Integer-Forcing Interference Alignment

Vasilis Ntranos
USC
ntranos@usc.edu

Viveck R. Cadambe
MIT / BU
viveck@mit.edu

Bobak Nazer
BU
bobak@bu.edu

Giuseppe Caire
USC
caire@usc.edu

***Abstract*—In this paper, we propose a novel framework, integer-forcing interference alignment, that can simultaneously exploit both signal-space and signal-scale alignment. We consider receivers that can decode integer-linear combinations of desired and interfering streams and then solve for their desired symbols. This is possible by using appropriate lattice codes at the transmitters and can be applied to the class of wireless communication systems that use linear beamforming. At the core of our architecture lies the compute-and-forward framework, which we extend here to encompass asymmetric power allocations. We evaluate the performance of our scheme in the context of the three-user interference channel through simulation results.**

I. Introduction

Since the discovery of interference alignment [1], [2], there has been significant interest in realizing its promise at finite signal-to-noise ratios (SNRs) using limited channel diversity (e.g., time extensions). A significant body of work on this topic has focused on studying and enhancing the performance of linear strategies, i.e., strategies in which transmitters encode their data streams onto beamforming vectors chosen to induce alignment and receivers null out the (aligned) interference via linear projections prior to decoding their desired data streams [2]. The interest in the study of linear strategies perhaps comes from the observation that linear strategies are generally sufficient to achieve the full degrees-of-freedom in a wide variety of network topologies, when the channel diversity is sufficiently large [2], [13]. In parallel, another line of work has highlighted the potential of alignment on the signal scale via lattice coding, an approach that has been especially useful in settings where there is limited (or no) scope for space alignment because of limited channel diversity, such as in static channels. The key property that enables signal-scale alignment is that integer-linear combinations of lattice codewords are themselves lattice codewords; this property can be exploited to collapse the interfering users into a single effective interferer from the perspective of the receiver. This approach has been used to characterize the degrees-of-freedom (DoF) for static interference networks [3] as well as approximate the capacities of the symmetric [4], [5] and many-to-one [6] Gaussian interference channels. The scope of prior work may lead one to believe that signal-scale alignment is useful only at very high SNR or for a very limited set of channel gains.

In this paper, we propose a novel framework, *integer-forcing interference alignment*, that can *simultaneously exploit signal-space and signal-scale alignment* for a wide array of channels. In particular, we develop an integer-forcing architecture for scenarios where the directions of beamforming vectors are aligned stream-by-stream. In these settings, it is often possible to guarantee signal-scale alignment without imposing any additional constraints. For instance, a careful inspection of the Cadambe-Jafar asymptotic alignment scheme [1] will reveal that each receiver sees the sum of interferers along each beamforming vector. If each data stream is encoded using the same lattice code, then the receiver can decode the interference aligned along each direction as if it were a single data stream. This enables us to go beyond treating aligned interference as noise, and decode some of the effective interferers along with our desired data streams (See [7] for an application to single-user MIMO channels). As we will demonstrate, this approach can yield higher rates than existing linear strategies, which are only able to exploit signal-space alignment. In our considerations, integer-forcing plays an important role beyond reducing the implementation complexity of the decoder. As recently shown in [5], decoding integer-linear combinations of the desired codewords and effective interferers suffices to approximate the sum capacity of symmetric interference channels within a constant number of bits. The result of [5] means that when lattice coding is used at the transmitters, integer-forcing can act as a good proxy for joint ML decoding at the receivers. Motivated by this result, our framework enables each receiver to decode by first recovering integer-linear combinations of its desired data streams and the aligned interference.

In the sequel, we will use the following notation. We denote the $m \times m$ identity matrix by $\mathbf{I}_m$ and the $m \times n$ all-zeroes matrix by $\mathbf{0}_{m\times n}$. Lowercase bold font (e.g., $\mathbf{x}$) denotes column vectors and uppercase bold font (e.g., $\mathbf{H}$) denotes matrices.

II. Architecture Overview

We now overview our framework for achievability. For the sake of exposition, we focus here on channels with real-valued inputs, outputs, and channel-gains, and describe the rate constraints imposed by a single receiver. Our framework naturally extends to complex-valued channels by viewing a complex number as a two-dimensional real vector. In Section IV, we will use the context of the three-user interference channel to show how the constraints posed by a single receiver combine across multiple receivers in a network to yield a rate region.

Consider a receiver that observes L desired signals and JL interfering signals. The JL interfering signals are par-

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titioned into J *alignment groups*, each group containing L interfering signals. Signals within each group are assumed to be aligned spatially and, as we will describe next, can be exploited for signal-scale alignment. The receiver observes an M-dimensional output

$$\mathbf{y}(\tau) = \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{D}}^{[\ell]} \mathbf{x}_{\mathrm{D}}^{[\ell]}(\tau) + \sum_{j=1}^{J} \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{I}}^{[j,\ell]} \mathbf{x}_{\mathrm{I}}^{[j,\ell]}(\tau) + \mathbf{z}(\tau) \quad (1)$$

where $\tau \in \{1, 2, \ldots, T\}$ represents the symbol index, $\mathbf{x}_{\mathrm{D}}^{[\ell]}(\tau)$ represents the ℓth desired signal which is a function of message $W_{\mathrm{D}}^{[\ell]}$ (which must be decoded) and $x_{\mathrm{I}}^{[j,\ell]}(\tau)$ represents the ℓth interfering signal in the jth alignment group which is a function of message $W_{\mathrm{I}}^{[j,\ell]}$. We assume that the dimensions of $\mathbf{x}_{\mathrm{D}}^{[\ell]}, \mathbf{x}_{\mathrm{I}}^{[j,\ell]}$ are arbitrary: they could represent the number of transmit antennas, the number of independent channel realizations, etc. We assume that each of these signals is associated with a channel gain (matrix) denoted respectively by $\mathbf{H}_{\mathrm{D}}^{[\ell]}$ and $\mathbf{H}_{\mathrm{I}}^{[j,\ell]}$. The variable $\mathbf{z}$ denotes the additive white Gaussian noise at the receiver, whose variance is assumed to be unity without loss of generality. We consider scenarios where the signals are encoded as

$$\mathbf{x}_{\mathrm{D}}^{[\ell]}(\tau) = \mathbf{v}_{\mathrm{D}}^{[\ell]} s_{\mathrm{D}}^{[\ell]}(\tau) \qquad \mathbf{x}_{\mathrm{I}}^{[j,\ell]}(\tau) = \mathbf{v}_{\mathrm{I}}^{[j,\ell]} s_{\mathrm{I}}^{[j,\ell]}(\tau)$$

where $\mathbf{v}_{\mathrm{D}}^{[\ell]}$ and $\mathbf{v}_{\mathrm{I}}^{[j,\ell]}$ are (message-independent) beamforming vectors and $s_{\mathrm{D}}^{[\ell]}(\tau)$ and $s_{\mathrm{I}}^{[j,\ell]}(\tau)$ denote the τth symbols from the codewords associated with the desired and interfering messages, respectively. We will denote the average powers[1] of the signals by

$$P_{\mathrm{D}}^{[\ell]} \triangleq \frac{1}{T} \sum_{\tau=1}^{T} \mathbb{E}\Big[\big|s_{\mathrm{D}}^{[\ell]}(\tau)\big|^2\Big] \qquad P_{\mathrm{I}}^{[j,\ell]} \triangleq \frac{1}{T} \sum_{\tau=1}^{T} \mathbb{E}\Big[\big|s_{\mathrm{I}}^{[j,\ell]}(\tau)\big|^2\Big]$$

We assume that the beamforming vectors satisfy

$$\mathbf{H}_{\mathrm{I}}^{[j,\ell]} \mathbf{v}_{\mathrm{I}}^{[j,\ell]} = \mathbf{g}_{\mathrm{I}}^{[j]}, \forall \ell = 1, 2, \ldots, L,$$

for some $M \times 1$ vector $\mathbf{g}_{\mathrm{I}}^{[j]}$. Intuitively, the above condition means that *all the interferers within one alignment group align in signal space, and can be potentially aligned in signal scale via lattice coding*. The received signal can thus be written as

$$\mathbf{y}(\tau) = \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{D}}^{[\ell]} \mathbf{v}_{\mathrm{D}}^{[\ell]} s_{\mathrm{D}}^{[\ell]}(\tau) + \sum_{j=1}^{J} \mathbf{g}_{\mathrm{I}}^{[j]} \sum_{\ell=1}^{L} s_{\mathrm{I}}^{[j,\ell]}(\tau) + \mathbf{z}(\tau) \quad (2)$$

Remark 1: The constraint that each alignment group carries an equal number (L) of streams is made for notational convenience. This assumption is not restrictive and can be easily generalized. Also note that $\mathbf{g}_{\mathrm{I}}^{[1]}, \mathbf{g}_{\mathrm{I}}^{[2]}, \ldots, \mathbf{g}_{\mathrm{I}}^{[J]}$ need not necessarily be linearly independent; all that is required is that the received signal can be expressed of the form of (2). An example of a scenario where the vectors corresponding to different alignment groups are linearly *dependent* occurs in the context of the symmetric interference channel. Indeed, in that context, our framework specializes to the framework of [5] and therefore can be used to approximate the sum capacity.

Below, we discuss the conventional approach to interference alignment that relies on i.i.d. random codes as well as our proposed integer-forcing framework that relies on lattice codes. The following notation will be helpful to write the achievable rates in a compact form. Define $\mathbf{F} = [\mathbf{F}_{\mathrm{D}} \;\; \mathbf{F}_{\mathrm{I}}]$ where

$$\mathbf{F}_{\mathrm{D}} = \Big[\mathbf{H}_{\mathrm{D}}^{[1]} \mathbf{v}_{\mathrm{D}}^{[1]} \; \cdots \; \mathbf{H}_{\mathrm{D}}^{[L]} \mathbf{v}_{\mathrm{D}}^{[L]}\Big] \qquad \mathbf{F}_{\mathrm{I}} = \Big[\mathbf{g}_{\mathrm{I}}^{[1]} \; \cdots \; \mathbf{g}_{\mathrm{I}}^{[J]}\Big] \; .$$

We also define $\mathbf{P}_{\mathrm{D}} = \operatorname{diag}\big(P_{\mathrm{D}}^{[1]}, \ldots, P_{\mathrm{D}}^{[L]}\big)$ and

$$\mathbf{P}_{\mathrm{I}} = \operatorname{diag}\left(\sum_{\ell=1}^{L} P_{\mathrm{I}}^{[1,\ell]}, \ldots, \sum_{\ell=1}^{L} P_{\mathrm{I}}^{[J,\ell]} \right) .$$

A. Treating Interference as Noise

We now briefly examine the classical receiver architecture of treating interference as noise. Assuming that the data streams $\mathbf{s}_{\mathrm{D}}^{[\ell]} = \big[s_{\mathrm{D}}^{[\ell]}(1) \; \cdots \; s_{\mathrm{D}}^{[\ell]}(T)\big]$ and $\mathbf{s}_{\mathrm{I}}^{[j,\ell]} = \big[s_{\mathrm{I}}^{[j,\ell]}(1) \; \cdots \; s_{\mathrm{I}}^{[j,\ell]}(T)\big]$ are drawn from i.i.d. Gaussian codebooks, it can be shown that there exist rate tuples $\big(R_{\mathrm{D}}^{[1]}, \ldots, R_{\mathrm{D}}^{[L]}\big)$ that meet the following sum rate bound.

$$\sum_{\ell=1}^{L} R_{\mathrm{D}}^{[\ell]} \leq \frac{1}{2} \log \left(\frac{\det\big(\mathbf{I} + \mathbf{F}_{\mathrm{D}} \mathbf{P}_{\mathrm{D}} \mathbf{F}_{\mathrm{D}}^{\mathsf{T}} + \mathbf{F}_{\mathrm{I}} \mathbf{P}_{\mathrm{I}} \mathbf{F}_{\mathrm{I}}^{\mathsf{T}}\big)}{\det\big(\mathbf{I} + \mathbf{F}_{\mathrm{I}} \mathbf{P}_{\mathrm{I}} \mathbf{F}_{\mathrm{I}}^{\mathsf{T}}\big)} \right) . \quad (3)$$

More precisely, the receiver imposes this sum rate bound on the rates of its desired messages as well as standard multiple-access bounds on the sum rates of all possible subsets of desired messages. This approach can be improved by enabling the receiver to decode subsets of the interfering messages and optimizing across all possible message subsets, depending on the channel realization. Due to space constraints, we do not describe its rate region.

B. Integer-Forcing Achievable Rates

The basic idea underpinning our integer-forcing architecture is to decode *integer-linear combinations* of the lattice-coded data streams which can be solved for the desired messages. For instance, consider a receiver that wishes to decode[2]

$$\mathbf{r}_m = \sum_{\ell=1}^{L} a_{\mathrm{D},m}^{[\ell]} \mathbf{s}_{\mathrm{D}}^{[\ell]} + \sum_{j=1}^{J} a_{\mathrm{I},m}^{[j]} \sum_{\ell=1}^{L} \mathbf{s}_{\mathrm{I}}^{[j,\ell]}$$

where $a_{\mathrm{D},m}^{[\ell]}, a_{\mathrm{I},m}^{[j]} \in \mathbb{Z}$. To do so, the receiver first projects its channel output $\mathbf{y}$ to get an effective channel output $\tilde{\mathbf{y}}_m = \mathbf{u}_m^{\mathsf{T}} \mathbf{y}$, which can be written as

$$\tilde{\mathbf{y}}_m = \mathbf{r}_m + \big(\mathbf{u}_m^{\mathsf{T}} \mathbf{F}_{\mathrm{D}} - \mathbf{a}_{\mathrm{D},m}^{\mathsf{T}}\big) \mathbf{S}_{\mathrm{D}} + \big(\mathbf{u}_m^{\mathsf{T}} \mathbf{F}_{\mathrm{I}} - \mathbf{a}_{\mathrm{I},m}^{\mathsf{T}}\big) \mathbf{S}_{\mathrm{I}} + \mathbf{u}_m^{\mathsf{T}} \mathbf{z}$$
$$\mathbf{a}_{\mathrm{D},m}^{\mathsf{T}} = \big[a_{\mathrm{D},m}^{[1]} \; \cdots \; a_{\mathrm{D},m}^{[L]}\big] \qquad \mathbf{a}_{\mathrm{I},m}^{\mathsf{T}} = \big[a_{\mathrm{I},m}^{[1]} \; \cdots \; a_{\mathrm{I},m}^{[J]}\big]$$
$$\mathbf{S}_{\mathrm{D}} = \big[\mathbf{s}_{\mathrm{D}}^{[1]} \; \cdots \; \mathbf{s}_{\mathrm{D}}^{[L]}\big]^{\mathsf{T}} \qquad \mathbf{S}_{\mathrm{I}} = \left[\sum_{\ell=1}^{L} \mathbf{s}_{\mathrm{I}}^{[1,\ell]} \; \cdots \; \sum_{\ell=1}^{L} \mathbf{s}_{\mathrm{I}}^{[J,\ell]} \right]^{\mathsf{T}} .$$

[1]These are not power *constraints*; they simply represent the power of each stream.

[2]More precisely, the receiver will decode the integer-linear combination of the underlying lattice codewords, modulo a shaping lattice, with the dithers removed. See Section III for more details.

This effective channel can be viewed simply as the desired integer-linear combination plus effective noise of variance

$$N_{\text{EFFEC}}(\mathbf{F}, \mathbf{a}_m, \mathbf{u}_m) \triangleq \left(\mathbf{F}^{\mathsf{T}}\mathbf{u}_m - \mathbf{a}_m\right)^{\mathsf{T}} \mathbf{P} \left(\mathbf{F}^{\mathsf{T}}\mathbf{u}_m - \mathbf{a}_m\right) \quad (4)$$

$$\mathbf{a}_m = \begin{bmatrix} \mathbf{a}_{\text{D},m} \\ \mathbf{a}_{\text{I},m} \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} \mathbf{P}_{\text{D}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\text{I}} \end{bmatrix} . \quad (5)$$

Substituting in for the optimal projection $\mathbf{u}_m = (\mathbf{I} + \mathbf{F}\mathbf{P}\mathbf{F}^{\mathsf{T}})^{-1}\mathbf{F}\mathbf{P}\mathbf{a}_m$ the effective noise variance becomes

$$N^*_{\text{EFFEC}}(\mathbf{F}, \mathbf{a}_m) \triangleq \mathbf{a}_m^{\mathsf{T}} \left(\mathbf{P}^{-1} + \mathbf{F}^{\mathsf{T}}\mathbf{F}\right)^{-1} \mathbf{a}_m \ .$$

Notice that if $\mathbf{P} = P\mathbf{I}$, decoding $\mathbf{r}_m$ falls directly in the basic compute-and-forward framework [8] that makes the tacit assumption that each data stream is allocated the same power. This is a sensible assumption in scenarios where the transmitters do not have channel state information. However, in our context, the transmitters know the channel realization well enough to perform interference alignment. Therefore, to achieve the highest rates, we should allow for asymmetric power allocation across data streams. Towards this end, we describe in Section III-B how the compute-and-forward framework can be generalized to permit asymmetric powers.

In order to solve for its desired messages, the receiver must recover M integer-linear combinations whose coefficient vectors $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_M]$ satisfy $[\mathbf{I}_L \ \mathbf{0}_{L \times J}] \in \text{rowspan}(\mathbf{A}^{\text{T}})$. As shown in [5] in a slightly different context, after each integer-linear combination is decoded, it can be used to cancel out the contribution of one data stream from the remaining integer-linear combination (i.e., successive cancellation). Let π denote a decoding order, i.e, an order in which streams can be canceled from subsequent integer-linear combination such that $\pi(\ell)$th stream does not participate in integer-linear combinations with indices $m > \ell$. The important feature of this procedure is that the rate of the $\pi(\ell)$th stream is not bounded from any subsequent computation rates. Combining [5, Theorem 5] and our asymmetric compute-and-forward scheme from Theorem 1, it can be shown that decoding is possible if

$$\begin{aligned} R_{\text{D}}^{[\ell]} &< \frac{1}{2} \log^+ \left(\frac{P_{\text{D}}^{[\ell]}}{N^*_{\text{EFFEC}}\left(\mathbf{F}, \mathbf{a}_{\pi(\ell)}\right)} \right) \\ R_{\text{I}}^{[j,\ell]} &< \frac{1}{2} \log^+ \left(\frac{P_{\text{I}}^{[j,\ell]}}{N^*_{\text{EFFEC}}\left(\mathbf{F}, \mathbf{a}_{\pi(j+L)}\right)} \right) \end{aligned} \quad (6)$$

where we have assumed without loss of generality that $N^*_{\text{EFFEC}}(\mathbf{F}, \mathbf{a}_1) \leq \cdots \leq N^*_{\text{EFFEC}}(\mathbf{F}, \mathbf{a}_M)$.

III. Asymmetric Compute-and-Forward

In this section, we demonstrate how compute-and-forward can be generalized to allow for asymmetric power allocations. For notational ease, we will assume in the following that the receiver has already projected its observation.

A. Lattice Preliminaries

We will work with a variation on the nested lattice codes used by Erez and Zamir to achieve the capacity of the point-to-point AWGN channel [9]. A *lattice* Λ is a discrete additive subgroup of $\mathbb{R}^T$. Let $Q_\Lambda(\mathbf{x}) = \arg\min_{\mathbf{t} \in \Lambda} \|\mathbf{x} - \mathbf{t}\|$ denote the nearest neighbor quantizer for Λ. Using this, we define the (fundamental) *Voronoi region* $\mathcal{V}$ of Λ to be the set of all points in $\mathbb{R}^T$ which are quantized to the zero vector and the *modulo operation* $[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$ which outputs the error from quantizing $\mathbf{x}$ onto Λ. We say that the lattice $\Lambda^{[1]}$ is *nested* in the lattice $\Lambda^{[2]}$ if $\Lambda^{[1]} \subseteq \Lambda^{[2]}$. Note that a nested lattice pair $\Lambda^{[1]} \subseteq \Lambda^{[2]}$ satisfies the following properties

$$\left[a[\mathbf{x}] \bmod \Lambda^{[1]} + \mathbf{y}\right] \bmod \Lambda^{[2]} = [a\mathbf{x} + \mathbf{y}] \bmod \Lambda^{[2]}$$
$$\left[Q_{\Lambda^{[1]}}(\mathbf{x})\right] \bmod \Lambda^{[2]} = \left[Q_{\Lambda^{[1]}}\left([\mathbf{x}] \bmod \Lambda^{[2]}\right)\right] \bmod \Lambda^{[2]}$$

for any integer $a \in \mathbb{Z}$. A *nested lattice codebook* $\mathcal{C}$ is generated using a nested lattice pair $\Lambda \subseteq \Lambda_{\text{FINE}}$. The codebook is comprised of all elements of the fine lattice Λ_{FINE} that fall within the Voronoi region of the coarse lattice, $\mathcal{C} = \Lambda_{\text{FINE}} \cap \mathcal{V}$. It follows that the rate of the codebook is $R = \frac{1}{T} \log \left|\mathcal{C}\right| = \frac{1}{T} \log(\text{Vol}(\mathcal{V})/\text{Vol}(\mathcal{V}_{\text{FINE}}))$. We use the variant of Construction A as described in [8] to create nested lattice codes.

B. Achievable Scheme

Here, we combine ideas from [10] and [8] to create a compute-and-forward framework that permits asymmetric power allocation. Consider a receiver that observes (real-valued) channel output $\mathbf{y} = \sum_{\ell=1}^{L} h^{[\ell]} \mathbf{s}^{[\ell]} + \mathbf{z}$ where the $h^{[\ell]} \in \mathbb{R}$ are effective channel coefficients, $\mathbf{s}^{[\ell]} \in \mathbb{R}^T$ denotes the ℓ^{th} data stream, and $\mathbf{z}$ is an i.i.d. Gaussian noise vector with mean zero and variance one. Let $\mathbf{h} = \left[h^{[1]} \cdots h^{[L]}\right]^{\mathsf{T}}$ denote the channel coefficient vector. Each data stream must meet its own power constraint $\|\mathbf{s}^{[\ell]}\|^2 \leq TP^{[\ell]}$. Let $\mathbf{P} = \text{diag}\left(P^{[1]}, P^{[2]}, \ldots, P^{[L]}\right)$.

Each encoder creates a nested lattice codebook $\mathcal{C}^{[\ell]}$ with rate $R^{[\ell]}$ using its own coarse and fine lattices $\Lambda^{[\ell]}$ and $\Lambda^{[\ell]}_{\text{FINE}}$. All lattices are nested $\Lambda^{[\sigma(1)]} \subseteq \cdots \subseteq \Lambda^{[\sigma(L)]} \subseteq \Lambda^{[\theta(1)]}_{\text{FINE}} \subseteq \cdots \subseteq \Lambda^{[\theta(L)]}_{\text{FINE}}$ according to some permutations $\sigma : \{1, \ldots, L\} \rightarrow \{1, \ldots, L\}$ and $\theta : \{1, \ldots, L\} \rightarrow \{1, \ldots, L\}$. Encoder ℓ maps its message $W^{[\ell]}$ (in a one-to-one fashion) to a lattice codeword $\mathbf{t}^{[\ell]} \in \mathcal{C}^{[\ell]}$. The data stream $\mathbf{s}^{[\ell]}$ is generated by dithering the lattice codeword $\mathbf{s}^{[\ell]} = \left[\mathbf{t}^{[\ell]} + \mathbf{d}^{[\ell]}\right] \bmod \Lambda^{[\ell]}$ where $\mathbf{d}^{[\ell]}$ is a random dither vector drawn independently and uniformly over $\mathcal{V}^{[\ell]}$, the Voronoi region of $\Lambda^{[\ell]}$.

The goal of the receiver is to recover an integer-linear combination with coefficient vector $\mathbf{a} = [a^{[1]} \ \cdots \ a^{[L]}] \in \mathbb{Z}^L$,

$$\mathbf{r}^{[k]} = \left[\sum_{\ell=1}^{L} a^{[\ell]} \tilde{\mathbf{t}}^{[\ell]}\right] \bmod \Lambda^{[k]} \quad (7)$$

$$\tilde{\mathbf{t}}^{[\ell]} = \begin{cases} \mathbf{t}^{[\ell]} & \Lambda^{[\ell]} \subseteq \Lambda^{[k]} \ , \\ \mathbf{t}^{[\ell]} - Q_{\Lambda^{[\ell]}}(\mathbf{t}^{[\ell]} + \mathbf{d}^{[\ell]}) & \Lambda^{[\ell]} \supset \Lambda^{[k]} \ . \end{cases} \quad (8)$$

Note that not all of the lattice codewords in the integer-linear combination correspond exactly to the transmitted lattice codewords. That is, if $\Lambda^{[k]}$ is coarser than $\Lambda^{[\ell]}$, then the effect of the dither will not be entirely removed. It can be argued this residual dither is not an obstacle to recovering the original messages. For instance, if the receiver decodes a full rank set of equations modulo $\Lambda^{[k]}$, then it can solve for $\mathbf{t}^{[k]}$.

Theorem 1: For any $\epsilon > 0$ and T large enough, there exist nested lattice codebooks $\mathcal{C}^{[1]}, \ldots, \mathcal{C}^{[L]}$ such that for any channel coefficient vector $\mathbf{h} \in \mathbb{R}^L$, a receiver can decode any integer-linear combination of codewords with coefficient vector $\mathbf{a} \in \mathbb{Z}^L$ with probability of error at most ϵ so long as each codebook satisfies

$$R^{[\ell]} < \frac{1}{2}\log^+\left(\frac{P^{[\ell]}}{1+(\mathbf{h}-\mathbf{a})^\mathsf{T}\mathbf{P}(\mathbf{h}-\mathbf{a})}\right) \tag{9}$$

for all ℓ such that $a^{[\ell]} \neq 0$.

Proof Sketch: The decoder is essentially unchanged from the symmetric case. First, the receiver removes the dithers. Then, it quantizes the result onto the finest lattice that participates in the equation, $\Lambda_{\text{FINE}}^{[\kappa]}$ where $\kappa = \arg\max_{\ell : a^{[\ell]} \neq 0} \theta(\ell)$. Finally, it applies the modulo operation with respect to $\Lambda^{[k]}$. Let $\mathbf{z}_{\text{EFFEC}} = \sum_{\ell=1}^{L}(h^{[\ell]} - a^{[\ell]})\mathbf{s}^{[\ell]} + \mathbf{z}$ denote the effective noise. Using the properties of nested lattices from Section III-A, it follows that the receiver's estimate $\hat{\mathbf{r}}$ can be written as

$$\begin{aligned}
\hat{\mathbf{r}} &= \left[Q_{\Lambda_{\text{FINE}}^{[\kappa]}}\left(\sum_{\ell=1}^{L} a^{[\ell]}(\mathbf{s}^{[\ell]} - \mathbf{d}^{[\ell]}) + \mathbf{z}_{\text{EFFEC}}\right)\right] \bmod \Lambda^{[k]} \\
&= \left[Q_{\Lambda_{\text{FINE}}^{[\kappa]}}\left(\left[\sum_{\ell=1}^{L} a^{[\ell]}\tilde{\mathbf{t}}^{[\ell]} + \mathbf{z}_{\text{EFFEC}}\right] \bmod \Lambda^{[k]}\right)\right] \bmod \Lambda^{[k]} \\
&= \left[Q_{\Lambda_{\text{FINE}}^{[\kappa]}}\left(\mathbf{r} + \mathbf{z}_{\text{EFFEC}}\right)\right] \bmod \Lambda^{[k]} \ .
\end{aligned} \tag{10}$$

Thus, if the fine lattice $\Lambda_{\text{FINE}}^{[\kappa]}$ is able to tolerate additive noise of variance $N_{\text{EFFEC}} = \frac{1}{T}\mathbb{E}[\|\mathbf{z}_{\text{EFFEC}}\|^2] = 1 + \sum_{\ell=1}^{L} P^{[\ell]}(h^{[\ell]} - a^{[\ell]})^2$, then the decoder can recover the equation $\mathbf{r}$ successfully, $\mathbb{P}(\hat{\mathbf{r}} \neq \mathbf{r}) < \epsilon$. Following the arguments of [8], [9], it can be shown that, for any $\epsilon > 0$ and T large enough, there exist good nested lattice codebooks with rates $R^{[\ell]} = \frac{1}{2}\log^+(P/N_{\text{EFFEC}}) - \epsilon$.

IV. Application: Three-User Time-Varying Interference Channel

We apply our integer-forcing interference alignment architecture to the asymptotic interference alignment scheme of Cadambe and Jafar [1]. The asymptotic alignment scheme used a $2N+1$ symbol extension over the interference channel and achieved $3N+1$ degrees of freedom over this extended channel. We provide an overview of the scheme of [1] here. The symbol-extended channel can be described as

$$\mathbf{y}^{[j]}(\tau) = \sum_{i=1}^{3} \mathbf{H}^{[ji]}\mathbf{x}^{[i]}(\tau) + \mathbf{z}^{[j]}(\tau)$$

where τ represents the (extended) symbol index, $\mathbf{y}^{[j]}(\tau), \mathbf{z}^{[j]}(\tau), \mathbf{x}^{[i]}(\tau)$ are all $(2N+1) \times 1$ vectors and $\mathbf{H}^{[ji]}$ is a $(2N+1) \times (2N+1)$ diagonal channel matrix[3]. In the sequel, the channel index τ is suppressed for brief notation when there is no ambiguity. The input powers are constrained as $\frac{1}{T}\sum_{\tau=1}^{T} E[||\mathbf{x}^{[i]}(\tau)||^2] \leq P^{[i]}$, where T denotes the size of the codeword.

[3]We assume that $\mathbf{H}^{[ji]}$, the channel gain matrix over the extended channel, is independent of τ, the index of the extended channel symbol. Physically, this translates to coding over $2N+1$ coherence blocks, and the assumption that each coherence block is sufficiently large for channel coding.

Over this channel, Transmitter 1 sends $N+1$ streams and transmitters 2 and 3 each send N streams as follows:

$$\mathbf{x}^{[i]}(\tau) = \left\{ \begin{array}{ll} \sum_{\ell=1}^{N} \mathbf{v}^{[i,\ell]}s^{[i,\ell]}(\tau) & i = 2,3 \\ \sum_{\ell=1}^{N+1} \mathbf{v}^{[i,\ell]}s^{[i,\ell]}(\tau) & i = 1 \end{array} \right\}$$

where $\mathbf{s}^{[i,\ell]} = \begin{bmatrix} s^{[i,\ell]}(1) & \cdots & s^{[i,\ell]}(T) \end{bmatrix}^\mathsf{T}$ is a codeword chosen based on the ith transmitter's message. The beamforming vectors $\mathbf{V}^{[1]} \triangleq \begin{bmatrix}\mathbf{v}^{[1,1]} & \ldots & \mathbf{v}^{[1,N+1]}\end{bmatrix}$ and $\mathbf{V}^{[i]} \triangleq \begin{bmatrix}\mathbf{v}^{[i,1]} & \ldots & \mathbf{v}^{[i,N]}\end{bmatrix}$, $i = 2, 3$, are chosen as

$$\begin{aligned}
&\mathbf{V}^{[1]} = \begin{bmatrix}\mathbf{w} & \mathbf{T}\mathbf{w} & \cdots & \mathbf{T}^N\mathbf{w}\end{bmatrix} \\
&\mathbf{V}^{[2]} = \mathbf{H}^{[31]}(\mathbf{H}^{[32]})^{-1}\mathbf{V}^{[1]} \quad , \quad \mathbf{V}^{[3]} = \mathbf{H}^{[21]}(\mathbf{H}^{[23]})^{-1}\mathbf{V}^{[1]}
\end{aligned}$$

where $\mathbf{T} = \mathbf{H}^{[12]}(\mathbf{H}^{[21]})^{-1}\mathbf{H}^{[23]}(\mathbf{H}^{[32]})^{-1}\mathbf{H}^{[31]}(\mathbf{H}^{[13]})^{-1}$, and $\mathbf{w}$ is a $(2N+1) \times 1$ random vector whose choice will be discussed later in this section. As shown in [1], the signals $\mathbf{y}^{[1]}, \mathbf{y}^{[2]}$ and $\mathbf{y}^{[3]}$ can be respectively written as

$$\sum_{\ell=1}^{N+1} s^{[1,\ell]}\mathbf{H}^{[11]}\mathbf{v}^{[1,\ell]} + \sum_{\ell=1}^{N}\left(s^{[2,\ell]} + s^{[3,\ell]}\right)\mathbf{H}^{[12]}\mathbf{v}^{[2,\ell]} + \mathbf{z}^{[1]},$$

$$\sum_{\ell=1}^{N} s^{[2,\ell]}\mathbf{H}^{[22]}\mathbf{v}^{[2,\ell]} + \sum_{\ell=1}^{N+1}\left(s^{[1,\ell]} + s^{[3,\ell-1]}\right)\mathbf{H}^{[21]}\mathbf{v}^{[1,\ell]} + \mathbf{z}^{[2]}$$

and

$$\sum_{\ell=1}^{N} s^{[3,\ell]}\mathbf{H}^{[33]}\mathbf{v}^{[3,\ell]} + \sum_{\ell=1}^{N+1}\left(s^{[1,\ell]} + s^{[2,\ell]}\right)\mathbf{H}^{[31]}\mathbf{v}^{[1,\ell]} + \mathbf{z}^{[3]},$$

where we assume that $s^{[2,N+1]} = s^{[3,0]} = 0$ and suppressed the time index τ due to space limitations. Comparing the above equations with (2), it is clear that the asymptotic alignment scheme of [1] is amenable to our architecture. The symbols $s^{[i,\ell]}$ are chosen from a codeword with an appropriate power constraint, i.e., $\frac{1}{T}\sum_{\tau=1}^{T} ||\mathbf{v}^{[i,\ell]}||^2 E\left[|s^{[i,\ell]}(\tau)|^2\right] = P^{[i,\ell]}$ where $P^{[i,\ell]}$ can be any set of powers such that $\sum_\ell P^{[i,\ell]} \leq P^{[i]}$. We next briefly discuss the integer-forcing architecture for this context.

Remark 2: It is the extension of the compute-and-forward technique to asymmetric powers that allows us to chose $\mathbf{P}^{[i,\ell]}$ freely and still maintain lattice alignment, as long as the overall power constraint is satisfied.

A. Integer-Forcing Achievable Rates

The integer-forcing architecture is applicable here via the employment of lattice coding. In particular, at Receiver 1, there are $N+1$ desired streams, and N alignment groups. At receivers 2 and 3, there are N desired streams, and $N+1$ alignment groups. Each receiver can choose a set of integer-linear combinations to decode and a decoding order. We denote the number of integer-linear combinations decoded at Receiver j as $M^{[j]}$, the corresponding coefficients as the $M^{[j]} \times (2N+1)$ matrix $\mathbf{A}^{[j]}$ and the decoding order as $\pi^{[j]}$. In particular, recall that as per Section II-B, the matrix $\mathbf{A}^{[j]}$ should be chosen so that $[\mathbf{I} \ \ \mathbf{0}]$ lies in the rowspan of $\mathbf{A}^{[j]\mathsf{T}}$. Then, the rate region imposed by this receiver on to each stream is denoted as $\mathcal{R}(\mathbf{A}^{[j]}, \pi^{[j]})$, which is the region described in (6).

The set of all rates achievable in the interference channel within the integer-forcing framework is then given by

$$\mathcal{R} = \bigcup_{\Pi, \mathcal{A}} \bigcap_{j=1}^{3} \mathcal{R}(\mathbf{A}^{[j]}, \pi^{[j]}) \tag{11}$$

where $\mathcal{A}$ is the set of all possible tuples $(\mathbf{A}^{[1]}, \mathbf{A}^{[2]}, \mathbf{A}^{[3]})$ which allow resolution of the desired streams, and Π is the set of all decoding orders that are possible.

B. Numerical Results

We now evaluate the performance of the proposed integer-forcing architecture in terms of its average achievable sum rate in the three-user time-varying interference channel. We consider channel gains that are drawn from a standard complex normal distribution $\mathcal{CN}(0,1)$ and assume an equal average power constraint among all transmitters, $P^{[k]} = P$, $\forall k$. To demonstrate the potential of the architecture, we choose $N = 1$ and compare the following schemes.

1) Integer-Forcing Interference Alignment: Recall that each receiver chooses the set of integer-linear combinations and a decoding order. Optimizing over this set is a challenging open problem. Here, for a given number of integer-linear combinations, we choose the corresponding coefficients via the LLL algorithm. We apply a brute-force search over the number of integer-linear combinations $M^{[j]}$ and the set of all decoding orders Π. In addition, $\mathbf{w}$ is chosen based on a Monte Carlo sampling over the space of all 3-dimensional vectors.

2) Treating Interference as Noise: The optimal beamforming choice for treating interference as noise for this channel is a challenging open problem, especially because several interference alignment heuristics which are applicable, for example, for some MIMO scenarios, are not applicable in the time-varying context (see [11]). Here, we use the DoF optimal construction of [1] as a heuristic for beamforming we exploit the potential for subspace alignment. Specifically, we choose our beamforming matrices as $\mathbf{V}^{[i]}\mathbf{C}^{[i]}$, where $\mathbf{C}^{[i]}$ is designed to water-fill over the direct channels. The vector $\mathbf{w}$ is optimized to maximize the chordal distance between signal and interfering subspaces (see [12]).

3) Decode all: We plot the performance when receivers decode all the streams (desired and interfering), assuming the same beamforming vectors associated with treating interference as noise.

4) Time Sharing: We plot orthogonalization as a baseline scheme, in which the transmitters use the channel one at a time in a round robin fashion.

The performance of the above schemes is plotted in Figure 1. As shown by the plots, the integer-forcing architecture of our paper clearly outperforms the competing schemes for a relevant range of signal-to-noise ratios. This performance demonstrates the enormous potential of integer-forcing interference alignment. The proposed architecture opens several lines of investigation relevant to the context of wireless communication systems including the optimization of beamformers for the purposes of integer-forcing and the development of efficient algorithms to select integer coefficients.

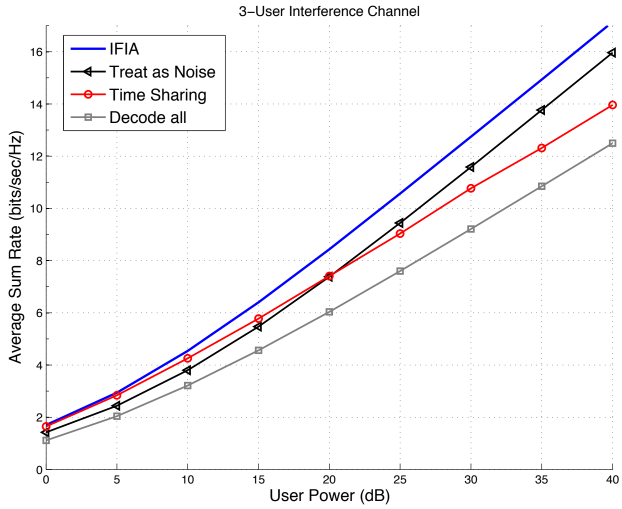

Fig. 1. Comparison of decoding strategies for the three-user time-varying interference channel with $N = 1$. We consider $\mathcal{CN}(0,1)$ distributed channel gains and equal power constraints, $P^{[k]} = P$, $\forall k$.

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