Capacities of Gaussian Classical-Quantum Channels

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***Abstract*—In the present paper we introduce and study Bosonic Gaussian classical-quantum (c-q) channels; the embedding of the classical input into quantum is always possible and therefore the classical entanglement-assisted capacity C_{ea} under appropriate input constraint is well defined. We prove a general property of entropy increase for weak complementary channel, that implies the equality $C = C_{ea}$ (where C is the unassisted capacity) for certain class of c-q Gaussian channels under appropriate energy-type constraint. On the other hand, we show by explicit example that the inequality $C < C_{ea}$ is not unusual for constrained c-q Gaussian channel.**

I. Introduction

In finite dimension a classical-quantum or quantum-classical channel can always be represented as a quantum channel, by embedding the classical input or output into quantum system. Then it makes sense to speak about entanglement-assisted capacity C_{ea} [1], [2] of such a channel, in particular, to compare it with the unentangled classical capacity C. In the paper [6] we considered the case of quantum-classical (measurement) channels, showing that generically $C < C_{ea}$ for such channels. For infinite dimensional (in particular, continuous variable) systems an embedding of the classical output into quantum is not always possible, however entanglement-assisted transmission still makes sense [6]; in particular this is the case for Bosonic Gaussian q-c channels. The measurement channels demonstrate the gain of entanglement assistance in the most spectacular way.

On the contrary, as shown in [9], (unconstrained) finite dimensional c-q channels (preparations) are *essentially* characterized by the property of having no gain of entanglement assistance, in this sense being "more classical" than measurements. However introducing certain input constraint may restore the gain. While constrained finite dimensional channels look artificial, an input (typically energy) constraint is mandatory for infinite dimensional channels. In the present paper we study Bosonic Gaussian c-q channels; we observe that the embedding of the classical input into quantum is always possible and C_{ea} under the input constraint is thus well defined. We prove a general property of entropy increase for the weak complementary channel, that implies equality $C = C_{ea}$ for certain class of c-q Gaussian channels under appropriate energy-type constraint. On the other hand, we show by explicit example that the inequality $C < C_{ea}$ is not unusual for *constrained* c-q Gaussian channels.

II. Bosonic Gaussian Systems

The main applications of infinite-dimensional quantum information theory are related to Bosonic systems, for detailed description of which we refer to Ch. 12 in [4]. Let $\mathcal{H}_A$ be the representation space of the Canonical Commutation Relations

$$W(z_A)W(z_A') = \exp\left(-\frac{i}{2}z_A^t\Delta_A z_A'\right)W(z_A'+z_A) \quad (1)$$

with a coordinate symplectic space (Z_A, Δ_A) and the Weyl system $W_A(z) = \exp(iR_A \cdot z_A)$; $z_A \in Z_A$. Here R_A is the row-vector of the canonical variables in $\mathcal{H}_A$, and Δ_A is the canonical skew-symmetric commutation matrix of the components of R_A,

$$\Delta = \mathrm{diag}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{j=1,\dots,s}. \quad (2)$$

Let $(Z_A, \Delta_A), (Z_B, \Delta_B)$ be the symplectic spaces of dimensions $2s_A, 2s_B$, which will describe the input and the output of the channel (here Δ_A, Δ_B have the canonical form (2)), and let $W_A(z_A), W_B(z_B)$ be the Weyl operators in the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ of the corresponding Bosonic systems. A centered Gaussian channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is defined via the action of its dual Φ^* on the Weyl operators:

$$\Phi^*[W_B(z_B)] = W(Kz_B)\exp\left[-\frac{1}{2}z_B^t\alpha z_B\right], \quad (3)$$

where K is a matrix of linear operator $Z_B \to Z_A$, and α is real symmetric matrix satisfying

$$\alpha \geq \pm\frac{i}{2}\left(\Delta_B - K^t\Delta_A K\right), \quad (4)$$

where $\Delta_B - K^t\Delta_A K \equiv \Delta_K$ is a real skew-symmetric matrix.

We will make use of the unitary dilation of the channel Φ constructed in [3] (see also [4]). Consider the composite Bosonic system $AD = BE$ with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E$ corresponding to the symplectic space $Z = Z_A \oplus Z_D = Z_B \oplus Z_E$, where $(Z_E, \Delta_E) \simeq (Z_A, \Delta_A)$. Thus $[R_A\, R_D] = [R_B\, R_E]$ describe two different splits of the set of canonical observables for the composite system. The channel Φ is then described by the linear input-output relation (preserving the commutators)

$$R_B' = R_A K + R_D K_D, \quad (5)$$

where the system D (input environment) is in a Gaussian state ρ_D with the covariance matrix α_D such that

$$\alpha = K_D^t \alpha_D K_D.$$

(for simplicity of notations we write $R_A, \ldots$ instead of $R_A \otimes I_D, \ldots$). It is shown that the commutator-preserving relation (5) can be complemented to the full linear canonical transformation by putting

$$R_E' = R_A L + R_D L_D, \tag{6}$$

where $(2s_A) \times (2s_E)-$ matrix L and $(2s_D) \times (2s_A)-$ matrix L_D are such that the square $2(s_A + s_D) \times 2(s_B + s_E)-$ matrix

$$T = \begin{bmatrix} K & L \\ K_D & L_D \end{bmatrix} \tag{7}$$

is symplectic, i.e. satisfies the relation

$$T^t \begin{bmatrix} \Delta_A & 0 \\ 0 & \Delta_D \end{bmatrix} T = \begin{bmatrix} \Delta_B & 0 \\ 0 & \Delta_E \end{bmatrix},$$

which is equivalent to

$$\begin{aligned} \Delta_B &= K^t \Delta_A K + K_D^t \Delta_D K_D, & (8)\\ 0 &= K^t \Delta_A L + K_D^t \Delta_D L_D, & (9)\\ \Delta_E &= L^t \Delta_A L + L_D^t \Delta_D L_D. & (10) \end{aligned}$$

Denote by the U_T the unitary operator in $\mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E$ implementing the symplectic transformation T so that

$$[R_B' \, R_E'] = U_T^* [R_B \, R_E] U_T = [R_A \, R_D] T. \tag{11}$$

Then we have the unitary dilation

$$\Phi^*[W_B(z_B)] = \mathrm{Tr}_D \, (I_A \otimes \rho_D) \, U_T^* \, (W_B(z_B) \otimes I_E) \, U_T. \tag{12}$$

The *weakly complementary* channel [3] is then

$$\left(\tilde{\Phi}^w\right)^* [W_E(z_E)] = \mathrm{Tr}_D \, (I_A \otimes \rho_D) \, U_T^* \, (I_B \otimes W_E(z_E)) \, U_T.$$

The equation (6) is nothing but the input-output relation for the weakly complementary channel which thus acts as

$$\left(\tilde{\Phi}^w\right)^* [W_E(z_E)] = W_A(L z_E) \exp\left[-\frac{1}{2} z_E^t L_D^t \alpha_D L_D z_E\right]. \tag{13}$$

In the case of pure state $\rho_D = |\psi_D\rangle\langle\psi_D|$ the relation (12) amounts to the Stinespring representation for the channel Φ with the isometry $V = U_T |\psi_D\rangle$, implying that $\tilde{\Phi}^w$ is the complementary channel $\tilde{\Phi}$ (see e.g. [4]).

III. Gaussian classical-quantum channels

We call a quantum channel Φ *classical-quantum* (c-q) if the range of Φ^* consists of commuting operators. It follows from (1) that the necessary and sufficient condition for the Gaussian channel (3) to be c-q is

$$K^t \Delta_A K = 0. \tag{14}$$

Thus $\Delta_K = \Delta_B$ and therefore $\det \Delta_K \neq 0$. Under this condition it was shown in [7] that in the unitary dilation described above one can take $s_E = s_A$, $s_D = s_B$ (and in fact $E = A, D = B$). We call such a dilation "minimal" as it is indeed such at least in the case of the pure state ρ_D, as follows from [3]. The condition (4) then amounts to

$$\alpha \geq \pm \frac{i}{2} \Delta_B, \tag{15}$$

saying that α is a covariance matrix of centered Gaussian state ρ_D. We say that the channel has *minimal noise* if ρ_D is a pure state, which is equivalent to the fact that α is a minimal solution of the inequality (15). In quantum optics such channels are called quantum-limited.

Let us explain how our definition of c-q channel agrees with the usual one (see e.g. [4]) in the case of Gaussian channels. The condition (14) means that the components of the operator $R_A K$ all commute, hence their joint spectral measure is a sharp observable, and their probability distribution $\mu_\rho(d^{2n}z)$ can be arbitrarily sharply peaked around any point $z = \mathsf{E}_\rho(R_A K)^t = K^t m$ in the support $\mathcal{M}$ of this measure by appropriate choice of the state ρ. Here E_ρ denotes expectation with respect to ρ and $m = \mathsf{E}_\rho(R_A)^t$, hence $\mathcal{M} = \mathbf{Ran} K^t \subseteq Z_B$. Thus in this case it is natural to identify Φ as c-q channel determined by the family of states $z \to W(z)\rho_B W(z)^*; z \in \mathcal{M}$.

Proposition 1: Let Φ be a Gaussian c-q channel, then the weak complementary $\tilde{\Phi}^w$ in the minimal unitary dilation has nonnegative entropy gain:

$$S(\tilde{\Phi}^w[\rho]) - S(\rho) \geq 0 \quad \text{for all} \quad \rho.$$

In particular if Φ has minimal noise, then this holds for the complementary channel $\tilde{\Phi}$, implying

$$I(\rho, \Phi) \leq S(\Phi[\rho]), \tag{16}$$

where

$$I(\rho, \Phi) = S(\rho) + S(\Phi[\rho]) - S(\tilde{\Phi}[\rho])$$

is the quantum mutual information.

Proof: Taking into account (14), the relation (8) becomes

$$\Delta_B = K_D^t \Delta_D K_D. \tag{17}$$

We consider the minimal dilation for which $\Delta_D = \Delta_B$, hence K_D is a symplectic $2s_B \times 2s_B-$ matrix. Then (9) implies

$$L_D = -\left(K_D^t \Delta_D\right)^{-1} K^t \Delta_A L.$$

Substituting (10) gives $\Delta_E = L^t M L$, where

$$\begin{aligned} M &= \Delta_A + \Delta_A K \left(\Delta_D K_D\right)^{-1} \Delta_D \left(K_D^t \Delta_D\right)^{-1} K^t \Delta_A \\ &= \Delta_A + \Delta_A K \Delta_B^{-1} K^t \Delta_A. \end{aligned}$$

Therefore $1 = (\det L)^2 \det M$, where

$$\begin{aligned} \det M &= \det\left(\Delta_A + \Delta_A K \Delta_B^{-1} K^t \Delta_A\right) \\ &= \det\left(I_{2s_A \times 2s_A} + K \Delta_B^{-1} K^t \Delta_A\right). \end{aligned}$$

Due to (14) the matrix $N = K\Delta_B^{-1} K^t \Delta_A$ satisfies $N^2 = 0$, hence its has only zero eigenvalues. Therefore $I_{2s_A \times 2s_A} + N$ has only unit eigenvalues, implying $\det M = 1$ and hence $|\det L| = 1$.

By relation (13), the channel $\tilde{\Phi}^w$ is the Gaussian channel with the operator L playing the role of K. By using a result of [5], we have

$$S(\tilde{\Phi}^w[\rho]) - S(\rho) \geq \log|\det L| = 0.$$

■

Proposition 2: Let Φ be a Gaussian c-q channel with minimal noise α, such that $\mathbf{Ran}K^t = Z_B$, satisfying the input constraint

$$\mathrm{Tr}\rho H \leq E, \tag{18}$$

where $H = RK\epsilon K^t R^t$ and ϵ is real symmetric strictly positive definite matrix. Then denoting $C(E)$ (resp. $C_{ea}(E)$) the classical (resp. entanglement-assisted) capacity of the channel under the constraint (18),

$$C(E) = C_{ea}(E) = \sup_{\rho:\mathrm{Tr}\rho H \leq E} S(\Phi[\rho]).$$

An important condition here is $\mathbf{Ran}K^t = Z_B$, as we shall see in the next Section, the others could be relaxed.

Proof: Due to the formula for the entanglement-assisted capacity and to (16),

$$C_{ea}(E) = \sup_{\rho:\mathrm{Tr}\rho H \leq E} I(\rho, \Phi) \leq \sup_{\rho:\mathrm{Tr}\rho H \leq E} S(\Phi[\rho]),$$

hence it is sufficient to show that

$$C(E) \geq \sup_{\rho:\mathrm{Tr}\rho H \leq E} S(\Phi[\rho]).$$

We first consider the supremum in the right-hand side. Since the constraint operator $H = RK\epsilon K^t R^t$ is quadratic in the canonical variables R, the supremum can be taken over (centered) Gaussian states. Since the entropy of Gaussian state with covariance matrix α is equal to

$$\frac{1}{2}\mathrm{Sp}g\left(\mathrm{abs}\left(\Delta^{-1}\alpha\right) - I/2\right) = \frac{1}{2}\sum_{j=1}^{2s} g(|\lambda_j| - \frac{1}{2}), \tag{19}$$

where $g(x) = (x+1)\log(x+1) - x\log x$, Sp denotes trace of the matrices as distinct from that of operators in $\mathcal{H}$, and λ_j are the eigenvalues of $\Delta^{-1}\alpha$ (see e.g. Sec. 12.3 of [4]), we have

$$\sup_{\rho:\mathrm{Tr}\rho H \leq E} S(\Phi[\rho]) = \tag{20}$$

$$= \frac{1}{2}\max_{\mu:\mathrm{Sp}\epsilon\mu \leq E} \mathrm{Sp}g\left(\mathrm{abs}\left(\Delta_B^{-1}(\mu + \alpha)\right) - I/2\right).$$

Here we used the formula (19) for the output state with the covariance matrix $K^t\beta K + \alpha$, denoted $\mu = K^t\beta K$ and used the fact that for every μ such a β exists due to the condition $\mathbf{Ran}K^t = Z_B$. In the second expression the supremum is attained on some μ_0 due to nondegeneracy of ϵ. Denote by β_0 a solution of the equation $\mu_0 = K^t\beta_0 K$.

We construct a sequence of suboptimal ensembles by using the following Lemma, the idea behind the proof can be seen from examples in the next Section.

Lemma 1: Assuming (14), there exist a sequence of real symmetric $(2s_A) \times (2s_A)$ −matrices γ_n satisfying:

1) $\gamma_n \geq \pm\frac{i}{2}\Delta_A$;
2) $K^t\gamma_n K \to 0$.

Using the condition 1, let ρ_n be a centered Gaussian state in $\mathcal{H}_A$ with the covariance matrices γ_n, and $\rho_n(z) = D(z)\rho_n D(z)^*, z \in Z_A$, be the family of the displaced states, where $D(z)$ are the displacement operators obtained by re-parametrization of the Weyl operators $W(z)$. Define the Gaussian probability density $p_n(z)$ with zero mean and the covariance matrix $k_n\beta_0$, where $k_n = 1 - \mathrm{Sp}\gamma_n K\epsilon K^t/E > 0$ for large enough n by the condition 2. The average state of this ensemble is centered Gaussian with the covariance matrix $\gamma_n + k_n\beta_0$. Taking into account that $S(\rho_n(z)) = S(\rho_0)$, the $\chi-$quantity of this ensemble is equal to

$$\chi_n = \frac{1}{2}\mathrm{Sp}\,g\left(\mathrm{abs}\left(\Delta_B^{-1}\left(K^t\gamma_n K + k_n K^t\beta_0 K + \alpha\right)\right) - I/2\right)$$

$$-\frac{1}{2}\mathrm{Sp}\,g\left(\mathrm{abs}\left(\Delta_B^{-1}\left(K^t\gamma_n K + \alpha\right)\right) - I/2\right).$$

By the condition 2 this converges to

$$\frac{1}{2}\mathrm{Sp}\,g\left(\mathrm{abs}\left(\Delta_B^{-1}\left(K^t\beta_0 K + \alpha\right)\right) - I/2\right)$$

$$-\frac{1}{2}\mathrm{Sp}\,g\left(\mathrm{abs}\left(\Delta_B^{-1}\alpha\right) - I/2\right).$$

By minimality of the noise the second term is entropy of a pure state, equal to zero, and the first term is just the maximum in (20). Thus

$$C(E) \geq \limsup_{n\to\infty} \chi_n = \sup_{\rho:\mathrm{Tr}\rho H \leq E} S(\Phi[\rho]).$$

■

IV. One mode

Let q, p be a Bosonic mode, $W(z) = \exp i(xq + yp)$ the corresponding Weyl operator. We give two examples where the channel describes classical signal with additive Gaussian (minimal) quantum noise, in the first case the signal being two-dimensional while in the second – one-dimensional. As we have seen, a c-q channel can be described in two equivalent ways: as a mapping $m \to \rho_m$, where m is the classical signal, and as an extended quantum channel satisfying (14).

1. We first consider the minimal noise c-q channel with two-dimensional real signal and show the coincidence of the classical entanglement-assisted and unassisted capacities of this channel under appropriate input constraint, by using result of Sec. III. Such a coincidence is generic for unconstrained finite-dimensional channels [2], but in infinite dimensions, as we will see in the second example, situation is different. Some sufficient conditions for the equality $C = C_{ea}$ were given in [9], however they do not apply to our example.

Let $m = (m_q, m_p) \in \mathbf{R}^2$ and consider the mapping $m \to \rho_m$, where ρ_m is the state with the characteristic function

$$\mathrm{Tr}\rho_m W(z) = \exp\left[i(m_q x + m_p y) - \frac{1}{4}(x^2 + y^2)\right]. \tag{21}$$

The mapping $m \to \rho_m$ can be considered as transmission of the two-dimensional classical signal $m = (m_q, m_p)$ with the additive minimal quantum Gaussian noise q,p. The classical capacity of this channel with the input constraint

$$\frac{1}{2}\int \|m\|^2 \; p(m)d^2m \leq E \tag{22}$$

is given by the relation (see e.g. [4], Sec. 12.1.4)

$$C(E)=g(E), \tag{23}$$

with the optimal distribution

$$p(m)=\frac{1}{2\pi E}\exp\left(-\frac{\|m\|^2}{2E}\right).$$

Another way to describe this channel is to represent it as a quantum Gaussian channel Φ in the spirit of previous Section. Since the input $m=(m_q,m_p)$ is two-dimensional classical, one has to use two Bosonic input modes $q_1,p_{1,},q_2,p_2$ to describe it quantum-mechanically, so that e.g. $m_q=q_1, m_p=q_2$. The environment is one mode q,p in the Gaussian state ρ_0 so the output is given by the equations

$$\begin{aligned} q' &= q+q_1=q+m_q; \\ p' &= p+q_2=p+m_p, \end{aligned} \tag{24}$$

and the channel Φ parameters are

$$K=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \alpha=\frac{1}{2}I_2.$$

The equations for the environment modes describing the complementary channel $\tilde{\Phi}$ are (see [4])

$$\begin{aligned} q_1' &= q_1, \\ p_1' &= p_1-p-q_2/2, \\ q_2' &= q_2, \\ p_2' &= p_2+q+q_1/2. \end{aligned} \tag{25}$$

Having realized the c-q channel as a quantum one (i.e. a channel with quantum input and output), it makes sense to speak of its entanglement-assisted capacity. Under the same constraint it is given by the expression

$$C_{ea}(E)=\sup_{\rho_{12}\in\mathfrak{S}_E} I(\rho_{12},\Phi), \tag{26}$$

where

$$\mathfrak{S}_E=\left\{\rho_{12}:\mathrm{Tr}\rho_{12}\left(\frac{q_1^2+q_2^2}{2}\right)\leq E\right\}$$

corresponds to the constraint (22). Notice that the constraint operator $H=\frac{q_1^2+q_2^2}{2}$ is unusual in that it is given by *degenerate* quadratic form in the input variables $q_1,p_{1,},q_2,p_2$. In this case the set $\mathfrak{S}_E$ is not compact, the supremum in (26) is not attained and to obtain this formula we need to use a result from [8].

Now assume the minimal noise $N=0$ and let us show that

$$C_{ea}(E)=C(E)=g(E). \tag{27}$$

Proposition 1 of Sec. III implies

$$C_{ea}(E)\leq \sup_{\rho_{12}\in\mathfrak{S}_E} S(\Phi[\rho_{12}]).$$

But

$$\Phi[\mathfrak{S}_E]=\left\{\bar{\rho}_p : p\in\mathcal{P}_E\right\},$$

where $\mathcal{P}_E$ is defined by (22), as can be seen from the equations of the channel (24) and the identification of the probability density $p(m_q\,,m_p)$ with that of observables q_1,q_2 in the state ρ_{12}. Invoking (23) gives $\sup_{\rho_{12}\in\mathfrak{S}_E} H(\Phi[\rho_{12}])=g(E)$ and hence the equality (27). This example is a special case of the Proposition 2 in Sec. III, all the conditions of which are fulfilled with $\mathbf{Ran}K^t=Z_B=\mathbf{R}^2$ and $(\varepsilon_n\to 0)$

$$\gamma_n=\begin{bmatrix} \varepsilon_n & 0 & 0 & 0 \\ 0 & \frac{1}{4\varepsilon_n} & 0 & 0 \\ 0 & 0 & \varepsilon_n & 0 \\ 0 & 0 & 0 & \frac{1}{4\varepsilon_n} \end{bmatrix}.$$

2. Now we give an example with $C(E)<C_{ea}(E)$. Let $m\in\mathbf{R}$ be a real one-dimensional signal and the channel is $m\to\rho_m$, where ρ_m is the state with the characteristic function

$$\mathrm{Tr}\rho_m W(z)=\exp\left[imx-\frac{1}{2}(\sigma^2x^2+\frac{1}{4\sigma^2}y^2)\right]. \tag{28}$$

The mapping $m\to\rho_m$ can be considered as transmission of the classical signal m with the additive noise arising from the q-component of quantum Gaussian mode q,p with the variances $\mathsf{D}q=\sigma^2, \mathsf{D}p=\frac{1}{4\sigma^2}$ and zero covariance between q and p. The state ρ_0 is pure (squeezed vacuum) corresponding to a minimal noise.

The constraint on the input probability distribution $p(m)$ is defined as

$$\int m^2\; p(m)dm\leq E, \tag{29}$$

where E is a positive constant. As the component p is not affected by the signal, from information-theoretic point this channel is equivalent to the classical additive Gaussian noise channel $m\to m+q$, and its capacity under the constraint (29) is given by the Shannon formula

$$C(E)=\frac{1}{2}\log\left(1+r\right), \tag{30}$$

where $r=E/\sigma^2$is the *signal-to-noise ratio*.

A different way to describe this channel is to represent it as a quantum Gaussian channel Φ. Introducing the input mode q_1,p_1, so that $m=q_1$, with the environment mode q,p in the state ρ_0, the output is given by the equations

$$\begin{aligned} q_1' &= q_1+q; \\ p_1' &= p, \end{aligned} \tag{31}$$

and the channel Φ parameters are

$$K=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \alpha=\begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{1}{4\sigma^2} \end{bmatrix}.$$

The equations for the environment mode describing the complementary channel $\tilde{\Phi}$ are (see [4])

$$\begin{aligned} q' &= q_1, \\ p' &= p_1 - p, \end{aligned} \tag{32}$$

and the set of equations (31), (32) describes the canonical transformation of the composite system = system+environment.

The classical entanglement-assisted capacity of this channel under the same constraint is given by the expression

$$C_{ea}(E) = \sup_{\rho_1 \in \mathfrak{S}_E^{(1)}} I(\rho_1, \Phi), \tag{33}$$

where $\mathfrak{S}_E^{(1)} = \left\{ \rho_1 : \mathrm{Tr}\rho_1 q_1^2 \leq E \right\}$. To compute it, consider the values of $I(\rho_A, \Phi)$ for centered Gaussian states $\rho_A = \rho_1$ with covariance matrices

$$\alpha_1 = \begin{bmatrix} E & 0 \\ 0 & E_1 \end{bmatrix},$$

satisfying the uncertainty relation $EE_1 \geq \frac{1}{4}$ and belonging to the set $\mathfrak{S}_E^{(1)}$ with the equality.

We use the formula (19) implying

$$S(\rho_A) = g\left(\sqrt{EE_1} - \frac{1}{2}\right),$$

By using (31), we obtain the entropy of the output state $\rho_B = \Phi[\rho_A]$

$$S(\rho_B) = g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4}} - \frac{1}{2}\right).$$

Similarly, according to (32) the state $\rho_E = \tilde{\Phi}[\rho_A]$ has the entropy

$$S(\rho_E) = g\left(\sqrt{EE_1 + \frac{E}{4\sigma^2}} - \frac{1}{2}\right).$$

Summing up,

$$I(\rho_A, \Phi) = g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4}} - \frac{1}{2}\right) - \delta_1(E_1),$$

where

$$\delta_1(E_1) = g\left(\sqrt{EE_1 + \frac{E}{4\sigma^2}} - \frac{1}{2}\right) - g\left(\sqrt{EE_1} - \frac{1}{2}\right)$$

is a positive decreasing function in the range $[\frac{1}{4E}, \infty)$. Thus

$$C_{ea}(E) \geq g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4}} - \frac{1}{2}\right).$$

Let us show that in fact there is equality here, by using the concavity of the quantum mutual information. For a given input state ρ with finite second moments consider the state $\tilde{\rho} = \frac{1}{2}\left(\rho + \rho^{\top}\right)$, where the transposition $^{\top}$ corresponds to the antiunitary conjugation $q, p \to q, -p$. The state $\tilde{\rho}$ has the same variances $\mathsf{D}q, \mathsf{D}p$ and zero covariance between q and p. The channel (31) is covariant with respect to the transposition; by the aforementioned concavity, $I(\tilde{\rho}, \Phi) \geq I(\rho, \Phi)$, moreover, $I(\tilde{\rho}_G, \Phi) \geq I(\tilde{\rho}, \Phi)$, where $\tilde{\rho}_G$ is the Gaussian state with the same first and second moments as $\tilde{\rho}$. Thus

$$C_{ea}(E) = g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4}} - \frac{1}{2}\right) = g\left(\frac{\sqrt{1+r} - 1}{2}\right).$$

Comparing this with (30), one has $C_{ea}(E) > C(E)$ for $E > 0$, with the entanglement-assistance gain $C_{ea}(E)/C(E) \sim -\frac{1}{2}\log r$, as $r \to 0$ and $C_{ea}(E)/C(E) \to 1$, as $r \to \infty$.

As it is to be expected, Proposition 2 is not applicable, as $\mathrm{rank}K^t = 1 < \dim Z_B$ here, while

$$\gamma_n = \begin{bmatrix} \varepsilon_n & 0 \\ 0 & \frac{1}{4\varepsilon_n} \end{bmatrix}$$

still satisfies the conditions 1-3.

V. Conclusion

We have shown that contrary to unconstrained finite dimensional classical-quantum channels which are characterized by the property of having no gain of entanglement assistance, the constrained c-q channels have more complicated behavior depending on the constraint. We gave general conditions that imply the equality $C = C_{ea}$ for a class of c-q Bosonic Gaussian channels with appropriate energy-type constraint and demonstrated that the inequality $C < C_{ea}$ is possible under certain degeneracy of the channel and the constraint. While the results are obtained for the minimal-noise channels, we anticipate that this restriction could be relaxed; the work in that direction is in progress.

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