Parallel Linear Deterministic Interference Channels with Feedback: Combinatorial Structure and Separability

Suvarup Saha and Randall A. Berry
Dept. of EECS, Northwestern University
e-mail: {ssa562,rberry}@eecs.northwestern.edu

***Abstract*—The sum-capacity of a 2-user linear deterministic interference channel (LDIC) can always be achieved with simple deterministic codes. The existence and design of such codes has been shown to be related to an underlying combinatorial structure of the channel. This is used here to explore the capacity of parallel or ergodic LDICs and LDICs with output feedback. We present simple algorithms that generate sum-rate optimal schemes in these cases. In the case of parallel LDICs this approach gives insight into when the channels are separable, i.e. when coding over component channels is not required. We further demonstrate that output feedback can change the separability of such channels.**

I. Introduction

Determining the capacity of interference channels has remained a challenging problem for decades. However, a recent approximation approach [4] has been instrumental in evaluating this region to within 1 bit per channel use in the Gaussian case. This positive result coincides with and is partly influenced by the 'linear deterministic interference channel' (LDIC) analyzed in [2]. The 'linear deterministic' model, first introduced in the context of the relay channel [3], is essentially a high signal-to-noise ratio (SNR) model that focuses on signal interaction by abstracting out the randomness of Gaussian noise. Exact capacity analysis using this channel model has in many cases yielded approximate capacity characterization of the corresponding Gaussian case. It is often possible to find simple constructive achievable schemes for LDICs, which can further simplify their analysis [6]. In [13] a simple algorithm for constructing sum capacity achievable schemes for 2-user LDICs was given. This algorithm arises out of an alternative view of the LDIC in terms of a collection of 'interference chains'. Sum-capacity of the channel bears a direct relationship with the length of these chains. In this work, we use the same chain structure to analyze two other interesting interference channel scenarios: 2-user parallel LDICs and 2-user LDICs with output feedback. Though the capacity of the latter is known already [10], that of the former, to the best of our knowledge, has not yet been addressed.

Parallel Gaussian interference channels (PGICs) have been considered as models of frequency selective interference channels. Under the assumption that independent coding is used on each sub-channels, optimal power allocation for a PGIC was derived in the strong interference regime [5]. Also, PGICs can be used to model interference channels with ergodic fading [12]. There, it has been shown that it is not always sum-rate optimal to independently code across the sub-channels; joint coding and decoding over the parallel channels can yield higher rates. There has been some progress in characterizing the sum-capacity of PGICs [7]–[9] and parallel discrete memoryless interference channels [11]. In particular, certain interference regimes have been identified where independent coding on sub-channels is sufficient to achieve sum-capacity, thus establishing 'separability'. In this work, we look at the linear deterministic version of such parallel channels and use the aforementioned chain representation to characterize the conditions under which such channels are separable. The chain representation also enables an alternative view of how feedback can increase the capacity of an LDIC [10]. Finally, the chain representation can be applied to parallel LDICs with output feedback where we show that in some cases feedback can change the separability of the channel. We also give exact characterization as to when this occurs.

The rest of the paper is organized as follows: In Section II, we consider the parallel LDIC and discuss when and how coding across channels is needed to achieve sum-capacity. The case of LDICs with output feedback is considered in Section III where construction of sum-capacity achieving schemes is discussed in the context of the chain representation. The effect of feedback on the capacity of parallel channels is considered in Section IV. Concluding remarks are provided in Section V.

II. Parallel LDIC

To begin, we consider a variation of the 2-user LDIC in [2] with L parallel channels. We will call this a 2-user L-parallel LDIC. The kth sub-channel will be specified by the 4-tuple $(n_{11}^k, n_{12}^k, n_{21}^k, n_{22}^k)$, where n_{ij}^k is the number of 'bit-levels' from transmitter j to receiver i in the kth channel for $k = 1, \ldots, L$. In a symmetric situation when $n_{11}^k = n_{22}^k = n_d^k$ and $n_{12}^k = n_{21}^k = n_c^k$, it is often useful to parametrize the channel in terms of the 'coupling parameter' $\alpha^k = \frac{n_c^k}{n_d^k}$, the value of which determines the strength of interference. Details on the LDIC model and its relation to the Gaussian model can be found in [2].

This research was supported in part by the National Science Foundation under awards CNS-0905407, CCF-0905224, and CNS-1147786.

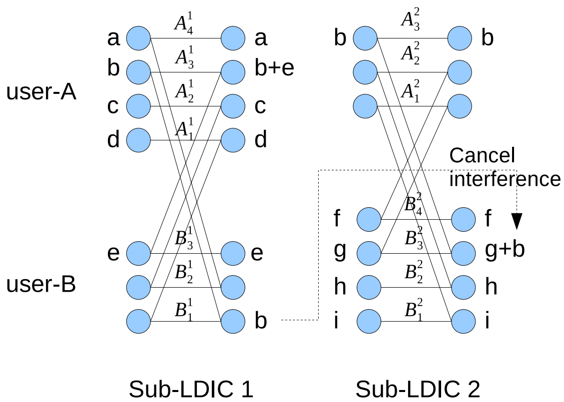

Fig. 1. Coding across channels in parallel LDIC

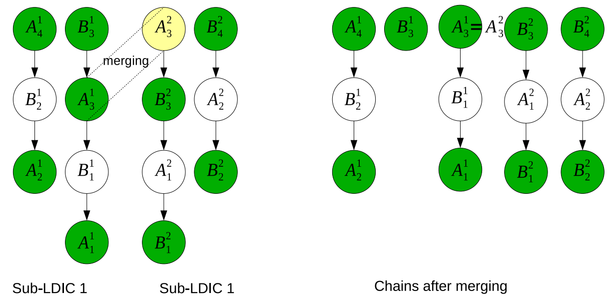

Fig. 2. Chain representation of parallel LDIC; as in [13], each node represents a bit level for one of the users and directed edges indicate that levels interfere with each other (e.g. A_4^1 denotes the most significant bit level of user A, in sub-LDIC 1, which interferes with the second most significant bit of user B).

Denote by $\mathcal{M}_1 = \{1, \ldots, M_1\}$ and $\mathcal{M}_2 = \{1, \ldots, M_2\}$ the message sets of users 1 and 2. Let the encoding functions $f_i : \mathcal{M}_i \to X_i$ with $f_i(m) = X_i(m)$ map the message m generated at user i into the length N codeword $X_i(m)$. Let the decoding functions $g_i(Y_i)$ map the received signal Y_i to the message m if $Y_i \in D_{ij}$, where D_{im} is the decoding set of message m for user i. An (N, M_1, M_2, μ) code consists of M_i codewords $X_i(m)$ and M_i decoding sets D_{im} such that the average probability of decoding error satisfies

$$\frac{1}{M_1 M_2} \sum_{mn} P(D_{1m} | (X_1(m), X_2(n)) \geq 1 - \mu, \tag{1}$$

$$\frac{1}{M_1 M_2} \sum_{mn} P(D_{2n} | (X_1(m), X_2(n)) \geq 1 - \mu. \tag{2}$$

Note that, in the above formulation, a single codeword spans over all the L parallel sub-LDICs. Thus coding and decoding is allowed to be jointly carried out.

A pair of nonnegative real numbers (R_1, R_2) is called an achievable rate for the parallel LDIC if for any $\epsilon > 0$, $0 < \mu < 1$, and for any sufficiently large N, there exists an (N, M_1, M_2, μ) code such that

$$\frac{1}{N} \log M_i \geq R_i - \epsilon. \tag{3}$$

The capacity region of this channel is the closure of the set of all achievable rate tuples. The sum-capacity is given by $\max(R_1 + R_2)$, where (R_1, R_2) is achievable.

Let us start off with a concrete example. Consider a 2-user parallel LDIC with 2 parallel sub-LDICs defined by parameter sets $n_{11}^1 = 4, n_{12}^1 = 3, n_{21}^1 = 2, n_{22}^1 = 3$ and $n_{11}^2 = 3, n_{12}^2 = 2, n_{21}^2 = 3, n_{22}^2 = 4$, respectively. This channel is illustrated in Fig. 1. As in [2], the sum-capacity of the sub-LDICs can be computed as $(n_{22}^1 - n_{12}^1)^+ + \max(n_{12}^1, n_{11}^1)$ = $(3-3)^+ + \max(3, 4)$= 4 bits and $(n_{11}^2 - n_{21}^2)^+ + \max(n_{21}^2, n_{22}^2)$ = 4 bits, respectively. Thus, if each user uses the parallel sub-LDICs independently, a maximum sum-rate of $4 + 4 = 8$ bits can be achieved. Now consider the following joint coding-decoding scheme. On the first sub-LDIC, let user 1 transmit independent bits $\{a, b, c, d\}$ from all its 4 levels, while user 2 only transmits one bit, say e, from the top most of its 3 levels. Though the receiver of user 2 can easily decode bit e, the receiver of user 1 can only decode 3 bits a, c, d while bit b cannot be decoded as the received signal is corrupted by interfering bit e.[1] Now, on the second sub-LDIC, let user 1 only re-transmit bit b from the top-most level while user 2 transmits 4 independent bits $\{f, g, h, i\}$ from its 4 levels. For user 1, the bit b that wasn't decoded in sub-LDIC 1 can be decoded in sub-LDIC 2. For user 2, bits f, h, i can be easily decoded but bit g gets interfered by the transmission of bit b by user 1. However, in sub-LDIC 1, user 2's receiver can decode b, which can be used to cancel the interference and decode bit g. Thus we see that by simple coding across the sub-LDICs it is possible to achieve a sum-rate of 9, which is better than that with independent coding.

Let us now look at the same channel in terms of interference chains as introduced in [13]. The chain representation for the discussed parallel LDIC is illustrated in Fig. 2. In [13] it was shown that the sum-capacity of a single LDIC can either be achieved with uncoded transmission using a maximal interference free set (MIFS) of nodes (i.e. a maximal set of node so that no two are adjacent) or by simple repetition coding across levels, which can be viewed as *merging* of two chains. This merging operation is only beneficial given two even length chains, with length greater than 2. In Fig. 2, each sub LDIC only has one even chain and so its capacity can be achieved using uncoded transmissions from a MIFS. However, when we consider the pair of channels, we can find a pair of even chains and merge them to get two odd length chains and a single node chain as shown in the figure.

The following theorem shows that this example generalizes to any 2-user L-parallel LDIC and moreover that exhausting all such merging operations yields the sum-capacity.

Theorem 1: Given any 2-user L-parallel LDIC defined by the tuples $\{n_{11}^k, n_{12}^k, n_{21}^k, n_{22}^k\}_1^L$, the following algorithm yields transmission strategies that are sum-rate optimal:

1) Find the chain representation of the parallel LDIC by simply forming chain representations of the component

[1] On the remaining levels each transmitter would simply send a known signal, conveying no information.

sub-LDICs. If a chain starts with a level belonging to user A(B), call it an A(B)-chain.

2) Let $S_A(S_B)$ be the set of A(B)-chains that are of even length greater than 2. If $n_{12}^k > n_{11}^k$ and $n_{21}^k > n_{22}^k$, for any $k = 1, \dots L$, then let $S_A(S_B)$ also include A(B)-chains of length 2 from those sub-LDICs. Otherwise, let $S_A(S_B) = \emptyset$.
3) Take an element c_A from S_A and c_B from S_B and merge them creating three chains.
4) $S_A \leftarrow S_A \setminus c_A$ and $S_B \leftarrow S_B \setminus c_B$
5) Repeat this process till $S_A = \emptyset$ or $S_B = \emptyset$.
6) Transmit independent bits from a MIFS in the resulting set of chains.

In [13], essentially the same algorithm was shown to always achieve sum-capacity of a single ($L = 1$) LDIC. The proof of this was based starting with the chain representation of the channel and iteratively showing that the proposed approach is optimal for larger and larger groups of chains. The same proof technique of [13] applies in this case. This is because when a parallel LDIC is represented in terms of interference chains, all the sub-LDICs, with their own chains, together form a bigger pool of chains. The ability to code across sub-LDICs is facilitated by the fact that any A-chain and B-chain can be merged irrespective of whether they come from the same sub-LDIC or different ones. The proof of optimality of the above algorithm thus exactly mimics that given in [13] and so is omitted here. The computational complexity of the algorithm clearly is polynomial in L.

Note that, in a symmetric setting, where each of the component sub-LDICs are symmetric LDICs (parametrized by the single coupling parameter α^k), separability is always guaranteed. This is because for any sub-LDIC, each even A-chain has a corresponding even B-chain, so that there is no need for merging across sub-LDICs. As a consequence, a single parameter generalized degrees of freedom (GDoF) cannot reveal the gains from 'coding across channels'.

We remark that Theorem 1 is not needed to determine the capacity of a 2-user L-parallel LDIC; this can be done directly as in the $L = 1$ case by viewing this as a special case of the general deterministic model in [1]. Whether such a channel is separable or not can then be seen by comparing the sum-capacity to the maximum sum-rate achieved with independent coding. The usefulness of Theorem 1 is that the resulting simple coding schemes provide more insight into when this occurs.

III. LDIC WITH OUTPUT FEEDBACK

The capacity region of the LDIC with complete output feedback was characterized in [10]. There, in a symmetric case, two regimes were identified where feedback increases capacity compared to the non-feedback case. The first is a weak interference regime ($\alpha < \frac{2}{3}$) where a 'resource-hole interpretation' attributes the sum-rate gain due to feedback to availability of signal levels not usable in the non-feedback case. The second regime is a very strong interference situation

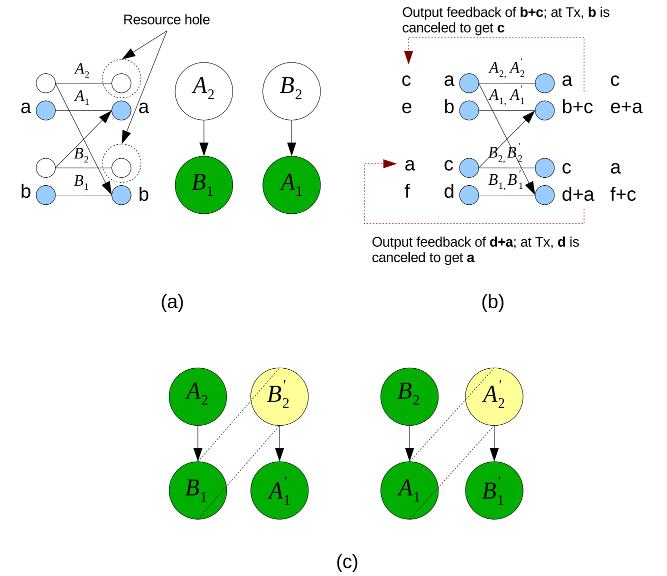

Fig. 3. LDIC with output feedback at $\alpha = \frac{1}{2}$. This example is taken from [10].

($\alpha > 2$) where availability of more cross levels than direct levels is put to good use by 'routing' ones own signal via the other user and using feedback. Our interference chain representation also captures these two phenomena as illustrated next. Further, our simple algorithm to design achievable scheme also applies in case of LDIC with output feedback, albeit with a slight modification in the definition of merging when it involves even chains of length 2.

Consider a symmetric 2-user LDIC with $\alpha = \frac{1}{2}$ as shown in Fig. 3(a). An optimal achievable scheme without feedback is shown in that figure which has resource holes in the sense that there exist signal levels that are used neither for sending data nor for accommodating interference. This predicament can also be understood in terms of interference chains. Indeed, whenever an interference chain has an even length, then there always exists a MIFS for that chain that leaves a 'resource hole'. When the length of the chain is greater than 2, then the merge operation described previously can remove these holes. However, when the length of the chain is 2, as in this example, the two chains can not be merged because the copy of a bit sent from a node at the end of a chain does not arrive at the other receiver. Output feedback remedies this by enabling chains to be 'merged' over time. This merging mimics the scheme in [13] which is shown in Fig. 3(b) and described next. In the first time instant, user 1 transmits independent bits $\{a, b\}$. Similarly, user 2 transmits $\{c, d\}$. At the respective receivers, only the top-most bit can be decoded correctly. However, the bottom interfered bit is fed back to the corresponding transmitters. This gives user 1's transmitter access to $b + c$ and since b is already known, c can be determined. In the second time instant, user 1 transmits $\{c, e\}$, where e is a new independent information bit. Similarly, user 2 now transmits $\{a, f\}$. Now, at receiver 1, c received at the top level can be

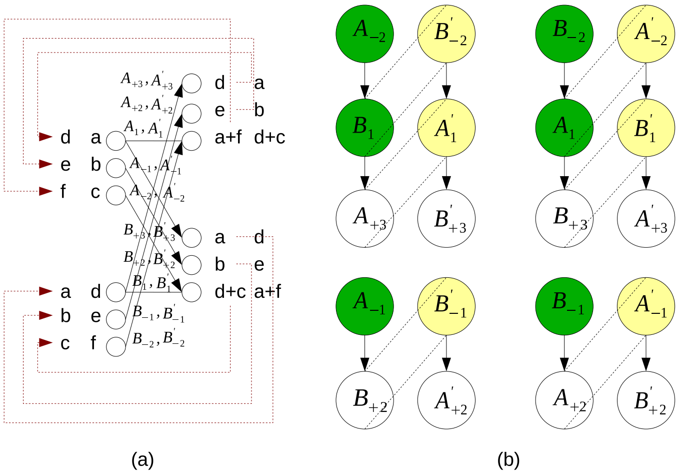

Fig. 4. LDIC with output feedback at $\alpha = 3$. This example is also taken from [10].

used to recover bit b from the previous time instant. Also, bit e can be decoded as the interfering bit a is known. Similarly, receiver 2 can decode bit d from the previous time instant and bit f from the current. Thus, a total of 6 bits over 2 time instants can be communicated, achieving sum-capacity. The interference chain representation of the same channel over two time instants is shown in Fig. 3(c), with the variables for the second time instant being denoted as primes. With the availability of output feedback, a length 2 even chain can now participate in merging. However, the definition of merging, when one of the chains is length 2, should now be: "use feedback from the received level (in the first time instant) to transmit the other user's signal (in the second time instant) by canceling off one's own signal".

In [13], the proof of optimality of the algorithm as in Theorem 1 involved a key step where a 'bounding chain' obtained from two length 2 chains was shown to have a maximum capacity of 2 bits. When feedback is present, it can be shown that the bounding chain has a capacity of 3 bits, thus accounting for the ability to merge two even chains, one of them being of length 2 and also showing that we cannot do any better. The proof of the latter observation can be found in the extended version of this paper [14].

In the very strong interference regime ($\alpha > 2$), the gain due to feedback can be potentially unlimited. This is illustrated in Fig. 4(a) for $\alpha = 3$, where feedback increases the sum-capacity from 2 bits (obtainable by simply transmitting 1 bit from each user) to 3 bits (where information bits from one user are routed through the other making good use of stronger cross links). In this interference regime, the chain formulation now needs to include levels that may now be marked by $+$ in the subscript denoting levels that are only used for receiving (and not transmitting) or by $-$, denoting levels that are only used for transmitting (and not receiving). Under these circumstances, there are now more merge opportunities between two chains, and even between two odd chains as illustrated in Fig. 4(b). It should be noted that in all these merges, the extended definition of a merge applies.

IV. Parallel LDIC with output Feedback

In the previous two sections, the interference chain representation has been exploited to design sum-rate achievable schemes for parallel LDICs and LDICs with output feedback. In this section, we combine these and consider parallel LDICs with output feedback. As discussed in the previous section, for weak interference, for output feedback to increase capacity, there must be at least one length 2 chain starting with one user and a corresponding 'un-matched' even chain for the other. As the next example shows, in a parallel LDIC with output feedback, the opportunities for finding such pairs are not limited to a single channel.

Consider a 2-parallel LDIC with parameters $n_{11}^1 = 4, n_{12}^1 = 3, n_{21}^1 = 2, n_{22}^1 = 3$ and $n_{11}^2 = 2, n_{12}^2 = 1, n_{21}^2 = 2, n_{22}^2 = 3$ as shown in Fig. 5(a). In the corresponding chain representation, the first sub-LDIC has one A-chain of length 3 and one B-chain of length 4, while the second sub-LDIC has one A-chain of length 2 and one B-chain of length 3. Without output feedback, there is no opportunity of merging even when we are allowed to code across sub-LDICs giving a sum-capacity of 7 bits. Now suppose we consider output feedback but require independent coding on each sub-LDIC. Again, since each channel has only one even length chain, there is no opportunity for merging, so the sum-capacity is again 7 bits.

Next consider both coding across sub-LDICs and using output feedback. In this case we can 'merge' the B-chain of length 4 on the first channel (at first time instant) with the length 2 A-chain on the second channel (at second time instant) as well as the B-chain of length 4 on the first channel (at second time instant) with the length 2 A-chain on the second channel (at first time instant) to increase the sum-capacity. The result of doing this is shown in Fig. 5(b); the corresponding coding scheme is shown in Fig. 5(a) and will be described next. At time instant 1, in sub-LDIC 1, user 1 transmits independent bits $\{a, b, c, d\}$, while user 2 transmits only a bit f from its top level. In sub-LDIC 2, user 1 transmits only a bit e from its top level while user 2 transmits independent bits $\{g, h, i\}$. Receiver 1 can decode bits $\{a, c, d, e\}$ but not b as it gets an interfered bit $b + f$ instead. Receiver 2 can decode bits $\{f, g, i\}$ but not bit h as it receives $h+e$. However, making use of output feedback, receiver 1 can feed back bit $b + f$ and receiver 2 can feed back bit $h + e$. Since transmitter 1 knows bit b, it can extract bit f. Similarly, transmitter 2 can also get bit e using its knowledge of h. At time instant 2, in sub-LDIC 1, user 1 transmits $\{j, k, l, m\}$ while user 2 transmits e. In sub-LDIC 2, user 1 transmits bit f while user 2 transmits 3 new independent bits $\{n, o, p\}$. At receiver 1, $\{j, k, l, m\}$ can be decoded as receiver 1 already knows bit e. Further, receiver 1 can can decode f on sub-LDIC 2 and use this to extract b. At receiver 2, $\{n, o, p\}$ can be decoded as f is known from previous time instant. On the other hand, the new knowledge of e enables it to decode bit h from the first time instant. This scheme results in a sum-rate of 8, which is larger than what is achievable with separate

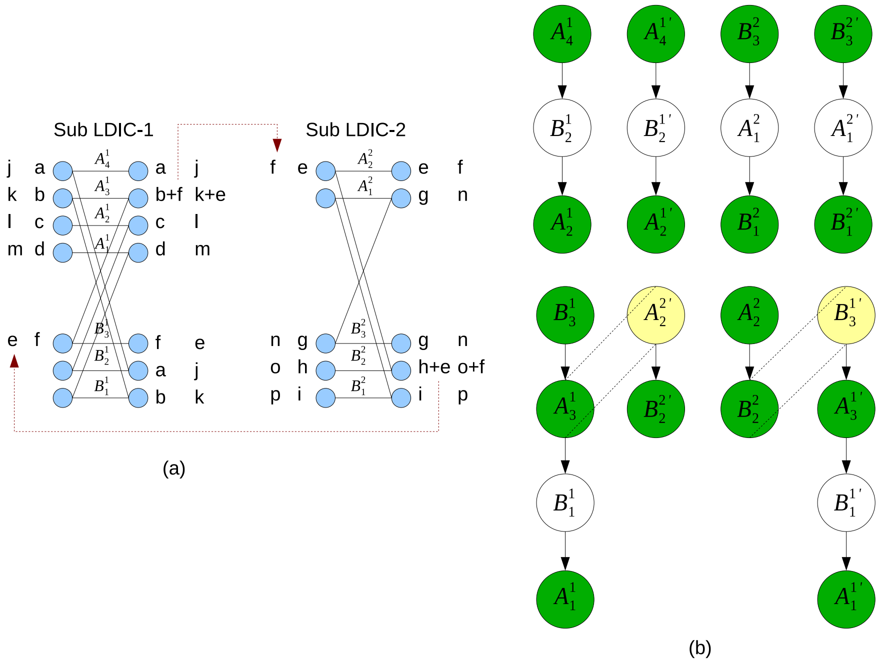

Fig. 5. A 2-parallel LDIC where output feedback increases capacity.

encoding, with or without feedback. More interestingly, note that, without feedback, this channel was separable, while with feedback it is not, showing that feedback can impact the separability of parallel interference channels.

For a general 2-user L-parallel LDIC, Theorem 1 can again be applied using the 'extended' definition of merging and shown to yield the sum-capacity. Applying this to the previous example, it follows that 8 bits is indeed the sum-capacity.

V. CONCLUSION

The interference chain representation of LDICs and interpretation of simple coding across signal levels in terms of chain merges provides a new way of looking at these channels. This novel view can not only produce simple sum-capacity achievable schemes in cases where the capacity region is already known but also in cases where the capacity has not yet been explored. Moreover, this approach provides insights into when L-parallel LDICs are separable, and when feedback can increase the capacity of such channels.

REFERENCES

[1] A. El Gamal and M. Costa, "The Capacity Region of a Class of Deterministic Interference Channels," *in IEEE Transactions on Information Theory*, Vol. IT-28, No. 2, pp. 343-346, Mar. 1982.

[2] G. Bresler and D. Tse, "The two-user Gaussian interference channel: a deterministic view,"in *Euro. Trans. Telecomm., vol. 19, no. 4, pp. 333-354*, June 2008.

[3] S. Avestimehr, S. Diggavi, and D. Tse, "A Deterministic Approach to Wireless Relay Networks," *Allerton Conference on Communication, Control, and Computing*, Monticello, IL, September 2007.

[4] R. Etkin, D. Tse, and H. Wang, "Gaussian Interference Channel Capacity to within One Bit", *IEEE Trans. on Information Theory*, vol. 54, no. 12, pp. 5534-5562, 2008.

[5] S.T. Chung, and J.M. Cioffi, "The Capacity Region of Frequency-Selective Gaussian Interference Channels Under Strong Interference, in *IEEE Transactions on Communications, Vol. 55, NO. 9, Sep. 2007.*

[6] R. Berry and D. Tse, "Information Theoretic Games on Interference Channels",in *Proceedings of the 2008 IEEE International Symposium on Information Theory (ISIT)*, Toronto, Canada, July, 2008.

[7] X. Shang, B. Chen, and G. Kramer, "On Sum-Rate Capacity of Parallel Gaussian Symmetric Interference Channels", in *Proceedings of the 2008 IEEE GLOBECOM Conference*, New Orleans, LA, Dec., 2008.

[8] S.W. Choi, and S.Y. Chung, "On the Separability of Parallel Gaussian Interference Channels, arXiv:0905.1537

[9] X. Shang, B. Chen, G. Kramer and H.V. Poor, "Noisy-Interference Sum-Rate Capacity of Parallel Gaussian Interference Channels", in *Proceedings of the 2009 IEEE International Symposium on Information Theory (ISIT)*, Seoul, Korea, July, 2009.

[10] C. Suh and D. Tse, "Feedback capacity of the Gaussian interference channel to within 2 bits", in *IEEE Transactions on Information Theory, vol. 57, no. 5, pp. 2667-2685, May 2011.*

[11] J. Xu, X. Shang, B. Chen, and H.V. Poor, "Parallel Discrete Memoryless Interference Channels Under Strong Interference: Separability and Capacity Region Results", in *Proceedings of the 2010 IEEE Information Theory Workshop (ITW)*, Cairo, Egypt, Jan., 2010.

[12] L. Sankar, X. Shang, E. Erkip, and H.V. Poor, "Ergodic Fading Interference Channels: Sum-Capacity and Separability", arXiv:0906.0744v2

[13] S. Saha, and R. Berry, "The Combinatorial Structure of Linear Deterministic Interference Channels", in *Proceedings of the 2012 IEEE Information Theory Workshop (ITW)*, Lausanne, Switzerland, Sep., 2012.

[14] S. Saha, and R. Berry, "A Combinatorial Look into Deterministic Interference Channels", *in preparation*.