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# Robust Multiple Description Coding – Joint Coding for Source and Storage

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***Abstract*—We propose a framework for robust content distribution in networks such that each network node stores a description of the source for users to access and it is robust against any single node failure. The fundamental problem is to identify the tradeoff among various parameters such as storage size, repair bandwidth and the level of distortion in the reconstructed estimate. We show that when we design a robust multiple description code, it is usually favourable that the descriptions should be as correlated as possible to reduce the amount of repair bandwidth in our network.**

## I. Introduction

The idea of Multiple Description (MD) coding dates back to the late 70's where the following source coding problem was studied [1]: Consider a random source $X_1, \ldots, X_N$, each of which is independently and identically distributed according to a known probability distribution $p(x)$. The transmitter (who has access to the source) encodes $X_1, \ldots, X_N$ into two descriptions, which will be transmitted across a network. Users in a network may receive either one or both of these descriptions, depending on the quality of network connections[1].

In the proposed scheme, it is required that any user in the network, having received a subset of these descriptions, can reconstruct a "noisy" estimate of $X_1, \ldots, X_N$. Clearly, the level of distortion between the estimate and the original source $X_1, \ldots, X_N$ will depend on the actual subset of descriptions that the user received. In the context of video streaming application, users with a better network connection may receive more descriptions and hence a less noisy estimate, while other users may only receive one description and hence a noisier estimate.

The model used in [1] is a fairly common and popular model for multiple description coding. The fundamental problem being of interest is about the tradeoff between the sizes of the descriptions and the levels of distortion achieved between the noisy estimate and the original source. The complete characterisation of the tradeoff is still unknown. Examples of achievable schemes and converses however can be found in [1], [2]. Extension of the problem to more than two descriptions are given in [2], [3].

The sizes of the descriptions are definitely related to the amount of network resources needed to transmit the two descriptions via a network. However, there are also other relevant parameters that are of interest but are also often ignored in the literature. One example is the amount of correlation between the descriptions. Each description is a "coarse" representation of the original source. It is thus expected that the two descriptions (at least in the low distortion regime) should be correlated. In practice, this correlation between the two descriptions can and should be exploited to further improve the network efficiency.

Correlation between descriptions can also be useful to ensure the robustness of the system against network node failure. Consider a very simple scenario as follows: The two descriptions of a video are stored at two geographically separated servers, where users in the network can establish connections with the two servers in order to retrieve (an estimate of) the video. Some users may only retrieve only one of the descriptions while some can retrieve both.

Now, suppose one of the network nodes storing a description is corrupted and all its stored data are erased. To allow recovery from such a disaster, the system must have a mechanism to regenerate the content of the corrupted node. Obviously, one can directly create a few copies (or mirrors) of each description and store them separately. Whenever a node fails, the system can regenerate the content of the failed node by copying the content from the corresponding mirrors. In fact, this is a common approach to protect system from failures.

In data regeneration, the fundamental problem of interest is to determine the amount of extra data that is needed to store in the mirrors (or more generally, the helper nodes) and the amount of data traffic needed to transmit from the surviving nodes to the failed node. It turns out that directly creating a mirror is not necessarily the most efficient solution. A better approach is via the use of coding [4]. The idea is very simple. Instead of directly copying the data, one can partition the data into $k$ equal parts (or called *packets*). Then one encode these $k$ packets into $n$ packets by using a $(n, k)$ maximum distance separable code. Each of the packet will be stored in a data centre (DC). In this case, the original data can be regenerated even up to $n - k$ (DC) are corrupted.

These data regeneration approaches minimise the amount of extra storage needed for recovery but ignores the communication costs (e.g., the amount of data needed to transmit from DCs to regenerate the data). How to achieve the optimal tradeoff between the cost of storage and the amount of repair

[1]In some scenarios, the qualities of services users received can also be differentiated by how much a user pays. Clearly, the higher a user pays for the service, the more descriptions (and hence the better quality of service) he/she receives.

bandwidth is one of most actively research topics recently. Further details on these topics can be founded in many existing works such as [4]. Even if we back up each description separately and independently, this approach is still not optimal. The reason is in fact quite simple. All descriptions are "noisy estimates" of the same source, and hence are most likely correlated. Therefore, when one of the descriptions is corrupted, we can exploit the correlation between the descriptions to regenerate the content of the failed node.

*Notations:* For brevity, we use the following convention to simplify our notation. For any positive integer $i$, let $\langle i \rangle = \{1, \ldots, i\}$. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a function. The set $\mathcal{Y}$ is the codomain of the mapping $f$. We will use $|f|$ to denote the size of the codomain $\mathcal{Y}$. In many cases, we will simply assume that $\mathcal{Y} = \{1, \ldots, |f|\} = \langle |f| \rangle$ and hence directly write

$$f : \mathcal{X} \longrightarrow \langle |f| \rangle.$$

In this paper, we will denote the source (say a video clip) by a sequence of random variables $\{X_1, X_2, \ldots, \}$, which are independently and identically distributed according to a probability distribution $p(x)$. Let $\delta : \mathcal{X} \times \hat{\mathcal{X}} \longrightarrow \mathbb{R}$ be a bounded distortion function where $\mathcal{X}'$ is the alphabet set of the reconstruction symbols. In many cases, $\mathcal{X}'$ and $\mathcal{X}$ are simply the same. Abusing our notation, for any sequences $X^N = (X_1, \ldots, X_N)$ and $\hat{X}^N = (\hat{X}_1, \ldots, \hat{X}_N)$, we will define

$$\delta(X^N, \hat{X}^N) = \sum_{i=1}^{N} \delta(X_i, \hat{X}_i).$$

*Definition 1 (Multiple description codes):* A multiple description (MD) code $\mathcal{C}$ for $X^N$ is specified by the tuple

$$\mathcal{C} = (N, f_1, f_2, g_1, g_2, g_{1,2})$$

such that $N$ is a positive integer, and $(f_1, f_2)$, $(g_1, g_2, g_{1,2})$ are respectively the "encoding" and "decoding" functions

$$f_1 : \mathcal{X}^N \longrightarrow \langle |f_1| \rangle \tag{1}$$
$$f_2 : \mathcal{X}^N \longrightarrow \langle |f_2| \rangle \tag{2}$$
$$g_1 : \langle |f_1| \rangle \longrightarrow \hat{\mathcal{X}}^N \tag{3}$$
$$g_2 : \langle |f_2| \rangle \longrightarrow \hat{\mathcal{X}}^N \tag{4}$$
$$g_{1,2} : \langle |f_1| \rangle \times \langle |f_2| \rangle \longrightarrow \hat{\mathcal{X}}^N. \tag{5}$$

Roughly speaking, for a given MD code $\mathcal{C}$, it will encode $N$ source symbols $X^N = \{X_1, \ldots, X_N\}$ into two descriptions, denoted by $f_1(X^N)$ and $f_2(X^N)$ respectively. If a receiver receives the description $f_1(X^N)$, it can construct a noisy estimate $g_1(f_1(X^N))$ of $X^N$. Similar can be interpreted if the receiver receives the second description. If the receiver receives both descriptions, then its estimate for $X^N$ will be given by $g_{1,2}(f_1(X^N), f_2(X^N))$. It is a fundamental problem to understand the tradeoff between distortion levels in reconstruction and the size of each description [1].

To simplify our notations, we will use the following convention: For any MD code $\mathcal{C} = (N, f_1, f_2, g_1, g_2, g_{1,2})$, let

$$F_i^N \triangleq f_i(X^N), \quad i = 1, 2 \tag{6}$$
$$D_\alpha(\mathcal{C}) = \frac{E[\delta(X^N, g_\alpha(F_i^N, i \in \alpha))]}{N}, \ \alpha \subseteq \{1, 2\}. \tag{7}$$

The decoding functions $g_1, g_2$ and $g_{1,2}$ are assumed optimally chosen to minimise the distortions $D_1(\mathcal{C})$, $D_2(\mathcal{C})$ and $D_{1,2}(\mathcal{C})$. Hence, we can simply denote the code by $\mathcal{C} = (N, f_1, f_2)$ and

$$D_\alpha(\mathcal{C}) \triangleq \inf_{g_\alpha} \frac{E[\delta(X^N, g_\alpha(F_j^N, j \in \alpha))]}{N} \tag{8}$$

*Remark 1:* It is worth to mention that the encoding functions $f_1$ and $f_2$ can essentially be determined from the joint distribution of $(X^N, F_1^N, F_2^N)$. Therefore, it is often instrumental and also simpler to refer to an MD code directly by the set of random variables $(X^N, F_1^N, F_2^N)$.

*Definition 2 (Achievability):* A tuple $(r_1, r_2, d_1, d_2, d_{1,2})$ is called *md-achievable* if there exists a sequence of MD codes

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N)$$

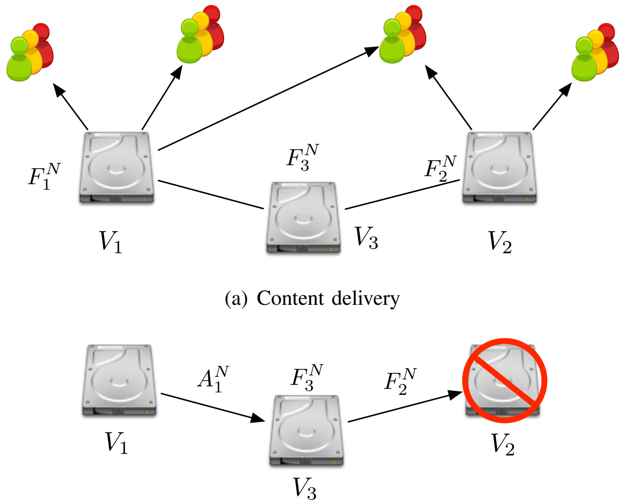
such that

$$\limsup_{N\to\infty} \frac{H(F_i^N)}{N} \leq r_i, \ i = 1, 2 \tag{9}$$
$$\limsup_{N\to\infty} D_\alpha(\mathcal{C}^N) \leq d_\alpha, \ \alpha \subseteq \{1, 2\}. \tag{10}$$

## II. Robust multiple description codes

Now, consider the following scenario that the two descriptions (i.e., $F_1$ and $F_2$) of a MD code $\mathcal{C}$ are stored respectively in network nodes $V_1$ and $V_2$. Users can connect to either one or both of these nodes to retrieve the descriptions. Surely, the amount of distortion between the original source and the estimate the user reconstructs will depend on which subset of descriptions it receives. Now, suppose the node $V_1$ is corrupted and all its stored content is lost. It is of critical importance that the system should be capable to recover from such a failure. In our case, it means the regeneration of the content $F_1$ (and $F_2$) in $V_1$ (and $V_2$).

In this paper, we will consider a very simple network consisting of three nodes, $V_1, V_2$ and a "helper node" $V_3$. We assume that the nodes $V_1$ and $V_2$ are connected via only the helper node. In other words, there are no direct links between $V_1$ and $V_2$ and that all the data communicated between the two nodes must pass through the helper node $V_3$. The helper node is not an arbitrary network node. In fact, it stores a "compressed/coded" version of the contents stored in nodes $V_1$ and $V_2$. Thus, when one of the nodes $V_1$ and $V_2$ is corrupted, this helper node will assist repairing the failed node. As the helper node itself may also be corrupted, it is also important that the helper node can be repaired from $V_1$ and $V_2$ in case of its failure.




(a) Content delivery

(b) Repairing node $V_2$

Fig. 1. Robust multiple descriptions code.

*Definition 3 (Robust MD codes):* Following our convention, a *Robust MD* code (for a block of $N$ source symbols) is defined by a tuple of random variables

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N, F_3^N, A_1^N, A_2^N, A_{*,1}^N, A_{*,2}^N)$$

such that $H(F_3^N|X^N) = 0$ and $H(A_{*,i}^N, A_i^N|F_i^N) = 0$ for $i = 1, 2$.

The physical meanings of the random variables are as follows: $F_3^N$ is the "message" stored in the helper node $V_3$. Suppose the description $F_2^N$ stored at node $V_2$ is erased. To recover the content stored in $V_2$, the message $A_1^N$ will be transmitted from $V_1$ to the helper node $V_3$. Then the helper node will regenerate the second description $F_2^N$ (i.e., the content stored in $V_2$) and send it to $V_2$. Similar reconstruction process will be required if the node $V_1$ fails. Besides recovering a single node failure at $V_1$ or $V_2$, we also need to manage data recovery when node $V_3$ fails. In this case, messages $A_{*,i}^N$ will be sent to node $V_3$ from node $V_i$, for $i = 1, 2$.

We will always assume that the repairing process is optimally chosen to minimise the probability of repairing errors, which are given as follows:

$$P_e^1(\mathcal{C}) \triangleq \min_{\chi_1} \Pr[F_2^N \neq \chi_1(A_1^N, F_3^N)] \tag{11}$$

$$P_e^2(\mathcal{C}) \triangleq \min_{\chi_2} \Pr[F_1^N \neq \chi_2(A_2^N, F_3^N)] \tag{12}$$

$$P_e^3(\mathcal{C}) \triangleq \min_{\chi_3} \Pr[F_3^N \neq \chi_3(A_{*,1}^N, A_{*,2}^N)]. \tag{13}$$

Here, $\chi_1, \chi_2$ and $\chi_3$ are the repairing functions.

*Definition 4 (rmd-Achievability):* A tuple

$$[\mathbf{r}, \mathbf{d}, \mathbf{w}] = [(r_1, r_2, r_3), (d_1, d_2, d_{1,2}), (w_1, w_2, w_{*,1}, w_{*,2})]$$

is called *rmd-achievable*[2] if there exists a sequence of robust MD codes $\mathcal{C}^N = (X^N, F_1^N, F_2^N, F_3^N, A_1^N, A_2^N, A_{*,1}^N, A_{*,2}^N)$ such that

$$\lim_{N\to\infty} \frac{H(F_i^N)}{N} \le r_i, \ i = 1, 2, 3 \tag{14}$$

$$\lim_{N\to\infty} \frac{H(A_j^N)}{N} \le w_j, \ j = 1, 2 \tag{15}$$

$$\lim_{N\to\infty} \frac{H(A_{*,j}^N)}{N} \le w_{*,j}, \ j = 1, 2 \tag{16}$$

$$\lim_{N\to\infty} P_e^i(\mathcal{C}^N) = 0, \ i = 1, 2, 3 \tag{17}$$

$$\lim_{N\to\infty} D_\alpha(\mathcal{C}^N) \le d_\alpha, \ \alpha \subseteq \{1, 2\}. \tag{18}$$

where

$$D_\alpha(\mathcal{C}^N) \triangleq \inf_{g_\alpha} \frac{E[\delta(X^N, g_\alpha(F_j^N, j \in \alpha))]}{N} \tag{19}$$

*Lemma 1:* Suppose $[\mathbf{r}, \mathbf{d}, \mathbf{w}]$ is *rmd*-achievable. Then

$$[\mathbf{r}, \mathbf{d}, \mathbf{w}] + [\epsilon_\mathbf{r}, \epsilon_\mathbf{d}, \epsilon_\mathbf{w}]$$

is also *rmd*-achievable for all nonnegative tuples[3] $[\epsilon_\mathbf{r}, \epsilon_\mathbf{d}, \epsilon_\mathbf{w}]$.

*Definition 5 (Pareto-optimality):* An *rmd*-achievable tuple $(\mathbf{r}, \mathbf{d}, \mathbf{w})$ is called *pareto-optimal* if there does not exists another *rmd*-achievable tuple $(\mathbf{r}', \mathbf{d}', \mathbf{w}') \neq (\mathbf{r}, \mathbf{d}, \mathbf{w})$ and a nonnegative tuple $(\epsilon_\mathbf{r}, \epsilon_\mathbf{d}, \epsilon_\mathbf{w})$ such that

$$(\mathbf{r}, \mathbf{d}, \mathbf{w}) = (\mathbf{r}', \mathbf{d}', \mathbf{w}') + (\epsilon_\mathbf{r}, \epsilon_\mathbf{d}, \epsilon_\mathbf{w}).$$

Roughly speaking, pareto-optimal *rmd*-achievable tuples $(\mathbf{r}, \mathbf{d}, \mathbf{w})$ are "on the boundary" of the set of all *rmd*-achievable tuples. In fact, to characterise the set of *rmd*-achievable tuples, it is necessary and sufficient to characterise only those pareto-optimal *rmd*-achievable tuples.

*Lemma 2 (Property of Pareto-optimal tuples):* Suppose $(\mathbf{r}, \mathbf{d}, \mathbf{w})$ is a pareto-optimal *rmd*-achievable tuples. If

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N, F_3^N, A_1^N, A_2^N, A_{*,1}^N, A_{*,2}^N)$$

is a sequence of robust MD codes satisfying (14)-(18), then

$$\lim_{N\to\infty} \frac{H(F_i^N)}{N} = r_i, \quad i = 1, 2, 3, \tag{20}$$

$$\lim_{N\to\infty} \frac{H(A_1^N)}{N} = \lim_{N\to\infty} \frac{H(A_2^N)}{N} = w_1 = w_2. \tag{21}$$

and

$$\lim_{N\to\infty} \frac{H(F_1^N, F_2^N)}{N} = r_3 + w_1. \tag{22}$$

*Sketch of proof:* Equality (20) follows directly from that $(\mathbf{r}, \mathbf{d}, \mathbf{w})$ is pareto-optimal. It remains to prove (21) and (22). Due to (17), it can be proved that

$$\lim_{N\to\infty} \frac{H(F_1^N|F_2^N, F_3^N)}{N} = \lim_{N\to\infty} \frac{H(F_2^N|F_1^N, F_3^N)}{N} = \lim_{N\to\infty} \frac{H(F_3^N|F_1^N, F_2^N)}{N} = 0. \tag{23}$$

[2] *rmd* is a mnemonic for "robust multiple description coding".

[3] A nonnegative tuple means that all its elements are nonnegative real numbers.

When $V_1$ fails, the repairing process is equivalent to that $F_1^N$ (and equivalently $F_2^N$ by (23)) is reconstructed at the helper node. The pareto-optimality of $(\mathbf{r}, \mathbf{d}, \mathbf{w})$ thus implies that the size of $A_2^N$ must be as small as possible. By using standard information theoretic argument, it implies that

$$w_2 = \lim_{N\to\infty} \frac{H(A_2^N)}{N} = \lim_{N\to\infty} \frac{H(F_1^N, F_2^N | F_3^N)}{N}. \tag{24}$$

The lemma thus follows. ■

## III. Characterisation of achievable tuples

A fundamental question to be answered here is the characterisation of the set of all *rmd*-achievable tuples. We will show that this problem can be reduced to a multiple description coding problem, subject to an additional constraint on the amount of correlations between descriptions.

*Definition 6:* A tuple $(r_1, r_2, d_1, d_2, d_{1,2}, \sigma)$ is called *c-achievable*[4] if there exists a sequence of MD codes

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N)$$

such that

$$\limsup_{N\to\infty} \frac{H(F_i^N)}{N} \le r_i, \ i = 1, 2 \tag{25}$$

$$\limsup_{N\to\infty} D_\alpha(\mathcal{C}^N) \le d_j, \ \alpha \subseteq \{1, 2\} \tag{26}$$

$$\lim_{N\to\infty} \frac{I(F_1^N; F_2^N)}{N} \ge \sigma. \tag{27}$$

*Theorem 1:* Suppose $(r_1, r_2, d_1, d_2, d_{1,2}, \sigma)$ is $c$-achievable, then $(r_1, r_2, r_3, d_1, d_2, d_{1,2}, w_1, w_2, w_{*,1}, w_{*,2})$ is *rmd*-achievable if

$$r_3 \ge r_1 - \sigma \tag{28}$$

$$r_3 \ge r_2 - \sigma \tag{29}$$

$$w_1 + r_3 \ge r_1 + r_2 - \sigma \tag{30}$$

$$w_2 + r_3 \ge r_1 + r_2 - \sigma \tag{31}$$

$$w_{*,1} \ge r_1 - \sigma \tag{32}$$

$$w_{*,2} \ge r_2 - \sigma \tag{33}$$

$$w_{*,1} + w_{*,2} \ge r_3. \tag{34}$$

*Sketch of proof:* Consider a sequence of codes $(X^N, F_1^N, F_2^N)$ satisfying (25)-(27). Assume without loss of generality that $F_1^N$ and $F_2^N$ are all binary row vectors (whose lengths depend on the entropies of the random variables). The random variable $F_3^N$ is constructed as follows. First, nodes $V_1$ and $V_2$ will generate matrices $M_1$ and $M_2$, each of which has $Nw_{*,1}$ and $Nw_{*,2}$ columns. Entries in the matrices are randomly generated. Then node $V_i$ will send $A_{*,i}^N \triangleq F_i^N M_i$ to the helper node. At the helper node, another matrix $M$ of size $(Nw_{*,1} + Nw_{*,1}) \times Nr_3$ will be randomly constructed. Then

$$F_3^N \triangleq [F_1^N M_1, F_3^N M_3] M.$$

[4] $c$ is a mnemonic for "correlation".

Finally,

$$A_i^N \triangleq F_i^N M_i^*$$

where $M^*$ is a random matrix which has $Nw_i$ columns. Then, it can be verified using standard information theoretic arguments that for such a class of codes

$$(X^N, F_1^N, F_2^N, F_3^N, A_1^N, A_2^N, A_{*,1}^N, A_{*,2}^N),$$

(14) - (18) hold. The theorem then follows. ■

*Theorem 2 (Converse):* Suppose a tuple

$$(r_1, r_2, r_3, d_1, d_2, d_{1,2}, w_1, w_2, w_{*,1}, w_{*,2})$$

is *rmd*-achievable and pareto-optimal. Let

$$\sigma = r_1 + r_2 - (r_3 + w_1). \tag{35}$$

Then $(\mathbf{r}, \mathbf{d}, \sigma)$ is $c$-achievable and satisfies (28)-(34).

*Sketch of proof:* Consider a sequence of robust MD codes $C^N$ satisfying (14)-(18). By Lemma 2, (20)-(22) hold. Consequently, we proved (28)-(31) and that $(\mathbf{r}, \mathbf{d}, \sigma)$ is $c$-achievable by the sequence of MD codes

$$(X^N, F_1^N, F_2^N).$$

Finally, (32)-(34) follow from that

$$\lim_{N\to\infty} \frac{H(A_{*,1}^N)}{N} \ge \lim_{N\to\infty} \frac{H(F_{*,1}^N | F_{*,2}^N)}{N} \tag{36}$$

$$\lim_{N\to\infty} \frac{H(A_{*,2}^N)}{N} \ge \lim_{N\to\infty} \frac{H(F_{*,2}^N | F_{*,1}^N)}{N} \tag{37}$$

$$\lim_{N\to\infty} \frac{H(A_{*,1}^N, A_{*,2}^N)}{N} \ge \lim_{N\to\infty} \frac{H(F_{*,3}^N)}{N}. \tag{38}$$

■

By Theorems 1 and 2, the set of pareto-optimal *rmd*-achievable tuples (and hence the set of all *rmd*-achievable tuples) can be determined if the set of $c$-achievable tuples is determined. In the next theorem, we will give an inner bound for the set of $c$-achievable tuples.

*Theorem 3 (Achievability):* Consider any probably mass functions $p(x)p(\hat{x}_0, \hat{x}_1, \hat{x}_2 | x)$ where $\hat{x}_0, \hat{x}_1, \hat{x}_2 \in \hat{\mathcal{X}}$. The tuple $(r_1, r_2, d_1, d_2, d_{1,2}, \sigma)$ is $c$-achievable if

$$r_1 > I(X; \hat{X}_1) \tag{39}$$

$$r_2 > I(X; \hat{X}_2) \tag{40}$$

$$r_1 + r_2 > I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \tag{41}$$

$$d_i \ge E[d(X, \hat{X}_i)], \ i = 0, 1, 2 \tag{42}$$

$$\sigma = I(\hat{X}_1; \hat{X}_2). \tag{43}$$

*Proof:* In [1, Theorem 1], it was proved that if (39)-(42) are satisfied, then $(r_1, r_2, d_1, d_2, d_{1,2})$ is achievable with a sequence of random MD codes $\mathcal{C}^N = (N, F_1^N, F_2^N)$. In fact, for the sequence of codes, it can be proved that (27) holds. Hence, the theorem is proved. ■

## IV. Extensions

In the previous section, descriptions of a source are stored in two nodes $V_1$ and $V_2$, with "side information" (for the purpose of data regeneration) stored in the helper node $V_3$. In that model, the helper node is not accessible by users. Suppose users can also access $V_3$. In this case, it will be desirable that the content stored there is also a description itself so that those users who can only access $V_3$ can also reconstruct an estimate of the source. As before, the three descriptions will be stored in three different servers and we will insist that the contents of each node can be regenerated from the other two nodes. As the first step answering the question, we assume that the amount of repair bandwidth is abundant. In other words, we are only interested in the tradeoff between the levels of distortion between the estimates and the original source, and the size of the descriptions (hence, the required storage size).

*Definition 7 (Self-repairable MD codes):* A self-repairable MD code is specified by a tuple

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N, F_3^N).$$

For such a MD code and $\alpha \subseteq \{1,2,3\}$, $D_\alpha(\mathcal{C}^N)$ is defined as in (8). Intuitively, for a receiver which has access to descriptions $(F_i^N, i \in \alpha)$, $D_\alpha(\mathcal{C}^N)$ will then be the level of distortion of the reconstructed message.

Similarly, in case of node failures, we will need the node to be repaired optimally in the sense of minimising error probability

$$P_e^1(\mathcal{C}^N) \triangleq \inf_{\chi_1} \Pr[F_1 \neq \chi_1(F_2^N, F_3^N)] \quad (44)$$

$$P_e^2(\mathcal{C}^N) \triangleq \inf_{\chi_2} \Pr[F_2 \neq \chi_2(F_1^N, F_3^N)] \quad (45)$$

$$P_e^3(\mathcal{C}^N) \triangleq \inf_{\chi_3} \Pr[F_3 \neq \chi_3(F_1^N, F_2^N)]. \quad (46)$$

Again, $F_i$ is a description stored in node $V_i$ and $\chi_i$ is used to regenerate the content in node $V_i$ when it failed. As we assume an abundance of repair bandwidth, we can directly copy the contents stored in those surviving nodes to the failed node for data recovery.

*Definition 8:* A tuple $[r, d_1, d_2]$ is called *$s$-achievable*[5] if there exists a sequence of robust MD codes

$$\mathcal{C}^N = (X^N, F_1^N, F_2^N, F_3^N)$$

such that

$$\limsup_{N\to\infty} \frac{H(F_i^N)}{N} \leq r,\ i = 1,2,3 \quad (47)$$

$$\limsup_{N\to\infty} D_\alpha(\mathcal{C}^N) \leq d_1,\ |\alpha| = 1 \quad (48)$$

$$\limsup_{N\to\infty} D_\alpha(\mathcal{C}^N) \leq d_2,\ |\alpha| \geq 2 \quad (49)$$

$$\limsup_{N\to\infty} P_e^j(\mathcal{C}^N) = 0,\ j = 1,2,3. \quad (50)$$

[5]Here, $s$ is a mnemonic for "self-repairable".

Clearly, if $[r, d_1, d_2]$ is $s$-repairable, then there exists a sequence of robust MD codes $\mathcal{C}^N$ such that asymptotically, the size of the $i^{th}$ description is at most $rN$ and a user who receives descriptions $F_i^N$ for $j \in \alpha$ can reconstruct an estimate of the source with average distortion at most $d_{|\alpha|}$ for $|\alpha| \leq 2$. Note also that as the content in every node can be regenerated from the other two nodes, the estimate reconstructed from three descriptions will have the same level of distortion as the one reconstructed from two descriptions.

*Theorem 4:* Let $\eta \triangleq \min_{x' \in \mathcal{X}'} \sum_{x \in \mathcal{X}} p(x)\delta(x, x')$. Consider any probably mass functions $p(x)p(\hat{x}_0, \hat{x}_1, \hat{x}_2|x)$. The tuple $(r, d_1, d_2)$ is $s$-achievable if

$$r \geq \frac{\ell + I(\hat{X}_1; \hat{X}_2)}{3} \quad (51)$$

$$d_1 \geq \frac{E[d(X,\hat{X}_1)] + E[d(X,\hat{X}_2)] + \eta}{3} \quad (52)$$

$$d_2 \geq E[d(X,\hat{X}_0)]. \quad (53)$$

where

$$\ell = \max(I(X;\hat{X}_1,\hat{X}_2,\hat{X}_0) + I(\hat{X}_1;\hat{X}_2), I(X;\hat{X}_1) + I(X;\hat{X}_2)).$$

*Remark 2:* The construction of the self-repairable code is based on the MD codes in Theorem 3 by adding redundancies in the descriptions to ensure the self-repairable property. Due to page limit, the proof will be omitted.

## V. Conclusion

This paper proposed a robust content distribution network for distributing contents subject to distortion constraints and robustness constraint. Specifically, nodes in the network are accessible by users to retrieve descriptions for a source. In addition, it is required that the descriptions stored in the node are robust against node failures (causing lost of data stored in one of the nodes). Parameters of interest include storage size, repair bandwidth and distortion levels in the estimates. We are interested in the tradeoff amoung these parameters. We showed that we can construct a robust content distribution scheme from an ordinary multiple description codes. Bounds on the set of achievable tuples were also obtained.

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