Energy Efficiency of Gaussian Channel with Random Data Arrival

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***Abstract*—A point-to-point communication system with an AWGN channel is considered. It is assumed that data randomly arrive at the system and the transmitter has a finite buffer. Supposing that, to save energy, the transmitter is able to adapt the rate depending on the number of bits existing in the buffer, the stationary probability distribution of the number of bits in the buffer is obtained and an approximate relation for the average of saving in energy is found.**

I. Introduction

Today, the reduction of energy consumption in both wireless and wired transmission technology has become an economic and environmental concern. However, since most communication networks are designed to decrease data loss in peak usage time, network capacity is usually not fully utilized. Most of the time, transmitters are idle or sending dummy messages in order to be synchronized with the receiver.

Considering the variation and burstiness of traffic in networks, it seems that a fixed transmission strategy, which corresponds to the fixed transmission power and fixed coding scheme, is not always energy efficient. Most existing methods for enhancing the energy efficiency propose to put transmitters into sleep mode, when they are idle. However, putting transmitters into sleep mode may increase the delay and probability of overflow. Furthermore, these methods do not pay much attention to the cost of switching between active and sleep mode.

From information-theoretic perspective, it is clear that decreasing the communication rate can reduce energy consumption. Transmitters can reduce the rate by either sending less data over the same period of time or sending the same amount of data over a longer period of time. Based on the fact that it is possible to reduce energy consumption by lowering the transmission power and transmitting the packet over a longer period of time, a packet transmission schedule which minimizes the energy consumption, subject to a delay constraint, has been presented in [1], [2]. It has also been proven that this scheduling method, which changes the transmission time of the packet according to the backlog, is optimal when the time of packet arrivals are non-causally known at the transmitter.

In contrast to most models used in information theory, it is assumed that data is always available at the transmitters and each transmitter node has an infinite buffer; in this paper, a point-to-point communication system with a finite buffer at the transmitter and random data arrivals is considered. We assume that the transmitter is always active, but to reduce transmission energy the encoder is able to change the channel coding rate depending on the amount of data existing in the buffer.

An indicator is introduced, showing how much energy is saved by changing the rate in each message transmission. Second, the stability condition of the system, using queuing theory, is established. Next, the exact and approximate stationary probability distribution of the number of bits existing in the buffer is found. Finally, using this probability distribution, we find the average energy saving of this transmission strategy.

The rest of this paper is organized as follows: section II defines the system model and notations used throughout the paper. Section III presents two theorems, which are the main result of this paper. The first theorem gives the stationary probability distribution of the number of bits in the buffer. The second theorem presents an approximate result on how much energy can be saved in the AWGN channel, with variable transmission rate that is changed based on the backlog. In section IV, the exact and approximate results are compared and finally, conclusions are presented in section V.

II. System Model

Consider the point-to-point communication system depicted in Fig. 1. This system consists of a buffer, an encoder, an additive white Gaussian noise (AWGN) channel and a decoder. To capture the traffic burstiness, it is assumed that data packets randomly arrive at the transmitter. Consequently, data either might not always be available at the transmitter; or the amount of data arriving might exceed the buffer size of the transmitter causing data loss even before transmission.

We adopted the model developed in [3] for a communication system with random data arrival. Assume that time is slotted and the length of a time slot is the same as a transmission. Let $a(i)$ be the number of bits arrive at the transmitter at the *end* of i^{th} time slot, $i \in \{0, 1, 2, \dots\}$. Moreover, assume that

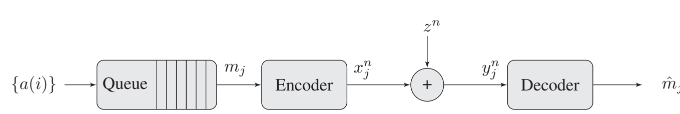

Fig. 1. Point-to-point communication system with AWGN channel and a buffer at the transmitter

the data arrival process, $\{a(i)\}$, be an i.i.d random process as follows

$$a(i) = \begin{cases} k \text{ (bits)} & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases} \tag{1}$$

which indicates that at the end of each transmission, a sequence with k bits information arrives at the transmitter with probability p, and with probability $(1-p)$ no packet arrives at the transmitter. The *arrival rate* is defined as kp which is the product of packet arrival rate $p \in [0,1]$ times the packet size k bits.

At the *beginning* of time slot $i = jn,\ j \in \{1,2,3,\dots\}$, the encoder takes out nR_j bits from the buffer. This sequence of bits makes a j^{th} message called m_j. To encode this message, $m_j \in [1 : 2^{nR_j}]$, the encoder uses a codebook generated by selecting Gaussian codewords with variance P_j where $P_j = \sigma_n^2(2^{2R_j} - 1)$, and σ_n^2 is the noise variance.

Finally, the transmitter sends the codeword corresponding to m_j over the AWGN channel in n time slots. The decoder maps the received signal to the sequence of bits with length nR_j, and by increasing n, the decoding error can be made as small as desired. Therefore, the communication rate between transmitter and receiver to send the j^{th} message is R_j which is not constant and depends on the number of bits existing in the buffer as follows

$$R_j = \begin{cases} R_{\min} & \text{if} \quad l_{jn} \le nR_{\min} \\ \frac{l_{jn}}{n} & \text{if} \quad nR_{\min} < l_{jn} < nR_{\max} \\ R_{\max} & \text{if} \quad l_{jn} \ge nR_{\max}, \end{cases} \tag{2}$$

where $R_{\max}$ is the maximum rate that the transmitter can reliably communicate, with the receiver, due to the power constraint. $R_{\min}$ is the minimum rate that the transmitter must send data with, for example it is the minimum rate that keeps the transmitter synchronized with the receiver. l_i is the number of bits existing in the buffer at the beginning of time slot i. If $i = jn, j \in \{1,2,3,\dots\}$, then l_i is the number of bits existing in the buffer before j^{th} block transmission.

To make it more clear, the transmitter is always *active* and must transmit a message. If the number of bits in the buffer is less than $nR_{\min}$, the encoder adds some dummy bits such as all zeros or ones, to make a message with the rate at least $R_{\min}$. On the other hand, if the number of bits in the buffer is greater than $nR_{\max}$, the transmitter is only able to send $nR_{\max}$ bits with the remaining bits being stored in the buffer to be sent in the future. Using this transmission strategy, it is clear that the length of the queue in the consecutive time slots are related by

$$l_{i+1} = \begin{cases} a(i) + (l_i - nR_{\max})^+ & \text{if } i = jn \\ a(i) + l_i & \text{otherwise}, \end{cases} \tag{3}$$

where $a^+ = \max(a, 0)$. Next, we need a factor to indicate how much energy is saved, by communicating with the rate R_j instead of $R_{\max}$. So, we define $\mathcal{G}_j$ as

$$\mathcal{G}_j = 1 - \frac{R_j}{R_{\max}} \frac{\mathcal{E}(R_j)}{\mathcal{E}(R_{\max})}, \tag{4}$$

where $\mathcal{E}(R)$ is the consumed energy per bit, when the message is transmitted with the rate R. For the rate near capacity in the AWGN channel, $\mathcal{E}(R)$ is as follows

$$\mathcal{E}(R) = \frac{P}{R} = \frac{\sigma_n^2(2^{2R} - 1)}{R}, \tag{5}$$

where P is the power constraint of the transmitter and σ_n^2 is the noise variance. By combining (2), (4) and (5), $\mathcal{G}_j$ is converted to

$$\mathcal{G}_j = \begin{cases} 1 - \frac{2^{2R_{\min}} - 1}{2^{2R_{\max}} - 1} & \text{if} \quad l_{jn} \le nR_{\min} \\ 1 - \frac{2^{2\frac{l_{jn}}{n}} - 1}{2^{2R_{\max}} - 1} & \text{if} \quad nR_{\min} < l_{jn} < nR_{\max} \\ 0 & \text{if} \quad l_{jn} \ge nR_{\max}, \end{cases} \tag{6}$$

and the average of $\mathcal{G}_j$ can be obtained from

$$\mathbb{E}(\mathcal{G}_j) = \left(1 - \frac{2^{2R_{\min}} - 1}{2^{2R_{\max}} - 1}\right) \sum_{b=0}^{nR_{\min}} \Pr\{l_{jn} = b\} + \sum_{b=nR_{\min}+1}^{nR_{\max}-1} \left(1 - \frac{2^{2b/n} - 1}{2^{2R_{\max}} - 1}\right) \Pr\{l_{jn} = b\}, \tag{7}$$

where $\Pr\{X\}$ is the probability of event X. To calculate the gain as indicated in (7), we are only interested knowing the probability of the number of bits in the buffer at the time slots $jn,\ j \in \{0,1,2,3,\dots\}$. So, we define

$$L_j = l_{jn} \tag{8}$$

$$A(j) = \sum_{i=jn}^{(j+1)n-1} a(i), \tag{9}$$

By using (3), we have

$$L_{j+1} = (L_j - nR_{max})^+ + A(j), \tag{10}$$

and $L_0 = l_0 = 0$. Recalling that $a(i)$ is independent of i. So $A(j)$ is independent of j and has a binomial distribution. $\{L_j\}$ is a Markov process, and from queuing theory the buffer is said to be stable if $\sup_j \mathbb{E}(L_j) < \infty$. As long as the buffer is stable, the probability of data loss due to the buffer overflow can be as small as desired with a finite buffer size.

Finally, the total energy saving gain is defined as the time average of all $\mathcal{G}_j$'s,

$$\bar{\mathcal{G}} = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(\mathcal{G}_j). \tag{11}$$

If the buffer is stable, there is a stationary probability mass function (pmf) for the number of bits in the buffer at time slots jn, such that $\pi_b = \lim_{j\to\infty} \Pr\{L_j = b\}$, and the total energy saving gain for a stable system is obtained from

$$\bar{\mathcal{G}} = \left(1 - \frac{2^{2R_{\min}} - 1}{2^{2R_{\max}} - 1}\right) \sum_{b=0}^{nR_{\min}} \pi_b + \sum_{b=nR_{\min}+1}^{nR_{\max}-1} \left(1 - \frac{2^{2b/n} - 1}{2^{2R_{\max}} - 1}\right) \pi_b. \tag{12}$$

III. Main Result

The main result of this paper is summarized in the following theorems. We assume that the length of each packet is equal to one, $k = 1$, but all the results can easily be extended to other cases, when $k > 1$, by assuming that $R_{\max}$ and $R_{\min}$ are multiples of k.

Moreover, from the queuing theory, we know that for the queue with bulk service, if the arrival rate is much less than the service rate, in this case $nkp << nR_{\max}$, then the stationary probability distribution of the state of the queue follows the distribution of the arrival sequence. Therefore, finding (12) is straightforward. In this paper we restrict our result to the heavy traffic case when kp is close to $R_{\max}$.

Theorem 1. *For the communication system defined in section II; when $nR_{\max} = r_{\max} \in \mathbb{N}$, $k = 1$ and $p < R_{\max}$, the stationary probability mass function (pmf) of existing b bits in the buffer is calculated from*

$$\pi_b = \sum_{j=0}^{\min(b,n)} \sum_{m=r_{\max}+1}^{n} \binom{n}{j} \frac{a_m}{z_m^{b-j+1}} p^j (1-p)^{n-j}, \quad (13)$$

and the average number of bits in the buffer is obtained from

$$\mathbb{E}(L) = \sum_{b=1}^{\infty} b\pi_b = np + \sum_{m=r_{\max}+1}^{n} \frac{1}{z_m - 1}, \quad (14)$$

where a_m's are calculated from

$$a_m = \frac{\prod_{i=r_{\max}+1}^{n} (z_i - 1)}{\prod_{\substack{i=r_{\max}+1 \\ i \neq m}}^{n} (z_i - z_m)}, \; m \in \{r_{\max}+1, \ldots, n\}, \quad (15)$$

and $z_m,\ m \in \{r_{\max}+1, \ldots, n\}$ are roots of

$$z^{r_{\max}} - (1 - p + pz)^n = 0, \quad (16)$$

which have absolute value greater than one.

The proof is presented in section IV. It is evident from (13) and (14) that calculating exact values of probabilities and average number of bits in the buffer is complicated and is primarily of mathematical interest. Therefore, in the next corollary, we are aiming to find an approximate relation for both (13) and (14), without having to calculate all the roots of (16).

Corollary 1. *For the communication system defined in section II; when $k = 1$, $nR_{\max} = r_{\max} \in \mathbb{N}$ and $p < R_{\max}$, if p is close to $R_{\max}$, the stationary pmf of existing b bits in the buffer is approximated by*

$$\pi_b \approx \frac{2(R_{\max} - p)}{R_{\max} - p^2} \sum_{j=0}^{\min(b,n)} \frac{\binom{n}{j} p^j (1-p)^{n-j}}{[1 + \frac{2(R_{\max}-p)}{R_{\max}-p^2}]^{b-j+1}}, \quad (17)$$

and the average number of bits in the buffer is approximated by

$$\mathbb{E}(L) \approx np + \frac{R_{\max} - p^2}{2(R_{\max} - p)}. \quad (18)$$

Proof: In section IV, we show that if $nR_{\max} = r_{\max} \in \mathbb{N}$, (16) has exactly $n - r_{\max}$ roots which have absolute values greater than one. Let them be represented by $\{z_m,\ m \in \{r_{\max}+1, \ldots, n\},\ |z_m| > 1\}$. Also, assume that $z_{r_{\max}+1}$ is the smallest element of this set in absolute value; and suppose that $z_{r_{\max}+1}$ is a real number and equal to $1 + \delta$, where δ is small and positive. Therefore,

$$(1 + \delta)^{R_{\max}} = 1 + p\delta,$$

which leads to

$$R_{\max} \ln(1 + \delta) = \ln(1 + p\delta). \quad (19)$$

When x is small, $\ln(1+x)$ can be approximated by $x - x^2/2$. We use this to solve (19), so δ can be approximated by

$$\delta \approx \frac{2(R_{\max} - p)}{R_{\max} - p^2}.$$

We assumed that δ is small. This is a valid assumption as long as $p \to R_{\max}^+$. Thus, when p is close to $R_{\max}$, there exists a root close to one; as a result, the proportionate contributions of other roots $z_m,\ m \in \{r_{\max}+2, \ldots, n\}$ in $\mathbb{E}(L)$ and π_b is small. By substituting $z_{r_{\max}+1}$ in (14) and ignoring the contribution of all other roots of (16), a satisfactory approximation for the average number of bits stored in the buffer is as follows

$$\mathbb{E}(L) \approx np + \frac{1}{z_{r_{\max}+1} - 1} \approx np + \frac{R_{\max} - p^2}{2(R_{\max} - p)}. \quad (20)$$

Since $z_{r_{\max}+1} = 1 + \delta$, we have

$$\prod_{i=r_{\max}+2}^{n} \frac{z_i - 1}{z_i - z_{r_{\max}+1}} \approx 1. \quad (21)$$

Next, from (15) and (21) we have

$$a_{r_{\max}+1} \approx z_{r_{\max}+1} - 1 \approx \frac{2(R_{\max} - p)}{R_{\max} - p^2}.$$

Finally, substituting $a_{r_{\max}+1}$ in (13) and ignoring the contributions of the other roots, π_b is approximated by (17). ■

Theorem 2. *For the stable communication system with the AWGN channel described in section II, if $k = 1$ and where p is close to $R_{\max}$, the total energy saving gain defined in* (12) *is approximated by (22).*

Proof: This is the direct result of Corollary 1 and the definition of $\bar{\mathcal{G}}$. ■

IV. Proof of Theorem 1

To prove Theorem 1, we use the same method introduced in [4], [5]. The next lemma gives the condition in which the buffer is stable. Otherwise, it is clear that the probability of data loss, because of buffer overflow, goes to one and the reliable communication is impossible. Furthermore, the total energy saving gain goes to zero.

Lemma 1. *The queue, with the arrival process defined in* (9) *and the transient states related by* (10)*, is stable if and only if $kp < R_{\max}$. (We will skip the proof due to the lack of space.)*

$$\bar{\mathcal{G}} \approx \Big(1-\frac{2^{2R_{\min}}-1}{2^{2R_{\max}}-1}\Big)\sum_{b=0}^{nR_{\min}}\binom{n}{b}p^b(1-p)^{n-b}\Bigg(1-\frac{1}{\big(1+\frac{2(R_{\max}-p)}{R_{\max}-p^2}\big)^{nR_{\min}-b+1}}\Bigg)$$
$$+\frac{2(R_{\max}-p)}{R_{\max}-p^2}\sum_{b=nR_{\min}+1}^{nR_{\max}-1}\sum_{j=0}^{b}\Big(1-\frac{2^{2b/n}-1}{2^{2R_{\max}}-1}\Big)\frac{\binom{n}{j}p^j(1-p)^{n-j}}{[1+\frac{2(R_{\max}-p)}{R_{\max}-p^2}]^{b-j+1}}. \tag{22}$$

For $k=1$ if $p<R_{\max}$, the buffer is stable. There exists a stationary pmf for the number of bits in the buffer. To find a closed form expression for π_b, we first calculate the probability generating function, which is defined as $\mathcal{L}(z)=\sum_{b=0}^{\infty}\pi_b z^b$. It is clear that $\mathcal{L}(z)|_{z=1}=1$ and $\mathcal{L}'(z)|_{z=1}=\mathbb{E}(L)$. The next lemma gives an expression for $\mathcal{L}(z)$.

Lemma 2. *The probability generating function for the number of bits in the buffer is obtained from*

$$\mathcal{L}(z)=\frac{(1-p+pz)^n}{z^{r_{\max}}-(1-p+pz)^n}\sum_{b=0}^{r_{\max}-1}\pi_b(z^{r_{\max}}-z^b), \tag{23}$$

where $r_{\max}=nR_{\max}$. (For the proof see Appendix A)

Obviously, $\mathcal{L}(z)$ can not be determined from (23), since it depends on the $r_{\max}$ probabilities, $\pi_b,\ b\in\{0,\dots,r_{\max}-1\}$. Considering that $\mathcal{L}(z)=\frac{\mathcal{H}_n(z)}{\mathcal{H}_d(z)}$, and recalling that $|\mathcal{L}(z)|<\infty$ in the region $|z|\le 1$, so $\mathcal{H}_n(z)$ and $\mathcal{H}_d(z)$ should have common zeros when $|z|\le 1$. In the next lemma, to find a deterministic expression for $\mathcal{L}(z)$, we will discuss the zeros of $\mathcal{H}_d(z)$.

Lemma 3. *The function $\mathcal{H}_d(z)=z^{r_{\max}}-(1-p+pz)^n$ has a set of zeros $\{z_i,\ i=1,2,\dots,n\}$ such that the number of zeros in the region $|z|\le 1$ is $r_{\max}$. Also, the zeros in that region are simple, if $np<r_{\max}$ and $r_{\max}=nR_{\max}\le n$. (For the proof see Appendix B)*

Using the result of lemma 3, the zeros of $\mathcal{H}_d(z)$ can be sorted as follows

$$|z_1|<\dots<|z_{r_{\max}}|=1<|z_{r_{\max}+1}|\le\dots\le|z_n|. \tag{24}$$

$\mathcal{H}_n(z)$ has a zero of the order n at $z=1-1/p$, which is not of interest because it is not a zero of $\mathcal{H}_d(z)$. As described earlier, all other zeros of $\mathcal{H}_n(z)$ are common with the $r_{\max}-1$ zeros of $\mathcal{H}_d(z)$ which are in the region $|z|<1$. So we have

$$\sum_{b=0}^{r_{\max}-1}\pi_b(z_m^{r_{\max}}-z_m^b)=0,\ m\in\{1,2,\dots,r_{\max}-1\}. \tag{25}$$

Furthermore, we know that $\lim_{z\to 1}\mathcal{L}(z)=1$ and

$$\lim_{z\to 1}\mathcal{L}(z)=\frac{\mathcal{H}_n'(z)|_{z=1}}{\mathcal{H}_d'(z)|_{z=1}}=\frac{\sum_{b=0}^{r_{\max}-1}\pi_b(r_{\max}-b)}{r_{\max}-np} \tag{26}$$

which gives us

$$\sum_{b=0}^{r_{\max}-1}\pi_b(r_{\max}-b)=r_{\max}-np. \tag{27}$$

The next step is to show that the $r_{\max}$ equations in (25) and (27) are independent. It can easily be shown that the determinant of the coefficients matrix cannot vanish, since all the $z_m,\ m\in\{1,2,\dots r_{\max}-1\}$ are different and not equal to one. Therefore, there is a solution for equations in (25) and (27).

Finally, by cancelling the common terms from the numerator and denominator of $\mathcal{L}(z)$ and recalling that $\mathcal{L}(z)|_{z=1}=1$. $\mathcal{L}(z)$ can be written as

$$\mathcal{L}(z)=(1-p+pz)^n\prod_{m=r_{\max}+1}^{n}\frac{z_m-1}{z_m-z} \tag{28}$$

where $z_m,\ m\in\{r_{\max}+1,\dots,n\}$ are the $n-r_{\max}$ zeros of $\mathcal{H}_d(z)$ such that their absolute values are greater than one. Having $\mathcal{L}(z)$, the average number of bits in the buffer is obtained from

$$\mathbb{E}(L)=\mathcal{L}'(z)|_{z=1}=np+\sum_{m=r_{\max}+1}^{n}\frac{1}{z_m-1}. \tag{29}$$

Furthermore, $\mathcal{L}(z)$ can easily be written in a summation form as follows

$$\mathcal{L}(z)=(1-p+pz)^n\sum_{m=r_{\max}+1}^{n}\frac{a_m}{z_m-z}, \tag{30}$$

where a_m's are calculated from

$$a_m=\frac{\prod_{i=r_{\max}+1}^{n}(z_i-1)}{\prod_{\substack{i=r_{\max}+1\\ i\neq m}}^{n}(z_i-z_m)},\ m\in\{r_{\max}+1,\ r_{\max}+2,\dots,n\}. \tag{31}$$

Now, we expand (31) to the following

$$\begin{aligned}\mathcal{L}(z)&=(1-p+pz)^n\sum_{m=r_{\max}+1}^{n}\Big(\frac{a_m}{z_m-z}\Big)\\ &=(1-p+pz)^n\sum_{m=r_{\max}+1}^{n}\Bigg(\frac{a_m}{z_m}\sum_{i=0}^{\infty}\Big(\frac{z}{z_m}\Big)^i\Bigg)\\ &=(1-p+pz)^n\sum_{i=0}^{\infty}\sum_{m=r_{\max}+1}^{n}\frac{a_m}{z_m^{i+1}}z^i.\end{aligned} \tag{32}$$

By comparing (32) with the definition of $\mathcal{L}(z)$, (13) results.

V. Numerical Results

We simulated the system for $n=128$, $R_{\min}=0.5$ and $R_{\max}=0.75$. It is evident from Fig. 2 that the approximate pmf found in Corollary 1 gives an excellent fit to the simulation results. Fig. 3 shows that the expression found in Theorem 2 , Eq. (22) , fairly approximates the energy saving gain.

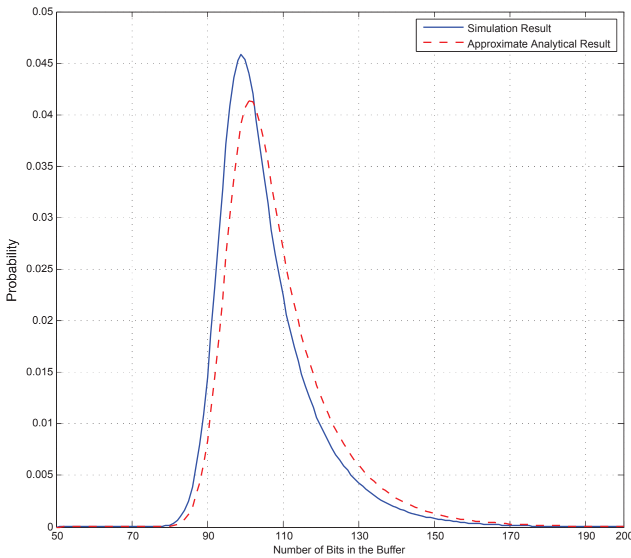

Fig. 2. Simulation and approximate analytical result for the stationary pmf of the number of bits in the buffer when $n = 128$, $R_{\max} = 0.75$ and probability of packet arrival, $p = \frac{95}{128}$

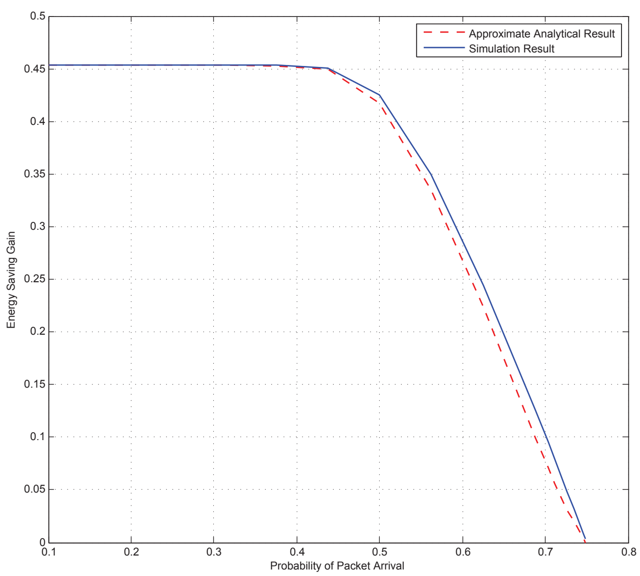

Fig. 3. Simulation and approximate analytical result for the energy saving gain in terms of probability of packet arrival when $n = 128$, $R_{\min} = 0.5$ and $R_{\max} = 0.75$

VI. Conclusion

We investigated a communication system consisting of an AWGN channel and a transmitter with a finite buffer. It was assumed that data would randomly arrive and the transmitter would be able to adjust its rate, depending on the number of bits existing in the buffer. It was shown that using this transmission strategy, the energy consumption could be reduced. To find the average amount of reduction in consumed energy, the stationary pmf of the number of bits in the buffer was obtained. The satisfactory approximate relation for the average amount of reduction in consumed energy was presented.

Appendix A
Proof of Lemma 2

We define $\tilde{L_j} = L_j - (L_j - r_{\max})^+$. So, from (10), we have $L_{j+1} = L_j - \tilde{L_j} + A(j)$, and

$$\begin{aligned}\mathcal{L}_{j+1}(z) &= \mathbb{E}[z^{L_j - \tilde{L_j} + A(j)}] = \mathcal{A}(z)\,\mathbb{E}[z^{L_j - \tilde{L_j}}] \\ &= \mathcal{A}(z)\Big(\sum_{b=0}^{r_{\max}} \Pr\{L_j = b\} + \sum_{b=r_{\max}+1}^{\infty} \Pr\{L_j = b\}\, z^{b-r_{\max}}\Big) \\ &= \mathcal{A}(z)\Big(\frac{1}{z^r_{\max}}\mathcal{L}_j(z) + \sum_{b=0}^{r_{\max}-1} (1 - \frac{z^b}{z^r_{\max}}) \Pr\{L_j = b\}\Big).\end{aligned}$$

We already assumed that the system is stable, therefore as j goes to infinity $\mathcal{L}_j(z) = \mathcal{L}_{j+1}(z) = \mathcal{L}(z)$, and $\Pr\{L_j = b\} = \pi_b$. Finally, to complete the proof we can obtain $\mathcal{A}(z)$ from

$$\mathcal{A}(z) = \sum_{b=0}^{n} \binom{n}{b} p^b (1-p)^{n-b} z^b = (1 - p + pz)^n. \quad (33)$$

Appendix B
Proof of Lemma 3

We first show that all the zeros of $\mathcal{H}_d(z)$, in addition to $z = 1$, in the region $|z| \le 1$ are simple. Otherwise, if z_i is a second order zero in the region $|z| \le 1$, we have

$$z_i^{r_{\max}} = (1 - p + pz_i)^n, \quad (34)$$

$$r_{\max} z_i^{r_{\max}-1} = np(1 - p + pz_i)^{n-1}. \quad (35)$$

Dividing both sides of the two equations in (34) and (35), gives

$$z_i = \frac{r_{\max}(1-p)}{p(n - r_{\max})}.$$

It is clear that we require $np \ge r_{\max}$ for $|z_i| \le 1$ which contradicts with the hypothesis. Second, by Rouche's theorem, $\mathcal{H}_d(z)$ will have $r_{\max}$ zeros in or on the unit circle if $|z^{r_{\max}}| > |(1 - p + pz)^n|$ on $|z| = 1 + \epsilon$ where ϵ is small and positive. Assuming $|z| = 1 + \epsilon$, we have

$$|1 - p + pz|^n \le (|1 - p| + |pz|)^n = (1 + p\epsilon)^n,$$

and by induction, it can easily be shown that when $R_{\max} > p$, for small and positive ϵ, $(1 + \epsilon)^{nR_{\max}} > (1 + p\epsilon)^n$.

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