On the Minimum Energy of Sending Gaussian Multiterminal Sources over the Gaussian MAC

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***Abstract*—We study the minimum energy of sending Gaussian multiterminal sources over the Gaussian multiple access channel (MAC). Distributed transmitters observe Gaussian multiterminal sources and describe their observations to a central decoder, which desires to reconstruct the sources under MSE constraints. We first lower bound the minimum energy by a cut-set argument which couples the transmitted signals and reconstruction errors. For achievability, separate source-channel coding is first studied as a benchmark. We then find out the minimum energy that can be achieved uncoded transmission. A hybrid digital/analog scheme is proposed to achieve the best known energy performance.**

I. Introduction

This paper studies the minimum energy of sending Gaussian multiterminal sources over the Gaussian MAC. Our problem falls in the framework of studying the capacity region of transmitting correlated information over the MAC, which remains open in general. The two-terminal case with matched bandwidth was considered by Lapidoth and Tinguely [1], which found the exact capacity region in certain cases. However, for energy considerations, bandwidths are not matched as the channel bandwidth is unlimited and can be optimized out. The energy problem was considered by Jain *et al.* [2] for the two-terminal case.

In this work, we consider the general case with an arbitrary number of positive symmetric Gaussian sources [3]. In Section III, we develop a new tighter lower bound on the minimum energy by following the cut-set argument in [2], but taking into account the distortion correlation in *all* cut-set components. Our approach is inspired by the lower bounding technique of optimization over the distortion matrix in [3]. For achievable schemes, we initially consider separate source-channel coding as a reference in section IV-A. In Section IV-B we study uncoded transmission, which was considered in the bandwidth matched case in [1] and shown to be optimum in certain cases (as it is in many other cases according to the work of Gastpar [4]). Uncoded transmission was also considered in terms of energy for the two-terminal case in [2]. Here we develop a new and generalized uncoded transmission scheme for the M-terminal case, which offers improvement in the low-distortion regime. In Section IV-C we develop a hybrid digital/analog scheme to achieve the best known energy efficiency. Lapidoth and Tinguely [1] proposed a superposition scheme that transmits a linear combination of a source and its quantized version. For the case of discrete messages, Lim *et al.* [5] provided a general hybrid source-channel coding approach, thus generalizing the work of [1]. Hybrid digital-analog schemes have been shown to be competitive for relevant joint source-channel coding problems as well (*cf.* [6]). Numerical results are given in Section V.

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We use the following notation. $S^{(K)} = \{S[k]\}_{k=1}^{K}, X^{(N)} = \{X[n]\}_{n=1}^{N}$ denote drawings of random variables in time. The set of positive integers $\{1, \cdots, M\}$ is denoted by $\mathcal{M}$ and the set of random variables $\{S_1, \cdots, S_M\}$ by $S_{\mathcal{M}}$, with subscript being the set of indices. Boldface, upper case letters denote matrices. $\boldsymbol{C}(a, b, M)$ denotes a symmetric circulant matrix in $\mathbb{R}^{M\times M}$ with diagonal elements a and off diagonal elements b. All logarithms are natural.

II. Problem Setup

1) Sources: Let $(S_1, S_2, \cdots, S_M)$ be a joint Gaussian random vector with zero mean and covariance matrix $\boldsymbol{\Sigma}_S$. The sources are i.i.d. in time. In this work, we restrict ourselves to positive symmetric sources (with equal variance and positive correlation $\rho > 0$), in which case

$$\boldsymbol{\Sigma}_S = \sigma_0^2 \cdot \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} \triangleq \sigma_0^2 \boldsymbol{C}(1, \rho, M)$$

with eigenvalues $\lambda_1 = 1 + (M-1)\rho$ and $\lambda_2 = 1 - \rho$ (with multiplicity $M-1$). We assume $\sigma_0^2 = 1$ and $\rho \in (0, 1)$ without loss of generality, since the cases with $\rho = 0$ and 1 can be handled with special transmission techniques.

2) Encoders: In each epoch, transmitter m observes and observes $S_m[k]$ to the decoder over the noisy channel. Denote the encoding function at transmitter m as $f_m^{(K,N)}(\cdot) : \mathbb{R}^K \to \mathbb{R}^N$, which maps $S_m^{(K)}$ to $X_m^{(N)}$.

3) The channel: We consider the Gaussian MAC for both practical and theoretical interests with the channel output being $Y[n] = \sum_{m=1}^{M} X_m[n] + Z[n]$, where $Z[n]$ is an AWGN. We normalize $\mathrm{E}[(Z[n])^2] = 1$ with double-sided noise power spectrum density $N_0/2 = 1$ W/Hz.

4) The decoder: The decoder reconstructs the sources from $Y^{(N)}$ with $g_m^{(K,N)}(\cdot) : \mathbb{R}^N \to \mathbb{R}^K$, yielding $\hat{S}_m^{(K)} =$

$g_m^{(K,N)}(Y^{(N)})$, with distortion measured by

$$d_m^{(K)} = \frac{1}{K}\mathrm{E}\left[|\hat{S}_m^K - S_m^K|^2\right] = \frac{1}{K}\sum_{k=1}^{K}\mathrm{E}\left[(S_m[k]-\hat{S}_m[k])^2\right].$$

5) Achievability: We define achievability as follows.

Definition 1: For a given distortion constraint $(D_1, \cdots, D_M)$, an energy tuple $(E_1, \cdots, E_M)$ is achievable if for any $\epsilon > 0$, there exist encoding functions $\left(f_1^{(K,N)}, \cdots, f_M^{(K,N)}\right)$ and decoding functions $\left(g_1^{(K,N)}, \cdots, g_M^{(K,N)}\right)$ such that $\forall m \in \mathcal{M}$,

$$\frac{1}{K}\mathrm{E}\left[|X_m^{(N)}|^2\right] \le E_m + \epsilon, \quad \text{and} \quad d_m^{(K)} \le D_m + \epsilon.$$

The achievable energy region is the convex hull of all energy tuples, i.e.,

$$\mathcal{E}(D_1,\cdots,D_M) = \{(E_1,\cdots,E_M) : d_m^{(K)} \le D_m+\epsilon, \forall m \in \mathcal{M}\}.$$

To simplify our analysis and exposition, we only give results on the *symmetric* case, with equal distortion constraints on individual components, and study the minimum energy $\underline{E}(D) \triangleq \min\{E|(E,\cdots,E) \in \mathcal{E}(D,\cdots,D)\}$. Our results apply to the asymmetric case as well.

III. Lower Bound

For the two-terminal case with infinite bandwidth, Jain *et al.* [2] provided a composite lower bound using a cut-set argument. In the following theorem, we present an improved lower bound using ideas from [3] by optimizing over the distortion matrix, which is defined as $\boldsymbol{D} \triangleq \frac{1}{K}\sum_{k=1}^{K}\boldsymbol{D}[k]$ with $\boldsymbol{D}[k] = \mathrm{cov}(S_1[k]-\hat{S}_1[k],\cdots, S_M[k]-\hat{S}_M[k])$.

Theorem 1: The minimum energy $\underline{E}(D)$ is lower bounded by

$$E^{lb}(D) \triangleq \frac{2\overline{R}(D,\theta^\star,M)}{M^2}\left[\frac{(M-1)^2\overline{R}(D,\theta^\star,1)}{\overline{R}(D,\theta^\star,M)-\overline{R}(D,\theta^\star,1)}+1\right],$$

where

$$\overline{R}(D,\,\theta^\star,\,M) \triangleq \frac{1}{2}\log\frac{(1-\rho)^M\,\phi(M,\rho)}{D^M(1-\theta^\star)^M\,\phi(M,\theta^\star)}, \tag{1}$$

$$\theta^\star \triangleq \frac{1}{D}\max\left(0, D-\lambda_2\right), \tag{2}$$

$$\phi(L,x) \triangleq \frac{1+(M-1)x}{1+(M-L-1)x}. \tag{3}$$

Before proving the lower bound, we give a lemma stating that there is no performance loss in assuming that the resultant distortion matrix is circulant. The lemma can be proved by a permutation and time-sharing argument using source and channel symmetry (see more details in [7]).

Lemma 1: Let $\tilde{E}(D)$ be the minimum energy achieved by a set of encoding and decoding functions such that the distortion matrix $\boldsymbol{D} = d\cdot\boldsymbol{C}(1,\,\theta,\,M)$ is circulant, with $\theta \in \left(-\frac{1}{M-1}, 1\right)$ and $d \le D+\epsilon$, then it holds that $\underline{E}(D) = \tilde{E}(D)$, i.e., the circulant distortion matrix is optimal.

Proof of Theorem 1: For any nontrivial cut of the MAC, we first upper bound the mutual information across the cut by a function of the transmission energy, and then lower bound it by a function of the distortion matrix. By connecting them, we have the lower bound on $\underline{E}(D)$. Note that any cuts with the same size $L\,(L \in \mathcal{M})$ are identical due to symmetry. More details are given below.

We first upper bound $I(Y^{(N)}; S_{\mathcal{L}}^{(K)}|S_{\mathcal{L}^C}^{(K)})$ by

$$\begin{aligned}
I(Y^{(N)}; S_{\mathcal{L}}^{(K)}|S_{\mathcal{L}^C}^{(K)}) &\le I(Y^{(N)}; X_{\mathcal{L}}^{(N)}|X_{\mathcal{L}^C}^{(N)}) \\
&\le \frac{1}{2}\sum\nolimits_{n=1}^{N}\mathrm{var}\left(\sum\nolimits_{l\in\mathcal{L}}X_l[n]\middle|\,X_{\mathcal{L}^C}[n]\right) \quad (4)\\
&= \frac{1}{2}\sum\nolimits_{n=1}^{N}L(1-\hat{\rho}_n)\,\phi(L,\hat{\rho}_n)\mathrm{var}(X_m[n]) \quad (5)\\
&\le \frac{1}{2}L(1-\hat{\rho})\,\phi(L,\hat{\rho})\sum\nolimits_{n=1}^{N}\mathrm{var}(X_m[n]) \quad (6)\\
&\le \frac{1}{2}KL(1-\hat{\rho})\,\phi(L,\hat{\rho})\underline{E}(D), \quad (7)
\end{aligned}$$

where

- (4) follows from standard inequalities as in [1], [2], [7].
- (5) is due to optimality of $\mathrm{cov}(X_{\mathcal{M}}[n]) = \mathrm{var}(X_m[n])\,\boldsymbol{C}\,(1,\hat{\rho}_n,M)$ with

$$\hat{\rho}_n \triangleq \frac{\mathrm{E}(X_1[n]X_2[n])}{\sqrt{\mathrm{var}(X_1[n])\mathrm{var}(X_2[n])}} \in \mathcal{R}_{\hat{\rho}} \triangleq \left(\max\left(-\frac{1}{M-1}, -\rho\right), \rho\right),$$

  which is bounded by the maximum correlation theory (*cf.* [1, Lemma B.2]). We define $\hat{\rho} \triangleq \frac{1}{N}\sum_{N=1}^{n}\hat{\rho}_n \in \mathcal{R}_{\hat{\rho}}$.
- (6) follows from Jensen's inequality.

We then lower bound $I(Y^{(N)}; S_{\mathcal{L}}^{(K)}|S_{\mathcal{L}^C}^{(K)})$ by

$$\begin{aligned}
&I(Y^{(N)}; S_{\mathcal{L}}^{(K)}|S_{\mathcal{L}^C}^{(K)}) \\
&= Kh(S_{\mathcal{L}}|S_{\mathcal{L}^C}) - \sum_{k=1}^{K}h(S_{\mathcal{L}}[k]|S_{\mathcal{L}}^{(k-1)}, S_{\mathcal{L}^C}^{(K)}, Y^{(N)}) \quad (8)\\
&= Kh(S_{\mathcal{L}}|S_{\mathcal{L}^C}) - \sum_{k=1}^{K}h\left(S_{\mathcal{L}}[k]\middle|S_{\mathcal{L}}^{(k-1)}, S_{\mathcal{L}^C}^{(K)}, Y^{(N)}, \hat{S}_{\mathcal{M}}[k]\right) \quad (9)\\
&= Kh(S_{\mathcal{L}}|S_{\mathcal{L}^C}) \\
&\quad - \sum_{k=1}^{K}h\left(\Delta_{\mathcal{L}|\mathcal{L}^C}^{\gamma}[k]\middle|S_{\mathcal{L}}^{(k-1)}, S_{\mathcal{L}^C}^{(K)}, Y^{(N)},\, \hat{S}_{\mathcal{L}}[k], \hat{S}_{\mathcal{L}^C}[k]\right) \quad (10)\\
&\ge Kh(S_{\mathcal{L}}|S_{\mathcal{L}^C}) - \sum\nolimits_{k=1}^{K}h\left(\Delta_{\mathcal{L}|\mathcal{L}^C}^{\gamma}[k]\right) \quad (11)\\
&\ge Kh(S_{\mathcal{L}}|S_{\mathcal{L}^C}) - \sum_{k=1}^{K}\frac{1}{2}\log\left[(2\pi e)^L\boldsymbol{\Gamma}\boldsymbol{D}[k]\boldsymbol{\Gamma}^T\right] \quad (12)\\
&\ge \frac{K}{2}\log\frac{|\mathrm{cov}(S_{\mathcal{L}}|S_{\mathcal{L}^C})|}{\left|\boldsymbol{\Gamma}\boldsymbol{D}\boldsymbol{\Gamma}^T\right|}, \quad (13)
\end{aligned}$$

in which

- (9) follows since $\hat{S}_{\mathcal{M}}[k]$ are functions of $Y^{(N)}$.
- In (10), we introduce $\Delta_{l|\mathcal{L}^C}^{\gamma}[k] \triangleq (S_l[k]-\hat{S}_l[k]) - \gamma\sum\limits_{l'\in\mathcal{L}^C}(S'_l[k]-\hat{S}'_l[k]),\ l \in \mathcal{L},\ \gamma \in \mathbb{R}$.
- (11) is true because conditioning reduces entropy.
- With $\boldsymbol{\Gamma} = \left[\boldsymbol{I}_{L\times L}\ \ -\gamma\boldsymbol{1}_{L\times(M-L)}\right]$, the covariance matrix of $\Delta_{\mathcal{L}|\mathcal{L}^C}^{\gamma}[k]$ is $\boldsymbol{\Gamma}\boldsymbol{D}[k]\boldsymbol{\Gamma}^T$, and hence in (12) we upper bound the differential entropy in the second term using the maximum entropy theorem.

- (13) follows from concavity of the logarithmic function.

Since the inequalities (8)-(13) hold for any $\gamma \in \mathbb{R}$, maximizing (13) over γ leads to

$$I(\boldsymbol{Y}^{(N)}; \boldsymbol{S}_{\mathcal{L}}^{(K)} | \boldsymbol{S}_{\mathcal{L}^C}^{(K)}) \geq \max_{\gamma} \frac{K}{2} \log \frac{|\mathrm{cov}(S_{\mathcal{L}}|S_{\mathcal{L}^C})|}{\left|\boldsymbol{\Gamma D \Gamma}^T\right|}$$

$$= \frac{K}{2} \log \frac{(1-\rho)^L \phi(L,\rho)}{d^L (1-\theta)^L \phi(L,\theta)} = K\overline{R}(d,\, \theta,\, L) \quad (14)$$

with the maximum achieved at $\gamma = \frac{\theta}{1+(M-L-1)\theta}$. We obtain a tighter lower bound over that of [2] via augmenting the conditioning in (9) with $\hat{S}_{\mathcal{L}^C}$. However, we only gain when $\theta^\star \neq 0$ so that conditioning of $\hat{S}_{\mathcal{L}^C}$ strictly reduces entropy, i.e., $D \in (\lambda_2, 1)$.

By connecting (12) and (14) and applying all nontrivial cut-set bounds ($L \in \mathcal{M}$) to $\underline{E}(D)$, we have

$$\underline{E}(D) \geq \inf \left\{ \max_L \frac{2\,\overline{R}(d,\, \theta,\, L)}{L(1-\hat{\rho})\, \phi(L,\hat{\rho})} : \mathbf{0} \preceq \boldsymbol{D} \preceq \boldsymbol{\Sigma}_S, d \leq D, \hat{\rho} \in \mathcal{R}_{\hat{\rho}} \right\}.$$

In the sequel, we evaluate the min-max by optimizing over d, θ, and $\hat{\rho}$.

We first optimize over θ and d. Lemma 1 translates the positive semi-definite constraint to one on the eigenvalues

$$\theta \in \mathcal{R}_\theta \triangleq \left(-\frac{1}{M-1}, 1 \right) \bigcap \left[\frac{d-\lambda_2}{d}, \frac{\lambda_1 - d}{(M-1)d} \right]. \quad (15)$$

It can be verified that $\frac{\partial}{\partial \theta} \overline{R}(d,\, \theta,\, L)$ has the same sign as θ, i.e., $\overline{R}(d,\, \theta,\, L)$ is minimized by the smallest nonnegative θ in (15), and also $\overline{R}(d, \theta, L)$ monotonically decreases with d. Since these monotonic properties are independent of L, the infimum holds at $d = D$, $\theta = \frac{1}{D} \max(0, D - \lambda_2)$.

The next step is to optimize over $\hat{\rho}$. Denote the cut-set bounds by

$$\hat{E}(D,\, \hat{\rho},\, L) \triangleq \frac{2\,\overline{R}(D,\, \theta^\star,\, L)}{L(1-\hat{\rho})\, \phi(L,\hat{\rho})}. \quad (16)$$

As a numerical example, $\hat{E}(D,\, \hat{\rho},\, L)$'s are evaluated over $\hat{\rho} \in [0, \rho]$ in Fig. 1 for the case with $M = 10, \rho = 0.5, D = 0.5$ (we only plot over positive $\hat{\rho}$, since we will show that the min-max is always achieved at some $\hat{\rho} \geq 0$).

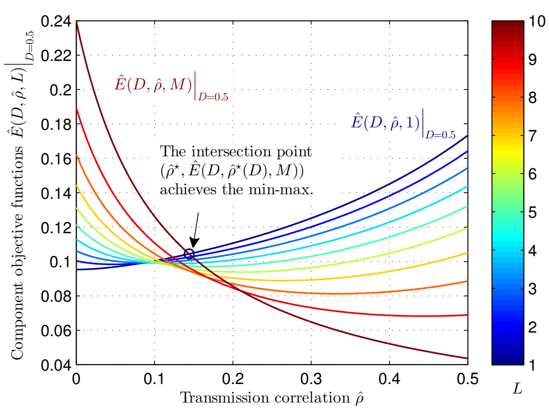

Figure 1. $\hat{E}(0.5, \hat{\rho}, L)\Big|_{D=0.5}$ for the case with $M = 10$ and $\rho = 0.5$.

We show next that there exists a unique $\hat{\rho} \in [0, \rho]$ that achieves the min-max

$$\hat{\rho}^\star(D) = \frac{\overline{R}(D,\, \theta^\star,\, M) - M\overline{R}(D,\, \theta^\star,\, 1)}{M(M-2)\overline{R}(D,\, \theta^\star,\, 1) + \overline{R}(D,\, \theta^\star,\, M)}, \quad (17)$$

where $\hat{E}(D,\, \hat{\rho},\, 1)$ and $\hat{E}(D,\, \hat{\rho},\, M)$ intersect.

1) When $\hat{\rho} < \hat{\rho}^\star(D)$, since $\hat{E}(D, \hat{\rho}, M)$ monotonically decreases with $\hat{\rho}$,

$$\max_L \hat{E}(D, \hat{\rho}, L) \geq \hat{E}(D, \hat{\rho},\, M) \geq \hat{E}(D, \hat{\rho}^\star(D), M). \quad (18)$$

2) When $\hat{\rho} \geq \hat{\rho}^\star(D)$, since $\hat{E}(D,\, \hat{\rho},\, 1)$ increases with $\hat{\rho}$,

$$\max_L \hat{E}(D, \hat{\rho}, L) \geq \hat{E}(D, \hat{\rho}, 1) \geq \hat{E}(D, \hat{\rho}^\star(D), 1) \quad (19)$$

3) When $\hat{\rho} = \hat{\rho}^\star(D)$, we need the following lemma, whose proof is given in [7] due to space limitations.

Lemma 2: $\hat{E}(D, \hat{\rho}^\star(D), L) \leq \hat{E}(D, \hat{\rho}^\star(D), M)$, $\forall L \neq 1$ or M, then $\max_L \hat{E}(D,\, \hat{\rho}^\star(D), L) = \hat{E}(D,\, \hat{\rho}^\star(D), M)$.

Combining (18), (19), and (2), we have

$$\inf_{\hat{\rho} \in \mathcal{R}_{\hat{\rho}}} \max_L \hat{E}(D,\, \hat{\rho},\, L) = \hat{E}(D,\, \hat{\rho}^\star(D),\, M).$$

The proof of Theorem 1 is completed by the existence and uniqueness of $\hat{\rho}^\star(D)$ in the following lemma, whose proof follows from continuity of $\hat{E}$. More details are given in [7].

Lemma 3: For any fixed D, there exists one and only one point $\hat{\rho}^\star(D) \in [0, \rho]$ such that $\hat{E}(D, \hat{\rho}, 1) = \hat{E}(D, \hat{\rho}, M)$, which achieves the min-max. ∎

We now make an observation on $\hat{\rho}^\star(D)$ as $D \to 0$ or 1 and state the following proposition with proof given in [7].

Proposition 1: The optimal transmission correlation for the limit case is either $\lim_{D \to 1} \hat{\rho}^\star(D) = \rho$, $\lim_{D \to 0} \hat{\rho}^\star(D) = 0$. That is, in the high-distortion regime the optimal transmitted signals tend to be maximally correlated, whereas in the low-distortion regime they tend to be uncorrelated.

IV. Upper Bounds

A. Separation Scheme

Motivated by its optimality in the point-to-point scenario, we study separate source-channel coding as our first achievable scheme. We give the upper bound from the separation scheme in the following theorem but omit the proof, which results from combining the sum rate of Gaussian multiterminal source coding [3] with the minimum energy of -1.59 dB to send each bit of independent information over the Gaussian MAC [8].

Theorem 2: The separation scheme achieves

$$E^s(D) \triangleq \frac{1}{M} \log \left[\left(1 + \frac{\lambda_1}{q(D)}\right) \left(1 + \frac{\lambda_2}{q(D)}\right)^{M-1} \right] \geq \underline{E}(D)$$

with

$$q(D) = \frac{\beta(D) + \sqrt{\beta(D)^2 + 4\lambda_1\lambda_2 D(1-D)}}{2(1-D)},$$

$$\beta(D) = (\lambda_1 + \lambda_2)D - \lambda_1\lambda_2.$$

The upper bound $E^s(D) \approx \log \frac{1}{D}$ is asymptotically optimal in the lower-distortion regime. We formalize this fact in the following proposition, which can be proved via taking limit on the ratio $\frac{E^s(D)}{E^{lb}(D)}$ as $D \to 0$.

Proposition 2: In the low-distortion regime with $D \to 0$, the separation scheme is asymptotically optimal in the sense that the ratio between the upper bound $E^s(D)$ from the separation scheme and the lower bound $E^{lb}(D)$ tends to one, *i.e.*, $\lim_{D\to 0} \frac{E^s(D)}{E^{lb}(D)} = 1$.

B. Uncoded Transmission

An uncoded scheme of sending bivariate Gaussian sources is considered for the bandwidth-matched case in [1] and for minimum energy in [2]. In the prior implementations of uncoded transmission, each encoder only sends one scaled version of its observation. When studying the minimum energy, we can do better by utilizing bandwidth expansion as in the following transmission scheme, since bandwidth is unlimited.

For each source sample $S_{\mathcal{M}}$, the encoders utilize the channel M' times, $M' \in \mathbb{Z}^+$ (*e.g.*, via time- or frequency-division multiplexing). We define a coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{M' \times M}$ with $\boldsymbol{A}_{ij}$ denoting the scale factor that encoder j adopts to scale its observed source for the i-th channel use. That is, encoder j sends $\boldsymbol{A}_{ij} S_j$, $i \in \mathcal{M}'$ and $j \in \mathcal{M}$, over the i-th channel use. The decoder receives M' noisy linear combinations

$$[Y_1, \cdots, Y_{M'}]^T = \boldsymbol{A} \cdot [S_1, \cdots, S_M]^T + [Z_1, \cdots, Z_{M'}]^T$$

of the source components (corrupted by i.i.d. Gaussian noises $[Z_1, Z_2, \cdots, Z_{M'}]$) before estimating the sources by computing the conditional expectations. When $M' = 1$, our scheme degenerates to the one in [1], [2]. We seek for the optimal $\boldsymbol{A}$ that gives the minimum energy. The resulting upper bound on $\underline{E}(D)$ from uncoded transmission is given in the following theorem.

Theorem 3: Optimal uncoded transmission of Gaussian multiterminal sources over the Gaussian MAC achieves the energy-distortion function

$$E^u(D) \triangleq \begin{cases} \frac{1}{D} - \frac{\lambda_1 - \rho}{\lambda_1 \lambda_2}, & D < \lambda_2; \\ \frac{1-D}{\lambda_1[MD-(M-1)\lambda_2]}, & D \geq \lambda_2. \end{cases} \tag{20}$$

Proof: Using the received signals $[Y_1, \cdots, Y_{M'}]$, the decoder estimates the sources as the conditional expectations $[\hat{S}_1, \cdots, \hat{S}_M]^T = \boldsymbol{\Sigma}_S \boldsymbol{A}^T \left(\boldsymbol{A}\boldsymbol{\Sigma}_S \boldsymbol{A}^T + I\right)^{-1} [Y_1, \cdots, Y_{M'}]^T$, and achieves a distortion matrix

$$\boldsymbol{D} = \mathrm{cov}\left(S_{\mathcal{M}} \,|\, Y_{\mathcal{M}}\right) = \left(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T\boldsymbol{A}\right)^{-1}. \tag{21}$$

Our goal is to solve the following optimization problem

$$\min_{\boldsymbol{A}} \left\{ \frac{\mathrm{tr}(\boldsymbol{A}^T\boldsymbol{A})}{M} : \boldsymbol{D} = \left(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T\boldsymbol{A}\right)^{-1}, \max(\mathrm{diag}(\boldsymbol{D})) \leq D \right\} \tag{22}$$

and find the optimal energy performance of uncoded transmission. However, it is cumbersome to directly optimizing the transmission energy over $\boldsymbol{A}$. Therefore, in order to find the optimal solution, we first make the following simplifications without loss of optimality.

1) Assuming that $\boldsymbol{A}^T\boldsymbol{A}$ is circulant does not lose optimality. This is due to (21), where both $\boldsymbol{\Sigma}_S$ and $\boldsymbol{D}$ (*cf.* Lemma 1) are circulant. We let $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Sigma}\boldsymbol{Q}^T$ be the eigenvalue decomposition of $\boldsymbol{A}^T\boldsymbol{A}$ with columns of $\boldsymbol{Q}$ being eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \cdots, \sigma_M)$, $\sigma_1, \cdots, \sigma_M \geq 0$. Owing to the circulant nature of $\boldsymbol{A}^T\boldsymbol{A}$, columns of $\boldsymbol{Q}$ are also eigenvectors of $\boldsymbol{D}$ and $\boldsymbol{\Sigma}_S$ since they are circulant.
2) The minimum energy in (22) is lower bounded by
$$\min_{\boldsymbol{A}} \left\{ \frac{\mathrm{tr}(\boldsymbol{A}^T\boldsymbol{A})}{M} : \boldsymbol{D} = \left(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T\boldsymbol{A}\right)^{-1}, \frac{\mathrm{tr}(\boldsymbol{D})}{M} \leq D \right\},$$
where we use the fact that $\frac{1}{M}\mathrm{tr}(\boldsymbol{D}) \leq \max(\mathrm{diag}(\boldsymbol{D}))$ to relax the distortion constraints on individual distortions. In fact, the lower bound above holds with equality, since we show that $\boldsymbol{D}$ is circulant.

With these simplifications, we turn the matrix optimization problem in (22) to one of

$$\min_{\sigma_1, \cdots, \sigma_M} \left\{ \sum_{i=1}^{M} \sigma_i : \frac{\lambda_1}{1 + \lambda_1 \sigma_1} + \sum_{i=2}^{M} \frac{\lambda_2}{1 + \lambda_2 \sigma_i} \leq MD \right\}$$

over the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$. The Karush-Kuhn-Tucker conditions lead to the optimal solution of

$$\sigma_1 = \left(\frac{1}{D} - \frac{1}{\lambda_1}\right)^+, \sigma_2 = \cdots = \sigma_M = \left(\frac{1}{D} - \frac{1}{\lambda_2}\right)^+.$$

The optimal $\boldsymbol{A}$ is not unique as long as $\boldsymbol{A}^T\boldsymbol{A}$ is optimal. However, one coefficient matrix that achieves optimality is $\boldsymbol{A} = \mathrm{diag}(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \cdots, \sqrt{\sigma_M}) \cdot \boldsymbol{Q}^T$, which is a unitary transform $\boldsymbol{Q}^T$ that decorrelates the sources (without explicit encoder cooperation) followed by a water-filling like energy allocation on $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_2}$. The optimal number of channel uses is

$$M' = \begin{cases} 1, & D \geq \lambda_2, \\ M, & D < \lambda_2. \end{cases}$$

We make two further observations on M' as follows:

1) When $D \geq \lambda_2$, $M' = 1$, that is, the uncoded scheme in [1], [2] is optimal in this case.
2) It suffices to utilize the channel M times for the M-terminal sources. ■

When $D < \lambda_2$, $E^u(D)$ improves the energy efficiency over the uncoded scheme in [1], [2]. By utilizing bandwidth expansion, an arbitrarily small distortion can be achieved with high enough energy. However, if the channel is used only once, the distortion cannot go lower than $\frac{(M-1)\lambda_2}{M}$ due to interference from the encoders.

C. Hybrid Digital/Analog Scheme

We propose the following hybrid scheme, in which digital and analog signals are transmitted over orthogonal channel uses. In the analog portion, each encoder transmits M scaled versions of its observation over M channel uses (cf. Section IV-B), and the decoder receives $[Y_1, \cdots, Y_M] = \boldsymbol{A} \cdot [S_1, \cdots, S_M]^T + [Z_1, \cdots, Z_M]^T$ with $\boldsymbol{A} \in \mathbb{R}^{M \times M}$. In digital transmission, using $\boldsymbol{Y}_a$ as side information, we employ Gaussian quantization and Slepian-Wolf coding to transmit the remaining fine information to meet the target distortion. We give the following theorem for the hybrid scheme and only an outline for the proof with more details given in [7].

Theorem 4: The hybrid scheme achieves the energy-distortion function

$$E^h(D) \triangleq \inf \left\{ \alpha^2 + \frac{1}{M} \log \left| \boldsymbol{I} + \frac{1}{q}(\boldsymbol{\Sigma}_S^{-1} + \alpha^2)^{-1} \right| : \right.$$
$$\left. \alpha, q \in \mathcal{R}^+, \left[\left(\boldsymbol{\Sigma}_S^{-1} + \alpha^2 + \boldsymbol{I}/q \right)^{-1} \right]_{ii} \le D, i = \in \mathcal{M} \right\},$$

which upper bounds $\underline{E}_{\mathrm{m}}(D)$. In the analog portion of the optimal hybrid scheme, the channel is utilized only once and all encoders use the same factor $\alpha \in \mathbb{R}^+$ to scale their respective sources.

Proof: In digital transmission, we introduce auxiliaries $\tilde{S}_m = S_m + Q_m$ with Q_m i.i.d. $\sim \mathcal{N}(0, q)$. Using $Y_{\mathcal{M}}$ as side information, we employ random binning and an optimal channel code to transmit the remaining fine information to meet the target distortion. This way, given that the energy each encoder uses in digital part is at least $\frac{N_0}{M} I(S_{\mathcal{M}}; \tilde{S}_{\mathcal{M}} | Y_{\mathcal{M}}) + \epsilon$, the decoder can recover $\tilde{S}_{\mathcal{M}}$ with arbitrarily small probability of error. Calculation of the conditional mutual information follows $I(S_{\mathcal{M}}; \tilde{S}_{\mathcal{M}} | Y_{\mathcal{M}}) = \frac{1}{2} \log \left| \boldsymbol{I} + \frac{1}{q}(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T \boldsymbol{A})^{-1} \right|$. Hence the decoder can estimate the sources using $\tilde{S}_{\mathcal{M}}$ and Y_a and achieve MSE distortion $d(\zeta_1, \zeta_2, q) \triangleq \mathrm{E}[(S_m - \hat{S}_m)^2] = [(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T \boldsymbol{A} + \boldsymbol{I}/q)^{-1}]_{ii}$ on S_m.

Without loss of optimality, we let $\boldsymbol{A}^T \boldsymbol{A} = \boldsymbol{C}(\zeta_1, \zeta_1 - \zeta_2, M)$ with $\zeta_1, \zeta_2 \in \mathbb{R}^+$ and formulate the following optimization problem

$$\min \quad \hat{E}^h(\zeta_1, \zeta_2, q) \triangleq \zeta_1 + \frac{1}{M} \log \left| \boldsymbol{I} + \frac{1}{q}(\boldsymbol{\Sigma}_S^{-1} + \boldsymbol{A}^T \boldsymbol{A})^{-1} \right|$$
$$\text{s.t.} \quad d(\zeta_1, \zeta_2, q) \le D, \quad \zeta_2 \ge 0.$$

The Karush-Kuhn-Tucker conditions necessarily leads to

$$\zeta_2 \cdot \left(\frac{\partial d}{\partial \zeta_1} \right)^{-1} \left(\frac{\partial \hat{E}^h}{\partial \zeta_2} \frac{\partial d}{\partial \zeta_1} - \frac{\partial \hat{E}^h}{\partial \zeta_1} \frac{\partial d}{\partial \zeta_2} \right) = 0.$$

It holds that $\frac{\partial d}{\partial \zeta_1} < 0$, $\frac{\partial \hat{E}^h}{\partial \zeta_2} \frac{\partial d}{\partial \zeta_1} - \frac{\partial \hat{E}^h}{\partial \zeta_1} \frac{\partial d}{\partial \zeta_2} < 0$, and thus we have $\zeta_2^\star = 0$, *i.e.*, $\boldsymbol{A}^T \boldsymbol{A} = \alpha^2 \mathbf{1}_{M \times M}$. Therefore, in analog transmission, each encoder only transmits one scaled observation per source sample *without needing bandwidth expansion*. This is because digital transmission is more efficient for sending fine information. The corresponding hybrid scheme achieves $E^h(D)$ given in the theorem. ∎

V. Numerical Results

In Fig. 2, we plot our lower bound and three upper bounds (in terms of energy to noise ratio) on $\underline{E}(D)$ for the cases of $M = 10, \rho = 0.5$ and $M = 20, \rho = 0.3$, together with both the lower and the upper bounds in [2] (after we extend them from the case of $M = 2$ in [2] to $M > 2$). It is clearly seen that $E^h(D)$ is the best upper bound, $E^{lb}(D)$ the tighter lower bound, and $\underline{E}(D)$ lies in between $E^h(D)$ and $E^{lb}(D)$. Note that the energy-distortion functions are convex, the curves in the figure do not appear so because we use the dB scale on the energy.

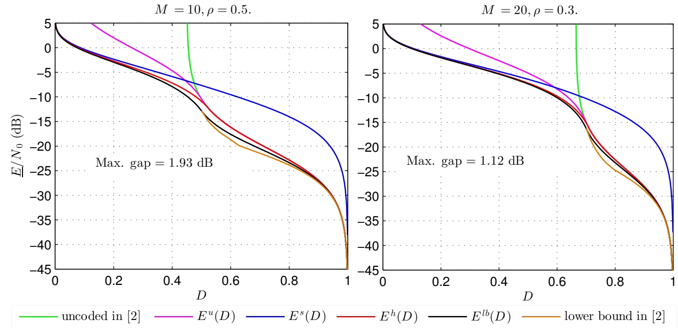

Figure 2. Upper and lower bounds on $\underline{E}(D)$.

As indicated in our upper and lower bounds, energy efficient transmission requires sufficient exploitation of the source correlation via not only statistic means (*e.g*, random binning) but also physical measures of introducing some of the source correlation into the channel codewords. But the optimal transmission correlation remains unknown. We plot the correlation of transmitted signals in the lower and upper bounds for the case with $M = 10, \rho = 0.5$ in Fig. 3. Higher correlation of transmitted signals gives beamforming gain; however, it is not efficient for the encoders to only transmit identical signals especially in the lower-distortion regime, where we should also convey the difference of the sources.

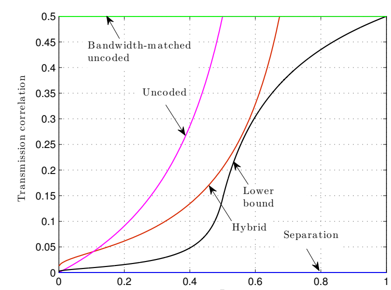

Figure 3. The correlation between transmitted signals in upper and lower bounds for the case with $M = 10$ and $\rho = 0.5$ plotted versus D.

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