Fast-Decodable MIDO Codes with Large Coding Gain

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***Abstract*—In this paper, a new method is proposed to obtain full-diversity, rate-2 (rate of 2 complex symbols per channel use) space-time block codes (STBCs) that are full-rate for multiple input, double output (MIDO) systems. Using this method, rate-2 STBCs for 4×2, 6×2, 8×2 and 12×2 systems are constructed and these STBCs are fast ML-decodable, have large coding gains, and STBC-schemes consisting of these STBCs have a non-vanishing determinant (NVD) so that they are DMT-optimal for their respective MIDO systems.**

I. Introduction and Background

Space-time block codes (STBCs) for asymmetric MIMO systems, where the number of receive antennas n_r is less than the number of transmit antennas n_t, have evoked a lot of interest in recent times. Asymmetric MIMO systems find application, for example, in the downlink transmission from a base station to a mobile phone, and in digital video broadcasting (DVB) from a TV broadcasting station to portable TV devices. Of particular interest is the 4×2 MIDO system for which a slew of rate-2 STBCs have been developed [1]-[7], with the particular aim of allowing fast-decodability (see Definition 2), a term that was first coined in [1]. Among these codes, those in [3]-[7] have been shown to have a minimum determinant that is bounded away from zero irrespective of the size of the signal constellation and hence, STBC-schemes that consist of these codes have the NVD property and are diversity-multiplexing gain tradeoff (DMT)-optimal for the 4×2 MIDO system [8]. A generalization of fast-decodable STBC construction for higher number of transmit antennas has been proposed in [3]. STBCs from nonassociative division algebras have also been proposed in [9].

The best performing code for the 4×2 MIDO system is the code of [2] which has the least ML-decoding complexity (of $\mathcal{O}(M^{4.5})$ for a square M-QAM constellation) and the best known normalized minimum determinant (see Definition 1) for 4-/16-QAM among comparable codes. However, this code was constructed using an ad hoc technique and had not been proven to have a non-vanishing determinant for arbitrary QAM constellations. In this paper, we propose a novel construction scheme to obtain rate-2 STBCs which have full-diversity, and STBC-schemes that employ these codes have the NVD property. These STBCs are obtained from nonassociative algebras. We then explicitly construct such STBCs for $n\times 2$ MIDO systems, $n=4,6,8,12$, and these codes are fast-decodable and have large normalized minimum determinants. The STBC constructed for 4 transmit antennas is shown to have the same algebraic structure, normalized minimum determinant, and ML-decoding complexity as that of the STBC of [2].

The paper is organized as follows. Section II gives the system model and some relevant definitions while Section III builds the theory needed to obtain rate-2 STBCs. Section IV deals with the construction of fast-decodable MIDO STBCs, and simulation results are given in Section V. Concluding remarks constitute Section VI.

Notation: Throughout the paper, bold, uppercase letters denote matrices. The determinant and Frobenius norm of $\mathbf{X}$ are denoted by $det(\mathbf{X})$ and $\|\mathbf{X}\|$, respectively. $\mathbf{I}$ and $\mathbf{O}$ denote the identity and the null matrix of appropriate dimensions. $\mathbb{Z}$ denotes the ring of rational integers while $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ denote the field of real, complex and rational numbers, respectively. $\mathbb{E}(X)$ denotes the expectation of the random variable X. Unless used as an index, a subscript or a superscript, i denotes $\sqrt{-1}$ and ω denotes the primitive third root of unity. For fields K and F, K/F denotes that K is an extension of F, and $[K:F]=m$ indicates that K is a finite extension of F of degree m. $Gal(K/F)$ denotes the Galois group of K/F. The elements 1 and 0 are understood to be the multiplicative identity and the additive identity, respectively, of the unit ring $\mathcal{R}$ in context.

Due to space constraint, the proofs of several claims have been omitted and provided instead in [10].

II. System Model and definitions

We consider an n_t transmit antenna, n_r receive antenna MIMO system ($n_t\times n_r$ system) with perfect channel-state information available at the receiver (CSIR) alone, the channel being quasi-static with Rayleigh fading. The system model is

$$\mathbf{Y}=\sqrt{\rho}\mathbf{H}\mathbf{S}+\mathbf{N}, \quad (1)$$

where $\mathbf{Y}\in\mathbb{C}^{n_r\times \mathrm{T}}$ is the received signal matrix, $\mathbf{S}\in\mathbb{C}^{n_t\times \mathrm{T}}$ is the codeword matrix that is transmitted over a block of T channel uses ($\mathrm{T}=n_t$ in this paper), $\mathbf{H}\in\mathbb{C}^{n_r\times n_t}$ and $\mathbf{N}\in\mathbb{C}^{n_r\times \mathrm{T}}$ are respectively the channel matrix and the noise matrix with entries independently and identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with zero mean and unit variance. The average signal-to-noise ratio

# Tx antennas	STBC $\mathcal{S}$	Constellation (average energy E)	$\delta_{min}(\mathcal{S})$	ML-decoding complexity (Worst case)
4	$\mathcal{S}_{4\times 2}$	QAM	$\frac{1}{25E^4}$	$\mathcal{O}\left(M^{4.5}\right)$
	Punctured[\$] perfect code [11]	QAM	$\frac{16}{1125E^4}$	$\mathcal{O}\left(M^{5.5}\right)$
	$\mathcal{C}_1$ [3, Sec. VIII-B]	QAM	$\frac{1}{25E^4}$	$\mathcal{O}\left(M^{6.5}\right)$
	A_4 [3, Sec. VIII-A]	QAM	Not Available	$\mathcal{O}\left(M^{5.5}\right)$
	Punctured $\mathcal{C}_4$ [14]	QAM	$\frac{1}{16E^4}$	$\mathcal{O}\left(M^{7}\right)$
6	$\mathcal{S}_{6\times 2}$	HEX	$\frac{1}{7^4E^6}$	$\mathcal{O}\left(M^{8.5}\right)$
	Punctured perfect code [11]	HEX	$\frac{1}{7^5E^6} \leq \delta_{min} \leq \frac{1}{7^4E^6}$	$\mathcal{O}\left(M^{11.5}\right)$
	Punctured $\mathcal{C}_6$ [14]	HEX	$\frac{1}{(3E)^6}$	$\mathcal{O}\left(M^{11.5}\right)$
	VHO-Code [3, Sec. X-C]	QAM	Not available[£]	$\mathcal{O}\left(M^{8.5}\right)$
	VHO-Code (Change of Basis)	QAM	Not available[¥]	$\mathcal{O}\left(M^{7}\right)$
8	$\mathcal{S}_{8\times 2}$	QAM	$\frac{1}{25(15)^4E^8}$	$\mathcal{O}\left(M^{9.5}\right)$
	Punctured perfect code [13]	QAM	$\frac{1}{5^7 2^{16}E^8}$	$\mathcal{O}\left(M^{15.5}\right)$
	VHO-Code [3]	QAM	Not Available[¥]	$\mathcal{O}\left(M^{9.5}\right)$
12	VHO-Code [3]	QAM	Not Available[¥]	$\mathcal{O}\left(M^{14.5}\right)$
	$\mathcal{S}_{12\times 2}$	HEX	$\delta_{min} \geq \frac{1}{(14E)^{12}}$	$\mathcal{O}\left(M^{17.5}\right)$

\$ Punctured STBCs for $n_r < n_t$ refer to rate-n_r STBCs obtained from rate-n_t STBCs (which transmit n_t^2 complex information symbols in n_t channel uses) by restricting the number of complex information symbols transmitted to be only $n_t n_r$.

£ The exact minimum determinant of this STBC has not been calculated, but it has been shown that the STBC has the NVD property [3].

¥ These STBCs are not available explicitly in [3]. However, it is possible to construct such STBCs with the ML-decoding complexities shown in corresponding row.

TABLE I
COMPARISON OF OUR STBCS WITH THE BEST KNOWN STBCS.

(SNR) at each receive antenna is denoted by ρ. It follows that

$$\mathbb{E}(\|\mathbf{S}\|^2) = \mathrm{T}. \tag{2}$$

Definition 1: (*Normalized minimum determinant*) For an STBC $\mathcal{S} = \{\mathbf{S}_i, i = 1, \cdots, |\mathcal{S}|\}$ that satisfies (2), the normalized minimum determinant $\delta_{min}(\mathcal{S})$ is defined as

$$\delta_{min}(\mathcal{S}) = \min_{\mathbf{S}_i, \mathbf{S}_j \in \mathcal{S}, i \neq j} \left| det\left(\mathbf{S}_i - \mathbf{S}_j\right)\right|^2. \tag{3}$$

Definition 2: (*Fast-decodable STBC* [1]) An STBC encoding k complex information symbols from a complex constellation of size M is said to be fast-decodable if the worst-case ML-decoding complexity of this STBC is $\mathcal{O}\left(M^p\right)$, $p < k$.

III. MATHEMATICAL FRAMEWORK

The reader is advised to refer to [3], [11] for a brief review of cyclic division algebras and their matrix representations. Let F and L be two distinct number fields and K a Galois extension of both F and L such that

1) $Gal(K/F) = \langle\sigma\rangle$ with $|Gal(K/F)| = m$,
2) $Gal(K/L) = \langle\tau\rangle$ with $|Gal(K/L)| = n$,
3) σ and τ commute, i.e., $\sigma\tau(a) = \tau\sigma(a)$, $\forall a \in K$.

Let $\mathcal{A} = (K/F, \sigma, \gamma)$ be a cyclic division algebra of degree m over F with $\{1, \mathbf{j}, \mathbf{j}^2, \cdots, \mathbf{j}^{m-1}\}$ being its basis as an m-dimensional right vector space over K.

We consider a non-commutative ring $\mathcal{M}_\mathcal{A}$ which is an n-dimensional bimodule over $\mathcal{A}$ (i.e., both a left $\mathcal{A}$-module and a right $\mathcal{A}$-module), but we will treat $\mathcal{M}_\mathcal{A}$ as a right $\mathcal{A}$-module in this paper. The structure of $\mathcal{M}_\mathcal{A}$ is as follows. We denote the elements of the basis of $\mathcal{M}_\mathcal{A}$ over $\mathcal{A}$ by $1, \mathbf{i}, \mathbf{i}^2, \cdots, \mathbf{i}^{n-1}$. The elements of $\mathcal{M}_\mathcal{A}$ are of the form $A_0 + \mathbf{i}A_1 + \cdots + \mathbf{i}^{n-1}A_{n-1}$ with $A_i \in \mathcal{A}$ and

$$A\mathbf{i} = \mathbf{i}\Upsilon(A), \quad \forall A \in \mathcal{A}, \tag{4}$$

$$\mathbf{i}^n = \gamma_{_M} \quad \text{for some } \gamma_{_M} \in \mathcal{A} \tag{5}$$

where

$$\Upsilon(A) \triangleq \tau(a_0) + \mathbf{j}\tau(a_1) + \cdots + \mathbf{j}^{m-1}\tau(a_{m-1}) \tag{6}$$

for $A = a_0 + \mathbf{j}a_1 + \cdots + \mathbf{j}^{m-1}a_{m-1}$, $a_1, \cdots, a_{m-1} \in K$. We further assume that $\gamma \in L$ so that $\tau(\gamma) = \gamma$. With this assumption and the fact that σ and τ commute, we have

$$\Upsilon(A)\Upsilon(B) = \Upsilon(AB), \quad A, B \in \mathcal{A}. \tag{7}$$

Now, forcing the relation $\mathbf{i}^a\mathbf{i}^b = \mathbf{i}^{a+b}$ for positive integral values of a and b, (5) implies that $\gamma_{_M}\mathbf{i} = \mathbf{i}\gamma_{_M}$ so that $\gamma_{_M}$ is invariant under Υ. Hence, we require $\gamma_{_M}$ to be of the form $a_0 + \mathbf{j}a_1 + \cdots + \mathbf{j}^{m-1}a_{m-1}$, $a_i \in L$, $i = 0, 1, \cdots, m-1$. In this paper, we only consider the case where $\gamma_{_M} \in L \subset K$.

Example 1: Consider $\mathcal{A}$ to be $(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(\sqrt{2}), \sigma : i \mapsto -i, -1)$ which is known to be a division algebra and is a subalgebra of Hamilton's quaternions. Next consider the Galois extension $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(i)$ whose Galois group is $\{1, \tau\}$ with $\tau : \sqrt{2} \to -\sqrt{2}$. Now, $\mathcal{M}_{\mathcal{A}} = \{A_0 + \mathbf{i}A_1 | A_0, A_1 \in \mathcal{A}, \mathbf{i}^2 = i\}$. If $A = a_0 + \sqrt{2}a_1 + i(a_2 + \sqrt{2}a_3) + \mathbf{j}\left(b_0 + \sqrt{2}b_1 + i(b_2 + \sqrt{2}b_3)\right)$ with $a_i, b_i \in \mathbb{Q}$, then $\Upsilon(A) = a_0 - \sqrt{2}a_1 + i(a_2 - \sqrt{2}a_3) + \mathbf{j}\left(b_0 - \sqrt{2}b_1 + i(b_2 - \sqrt{2}b_3)\right)$.

In $\mathcal{M}_{\mathcal{A}}$, we seek conditions under which every element of the form $A_0 + \mathbf{i}A_1$ has a unique right inverse, i.e., for every element of the form $A_0 + \mathbf{i}A_1$, $A_0, A_1 \in \mathcal{A}$, there exists a *unique* element $B \in \mathcal{M}_{\mathcal{A}}$ such that $(A_0 + \mathbf{i}A_1)B = 1$. Towards this end, we make use of the following lemma.

Lemma 1: A nonzero element A of $\mathcal{M}_{\mathcal{A}}$, when it has a right inverse, has a unique right inverse if and only if it is not a left zero divisor, i.e., there exists no nonzero element $B \in \mathcal{M}_{\mathcal{A}}$ such that $AB = 0$.

Proof: If A is not a left zero divisor, the uniqueness of the inverse follows, for if $AB = 1$ and $AB' = 1$, then $A(B - B') = 0 \Rightarrow B = B'$. Conversely, if A has a unique right inverse, it is not a left zero divisor, for if $AB = 1$ and $AC = 0$ for some $C \in \mathcal{A}$, then $A(B - C) = 1 \Rightarrow C = 0$. ■

In the following theorem which is a generalization of [7, Lemma 7], we establish conditions under which each element of $\mathcal{M}_{\mathcal{A}}$ of the form $A_0 + \mathbf{i}A_1$, $A_0, A_1 \in \mathcal{A}$, has a unique right inverse.

Theorem 1: Every nonzero element of $\mathcal{M}_{\mathcal{A}}$ of the form $A_0 + \mathbf{i}A_1$, $A_0, A_1 \in \mathcal{A}$, has a unique right inverse if and only if

$$C\Upsilon(C)\Upsilon^2(C)\cdots\Upsilon^{n-1}(C) \neq \gamma_{_M} \text{ for every } C \in \mathcal{A}. \tag{8}$$

Proof: The proof is given in [10, Appendix A]. ■

Remark 1: The condition in (8) is also necessary and sufficient for any element of the form $\mathbf{i}^k A_0 + \mathbf{i}^l A_1$, $A_0, A_1 \in \mathcal{A}$, $0 \leq k < l \leq n-1$, to have a unique right inverse. The proof is on similar lines to the proof of Theorem 1.

It is to be noted from (4) that $\mathbf{ij} = \mathbf{ji}$. So, we have the following possibilities for $\gamma_{_M}$ and γ (recall that $\gamma = \mathbf{j}^m \in F^\times \cap L$).

Case 1: $\gamma_{_M} \in L \cap F$. In this case $\mathcal{M}_{\mathcal{A}}$ is an associative algebra over $L \cap F$.

Case 2: $\gamma_{_M} \in L \backslash F$. In this case, $\mathcal{M}_{\mathcal{A}}$ is never an associative algebra over $L \cap F$ and hence does not have a matrix representation, for if $\mathcal{M}_{\mathcal{A}}$ is an associative algebra over $L \cap F$ with $\gamma_{_M} \notin L \cap F$, then we have $\mathbf{j}\mathbf{i}^n = \mathbf{i}^n\mathbf{j}$ due to commutativity of $\mathbf{i}$ and $\mathbf{j}$, but $(\mathbf{ji})\mathbf{i}^{n-1} = \mathbf{j}(\mathbf{i}\mathbf{i}^{n-1}) = \mathbf{j}\mathbf{i}^n = \mathbf{j}\gamma_{_M} \neq \gamma_{_M}\mathbf{j} = \mathbf{i}^n\mathbf{j}$, leading to a contradiction.

In this paper, we consider the case $\gamma_{_M} \in L \setminus F$, $\gamma \in L \cap F$. Even though $\mathcal{M}_{\mathcal{A}}$ is now nonassociative and does not have a matrix representation, we still can make use of Theorem 1 to obtain invertible matrices, which are desirable from the point of view of constructing full-diversity STBCs. In this direction, we arrive at the following result.

Lemma 2: If every element C of a cyclic division algebra $\mathcal{A}$ of degree n is such that $C\Upsilon(C)\Upsilon^2(C)\cdots\Upsilon^{n-1}(C) \neq \gamma_{_M}$ where $\gamma_{_M}$ is some nonzero field element of $\mathcal{A}$, then the matrix $\mathbf{C}\Upsilon(\mathbf{C})\Upsilon^2(\mathbf{C})\cdots\Upsilon^{n-1}(\mathbf{C}) - \gamma_{_M}\mathbf{I}$ is invertible, where $\mathbf{C}$, $\Upsilon(\mathbf{C})$, $\cdots$, $\Upsilon^{n-2}(\mathbf{C})$ and $\Upsilon^{n-1}(\mathbf{C}) \in K^{m\times m}$ are respectively the matrix representations[1] of C, $\Upsilon(C)$, $\cdots$, $\Upsilon^{n-2}(C)$ and $\Upsilon^{n-1}(C)$ in $M_m(K)$.

Proof: The proof is given in [10, Appendix B]. ■

We make use of Lemma 2 to obtain the following result.

Theorem 2: Let $\mathcal{M}_{\mathcal{A}}$ be such that any element of the form $A_0 + \mathbf{i}A_1$ has a unique right inverse, and let $\mathbf{A}_0$ and $\mathbf{A}_1$ be matrix representations of A_0 and A_1, respectively, in $M_m(K)$. Consider the matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{O} & \cdots & \gamma_{_M}\Upsilon^{n-1}(\mathbf{A}_1) \\ \mathbf{A}_1 & \Upsilon(\mathbf{A}_0) & \cdots & \mathbf{O} \\ \mathbf{O} & \Upsilon(\mathbf{A}_1) & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \Upsilon^{n-1}(\mathbf{A}_0) \end{bmatrix}. \tag{9}$$

Then,

1) $\mathbf{M}$ is invertible.
2) $det(\mathbf{M}) \in L$.

Proof:

The proof is available in [10, Appendix C]. ■

Corollary 1: If all the elements of $\mathbf{M}$ are from $\mathcal{O}_K$, the ring of integers of K, then $det(\mathbf{M}) \in L \cap \mathcal{O}_K = \mathcal{O}_L$.

IV. STBC CONSTRUCTION

A. General design procedure

The general scheme to obtain invertible matrices as codewords of an STBC for nm transmit antennas is as follows.

1) L is chosen to be either $\mathbb{Q}(i)$ or $\mathbb{Q}(\omega)$, the reason being that a finite subset of $\mathbb{Z}[i]$ is the QAM constellation and that of $\mathbb{Z}[\omega]$ is the HEX constellation, both of practical significance.
2) A cyclic division algebra $\mathcal{A} = (K/F, \sigma, \gamma)$ of degree m over a number field F with $F \neq L$, and an element $\gamma_{_M}$ are chosen such that
   a) K/L is a Galois extension of degree n with $Gal(K/L) = \langle\tau\rangle$.
   b) σ and τ commute.
   c) $\gamma \in F \cap L$.
   d) $\gamma_{_M} \in L \setminus F$.
   e) $\prod_{i=0}^{n-1} \Upsilon^i(C) \neq \gamma_{_M}$, $\forall C \in \mathcal{A}$.

[1] Throughout the paper we denote by $\Upsilon(\mathbf{C})$ the matrix obtained by applying τ to each entry of $\mathbf{C}$, and for the special case of $\mathbf{C}$ being the matrix representation of $C \in \mathcal{A}$, $\Upsilon(\mathbf{C})$ happens to be the matrix representation of $\Upsilon(C)$.

$$\mathbf{A}_k = \left[\begin{array}{cccc} \sum_{i=1}^{n} s_{ki}\theta_i & \gamma\sigma\left(\sum_{i=1}^{n} s_{k(i+nm-n)}\theta_i\right) & \cdots & \gamma\sigma^{m-1}\left(\sum_{i=1}^{n} s_{k(i+n)}\theta_i\right) \\ \sum_{i=1}^{n} s_{k(i+n)}\theta_i & \sigma\left(\sum_{i=1}^{n} s_{ki}\theta_i\right) & \cdots & \gamma\sigma^{m-1}\left(\sum_{i=1}^{n} s_{k(i+2n)}\theta_i\right) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} s_{k(i+nm-n)}\theta_i & \sigma\left(\sum_{i=1}^{n} s_{k(i+nm-2n)}\theta_i\right) & \cdots & \sigma^{m-1}\left(\sum_{i=1}^{n} s_{ki}\theta_i\right) \end{array}\right], \quad k = 0, 1. \tag{10}$$

When $\mathcal{A}$ satisfies the above conditions, any nonzero matrix having the structure shown in (9) is invertible, with $\mathbf{A}_0$ and $\mathbf{A}_1$ being matrix representations of elements A_0 and A_1 of $\mathcal{A}$. If $\mathcal{A}$ is of degree m over F so that $[K : F] = m$, then $\mathbf{A}_0,\ \mathbf{A}_1 \in K^{m\times m}$ and so, $\mathbf{M} \in K^{nm\times nm}$. Each entry of $\mathbf{A}_0$ and $\mathbf{A}_1$ which belongs to K can be viewed as a linear combination of n independent elements over L (since $[K : L] = n$). We express each element of $\mathbf{A}_0$ and $\mathbf{A}_1$ as a linear combination of some chosen L-basis $\{\theta_i, i = 1, \cdots, n | \theta_i \in \mathcal{O}_K\}$ over $\mathcal{O}_L$. From the point of view of space-time coding, each codeword matrix of the STBC constructed using the proposed method has the structure shown in (9) where $\mathbf{A}_0$ and $\mathbf{A}_1$ specifically have the structure given in (10) at the top of the page with s_{ki}, $k = 0, 1$, $i = 1, \cdots, nm$, being the complex information symbols taking values from QAM (a finite subset of $\mathbb{Z}[i]$) or HEX (a finite subset of $\mathbb{Z}[\omega]$) constellations. Clearly, the symbol-rate of the codes constructed thus is 2 complex symbols per channel use.

Proposition 1: The STBC-scheme that is based on the STBCs constructed using the proposed method has the NVD property if $\gamma \in \mathcal{O}_F$ and $\gamma_{_M} \in \mathcal{O}_L$.

Proof: See [10, Proposition 2]. ■

While our proposed scheme can be applied to a wide range of MIMO configurations, we illustrate its application to 4 MIDO configurations - 4×2, 6×2, 8×2 and 12×2 systems. The reason for choosing these 4 configurations is easy to see - the existence of perfect codes [11] for $2, 3, 4, 6$ transmit antennas and the Alamouti code for 2 transmit antennas. The perfect codes of [11] are known for their large coding gain while the Alamouti code has the least ML-decoding complexity among STBCs from CDAs in addition to having the best coding gain among known rate-1 codes for the 2×1 system. We wish to combine the advantages of both these STBCs and so, we focus on the four mentioned MIDO systems. The STBC design procedure for these four MIMO configurations is briefly outlined as follows, and an explicit code construction is presented for 4 transmit antennas. For $n_t = 4, 8$, we choose L to be $\mathbb{Q}(i)$ while for $n_t = 6, 12$, we choose L to be $\mathbb{Q}(\omega)$. K and $\gamma_{_M}$ are respectively chosen to be the maximal subfield and the non-norm element of the division algebra used to construct the perfect codes for $n_t/2$ transmit antennas. So, K is of the form $L(\theta)$, $\theta \in \mathbb{R}$. Next, $\mathcal{A}$ is chosen to be $\mathcal{A} = (K/\mathbb{Q}(\theta), \sigma : i \mapsto -i, -1)$ which is a subalgebra of Hamilton's quaternion algebra $\mathcal{A}_{\mathbb{H}} = (\mathbb{C}/\mathbb{R}, \sigma : i \mapsto -i, -1)$.

B. 4×2 *MIDO system*

We choose $L = \mathbb{Q}(i)$, $K = \mathbb{Q}(i, \sqrt{5})$ and $\gamma_{_M} = i$. The Galois group of $\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i)$ is $\{1, \tau : \sqrt{5} \mapsto -\sqrt{5}\}$, and $(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i), \tau, i)$ is the CDA used to construct the Golden code for 2 transmit antennas. $\mathcal{A}$ is chosen to be $(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(\sqrt{5}), \sigma, -1)$. Note that $\gamma_{_M} = i \notin \mathbb{Q}(\sqrt{5})$. The STBC for the 4×2 system (unnormalized with respect to SNR) obtained upon application of the construction scheme depicted in the previous section is given as

$$\mathcal{S}_{4\times 2} = \left\{ \left[\begin{array}{cccc} a_0 & -\sigma(a_1) & i\tau(a_2) & -i\tau\sigma(a_3) \\ a_1 & \sigma(a_0) & i\tau(a_3) & i\tau\sigma(a_2) \\ a_2 & -\sigma(a_3) & \tau(a_0) & -\tau\sigma(a_1) \\ a_3 & \sigma(a_2) & \tau(a_1) & \tau\sigma(a_0) \end{array}\right] \right\}$$

where $a_0 = s_{01}\theta_1 + s_{02}\theta_2$, $a_1 = s_{03}\theta_1 + s_{04}\theta_2$, $a_2 = s_{11}\theta_1 + s_{12}\theta_2$, $a_3 = s_{13}\theta_1 + s_{14}\theta_2$ with $s_{kj} \in M$-QAM $\subset \mathbb{Z}[i]$, and $\{\theta_1, \theta_2 | \theta_i \in \mathcal{O}_K\}$ is a suitable $\mathbb{Q}(i)$-basis. From [11], we pick $\theta_1 = \alpha, \theta_2 = \alpha\theta$ where $\alpha = 1+i(1-\theta)$, $\theta = (1+\sqrt{5})/2$, and $\{\alpha, \alpha\theta\}$ is now a basis of a principal ideal of $\mathcal{O}_K$ generated by α. We now wish to prove that the STBC-scheme that is based on $\mathcal{S}_{4\times 2}$ has the NVD property. To do so, it is sufficient from Proposition 1 to prove that $A\Upsilon(A) \neq i$, $\forall A \in \mathcal{A}$.

Proposition 2: Let $\mathcal{A} = (\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(\sqrt{5}), \sigma, -1)$. Then, $A\Upsilon(A) \neq i$, $\forall A \in \mathcal{A}$.

Proof: See [10, Appendix D]. ■

Therefore, $\mathcal{S}_{4\times 2}$ is a rate-2 STBC with full-diversity and equipped with the property of non-vanishing determinant.

1) Minimum determinant: When s_{ki}, $k = 0, 1$, $i = 1, \cdots, 4$, take values from $\mathbb{Z}[i]$, from Corollary 1 the determinant of each of the codewords of $\mathcal{S}_{4\times 2}$ belongs to $\mathbb{Z}[i]$. Noting that the entries of the i^{th} column of a codeword matrix, $i = 1, \cdots, 4$, are all respectively multiples of α, $\sigma(\alpha)$, $\tau(\alpha)$, and $\sigma\tau(\alpha)$, the minimum determinant is a multiple of $|\alpha\sigma(\alpha)\tau(\alpha)\sigma\tau(\alpha)|^2 = |N_{K/L}(\alpha)|^4 = 25$. When s_{ki} take values from an M-QAM with average energy E units, a normalization factor of $\frac{1}{\sqrt{4E|\alpha|^2(1+\theta^2)}} = \frac{1}{\sqrt{20E}}$ has to be taken into account. Further, since the difference between any two signal points in a QAM constellation is a multiple of 2, the normalized minimum determinant of $\mathcal{S}_{4\times 2}$ is $\delta_{min}(\mathcal{S}_{4\times 2}) = 25\left(\frac{2}{\sqrt{20E}}\right)^8 = \frac{1}{25E^4}$.

Proposition 3: $\mathcal{S}_{4\times 2}$ has the same algebraic structure, normalized minimum determinant, and ML-decoding complexity as that of the STBC of [2].

Proof: See [10, Subsection IV-B]. ■

The constructions of fast-decodable MIDO STBCs for 6, 8 and 12 transmit antennas have been provided in Subsections IV-C, IV-D, and IV-E, respectively, of [10], and the ML-decoding complexity analysis has been provided in [10, Section V].

V. Comparison with existing STBCs

We compare the performance of the STBCs constructed in this paper with some of the best known STBCs. As rival codes for $\mathcal{S}_{4\times 2}$, we consider the following four STBCs - the punctured perfect code for 4 transmit antennas (two of its

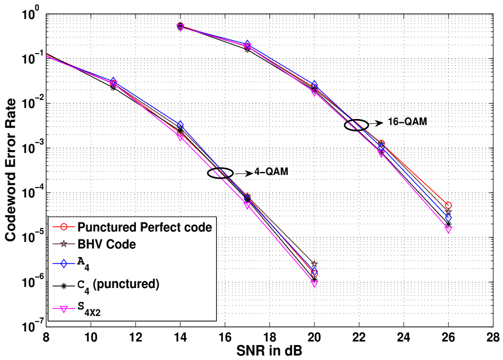

Fig. 1. CER performance of various rate-2 STBCs for the 4×2 system with 4-/16-QAM

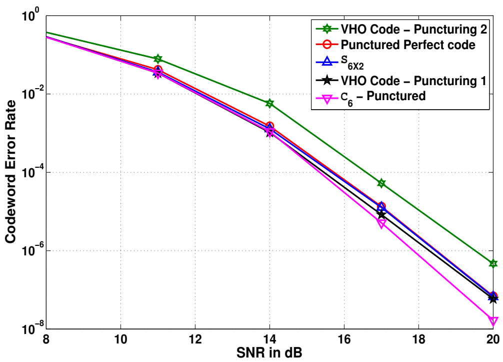

Fig. 2. CER performance of some well-known rate-2 STBCs for the 6×2 system at 4 bits per channel use

layers have zero entries), the BHV code [1], the rate-2 STBC called A_4 code which is obtained in [3, Section VIII-A], and a new STBC obtained by puncturing $\mathcal{C}_4$ [14]. Even though the BHV code is not a full-diversity STBC, it is considered here since it is the first fast-decodable STBC proposed for the 4×2 system, having an ML-decoding complexity of $\mathcal{O}(M^{4.5})$ for square M-QAM. The constellations used in our simulations are 4-QAM and 16-QAM. Fig. 1 reveals that $\mathcal{S}_{4\times 2}$ has the best error performance among all codes under comparison, although punctured $\mathcal{C}_4$ has the best coding gain.

For the 6×2 system, the rival codes for $\mathcal{S}_{6\times 2}$ are the punctured perfect code for 6 antennas [11] (4 layers punctured), punctured $\mathcal{C}_6$ [14], and two versions of the VHO-code for 6 transmit antennas [3, Section X-C]. The first version of the VHO-code for the 6×2 system is the one given in [3, Section X-C]. This STBC has an ML-decoding complexity of $\mathcal{O}\left(M^{8.5}\right)$. The second version of the rate-2 VHO-code is obtained by using the $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_7)$ to be $\{\zeta_7+\zeta_7^6, \zeta_7-\zeta_7^6, \zeta_7^2+\zeta_7^5, \frac{\zeta_7^2-\zeta_7^5}{2}, \frac{\zeta_7^3+\zeta_7^4}{2}, \zeta_7^3-\zeta_7^4\}$ instead of the integral basis $\{1, \zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5\}$. This change of basis results in the overall ML-decoding complexity being only $\mathcal{O}(M^7)$. Both versions of the VHO-code use 4-QAM while the other STBCs use 4-HEX constellation.

Fig 2 shows that $\mathcal{S}_{6\times 2}$, the punctured perfect code, and the first version of the VHO-code (marked as "VHO-code - Puncturing 1" in the figure) have a very similar error performance. The second version of the VHO-code has poorer error performance but lower ML-decoding complexity. The best performance is that of punctured $\mathcal{C}_6$ which has the largest normalized minimum determinant.

VI. Discussion

In this paper, we proposed a new method to obtain full-diversity, rate-2 STBCs. The obtained rate-2, fast-decodable STBCs for 4×2, 6×2, 8×2 and 12×2 systems have large normalized minimum determinants, and STBC-schemes consisting of these STBCs have a non-vanishing determinant (NVD) so that they are DMT-optimal for their respective MIDO systems. However, there is still scope for improvement. Firstly, with the exception of the STBC for the 4×2 MIDO system, the remaining STBCs constructed using the proposed method have a lot of zero entries and naturally, there is the issue of high peak to average power ratio (PAPR) which needs to be lowered. Secondly, it is natural to seek conditions that enable the construction of higher rate codes (rate > 2) with high coding gain and fast-decodability on the lines of the STBCs constructed in this paper. These are the possible directions for future research.

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