ISIT'2013 1569720357

On the tightness of the generalized network sharing bound for the two-unicast-Z network

Weifei Zeng
Research Laboratory of Electronics,
Massachusetts Institute of Technology
weifei@mit.edu

Viveck Cadambe
Research Laboratory of Electronics,
Massachusetts Institute of Technology
viveck@mit.edu

Muriel Médard
Research Laboratory of Electronics,
Massachusetts Institute of Technology
medard@mit.edu

***Abstract*—We study *two-unicast-Z networks*[1] - two-source two-destination (two-unicast) wireline networks over directed acyclic graphs, where one of the two destinations (say the second destination) is apriori aware of the interfering (first) source's message. For certain classes of two-unicast-Z networks, we show that the rate-tuple $(N, 1)$ is achievable as long as the individual source-destination cuts for the two source-destination pairs are respectively at least as large as N and 1, and the generalized network sharing cut - a bound previously defined by Kamath et. al. - is at least as large as $N + 1$. We show this through a novel achievable scheme which is based on random linear coding at all the edges in the network, except at the GNS-cut set edges, where the linear coding co-efficients are chosen in a structured manner to cancel interference at the receiver first destination.**

I. Introduction

The classical max-flow min-cut theorem states that, for a single-source single-destination wireline network, the *min-cut* - the minimum capacity over all the source-destination cuts - is equal to to the maximum capacity that can be achieved in the network. Subsequently, the min-cut has been identified to be the fundamental bottleneck for multicast networks where there are multiple destinations demanding the same, or a common set of sources [1], [2], and for certain classes of non-multicast networks where the sources or destinations are collocated (See for example [2]). In contrast, for two-unicast networks where the sources and destinations are distributed, the identification of the fundamental network bottlenecks remains a challenging open problem. In fact, the two-unicast problem is the smallest wireline network capacity open problem in terms of the number of sources and number of destinations. It is this problem that is the object of focus of our paper.

The two-unicast problem was studied under the specific context of the feasibility of rate pair $(1, 1)$ in reference [3]. The authors resolved this issue by showing that the feasibility of $(1, 1)$ is equivalent to a graph theoretical quantity - later crystallized to be the value of a minimum *generalized network sharing cut set* - having more than or equal to 2 edges. The generalized network sharing (GNS) cut set is a generalization of the classical concept of the cut set to multiple unicast settings. Loosely speaking, in the context of a two-unicast

This material is based upon work supported by the Air Force Office of Scientific Research (AFOSR) under award No. FA9550-13-1-0023.

[1]The nomenclature is inspired from Z-interference channels in wireless communications, where, like our network, only one destination faces "interference"

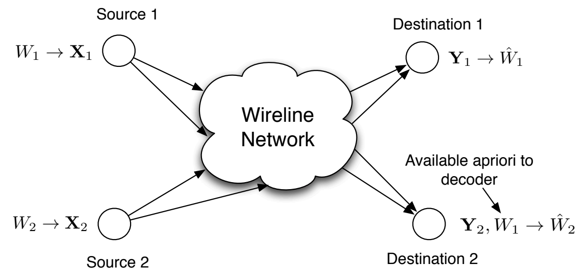

Fig. 1. The *two-unicast-Z* network.

networks, a GNS-cut set is a set of edges that (simultaneously) disconnects (1) the first source from the first destination, (2) the second source from the second destination and (3) the second source from the first destination or the first source from the second destination. Reference [4] showed that, much like a cut set in the context of a single unicast network, the sum of the capacities of the edges associated with any GNS-cut set is an outer bound on the *sum* capacity of a two-unicast network. However, the reference also showed that unlike the min cut bound which is achievable in single unicast networks, in two-unicast networks, the GNS-cut set bound is not tight in general. From the perspective of achievability, developments in interference management have inspired recent progress in two-unicast networks in both wireline and wireless systems. References [5]–[7] studied classes of layered two-unicast *wireless* networks from a degrees-of-freedom perspective. Notably, the technique of interference neutralization [8] was shown to play an integral role in certain wireless two-unicast networks. In addition, interference alignment has been recently used to formulate achievable schemes for wireline two unicast networks in [9]–[12]. However, the looseness of the GNS-cut set bound and the lack of a systematic approach to constructing achievable schemes is highlighted by a shortage of capacity results in the context of multi-source multi-sink networks, and specifically in the context of two-source two-sink networks. The motivation of our work is to find classes of networks where the GNS-cut set bound is tight.

A. Summary of Contributions

In this paper, we focus on *two-unicast-Z networks* over directed acyclic graphs, a communication scenario where one of the two destinations is apriori aware of the interfering source message (See Fig. 1). Since the two-unicast-Z network can be interpreted as a special class of two-unicast networks where there is an infinite capacity link from one of the source to the other destination, the GNS-cut set bound is an upper bound on the sum-capacity of this network. We study the feasibility of the rate pair $(N,1)$ in this network. for the class of two-unicast-Z networks where, there is a minimum GNS-cut set such that there is no communication between the GNS-cut set edges. In such networks, when a set of "richness" conditions is satisfied, the GNS-cut set bound along with the two individual source-destination cut-set constraints dictate the achievability of $(N,1)$. In other words, if the GNS-cut set bound is greater than or equal to $N+1$, then the rate point $(N,1)$ is achievable subject to the richness conditions specified in our main result (Theorem 2). Our achievable scheme is based on applying random coding on all edges of the network, except those in a minimum GNS-cut set. The coefficients associated with these GNS-cut set edges are chosen to neutralize the second source at the first receiver. Through a careful inspection of the algebraic properties of the transfer matrices, it is shown that the richness conditions imply that the neutralization can be carried out without cancelling the transfer matrices associated with the desired streams at the receivers. We start with a description of our system model before proceeding to our main result.

II. System Model

Consider a directed acyclic graph denoted by $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ denotes the set of edges. We allow multiple edges between the vertices, so $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}\times\mathbb{Z}_+$ where $\mathbb{Z}_+$ denotes the set of positive integers. An edge $e\in\mathcal{E}$ is denoted as (u,v,i), where $u,v\in\mathcal{V}, i\in\mathbb{Z}_+$. For an edge $e=(u,v,i)$, we denote $v=\text{Head}(e)$ and $u=\text{Tail}(e)$. Given $v\in\mathcal{V}$ we denote $\text{In}(v)=\{e\in\mathcal{E}:\text{Head}(e)=v,\}$, and $\text{Out}(v)=\{e\in\mathcal{E}:\text{Tail}(e)=v\}$. The set $\mathcal{S}=\{s_1,s_2\}$, where $s_1,s_2\in\mathcal{V}$ denotes the set of sources in the network. Similarly, the set $\mathcal{T}=\{t_1,t_2\}$ where $\{t_1,t_2\}\in\mathcal{V}$ denotes the set of destinations in the network. We assume that $\mathcal{S}$ and $\mathcal{T}$ are disjoint. Without loss of generality, we assume that there are no edges coming into the sources and no edges going out of the destinations, i.e., $\text{In}(s_i)=\phi$ and $\text{Out}(t_i)=\phi$ for $i=1,2$. There are two independent messages W_1,W_2 - one corresponding to each source - are encoded respectively using codebooks of rates R_1,R_2. In our model, each edge represents an error-free link with unit capacity. The codeword on the edges in $\text{Out}(s_i)$ is a function of W_i and the codeword on any other edge e is a function of codewords on the edges $\{e^{'}:e^{'}\in\text{In}(\text{Tail}(e))\}$. For simplicity, we assume that there is no delay associated with any of the links although the results of this paper are valid even if there are delays in the network. There are two decoders, one associated with each destination. In the two-unicast-Z network, the decoder at t_1 intends to resolve message W_1 using the codewords associated with the edges in $\text{In}(t_1)$. The decoder at t_2 intends to resolve W_2 using its apriori side information W_1 and the codewords associated with edges in $\text{In}(t_2)$.

A rate pair (R_1,R_2) is *achievable* if, for every $\epsilon>0,\delta>0$, there exists a coding scheme which encodes message W_i at rate $R_i-\delta_i$, for some $0\le\delta_i\le\delta$, such that the average probability of its decoding error is smaller than ϵ. The capacity region is the closure of the set of all achievable rate pairs.

Cut set and GNS-cut set: For any sets $\mathcal{C}\subseteq\mathcal{E}, A\subseteq\mathcal{V}, B\subseteq\mathcal{V}$, we say $\mathcal{C}$ is an $A-B$ cut set if there exists no directed path from any vertex in A to any vertex in B on the graph $(\mathcal{V},\mathcal{E}\backslash\mathcal{C})$. We define function $c^{\mathcal{G}}(A;B)\triangleq\min_{\mathcal{C}}\{|\mathcal{C}|:\mathcal{C}\text{ is a }A-B\text{ cut set}\}^2$. For convenience, we mildly abuse notation when A, and/or B is a singleton; we write $c(\{v\};\{w\})$ as simply $c(v;w)$.

In the context of the two-unicast-Z networks studied in this paper, a *GNS-cut set* is defined as a set $\mathcal{Q}\subseteq\mathcal{E}$, such that $\mathcal{Q}$ is a s_1-t_1 cut-set, a s_2-t_2 cut-set, and, a s_2-t_1 cut-set. The smallest GNS-cut set denoted as $c^{\mathcal{G}}_{\text{GNS}}$ is defined as $c^{\mathcal{G}}_{\text{GNS}}=\min_{\mathcal{Q}\subseteq\mathcal{E}}\{|\mathcal{Q}|:\mathcal{Q}\text{ is a GNS-cut set}\}^2$. The subscript $\mathcal{G}$ is dropped from $c^{\mathcal{G}}_{\text{GNS}}$ when there is no ambiguity. Before proceeding to describing our main results, the following theorem shown in [4] is restated here for completeness.

Theorem 1 (Theorem 2 in [4])**.** *A rate pair (R_1,R_2) that is achievable in the two-unicast-Z network satisfies*

$$R_1\le c(s_1;t_1)\quad,\quad R_2\le c(s_2;t_2)\tag{1}$$
$$R_1+R_2\le c_{GNS}\tag{2}$$

III. Main Results

The following notation will be useful to describe our results. In our network graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, if $u,v\in\mathcal{V}$, then $u\rightsquigarrow v$ indicates that u communicates with v, i.e. there is a path from u to v on $\mathcal{G}$. $u\not\rightsquigarrow v$ means that u does not communicate with v on $\mathcal{G}$. For $i,j\in\mathcal{E}$, $i\rightsquigarrow j$ is equivalent to $\text{Head}(i)\rightsquigarrow\text{Tail}(j)$.

Theorem 2. *Consider a two-unicast-Z network with* $c(s_1;t_1)\ge N$, $c(s_2;t_2)\ge 1$ *and* $c_{GNS}\ge N+1$. *If,*

- *there exists a minimum GNS-cut set $\mathcal{Q}$ such that, $\forall e_i,e_j\in\mathcal{Q}$, $e_i\not\rightsquigarrow e_j$, i.e., the GNS-cut set edges do not communicate with each other.*
- *Let $\tilde{\mathcal{G}}=\mathcal{V}\cup\{\tilde{t}_1\},\tilde{\mathcal{E}}$ be a graph with $\mathcal{G}$ appended with a virtual node $\tilde{t}_1$ such that there are N incoming edges from t_1 to $\tilde{t}_1$. Denoting $\mathcal{U}=\{e\in\mathcal{Q}:s_2\rightsquigarrow e\}$, Suppose that the following two conditions are satisfied,*
  - $c^{\tilde{\mathcal{G}}}(\mathcal{U};\tilde{t}_1)=|\mathcal{U}|-1$,
  - *if $\mathcal{U}_0$ is a proper subset of $\mathcal{U}$, i.e. $\mathcal{U}_0\subsetneq\mathcal{U}$, then* $c^{\tilde{\mathcal{G}}}(\mathcal{U}_0;\tilde{t}_1)=|\mathcal{U}_0|$.

Then, the rate pair of $(N,1)$ is achievable.

Since the c_{GNS} is an outer bound on the sum capacity of the network, if $c_{\text{GNS}}=N+1$ and the conditions of the above network are satisfied, then, the point $(N,1)$ is sum-capacity optimal. The proof of achievability, which is the main technical result of this paper, is placed in Section IV. Here, we make some initial observations to help parse and interpret the above result. We shall also illustrate the result via an example.

A. Discussion

Richness in the Network: The conditions of the theorem can be interpreted as the requirement of a *richness* constraint on the network. It can be illustrated with the following corollary.

[2] When the graph in consideration can be gleaned unambiguously from the context, we drop the superscript $\mathcal{G}$.

Corollary 1. *For a two-unicast-Z network, satisfying $c(s_1;t_1) \geq N, c(s_2;t_2) \geq 1$ and $c_{GNS} \geq N+1$. The rate pair $(N,1)$ is achievable, if*

- *there exists a minimum GNS-cut set $\mathcal{Q}$, such that s_2 communicates with all the edges in $\mathcal{Q}$,*
- *from any subset of m edges in the GNS-cut set $\mathcal{Q}$, there are m edge independent paths to destination t_1.*

The above follows from Theorem 2 because $c^{\tilde{\mathcal{G}}}(\mathcal{U};\tilde{t}_1) = c^{\tilde{\mathcal{G}}}(\mathcal{Q};\tilde{t}_1) = N = |\mathcal{U}|-1$, since the N edges from t_1 to $\tilde{t}_1$ forms a cut set that separates $\mathcal{U}$ from $\tilde{t}_1$. In addition, the condition $c^{\mathcal{G}}(s_1;t_1) \geq N$ implies $c^{\tilde{\mathcal{G}}}(\mathcal{Q};\tilde{t}_1) \geq N$ because any set that is a *minimum* $s_1 - \tilde{t}_1$ cut set either separates s_1 from t_1 or includes the N edges from t_1 to $\tilde{t}_1$. The second condition above can be interpreted as the existence of sufficient diversity in the network, since, for any subset of GNS-cut set edges, the number of paths between this set and the first destination, t_1 is as large as possible.

Distributed versus Joint Processing at the GNS-cut set edges: Consider the following theorem.

Theorem 3. *Consider a two-unicast network there exists a minimum GNS-cut set whose all the edges have a common head vertex, i.e., a minimum GNS-cut set $\mathcal{Q}$ and a vertex $v \in \mathcal{V}$ where $Head(e) = v, \forall e \in \mathcal{Q}$. Then, the capacity region of this two-unicast network is characterized by (1)-(2).*

Proof: See Appendix. ■

Note that the theorem applies to the general two-unicast networks, not just two-unicast-Z networks. The implication of the above theorem is worth examining here. Consider a two-unicast-Z network where $c_{\text{GNS}} = N+1$ and $c(s_1;t_1) \geq N$ and $c(s_2;t_2) \geq 1$, and there exists a minimum GNS-cut set edges whose edges do not communicate with each other. Now, suppose that, hypothetically speaking, all the GNS-cut set edges could process the information *jointly*, then, the above theorem implies that the rate $(N,1)$ is achievable and is the sum capacity of the network. Theorem 2 implies that, under the specified (richness) conditions, *there is no loss in distributed processing at the GNS-cut set edges*.

B. An Example

Consider the two-unicast-Z network shown in Figure 2, where $\mathcal{Q} = \{g_1, g_2, g_3\}$ is a minimum GNS-cut set. It can be verified that all the conditions in Corollary 1 are satisfied. We demonstrate that the rate pair $(R_1,R_2) = (2,1)$ is achievable by constructing a coding solution over the field $\mathbb{F}_5 = \{0,1,2,3,4\}$.

The symbols sent on the edges are shown in the Figure, in which X_{11}, X_{12} and X_2 are the input symbols from t_1 and t_2 respectively, while Z_i is the symbol transmitted on g_i. Consider the following encoding at node v_i for edge g_i,

$$Z_1 = 4X_2\,, \quad Z_2 = X_{12} + 2(X_{11}+X_2)\,, \quad Z_3 = 4X_{11} + 4X_2.$$

Note that t_2 receives Z_2 while t_1 receives Y_{11} and Y_{12},

$$Y_{11} = Z_1 + Z_2 + Z_3 = X_{11} + X_{12}\,, \quad Y_{12} = Z_2 + 2Z_3 = X_{12}$$

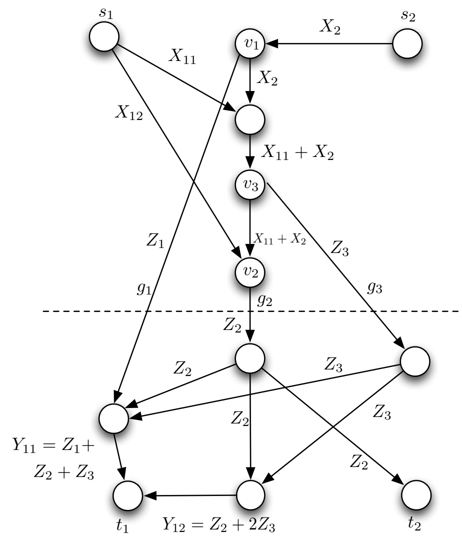

Fig. 2. An example graph in which conditions in Theorem 2 are satisfied by : GNS-cut set $\mathcal{Q} = \{g_1,g_2,g_3\} = \mathcal{U}$ and $c(\mathcal{U};t_1) = |\mathcal{U}|$, $\forall \mathcal{U}_0 \subset \mathcal{U}$, $c(\mathcal{U}_0;t_1) = |\mathcal{U}_0|$. The vector besides a node indicates the local coding vector at the node.

Since in the two-unicast-Z network, destination node t_2 is aware of X_{11} and X_{12} apriori, it can recover X_2 by subtracting the interference terms in Z_2. For destination t_1, clearly, both X_{11} and X_{12} can be recovered from Y_{11} and Y_{12} under this coding scheme. Therefore, $(R_1,R_2) = (2,1)$ is achievable and the sum rate capacity is achieved.

In fact, it can be readily verified neither random linear network coding nor routing can achieve the rate-pair $(2,1)$. Indeed, its achievability is associated with the careful design of network coding co-efficients. *The contribution of our approach is to create, over a sufficiently large field, a systematic way of creating coding solutions for the class of graphs of Theorem 2*. We proceed to describe our approach next.

IV. Achievable Scheme for Theorem 2

A. Proof of Theorem 2

Let $\tilde{t}_1$ and t_2 be the destinations on $\tilde{\mathcal{G}}$ for source s_1 and s_2 respectively. It can be verified that $\mathcal{Q}$ - a minimum GNS-cut set for $\mathcal{G}$ - is a minimum GNS-cut set for the extended graph $\tilde{\mathcal{G}}$ as well. Moreover, since the received codewords at $\tilde{t}_1$ are functions of received codewords at t_1, t_1 is able to decode any message that can be decoded at $\tilde{t}_1$. Therefore, it suffices to design a coding scheme to achieve rate pair $(N,1)$ on the extended graph $\tilde{\mathcal{G}}$. We consider exclusively the graph $\tilde{\mathcal{G}}$ for the rest of this section.

At a high level, our achievable scheme uses random linear coding at all the edges in the network, except the GNS-cut set edges in $\mathcal{Q}$. To achieve rate pair $(N,1)$, source s_1 sends N data streams, source s_2 sends 1 data stream. The coding co-efficients at the GNS-cut set edges are chosen to ensure that stream from source s_2 is nulled at all N edges associated with destination $\tilde{t}_1$. This nulling has to be done in a manner that ensures that none of the N desired streams are nulled at destination $\tilde{t}_1$, i.e., they have to appear with full rank of N. Note that there are at least $N+1$ edges in the minimum

GNS-cut set $\mathcal{Q}$. We demonstrate that nulling the stream of s_2 at N edges of $\tilde{t}_1$ corresponds to solving N linear equations in $|\mathcal{Q}| > N$ variables, i.e., solving an underconstrained linear system. Under the richness conditions of the theorem, we will show that a solution to this linear system suffices to achieve the rate point $(N, 1)$. We begin with a brief description of the linear framework we use, which is based on [2].

Our achievable scheme uses a scalar linear coding achievable scheme over Galois field $\mathbb{F}_q$, which is chosen to be sufficiently large for purposes that will be described shortly. We assume that encoded symbol on edge $e \in \tilde{\mathcal{E}}$ denoted as χ_e can be expressed as

$$\chi_e = \sum_{e' \in \text{In}(\text{Tail}(e))}^{k} f_{e_k, e'} \chi_{e_k},$$

where f_{e_i,e_j} represents the linear coding co-efficient on to edge e_j from edge e_i. Note that because of adjacency matrix we have $f_{e_i,e_j} = 0$ if $(\text{Head}(e_i), \text{Tail}(e_j)) \notin \tilde{\mathcal{E}}$. With linear coding, the relation between the inputs symbols $\mathbf{X}_i$ and the outputs symbols $\mathbf{Y}_i$ can be described as

$$\mathbf{Y}_1 = \mathbf{X}_1\mathbf{A}_1 + \mathbf{X}_2\mathbf{A}_2, \tag{3}$$
$$\mathbf{Y}_2 = \mathbf{X}_1\mathbf{B}_1 + \mathbf{X}_2\mathbf{B}_2, \tag{4}$$

where, for $i = 1, 2,$, matrix $\mathbf{A}_i$ is a $|\text{Out}(s_i)| \times |\text{In}(\tilde{t}_1)|$ matrix and similarly $\mathbf{B}_i$ is a $|\text{Out}(s_i)| \times |\text{In}(t_2)|$. The matrices $\mathbf{A}_i, \mathbf{B}_i$ can be explicitly expressed as submatrices of the matrix $\mathbf{M} \triangleq (\mathbf{I} - \mathbf{F})^{-1}$, where $\mathbf{F}$ - the *adjacency coding matrix* - is a matrix whose (i, j)th entry is f_{e_i,e_j}. We assume an ancestral topological ordering so that the adjacency matrix, and hence the matrix $\mathbf{F}$ is upper triangular (because our graph is directed and acyclic).

$$\mathbf{A}_i = \mathbf{M}_{\text{Out}(s_i),\text{In}(\tilde{t}_1)}, \qquad \mathbf{B}_i = \mathbf{M}_{\text{Out}(s_i),\text{In}(t_2)}, \tag{5}$$

where the notation $\mathbf{M}_{P,Q}$ is used to denote the submatrix of $\mathbf{M}$ formed by rows indexed by P and columns indexed by Q, according to their topological order. Next, we aim to create a coding solution for the two-unicast-Z networks considered, i.e., design $\mathbf{F}$, such that $\mathbf{A}_2 = \mathbf{0}$ and $\text{rank}(\mathbf{A_1}) = N$ and $\text{rank}(\mathbf{B_2}) = 1$.

Design of Coding Coefficients: Since the GNS-cut set edges do not communicate with each other, without loss of generality, we can assume that the GNS-cut set are $|\mathcal{Q}|$ consecutive edges on the ancestral topological ordering, indexed from $h + 1$ to $h + |\mathcal{Q}|$. Such an ancestral order always exists, since none of the edges in $\mathcal{Q}$ is an ancestor of another edge in $\mathcal{Q}$. Consider a random coding solution to the network, i.e., consider matrix $\tilde{\mathbf{F}}$ chosen randomly from a sufficiently large field. We choose coding co-efficients for our solution as

$$\mathbf{f}_j = \begin{cases} \tilde{\mathbf{f}}_j & j \notin \{h+1, h+2, \ldots, h+|\mathcal{Q}|\} \\ \lambda_j \tilde{\mathbf{f}}_j & j \in \{h+1, h+2, \ldots, h+|\mathcal{Q}|\} \end{cases}$$

where $\tilde{\mathbf{f}}_j$ and $\mathbf{f}_j$ denote the jth columns of $\tilde{\mathbf{F}}$ and $\mathbf{F}$ respectively. The above operation can be visualized as a virtual node sitting in the middle of the ith GNS-cut set edge (i.e. edge indexed $j = i + h$ in $\tilde{\mathcal{G}}$), and scaling the input by λ_{h+i}.

Let the $\mathcal{G}_1$ be the subgraph of $\tilde{\mathcal{G}}$, induced by the edges indexed by $\{1, 2, \ldots, h + |\mathcal{Q}|\}$, and $\mathcal{G}_2$ be the subgraph of $\tilde{\mathcal{G}}$ induced by edges indexed by $\{h+1, h+2, \ldots, |\tilde{\mathcal{E}}|\}$. We denote the network transfer matrix of $\mathcal{G}_i$ using $\mathbf{M}^{(i)}$ - so $\mathbf{M}^{(1)}$ is a $(h+|\mathcal{Q}|) \times (h+|\mathcal{Q}|)$ matrix and $\mathbf{M}^{(2)}$ is a $(|\tilde{\mathcal{E}}|-h)) \times (|\tilde{\mathcal{E}}|-h)$. Define $\mathbf{H}_i$ and $\overline{\mathbf{H}}_i$, $i = 1, 2$ as follows,

$$\mathbf{H}_i = \mathbf{M}^{(1)}_{\text{Out}(s_i),\mathcal{Q}}, \quad \overline{\mathbf{H}}_1 = \mathbf{M}^{(2)}_{\mathcal{Q},\text{In}(\tilde{t}_1)}, \quad \overline{\mathbf{H}}_2 = \mathbf{M}^{(2)}_{\mathcal{Q},\text{In}(t_2)}.$$

That is, $\mathbf{H}_i$ is the transfer matrix from source s_i to the GNS-cut set edges $\mathcal{Q}$ on $\mathcal{G}_1$, while $\overline{\mathbf{H}}_1$ is the transfer matrix from $\mathcal{Q}$ to destination $\tilde{t}_1$ on $\mathcal{G}_2$, and $\overline{\mathbf{H}}_2$ is the transfer matrix from $\mathcal{Q}$ to t_2 on $\mathcal{G}_2$. The following lemma is useful in the construction of our coding solution.

Lemma 1. *Given a GNS-cut set $\mathcal{Q} = \{g_1, \ldots, g_{|\mathcal{Q}|}\}$, such that $\forall g_i, g_j \in \mathcal{Q}, g_i \nrightarrow g_j$, and let $\mathcal{G}_i, \mathbf{H}_i, \overline{\mathbf{H}}_i, i = 1, 2$ be defined as above. If the transmitted symbol on the g_i is scaled by λ_{h+i}, then*

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{D} \\ \mathbf{A}_2 & \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} \overline{\mathbf{H}}_1 & \overline{\mathbf{H}}_2 \end{bmatrix}, \tag{6}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{B}_2$ are as defined in (5). *$\mathbf{D}$ is some Out$(s_1) \times$ In(t_2) matrix. $\mathbf{\Lambda}$ = diag$(\lambda_{h+1}, \lambda_{h+2}, \ldots, \lambda_{h+|\mathcal{Q}|})$ is a $|\mathcal{Q}| \times |\mathcal{Q}|$ diagonal matrix.*

The proof is placed in the extended version [13] of the paper. Since $\text{In}(\tilde{t}_1) = N$, $\mathbf{H}_2$ is a length $|\mathcal{Q}|$ row vector. Let $\mathbf{H}_2 = \begin{bmatrix} c_1 & c_2 & \ldots & c_{|\mathcal{Q}|} \end{bmatrix}$ and Let

$$\overline{\mathbf{H}}_1 = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_{|\mathcal{Q}|} \end{bmatrix},$$

where $\mathbf{r}_i$ is a length N row vector denotes the i-th row of $\overline{\mathbf{H}}_1$. Note that c_i indicates the transfer coefficient between s_2 and edge $i + h$ in $\mathcal{Q}$, while $\mathbf{r}_i$ represents the transfer matrix between GNS-cut set edge $i + h$ to the N sink edges at $\tilde{t}_1$.

To set $\mathbf{A}_2 = \mathbf{0}$, we need $\mathbf{H}_2\mathbf{\Lambda}\overline{\mathbf{H}}_1 = \mathbf{0}$. Equivalently, we need to design $\lambda_{h+1}, \lambda_{h+2}, \ldots, \lambda_{h+|\mathcal{Q}|}$, so that

$$\sum_{i=1}^{|\mathcal{Q}|} \lambda_{h+i} c_i \mathbf{r}_i = \mathbf{0}. \tag{7}$$

We have the following lemma.

Lemma 2. *In a sufficiently large field, given that the richness conditions in Theorem 2 are satisfied, using random linear network coding on $\mathcal{G}_i, i = 1, 2$, Equation* (7) *has at least one solution. Furthermore, all its solutions satisfy that $\lambda_{h+i} \neq 0, i = 1, 2, \ldots, |\mathcal{Q}|$.*

Proof: From the definition of $\mathcal{U}$, (7) is equivalent to

$$\sum_{i+h \in \mathcal{U}} \lambda_{h+i} c_i \mathbf{r}_i = \mathbf{0}, \tag{8}$$

and for $i + h \notin \mathcal{U}$, λ_{i+h} can be set to an arbitrary non-zero value. Since $c(\mathcal{U}; \tilde{t}_1) = |\mathcal{U}| - 1$ and random coding is used in $\mathcal{G}_2$, the maximum rank of the transfer matrix between $\mathcal{U}$ and $\tilde{t}_1$ is $|\mathcal{U}| - 1$. Equivalently, the row vectors $\{\mathbf{r}_i, i + h \in \mathcal{U}\}$ are linearly dependent. Therefore, (8) and thus, (7) has at least one non-trivial solution.

For any subset $\mathcal{U}_0$ of $\mathcal{U}$, with the richness conditions, we have $c(\mathcal{U}_0;\tilde{t}_1) = |\mathcal{U}_0|$. Hence, the vectors $\{\mathbf{r}_i, i+h \in \mathcal{U}_0\}$, which are rows of transfer matrix from $\mathcal{U}_0$ to $\tilde{t}_1$ are linearly independent. Consequently, any solution of (8) will satisfy $\lambda_{h+i} \neq 0, \forall i+h \in \mathcal{U}$. Otherwise, there exists a subset $\mathcal{Z} \subseteq \mathcal{U}$, such that the rows $\{\mathbf{r}_i,\ i+h \in \mathcal{Z}\}$ are linearly dependent, which is a contradiction. ■

Finally, we show that given a solution for (7), with $\lambda_{h+i} \neq 0\ \forall i = 1,2,\ldots,|\mathcal{Q}|$, we have $\text{rank}(\mathbf{A}_1) = \text{rank}(\mathbf{H}_1\mathbf{\Lambda}\overline{\mathbf{H}}_1) = N$, and $\mathbf{B}_2 \neq 0$. To show this, recall that if $\mathbf{P}$ is a $\ell\times m$ matrix, $\mathbf{Q}$ is a $m\times n$ matrix, and rank $(\mathbf{P}) = m$, then rank $(\mathbf{PQ})$ = rank $(\mathbf{Q})$. With random coding in a sufficiently large field, the rank of the transfer matrix between both sources and the GNS-cut set edges achieves the min cut with high probability, i.e.

$$\text{rank}\left(\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}\right) = c(\{s_1;s_2\};\mathcal{Q}) = |\mathcal{Q}|,$$

The last equality holds since any $\{s_1,s_2\}-\mathcal{Q}$ cut $\mathcal{Q}'$ is itself a GNS-cut set. Thus, $|\mathcal{Q}'| \geq |\mathcal{Q}|$ by the assumption that $\mathcal{Q}$ is a minimum GNS-cut set. Also, since $c(\mathcal{Q};\tilde{t}_1) = N$, with random coding,

$$\text{rank}\left(\left[\overline{\mathbf{H}}_1\right]\right) = N\ .$$

Now, note that $\lambda_{h+i} \neq 0,\ \forall\, i, \text{rank}\,(\mathbf{\Lambda}) = |\mathcal{Q}|$, and $\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}$ is a $(\text{Out}(s_1)+\text{Out}(s_2))\times|\mathcal{Q}|$ matrix with rank $|\mathcal{Q}|$. We have,

$$\text{rank}\left(\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}\mathbf{\Lambda}\overline{\mathbf{H}}_1\right) = \text{rank}\left(\left[\mathbf{\Lambda}\overline{\mathbf{H}}_1\right]\right) = \text{rank}\left[\overline{\mathbf{H}}_1\right] = N$$

On the other hand, as λ_{h+i} satisfies (7), we have,

$$N = \text{rank}\left(\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}\mathbf{\Lambda}\overline{\mathbf{H}}_1\right) = \text{rank}\left(\begin{bmatrix}\mathbf{H}_1\mathbf{\Lambda}\overline{\mathbf{H}}_1\\ \mathbf{0}\end{bmatrix}\right),$$

therefore, $\text{rank}\left(\left[\mathbf{H}_1\mathbf{\Lambda}\overline{\mathbf{H}}_1\right]\right) = N$. Finally, since $c(s_2;t_2) \geq 1$, there exists at least one path from s_2 to t_2, $\mathbf{B}_2 \neq 0$. ■

V. Discussion

We studied the feasibility of $(N,1)$ over a class of two-unicast-Z networks. A natural question of interest is whether the feasibility of arbitrary rate pairs can be resolved for this class. This question is arguably more challenging than the problem solved here since an application of the approach of this paper leads to the requirement of resolving over-constrained equations (in the scaling of the GNS-cut set edges). However, the over-constrained nature in itself does not necessarily make it infeasible as demonstrated recently in [14]. Among other open research questions motivated by this work, those of particular interest are the exploration of analogous results for (non-layered) wireless networks, and the study of two-unicast-Z networks with one-sided source co-operation.

Appendix

A. Proof of Theorem 3

We only need to show that the region is achievable. To see this, since $\mathcal{Q}$ is a minimum GNS-cut set and $\text{Head}(e) = v, \forall\, e \in \mathcal{Q}$, we have, for $i = 1,2$,

$$c(s_i;v) \geq c(s_i;t_i) \quad , \quad c(\{s_1,s_2\};v) = c_{\text{GNS}} \qquad (9)$$

$$c(v;t_i) \geq c(s_i;t_i) \quad , \quad c(v;\{t_1,t_2\}) \geq c_{\text{GNS}} \qquad (10)$$

Ignore all the edges in $\mathcal{E}$ that do not communicate or can not be communicated from any edges in $\mathcal{Q}$. Consider a scheme where v recovers all messages transmitted by s_1 and s_2, before sending them to t_1 and t_2. From Theorem 8 in [2] and (9), it is clear that v can recover all the message sent by s_1 and s_2, if R_1 and R_2 satisfies (1)-(2), since $R_1 \leq c(s_1;v), R_2 \leq c(s_2;v)$ and $R_1+R_2 \leq c(\{s_1,s_2\};v)$.

Now, suppose v acts as a source and transmits message W_1 and W_2 at rate R_1' and R_2' to t_1 and t_2. For this disjoint multicast scenario, from Theorem 9 in [2], (R_1', R_2') is achievable, if $R_1' \leq c(v;t_1), R_2' \leq c(v;t_2)$ and $R_1'+R_2' \leq c(v;\{t_1,t_2\})$. Simply set $R_1' = R_1$ and $R_2' = R_2$ and the conditions are satisfied because of (10). Therefore, the overall rate pair (R_1,R_2) characterized by (1)-(2) can be achieved. ■

References

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1204–1216, Jul. 2000.

[2] R. Koetter and M. Médard, "An algebraic approach to network coding," in *IEEE/ACM Trans. Networking*, vol. 11, pp. 782–295, 2003.

[3] C.-C. Wang and N. B. Shroff, "Pairwise intersession network coding on directed networks.," *IEEE Transactions on Information Theory*, vol. 56, no. 8, pp. 3879–3900, 2010.

[4] S. Kamath, D. Tse, and V. Anantharam, "Generalized network sharing outer bound and the two-unicast problem," in *International Symposium Network Coding (NetCod)*, pp. 1 –6, July 2011.

[5] T. Gou, S. Jafar, C. Wang, S.-W. Jeon, and S.-Y. Chung, "Aligned interference neutralization and the degrees of freedom of the 2 2 2 interference channel," *IEEE Transactions on Information Theory*, vol. 58, no. 7, pp. 4381–4395, 2012.

[6] I. Shomorony and A. Avestimehr, "Two-unicast wireless networks: Characterizing the degrees of freedom," *IEEE Transactions on Information Theory*, vol. 59, no. 1, pp. 353–383, 2013.

[7] I.-H. Wang, S. U. Kamath, and D. N. C. Tse, "Two unicast information flows over linear deterministic networks," *CoRR*, vol. abs/1105.6326, 2011. http://arxiv.org/abs/1105.6326.

[8] S. Mohajer, S. Diggavi, C. Fragouli, and D. Tse, "Approximate capacity of a class of Gaussian interference-relay networks," *Information Theory, IEEE Transactions on*, vol. 57, no. 5, pp. 2837–2864, 2011.

[9] S. Huang and A. Ramamoorthy, "An achievable region for the double unicast problem based on a minimum cut analysis," *CoRR*, vol. abs/1111.0595, 2011. http://arxiv.org/abs/1111.0595.

[10] W. Zeng, V. Cadambe, and M. Medard, "An edge reduction lemma for linear network coding and an application to two-unicast networks," in *50th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 509–516, 2012.

[11] A. Das, S. Vishwanath, S. Jafar, and A. Markopoulou, "Network coding for multiple unicasts: An interference alignment approach," in *2010 IEEE International Symposium on Information Theory Proceedings (ISIT)*, pp. 1878 –1882, June 2010.

[12] C. Meng, A. Ramakrishnan, A. Markopoulou, and S. Jafar, "On the feasibility of precoding-based network alignment for three unicast sessions," in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*, pp. 1907 –1911, July 2012.

[13] W. Zeng, V. R. Cadambe, and M. Médard, "On the tightness of the generalized network sharing bound for the two-unicast-z network." Available on the authors' website: http://goo.gl/O3Qfj.

[14] I. Shomorony and A. S. Avestimehr, "Degrees of freedom of two-hop wireless networks: "everyone gets the entire cake"," *CoRR*, vol. abs/1210.2143, 2012. http://arxiv.org/abs/1210.2143.