Communication over Fiber-optic Channels using the Nonlinear Fourier Transform

Mansoor I. Yousefi and Frank R. Kschischang
Dept. of Electrical & Computer Engineering, University of Toronto
Email: {mansoor,frank}@comm.utoronto.ca

***Abstract*—Motivated by the looming "capacity crunch" in current fiber-optic systems, we recently suggested using the nonlinear Fourier transform (NFT) to transmit information over integrable communication channels such as the optical fiber channel, which is governed by the generalized nonlinear Schrödinger equation. In this transmission scheme information is encoded in the nonlinear Fourier transform of the signal, consisting of two components: a discrete and a continuous spectral function. In this paper, we restrict to discrete spectrum modulation and provide some simple examples of achievable spectral efficiencies. With this new method, deterministic distortions arising from the dispersion and nonlinearity, such as inter-symbol and inter-channel interference are zero for a single user channel or all users of a multiple user network.**

I. Introduction

This paper uses the method introduced in Part I [1] and Part II [2] for data transmission using the nonlinear Fourier transform (NFT). [Part I] describes the mathematical tools underlying this approach to communications. Numerical methods for implementing the NFT at the receiver are discussed in [Part II]. The aims of this paper are to provide methods for implementing the inverse NFT at the transmitter, to discuss the influence of noise on the received spectra, and to provide some example transmission schemes, which illustrate some of the spectral efficiencies achievable by this method.

The proposed nonlinear frequency-division multiplexing (NFDM) scheme can be considered as a generalization of orthogonal frequency-division multiplexing (OFDM) to integrable nonlinear dispersive communication channels [Part I]. The advantages of NFDM stem from the following:

1) NFDM removes deterministic inter-channel interference (cross-talk) between users of a network sharing the same fiber channel;
2) NFDM removes deterministic inter-symbol interference (ISI) (intra-channel interactions) for each user;
3) spectral invariants as carriers of data are remarkably stable and noise-robust features of the nonlinear Schrödinger (NLS) flow;
4) with NFDM, information in each channel of interest can be conveniently read anywhere in a network independently of the optical path length(s).

As described in [Part I], the nonlinear Fourier transform of a signal with respect to a Lax operator consists of discrete and continuous spectral functions, in one-to-one correspondence with the signal. Information transmission over optical fibers by modulation of the eigenvalues was first proposed by Hasegawa and Nyu [3], who presented a simple special case. In this paper we focus mainly on discrete spectrum modulation, which captures a large class of input signals of interest. For this class of signals, the inverse NFT is a map from $2N$ complex parameters (discrete spectral degrees of freedom) to an N-soliton pulse in the time domain. This special case corresponds to an optical communication system employing multisoliton transmission and detection in the focusing regime.

A physically (and commercially) important integrable channel is the optical fiber channel. Despite substantial effort, fiber-optic communications using fundamental solitons (*i.e.*, 1-solitons) has faced numerous challenges in the past decades, partly because the spectral efficiency of conventional soliton systems is typically quite low ($\rho \sim 0.2$ bits/s/Hz). Although solutions have been suggested to alleviate these limitations [4], most current research is focused on the use of sinc-like spectrally-efficient pulse shapes, such as raised-cosine pulses, with digital backpropagation at the receiver [5]. Although these methods provide a substantial spectral efficiency at low to moderate signal-to-noise ratios (SNRs), their achievable rates decline after at a finite SNR ~ 25 dB where $\rho \sim 5$ bits/s/Hz. We believe that this is due to the incompatibility of linear wavelength-division multiplexing (WDM) with the flow of the nonlinear Schrödinger equation, causing severe inter-channel interference.

While a fundamental soliton can be modulated, detected and analyzed in the time domain, N-solitons are best understood via their spectrum in the complex plane. In this paper, these pulses are obtained by implementing a simplified inverse NFT at the transmitter and are demodulated at the receiver by recovering their spectral content using the forward NFT. Since the spectral parameters of a multisoliton naturally do not interact with one another (at least in the absence of the noise), there is potentially a great advantage in directly modulating these non-interacting degrees of freedom. Sending an N-soliton train for large N and detecting it at the receiver—a daunting task in the time-domain due to the interaction of the individual components—can be efficiently accomplished, with the help of the NFT, in the nonlinear frequency domain.

II. The Discrete Spectral Function

The evolution of the slowly-varying part $q(t,z)$ of a narrowband signal propagating in an optical fiber as a function of retarded time t and distance z, after appropriate normalization, is modeled by the stochastic NLS equation [Part I]

$$jq_z(t,z) = q_{tt} + 2|q(t,z)|^2 q(t,z) + \epsilon n(t,z), \tag{1}$$

in which $n(t,z)$ is a bandlimited white Gaussian noise, *i.e.*, with

$$\mathsf{E}\left\{n(t,z)n^*(t',z')\right\} = \delta_W(t-t')\delta(z-z'),$$

where $\delta_W(x) = 2W\mathrm{sinc}(2Wx)$ and ϵ is a small parameter. In (1) and throughout this paper subscripts denote partial differentiation with respect to the corresponding variable. It is assumed that the transmitter is bandlimited to W for all z, $0 \leq z \leq \mathcal{L}$, and power limited to $\mathcal{P}_0$

$$E\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} |q(t,0)|^2 dt = \mathcal{P}_0.$$

It is known that (1) in the absence of the noise is integrable [1]. The nonlinear Fourier transform of a signal in (1) arises via the spectral analysis of the Zakharov-Shabat operator

$$L = j\begin{pmatrix} \frac{\partial}{\partial t} & -q(t) \\ -q^*(t) & -\frac{\partial}{\partial t} \end{pmatrix}. \tag{2}$$

Let $v(t,\lambda)$ be an eigenvector of L with eigenvalue λ. Following [Part I], the discrete spectral function of the signal propagating according to (1) is obtained by solving the eigenproblem $Lv = \lambda v$ for (2), or equivalently

$$v_t = \begin{pmatrix} -j\lambda & q(t) \\ -q^*(t) & j\lambda \end{pmatrix} v, \quad v(-\infty,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-j\lambda t}, \tag{3}$$

where the initial condition was chosen based on the assumption that the signal $q(t)$ vanishes as $|t| \to \infty$. The system of ordinary differential equations (3) is solved from $t = -\infty$ to $t = +\infty$ to obtain $v(+\infty,\lambda)$. The nonlinear Fourier coefficients $a(\lambda)$ and $b(\lambda)$ are then defined as

$$a(\lambda) = \lim_{t\to\infty} v_1(t,\lambda)e^{j\lambda t}, \quad b(\lambda) = \lim_{t\to\infty} v_2(t,\lambda)e^{-j\lambda t}.$$

Finally, the discrete spectral function is defined on the upper half complex plane $\mathbb{C}^+ = \{\lambda : \Im(\lambda) > 0\}$:

$$\tilde{q}(\lambda_j) = \frac{b(\lambda_j)}{a'(\lambda_j)},\ j = 1,\cdots,N,$$

where the prime $'$ denotes differentiation and λ_j are the isolated zeros of $a(\lambda)$ in $\mathbb{C}^+$, *i.e.*, solutions of $a(\lambda_j) = 0$. The continuous spectral function is defined on the real axis $\lambda \in \mathbb{R}$ as $\hat{q}(\lambda) = b(\lambda)/a(\lambda)$.

III. Discrete Spectrum Modulation Using Darboux Transformation

Let the nonlinear Fourier transform of the signal $q(t)$ be represented by $q(t) \longleftrightarrow (\hat{q}(\lambda), \tilde{q}(\lambda_j))$. When the continuous spectrum $\hat{q}(\lambda)$ is set to zero, the nonlinear Fourier transform consists only of discrete spectral functions $\tilde{q}(\lambda_j)$, *i.e.*, N complex numbers $\lambda_1,\ldots,\lambda_N$ in $\mathbb{C}^+$ together with the corresponding N complex spectral amplitudes $\tilde{q}(\lambda_1),\ldots,\tilde{q}(\lambda_N)$. In this case, the inverse nonlinear Fourier transform can be worked out in closed-form, giving rise to N-soliton pulses [6]. In this section we give a method for the implementation of the inverse NFT at the transmitter when $\hat{q}(\lambda) = 0$.

Multisoliton solutions of the NLS equation can be constructed recursively using the Darboux transformation. The Darboux transformation, originally introduced in the context of the Sturm-Liouville differential equations and later used in nonlinear integrable systems, provides the possibility to construct a solution of a differential equation from another solution [7]. For instance, one can start from the trivial solution $q = 0$ of the NLS equation, and recursively obtain all higher order N-soliton solutions. This approach is particularly well suited for numerical implementation.

Let $x(t,\lambda;q)$ denote a solution of the system

$$x_t = P(\zeta,q)x, \quad x_z = M(\zeta,q)x, \tag{4}$$

for the signal q and complex number $\zeta = \lambda$ (not necessarily an eigenvalue of q), where the P and M are 2×2 matrix operators defined in [Part I]. It is clear that $\tilde{x} = [x_2^*, -x_1^*]^T$ satisfies (4) for $\zeta \to \zeta^*$, and furthermore, by cross-elimination $x_{tz} = x_{zt}$, q is a solution of the integrable equation underlying (4).

The Darboux theorem is stated as follows.

Theorem 1 (Darboux Transformation)**.** *Let $\phi(t,\lambda;q)$ be a known solution of* (4)*, and set $\Sigma = S\Gamma S^{-1}$, where $S = [\phi(t,\lambda), \tilde{\phi}(t,\lambda)]$ and $\Gamma = diag(\lambda,\lambda^*)$. If $v(t,\mu;q)$ satisfies* (4)*, then $u(t,\mu;\tilde{q})$ obtained from the Darboux transform*

$$u(t,\mu;\tilde{q}) = (\mu I - \Sigma)\, v(t,\mu;q), \tag{5}$$

satisfies (4) *as well, for*

$$\tilde{q} = q - 2j(\lambda^* - \lambda)\frac{\phi_2^*\phi_1}{|\phi_1|^2 + |\phi_2|^2}. \tag{6}$$

Furthermore, both q and $\tilde{q}$ satisfy the integrable equation underlying the system (4)*.*

Proof: See Appendix A. ■

Theorem 1 immediately provides the following observations.

1) From $\phi(t,\lambda;q)$ and $v(t,\mu;q)$, we can obtain $u(t,\mu;\tilde{q})$ according to (5). If μ is an eigenvalue of q, then μ is an eigenvalue of $\tilde{q}$ as well. Furthermore, since $u(t,\mu = \lambda;\tilde{q}) = 0$, λ is also an eigenvalue of $\tilde{q}$. It follows that the eigenvalues of $\tilde{q}$ are the eigenvalues of q together with λ.
2) $\tilde{q}$ is a new solution of the equation underlying (4), obtained from q according to (6), and $u(t,\mu;\tilde{q})$ is one of its eigenvectors.

These observations suggest a two-step iterative algorithm to generate N-solitons. Denote a k-soliton pulse with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_k$ by $q(t;\lambda_1,\lambda_2,\cdots,\lambda_k) := q^{(k)}$. The update equations for the recursive Darboux method are given in Table I and illustrated in Figs. 1-2. Note that $v(t,\lambda_j,q^{(k+1)})$ can also be obtained directly by solving the Zakharov-Shabat system (3) for $q^{(k+1)}$. It is however more efficient to update the required eigenvector according to Table I. The algorithm

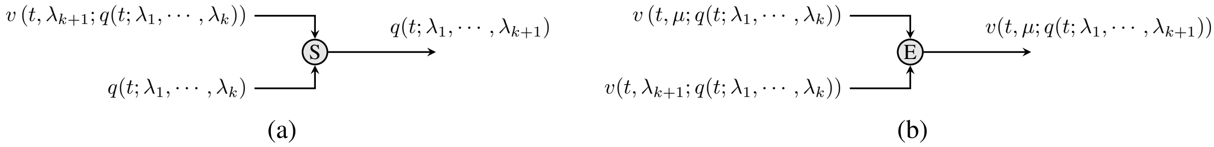

Fig. 1. (a) Signal update. (b) Eigenvector update.

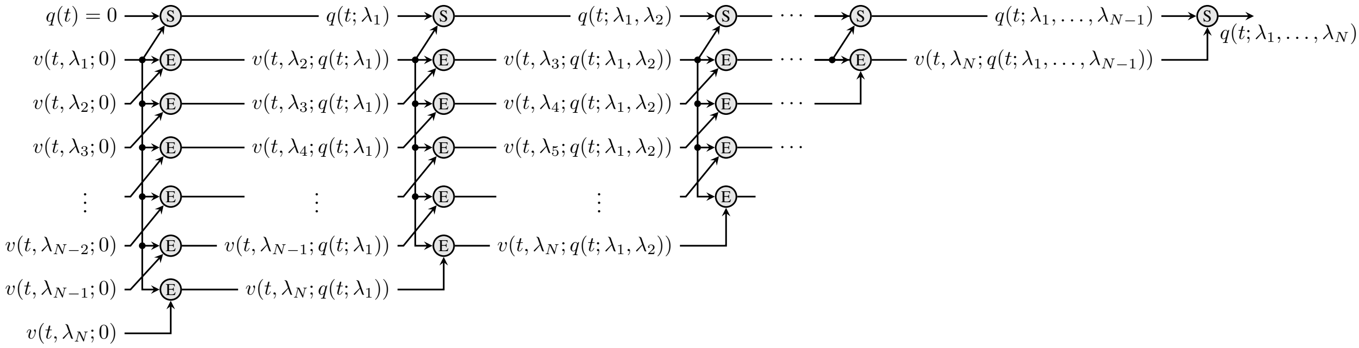

Fig. 2. Darboux iterations for the construction of an N-soliton.

is initialized from the trivial solution $q = 0$. The initial eigenvectors $v(t, \lambda_k; 0)$ in Fig. 2 are chosen to be the (non-canonical) eigenvectors $v(t, \lambda_j) = [Ae^{-j\lambda_j t}, Be^{j\lambda_j t}]^T$. The coefficients A and B control the spectral amplitudes and the shape of the pulses. For a single soliton $A = \exp(j\angle\tilde{q})$ and $B = |\tilde{q}|$.

IV. Statistics of the Spectral Data

In this section we present a simple method to approximate the statistics of the eigenvalues at the receiver in the presence of the noise. Since the noise power in optical fiber systems is quite small compared to the signal power, we follow a perturbation theory approach.

Noise is injected continuously throughout the fiber as a result of the distributed Raman amplification. We discretize the fiber into a large number n of small fiber segments and add lumped noise at the end of each segment. Each such injection of noise then acts as a random perturbation of the initial data at the input of the next segment. We then take the limit $n \to \infty$.

As the signal $q(t)$ at the input of each small segment is perturbed to $q(t) + \epsilon n(t)$, for some small parameter ϵ and (normalized) noise process $n(t)$, the (discrete) eigenvalues and spectral amplitudes deviate slightly from their nominal values. Separating the signal and noise terms, the perturbed v and λ satisfy

$$(L + \epsilon R)v = \lambda v, \qquad R = \begin{pmatrix} 0 & n \\ -n^* & 0 \end{pmatrix}, \tag{7}$$

where R is the matrix containing the noise. The study of the nonlinear Fourier transform in the presence of (small) input noise is thus a perturbation theory of the non-self-adjoint operator $L + \epsilon R$.

Perturbation theory of self-adjoint operators is well-studied (*e.g.*, in quantum mechanics). The Zakharov-Shabat operator (7) is however non-self-adjoint. Unfortunately many useful properties of self-adjoint operators (in particular, the existence of a complete orthogonal basis for the solution space) do not carry over to non-self-adjoint operators. For either type of operator, deterministic perturbation analysis already exists in the literature [8]–[11]. These results, however, are non-stochastic and the distribution of the scattering data is still lacking [12].

For the non-self adjoint operators L, the orthogonality that we require is between the space of left and right eigenvectors of L associated with distinct eigenvalues; that is to say, between eigenvectors of L associated with λ and eigenvectors of the adjoint operator L^* associated with $\mu \neq \lambda^*$.

We use a *small noise approximation* method, expanding unknown variables in noise parameter ϵ

$$v(t) = v^{(0)}(t) + \epsilon v^{(1)}(t) + \epsilon^2 v^{(2)}(t) + \cdots, \tag{8}$$
$$\lambda = \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \cdots. \tag{9}$$

We assume these variables are analytic functions of ϵ so that the above series are convergent. Plugging (8)-(9) into (7) and equating like powers of ϵ, we obtain

$$Lv^{(0)} = \lambda^{(0)}v^{(0)}, \tag{10}$$
$$(L - \lambda^{(0)})v^{(1)} = -(R - \lambda^{(1)})v^{(0)}, \tag{11}$$
$$(L - \lambda^{(0)})v^{(2)} = -(R - \lambda^{(1)})v^{(1)} + \lambda^{(2)}v^{(0)},$$

and so on. The first equation (10) implies that $v^{(0)}$ and $\lambda^{(0)}$ are eigenvalue and eigenvector of the (nominal) operator L. To eliminate v_1 from the second equation, we take the (usual L^2) inner product on both sides of (11) with some vector u; the left hand side of the resulting expression is

$$\begin{aligned}\langle u, (L - \lambda^{(0)})v^{(1)}\rangle &= \langle (L - \lambda^{(0)})^* u, v^{(1)}\rangle \\ &= \langle (L^* - \lambda^{(0)*})u, v^{(1)}\rangle. \end{aligned} \tag{12}$$

TABLE I
UPDATE EQUATIONS FOR THE RECURSIVE DARBOUX METHOD

Eigenvector update (E operation):

$$v_1(t,\lambda_j,q^{(k+1)}) = -\Big\{(\lambda_j-\lambda_{k+1})|v_1(t,\lambda_{k+1},q^{(k)})|^2+(\lambda_j-\lambda_{k+1}^*)|v_2(t,\lambda_{k+1},q^{(k)})|^2\Big\}v_1(t,\lambda_j,q^{(k)}) + (\lambda_{k+1}^*-\lambda_{k+1})v_1(t,\lambda_{k+1},q^{(k)})v_2^*(t,\lambda_{k+1},q^{(k)})v_2(t,\lambda_j,q^{(k)}),$$

$$v_2(t,\lambda_j,q^{(k+1)}) = -\Big\{(\lambda_j-\lambda_{k+1}^*)|v_1(t,\lambda_{k+1},q^{(k)})|^2+(\lambda_j-\lambda_{k+1})|v_2(t,\lambda_{k+1},q^{(k)})|^2\Big\}v_2(t,\lambda_j,q^{(k+1)}) - (\lambda_{k+1}-\lambda_{k+1}^*)v_1^*(t,\lambda_{k+1},q^{(k)})v_2(t,\lambda_{k+1},q^{(k)})v_1(t,\lambda_j,q^{(k)}),$$

for $k=1,\ldots,N$ and $j=k,\ldots,N$.

Signal update (S operation):

$$q^{(k+1)} = q^{(k)} - 2j(\lambda_{k+1}^*-\lambda_{k+1})\cdot\frac{v_1(t,\lambda_{k+1},q^{(k)})v_2^*(t,\lambda_{k+1},q^{(k)})}{\|v(t,\lambda_{k+1},q^{(k)})\|^2}.$$

To have the right-hand side of (12) vanish, we can choose u to be an eigenvector of the adjoint operator L^* associated with an eigenvalue $\mu=\lambda^{(0)*}$, *i.e.*, $(L^*-\lambda^{(0)*})u=0$. Since $L^*(q)=L(-q)$, if $Lv=\lambda^{(0)}v$, it can be verified that $L^*u=\lambda^{(0)}u$ for $u=[v_1,-v_2]=\Sigma_3 v$, where $\Sigma_3=\mathrm{diag}(1,-1)$. Setting $u=u^{(0)}=\Sigma_3 v^{(0)}(t,\lambda^*)$

$$\lambda^{(1)}=\frac{\langle u^{(0)},Rv^{(0)}\rangle}{\langle u^{(0)},v^{(0)}\rangle}.$$

Using similar calculations we obtain $\lambda^{(2)}$

$$\lambda^{(2)}=\frac{\langle u,Rv^{(1)}\rangle}{\langle u,v^{(0)}\rangle}-\lambda^{(1)}\frac{\langle u,v^{(1)}\rangle}{\langle u,v^{(0)}\rangle},$$

and so on.

To summarize, the fluctuations of discrete eigenvalues λ_n are given by

$$\hat{\lambda}_n=\lambda_n+\epsilon\frac{\langle u_n,Rv_n\rangle}{\langle u_n,v_n\rangle}+O(\epsilon^2),\quad n=1,2,\cdots,N.$$

Taking the limit $n\to\infty$, we also have $(\lambda_n)_z=\epsilon\langle u_n,Rv_n\rangle/\langle u_n,v_n\rangle+\mathcal{O}(\epsilon^2)$. It follows that the perturbation of the eigenvalues is distributed, to the first order, according to a zero-mean complex Gaussian distribution.

Remark 1. The NLS equation in the presence of a loss term does not have conserved quantities and consequently is not integrable. We have not explicitly included the effects of the signal attenuation in our model; however spectrum perturbation caused by loss is indeed traded with noise perturbation (noise power is proportional to the loss coefficient); see [2].

V. SOME ACHIEVABLE SPECTRAL EFFICIENCIES USING NFT

To illustrate how the NFT method works, we give two simple examples. The first example is intended to explain the details of transmission and detection using the NFT, while the second example is better optimized for performance.

a) Example 1.: Consider the following signal set with 4 elements:

$$\begin{aligned} S_1 &: \quad 0,\\ S_2 &: \quad \tilde{q}(0.5j)=1,\\ S_3 &: \quad \tilde{q}(0.25j)=0.5,\\ S_4 &: \quad \tilde{q}(0.25j,0.5j)=(1,1). \end{aligned}\tag{13}$$

Table II shows the energy, duration, power and the bandwidth of these signals.

We compare this with a standard on-off keying (OOK) soliton transmission system, consisting of S_1 and S_2. From the signal parameters given in Table II, it follows that the OOK system provides about $\rho_0=0.33$ bits/s/Hz spectral efficiency at $P_0=0.1876$ mW and $R_0=7.42$ Gbits/s data rate on a standard single mode fiber. Note that the noise level is so small compared to the imaginary part of the eigenvalues that this scheme essentially achieves a transmission rate of 2 bits/symbol.

The full constellation defined in (13) has average power $0.46P_0$ and average time duration $1.65T_1$, where P_0 and T_1 are the power and the time duration of the fundamental soliton. The new signal set therefore provides a spectral efficiency of about $\frac{\log 4}{1.65T_1W_0}=1.2121\times\rho_0$ bits/s/Hz and operates at $R=1.2121\times R_0$ for about the same average power $(0.5P_0)$. Note that without S_4 the average power would be higher and in addition the improvement in the spectral efficiency would be slightly smaller compared to the on-off keying system. Signal S_4 is the new signal (a 2-soliton) that goes beyond conventional pulses. Such signals do not cost much in terms of time $\times$ maximum bandwidth product, while they add additional elements to the signal set. These additional signals can generally be best decoded with the help of the nonlinear Fourier transform.

b) Example 2.: To achieve greater spectral efficiencies, a dense constellation in the upper-half complex plane needs to be considered. A spectral constellation with n possible eigenvalues in $\mathbb{C}^+$ (from which k eigenvalues are chosen, $0\le k\le n$) and m possible values for spectral amplitudes

TABLE II
PARAMETERS OF THE SIGNAL SET IN EXAMPLE 1). HERE $E_0=4\times0.5=2$, $T_0=1.763$ AT FWHM POWER, $T_1=5.2637$ (99% ENERGY), $P_0=0.38$ AND $W_0=0.5714$. THE SCALE PARAMETERS ARE $T_0'=25.246$ ps AND $P_0'=0.5$ mW AT DISPERSION 0.5 ps/(nm − km).

signal	energy	duration FWHM	99% duration	power	bandwidth
S_1	0	T_0	T_1	0	W_0
S_2	E_0	T_0	T_1	P_0	W_0
S_3	0.5 E_0	$2T_0$	$2T_1$	$0.25P_0$	$0.5W_0$
S_4	1.5 E_0	$4.25T_0$	$2.58T_1$	$0.58P_0$	$0.5W_0$

provides up to

$$\log\left(\sum_{k=0}^{n}\binom{n}{k}m^k\right) = n\log\left(m+1\right),$$

bits per symbol (fewer if a subset is chosen). One can continue the approach presented in the previous examples by increasing n and m. The NFT receiver architecture presented in [Part I] is fairly simple and is able to decode signals rather efficiently.

Some pulses may have a large peak to average power, large (99%) bandwidth, or large (99%) duration during their propagation; hence, the signal set should be expurgated to avoid such undesirable signals. We have not yet found rules for modulating the spectrum so that such undesirable signals are not generated. For the small examples given here, we can check pulse properties directly; however, appropriate design criteria for the spectral data (particularly the discrete spectral amplitudes) should be developed.

In this simulation, we assume a constellation with 30 points uniformly chosen in the interval $0 \leq \lambda \leq 2$ on the imaginary axis and create all N-solitons, $1 \leq N \leq 6$. We then prune signals with undesirable bandwidth or duration from this large signal set. The remaining multi-solitons are used as carriers of data in the typical fiber system considered earlier. Here a spectral efficiency of 1.5 bits/s/Hz is achieved. For this calculation, we take the maximum pulse width (containing 99% of the signal energy) and the maximum bandwidth of the signal set. Since pulse widths are large, the shift of the signal energy from the symbol period, due to the Gordon-Haus effect, is insignificant here. By increasing n and m, the Gordon-Haus effect is as important as it is for sinc-like pulse transmission and backpropagation.

VI. Conclusions

Motivated by recent studies showing that the spectral efficiency of optical fiber networks asymptotically vanishes at high powers due to the impacts of the nonlinearity [5], we revisited information transmission in such nonlinear systems. The capacity limitation in the prior work is largely due to the use of methods suited for linear systems, such as pulse train transmission and wavelength-division multiplexing, for the nonlinear optical channel. Exploiting the integrability of the NLS equation, we presented a nonlinear frequency-division multiplexing method which directly modulates the non-interacting signal degrees of freedom under the NLS propagation. The distinction with the previous methods is that NFDM is now able to handle the nonlinearity as well, and thus as the the signal power or transmission distance is increased, the new method does not suffer from the deterministic cross talk between signal components, which has severely limited the performance of the prior work. The scheme has numerous other advantages desired in a communication network.

Appendix A
Proof of the Darboux Theorem

Let $\phi(t,\lambda;q)$ be a known eigenvector associated with λ and q, *i.e.*, satisfying $\phi_t = P(q,\lambda)\phi$. Its adjoint $\tilde{\phi}(t,\lambda) = [\phi_2^*, -\phi_1^*]$ satisfies $\tilde{\phi}_t(t,\lambda) = P(q,\lambda^*)\tilde{\phi}(t,\lambda)$. Using this known solution, set $S = [\phi,\tilde{\phi}]$, $\Gamma = \operatorname{diag}(\lambda,\lambda^*)$, and $\Sigma = S\Gamma S^{-1}$. We can verify that $S_t = JS\Gamma + QS$, where $J = \operatorname{diag}(j,-j)$ and $Q = \operatorname{offdiag}(q,-q^*)$. In addition we have $\Sigma_t = [J\Sigma + Q, \Sigma]$.

Given that $\phi(t,\lambda;q)$ is known, the Darboux transformation maps $\{v(t,\mu;q),\tilde{v}(t,\mu;q)\}$ to $\{u(t,\mu;\tilde{q}),\tilde{u}(t,\mu;\tilde{q})\}$ according to

$$U = V\Lambda - \Sigma V,$$

where $V = [v,\tilde{v}]$, $U = [u,\tilde{u}]$, $\Lambda = \operatorname{diag}(\mu,\mu^*)$.

We have $V_t = JV\Lambda + QV$ and

$$\begin{aligned}
U_t &= V_t\Lambda - (\Sigma_t V + \Sigma V_t)\\
&= (JV\Lambda + QV)\Lambda - ([J\Sigma + Q,\Sigma]V + \Sigma(JV\Lambda + QV))\\
&= (JV\Lambda + QV)\Lambda - \Sigma JV\Lambda - \{[J\Sigma + Q,\Sigma] + \Sigma Q\}V\\
&= J(V\Lambda - \Sigma V)\Lambda + J\Sigma V\Lambda - \Sigma JV\Lambda\\
&\quad + QV\Lambda - ([J\Sigma + Q,\Sigma] + \Sigma Q)V\\
&= JU\Lambda + [J,\Sigma]V\Lambda - ([J\Sigma + Q,\Sigma] + \Sigma Q)V + QV\Lambda\\
&= JU\Lambda + [J,\Sigma]V\Lambda - \left(J\Sigma^2 + Q\Sigma - \Sigma J\Sigma\right)V + QV\Lambda\\
&= JU\Lambda + [J,\Sigma]V\Lambda - [J,\Sigma]\Sigma V - Q\Sigma V + QV\Lambda\\
&= JU\Lambda + [J,\Sigma](V\Lambda - \Sigma V) - Q\Sigma V + QV\Lambda\\
&= JU\Lambda + [J,\Sigma]U + Q(V\Lambda - \Sigma V)\\
&= JU\Lambda + (Q + [J,\Sigma])U\\
&= JU\Lambda + \tilde{Q}U,
\end{aligned}$$

where $\tilde{Q} = Q + [J,\Sigma]$. In the same manner we can show that u and $\tilde{u}$ satisfy the M-equation $v_z = M(\lambda,\tilde{q})$ and $\tilde{v}_t = M(\lambda^*,\tilde{q})$.

References

[1] M. I. Yousefi and F. R. Kschischang, "Information transmission using the nonlinear Fourier transform, Part I: Mathematical tools," submitted for publication to the *IEEE Trans. Inf. Theory*, Feb. 2012.

[2] ——, "Information transmission using the nonlinear Fourier transform, Part II: Numerical methods," submitted for publication to the *IEEE Trans. Inf. Theory*, Apr. 2012.

[3] A. Hasegawa and T. Nyu, "Eigenvalue communication," *IEEE J. Lightw. Technol.*, vol. 11, no. 3, pp. 395–399, 1993.

[4] L. F. Mollenauer and J. P. Gordon, *Solitons in Optical Fibers: Fundamentals and Applications*. Academic Press, 2006.

[5] R. J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, "Capacity limits of optical fiber networks," *IEEE J. Lightw. Technol.*, vol. 28, no. 4, pp. 662–701, Feb. 2010.

[6] R. Hirota, *The Direct Method in Soliton Theory*. Cambridge University Press, 2004.

[7] V. Matveev and M. Salle, *Darboux Transformations and Solitons*. Springer-Verlag Berlin, 1991.

[8] V. I. Karpman and V. V. Solov'ev, "A perturbational approach to the two-soliton systems," *Physica D: Nonlinear Phenomena*, vol. 3, no. 3, pp. 487–502, 1981.

[9] V. I. Karpman and E. M. Maslov, "A perturbation theory for the Korteweg-de Vries equation," *Phys. Lett. A*, vol. 60, no. 4, pp. 307–308, 1977.

[10] D. J. Kaup, "A perturbation expansion for the Zakharov-Shabat inverse scattering transform," *SIAM J. Appl. Math.*, vol. 31, no. 1, pp. 121–133, 1976.

[11] V. Konotop and L. Vázquez, *Nonlinear Random Waves*. World Scientific Publishing Company Inc., 1994.

[12] P. Kazakopoulos and A. L. Moustakas, "Nonlinear Schrödinger equation with random Gaussian input: Distribution of inverse scattering data and eigenvalues," *Phys. Rev. E*, vol. 78, no. 1, p. 016603 (7), 2008.