}\Big[X_k - \widehat{X}_k \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big]\Big|^2 \\
&= \mathbb{E}\Big[\Big(X_k - \widehat{X}_k\Big)\Big(X_k - \widehat{X}_k\Big)^* \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big] \\
&\overset{(a)}{=} \mathbb{E}\Big[X_k\Big(X_k - \widehat{X}_k\Big)^* \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big] \\
&= P - \mathbb{E}\Big[X_k\Big(\widehat{X}_k\Big)^* \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big] \\
&\overset{(b)}{=} P - \frac{\mathbb{E}\Big[P\tilde{h}_{kk}X_k\left(\overline{Y}_k\right)^* \Big|\widetilde{\mathbf{H}}, \overline{Y}_k\Big]}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2 + 2\left|\tilde{h}_{kk}\right|^2\right)P} \\
&\overset{(c)}{=} P - \frac{2\left|\tilde{h}_{kk}\right|^2 P^2}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\,\sigma_{\varepsilon,\mathrm{II}}^2 + 2\left|\tilde{h}_{kk}\right|^2\right)P} \\
&= \frac{P}{1 + \frac{2\left|\tilde{h}_{kk}\right|^2 P}{1 + \left(\sigma_{\varepsilon,\mathrm{I}}^2 + (K-1)\sigma_{\varepsilon,\mathrm{II}}^2\right)P}} \qquad (16)
\end{aligned}$$

where (a) follows the orthogonality of the estimated signal to the estimation error of the MMSE estimator; (b) follows the substitution of $\widehat{X}_k$ given in (11); and (c) follows substituting $\overline{Y}_k$ given in (6), and noting that X_k is mutually independent of Z_k^m, $Z_k^{m_p}$ and X_l ($\forall l \in \{1, 2, ..., K\}, l \neq k$).

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