Impact of Topology on Interference Networks with No CSIT

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***Abstract*—We study the symmetric degrees-of-freedom (DoF) of partially connected K-user interference networks in which the transmitters are unaware of the actual channel gain values. Several linear algebraic and graph theoretic concepts are introduced to derive new outer and inner bounds on the symmetric DoF for arbitrary network topologies. We evaluate the bounds for a class of networks, showing that the bounds are tight for most topologies in that class, and we also quantify the gains obtained over benchmark schemes.**

I. Introduction

Channel state information (CSI) plays a central role in the design of physical layer interference management strategies for wireless networks. Thus, training-based channel estimation techniques are commonly used in today's wireless networks to estimate the channel parameters at the receivers and then to propagate the estimates to other nodes in the network via feedback links. However, as wireless networks grow in size and mobility increases, the availability of CSI at the transmitters (CSIT) becomes a challenging task to accomplish.

Consequently, there has been a growing interest in understanding how the lack of CSI would have an impact on fundamental limits of interference management in wireless networks. In particular, in the context of interference channels, various settings for CSIT have been studied in the literature (e.g., no CSIT, local CSIT, and delayed CSIT).

In this work, we focus on the impact of knowledge about network topology on interference management in interference networks. We consider an interference network with K transmitters and K receivers, where each transmitter intends to deliver a message to its corresponding receiver. In order to model propagation path loss and interference topology, the network is considered to be partially connected, and the network topology is represented by the adjacency matrix of the network connectivity graph. However, in this work, we assume that the transmitters are only aware of network topology, and they do not know the actual channel gain values. This is partially motivated by the fact the network connectivity often changes at a slower pace than the channel gains, hence it is plausible to acquire them at the transmitters.

This problem has been also considered in some prior works in the literature. In [1], a slow fading scenario is considered, and the authors have used a "normalized sum-capacity" metric to characterize the largest achievable fraction of the sum-capacity-with-full-CSI when transmitters only know network topology and the gains of some local channels. In [2], a fast fading scenario is considered in which the channel gains are i.i.d. over time and also across the users, with a sufficiently large coherence time. It has been shown that the degree-of-freedom (DoF) region of this problem is bounded above by the DoF region of a wireless index coding problem, and (with sufficiently large coherence time) they are equivalent if the problems are restricted to linear solutions. Moreover in [3], a slow fading scenario is considered and the channel gains are assumed to be sufficiently large to satisfy a minimum signal-to-noise ratio (SNR) at each receiver. Several outer bounds on the symmetric DoF were derived and an interference-alignment-based achievable scheme has been developed.

In this paper, we focus on the fast fading scenario with only network topology knowledge at the nodes and i.i.d. fading across all existing channels in the network. The assumption of coherence time of 1 is an extreme case in the spectrum of topological interference management problem [2,3], in which, as opposed to [2,3], we do not make use of temporal alignment. We develop new graph theoretic and linear algebraic inner and outer bounds on the symmetric DoF. The outer bounds are based on two novel linear algebraic concepts of "generators" and "fractional generators". The key idea in the outer bounds is that we seek for a number of signals from which we can decode the messages of all the users, and then we will find the tightest upper bound for the entropy of those signals. These outer bounds can be applied to any channel coherence time.

Moreover, we present our inner bounds by proposing an achievable scheme, called "structured repetition coding". The main idea is to enable neutralization of interference at the receivers by repeating the symbols based on a carefully-chosen structure at the transmitters. We derive graph theoretic conditions, based on the matching number of bipartite graphs induced by network topology and the repetition structure of transmitters, that characterize the symmetric DoF achieved by structured repetition coding. This scheme can also be applied to any channel coherence time by means of interleaving. Thus, the coherence time of 1 is the worst case in this sense.

Finally, we evaluate our inner and outer bounds to characterize the symmetric DoF of 6-user networks with 6 square cells. Interestingly, our bounds are tight for all topologies, except 16, implying that temporal alignment is not needed in most of these networks and tight DoF results can be obtained even with a coherence time of 1. We also illustrate the gain of structured repetition coding over two benchmark schemes and study the impact of network density on these gains.

II. Problem Settings and Notations

A K-user interference network ($K \in \mathbb{N}$) is defined as a set of K transmitter nodes $\{\mathrm{T}_i\}_{i=1}^K$ and K receiver nodes $\{\mathrm{D}_i\}_{i=1}^K$. We consider the network to be partially connected, represented by the adjacency matrix $\mathbf{M} \in \{0,1\}^{K\times K}$, such that $\mathbf{M}_{ij} = 1$ iff transmitter T_i is connected to receiver D_j. For all $i \in [1:K]$ (we use the notation $[1:m]$ to denote $\{1,2,...,m\}$ for $m \in \mathbb{N}$), we assume the direct link exists between T_i and D_i; i.e. $\mathbf{M}_{ii} = 1$, $\forall i \in [1:K]$. We also define the set of interfering nodes to D_j as $\mathcal{IF}_j := \{i : \mathbf{M}_{ij} = 1,\ i \neq j\}$. Furthermore, if $\mathcal{S} \subseteq [1:K]$, then $\mathbf{M}^{\mathcal{S}}$ denotes the adjacency matrix of the corresponding subgraph. Moreover,

- If $n \in \mathbb{N}$, then $\mathbf{I}^n$ denotes the $n \times n$ identity matrix.
- For an $m \times n$ matrix $\mathbf{A}$ and $\mathcal{N} \subseteq [1:n]$, $\mathbf{A}_{\mathcal{N}}$ denotes the submatrix of $\mathbf{A}$ composed of the columns whose indices are in $\mathcal{N}$. Also, if $i \in [1:m]$ and $j \in [1:n]$, $\mathbf{A}_{i,*}$ and $\mathbf{A}_j$ denote the i^{th} row and j^{th} column of $\mathbf{A}$, respectively. Besides, $c(\mathbf{A})$ denotes the number of columns of $\mathbf{A}$.

At each time l ($l \in \mathbb{N}$), the transmit signal of T_i is denoted by $X_i[l] \in \mathbb{C}$ and the received signal of D_j ($Y_j[l]$) is given by

$$Y_j[l] = g_{jj}[l]X_j[l] + \sum_{i\in\mathcal{IF}_j} g_{ij}[l]X_i[l] + Z_j[l],$$

where $Z_j[l] \sim \mathcal{CN}(0,1)$ is the additive noise (i.i.d. among the users and time slots, and independent of transmit symbols and channel gains) and $g_{ij}[l] \in \mathbb{C}$ is the channel gain from T_i to D_j. If T_i is not connected to D_j, then $g_{ij}[l] = 0$, $\forall l$. The non-zero channel gains are i.i.d. (with a continuous distribution $f_G(g)$) through time and across the users, and independent of transmit symbols. The distribution $f_G(g)$ is assumed to satisfy three conditions: $\mathbb{E}[\log(1+|g|)] < \infty$, $f_G(g) = f_G(-g), \forall g \in \mathbb{C}$, and $\exists f_{max}$ s.t. $f_{|G|}(r) \leq f_{max}, \forall r \in \mathbb{R}^+$, where $f_{|G|}(.)$ is the distribution of $|g|$. The transmitters are assumed to only know $\mathbf{M}$ and f_G. We refer to this assumption as no CSIT. As for receivers, each D_j knows $\mathbf{M}$ and $\{g_{ij}[l] : i \in [1:K], \forall l\}$.

In this network, every transmitter T_i intends to deliver a message W_i to its receiver D_i. W_i is encoded to a vector $X_i^n = [X_i[1]\ X_i[2]\ \ldots\ X_i[n]]^T \in \mathbb{C}^n$ through an encoding function $e_i(W_i|\mathbf{M}, f_G)$, with a transmit power constraint $\mathbb{E}\left[\frac{1}{n}\|X_i^n\|\right] \leq P$, $\forall i \in [1:K]$. Each receiver D_j receives the vector $Y_j^n = [Y_j[1]\ Y_j[2]\ \ldots\ Y_j[n]]^T$ and uses a decoding function $e'_j(Y_j^n|\mathbf{M}, \mathcal{G}_j^n)$ to recover its desired message W_j. Here, $\mathcal{G}_j^n := \{g_{ij}^n : i \in [1:K]\}$ where $g_{ij}^n := [g_{ij}[1]\ g_{ij}[2]\ \ldots\ g_{ij}[n]]^T$. We also denote the set of all channel gains in all time slots by $\mathcal{G}^n = \{\mathcal{G}_1^n, ..., \mathcal{G}_K^n\}$.

The rate of user i is denoted by $R_i(P) := \frac{\log|W_i(P)|}{n}$ where $|W_i(P)|$ is the size of the message set of user i. A rate tuple $(R_1(P), ..., R_K(P))$ is achievable if the maximum error probability at the receivers goes to zero as $n \to \infty$. The considered metric in this paper is the symmetric degrees-of-freedom (DoF), denoted by d_{sym}, which is defined as the supremum d such that for all $i \in [1:K]$, $d = \lim\limits_{P\to\infty} \frac{R_i(P)}{\log(P)}$ and the rate tuple $(R_1(P), ..., R_K(P))$ is achievable.

III. Outer Bounds on d_{sym}

A. Outer Bounds Based on the Concept of Generators

To illustrate our first bound, consider the network in Fig. 1. We claim that in this network, $d_{sym} \leq \frac{2}{6}$. Suppose rates R_i, $i \in [1:6]$ are achievable. Define

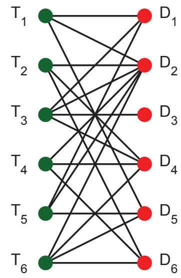

Fig. 1: A 6-user interference network for illustrating the outer bounds.

$$\tilde{Y}_1^n := g_1^n X_1^n + g_3^n X_3^n + g_5^n X_5^n + Z_1^n$$
$$\tilde{Y}_4^n := g_2^n X_2^n + g_3^n X_3^n + g_4^n X_4^n + g_6^n X_6^n + Z_4^n,$$

where $g_i^n = g_{ii}^n$, $i \in [1:6]$. We will show, through the following steps, that $H(W_1, ..., W_6|\tilde{Y}_1^n, \tilde{Y}_4^n, \mathcal{G}^n) \leq no(\log P) + n\epsilon_n$. In all the steps, the $n\epsilon_n$ term is due to Fano's inequality.

- $H(W_1, W_4|\tilde{Y}_1^n, \tilde{Y}_4^n, \mathcal{G}^n) \leq n\epsilon_n$, because $\tilde{Y}_1^n$ and $\tilde{Y}_4^n$ are statistically the same as Y_1^n and Y_4^n, respectively.
- $H(W_2|\tilde{Y}_1^n, \tilde{Y}_4^n, W_1, W_4, \mathcal{G}^n) \leq n\epsilon_n$, because $\tilde{Y}_4^n$ is a less-interfered version of Y_2^n. In other words,

$$H(W_2|\tilde{Y}_4^n, \mathcal{G}^n) = H(W_2|\tilde{Y}_4^n, X_1^n, X_5^n, \mathcal{G}^n)$$
$$\leq H(W_2|\tilde{Y}_4^n + g_1^n X_1^n + g_5^n X_5^n, \mathcal{G}^n) \leq n\epsilon_n,$$

  since $\tilde{Y}_4^n + g_1^n X_1^n + g_5^n X_5^n$ is statistically the same as Y_2^n.
- $H(W_5|\tilde{Y}_1^n, \tilde{Y}_4^n, W_1, W_4, W_2, \mathcal{G}^n) \leq no(\log P) + n\epsilon_n$, because from the terms in the conditioning, one can generate $\tilde{Y}_1^n - \tilde{Y}_4^n + g_2^n X_2^n + g_4^n X_4^n$, which is statistically the same as Y_5^n (because the distribution of g_i's is symmetric around zero) with a bounded difference in noise variance, which is taken care for by $no(\log P)$ term (see Lemma 1).
- $H(W_3|\tilde{Y}_1^n, \tilde{Y}_4^n, W_1, W_4, W_2, W_5, \mathcal{G}^n) \leq n\epsilon_n$, because from the terms in the conditioning, one can generate $\tilde{Y}_1^n - g_1^n X_1^n - g_5^n X_5^n$, which is statistically the same as Y_3^n.
- $H(W_6|\tilde{Y}_1^n, \tilde{Y}_4^n, W_1, W_4, W_2, W_5, W_3, \mathcal{G}^n) \leq n\epsilon_n$, since from the terms in the conditioning, one can build $\tilde{Y}_4^n - g_3^n X_3^n$, which is statistically the same as Y_6^n.

Adding the above inequalities, together with chain rule, yields $H(W_1, ..., W_6|\tilde{Y}_1^n, \tilde{Y}_4^n, \mathcal{G}^n) \leq no(\log P) + n\epsilon_n$. Hence,

$$\begin{aligned} n\sum_{i=1}^{6} R_i &\leq I(W_1, ..., W_6; \tilde{Y}_1^n, \tilde{Y}_4^n|\mathcal{G}^n) + no(\log P) + n\epsilon_n \\ &= h(\tilde{Y}_1^n, \tilde{Y}_4^n|\mathcal{G}^n) + no(\log P) + n\epsilon_n \qquad (1) \\ &\leq 2n\log P + no(\log P) + n\epsilon_n, \end{aligned}$$

which, after letting n and P go to infinity, yields $d_{sym} \leq \frac{2}{6}$.

The above steps can also be described in a linear algebraic context, by noting that each of the above signals, ignoring the noise term, can be represented by a vector in $\{0, \pm 1\}^6$ whose i^{th} element equals the coefficient of $g_i^n X_i^n$ in that signal. For instance, the signals $\{\tilde{Y}_1^n, \tilde{Y}_4^n\}$ correspond to the 6×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}^T.$$

To mention the linear algebraic conditions equivalent to the previous steps, we need the following definition.

Definition 1. If $\mathbf{v} \in \{0, \pm1\}^{n\times 1}$, $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, and $i \in [1:n]$, then $\mathbf{v} \in_i^{\pm} \mathcal{V}$ is defined as

$$\mathbf{v} \in_i^{\pm} \mathcal{V} \Leftrightarrow \exists \tilde{\mathbf{v}} \in \mathcal{V} \text{ s.t. } |\tilde{\mathbf{v}}_i| = |\mathbf{v}_i| \text{ and } \tilde{\mathbf{v}}_j(|\tilde{\mathbf{v}}_j| - |\mathbf{v}_j|) = 0,\ \forall j \in [1:n] \setminus \{i\},$$

implying that there exists $\tilde{\mathbf{v}} \in \mathcal{V}$ such that $\tilde{\mathbf{v}}_i$ is the same as $\mathbf{v}_i$ up to its sign, while every other element of $\tilde{\mathbf{v}}$ either is zero or matches the corresponding element of $\mathbf{v}$ up to its sign. △

Setting $i_1 = 1, i_2 = 4, i_3 = 2, i_4 = 5, i_5 = 3$ and $i_6 = 6$, the above steps are equivalent to the following.

$$\mathbf{M}_{i_j}^{\mathcal{S}} \in_{i_j}^{\pm} \text{span } (\mathbf{A}, \mathbf{I}_{\{i_1,...,i_{j-1}\}}^{|\mathcal{S}|}),\ \forall j \in [1:6],$$

meaning that in step j, by combining the previously decoded symbols $(X_{i_1}^n, ..., X_{i_{j-1}}^n)$ and the signals corresponding to $\mathbf{A}$, we can generate a statistically-similar or less-interfered version of the signal at D_j. This motivates the definition of generators.

Definition 2. Consider a K-user interference network with adjacency matrix $\mathbf{M}$ and assume $\mathcal{S} \subseteq [1:K]$ is a subset of users. $\mathbf{A} \in \{\pm1, 0\}^{|\mathcal{S}|\times r}$ $(r \in \mathbb{N})$ is called a *generator* of $\mathcal{S}$ if there exists a sequence $\Pi_{\mathcal{S}} = (i_1, ..., i_{|\mathcal{S}|})$ of users in $\mathcal{S}$ s.t.

$$\mathbf{M}_{i_j}^{\mathcal{S}} \in_{i_j}^{\pm} \text{span } (\mathbf{A}, \mathbf{I}_{\{i_1,...,i_{j-1}\}}^{|\mathcal{S}|}),\ \forall j \in [1:|\mathcal{S}|].$$

We use $\mathcal{J}(\mathcal{S})$ to denote the set of all generators of $\mathcal{S}$. △

We now state our first outer bound based on above definitions.

Theorem 1. *The symmetric DoF of a K-user interference network with no CSIT is upper bounded by*

$$d_{sym} \leq \min_{\mathcal{S}\subseteq[1:K]} \min_{\mathbf{A}\in\mathcal{J}(\mathcal{S})} \frac{c(\mathbf{A})}{|\mathcal{S}|}.$$

To prove Theorem 1, we need the following lemma, whose proof can be found in Appendix A of [4].

Lemma 1. *For a discrete random variable W, continuous random vector Y^n, and two complex Gaussian noise vectors Z_1^n and Z_2^n, where each element of Z_1^n and Z_2^n are $\mathcal{CN}(0,1)$ and $\mathcal{CN}(0,N)$ random variables, respectively and all the random variables are mutually independent, if $H(W|Y^n + Z_1^n) \leq n\epsilon$, then $H(W|Y^n + Z_2^n) \leq n\epsilon + n\log(N+1)$.*

Proof Sketch of Theorem 1. If $\mathbf{A} \in \mathcal{J}(\mathcal{S})$, we can show that $H(W_1, ..., W_{|\mathcal{S}|}|\tilde{Y}_1^n, ..., \tilde{Y}_{c(\mathbf{A})}^n, \mathcal{G}^n) \leq no(\log P) + n\epsilon_n$, similar to the network of Fig. 1, where the signals $\tilde{Y}_i^n$'s correspond to the columns of $\mathbf{A}$. This implies $d_{sym} \leq \frac{c(\mathbf{A})}{|\mathcal{S}|}$. Optimizing over $\mathbf{A} \in \mathcal{J}(\mathcal{S}), \mathcal{S} \subseteq [1:K]$ yields the bound in Theorem 1. The complete proof can be found in [4] (Section III). □

B. Outer Bounds Based on Fractional Generators

Consider again the network in Fig. 1. We will now illustrate how a tighter outer bound of $d_{sym} \leq \frac{2}{7}$ can be obtained. Following (1), we can write

$$\begin{aligned} n\sum_{i=1}^{6} R_i &\leq h(\tilde{Y}_1^n, \tilde{Y}_4^n|\mathcal{G}^n) + no(\log P) + n\epsilon_n \\ &\leq h(\tilde{Y}_1^n|\mathcal{G}^n) + h(\tilde{Y}_4^n|\mathcal{G}^n) + no(\log P) + n\epsilon_n \\ &\leq h(\tilde{Y}_1^n|\mathcal{G}^n) + n\log P + no(\log P) + n\epsilon_n. \quad (2) \end{aligned}$$

Now, we can find a tighter upper bound on $h(\tilde{Y}_1^n|\mathcal{G}^n)$ than $n\log P + no(\log P)$. The intuitive reason is that in the network of Fig. 1, D_1 receives signals from $\mathrm{T}_1, \mathrm{T}_3$ and T_5. However, $\{1,3,5\} \subseteq \mathcal{IF}_2$, leading to the upper bound $h(\tilde{Y}_1^n|\mathcal{G}^n) \leq n(\log P - R_2) + no(\log P) + n\epsilon_n$. To prove this more rigorously, first note that

$$\begin{aligned} &H(W_2) - H(W_2|g_{22}^n X_2^n + \tilde{Y}_1^n, \mathcal{G}^n) \\ =&h(g_{22}^n X_2^n + \tilde{Y}_1^n|\mathcal{G}^n) - h(g_{22}^n X_2^n + \tilde{Y}_1^n|W_2, \mathcal{G}^n), \end{aligned}$$

since both sides are equal to $I(g_{22}^n X_2^n + \tilde{Y}_1^n; W_2|\mathcal{G}^n)$. Thus,

$$\begin{aligned} h(g_{22}^n X_2^n + \tilde{Y}_1^n|W_2, \mathcal{G}^n) &= H(W_2|g_{22}^n X_2^n + \tilde{Y}_1^n, \mathcal{G}^n) \\ &\quad + h(g_{22}^n X_2^n + \tilde{Y}_1^n|\mathcal{G}^n) - H(W_2) \\ &\leq n\epsilon_n + h(g_{22}^n X_2^n + \tilde{Y}_1^n|\mathcal{G}^n) - H(W_2) \quad (3) \\ &\leq n\log P - nR_2 + no(\log P) + n\epsilon_n, \quad (4) \end{aligned}$$

where (3) holds because $g_{22}^n X_2^n + \tilde{Y}_1^n$ is a less-interfered version of Y_2^n, hence able to decode W_2. On the other hand, since X_2^n is a function of W_2, we have

$$h(g_{22}^n X_2^n + \tilde{Y}_1^n|W_2, \mathcal{G}^n) = h(\tilde{Y}_1^n|\mathcal{G}^n),$$

which together with (4) yields $h(\tilde{Y}_1^n|\mathcal{G}^n) \leq n(\log P - R_2) + no(\log P) + n\epsilon_n$. Combining this with (2) leads to $d_{sym} \leq \frac{2}{7}$.

This example motivates the notion of fractional generators.

Definition 3. Consider a K-user interference network with adjacency matrix $\mathbf{M}$ and suppose $\mathcal{S}' \subseteq \mathcal{S} \subseteq [1:K]$. A vector $\mathbf{c} \in \{\pm1, 0\}^{|\mathcal{S}|}$ is called a *fractional generator of $\mathcal{S}'$ in $\mathcal{S}$* if there exists a sequence $\Pi_{\mathcal{S}'} = (i_1, ..., i_{|\mathcal{S}'|})$ of users in $\mathcal{S}'$ s.t.:

$$\mathbf{M}_{i_j}^{\mathcal{S}} \in_{i_j}^{\pm} \text{span } (\mathbf{c} + \sum_{k\in\mathcal{S}'} \mathbf{I}_k^{|\mathcal{S}|}, \mathbf{I}_{\{i_1,...,i_{j-1}\}}^{|\mathcal{S}|}),\ \forall j \in [1:|\mathcal{S}'|].$$

We use $\mathcal{J}_{\mathcal{S}}(\mathcal{S}')$ to denote the set of all fractional generators of $\mathcal{S}'$ in $\mathcal{S}$. △

For example, if $\mathcal{S}' = \{2\}$ and $\mathcal{S} = [1:6]$, then $\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^T \in \mathcal{J}_{\mathcal{S}}(\mathcal{S}')$. Definition 3 allows us to state the following lemma, proved in Appendix B of [4].

Lemma 2. *Consider a subset of users $\mathcal{S} \subseteq [1:K]$ in a K-user interference network, and suppose $\mathbf{c} \in \mathcal{J}_{\mathcal{S}}(\mathcal{S}')$. If rates R_i are achievable for all $i \in \mathcal{S}'$, then*

$$\begin{aligned} &h(\sum_{j\in\mathcal{S}} \mathbf{c}_j g_j^n X_j^n + Z^n|\mathcal{G}^n) \\ &\qquad \leq n(\log P - \sum_{i\in\mathcal{S}'} R_i) + no(\log P) + n\epsilon_n, \end{aligned}$$

where $g_j^n = g_{jj}^n$, $\forall j \in \mathcal{S}$ and each element in Z^n is a $\mathcal{CN}(0,1)$ random variable.

We also define $n_{\mathcal{S}}(\mathbf{c})$ for a vector $\mathbf{c} \in \{\pm1, 0\}^{|\mathcal{S}|}$ as follows.

Definition 4. Consider a subset of users $\mathcal{S} \subseteq [1:K]$ in a K-user interference network. For a vector $\mathbf{c} \in \{\pm1, 0\}^{|\mathcal{S}|}$,

$$n_{\mathcal{S}}(\mathbf{c}) := \max_{\mathcal{S}'} |\mathcal{S}'| \text{ s.t. } \mathbf{c} \in \mathcal{J}_{\mathcal{S}}(\mathcal{S}').$$

△

For example, $n_{\mathcal{S}}(\mathbf{M}_1) = 1$. Using Lemma 2 and the above definition we can now state our second converse.

Theorem 2. *The symmetric DoF of a K-user interference network with no CSIT is upper bounded by*

$$d_{sym} \leq \min_{\mathcal{S}\subseteq[1:K]} \min_{\mathbf{A}\in\mathcal{J}(\mathcal{S})} \frac{c(\mathbf{A})}{|\mathcal{S}| + \sum_{i=1}^{c(\mathbf{A})} n_{\mathcal{S}}(\mathbf{A}_i)}.$$

Proof Sketch of Theorem 2. For each signal $\tilde{Y}_i^n$ corresponding to the vector $\mathbf{A}_i$, if $\mathbf{A}_i \in \mathcal{J}_{\mathcal{S}}(\mathcal{S}')$, then Lemma 2 implies

$$h(\tilde{Y}_i^n) \leq n(\log P - \sum_{i\in\mathcal{S}'} R_i) + no(\log P) + n\epsilon_n.$$

Thus, finding the largest $\mathcal{S}'$ s.t. $\mathbf{A}_i \in \mathcal{J}_{\mathcal{S}}(\mathcal{S}')$ leads to the tightest bound on $h(\tilde{Y}_i^n)$, hence on d_{sym}. In fact, if $\mathbf{A} \in \mathcal{J}(\mathcal{S})$, then $|\mathcal{S}|d_{sym} \leq c(\mathbf{A}) - \sum_{i=1}^{c(\mathbf{A})} n_{\mathcal{S}}(\mathbf{A}_i)d_{sym}$. Optimizing over $\mathbf{A} \in \mathcal{J}(\mathcal{S}), \mathcal{S} \subseteq [1:K]$ yields the bound in Theorem 2. Proof details can be found in [4] (Section III). □

IV. Inner Bounds on d_{sym}

In this section, we provide inner bounds on d_{sym} based on the structured repetition coding scheme. To motivate this achievable scheme, consider the 6-user network in Fig. 2.

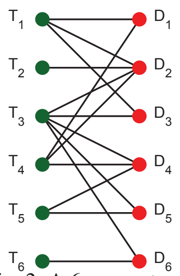

Fig. 2: A 6-user network.

There are two benchmark schemes that can be considered when there is no CSIT, namely interference avoidance (in which interfering users avoid transmitting data at the same time) and random Gaussian coding with interference decoding (in which users communicate at sufficiently low rates such that all the interference is decoded at each receiver). As shown in [4] (Section IV), the d_{sym} achieved by these schemes are $\frac{1}{\chi_f}$ and $\frac{1}{\Delta_R}$, respectively, where χ_f is the fractional chromatic number of the corresponding conflict graph, and Δ_R is the maximum receiver degree. For the network of Fig. 2, $\Delta_R = \chi_f = 4$, hence both schemes achieve $d_{sym} = \frac{1}{4}$.

However, d_{sym} of $\frac{1}{3}$ can be achieved in this network by using a scheme in which each transmitter repeats its symbol according to a carefully designed structure. More specifically, consider a scheme in which each T_i intends to send a symbol X_i to receiver D_i in 3 time slots, by following a transmission pattern according to a transmission matrix $\mathbf{T}$ defined as

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}^T,$$

where $\mathbf{T}_{ik} = 1$ if transmitter i sends X_i in time slot k and $\mathbf{T}_{ik} = 0$ if transmitter i is silent in time slot k, $\forall i \in [1:6], \forall k \in [1:3]$. It is then easy to verify that, for almost all channel gain values, each receiver D_j can create an interference-free version of X_j by a linear combination of the signals it receives in 3 time slots. For instance, D_1 receives

$$Y_1[1] = g_{11}[1]X_1 + g_{41}[1]X_4 + Z_1[1],$$
$$Y_1[2] = Z_1[2], \quad Y_1[3] = g_{41}[3]X_4 + Z_1[3],$$

and the combination $\frac{1}{g_{11}[1]}Y_1[1] - \frac{g_{41}[1]}{g_{41}[3]g_{11}[1]}Y_1[3]$ yields $X_1 + \tilde{Z}_1$, where variance of $\tilde{Z}_1$ depends only on the channel gains.

The existence of this linear combination can also be viewed in terms of the matching number of a bipartite graph that is induced by network topology and the transmission matrix T. We illustrate the procedure for receiver D_1. The idea is to first create an "effective" transmission matrix $\bar{\mathbf{T}}^1$ for receiver 1, which is defined as a 6×3 matrix, where $\bar{\mathbf{T}}^1_{ik} = \mathbf{M}_{i1}\mathbf{T}_{ik}, \forall i \in [1:6], k \in [1:3]$, as shown in (5). In words, $\bar{\mathbf{T}}^1$ is the same as $\mathbf{T}$ with the distinction that the rows corresponding to the transmitters which are not connected to D_1 are set to zero. This matrix corresponds to a bipartite graph $\bar{G}^1$, shown in Fig. 3, with vertices $\{v_1, ..., v_6\} \cup \{v'_1, v'_2, v'_3\}$, where v_i is connected to v'_k iff $\bar{\mathbf{T}}^1_{ik} = 1$, $\forall i \in [1:6], k \in [1:3]$.

$$\bar{\mathbf{T}}^1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T. \tag{5}$$

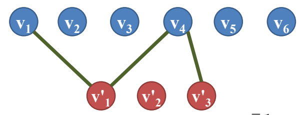

Fig. 3: The bipartite graph $\bar{G}^1$.

Now, note that the matching number of $\bar{G}^1$ (the largest number of edges without common vertices) is 2. However, it reduces to 1 if v_1 and its corresponding edge are removed from $\bar{G}^1$. As we show later in Lemma 3, this reduction in the matching number implies the existence of a linear combination of the signals at D_1 which is an interference-free version of X_1 (for almost all channel gain values). Likewise, the same property (i.e., reduction in the matching number of the induced bipartite graph after removing the receiver node and its incident edges) holds at other receivers, implying that they can create interference-free versions of their desired symbols.

This example demonstrated how the transmitters can enable the receivers to neutralize interference by repeating the symbols based on a carefully-chosen structure, and illustrated how the success of the scheme is equivalent to the satisfaction of conditions based on bipartite matching. Motivated by the example, we now formally define structured repetition coding.

Definition 5. Consider a K-user interference network with adjacency matrix $\mathbf{M}$ and a matrix $\mathbf{T} \in \{0,1\}^{mK\times n}$ $(m, n \in \mathbb{N})$ satisfying $\sum_{l=(i-1)m+1}^{im} \mathbf{T}_{lk} \leq 1$, $\forall k \in [1:n]$, $\forall i \in [1:K]$. Then, *structured repetition coding with transmission matrix* $\mathbf{T}$ is defined as a scheme, in which T_i $(i \in [1:K])$ intends to deliver m symbols $\left\{\tilde{X}_l\right\}_{l=(i-1)m+1}^{im}$ to D_i in n time slots, using the following encoding and decoding procedure.

- T_i $(i \in [1:K])$ transmits $X_i^n = \sum_{l=(i-1)m+1}^{im} \mathbf{T}_{l,*}^T \tilde{X}_l$.
- D_j $(j \in [1:K])$ receives

$$Y_j^n = \sum_{i=1}^{K} \mathbf{M}_{ij} g_{ij}^n \left(\sum_{l=(i-1)m+1}^{im} \mathbf{T}_{l,*}^T \tilde{X}_l \right) + Z_j^n,$$

and looks for vectors $\mathbf{u}_l$, $l \in [(j-1)m+1 : jm]$ s.t.

$$(Y_j^n)^T \mathbf{u}_l = \tilde{X}_l + (Z_j^n)^T \mathbf{u}_l, \ \forall l \in [(j-1)m+1 : jm]. \quad (6) \triangle$$

The remaining step is to state the conditions that $\mathbf{T}$ must satisfy to assure the existence of $\mathbf{u}_l$'s satisfying (6), so that the receivers are able to neutralize all the interference. But before that, we need the following definition.

Definition 6. Consider a K-user interference network with adjacency matrix $\mathbf{M}$ and structured repetition coding with transmission matrix $\mathbf{T} \in \{0,1\}^{mK \times n}$. For each $j \in [1:K]$, $\bar{\mathbf{T}}^j$ is the $mK \times n$ matrix where $\bar{\mathbf{T}}^j_{ik} = \mathbf{T}_{ik}\mathbf{M}_{\lceil \frac{i}{m} \rceil j}, i \in [1:mK], k \in [1:n]$. Moreover, $\bar{G}^j$ denotes the bipartite graph with the set of vertices $\mathcal{V} = \{v_1, ..., v_{mK}\} \cup \{v'_1, ..., v'_n\}$ where v_i is connected to v'_k iff $\bar{\mathbf{T}}^j_{ik} = 1, \forall i \in [1:mK], \forall k \in [1:n]$. Also, for all $i \in [1:mK]$, $\bar{G}^j \setminus i$ denotes the subgraph of $\bar{G}^j$ with node v_i and its incident edges removed. $\triangle$

Equipped with Definition 6, we now state the following theorem that mentions the conditions that the transmission matrix $\mathbf{T}$ should satisfy to guarantee that the vectors $\mathbf{u}_l$ satisfying (6) exist, hence characterizing the symmetric DoF achievable by structured repetition coding.

Theorem 3. *Consider a K-user interference network with adjacency matrix $\mathbf{M}$. If a transmission matrix $\mathbf{T} \in \{0,1\}^{mK \times n}$ satisfies the following conditions*

$$\mu(\bar{G}^j) - \mu(\bar{G}^j \setminus l) = 1, \forall l \in [(j-1)m+1 : jm], \forall j \in [1:K],$$

where $\mu(G)$ denotes the matching number of the graph G [4], then structured repetition coding with transmission matrix $\mathbf{T}$ achieves the symmetric DoF of $\frac{m}{n}$.

Proof Sketch of Theorem 3. For a vector $\mathbf{u}_l$ to satisfy (6), it is sufficient if it satisfies $\mathbf{G}^j\mathbf{u}_l = \mathbf{I}^{mK}_l$, where $\mathbf{G}^j$ is an $mK \times n$ matrix whose entries are either zero or i.i.d. random variables (corresponding to g_{ij}'s). Then, we use the following lemma, proved in [4] (Appendix C), which addresses the existence of $\mathbf{u}_l$'s satisfying $\mathbf{G}^j\mathbf{u}_l = \mathbf{I}^{mK}_l$ for such random matrices.

Lemma 3. *Consider a bipartite graph $G = (\{v_1, ..., v_m\} \cup \{v'_1, ..., v'_n\}, \mathcal{E})$ with corresponding adjacency matrix $\mathbf{T}^{m \times n}$, where $\mathbf{T}_{ij} = 1$ iff v_i is connected to v'_j. Also, define matrix $\tilde{\mathbf{T}}^{m \times n}$ s.t. $\tilde{\mathbf{T}}_{ij} = g_{ij}\mathbf{T}_{ij}$, where g_{ij}'s are i.i.d. random variables drawn from a continuous distribution. If for $l \in [1:m]$, $\mu(G) - \mu(G \setminus l) = 1$ (where $G \setminus l$ denotes the subgraph of G with v_l and its incident edges removed), then for almost all values of g_{ij}'s, there exists a vector $\mathbf{u} \in \mathbb{C}^n$ s.t. $\tilde{\mathbf{T}}\mathbf{u} = \mathbf{I}^m_l$.*

The only remaining issue is the finiteness of noise variance in (6), which is proved in Appendix D of [4]. $\square$

V. Numerical Analysis

We have so far discussed several outer and inner bounds on the symmetric DoF of interference networks with arbitrary topology. The main feature of our bounds is that they are reduced to combinatorial optimization problems and, therefore, can be evaluated systematically for any topology. In particular, we intend to check the tightness of our bounds, compare structured repetition coding with benchmark schemes, introduced in subsection IV-A of [4] (which achieve $\frac{1}{\chi_f}$ and $\frac{1}{\Delta_R}$ as mentioned in section IV), and study the effect of network density on the gains over the benchmark schemes. As shown in Fig. 4, we consider 6-user networks with 6 square cells, each one having a base station (BS) and a user in the cell area. We

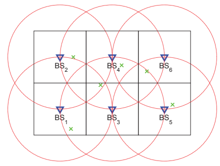

Fig. 4: A 6-cell network realization where the blue triangles, green crosses, black squares and red circles represent base stations, mobile users, cell boundaries and coverage area of base stations, respectively.

consider the cases in which each user can receive interference from any nonempty subset of three adjacent cells. For instance in Fig. 4, user 2 can receive interference from any nonempty subset of $\{\mathrm{BS}_1, \mathrm{BS}_3, \mathrm{BS}_4\}$.

Ignoring isomorphic topologies, there are 22336 unique topologies in this class. Interestingly, in all cases except 16, our bounds are tight. Fig. 5 shows the distribution of the gain of structured repetition coding over random Gaussian coding with interference decoding and interference avoidance.

% Gain over random Gaussian coding with interference decoding	20%	33%	50%	67%	100%
Number of topologies	104	9925	847	11	100

% Gain over interference avoidance	20%	25%	33%	50%
Number of topologies	25	59	1499	230

Fig. 5: Distribution of gain of structured repetition coding over random Gaussian coding with interference decoding and interference avoidance.

Fig. 6 shows the impact of network density, measured by the number of cross links in the network, on the fraction of topologies in which structured repetition coding outperforms benchmark schemes. We observe that in denser networks, there is few chance for structured repetition coding to go beyond benchmark schemes, while in moderate levels of density, there are many topologies yielding gains over benchmark schemes.

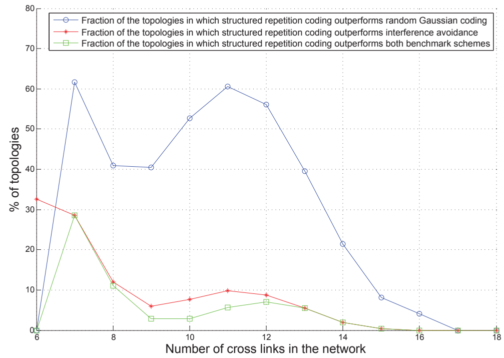

Fig. 6: Comparing structured repetition coding with benchmark schemes based on network density in 6-cell networks.

One of the challenges in extending this analysis to larger networks is the computational complexity for evaluating our inner and outer bounds. Thus, finding efficient algorithms for these optimization problems is an interesting future direction.

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