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An Upper Bound on the Partial-Period Correlation of Zadoff-Chu Sequences

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***Abstract*—In this paper, we investigate the partial-period correlation of Zadoff-Chu sequences. For a pair of Zadoff-Chu sequences, we define the linear phase-shifting sequences of one of them and analyze their full-period correlation properties with the other one. By linking them to the partial-period correlation of the given pair, we derive an upper bound on the magnitude of the partial-period correlation of Zadoff-Chu sequences.**

I. Introduction

Pseudo-random sequences with good correlation properties have many applications to communication systems [1]. In order to improve their performance, it is required to employ a family of pseudo-random sequences with large size and low correlation. There are two types of correlation: full-period correlation and partial-period correlation. Full-period correlation has been widely studied so far, but partial-period correlation has not received much attention in spite of its importance in real situations [2]-[5]. In code-division multiple-access (CDMA) systems, each data symbol is divided into B chips by employing a sequence of period N as a spreading sequence. When N is much larger than B, the actual correlation computed at the receiver is not full-period correlation, but partial-period correlation [2]. In such a case, the absolute values of out-of-phase partial-period correlation are needed to be low. Furthermore, partial-period correlation plays an important role in synchronization problems [4], [6].

In [5] Paterson and Lothian linked the partial-period correlation of sequences to the discrete Fourier transform (DFT) of the so-called window sequence, and derived several bounds on partial-period correlations of well known sequences such as m-sequences, Kasami sequences, and d-form sequences, etc. Their approach is particularly useful in the sense that the partial-period correlation of sequences can be analyzed in terms of full-period correlation.

Zadoff-Chu sequences are one of the best known and most widely used families of polyphase sequences [7], [8], because they have ideal full-period autocorrelation as well as they exist for any period N. In addition, the full-period cross-correlation between two Zadoff-Chu sequences is optimal with respect to the Sarwate bound [9] under some specific conditions. Recently, Kang *et al.* [10] presented general full-period cross-correlation properties of Zadoff-Chu sequences. However, there have been very few theoretical results on the partial-period correlation of Zadoff-Chu sequences, despite their wide usage in communication systems.

In this paper, we investigate the partial-period correlation of Zadoff-Chu sequences. For a pair of Zadoff-Chu sequences, we first define the linear phase-shifting sequences of one of them and analyze their full-period correlation properties with the other one. We then link them to the partial-period correlation of the given pair by generalizing the approach in [5]. Analyzing linear phase-shifting sequences and the DFT of the window sequence, we derive an upper-bound on the magnitude of the partial-period correlation of Zadoff-Chu sequences.

The outline of the paper is as follows. We give some preliminaries in Section II. In Section III, we review Zadoff-Chu sequences and investigate their properties related to the concepts introduced in Section II. We then present an upper bound on the magnitude of the partial-period correlation of Zadoff-Chu sequences in Section IV. Finally, we give some concluding remarks in Section V.

II. Preliminaries

Throughout the paper, we denote by $\langle x \rangle_y$ the least nonnegative residue of x modulo y for an integer x and a positive integer y, and denote by $\mathbb{Z}_n$ the set of nonnegative residues modulo n for a positive integer n.

Let $\{a(n)\}$ be a complex-valued sequence of period N. The discrete Fourier transform (DFT) sequence $\{\hat{a}(l)\}$ of $\{a(n)\}$ is defined as

$$\hat{a}(l) = \sum_{n=0}^{N-1} a(n) W_N^{nl}, \quad 0 \le l \le N-1$$

where $W_N = \exp(2\pi\sqrt{-1}/N)$. The sequence $\{\hat{a}(l)\}$ is simply called the spectrum of $\{a(n)\}$. Conversely, $\{a(n)\}$ can be obtained from $\{\hat{a}(l)\}$ by the inverse discrete Fourier transform (IDFT) as follows:

$$a(n) = \frac{1}{N} \sum_{l=0}^{N-1} \hat{a}(l) W_N^{-nl}, \quad 0 \le n \le N-1.$$

Let $\{a(n)\}$ and $\{b(n)\}$ be two complex-valued sequences of period N such that $|a(n)| = |b(n)| = 1$ for all $n \in \mathbb{Z}$. The (full-period) cross-correlation $\theta_{a,b}(\tau)$ between $\{a(n)\}$ and $\{b(n)\}$ is defined as

$$\theta_{a,b}(\tau) = \sum_{n=0}^{N-1} a(n)b^*(n+\tau)$$

for an integer $0 \le \tau \le N-1$, where $*$ denotes the complex conjugation and all the operations among the indices are computed modulo N. In particular, $\theta_{a,b}(\tau)$ is called the (full-period) autocorrelation of $\{a(n)\}$ if $a(n) = b(n)$ for all $n \in \mathbb{Z}$, and is denoted by $\theta_a(\tau)$.

On the other hand, the partial-period correlation is computed over a window with a specific size, instead of the full-period. For an integer $0 \le k < N$ and an integer $0 < B \le N$, the partial-period cross-correlation $R_{a,b}(\tau, k; B)$ between $\{a(n)\}$ and $\{b(n)\}$ is defined as

$$R_{a,b}(\tau, k; B) = \sum_{n=k}^{k+B-1} a(n)b^*(n+\tau) \qquad (1)$$

where all the operations among the indices are computed modulo N. In particular, $R_{a,b}(\tau, k; B)$ is called the partial-period autocorrelation of $\{a(n)\}$ if $a(n) = b(n)$ for all $n \in \mathbb{Z}$, and is denoted by $R_a(\tau, k; B)$. Note that k represents the initial position, and B denotes the size of the correlation window.

In [5], Paterson and Lothian gave another expression of the partial-period correlation by introducing the so-called window sequence. For two integers k and B, where $0 \le k \le N-1$ and $0 < B \le N$, define the window sequence $\{\lambda_{k,B}(n)\}$ as

$$\lambda_{k,B}(n) = \mathbf{1}_{A(k,B)}(n)$$

where $A(k, B)$ is the subset of $\mathbb{Z}_N$ defined by

$$A(k,B) = \begin{cases} \{n \,|\, k \le n \le k+B-1\}, & \\ \qquad\qquad \text{if } k+B-1 \le N-1 & \\ \{n \,|\, 0 \le n \le \langle k+B-1\rangle_N & \\ \quad \text{or } k \le n \le N-1\}, & \\ \qquad\qquad \text{if } k+B-1 > N-1 & \end{cases}$$

and $\mathbf{1}_S(\cdot)$ denotes the indicator function of S defined by

$$\mathbf{1}_S(n) = \begin{cases} 1, & \text{if } n \in S \\ 0, & \text{otherwise.} \end{cases}$$

Then, $R_{a,b}(\tau, k; B)$ in (1) can be rewritten as

$$\begin{aligned} R_{a,b}(\tau, k; B) &= \sum_{n=0}^{N-1} \lambda_{k,B}(n)a(n)b^*(n+\tau) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} \hat{\lambda}_{k,B}(l) \sum_{n=0}^{N-1} a(n)W_N^{-nl} b^*(n+\tau). \end{aligned}$$

Note that the inner sum is the full-period cross-correlation between $\{a(n)W_N^{-nl}\}$ and $\{b(n)\}$.

For a clear presentation, we define the lth linear phase-shifting sequence $\{a^l(n)\}$ of a complex-valued sequence of period N, $\{a(n)\}$, as

$$a^l(n) \triangleq a(n)W_N^{-nl}, \quad 0 \le n \le N-1.$$

Then the partial-period cross-correlation is given by

$$R_{a,b}(\tau, k; B) = \frac{1}{N}\sum_{l=0}^{N-1} \hat{\lambda}_{k,B}(l)\theta_{a^l,b}(\tau).$$

Hence, $R_{a,b}(\tau, k; B)$ can be computed if the spectrum of the window sequence $\{\lambda_{k,B}(n)\}$ and the full-period correlations at displacement τ between the lth linear phase-shift sequence $\{a^l(n)\}$ and $\{b(n)\}$ for $0 \le l \le N-1$ are known. In other words, it is characterized only by the full-period cross-correlations $\theta_{a^l,b}(\tau)$ for $0 \le l \le N-1$ since $\{\hat{\lambda}_{k,B}(l)\}$ has nothing to do with either $\{a(n)\}$ or $\{b(n)\}$. Note that the full-period correlations $\theta_{a^l,b}(\tau)$ for $0 \le l \le N-1$ and $0 \le \tau \le N-1$ can be expressed equivalently as the $N \times N$ matrix $\Theta_{a,b}$ defined by

$$\Theta_{a,b} = \begin{bmatrix} \theta_{a^0,b}(0) & \theta_{a^0,b}(1) & \cdots & \theta_{a^0,b}(N-1) \\ \theta_{a^1,b}(0) & \theta_{a^1,b}(1) & \cdots & \theta_{a^1,b}(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{a^{N-1},b}(0) & \theta_{a^{N-1},b}(1) & \cdots & \theta_{a^{N-1},b}(N-1) \end{bmatrix}. \qquad (2)$$

Jenson's inequality [11] is one of the most widely used inequalities in mathematics. It will also be useful for deriving an upper bound on the partial-period correlation in the next section.

Lemma 1 ([11]): If $f(\cdot)$ is a convex function and X is an arbitrary random variable, then

$$E[f(X)] \le f(E[X]) \qquad (3)$$

where $E[\cdot]$ is the expectation operator. Moreover, if $f(\cdot)$ is strictly convex, the equality in (3) implies that X is a constant.

III. Properties of $\Theta_{a,b}$ of Zadoff-Chu Sequences

As shown in the previous section, $\Theta_{a,b}$ plays a key role in analyzing the partial-period cross-correlation between $\{a(n)\}$ and $\{b(n)\}$. In this section we will investigate $\Theta_{a,b}$ when both $\{a(n)\}$ and $\{b(n)\}$ are Zadoff-Chu sequences of the same period.

Definition 2 ([12], [13]): For two positive integers N and r such that $\gcd(N, r) = 1$ and any integer q, a Zadoff-Chu sequence $\{c_r(n)\}$ of period N is defined as

$$c_r(n) = W_N^{\frac{rn(n+\langle N\rangle_2)}{2}+qn}. \qquad (4)$$

In the definition of a Zadoff-Chu sequence in (4), q is assumed to be zero, unless otherwise specified. The properties of the full-period correlation of Zadoff-Chu sequences were widely studied in [10].

Theorem 3 ([10]): Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of period N. Then the magnitude of the full-period cross-correlation $\theta_{c_r,c_s}(\tau)$ between $\{c_r(n)\}$ and $\{c_s(n)\}$ is given by

$$|\theta_{c_r,c_s}(\tau)| = \begin{cases} \sqrt{Ng}\delta_K(\tau_2), \\ \qquad \text{if } \langle N\rangle_2 = \langle uv\rangle_2 = 0 \text{ or } \langle N\rangle_2 = 1 \\ \sqrt{Ng}\delta_K(\tau_2 - \frac{g}{2}), \\ \qquad \text{if } \langle N\rangle_2 = 0 \text{ and } \langle uv\rangle_2 = 1, \end{cases}$$

where $g = \gcd(N, r-s)$, $u = N/g$, $v = (r-s)/g$, $\tau_2 = \langle\tau\rangle_g$ and $\delta_K(\cdot)$ denotes the Kronecker delta function defined by

$$\delta_K(x-a) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise.} \end{cases}$$

It should be mentioned that the magnitude of each component of the first row of Θ_{c_r,c_s} is given in Theorem 3. As a first step, we will extend Theorem 3 to all the other rows of Θ_{c_r,c_s}.

Lemma 4: Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of period N. Then the squared magnitude of the full-period cross-correlation $\theta_{c_r^l,c_s}(\tau)$ between the lth linear phase-shifting sequence of $\{c_r(n)\}$ and $\{c_s(n)\}$ is given by

$$|\theta_{c_r^l,c_s}(\tau)|^2 = \begin{cases} N\sum_{m=0}^{g-1} W_g^{-m(s\tau_2+l)}, \\ \qquad \text{if } \langle N\rangle_2 = \langle uv\rangle_2 = 0 \text{ or } \langle N\rangle_2 = 1 \\ N\sum_{m=0}^{g-1} W_g^{-m(s(\tau_2-\frac{g}{2})+l)}, \\ \qquad \text{if } \langle N\rangle_2 = 0 \text{ and } \langle uv\rangle_2 = 1, \end{cases}$$

where $g = \gcd(N, r-s)$, $u = N/g$, $v = (r-s)/g$, and $\tau_2 = \langle\tau\rangle_g$.

The proof of Lemma 4 is omitted here due to the limit of the space. Based on Lemma 4, we get the following theorem.

Theorem 5: Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of period N. Let $g = \gcd(N, r-s)$, $u = N/g$ and $v = (r-s)/g$. When $\langle N\rangle_2 = \langle uv\rangle_2 = 0$ or $\langle N\rangle_2 = 1$,

$$|\theta_{c_r^l,c_s}(\tau)| = \begin{cases} \sqrt{Ng}\delta_K(\tau_2), & \text{if } \langle l\rangle_g = 0 \\ \sqrt{Ng}\delta_K(\langle s\tau_2 + l_2\rangle_g), & \text{otherwise.} \end{cases}$$

When $\langle N\rangle_2 = 0$ and $\langle uv\rangle_2 = 1$,

$$|\theta_{c_r^l,c_s}(\tau)| = \begin{cases} \sqrt{Ng}\delta_K(\tau_2 - \frac{g}{2}), & \text{if } \langle l\rangle_g = 0 \\ \sqrt{Ng}\delta_K(\langle s(\tau_2 - \frac{g}{2}) + l_2\rangle_g), & \text{otherwise.} \end{cases}$$

Let $|\mathbf{A}|$ denote the magnitude matrix of an $M \times N$ matrix $\mathbf{A}$ defined by $|A|_{ij} = |A_{ij}|$ for all i and j. It can be easily checked that $|\Theta_{c_r,c_s}|$ has the following properties:

a) Every entry of $|\Theta_{c_r,c_s}|$ is either $\sqrt{Ng}$ or 0;
b) Each row of $|\Theta_{c_r,c_s}|$ has exactly u entries taking on $\sqrt{Ng}$. Moreover, the u entries are at a distance of g from each other;
c) Each column of $|\Theta_{c_r,c_s}|$ has exactly u entries taking on $\sqrt{Ng}$. Moreover, the u entries are at a distance of g from each other;
d) The ith row is the cyclic-shifted version (to the left) of the first row by $(si \mod g)$;
e) The jth column is the cyclic-shifted version (to the top) of the first column by $(s^{-1}j \mod g)$.

Example 6: Let $N = 15$, $r = 7$, and $s = 4$ such that $g = 3$, $u = 5$, and $v = 1$. Then two Zadoff-Chu sequences $\{c_7(n)\}$ and $\{c_4(n)\}$ are given by

$$\begin{aligned} \{c_7(n)\} &= \{W_{15}^0, W_{15}^7, W_{15}^6, W_{15}^{12}, W_{15}^{10}, W_{15}^0, W_{15}^{12}, \\ &\qquad W_{15}^1, W_{15}^{12}, W_{15}^0, W_{15}^{10}, W_{15}^{12}, W_{15}^6, W_{15}^7, W_{15}^0\}, \\ \{c_4(n)\} &= \{W_{15}^0, W_{15}^4, W_{15}^{12}, W_{15}^9, W_{15}^{10}, W_{15}^0, W_{15}^9, \\ &\qquad W_{15}^7, W_{15}^9, W_{15}^0, W_{15}^{10}, W_{15}^9, W_{15}^{12}, W_{15}^4, W_{15}^0\}. \end{aligned}$$

The matrix $|\Theta_{c_7,c_4}|$ can be represented as

$$|\Theta_{c_7,c_4}| = \begin{bmatrix} \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \end{bmatrix},$$

where $\mathbf{A}$ is the 3×3 matrix given by

$$\mathbf{A} = \begin{bmatrix} \sqrt{45} & 0 & 0 \\ 0 & 0 & \sqrt{45} \\ 0 & \sqrt{45} & 0 \end{bmatrix}. \qquad \square$$

Note that $|\Theta_{c_r,c_s}|$ in Example 6 is a matrix of a special form. In general, $|\Theta_{c_r,c_s}|$ can be represented as

$$|\Theta_{c_r,c_s}| = \mathbf{1}_{u\times u} \otimes \mathbf{A}$$

where $\otimes$ denotes the Kronecker product, $\mathbf{1}_{u\times u}$ is $u \times u$ all-one matrix and $\mathbf{A}$ is the $g \times g$ matrix determined by Theorem 5 without taking modulo g operation.

IV. Upper Bound on the Partial-Period Correlation of Zadoff-Chu Sequences

Based on the results obtained in the previous section, we will give an upper bound on the partial-period correlation of Zadoff-Chu sequences in this section. Consider two Zadoff-Chu sequences $\{c_r(n)\}$ and $\{c_s(n)\}$ of period N and let $\hat{\boldsymbol{\lambda}}_{k,B}$ be the vector of length N given by $\hat{\boldsymbol{\lambda}}_{k,B} = (\hat{\lambda}_{k,B}(0), \hat{\lambda}_{k,B}(1), \cdots, \hat{\lambda}_{k,B}(N-1))$. Throughout the section, we assume that $g = \gcd(N, r-s)$, $u = N/g$ and $v = (r-s)/g$. Then the partial-period cross-correlation $R_{c_r,c_s}(\tau, k; B)$ is equal to the inner product between $\hat{\boldsymbol{\lambda}}_{k,B}$ and the τth column of Θ_{c_r,c_s} scaled by $\frac{1}{N}$, that is,

$$R_{c_r,c_s}(\tau, k; B) = \frac{1}{N}\sum_{l=0}^{N-1} \hat{\lambda}_{k,B}(l)\theta_{c_r^l,c_s}(\tau).$$

Hence,

$$\begin{aligned} &|R_{c_r,c_s}(\tau, k; B)| \\ &= \frac{1}{N}\left|\sum_{l=0}^{N-1} \hat{\lambda}_{k,B}(l)\theta_{c_r^l,c_s}(\tau)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{N}\sum_{l=0}^{N-1}|\hat{\lambda}_{k,B}(l)||\theta_{c_r^l,c_s}(\tau)| \\
&= \frac{1}{N}\sum_{l=0}^{N-1}|\hat{\lambda}_{0,B}(l)||\theta_{c_r^l,c_s}(\tau)| \\
&= \begin{cases} \sqrt{\frac{1}{u}}\sum_{m=0}^{u-1}|\hat{\lambda}_{0,B}(\langle -s\tau_2\rangle_g+mg)|, \\ \qquad \text{if } \langle N\rangle_2=\langle uv\rangle_2=0 \text{ or } \langle N\rangle_2=1 \\ \sqrt{\frac{1}{u}}\sum_{m=0}^{u-1}|\hat{\lambda}_{0,B}(\langle -s(\tau_2-\frac{g}{2})\rangle_g+mg)|, \\ \qquad \text{if } \langle N\rangle_2=0 \text{ and } \langle uv\rangle_2=1 \end{cases} \qquad (5)
\end{aligned}
$$

where $\tau_2=\langle\tau\rangle_g$, the second equality comes from the fact that the magnitude of the spectrum is invariant under the cyclic-shift in the time domain, and the third equality comes from the structure of $|\Theta_{c_r,c_s}|$. By the definition of the window sequence $\{\lambda_{k,B}(n)\}$, we get

$$
\begin{aligned}
|\hat{\lambda}_{0,B}(l)| &= \left|\sum_{n=0}^{N-1}\lambda_{0,B}(n)W_N^{nl}\right| \\
&= \left|\sum_{n=0}^{B-1}W_N^{nl}\right| \\
&= \frac{\left|\sin\left(\frac{\pi}{N}Bl\right)\right|}{\sin\left(\frac{\pi}{N}l\right)}
\end{aligned}
$$

for $0\leq l\leq N-1$ and $0<B\leq N$.

Lemma 7 ([14]): $|\hat{\lambda}_{0,B}(l)|$ has the following properties:

a) If $B=1$, then $|\hat{\lambda}_{0,1}(l)|=1$ for all l;
b) If $B=N-1$, then $|\hat{\lambda}_{0,1}(0)|=N-1$ and $|\hat{\lambda}_{0,1}(l)|=1$ for $1\leq l\leq N-1$;
c) If $B=N$, then $|\hat{\lambda}_{0,1}(0)|=N$ and $|\hat{\lambda}_{0,1}(l)|=0$ for $1\leq l\leq N-1$;
d) If $l=0$, then $|\hat{\lambda}_{0,B}(0)|=B$ for all B;
e) $|\hat{\lambda}_{0,B}(l)|=0$ if and only if $l\neq 0$ and $N|Bl$;
f) $|\hat{\lambda}_{0,B}(l)|=1$ if and only if $N|(B+1)l$ or $N|(B-1)l$;
g) $|\hat{\lambda}_{0,B}(l)|=|\hat{\lambda}_{0,B}(N-l)|$ for all l and B; and
h) $|\hat{\lambda}_{0,B}(l)|=|\hat{\lambda}_{0,N-B}(l)|$ for $1\leq l\leq N-1$ and $1\leq B\leq N-1$.

For a simple presentation, we define the function $f_{s,g,B}(\tau_2)$ for $0\leq\tau_2\leq g-1$ as

$$
f_{s,g,B}(\tau_2)=\sum_{m=0}^{u-1}|\hat{\lambda}_{0,B}(\langle -s\tilde{\tau}_2\rangle_g+mg)|,
$$

where

$$
\tilde{\tau}_2=\begin{cases}\tau_2, & \text{if } \langle N\rangle_2=\langle uv\rangle_2=0 \text{ or } \langle N\rangle_2=1 \\ \tau_2-\frac{g}{2}, & \text{if } \langle N\rangle_2=0 \text{ and } \langle uv\rangle_2=1.\end{cases}
$$

Then, the bound in (5) can be simply represented as

$$
|R_{c_r,c_s}(\tau,k;B)|\leq\frac{1}{\sqrt{u}}f_{s,g,B}(\tau_2).
$$

The properties of $f_{s,g,B}(\tau_2)$ are given in the following lemma.

Lemma 8: Assuming the above notation, the function $f_{s,g,B}(\tau_2)$ has the following properties:

a) If $B=1$, then $f_{s,g,1}(\tau_2)=u$;
b) If $B=N-1$, then

$$
f_{s,g,N-1}(\tau_2)=\begin{cases}N+u-2, & \text{if } \tilde{\tau}_2=0\\ u, & \text{if } \tilde{\tau}_2\neq 0;\end{cases}
$$

c) If $B=N$, then

$$
f_{s,g,N}(\tau_2)=\begin{cases}N, & \text{if } \tilde{\tau}_2=0\\ 0, & \text{if } \tilde{\tau}_2\neq 0;\end{cases}
$$

d) If N is even and $B=N/2$, then

$$
f_{s,g,N/2}(\tau_2)=\begin{cases}B, & \text{for } \tilde{\tau}_2=0\\ 0, & \text{for } \tilde{\tau}_2\neq 0 \text{ and } \langle\tilde{\tau}_2\rangle_2=0;\end{cases}
$$

e) If $2\leq B\leq N-2$ with $(N,B)=1$, then

$$
f_{s,g,B}(\tau_2)=\begin{cases}B+1+2\sum_{m=1}^{(u-2)/2}|\hat{\lambda}_{0,B}(mg)|, \\ \qquad \text{for } \tilde{\tau}_2=0 \text{ and } \langle u\rangle_2=0\\ B+2\sum_{m=1}^{(u-1)/2}|\hat{\lambda}_{0,B}(mg)|, \\ \qquad \text{for } \tilde{\tau}_2=0 \text{ and } \langle u\rangle_2=1;\end{cases}
$$

f) If $2\leq B\leq N-2$ with $(N,B)\neq 1$, then

$$
f_{s,g,B}(\tau_2)=\begin{cases}B, \\ \quad \text{if } \tilde{\tau}_2=0 \text{ and } \langle Bg\rangle_N=0\\ B+1+2\sum_{m=1}^{(u-2)/2}|\hat{\lambda}_{0,B}(mg)|, \\ \quad \text{if } \tilde{\tau}_2=0, \langle Bg\rangle_N\neq 0 \text{ and } \langle u\rangle_2=0\\ B+2\sum_{m=1}^{(u-1)/2}|\hat{\lambda}_{0,B}(mg)|, \\ \quad \text{if } \tilde{\tau}_2=0, \langle Bg\rangle_N\neq 0 \text{ and } \langle u\rangle_2=1.\end{cases}
$$

Sarwate [15] derived an upper bound on $\sum_{l=0}^{N-1}|\hat{\lambda}_{0,B}(l)|$ by using Jensen's inequality. Following a similar approach to Sarwate's, it is possible to derive an upper bound on $|R_{c_r,c_s}(\tau,k;B)|$ in the following theorem.

Theorem 9: Let $\{c_r(n)\}$ and $\{c_s(n)\}$ be two Zadoff-Chu sequences of period N. Then

$$
|R_{c_r,c_s}(\tau,k;B)|\leq
$$

$$
\begin{cases}1, & \text{if } B=1\\ \sqrt{Ng}, & \text{if } B=N\\ \csc\left(\frac{\pi}{N}\right)\sin\left(\frac{\pi}{N}B\right), & \text{if } 2\leq B\leq N-1 \text{ and } u=1\\ \max\{\frac{1}{\sqrt{u}}\left(B+1+\frac{2u}{\pi}\ln\frac{4u}{\pi}\right), \\ \qquad \frac{1}{\sqrt{u}}\left(\sec\left(\frac{\pi}{2u}\right)+2\left(\csc\left(\frac{\pi}{N}\right)+\frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right)\right)\right)\}, \\ & \text{if } 2\leq B\leq N-1 \text{ and } u\neq 1.\end{cases}
$$

Proof. We prove only the case that $\langle N\rangle_2=\langle uv\rangle_2=0$ or $\langle N\rangle_2=1$, since the other cases can be similarly proved. Clearly, $|R_{c_r,c_s}(\tau,k;1)|=1$. Also, we get $|R_{c_r,c_s}(\tau,k;N)|=\sqrt{Ng}$ by Theorem 3. Now, we will focus on the case that $2\leq B\leq N-1$. Our problem can be divided into four cases:

Case 1) $\tilde{\tau}_2 \neq 0$ and $\langle u \rangle_2 = 0$:

$$\begin{aligned}
&\sum_{m=0}^{u-1} \left| \hat{\lambda}_{0,B}(\langle -s\tilde{\tau}_2 \rangle_g + mg) \right| \\
&\le \sum_{m=0}^{u-1} \left| \frac{1}{\sin\left(\frac{\pi}{N}(\langle -s\tilde{\tau}_2 \rangle_g + mg)\right)} \right| \\
&= \sum_{m=0}^{u-1} \csc\left(\frac{\pi}{N}(\langle -s\tilde{\tau}_2 \rangle_g + mg)\right) \\
&< \left(\csc\left(\frac{\pi}{N}\right) + \sum_{m=1}^{u/2-1} \csc\left(\frac{\pi}{N}mg\right) \right) \\
&\quad + \left(\csc\left(\frac{\pi}{N}(N-1)\right) + \sum_{m=u/2+1}^{u-1} \csc\left(\frac{\pi}{N}mg\right) \right) \\
&= 2\left(\csc\left(\frac{\pi}{N}\right) + \sum_{m=1}^{u/2-1} \csc\left(\frac{\pi}{N}mg\right) \right) \qquad (6)
\end{aligned}$$

where the second inequality and the second equality come from the convexity and symmetry of the cosecant function, respectively. Define $\frac{u}{2}-1$ random variables X_m, $1 \le m \le \frac{u}{2}-1$, whose probability density functions are given by

$$f_{X_m}(x) = \begin{cases} 1, & m - \frac{1}{2} \le x \le m + \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Then $E[\frac{\pi}{N}X_m g] = \frac{\pi}{N}mg$, which leads to $\csc\left(\frac{\pi}{N}gm\right) \le E[\csc\left(\frac{\pi}{N}X_m g\right)]$ by Jenson's inequality. After some manipulation, we have

$$\sum_{m=1}^{u/2-1} \csc\left(\frac{\pi}{N}mg\right) \le \frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right). \qquad (7)$$

By combining (6) and (7), we get

$$\sum_{m=0}^{u-1} |\hat{\lambda}_{0,B}(\langle -s\tilde{\tau}_2 \rangle_g + mg)| < 2\left(\csc\left(\frac{\pi}{N}\right) + \frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right) \right).$$

Case 2) $\tilde{\tau}_2 \neq 0$ and $\langle u \rangle_2 = 1$: In a similar way, we have

$$\begin{aligned}
\sum_{m=0}^{u-1} |\hat{\lambda}_{0,B}(\langle -s\tilde{\tau}_2 \rangle_g + mg)| &< \sec\left(\frac{\pi}{2u}\right) \\
&\quad + 2\left(\csc\left(\frac{\pi}{N}\right) + \frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right) \right),
\end{aligned}$$

where $u \neq 1$.

Case 3) $\tilde{\tau}_2 = 0$ and $\langle u \rangle_2 = 0$: Similarly, we get

$$\sum_{m=1}^{(u-2)/2} |\hat{\lambda}_{0,B}(mg)| \le \frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right).$$

Case 4) $\tilde{\tau}_2 = 0$ and $\langle u \rangle_2 = 1$: Similarly, we have

$$\sum_{m=1}^{(u-1)/2} |\hat{\lambda}_{0,B}(mg)| \le \frac{u}{\pi}\ln\left(\frac{4u}{\pi}\right).$$

Combining the results of the above four cases, we get an upper bound when $r \neq s$, i.e., $u \neq 1$. To complete the proof, we need to show that

$$|R_{c_r}(\tau, k; B)| \le \csc\left(\frac{\pi}{N}\right)\sin\left(\frac{\pi}{N}B\right),$$

but we omit its proof due to the limit of the space. □

V. Conclusion

We investigated the partial-period correlation of Zadoff-Chu sequences by introducing linear phase-shifting sequences and linking the full-period correlation with the partial-period correlation. As a generalization of the approach in [5], our method can be applied to the partial-period correlation between any pair of sequences $\{a(n)\}$ and $\{b(n)\}$ as long as the full-period correlation properties between the lth linear phase-shifting sequence $\{a^l(n)\}$ and $\{b(n)\}$, which are represented as an $N \times N$ matrix $\Theta_{a,b}$, are available. Analysis of $\Theta_{a,b}$ for two Zadoff-Chu sequences $\{a(n)\}$ and $\{b(n)\}$ led us to an upper bound on their partial-period cross-correlation, which has not yet been known.

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