Spatially-Coupled Multi-Edge Type LDPC Codes with Bounded Degrees that Achieve Capacity on the BEC under BP Decoding

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***Abstract*—Convolutional (or spatially-coupled) low-density parity-check (LDPC) codes have now been shown to approach capacity for a variety of problems. Yet, most of these results require sequences of regular LDPC ensembles with increasing variable and check degrees. Previously, Kasai and Sakaniwa showed empirically that, for the BEC, this limitation can be overcome by using spatially-coupled MacKay-Neal (MN) and Hsu-Anastasopoulos (HA) ensembles. In this paper, we prove this analytically for (k,2,2)-MN and (2,k,2)-HA ensembles when k is at least 3. The proof is based on the simple approach to threshold saturation, introduced by Yedla et al., which relies on potential functions. The key step is verifying the non-negativity of a potential function associated with the uncoupled system. Along the way, we derive the potential function general multi-edge type (MET) LDPC ensembles and establish a duality relationship between dual ensembles of MET LDPC codes.**

***Index Terms*—multi-edge type LDPC codes, MacKay-Neal codes, spatial coupling, density evolution, potential functions**

I. Introduction

Low-density parity-check (LDPC) convolutional codes, or spatially-coupled (SC) LDPC codes, were introduced in [1] and observed to have excellent belief-propagation (BP) thresholds in [2]. Recently, they have been observed to approach capacity for a variety of problems. The principle behind their excellent performance is described in [3], where it is shown analytically for the BEC that the BP threshold of a regular SC ensemble converges to the maximum-a-posteriori (MAP) threshold of the uncoupled ensemble. This phenomenon is now called *threshold saturation*.

MacKay-Neal (MN) codes [4] are non-systematic two-edge type LDPC codes that are conjectured to achieve capacity on BMS channels under ML decoding. Empirical evidence of this conjecture, for the BSC and AWGN channels, is reported in [5], [6] based on the non-rigorous statistical mechanics approach known as *replica method*. In [7], Hsu and Anastasopoulos prove that LDPC codes concatenated with low-density generator-matrix (LDGM) codes achieve the capacity of arbitrary BMS channels with bounded density under ML decoding. Therefore, we call these Hsu-Anastasopoulos (HA) codes. It is worth noting that this ensemble was also introduced independently by Martinian and Wainwright for lossy source coding in [8].

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In this paper, we prove that some SC-MN and SC-HA codes achieve capacity on the BEC under BP decoding. Previously, Kasai and Sakaniwa presented empirical evidence that suggests SC-MN and SC-HA codes, with small variable and check degrees, achieve the capacity on the BEC [9]. The proof is based on the recent theorem of Yedla et al. [10], which establishes threshold saturation for certain SC systems of vector recursions via potential theory. The theorem asserts that BP decoding of SC codes on the BEC has vanishing failure probability when the erasure rate is below the *potential threshold* of the uncoupled code. We show that potential threshold is equal to the Shannon limit for $(\mathtt{l},2,2)$-MN ensembles with $\mathtt{l}\geq 3$ and $(2,\mathtt{r},2)$-HA ensembles with $\mathtt{r}\geq 3$.

The potential function is also derived for the general class of multi-edge type (MET) LDPC codes. Its form leads naturally to two duality lemmas that establish simple relationships between the potential function of a MET ensemble and its dual ensemble. Since MN and HA ensembles are dual to each other [9], the main result is proven first for SC-MN ensembles and then extended to SC-HA ensembles via duality.

The following notation is used throughout this paper. We let $d\in\mathbb{N}$ be the dimension for the vector recursion $\mathcal{X}\triangleq[0,1]^d$ be the space on which the recursion is defined, and $\mathcal{E}\triangleq[0,1]$ be the parameter space of the recursive system. For convenience, we let $\mathcal{X}_\circ\triangleq\mathcal{X}\setminus\{\mathbf{0}\}$. Vectors are denoted in boldface lowercase (e.g. $\boldsymbol{x},\boldsymbol{y}$) and assumed to be row vectors with elements $[\boldsymbol{x}]_i=x_i$. They inherit the natural partial order $\boldsymbol{x}\preceq\boldsymbol{y}$ defined by $x_i\leq y_i$ for $1\leq i\leq d$. Matrices are denoted in boldface capital letters (e.g. $\boldsymbol{D}$). Standard-weight typeface is used for scalar-valued functions with a vector argument (e.g., $F(\boldsymbol{x})$) and boldface is used to denote a vector-valued function with a vector argument (e.g., $\boldsymbol{f}(\boldsymbol{x})=[f_1(\boldsymbol{x}),\cdots,f_d(\boldsymbol{x})]$).

II. Background

A. Multi-Edge Type LDPC Codes

A multi-edge type (MET) LDPC ensemble is a generalization of the standard irregular LDPC ensemble based on using $d\geq 2$ distinct permutations to define the ensemble [11], [12]. In this case, the degree of a variable (or check) node consists a vector $\boldsymbol{i}\in\mathbb{N}^d$, where i_j is the number of edges connecting the node to the j-th permutation, and a flag that indicates whether or not the node is "associated with an observation".

Let $\nu_{0,\boldsymbol{i}}$ (resp. $\nu_{1,\boldsymbol{i}}$) be the normalized number of punctured (resp. transmitted) variable nodes with degree $\boldsymbol{i} \in \mathbb{N}^d$ satisfying $\sum_j i_j \geq 2$. Variable nodes with degree-1 are instead counted as observations associated with their attached check node. Similarly, $\mu_{0,\boldsymbol{i}}$ (resp. $\mu_{1,\boldsymbol{i}}$) is the normalized number of standard check nodes (resp. check nodes with an attached degree-1 variable node) of degree $\boldsymbol{i} \in \mathbb{N}^d$. This is a slight modification to the standard definition of the MET ensemble which allows us to exploit the natural duality between MN and HA codes. A MET LDPC ensemble is compactly defined by its multivariable degree distribution, (ν, μ), where

$$\nu(\boldsymbol{x}; \varepsilon) \triangleq \sum_{\boldsymbol{i} \in \mathbb{N}^d} (\nu_{0,\boldsymbol{i}} + \nu_{1,\boldsymbol{i}} \varepsilon) \boldsymbol{x}^{\boldsymbol{i}},$$
$$\mu(\boldsymbol{x}; \varepsilon) \triangleq \sum_{\boldsymbol{i} \in \mathbb{N}^d} (\mu_{0,\boldsymbol{i}} + \mu_{1,\boldsymbol{i}} \varepsilon) \boldsymbol{x}^{\boldsymbol{i}},$$

$\boldsymbol{x}^{\boldsymbol{i}} \triangleq \prod_{j=1}^d x_j^{i_j}$, and $\nu_{q,\boldsymbol{i}}, \mu_{q,\boldsymbol{i}} \in \mathbb{R}_{\geq 0}$ for $\boldsymbol{i} \in \mathbb{N}^d, q \in \{0,1\}$. Counting the variable and check nodes shows that the design rate is given by

$$R(\nu, \mu) \triangleq \frac{\nu(\mathbf{1};1) - \mu(\mathbf{1};0)}{\nu(\mathbf{1};1) - \nu(\mathbf{1};0) + \mu(\mathbf{1};1) - \mu(\mathbf{1};0)} = \frac{1}{1 + \frac{\mu(\mathbf{1};1) - \nu(\mathbf{1};0)}{\nu(\mathbf{1};1) - \mu(\mathbf{1};0)}}.$$

For simplicity, we also define $\nu_j(\boldsymbol{x}; \varepsilon) \triangleq \frac{\mathrm{d}}{\mathrm{d}x_j} \nu(\boldsymbol{x}; \varepsilon)$, $\nu_j(\mathbf{1}) \triangleq \nu_j(\mathbf{1}; 1)$, $\mu_j(\boldsymbol{x}; \varepsilon) \triangleq \frac{\mathrm{d}}{\mathrm{d}x_j} \mu(\boldsymbol{x}; \varepsilon)$, and $\mu_j(\mathbf{1}) = \mu_j(\mathbf{1}; 1)$. The requirement that each permutation must have an equal number of variable and check edges implies that $\nu_j(\mathbf{1}) = \mu_j(\mathbf{1})$ for $j = 1, 2, \ldots, d$. Also, one can write the DE update equations for the BEC as

$$\begin{aligned} y_j^{(\ell)} &= \left[\boldsymbol{g}\left(\boldsymbol{x}^{(\ell)}; \varepsilon\right)\right]_j \triangleq 1 - \frac{\mu_j\left(\mathbf{1} - \boldsymbol{x}^{(\ell)}; 1 - \varepsilon\right)}{\mu_j(\mathbf{1}; 1)} \\ x_j^{(\ell+1)} &= \left[\boldsymbol{f}\left(\boldsymbol{y}^{(\ell)}; \varepsilon\right)\right]_j \triangleq \frac{\nu_j\left(\boldsymbol{y}^{(\ell)}; \varepsilon\right)}{\nu_j(\mathbf{1}; 1)}. \end{aligned} \tag{1}$$

Finally, one can use Forney's duality transform [13] for LDPC codes to see that codes in a MET LDPC ensemble with degree distribution (ν, μ) have their dual code in the MET LDPC ensemble with degree distribution (μ, ν) (i.e., the variable and check degree distributions are swapped). One implication of this is that the above DE equations are symmetric under the interchange $\boldsymbol{x} \mapsto \mathbf{1} - \boldsymbol{y}$, $\boldsymbol{y} \mapsto \mathbf{1} - \boldsymbol{x}$, $\mu \leftrightarrow \nu$, and $\varepsilon \leftrightarrow 1 - \varepsilon$. Likewise, the rate equation also satisfies $R(\nu, \mu) = 1 - R(\mu, \nu)$. This can be seen as an extension of earlier results for repeat-accumulate and accumulate-repeat-accumulate codes [14].

B. MacKay-Neal and Hsu-Anastasopoulos Codes

MN codes are a class of MET LDPC codes first introduced by MacKay and Neal [4], [15]. As observed in [9], they are actually equivalent to non-systematic repeat-accumulate codes [16]. The $(\mathtt{l}, \mathtt{r}, \mathtt{g})$-MN ensemble is equivalent to the MET LDPC ensemble [9] with multivariate degree distribution

$$\nu(\boldsymbol{x}; \varepsilon) = \frac{\mathtt{r}}{\mathtt{l}} x_1^{\mathtt{l}} + \varepsilon x_2^{\mathtt{g}}, \quad \mu(\boldsymbol{x}; \varepsilon) = x_1^{\mathtt{r}} x_2^{\mathtt{g}}. \tag{2}$$

HA codes are a class of MET LDPC codes introduced by Hsu and Anastasopoulos [17], [7]. The $(\mathtt{l}, \mathtt{r}, \mathtt{g})$-HA ensemble is equivalent to the MET LDPC ensemble [9] with multivariate degree distribution

$$\nu(\boldsymbol{x}; \varepsilon) = x_1^{\mathtt{l}} x_2^{\mathtt{g}}, \quad \mu(\boldsymbol{x}; \varepsilon) = \frac{\mathtt{l}}{\mathtt{r}} x_1^{\mathtt{r}} + \varepsilon x_2^{\mathtt{g}}. \tag{3}$$

C. Spatially-Coupled Codes and Threshold Saturation

Spatially-coupled constructions of MN and HA codes were first proposed by Kasai and Sakaniwa in [9, Sec. IV]. The performance of SC-MN and SC-HA ensembles was investigated using numerical density evolution and, in both cases, threshold saturation was observed. While the uncoupled ensembles do not have BP thresholds (i.e., the bit erasure rate does not vanish even when all transmitted bits are known), the BP thresholds of the SC-MN and SC-HA ensembles were found to be very close to the Shannon limit [9]. The design rates of SC-MN and SC-HA ensembles are also given in [9, Sec. IV] and it is observed that $R^{\mathrm{MN}}(\mathtt{l}, \mathtt{r}, \mathtt{g}, L, w) = \frac{\mathtt{r}}{\mathtt{l}} + O\left(L^{-1}\right)$ and $R^{\mathrm{HA}}(\mathtt{l}, \mathtt{r}, \mathtt{g}, L, w) = 1 - \frac{\mathtt{l}}{\mathtt{r}} + O\left(L^{-1}\right)$.

To establish this result rigorously, the simple proof of threshold saturation proposed by two of the authors in [10] is employed to prove threshold saturation for certain classes of SC-MN and SC-HA codes. The key steps involve verifying some mathematical properties of the uncoupled systems. The proof technique is based on potential functions for density-evolution (DE) recursions and is related to [18], [19].

Since our main result is quite dependent on [10], we start by stating some slight generalizations of definitions and results from [10]. First, we define a new admissible class of vector recursions and their corresponding potential functions.

Definition 1 (c.f., [10, Def. 1])**.** *Let $F : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ and $G : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ be functionals satisfying $G(\mathbf{0}, \varepsilon) = 0$ and $\boldsymbol{D}$ be a $d \times d$ positive diagonal matrix. Consider the recursion defined by*

$$\boldsymbol{x}^{(\ell+1)} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}^{(\ell)}; \varepsilon); \varepsilon), \tag{4}$$

where $\boldsymbol{f} : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$ and $\boldsymbol{g} : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$ are mappings defined by $F'(\boldsymbol{x}; \varepsilon) = \boldsymbol{f}(\boldsymbol{x}; \varepsilon)\boldsymbol{D}$ and $G'(\boldsymbol{x}; \varepsilon) = \boldsymbol{g}(\boldsymbol{x}; \varepsilon)\boldsymbol{D}$. Then, the pair $(\boldsymbol{f}, \boldsymbol{g})$ defines a vector admissible system *if*

i) $\boldsymbol{f}, \boldsymbol{g}$ are twice continuously differentiable,
ii) $\boldsymbol{f}(\boldsymbol{x}; \varepsilon), \boldsymbol{g}(\boldsymbol{x}; \varepsilon)$ are non-decreasing (w.r.t. $\preceq$) in x and ε,
iii) $\boldsymbol{f}(\boldsymbol{g}(\mathbf{0}; \varepsilon); \varepsilon) = \mathbf{0}$, and $F(\boldsymbol{g}(\mathbf{0}; \varepsilon); \varepsilon) = \mathbf{0}$.

Let $\boldsymbol{g}'(\boldsymbol{x}; \varepsilon)$ be the Jacobian matrix of $\boldsymbol{g}(\boldsymbol{x}; \varepsilon)$, defined by $[\boldsymbol{g}'(\boldsymbol{x}; \varepsilon)]_{i,j} \triangleq \partial g_i(\boldsymbol{x}; \varepsilon) / \partial x_j$. The potential function is defined by a line integral as follows.

Definition 2 (c.f., [10, Def. 2])**.** *The* potential function $U(\boldsymbol{x}; \varepsilon)$ *of a vector admissible system $(\boldsymbol{f}, \boldsymbol{g})$ is*

$$\begin{aligned} U(\boldsymbol{x}; \varepsilon) &\triangleq \int_0^{\boldsymbol{x}} \left[(\boldsymbol{z} - \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{z}; \varepsilon); \varepsilon)) \boldsymbol{D} \boldsymbol{g}'(\boldsymbol{z}; \varepsilon)\right] \cdot \mathrm{d}\boldsymbol{z} \\ &= \boldsymbol{g}(\boldsymbol{x}; \varepsilon) \boldsymbol{D} \boldsymbol{x}^{\top} - G(\boldsymbol{x}; \varepsilon) - F(\boldsymbol{g}(\boldsymbol{x}; \varepsilon); \varepsilon). \end{aligned}$$

Let $\mathcal{F}(\varepsilon) \triangleq \{\boldsymbol{x} \in \mathcal{X}_\circ \mid \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}; \varepsilon); \varepsilon)\}$ be the set of non-zero fixed points for an $\varepsilon \in \mathcal{E}$. The potential threshold is defined as follows.

Definition 3 (c.f., [10, Def. 7])**.** *The* potential threshold *is*

$$\varepsilon^* \triangleq \sup\{\varepsilon \in \mathcal{E} \mid \min_{\boldsymbol{x}\in\mathcal{F}(\varepsilon)} U(\boldsymbol{x};\varepsilon) > 0\}.$$

This is well defined only if $\min_{\boldsymbol{x}\in\mathcal{F}(\varepsilon)} U(\boldsymbol{x};0) > 0$*. Let* ε_s^* *be the uncoupled system threshold defined in [10, Def. 6]. For* $\varepsilon_s^* < \varepsilon < \varepsilon^*$*, the* energy gap $\Delta E(\varepsilon) \triangleq \min_{\boldsymbol{x}\in\mathcal{F}(\varepsilon)} U(\boldsymbol{x};\varepsilon)$ *is well defined and strictly positive.*

Remark 1. *This definition of the potential threshold is slightly different than the one given in [10]. If the* $\boldsymbol{f},\boldsymbol{g}$ *functions are strictly increasing in* ε*, as required by [10], then the two definitions are equivalent. But, that requirement is too strong for the systems considered in this paper. Therefore, the new definition appears to be more natural.*

We now consider a "spatial-coupling" of the single system recursion, (4), that gives rise to the recursion (5).

Definition 4 (c.f., [10, Def. 9])**.** *The basic* spatially-coupled vector system *is defined by placing* $2L+1$ $\boldsymbol{f}$*-systems at positions in the set* $\mathcal{L}_{\boldsymbol{f}} = \{-L,-L+1,\ldots,L\}$ *and coupling them with* $2L+w$ $\boldsymbol{g}$*-systems at positions in the set* $\mathcal{L}_{\boldsymbol{g}} = \{-L,-L+1,\ldots,L+(w-1)\}$*. For the coupled system, this leads to the recursion, for* $i \in \mathcal{L}_{\boldsymbol{g}}$*, given by*

$$\boldsymbol{x}_i^{(\ell+1)} = \frac{1}{w}\sum_{k=\max(0,i-L)}^{\min(w-1,i+L)} \boldsymbol{f}\left(\frac{1}{w}\sum_{j=0}^{w-1}\boldsymbol{g}(\boldsymbol{x}_{i+j-k}^{(\ell)};\varepsilon);\varepsilon\right), \quad (5)$$

where $\boldsymbol{x}_i^{(0)} = \boldsymbol{1}$ *for* $i \in \mathcal{L}_{\boldsymbol{g}}$ *and* $\boldsymbol{x}_i^{(\ell)} = \boldsymbol{0}$ *for* $i \notin \mathcal{L}_{\boldsymbol{g}}$ *and all* ℓ*.*

Let $K_{\boldsymbol{f},\boldsymbol{g}}$ be the constant, which depends only on $(\boldsymbol{f},\boldsymbol{g})$, defined in [10, Lem. 11]. The main theorem in [10], which is stated below, is implicitly generalized by the modified definitions above. Its proof is easily constructed from the arguments given in [10].

Theorem 1 (c.f., [10, Thm. 1])**.** *For a vector admissible system* $(\boldsymbol{f},\boldsymbol{g})$ *with* $\varepsilon < \varepsilon^*$ *and* $w > (dK_{\boldsymbol{f},\boldsymbol{g}})/(2\Delta E(\varepsilon))$*, the only fixed point of the spatially-coupled system (Def. 4) is* $\boldsymbol{0}$*.*

III. Potential Functions

A. Multi-Edge Type Ensembles

In this section, we consider two potential functions for MET LDPC ensembles. The first provides a natural stepping stone to the second, which satisfies the necessary conditions of Thm. 1. First, we define the bivariate potential function

$$\Phi(\boldsymbol{x},\boldsymbol{y};\varepsilon) \triangleq \mu(\boldsymbol{1},1-\varepsilon) - \mu(\boldsymbol{1}-\boldsymbol{x};1-\varepsilon) - \nu(\boldsymbol{y};\varepsilon) - \sum_{j=1}^{d} x_j(1-y_j)\mu_j(\boldsymbol{1}). \quad (6)$$

This type of potential function was introduced recently for the scalar case in [19] and the key property of this potential function is that setting the x_j, y_j derivatives to zero gives the DE update equations. This is analogous to the Bethe free energy of a factor graph because the associated sum-product updates can be generated by setting certain derivatives to zero [20]. Also, the convention $\Phi(\boldsymbol{0},\boldsymbol{0};\varepsilon) = 0$ implies that the constant term must equal $\mu(\boldsymbol{1},1-\varepsilon)$.

Using the bivariate potential function and the recursion in (1), one can easily obtain the univariate potential function

$$\begin{aligned} U(\boldsymbol{x};\varepsilon) &= \Phi(\boldsymbol{x},\boldsymbol{g}(\boldsymbol{x};\varepsilon);\varepsilon) \\ &= \mu(\boldsymbol{1},1-\varepsilon) - \mu(\boldsymbol{1}-\boldsymbol{x};1-\varepsilon) - \nu(\boldsymbol{g}(\boldsymbol{x};\varepsilon);\varepsilon) \\ &\quad - \sum_{j=1}^{d} x_j\mu_j(\boldsymbol{1}-\boldsymbol{x};1-\varepsilon). \end{aligned} \quad (7)$$

To verify that the DE update equations for a MET ensemble define a vector admissible system, we start by choosing $[\boldsymbol{D}]_{i,i} = \mu_i(\boldsymbol{1};1) = \nu_i(\boldsymbol{1};1)$, $F(\boldsymbol{x};\varepsilon) = \nu(\boldsymbol{x};\varepsilon)$, and

$$G(\boldsymbol{x};\varepsilon) = \mu(\boldsymbol{1}-\boldsymbol{x};1-\varepsilon) - \mu(\boldsymbol{1};1-\varepsilon) + \sum_{i=1}^{d} x_i\mu_i(\boldsymbol{1};1).$$

Using these definitions, it is easy to verify that the implied $\boldsymbol{f},\boldsymbol{g}$ satisfy all conditions of Def. 1 except iii). Therefore, the DE update equations for a MET ensemble define an admissible system as long as iii) is satisfied. It is also easy to check that (7) is the potential function in Def. 2 for the BEC DE update equations in (1).

We also note that swapping $\boldsymbol{f}$ and $\boldsymbol{g}$ in Def. 1 shifts the recursion by a half-iteration. The univariate potential function for the shifted recursion depends only on $\boldsymbol{y}$ and is given by

$$V(\boldsymbol{y};\varepsilon) \triangleq \Phi(\boldsymbol{f}(\boldsymbol{y};\varepsilon),\boldsymbol{y};\varepsilon).$$

B. Duality

Now, we consider the relationship between the potential functions of a MET ensemble and its dual ensemble. Since the dual ensemble is obtained by swapping ν and μ, it is easy to verify that BEC DE update equations for the dual ensemble are given by $\boldsymbol{f}^{\perp}(\boldsymbol{x};\varepsilon) = \boldsymbol{1} - \boldsymbol{g}(1-\boldsymbol{x};1-\varepsilon)$ and $\boldsymbol{g}^{\perp}(\boldsymbol{y};\varepsilon) = \boldsymbol{1} - \boldsymbol{f}(1-\boldsymbol{y};1-\varepsilon)$. Computing (6) for the dual ensemble (μ,ν) shows that

$$\begin{aligned} \Phi^{\perp}(\boldsymbol{x},\boldsymbol{y};\varepsilon) &\triangleq \nu(\boldsymbol{1},1-\varepsilon) - \nu(\boldsymbol{1}-\boldsymbol{x};1-\varepsilon) \\ &\quad - \mu(\boldsymbol{y};\varepsilon) - \sum_{j=1}^{d} x_j(1-y_j)\nu_j(\boldsymbol{1}) \\ &= \Phi(\boldsymbol{1}-\boldsymbol{y},\boldsymbol{1}-\boldsymbol{x};1-\varepsilon) + \nu(\boldsymbol{1},1-\varepsilon) - \mu(\boldsymbol{1},\varepsilon). \end{aligned} \quad (8)$$

Using this, we present the first duality lemma.

Lemma 1. *Let* $V^{\perp}(\boldsymbol{y};\varepsilon) \triangleq \Phi^{\perp}(\boldsymbol{f}^{\perp}(\boldsymbol{y};\varepsilon),\boldsymbol{y};\varepsilon)$ *be the univariate potential function of the dual ensemble for the shifted recursion. Then, we have*

$$V^{\perp}(\boldsymbol{y};\varepsilon) = U(\boldsymbol{1}-\boldsymbol{y};1-\varepsilon) + \nu(\boldsymbol{1},1-\varepsilon) - \mu(\boldsymbol{1},\varepsilon) \quad (9)$$

Proof: Using (8), we observe that

$$\begin{aligned} U(\boldsymbol{1}-\boldsymbol{y};1-\varepsilon) &= \Phi(\boldsymbol{1}-\boldsymbol{y},\boldsymbol{g}(\boldsymbol{1}-\boldsymbol{y};1-\varepsilon);1-\varepsilon) \\ &= \Phi^{\perp}(\boldsymbol{1}-\boldsymbol{g}(\boldsymbol{1}-\boldsymbol{y};1-\varepsilon),\boldsymbol{y};\varepsilon) - \nu(\boldsymbol{1},1-\varepsilon) + \mu(\boldsymbol{1},\varepsilon) \\ &= \Phi^{\perp}(\boldsymbol{f}^{\perp}(\boldsymbol{y};\varepsilon),\boldsymbol{y};\varepsilon) - \nu(\boldsymbol{1},1-\varepsilon) + \mu(\boldsymbol{1},\varepsilon) \\ &= V^{\perp}(\boldsymbol{y};\varepsilon) - \nu(\boldsymbol{1},1-\varepsilon) + \mu(\boldsymbol{1},\varepsilon). \end{aligned}$$

Rearranging terms completes the proof. ■

The second duality lemma links U and $U^{\perp}$ at fixed points.

Lemma 2. *Let $\tilde{\boldsymbol{x}}$ be a fixed point of the dual ensemble satisfying $\tilde{\boldsymbol{x}} = \boldsymbol{f}^{\perp}(\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon)$. Then, $\boldsymbol{x}_0 = \boldsymbol{f}(\boldsymbol{1}-\tilde{\boldsymbol{x}};1-\varepsilon)$ is a fixed point of the standard recursion and*

$$U^{\perp}(\tilde{\boldsymbol{x}};\varepsilon) = U(\boldsymbol{f}(\boldsymbol{1}-\tilde{\boldsymbol{x}};1-\varepsilon);1-\varepsilon) + \nu(\boldsymbol{1},1-\varepsilon) - \mu(\boldsymbol{1},\varepsilon).$$

Proof: It is easy to see that the univariate potential function of the dual code ensemble is $U^{\perp}(\tilde{\boldsymbol{x}};\varepsilon) \triangleq \Phi^{\perp}(\tilde{\boldsymbol{x}},\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon)$. Since $\tilde{\boldsymbol{x}}$ is a fixed point of the dual ensemble, one can see

$$\begin{aligned} U^{\perp}(\tilde{\boldsymbol{x}};\varepsilon) &= \Phi^{\perp}(\boldsymbol{f}^{\perp}(\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon),\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon) \\ &= V^{\perp}(\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon) \end{aligned} \tag{10}$$

This part of the lemma follows from expanding (10) with (9) and using the fact that $\boldsymbol{f}(1-\tilde{\boldsymbol{x}};1-\varepsilon) = \boldsymbol{1}-\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon)$. Finally, $\boldsymbol{x}_0$ is a fixed point of standard recursion because

$$\tilde{\boldsymbol{x}} = \boldsymbol{f}^{\perp}(\boldsymbol{g}^{\perp}(\tilde{\boldsymbol{x}};\varepsilon);\varepsilon) = \boldsymbol{1} - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{1}-\tilde{\boldsymbol{x}};1-\varepsilon);1-\varepsilon)$$

implies (by applying $\boldsymbol{f}(\boldsymbol{1}-*;1-\varepsilon)$ to both sides) that

$$\boldsymbol{f}(\boldsymbol{1}-\tilde{\boldsymbol{x}};1-\varepsilon) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{1}-\tilde{\boldsymbol{x}};1-\varepsilon);1-\varepsilon);1-\varepsilon)$$

and simplifying gives $\boldsymbol{x}_0 = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}_0;1-\varepsilon);1-\varepsilon)$. ■

IV. Achieving Capacity

A. MacKay-Neal Codes

In this section, we show that $(\mathtt{l},2,2)$-MN codes achieve capacity on the BEC for $\mathtt{l}\geq 3$. From the multivariate degree distributions in (2), the DE updates are given by (4) with

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x};\varepsilon) &= (x_1^{\mathtt{l}-1}, \varepsilon x_2^{\mathtt{g}-1}), \\ \boldsymbol{g}(\boldsymbol{x},\varepsilon) &= \big(1-(1-x_1)^{\mathtt{r}-1}(1-x_2)^{\mathtt{g}}, 1-(1-x_1)^{\mathtt{r}}(1-x_2)^{\mathtt{g}-1}\big). \end{aligned} \tag{11}$$

Using Def. 2, one finds the potential function is given by

$$\begin{aligned} U_{\text{MN}}(\boldsymbol{x};\varepsilon) = {}& 1 - \varepsilon\left(1-(1-x_1)^{\mathtt{r}}(1-x_2)^{\mathtt{g}-1}\right)^{\mathtt{g}} \\ & - \frac{\mathtt{r}}{\mathtt{l}}\left(1-(1-x_1)^{\mathtt{r}-1}(1-x_2)^{\mathtt{g}}\right)^{\mathtt{l}} \\ & - (1-x_1)^{\mathtt{r}}(1-x_2)^{\mathtt{g}}\Big(1+\frac{\mathtt{r}x_1}{1-x_1}+\frac{\mathtt{g}x_2}{1-x_2}\Big). \end{aligned}$$

Lemma 3. *The fixed points of the DE update* (11) *are given by the $(x_1,x_2;\varepsilon)$ triples that satisfy*

$$\begin{aligned} x_1 &= \left(1-(1-x_1)^{\mathtt{r}-1}(1-x_2)^{\mathtt{g}}\right)^{\mathtt{l}-1}, \\ x_2 &= \varepsilon(1-(1-x_1)^{\mathtt{r}}(1-x_2)^{\mathtt{g}-1})^{\mathtt{g}-1}. \end{aligned}$$

The points $(0,0;\varepsilon)$ and $(1,\varepsilon;\varepsilon)$ are the unique fixed points with $x_1=0$ and $x_1=1$. All other fixed points can be written parametrically as $(x_1,x_2(x_1);\varepsilon(x_1))$ with $x_1\in(0,1)$ and

$$\begin{aligned} x_2(x_1) &\triangleq 1-\left((1-x_1^{1/(\mathtt{l}-1)})/(1-x_1)^{\mathtt{r}-1}\right)^{1/\mathtt{g}}, \\ \varepsilon(x_1) &\triangleq \frac{x_2(x_1)}{(1-(1-x_1)^{\mathtt{r}}(1-x_2(x_1))^{\mathtt{g}-1})^{\mathtt{g}-1}}. \end{aligned}$$

Proof: It is easy to check that $(0,0;\varepsilon),(1,\varepsilon;\varepsilon)$ are fixed points for $\varepsilon\in[0,1]$. For $x_1\neq 0$ and $x_1\neq 1$, one can derive the fixed-point curve by solving for x_2 and ε in terms of x_1. ■

Lemma 4. *For $(\mathtt{l},2,2)$-MN codes with $\mathtt{l}\geq 3$, the potential satisfies $U_{\text{MN}}((x_1,x_2(x_1));\varepsilon(x_1))>0$ for all $x_1\in(0,1)$.*

Proof: First let us check if the potential at $x_1=0$ and $x_1=1$. For $\mathtt{r}=2$ and $\mathtt{g}=2$,

$$\begin{aligned} &\lim_{x_1\to 0} U_{\text{MN}}((x_1,x_2(x_1));\varepsilon(x_1)) = 0 \\ &\lim_{x_1\to 1} U_{\text{MN}}((x_1,x_2(x_1));\varepsilon(x_1)) = \sqrt{\frac{1}{\mathtt{l}-1}} - \frac{2}{\mathtt{l}} > 0. \end{aligned}$$

Consider the following reparameterized fixed-point potential

$$\tilde{U}(z) \triangleq \frac{1}{z^{\mathtt{l}-1}} U_{\text{MN}}((x_1,x_2(x_1));\varepsilon(x_1))\Big|_{\mathtt{r}=2,\mathtt{g}=2,z=x^{\frac{1}{\mathtt{l}-1}}}$$

The second derivative of $\tilde{U}(z)$ is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 z}\tilde{U}(z) = \frac{\tilde{U}_{\text{nu}}(z)}{4\,(1-z)^4\,(\sum_{i=0}^{\mathtt{l}-2} z^i)^{5/2}},$$

where $\tilde{U}_{\text{nu}}(z)$ is given by

$$\begin{aligned} \tilde{U}_{\text{nu}}(z) \triangleq {}& -2+5z^{\mathtt{l}-1} - (\mathtt{l}-2)\mathtt{l}z^{3\mathtt{l}-3} + 4(\mathtt{l}-2)\mathtt{l}z^{2\mathtt{l}-2} \\ & + 2(\mathtt{l}-1)(2\mathtt{l}-3)z^{2\mathtt{l}-4} - 2(\mathtt{l}-1)(4\mathtt{l}-5)z^{2\mathtt{l}-3} \\ & - (\mathtt{l}-3)(\mathtt{l}-1)z^{3\mathtt{l}-5} + 2(\mathtt{l}-2)(\mathtt{l}-1)z^{3\mathtt{l}-4}. \end{aligned}$$

For $\mathtt{l}=3$ and $z\in(0,1)$, it is easy to see that

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{U}(z) = \frac{-z(3z+8)-2}{4(z+1)^{5/2}} < 0.$$

One can show this for any $\mathtt{l}>3$, namely $\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{U}(z)<0$ for $z\in(0,1)$, in a similar way. We omit the details for space. ■

Theorem 2. *For $\mathtt{l}\geq 3$, spatially-coupled $(\mathtt{l},2,2)$-MN codes achieve the Shannon limit $\varepsilon^{\text{Sha}} = 1-\frac{\mathtt{r}}{\mathtt{l}}$ under BP decoding.*

Proof: Let $U_{\text{MN}}((x_1,x_2);\varepsilon)$ be the potential function of the stated MN ensembles. Working directly, one finds that

$$U_{\text{MN}}((1,\varepsilon);\varepsilon) = 1-\frac{\mathtt{r}}{\mathtt{l}}-\varepsilon. \tag{12}$$

Hence, from Lem. 4, (12), and Def. 3 it follows that $\varepsilon^* = 1-\frac{\mathtt{r}}{\mathtt{l}} = \varepsilon^{\text{Sha}}$. From Thm. 1, it follows that $(\mathtt{l},2,2)$-MN codes achieve the potential threshold ε^* under BP decoding. ■

B. Hsu-Anastasopoulos (HA) Codes

From (3) and (1), the DE updates are

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x};\varepsilon) &= \left(x_1^{\mathtt{l}-1}x_2^{\mathtt{g}}, x_1^{\mathtt{l}}x_2^{\mathtt{g}-1}\right), \\ \boldsymbol{g}(\boldsymbol{x};\varepsilon) &= \Big(1-(1-x_1)^{\mathtt{r}-1}, 1-(1-\varepsilon)\,(1-x_2)^{\mathtt{g}-1}\Big). \end{aligned}$$

To verify that $(\boldsymbol{f},\boldsymbol{g})$ is a vector admissible system, we must only check that condition iii) in Def. 1 is satisfied. Although $\boldsymbol{g}(\boldsymbol{0};\varepsilon)=(0,\varepsilon)\neq\boldsymbol{0}$, we know $\boldsymbol{0}$ is a fixed point because $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{0};\varepsilon);\varepsilon) = \boldsymbol{f}((0,\varepsilon);\varepsilon) = \boldsymbol{0}$. Since $F(\boldsymbol{x};\varepsilon)=\nu(\boldsymbol{x};\varepsilon)$ with $\nu(\boldsymbol{x};\varepsilon)$ defined in (3), one finds $F(\boldsymbol{g}(\boldsymbol{0};\varepsilon);\varepsilon) = F((0,\varepsilon);\varepsilon) = \boldsymbol{0}$. Using (3) and (7), the potential function is

$$\begin{aligned} U_{\text{HA}}(\boldsymbol{x};\varepsilon) = {}& \frac{\mathtt{l}}{\mathtt{r}} + (1-\varepsilon) - \frac{\mathtt{l}}{\mathtt{r}}(1-x_1)^{\mathtt{r}} - (1-\varepsilon)(1-x_2)^{\mathtt{g}} \\ & - \left(1-(1-\varepsilon)(1-x_2)^{\mathtt{g}-1}\right)^{\mathtt{g}}\left(1-(1-x_1)^{\mathtt{r}-1}\right)^{\mathtt{l}} \\ & - \mathtt{l}x_1(1-x_1)^{\mathtt{r}-1} - (1-\varepsilon)\mathtt{g}x_2(1-x_2)^{\mathtt{g}-1}. \end{aligned}$$

The BEC DE updates of HA codes are given by

$$x_1^{(\ell+1)} = \left(1-(1-x_1^{(\ell)})^{\mathtt{r}-1}\right)^{\mathtt{l}-1}\left(1-(1-\varepsilon)(1-x_2^{(\ell)})^{\mathtt{g}-1}\right)^{\mathtt{g}},$$
$$x_2^{(\ell+1)} = \left(1-(1-x_1^{(\ell)})^{\mathtt{r}-1}\right)^{\mathtt{l}}\left(1-(1-\varepsilon)(1-x_2^{(\ell)})^{\mathtt{g}-1}\right)^{\mathtt{g}-1}.$$

Consider the set of DE fixed-point triples, $(x_1, x_2; \varepsilon)$, for HA codes. It is easy to to verify that $(0, 0; \varepsilon)$ and $(1, 1; \varepsilon)$ are fixed points for $\varepsilon \in [0, 1]$. All other fixed points can be written parametrically in terms of $x_1 \in (0, 1)$ with

$$x_2(x_1) = \left(x^{\mathtt{g}-1}\left(1-(1-x_1^{(\ell)})^{\mathtt{r}-1}\right)^{\mathtt{l}+\mathtt{g}-1}\right)^{1/\mathtt{g}},$$
$$\varepsilon(x_1) = 1 - \frac{1-\frac{x_1}{x_2(x_1)}\left((1-(1-x_1)^{\mathtt{r}-1}\right)}{(1-x_2(x_1))^{\mathtt{g}-1}}.$$

The BEC DE updates and fixed points for HA ensembles can be derived via duality from the same for MN ensemble. Likewise, the potential function can be computed using Lem. 2. For that reason, we have the following theorem.

Theorem 3. *For $\mathtt{r} \geq 3$, spatially-coupled $(2, \mathtt{r}, 2)$-HA codes achieve the Shannon limit $\varepsilon^{\text{Sha}} = \frac{\mathtt{l}}{\mathtt{r}}$ under BP decoding.*

Proof: Consider $(\mathtt{l}_{\text{MN}}, \mathtt{r}_{\text{MN}}=2, \mathtt{g}_{\text{MN}}=2)$-MN codes, and $(\mathtt{l}_{\text{HA}} = \mathtt{r}_{\text{MN}}, \mathtt{r}_{\text{HA}} = \mathtt{l}_{\text{MN}}, \mathtt{g}_{\text{HA}} = \mathtt{g}_{\text{MN}})$-HA codes. Let $\boldsymbol{x}_{\text{MN}} = (x_{1_{\text{MN}}}, x_{2_{\text{MN}}})$ and $\boldsymbol{x}_{\text{HA}} = (x_{1_{\text{HA}}}, x_{2_{\text{HA}}})$ be fixed points the MN and HA recursions satisfying $\boldsymbol{x}_{\text{MN}} = \boldsymbol{f}(\mathbf{1} - \boldsymbol{x}_{\text{HA}}; 1-\varepsilon)$ from Lem. 2. Let $U_{\text{MN}}(\cdot)$ and $U_{\text{HA}}(\cdot)$ be the potential functions for the MN and HA recursions. One can show that if $x_{1_{\text{HA}}} \in (0, 1)$, then $x_{1_{\text{MN}}} \in (0, 1)$. Likewise, if $x_{1_{\text{HA}}} = 1$ (i.e., $\boldsymbol{x}_{\text{HA}} = \mathbf{1}$), then $\boldsymbol{x}_{\text{MN}} = \mathbf{0}$. Let $\varepsilon(\boldsymbol{x}_{\text{HA}})$ be the unique ε associated with the fixed point $\boldsymbol{x}_{\text{HA}}$. For $x_{1_{\text{HA}}} \in (0, 1)$ and $\varepsilon(\boldsymbol{x}_{\text{HA}}) < \varepsilon_{\text{HA}}^{\text{Sha}}$, combining Lem. 2 and Lem. 4 shows that

$$\begin{aligned} U_{\text{HA}}(\boldsymbol{x}_{\text{HA}}; \varepsilon(\boldsymbol{x}_{\text{HA}})) &= U_{\text{MN}}(\boldsymbol{f}(\mathbf{1}-\boldsymbol{x}_{\text{HA}}; 1-\varepsilon(\boldsymbol{x}_{\text{HA}})); 1-\varepsilon(\boldsymbol{x}_{\text{HA}})) \\ &\qquad + \nu(\mathbf{1}, 1-\varepsilon(\boldsymbol{x}_{\text{HA}})) - \mu(\mathbf{1}, \varepsilon(\boldsymbol{x}_{\text{HA}})) \\ &> \nu(\mathbf{1}, 1-\varepsilon(\boldsymbol{x}_{\text{HA}})) - \mu(\mathbf{1}, \varepsilon(\boldsymbol{x}_{\text{HA}})) \\ &= \frac{\mathtt{r}_{\text{MN}}}{\mathtt{l}_{\text{MN}}} - \varepsilon(\boldsymbol{x}_{\text{HA}}) = \frac{\mathtt{l}_{\text{HA}}}{\mathtt{r}_{\text{HA}}} - \varepsilon(\boldsymbol{x}_{\text{HA}}) > 0. \end{aligned}$$

For $x_{1_{\text{HA}}} = 1$ (i.e., $\boldsymbol{x}_{\text{HA}} = \mathbf{1}$), Lem. 2 implies that

$$U_{\text{HA}}(\mathbf{1}; \varepsilon) = U_{\text{MN}}(\mathbf{0}; 1-\varepsilon) + \nu(\mathbf{1}, 1-\varepsilon) - \mu(\mathbf{1}, \varepsilon) = \frac{\mathtt{l}_{\text{HA}}}{\mathtt{r}_{\text{HA}}} - \varepsilon.$$

Hence, $U_{\text{HA}}(\boldsymbol{x}_{\text{HA}}; \varepsilon(\boldsymbol{x}_{\text{HA}})) > 0$ for $\varepsilon(\boldsymbol{x}_{\text{HA}}) < \mathtt{l}_{\text{HA}}/\mathtt{r}_{\text{HA}}$ and $U_{\text{HA}}(\mathbf{1}; \varepsilon) < 0$ if $\varepsilon(\boldsymbol{x}_{\text{HA}}) > \mathtt{l}_{\text{HA}}/\mathtt{r}_{\text{HA}}$. From Def. 3, it follows that $\varepsilon^* = \mathtt{l}_{\text{HA}}/\mathtt{r}_{\text{HA}} = \varepsilon_{\text{HA}}^{\text{Sha}}$. The proof is completed by noting that Thm. 1 shows spatially-coupled $(2, \mathtt{r}, 2)$-HA codes achieve the potential threshold ε^* under BP decoding. ■

V. Conclusions

In this paper, spatially-coupled $(\mathtt{l}, 2, 2)$-MN and $(2, \mathtt{r}, 2)$-HA ensembles, with $\mathtt{l} \geq 3$ and $\mathtt{r} \geq 3$, are shown to achieve capacity on the BEC under BP decoding. Also, a modified representation of MET LDPC codes is introduced where there is a close relationship between the DE equations of an MET ensemble and its associated dual ensemble. The proof is based on the potential function technique introduced in [10]. By verifying the non-negativity of the potential functions associated with the uncoupled systems, the main results are obtained. In particular, the main result is proven directly for SC-MN ensembles and extended to SC-HA ensembles via duality since HA ensembles are dual to MN ensembles.

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