Energy-Efficient Communication in the Presence of Synchronization Errors

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***Abstract*—Communication systems are traditionally designed to have tight transmitter-receiver synchronization. This requirement has negligible overhead in the high-SNR regime. However, in many applications, such as wireless sensor networks, communication needs to happen primarily in the energy-efficient regime of low SNR, where requiring tight synchronization can be highly suboptimal.**

In this paper, we model the noisy channel with synchronization errors as an insertion/deletion/substitution channel. For this channel, we propose a new communication scheme that requires only loose transmitter-receiver synchronization. We show that the proposed scheme is asymptotically optimal for the Gaussian channel with synchronization errors in terms of energy efficiency as measured by the rate per unit energy. In the process, we also establish that the lack of synchronization causes negligible loss in energy efficiency. We further show that, for a general discrete memoryless channel with synchronization errors and a general cost function (with a zero-cost symbol) on the input, the rate per unit cost achieved by the proposed scheme is within a factor two of the information-theoretic optimum.

I. Introduction

Traditionally, data transmission in a communication system is based on tight synchronization between the transmitter and the receiver. This tight synchronization is usually achieved through either of two strategies. In the first strategy, synchronization is achieved through periodic transmission of pilot signals, followed by transmission of information over the synchronized channel. In the second strategy, data bits are modulated differentially which implicitly achieves tight synchronization.

The above strategies work well at high signal-to-noise ratios (SNRs) as the energy overhead of achieving tight synchronization is negligible compared to that of data transmission. However, in many applications, such as wireless sensor networks, space communication, or in general any communication system requiring high energy efficiency, communication by necessity has to primarily take place in the low-SNR regime (due to the concavity of the power-rate function). In such scenarios, the energy overhead to achieve tight synchronization becomes significant and can render the aforementioned strategies highly suboptimal in terms of energy efficiency. In fact, it can be shown that requiring tight transmitter-receiver synchronization can have arbitrarily large loss in performance in terms of energy efficiency.

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To mitigate this, in this paper, we develop and analyze a framework to perform data transmission while only requiring loose synchronization between the transmitter and the receiver. To focus on the energy-efficiency aspect, we choose the rate per unit cost (with energy being a prime example of the cost) as our performance metric. We model synchronization errors through channel insertions/deletions—an approach introduced in [1]. To motivate this model, consider a transmitter-receiver pair with unsynchronized clocks, as illustrated in Fig. 1. Due to the absence of synchronization, the value of the clock at the receiver exhibits drift and jitter with respect to the value of the reference clock at the transmitter. This leads to the receiver sampling the transmitted signal either faster than the transmitter, leading to channel insertions, or slower, leading to channel deletions.

Before we describe the contributions of this work in more detail in Section I-B, we provide a brief overview of related work on energy-efficient communication and on channels with synchronization errors.

A. Related Work

It is well known that the capacity per unit energy of a Gaussian channel with noise variance η^2 is $1/(2\eta^2 \ln 2)$, and that this can be achieved with appropriately designed pulse-position modulation [2]. For a general discrete memoryless channel (DMC), [3] has analyzed the reliability function of the rate per unit cost. Subsequently, [4] has obtained a succinct single-letter characterization for the capacity per unit cost of a DMC with a general cost function. These results, however, strongly depend on the channel being memoryless. As discussed next, synchronization errors introduce memory in the channel, and thus the aforementioned results do not apply.

The insertion/deletion/substitution channel was introduced by Dobrushin in 1967 as a model for a channel with synchronization errors [1]. Despite significant research effort since then, the capacity of this channel is still not known [5]–[12]. Indeed, even for one of the simplest versions of this problem, the noiseless binary deletion channel, only loose bounds on the capacity are known. For example, the recent paper [6] provides an approximation of the capacity of the binary deletion channel to within a factor 9. The main difficulty in analyzing these channels arises from the channel memory introduced by the

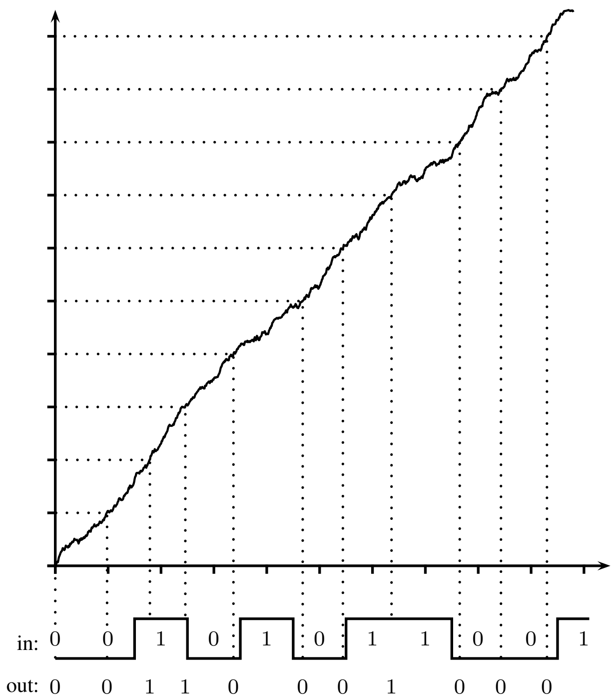

Fig. 1. An example of unsynchronized transmitter-receiver clocks. The figure plots the value of the receiver clock (y-axis) as a function of the value of the reference clock at the transmitter (x-axis). The drift and jitter of the receiver clock are visible. For a transmitted input sequence, the lack of synchronization leads to insertions/deletions in the corresponding sampled output sequence at the receiver (illustrated here for the case without receiver noise).

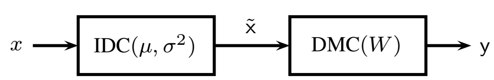

Fig. 2. The end-to-end communication channel between an unsynchronized transmitter-receiver pair is modeled by concatenating an insertion/deletion channel IDC(μ, σ^2) with a discrete memoryless channel DMC(W).

insertions and deletions, which prevents direct application of standard information-theoretic tools.

It is worth emphasizing that the synchronization errors considered here are those at the symbol level. There are other types of synchronization issues. One such issue is frame synchronization, where errors are caused by incorrect identification of the location of the "sync word" in the frame [13]. Energy-efficient communication in the presence of such frame asynchronism has been investigated in [14].

B. Summary of Results

In this paper, we consider communication channels which, in addition to synchronization errors, exhibit receiver noise. We model the end-to-end communication channel between the transmitter and the receiver as the concatenation of two sub-channels, as illustrated in Fig. 2. The first sub-channel is an insertion/deletion channel (IDC), which models synchronization errors. The second sub-channel is a noisy memoryless channel, which models errors due to the receiver noise. The concatenation of the two channels is an insertion/deletion/substitution channel. The details of this model are discussed in Section II.

We first study communication systems with synchronization errors operating over Gaussian channels. We propose a new communication scheme that requires only loose synchronization between the transmitter and the receiver. Specifically, we deal with the lack of transmitter-receiver synchronization by developing pulse-position-modulation waveforms where the signal energy is spread over increasing intervals and guard spaces of increasing lengths are introduced. Decoding at the receiver is based on a sequence of independent hypothesis tests. When the aforementioned durations are chosen appropriately, we show that the scheme asymptotically achieves the information-theoretically optimal performance in terms of the energy efficiency, i.e., the capacity per unit energy. In the process, we also establish that the lack of transmitter-receiver synchronization causes negligible loss in terms of energy efficiency.

We then analyze communication systems with synchronization errors operating over general DMCs and with a general cost function (with a zero-cost symbol). We generalize the proposed achievable scheme for the Gaussian case to DMCs, and we show that the scheme achieves a rate per unit cost within a factor two of the information-theoretic optimum. Thus, while only loose bounds are known for the *capacity* of the general insertion/deletion/substitution channel, we provide here a tight approximation for its *capacity per unit cost*. To establish this, we obtain an upper bound on the capacity per unit cost of the channel in Fig. 2 by considering the effect of the IDC as a specific way of encoding for the DMC with an appropriately modified cost function. The upper bound is then obtained by utilizing the characterization of the capacity per unit cost for memoryless channels in [4].

C. Organization

The remainder of the paper is organized as follows. Section II provides the detailed description of the channel model and the problem formulation. The main results of the paper are summarized in Section III. Due to space constraints, all proofs are omitted; they are reported in the full version of this paper [15].

II. Channel Model and Problem Statement

We consider an insertion/deletion/substitution channel with a cost constraint. The insertion/deletion/substitution channel consists of an IDC connected to a DMC as shown in Fig. 2 in Section I. The insertion/deletion part of the channel models synchronization errors (see Fig. 1 in Section I), the substitution part models noise.

The IDC maps the input sequence $(x[1], \dots, x[T]) \in \mathcal{X}^T$ to the output sequence $(\tilde{\mathsf{x}}[1], \dots, \tilde{\mathsf{x}}[\mathsf{L}]) \in \mathcal{X}^{\mathsf{L}}$ for some random length L, where here and in the following we use sans-serif font to denote random variables. The actions of the IDC are governed by the i.i.d. sequence of states $(\mathsf{s}[1], \dots, \mathsf{s}[T]) \in \{0, 1, 2, \dots\}^T$. State $s[t]$ describes how many times input symbol $x[t]$ appears at the output of the IDC.

Formally, the total number of output bits is given by

$$\mathsf{L} \triangleq \sum_{t=1}^{T} \mathsf{s}[t].$$

Observe that L is a random variable depending on the state sequence of the IDC. Define for each $\ell \in \{1, 2, \dots, \mathsf{L}\}$ the random variable

$$\mathsf{t}[\ell] \triangleq \min\Big\{t : \sum_{j=1}^{t} \mathsf{s}[j] \geq \ell\Big\}.$$

The relationship between the input and output of the IDC is then given by

$$\tilde{\mathsf{x}}[\ell] \triangleq x[\mathsf{t}[\ell]].$$

We illustrate the operation of the IDC with an example.

Example 1.

$$\begin{array}{ccccccccc} \boldsymbol{x} & = & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & \\ \mathbf{s} & = & 1 & 1 & 2 & 1 & 0 & 2 & \\ \mathbf{t} & = & 1 & 2 & 3 & 3 & 4 & 6 & 6 \\ \tilde{\mathbf{x}} & = & x[1] & x[2] & x[3] & x[3] & x[4] & x[6] & x[6] \end{array}$$

In the example, $T = 6$ and $\mathsf{L} = 7$. Here $\boldsymbol{x}$ is the vector of inputs and $\tilde{\mathbf{x}}$ is the vector of outputs of the IDC. $\mathbf{s}$ is the vector of states and the corresponding vector of sampling times is $\mathbf{t}$. For concreteness, assume

$$\boldsymbol{x} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then the output of the IDC is

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

see also Fig. 1 in Section I. $\Diamond$

We denote by

$$\mu \triangleq \mathbb{E}(\mathsf{s}[1]), \qquad \sigma^2 \triangleq \mathrm{var}(\mathsf{s}[1])$$

the mean and variance of the insertion/deletion process, respectively, and we refer to any IDC with those parameters as IDC(μ, σ^2). Here, μ and σ^2 can be interpreted as capturing the drift and jitter of the receiver clock, respectively. In most situations arising in practice, the parameter μ is close to 1, e.g., $\mu = 1 \pm 10^{-4}$.

Our results will be presented for arbitrary insertion/deletion processes with finite mean and variance. For illustrative purposes, we present a commonly used special case of this setting.

Example 2. One commonly used definition of the state process is

$$s[t] = \begin{cases} 1, & \text{w.p. } 1-d, \\ 0, & \text{w.p. } d, \end{cases}$$

for some parameter d. This results in the so-called *deletion channel*, which deletes each input symbol with probability d. $\Diamond$

The output of the IDC(μ, σ^2) is then fed into a discrete memoryless channel DMC(W) described by the distribution $W(\cdot|\tilde{x})$ of the channel output $\mathsf{y} \in \mathcal{Y}$ given the channel input $\tilde{x} \in \mathcal{X}$. The insertion/deletion/substitution channel is the (random) mapping from x to y described by the concatenation of the IDC(μ, σ^2) and the DMC(W).

The goal is to maximize the number of bits reliably transmitted per unit cost over this insertion/deletion/substitution channel formed by the concatenation of the IDC(μ, σ^2) with the DMC(W). We adopt the framework of [3], [4]. The *cost function* $c\colon \mathcal{X} \to \mathbb{R}_+$ associates with each input symbol $x \in \mathcal{X}$ the cost $c(x)$ incurred by transmitting x over the channel. We make the assumption that $\mathcal{X}$ contains a free input symbol, and without loss of generality we label this symbol as 0. In other words, $0 \in \mathcal{X}$ and $c(0) = 0$. For an input sequence $(x[1], \dots, x[T]) \in \mathcal{X}^T$, the cost is given by

$$c\big((x[1], \dots, x[T])\big) \triangleq \sum_{t=1}^{T} c(x[t]).$$

A (T, M, P, ε) *code* consists of M codewords

$$(x_m[1], \dots, x_m[T]), \quad m \in \{1, 2, \dots, M\}$$

each of length T and cost at most P, with average (assuming equiprobable messages) probability of decoding error at most ε.

Definition. A rate $\hat{R}$ per unit cost is *achievable* if for every $\varepsilon > 0$ and every large enough M there exists a (T, M, P, ε) code satisfying[1] $\log(M)/P \geq \hat{R}$. The *capacity per unit cost* $\hat{C}$ is the supremum of achievable rates per unit cost.

Throughout this paper, we are interested in $\hat{C}(\mu, \sigma^2, W)$, the capacity per unit cost of the insertion/deletion/substitution channel given by the IDC(μ, σ^2) concatenated with the DMC(W). We also consider the compound capacity per unit cost $\hat{C}([\mu_1, \mu_2], [\sigma_1^2, \sigma_2^2], W)$, for which the encoder and decoder have to be able to operate on any IDC(μ, σ) with $\mu \in [\mu_1, \mu_2]$ and $\sigma \in [\sigma_1^2, \sigma_2^2]$ without knowledge of the actual values of μ and σ^2. This compound setting is of practical relevance, since usually the mean clock drift μ is only specified as an interval (and might indeed be slowly time varying) and is hence not known exactly at the transmitter or the receiver. We also treat the Gaussian version of the problem, where the output of the IDC(μ, σ^2) is subject to additive Gaussian noise of mean zero and variance η^2. The cost function is in this case the signal energy, i.e., $c(x) = x^2$. With slight abuse of notation, we refer to the capacity per unit energy in this case as $\hat{C}(\mu, \sigma^2, \mathcal{N}(0, \eta^2))$.

III. Main Results

In this section, we summarize the main results. We start with the results for Gaussian channels with synchronization errors for the case where the statistics μ and σ^2 of the insertion/deletion channel are known at the transmitter and the receiver.

Theorem 1. *The Gaussian channel with synchronization errors having insertion/deletion process with mean μ and*

[1]Throughout this paper, $\log(\cdot)$ and $\ln(\cdot)$ denote the logarithms to the base 2 and e, respectively.

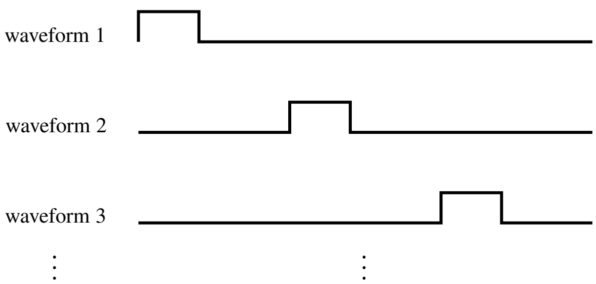

Fig. 3. Sketch of input waveforms for the Gaussian channel with synchronization errors.

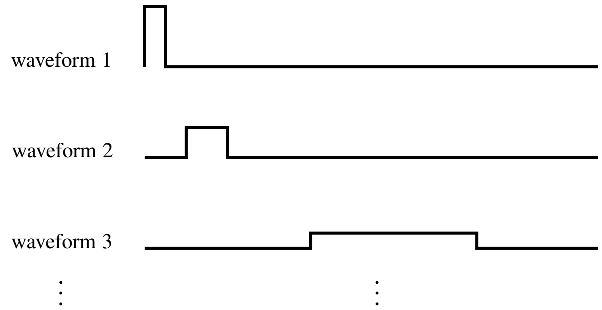

Fig. 4. Sketch of input waveforms for the compound Gaussian channel with synchronization errors.

variance σ^2 and having noise power η^2 has capacity per unit energy

$$\hat{C}(\mu, \sigma^2, \mathcal{N}(0, \eta^2)) = \frac{\mu}{2\eta^2 \ln 2}.$$

Recall that the capacity per unit energy of the Gaussian channel is $1/(2\eta^2 \ln 2)$. Furthermore, as discussed in Section II, the mean μ of the insertion/deletion process is typically close to 1. Hence, Theorem 1 implies that the lack of synchronization results in only negligible loss in the capacity per unit energy.

To establish achievability, we propose a new communication scheme that jointly performs data modulation and loose synchronization. To this end, we develop signaling waveforms where the signal energy is spread over increasing intervals and guard spaces of increasing lengths are introduced as sketched in Fig. 3. Decoding at the receiver is based on a sequence of independent hypothesis tests, which are carefully chosen to account for the uncertainty arising due to the lack of tight synchronization. By appropriately choosing the aforementioned durations, the probability of error can be made arbitrarily small for any rate per unit energy up to $\hat{C}$. The upper bound in Theorem 1 follows as a special case of the upper bound derived for a general DMC with synchronization errors, discussed in Theorem 3 below.

Next, consider the case where the exact statistical properties of the insertion/deletion process are not known a priori. Instead, we only know a range for each parameter, i.e., the mean μ is in $[\mu_1, \mu_2]$ and the variance in $[0, \sigma^2]$. We are interested in a communication scheme that works simultaneously for every possible set of parameters in this range. As pointed out in Section II, this compound setting is of practical relevance, since the precision of the transmitter and receiver clocks are usually only known to lie within some range.

Theorem 2. *The class of Gaussian channels with synchronization errors having insertion/deletion process with mean $\mu \in [\mu_1, \mu_2]$ and variance upper bounded by σ^2 and having noise power η^2 has compound capacity per unit energy*

$$\hat{C}([\mu_1, \mu_2], [0, \sigma^2], \mathcal{N}(0, \eta^2)) = \frac{\mu_1}{2\eta^2 \ln 2}.$$

Comparing Theorems 1 and 2, we see that

$$\hat{C}([\mu_1, \mu_2], [0, \sigma^2], \mathcal{N}(0, \eta^2)) = \min_{\mu \in [\mu_1, \mu_2]} \hat{C}(\mu, \sigma^2, \mathcal{N}(0, \eta^2)).$$

Since a scheme for the compound setting must work for any possible value of μ and σ^2 of the insertion/deletion process, it is clear that it must work for the worst one, so that

$$\begin{aligned}\hat{C}([\mu_1, \mu_2], [0, \sigma^2], \mathcal{N}(0, \eta^2)) &\leq \min_{\mu \in [\mu_1, \mu_2]} \hat{C}(\mu, \sigma^2, \mathcal{N}(0, \eta^2)) \\ &= \frac{\mu_1}{2\eta^2 \ln 2}.\end{aligned}$$

Theorem 2 thus shows that there is no further loss beyond this resulting from the lack of precise knowledge of the insertion/deletion statistics at the transmitter and the receiver. The waveforms achieving the lower bound in Theorem 2 are sketched in Fig. 4

Finally, consider an insertion/deletion channel $\mathrm{IDC}(\mu, \sigma^2)$ concatenated with a general discrete memoryless channel $\mathrm{DMC}(W)$ specified by its transition probability matrix $W : \mathcal{X} \to \mathcal{Y}$. Furthermore, consider an arbitrary cost function $c : \mathcal{X} \to \mathbb{R}_+$. As mentioned in Section II, we assume that $0 \in \mathcal{X}$ and $c(0) = 0$. Then, the following bounds hold on the capacity per unit cost.

Theorem 3. *The insertion/deletion/substitution channel consisting of an $IDC(\mu, \sigma^2)$ concatenated with a $DMC(W)$ has capacity per unit cost $\hat{C}(\mu, \sigma^2, W)$ satisfying*

$$\begin{aligned}\frac{\mu}{2} \sup_{x \in \mathcal{X} \setminus \{0\}} & \frac{D(W(\cdot|x) \| W(\cdot|0))}{c(x)} \\ &\leq \hat{C}(\mu, \sigma^2, W) \\ &\leq \mu \sup_{x \in \mathcal{X} \setminus \{0\}} \frac{D(W(\cdot|x) \| W(\cdot|0))}{c(x)}\end{aligned}$$

where $D(P\|Q)$ is the Kullback-Leibler divergence between distributions P and Q and where $0 \in \mathcal{X}$ is an input symbol with zero cost.

Theorem 3 approximates the capacity per unit cost of a general DMC with synchronization errors and general cost function with a zero-cost symbol to within a factor two. In contrast, recall from Section I that for the *capacity*, even of the noiseless deletion channel, only loose bounds are known despite over four decades since the introduction of the model in [1].

It was shown in [4] that the capacity per unit cost $\hat{C}(W)$

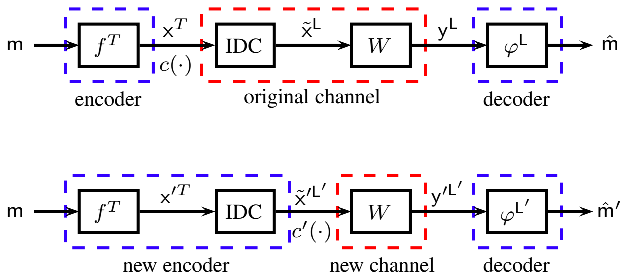

Fig. 5. The behavior of the original insertion/deletion/substitution channel (top figure) model can be simulated over the discrete memoryless channel W by modifying the encoder and the cost function (bottom figure).

of a DMC(W) is

$$\hat{C}(W) = \sup_{x \in \mathcal{X} \setminus \{0\}} \frac{D(W(\cdot|x) \| W(\cdot|0))}{c(x)}. \tag{1}$$

Thus, from the lower and upper bounds in Theorem 3, we obtain the following corollary, showing that the loss due to synchronization errors is within a factor between $\mu/2$ and μ.

Corollary 4. *The insertion/deletion/substitution channel consisting of an IDC(μ, σ^2) concatenated with a DMC(W) has capacity per unit cost $\hat{C}(\mu, \sigma^2, W)$ satisfying*

$$\frac{\mu}{2}\hat{C}(W) \le \hat{C}(\mu, \sigma^2, W) \le \mu \hat{C}(W).$$

The achievability in Theorem 3 is established by generalizing the proposed scheme for the Gaussian channel with synchronization errors to DMCs. For the upper bound on the capacity per unit cost in Theorem 3, we treat the effect of the IDC as a specific way of encoding for the DMC with an appropriately modified cost function (see Fig. 5). The upper bound is then obtained by utilizing the characterization of the capacity per unit cost for *memoryless* channels in [4].

We conclude this section by demonstrating through an example that the conventional schemes based on tight synchronization between the transmitter and the receiver can be highly suboptimal in terms of their rate per unit cost.

Example 3. Let us consider the simplest synchronized communication setting: a channel with binary input and no noise, i.e., $W(x|x) = 1$ for $x \in \{0, 1\}$. Further, let the cost function be the number of ones transmitted, i.e., $c(x) = x$. This is a DMC, and by (1), its capacity per unit cost is

$$\hat{C}(W) = \frac{D(W(\cdot|1) \| W(\cdot|0))}{c(1)} = \infty.$$

Let us now consider the scenario where the transmitter and the receiver are no longer perfectly synchronized. Specifically, the input signals are first corrupted by a deletion channel with deletion probability $d \in (0, 1)$ (see Example 2 in Section II for a formal definition of this special case of an IDC), before being sent over the aforementioned noiseless channel W.

Consider first the operation of conventional schemes based on tight synchronization. In this example, we take this to mean any scheme that detects and corrects deletions without letting them accumulate. This definition applies to schemes using pilots as well as to schemes using differential modulation. To maintain tight synchronization, the channel inputs cannot contain too many consecutive zeros (since otherwise deletions would accumulate without any way of correcting for them). On average, we expect to see about one deleted bit every $1/d$ transmitted bits. Thus, roughly every $1/d$ channel inputs needs to be a 1 at a cost of $c(1) = 1$. Now, over a block of $1/d$ bits, we can reliably transmit at most $1/d$ bits. Hence, the rate per unit cost achieved by any scheme based on tight synchronization is at most

$$\hat{R}_{\text{sync}}(d, W) \le \frac{1}{d} < \infty.$$

On the other hand, from Corollary 4, the communication scheme proposed in this paper achieves a rate per unit cost that is within a factor of $\mu/2 = (1-d)/2$ of the capacity per unit cost $\hat{C}(W)$ of the underlying DMC(W). Hence, the capacity per unit cost with synchronization errors is

$$\hat{C}(d, W) \ge \frac{1-d}{2}\hat{C}(W) = \infty.$$

Thus, even in the presence of synchronization errors, the rate per unit cost achieved by the proposed scheme is arbitrarily large. This illustrates that the improvement in the rate per unit cost achieved by the proposed scheme over schemes based on tight synchronization can be unbounded. $\Diamond$

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