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Structured Lattice Codes for $2 \times 2 \times 2$ MIMO Interference Channel

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***Abstract*—We consider the $2 \times 2 \times 2$ multiple-input multiple-output interference channel where two source-destination pairs wish to communicate with the aid of two intermediate relays. In this paper we present a novel lattice strategy called Precoded Compute-and-Forward (PCoF) with Channel Integer Alignment (CIA). This scheme consists of two phases: 1) Using the CoF framework based on CIA we convert the Gaussian network into a *deterministic* finite-field network. 2) Using linear precoding (over finite-field) we eliminate the end-to-end interference in the finite-field domain. Further, we exploit the algebraic structure of lattices to enhance the performance at finite SNR, such that beyond a degree of freedom result (also achievable by other means). We can also show that PCoF with CIA outperforms time-sharing in a range of reasonably moderate SNR, with increasing gain as SNR increases.**

I. Introduction

In recent years, significant progress has been made on the understanding of the theoretical limits of wireless communication networks. In [1], the capacity of multiple multicast network (where every destination desires all messages) is approximated within a constant gap independent of SNR and of the realization of the channel coefficients. Also, for multiple flows over a single hop, new capacity approximations were obtained in the form of degrees of freedom (DoF), generalized degrees of freedom (GDoF), and $O(1)$ approximations [2]–[4]. Yet, the study of multiple flows over multiple hops remains largely unsolved. The $2 \times 2 \times 2$ Gaussian interference channel (IC) has received much attention recently, being one of the fundamental building blocks to characterize the DoFs of two-flow networks [5]. One natural approach is to consider this model as a cascade of two ICs. In [6], the authors apply the Han-Kobayashi scheme [7] for the first hop to split each message into private and common parts. Relays can cooperate using the shared information (i.e., common messages) for the second hop in order to improve the data rates. This approach is known to be highly suboptimal at high signal-to-noise ratios (SNRs) since two-user IC can only achieve 1 DoF. In [8], Cadambe and Jafar show that $\frac{4}{3}$ DoF is achievable by viewing each hop as an X-channel. This is accomplished using the *interference alignment* scheme for each hop. Recently, the optimal DoF was obtained in [9] using *aligned interference neutralization*, which appropriately combines interference alignment and interference neutralization. Also, the $K \times K \times K$ Gaussian IC was recently studied in [10], where it is shown that the $K \times K$ MIMO cut-set upper bound (equal to K) can be effectively achieved using *aligned network diagonalization*.

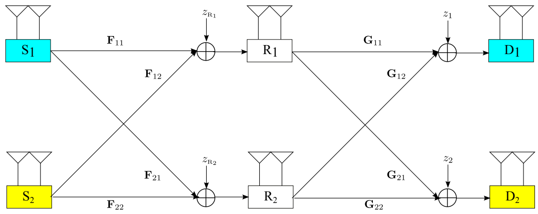

Fig. 1. $2 \times 2 \times 2$ MIMO Gaussian interference channel.

We consider the $2 \times 2 \times 2$ multiple-input multiple-output (MIMO) IC as shown in Fig. 1 consisting of two sources, two relays, and two destinations. All nodes have M multiple antennas. Each source k has a message for its intended destination k, for $k = 1, 2$. In the first hop, a block of n channel uses of the discrete-time complex MIMO IC is described by

$$\begin{bmatrix} \underline{\mathbf{Y}}_{\mathrm{R}_1} \\ \underline{\mathbf{Y}}_{\mathrm{R}_2} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_1 \\ \underline{\mathbf{X}}_2 \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{Z}}_{\mathrm{R}_1} \\ \underline{\mathbf{Z}}_{\mathrm{R}_2} \end{bmatrix} \quad (1)$$

where the matrices $\underline{\mathbf{X}}_k$ and $\underline{\mathbf{Y}}_{\mathrm{R}_k}$ contain, arranged by rows, the source channel input sequences $\underline{\mathbf{x}}_{k,\ell} \in \mathbb{C}^{1\times n}$ and the relay channel output sequences $\underline{\mathbf{y}}_{\mathrm{R}_{k,\ell}} \in \mathbb{C}^{1\times n}$. In the second hop, a block of n channel uses of the discrete-time complex MIMO IC is described by

$$\begin{bmatrix} \underline{\mathbf{Y}}_1 \\ \underline{\mathbf{Y}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_{\mathrm{R}_1} \\ \underline{\mathbf{X}}_{\mathrm{R}_2} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{Z}}_1 \\ \underline{\mathbf{Z}}_2 \end{bmatrix} \quad (2)$$

where the matrices $\underline{\mathbf{X}}_{\mathrm{R}_k}$ and $\underline{\mathbf{Y}}_k$ contain, arranged by rows, the relay channel input sequences $\underline{\mathbf{x}}_{\mathrm{R}_{k,\ell}} \in \mathbb{C}^{1\times n}$ and the destination channel output sequences $\underline{\mathbf{y}}_{k,\ell} \in \mathbb{C}^{1\times n}$. The matrix $\underline{\mathbf{Z}}_k$ (or $\underline{\mathbf{Z}}_{\mathrm{R}_k}$) contains i.i.d. Gaussian noise samples $\sim \mathcal{CN}(0, 1)$. We assume that the elements of $\mathbf{F}_{jk}$ and $\mathbf{G}_{jk}$ are drawn i.i.d. according to a continuous distribution (i.e., Gaussian distribution). The channel matrices are assumed to be constant over the whole block of length n and known to all nodes, and we consider a total power constraint equal to P_{sum} at each transmitter (both sources and relays).

Our Contribution: We present a novel lattice strategy named Precoded Compute-and-Forward (PCoF) with Channel Integer Alignment (CIA). CoF makes use of lattice codes so that each receiver can reliably decode a linear combination with integer coefficients of the interfering codewords. Thanks to the fact that lattices are modules over the ring of integers, the linear combinations translates directly into a linear combination of messages over a suitable finite field. In this way, each hop is transformed into a *deterministic* finite-field IC. The end-to-end interferences in the finite field domain are eliminated by distributed precoding (over finite field) at relays. Using this framework, we characterize a *symmetric* sum rate and prove that sum DoF of $(2M-1)$ is achievable by lattice coding (this DoF result is proven in [9] without resorting to CoF and lattice coding). Further, we use the lattice codes algebraic structure in order to obtain also good performance at finite SNRs. We use *integer-forcing receiver* (IFR) of [11] in order to minimize the impact of noise boosting at the receivers, and *integer-forcing beamforming* (IFB), proposed by the authors in [12], in order to minimize the power penalty at the transmitters. We provide numerical results showing that PCoF with CIA outperforms time-sharing even at reasonably moderate SNR, with increasing performance gain as SNR increases.

II. Preliminaries

In this section we provide some basic definitions and results that will be extensively used in the sequel.

A. Nested Lattice Codes

Let $\mathbb{Z}[j]$ be the ring of Gaussian integers and p be a prime. Let $\oplus$ denote the addition over $\mathbb{F}_q$ with $q = p^2$, and let $g : \mathbb{F}_q \to \mathbb{C}$ be the natural mapping of $\mathbb{F}_q$ onto $\{a + jb : a, b \in \mathbb{Z}_p\} \subset \mathbb{C}$. We recall the nested lattice code construction given in [13]. Let $\Lambda = \{\underline{\boldsymbol{\lambda}} = \underline{\mathbf{z}}\mathbf{T} : \underline{\mathbf{z}} \in \mathbb{Z}^n[j]\}$ be a lattice in $\mathbb{C}^n$, with full-rank generator matrix $\mathbf{T} \in \mathbb{C}^{n\times n}$. Let $\mathcal{C} = \{\underline{\mathbf{c}} = \underline{\mathbf{w}}\mathbf{G} : \mathbf{w} \in \mathbb{F}_q^r\}$ denote a linear code over $\mathbb{F}_q$ with block length n and dimension r, with generator matrix $\mathbf{G}$. The lattice Λ_1 is defined through "construction A" (see [14] and references therein) as

$$\Lambda_1 = p^{-1} g(\mathcal{C})\mathbf{T} + \Lambda, \tag{3}$$

where $g(\mathcal{C})$ is the image of $\mathcal{C}$ under the mapping g (applied component-wise). It follows that $\Lambda \subseteq \Lambda_1 \subseteq p^{-1}\Lambda$ is a chain of nested lattices, such that $|\Lambda_1/\Lambda| = p^{2r}$ and $|p^{-1}\Lambda/\Lambda_1| = p^{2(n-r)}$.

For a lattice Λ and $\underline{\mathbf{r}} \in \mathbb{C}^n$, we define the lattice quantizer $Q_\Lambda(\underline{\mathbf{r}}) = \operatorname{argmin}_{\underline{\boldsymbol{\lambda}}\in\Lambda} \|\underline{\mathbf{r}} - \underline{\boldsymbol{\lambda}}\|^2$, the Voronoi region $\mathcal{V}_\Lambda = \{\underline{\mathbf{r}} \in \mathbb{C}^n : Q_\Lambda(\underline{\mathbf{r}}) = \underline{\mathbf{0}}\}$ and $[\underline{\mathbf{r}}] \mod \Lambda = \underline{\mathbf{r}} - Q_\Lambda(\underline{\mathbf{r}})$. For Λ and Λ_1 given above, we define the lattice code $\mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda$ with rate $R = \frac{1}{n}\log|\mathcal{L}| = \frac{r}{n}\log q$. Construction A provides a *natural labeling* of the codewords of $\mathcal{L}$ by the information messages $\underline{\mathbf{w}} \in \mathbb{F}_q^r$. Notice that the set $p^{-1}g(\mathcal{C})\mathbf{T}$ is a *system of coset representatives* of the cosets of Λ in Λ_1. Hence, the natural labeling function $f : \mathbb{F}_q^r \to \mathcal{L}$ is defined by $f(\underline{\mathbf{w}}) = p^{-1}g(\underline{\mathbf{w}}\mathbf{G})\mathbf{T} \mod \Lambda$.

B. Compute-and-Forward and Integer-Forcing

We recall here the CoF scheme of [13] applied to a particular case of Gaussian MIMO channel with joint processing of the receiver antennas and independent lattice coding at each transmit antenna. Our reference model is given by

$$\underline{\mathbf{Y}} = \mathbf{HC}\underline{\mathbf{X}} + \underline{\mathbf{Z}} \tag{4}$$

where $\mathbf{H} \in \mathbb{C}^{M\times M}$, $\mathbf{C} \in \mathbb{Z}[j]^{M\times 2M}$, $\underline{\mathbf{X}} \in \mathbb{C}^{2M\times n}$, and $\underline{\mathbf{Z}}$ contains i.i.d. Gaussian noise samples $\sim \mathcal{CN}(0,1)$. For $k = 1, \ldots, 2M$, each k-th independent message $\underline{\mathbf{w}}_k \in \mathbb{F}_q^r$ is encoded as $\underline{\mathbf{t}}_k = f(\underline{\mathbf{w}}_k)$ of the same lattice code $\mathcal{L}$ of rate R and mapped to the channel input sequence

$$\underline{\mathbf{x}}_k = [\underline{\mathbf{t}}_k + \underline{\mathbf{d}}_k] \mod \Lambda \tag{5}$$

where the *dithering sequences* $\{\underline{\mathbf{d}}_k\}$ are mutually independent, uniformly distributed over $\mathcal{V}_\Lambda$, and known to the receiver. The encoded sequences $\{\underline{\mathbf{x}}_k\}$ are arranged by rows into the transmit signal matrix $\underline{\mathbf{X}}$. We also define the matrix $\underline{\mathbf{T}}$ of dimensions $2M\times n$ containing $\{\underline{\mathbf{t}}_k\}$ arranged by rows, and the dithering matrix $\underline{\mathbf{D}}$ with rows $\{\underline{\mathbf{d}}_k\}$. Channel matrices in the form $\mathbf{HC}$ as in (4) will appear in this paper as a consequence of *channel integer alignment*, explicitly designed such that $[\mathbf{C}\underline{\mathbf{T}}] \mod \Lambda$ has lattice codewords arranged by rows.

The decoder's goal is to recover M integer linear combinations of the $2M$ lattice codewords, given by the rows $\underline{\mathbf{s}}_\ell$ of the matrix $\underline{\mathbf{S}} = [\mathbf{B}^{\mathsf{H}}\mathbf{C}\underline{\mathbf{T}}] \mod \Lambda$, for some integer matrix $\mathbf{B} \in \mathbb{Z}[j]^{M\times M}$. Letting $\mathbf{b}_\ell$ denote the ℓ-th column of $\mathbf{B}$, the receiver computes

$$\begin{aligned}\hat{\underline{\mathbf{y}}}_\ell &= \left[\boldsymbol{\alpha}_\ell^{\mathsf{H}}\underline{\mathbf{Y}} - \mathbf{b}_\ell^{\mathsf{H}}\mathbf{C}\underline{\mathbf{D}}\right] \mod \Lambda \\ &= \left[\underline{\mathbf{s}}_\ell + \underline{\mathbf{z}}_{\text{eff}}(\mathbf{HC}, \mathbf{b}_\ell, \boldsymbol{\alpha}_\ell)\right] \mod \Lambda \end{aligned} \tag{6}$$

where $\boldsymbol{\alpha}_\ell \in \mathbb{C}^{M\times 1}$ and $\underline{\mathbf{z}}_{\text{eff}}(\mathbf{HC}, \mathbf{b}_\ell, \boldsymbol{\alpha}_\ell)$ is the ℓ-th effective noise sequence, distributed as:

$$\underbrace{(\boldsymbol{\alpha}_\ell^{\mathsf{H}}\mathbf{H} - \mathbf{b}_\ell^{\mathsf{H}})\mathbf{C}\underline{\mathbf{D}}}_{\text{non-integer penalty}} + \underbrace{\boldsymbol{\alpha}_\ell^{\mathsf{H}}\underline{\mathbf{Z}}}_{\text{Gaussian noise}} . \tag{7}$$

Choosing $\boldsymbol{\alpha}_\ell^{\mathsf{H}} = \mathbf{b}_\ell^{\mathsf{H}}\mathbf{H}^{-1}$, the variance of the effective noise is given by $\sigma^2_{\text{eff},\ell} = \|(\mathbf{H}^{-1})^{\mathsf{H}}\mathbf{b}_\ell\|^2$. This choice is referred to in [11] as the *exact* Integer Forcing Receiver (IFR). In this way, the non-integer penalty of CoF is completely eliminated. More in general, the decoding performance can be improved especially at low SNR by minimizing the effective noise variance with respect to $\boldsymbol{\alpha}_\ell$ for given $\mathbf{b}_\ell$. This yields

$$\sigma^2_{\text{eff},\ell} = \mathbf{b}_\ell^{\mathsf{H}}\mathbf{C}(\mathsf{SNR}^{-1}\mathbf{I} + \mathbf{C}^{\mathsf{H}}\mathbf{H}^{\mathsf{H}}\mathbf{HC})^{-1}\mathbf{C}^{\mathsf{H}}\mathbf{b}_\ell. \tag{8}$$

Since $\mathbf{b}_\ell$ and $\mathbf{C}$ are integer-valued, $\underline{\mathbf{s}}_\ell = [\mathbf{b}_\ell^{\mathsf{H}}\mathbf{C}\underline{\mathbf{T}}] \mod \Lambda$ is a codeword of $\mathcal{L}$. From [13], we know that by applying lattice decoding to $\hat{\underline{\mathbf{y}}}$ given in (6) there exist sequences of lattice codes $\mathcal{L}$ of rate R and increasing block length n such that $\underline{\mathbf{s}}_\ell$ can be decoded successfully with arbitrary high probability as $n \to \infty$, provided that

$$R < \log^+(\mathsf{SNR}/\sigma^2_{\text{eff},\ell}) \tag{9}$$

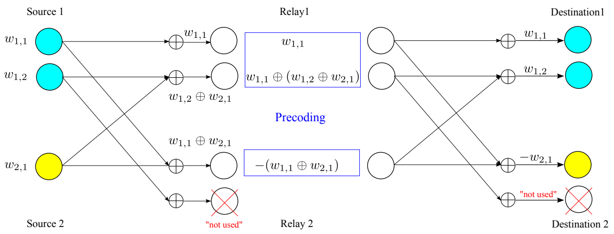

Fig. 2. A *deterministic* $2 \times 2 \times 2$ finite-field interference channel.

where $\log^+(x) \triangleq \max\{\log(x), 0\}$. All M linear combinations can be reliably decoded if

$$R \leq \min_{\ell}\{\log^+(\mathsf{SNR}/\sigma^2_{\text{eff},\ell})\} \triangleq R(\mathbf{H}, \mathbf{B}, \mathsf{SNR}). \tag{10}$$

Using the lattice encoding linearity, the corresponding M linear combinations over $\mathbb{F}_q$ for the messages are obtained as

$$\begin{aligned} \underline{\mathbf{U}} &= g^{-1}([\mathbf{B}^{\mathsf{H}}] \mod p\mathbb{Z}[j])g^{-1}([\mathbf{C}] \mod p\mathbb{Z}[j])\underline{\mathbf{W}} \\ &\overset{(a)}{=} [\mathbf{B}^{\mathsf{H}}]_q[\mathbf{C}]_q\underline{\mathbf{W}}, \end{aligned} \tag{11}$$

where we use the notation $[\mathbf{B}^{\mathsf{H}}]_q \triangleq g^{-1}([\mathbf{B}^{\mathsf{H}}] \mod p\mathbb{Z}[j])$.

III. PCoF with CIA

In this section we present a novel lattice strategy called Precoded CoF (PCoF) with Channel Integer Alignment (CIA). This consists of two phases: 1) Using the CoF framework, we convert the two-user MIMO IC into a *deterministic* finite-field IC. 2) A linear precoding scheme is used over finite-field to eliminate the end-to-end interferences (see Fig. 2). The main performance bottleneck of CoF consists of the non-integer penalty, which ultimately limits the performance of CoF at high SNR [15]. To overcome this bottleneck, we employ CIA in order to create an "aligned" channel matrix for which exact integer forcing is possible, similarly to what done in Section II-B. Specifically, we use alignment precoding matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ at the two sources such that

$$\left[\begin{array}{cc} \mathbf{F}_{k1}\mathbf{V}_1 & \mathbf{F}_{k2}\mathbf{V}_2 \end{array}\right] = \mathbf{H}_{\mathsf{R}_k}\mathbf{C}_{\mathsf{R}_k}, \tag{12}$$

where $\mathbf{H}_{\mathsf{R}_k} \in \mathbb{C}^{M\times M}$ and $\mathbf{C}_{\mathsf{R}_k} \in \mathbb{Z}[j]^{M\times 2M}$. Linear precoding over the complex field may produce a power-penalty due to the non-unitary nature of the alignment matrices, and this can degrade the performance at finite SNR. In order to counter this effect, we use *Integer Forcing Beamforming* (IFB) [12]. The main idea is that $\mathbf{V}_k$ can be pre-multiplied (from the left) by some appropriately chosen full rank integer matrix $\mathbf{A}_k$ since its effect can be undone by precoding over $\mathbb{F}_q$ using $[\mathbf{A}_k]_q$. Then, we can optimize the integer matrix in order to minimize the power penalty of alignment. The details of the coding scheme are given in the following sections.

A. CoF framework based on CIA

This section shows how to turn any two-user MIMO IC into a deterministic finite-field IC using the CoF framework. We focus on the first-hop of our $2 \times 2 \times 2$ IC since the same coding scheme is straightforwardly applied to the second hop. Consider the MIMO IC in (1). Let $\{\underline{\mathbf{w}}_{1,\ell} \in \mathbb{F}_q^r : \ell = 1, \ldots, M\}$ denote the messages of source 1 and $\{\underline{\mathbf{w}}_{2,\ell} \in \mathbb{F}_q^r : \ell = 1, \ldots, M-1\}$ denote the messages of source 2. Also, we let $\mathbf{V}_1 = [\mathbf{v}_{1,1} \cdots \mathbf{v}_{1,M}] \in \mathbb{C}^{M\times M}$ and $\mathbf{V}_2 = [\mathbf{v}_{2,1} \cdots \mathbf{v}_{2,M-1}] \in \mathbb{C}^{M\times M-1}$ denote the precoding matrices used at sources 1 and 2, respectively, chosen to satisfy the *alignment conditions*

$$\mathbf{F}_{11}\mathbf{v}_{1,\ell+1} = \mathbf{F}_{12}\mathbf{v}_{2,\ell} \tag{13}$$

$$\mathbf{F}_{21}\mathbf{v}_{1,\ell} = \mathbf{F}_{22}\mathbf{v}_{2,\ell} \tag{14}$$

for $\ell = 1, \ldots, M-1$. The feasibility of the above conditions is shown in [9] for any integer $M \geq 2$, almost surely with respect to the continuously distributed channel matrices. Let $\mathbf{A}_1 \in \mathbb{Z}[j]^{M\times M}$ and $\mathbf{A}_2 \in \mathbb{Z}[j]^{(M-1)\times(M-1)}$ denote full rank integer matrices (the optimization of which in order to minimize the transmit power penalty is discussed in Section IV). The transmitters make use of the same nested lattice codebook $\mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda$, where Λ has the second moment $\sigma^2_\Lambda = \mathsf{SNR}$. Then, CoF based on CIA proceeds as follows.

Encoding: Each source k precodes its messages over $\mathbb{F}_q$ as

$$\underline{\mathbf{W}}'_k = [\mathbf{A}_k]_q^{-1}\underline{\mathbf{W}}_k, \quad k = 1, 2. \tag{15}$$

Then, the precoded messages (rows of $\underline{\mathbf{W}}'_k$) are encoded using the nested lattice codes as $\underline{\mathbf{t}}'_{k,\ell} = f(\underline{\mathbf{w}}'_{k,\ell})$. Finally, the channel input sequences are given by the rows of:

$$\underline{\mathbf{X}}''_k = \mathbf{V}_k\mathbf{A}_k\underline{\mathbf{X}}'_k \tag{16}$$

where $\underline{\mathbf{X}}'_k$ has rows $\underline{\mathbf{x}}'_{k,\ell} = [\underline{\mathbf{t}}'_{k\ell} + \underline{\mathbf{d}}_{k,\ell}] \mod \Lambda$. Due to the sum-power constraint equal to P_{sum} at each source, the second moment of coarse lattice (i.e., SNR) must satisfy

$$\mathsf{SNR} \cdot \text{tr}(\mathbf{V}_k\mathbf{A}_k\mathbf{A}_k^{\mathsf{H}}\mathbf{V}_k^{\mathsf{H}}) \leq P_{\text{sum}} \text{ for } k = 1, 2. \tag{17}$$

Then, we can choose:

$$\mathsf{SNR} = \min\{P_{\text{sum}}/\text{tr}(\mathbf{V}_k\mathbf{A}_k\mathbf{A}_k^{\mathsf{H}}\mathbf{V}_k^{\mathsf{H}}) : k = 1, 2\}. \tag{18}$$

Decoding: Relay 1 observes:

$$\begin{aligned} \underline{\mathbf{Y}}_{\mathsf{R}_1} &= \mathbf{F}_{11}\underline{\mathbf{X}}''_1 + \mathbf{F}_{12}\underline{\mathbf{X}}''_2 + \underline{\mathbf{Z}}_{\mathsf{R}_1} \\ &\overset{(a)}{=} \underbrace{\mathbf{F}_{11}\mathbf{V}_1}_{\triangleq \mathbf{H}_{\mathsf{R}_1}}\left[\begin{array}{cc} \mathbf{I}_{M\times M} & \mathbf{C}_{12} \end{array}\right]\left[\begin{array}{c} \mathbf{A}_1\underline{\mathbf{X}}'_1 \\ \mathbf{A}_2\underline{\mathbf{X}}'_2 \end{array}\right] + \underline{\mathbf{Z}}_{\mathsf{R}_1} \\ &= \mathbf{H}_{\mathsf{R}_1}\mathbf{C}_{\mathsf{R}_1}\left[\begin{array}{c} \underline{\mathbf{X}}'_1 \\ \underline{\mathbf{X}}'_2 \end{array}\right] + \underline{\mathbf{Z}}_{\mathsf{R}_1} \end{aligned} \tag{19}$$

where (a) follows from the fact that the precoding vectors satisfy the alignment conditions in (14) and $\mathbf{C}_{\mathsf{R}_1} = \left[\begin{array}{cc} \mathbf{A}_1 & \mathbf{C}_{12}\mathbf{A}_2 \end{array}\right]$ with

$$\mathbf{C}_{12} \triangleq \left[\begin{array}{c} 0_{1\times M-1} \\ \mathbf{I}_{M-1\times M-1} \end{array}\right]. \tag{20}$$

Similarly, relay 2 observes the aligned signals:

$$\underline{\mathbf{Y}}_{\mathsf{R}_2} = \mathbf{H}_{\mathsf{R}_2}\mathbf{C}_{\mathsf{R}_2}\left[\begin{array}{c} \underline{\mathbf{X}}'_1 \\ \underline{\mathbf{X}}'_2 \end{array}\right] + \underline{\mathbf{Z}}_{\mathsf{R}_2} \tag{21}$$

where $\mathbf{C}_{\text{R}_2} = [\ \mathbf{A}_1 \quad \mathbf{C}_{22}\mathbf{A}_2\ ]$ with

$$\mathbf{C}_{22} \triangleq \begin{bmatrix} \mathbf{I}_{M-1\times M-1} \\ 0_{1\times M-1} \end{bmatrix}. \quad (22)$$

Notice that the channel matrices in (19) and (21) follow the particular form in (4). Following the CoF framework in (10) and (11), if $R \leq R(\mathbf{H}_{\text{R}_k}\mathbf{C}_{\text{R}_k}, \mathbf{B}_{\text{R}_k}, \mathsf{SNR})$, the relay k can decode the M linear combinations with full rank coefficients matrix $\mathbf{B}_{\text{R}_k}$:

$$\begin{aligned} \underline{\mathbf{U}}_k &= [\mathbf{B}_{\text{R}_k}]_q[\mathbf{C}_{\text{R}_k}]_q \begin{bmatrix} \underline{\mathbf{W}}_1' \\ \underline{\mathbf{W}}_2' \end{bmatrix} \\ &= [\mathbf{B}_{\text{R}_k}]_q \left[\ [\mathbf{A}_1]_q \quad [\mathbf{C}_{k2}]_q[\mathbf{A}_2]_q\ \right] \begin{bmatrix} \underline{\mathbf{W}}_1' \\ \underline{\mathbf{W}}_2' \end{bmatrix} \\ &\overset{(a)}{=} [\mathbf{B}_{\text{R}_k}]_q \left[\ \mathbf{I}_{M\times M} \quad [\mathbf{C}_{k2}]_q\ \right] \begin{bmatrix} \underline{\mathbf{W}}_1 \\ \underline{\mathbf{W}}_2 \end{bmatrix} \end{aligned}$$

where (a) is due to the precoding over $\mathbb{F}_q$ in (15). Let $\hat{\underline{\mathbf{W}}}_1 = [\mathbf{B}_{\text{R}_1}]_q^{-1}\underline{\mathbf{U}}_1$ and $\hat{\underline{\mathbf{W}}}_2$ be the first $(M-1)$ rows of $[\mathbf{B}_{\text{R}_2}]_q^{-1}\underline{\mathbf{U}}_2$. The mapping between $\{\underline{\mathbf{W}}_1, \underline{\mathbf{W}}_2\}$ and $\{\hat{\underline{\mathbf{W}}}_1, \hat{\underline{\mathbf{W}}}_2\}$ defines a *deterministic* finite-field IC given by:

$$\begin{bmatrix} \hat{\underline{\mathbf{W}}}_1 \\ \hat{\underline{\mathbf{W}}}_2 \end{bmatrix} = \mathbf{Q}_{\text{sys}} \begin{bmatrix} \underline{\mathbf{W}}_1 \\ \underline{\mathbf{W}}_2 \end{bmatrix} \quad (23)$$

where the so-called *system matrix* is defined by

$$\mathbf{Q}_{\text{sys}} \triangleq \begin{bmatrix} \mathbf{I}_{M\times M} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{I}_{(M-1)\times(M-1)} \end{bmatrix} \quad (24)$$

where

$$\begin{aligned} \mathbf{Q}_{12} &= \begin{bmatrix} 0_{1\times(M-1)} \\ \mathbf{I}_{(M-1)\times(M-1)} \end{bmatrix} \\ \mathbf{Q}_{21} &= [\ \mathbf{I}_{(M-1)\times(M-1)} \quad 0_{(M-1)\times 1}\ ]. \end{aligned}$$

Notice that the system matrix is fixed and independent of the channel matrices, since it is determined only by the alignment conditions.

B. Linear precoding over deterministic network

Eq. (23) defines the first hop of the deterministic finite-field $2\times 2\times 2$ IC. In the second hop, the relay k uses a precoded version of decoded linear combinations $\underline{\mathbf{W}}_{\text{R}_k} = \mathbf{M}_k\hat{\underline{\mathbf{W}}}_k$ as its messages. Operating in a similar way as for the first hop, the second hop deterministic finite-field IC is given by

$$\begin{bmatrix} \hat{\underline{\mathbf{W}}}_{\text{R}_1} \\ \hat{\underline{\mathbf{W}}}_{\text{R}_2} \end{bmatrix} = \mathbf{Q}_{\text{sys}} \begin{bmatrix} \underline{\mathbf{W}}_{\text{R}_1} \\ \underline{\mathbf{W}}_{\text{R}_2} \end{bmatrix}. \quad (25)$$

Concatenating (23) and (25), the end-to-end deterministic IC over finite-field is described by

$$\begin{bmatrix} \hat{\underline{\mathbf{W}}}_{\text{R}_1} \\ \hat{\underline{\mathbf{W}}}_{\text{R}_2} \end{bmatrix} = \mathbf{Q}_{\text{sys}} \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{bmatrix} \mathbf{Q}_{\text{sys}} \begin{bmatrix} \underline{\mathbf{W}}_1 \\ \underline{\mathbf{W}}_2 \end{bmatrix}. \quad (26)$$

Lemma 1 shows that the linear combinations decoded at destination 1 are equal to its desired messages and are equal to the messages with a change of sign (multiplication by -1 in the finite field) at destination 2 (see Fig. 2).

Lemma 1: Choosing precoding matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ as

$$\begin{aligned} \mathbf{M}_1 &= (\mathbf{I}_{M\times M} \oplus (-\mathbf{Q}_{12}\mathbf{Q}_{21}))^{-1} && (27) \\ \mathbf{M}_2 &= -(\mathbf{I}_{(M-1)\times(M-1)} \oplus (-\mathbf{Q}_{21}\mathbf{Q}_{12}))^{-1} && (28) \end{aligned}$$

the end-to-end system matrix becomes a diagonal matrix:

$$\mathbf{Q}_{\text{sys}} \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{bmatrix} \mathbf{Q}_{\text{sys}} = \begin{bmatrix} \mathbf{I}_{M\times M} & 0 \\ 0 & -\mathbf{I}_{(M-1)\times(M-1)} \end{bmatrix}.$$

Proof: See the long version of this paper [16]. ■

Based on the above, we proved the following:

Theorem 1: For the $2\times 2\times 2$ MIMO IC, PCoF with CIA can achieve the *symmetric* sum rate of $(2M-1)R$ with all message of the same rate given by

$$R = \min_{k=1,2}\{R(\mathbf{H}_{\text{R}_k}\mathbf{C}_{\text{R}_k}, \mathbf{B}_{\text{R}_k}, \mathsf{SNR}), R(\mathbf{H}_k\mathbf{C}_k, \mathbf{B}_k, \mathsf{SNR}')\}$$

for any full rank integer matrices $\mathbf{A}_1, \mathbf{A}_{\text{R}_1} \in \mathbb{Z}[j]^{M\times M}$, $\mathbf{A}_2, \mathbf{A}_{\text{R}_2} \in \mathbb{Z}[j]^{(M-1)\times(M-1)}$, and $\mathbf{B}_{\text{R}_k}, \mathbf{B}_k \in \mathbb{Z}[j]^{M\times M}, k = 1, 2$, and any alignment precoding matrices $\mathbf{V}_k, \mathbf{V}_{\text{R}_k}$ satisfying the *alignment conditions* in (14), where

$$\begin{aligned} \mathbf{H}_{\text{R}_k} &= \mathbf{F}_{k1}\mathbf{V}_1, \quad \mathbf{H}_k = \mathbf{G}_{k1}\mathbf{V}_{\text{R}_1} \\ \mathbf{C}_{\text{R}_k} &= [\ \mathbf{A}_1 \quad \mathbf{C}_{k2}\mathbf{A}_2\ ], \mathbf{C}_k = [\ \mathbf{A}_{\text{R}_1} \quad \mathbf{C}_{k2}\mathbf{A}_{\text{R}_2}\ ] \\ \mathsf{SNR} &= \min\{P_{\text{sum}}/\text{tr}(\mathbf{V}_k\mathbf{A}_k\mathbf{A}_k^{\mathsf{H}}\mathbf{V}_k^{\mathsf{H}}) : k = 1, 2\} \\ \mathsf{SNR}' &= \min\{P_{\text{sum}}/\text{tr}(\mathbf{V}_{\text{R}_k}\mathbf{A}_{\text{R}_k}\mathbf{A}_{\text{R}_k}^{\mathsf{H}}\mathbf{V}_{\text{R}_k}^{\mathsf{H}}) : k = 1, 2\}. \end{aligned}$$

■

Showing that R grows as $\log \mathsf{SNR}$ yields:

Corollary 1: PCoF with CIA achieves the sum DoF equal to $(2M-1)$ for the $2\times 2\times 2$ MIMO IC, when all nodes have M multiple antennas.

Proof: See the long version of this paper [16]. ■

IV. Optimization of Achievable Rates

We optimize the integer matrices in Theorem 1 by assuming that the precoding matrices $\mathbf{V}_k, \mathbf{V}_{\text{R}_k}$ are given. The dimensions of integer matrices can be either $M\times M$ or $(M-1)\times(M-1)$. Since this does not change the optimization problem, we will just consider the dimension M. The power-penalty optimization with respect to $\mathbf{A}$ takes on the form:

$$\begin{aligned} &\text{argmin} && \text{tr}\left(\mathbf{V}\mathbf{A}\mathbf{A}^{\mathsf{H}}\mathbf{V}^{\mathsf{H}}\right) = \sum_{\ell=1}^{M} \|\mathbf{V}\mathbf{a}_\ell\|^2 \\ &\text{subject to} && \mathbf{A} \text{ is full rank over } \mathbb{Z}[j] \qquad (29) \end{aligned}$$

where $\mathbf{a}_\ell$ denotes the ℓ-th column of $\mathbf{A}$. Also, the minimization of the effective noise variance with respect to $\mathbf{B}$ takes on the form:

$$\begin{aligned} &\text{argmin} && \max_\ell\{\|\mathbf{L}^{\mathsf{H}}\mathbf{b}_\ell\|^2\} \\ &\text{subject to} && \mathbf{B} \text{ is full rank over } \mathbb{Z}[j] \qquad (30) \end{aligned}$$

where $\mathbf{L}$ denotes a square-root factor of $(\mathsf{SNR}^{-1}\mathbf{I} + \mathbf{H}^{\mathsf{H}}\mathbf{H})^{-1}$, $\mathbf{H}$ denotes an aligned channel matrix, and $\mathbf{b}_\ell$ is the ℓ-th column of $\mathbf{B}$. We notice that problem (29) (resp., (30)) is equivalent to finding a reduced basis for the lattice generated by $\mathbf{V}$ (resp., $\mathbf{L}^{\mathsf{H}}$). In particular, the reduced basis takes on

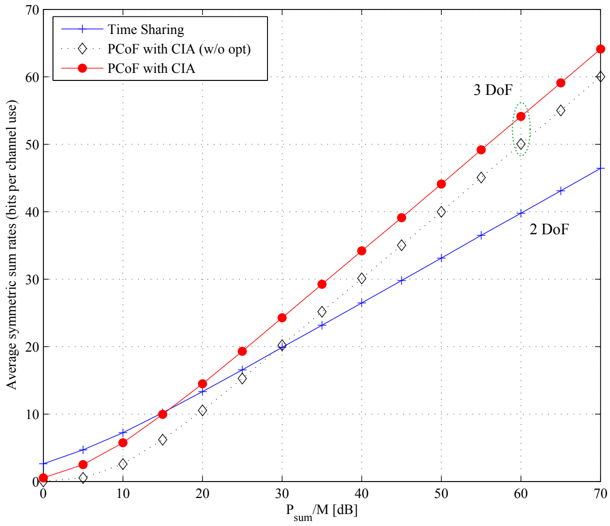

Fig. 3. Performance comparison of PCoF with CIA and time-sharing with respect to ergodic symmetric sum rates for $2 \times 2 \times 2$ MIMO IC with $M = 2$.

the form $\mathbf{VU}$ where $\mathbf{U}$ is a unimodular matrix over $\mathbb{Z}[j]$. Hence, choosing $\mathbf{A} = \mathbf{U}$ yields the minimum power-penalty subject to the full rank condition in (29). In practice we used the (complex) LLL algorithm, with refinement of the LLL reduced basis approximation by Phost or Schnorr-Euchner lattice search (see the long version for details [16]).

A. Numerical Results: Rayleigh fading

We evaluate the performance of PCoF with CIA in terms of its average achievable sum rates. We computed the *ergodic* sum rates by Monte Carlo averaging with respect to the channel realizations with i.i.d. Rayleigh fading $\sim \mathcal{CN}(0,1)$. For comparison, we considered the performance of *time-sharing* where IFR is used for each $M \times M$ MIMO IC. We used the IFR since it is known to almost achieve the performance of joint maximum likelihood receiver [11] and has a similar complexity with PCoF with CIA. In this case, an achievable symmetric sum rate is obtained as

$$R = \min\{R(\mathbf{F}_{11}, \mathbf{B}_1, P_{\text{sum}}), R(\mathbf{G}_{11}, \mathbf{B}_2, P_{\text{sum}})\} \quad (31)$$

for any full-rank integer matrices $\mathbf{B}_1$ and $\mathbf{B}_2$. The integer matrices are optimized in the same manner of PCoF with CIA. Also, in this case, we used the $2P_{\text{sum}}$ for power-constraint since each transmitter is active on every odd (or even) time slot. For PCoF with CIA, we need to find precoding matrices for satisfying the alignment condition in (14). For $M = 2$, the conditions are given by

$$\mathbf{F}_{11}\mathbf{v}_{1,2} = \mathbf{F}_{12}\mathbf{v}_{2,1} \text{ and } \mathbf{F}_{21}\mathbf{v}_{1,1} = \mathbf{F}_{22}\mathbf{v}_{2,1}. \quad (32)$$

For the simulation, we used the following precoding matrices to satisfy the above conditions:

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{F}_{21}^{-1}\mathbf{F}_{22}\mathbf{1} & \mathbf{F}_{11}^{-1}\mathbf{F}_{12}\mathbf{1} \end{bmatrix} \text{ and } \mathbf{v}_{2,1} = \mathbf{1}.$$

Also, the same construction method is used for the second hop. Since source 1 (or relay 1) transmits one more stream than source 2 (or relay 2), the former always requires higher transmission power. In order to efficiently satisfy the average power-constraint, the role of sources 1 and 2 (equivalently, relays 1 and 2) is alternatively reversed in successive time slots. In Fig. 3, we observe that PCoF with CIA can have the SNR gain about 5 dB by optimizing the integer matrices for IFR and IFB, comparing with simply using identity matrices. Also, PCoF with CIA provides a higher sum rate than time-sharing if SNR $\geq$ 15 dB, and its gain over time-sharing increases with SNR, showing that in this case the DoF result matters also at finite SNR.

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