Exact Capture Probability Analysis of GSC Receivers over i.n.d. Rayleigh Fading Channels

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***Abstract*—A closed-form expression of the capture probability of generalized selection combining (GSC) RAKE receivers was introduced in [1]. The idea behind this new performance metric is to quantify how the remaining set of uncombined paths affects the overall performance both in terms of loss in power and increase in interference levels. In this previous work, the assumption was made that the fading is both independent and identically distributed from path to path. However, the average strength of each path is different in reality. In order to derive a closed-form expression of the capture probability over independent and non-identically distributed (i.n.d.) fading channels, we need to derive the joint statistics of ordered non-identical exponential variates. With this motivation in mind, we first provide in this paper some new order statistics results in terms of both moment generating function (MGF) and probability density function (PDF) expressions under an i.n.d. assumption and then derive a new exact closed-form expression for the capture probability GSC RAKE receivers in this more realistic scenario.**

I. Introduction

Recently, [1] presented the exact performance analyses on the capture probability on GSC RAKE receivers over independent and identically distributed (i.i.d.) fading channels. The major difficulty in the analysis was to derive the joint statistics of ordered identical exponential variates. Capitalizing on some new order statistics results [2], [1] derived exact closed-form expressions for the capture probability of GSC RAKE receivers. In [1], the assumption was made that the fading is both independent and identically distributed from path to path. However, the average signal-to-noise ratio (SNR) of each path (or branch) is different for most practical channel models, especially for wide-band spread spectrum signals since the average fading power may vary from one path to the other. For example, experimental measurements indicate that the radio channel is characterized by an exponentially decaying multipath intensity profile (MIP) for indoor office buildings [3] as well as urban [4] and suburban areas [5].

In this paper, we extend the results of [1] by maintaining the assumption of independence among the diversity paths but relaxing the identically distributed assumption. The major difficulty is to derive the needed joint statistics of ordered non-identical exponential variates. With this motivation in mind, we derive some new interesting order statistics for non-identical random variables (RVs) and use them to offer a new exact closed-form expression of the capture probability over independent and non-identically distributed (i.n.d.) Rayleigh fading conditions. Note that, in the view of contribution to ordered statistics, this new statistical results can provide the potential solution of ordered statistics in advanced wireless communications research.

II. Capture Probability of GSC RAKE receiver

As introduced in [1] and [6], the capture probability refers to the probability that the ratio of the power of the combined paths to that of the power of all available diversity paths exceeds a certain threshold. In here, we consider a diversity system with N diversity paths. Let γ_i be the SNR of the i-th diversity path and u_i $(i = 1, 2, \ldots, N)$ be the order statistics obtained by arranging N $(N \geq 2)$ nonnegative i.n.d. RVs, $\{\gamma_i\}_{i=1}^{N}$, in decreasing order of magnitude such that $u_1 \geq u_2 \geq \cdots \geq u_N$. The capture probability, denoted by Prob_T, is then defined as [1]

$$\mathrm{Prob}_T = \Pr\left[\frac{\sum_{n=1}^{m} u_n}{\sum_{n=1}^{N} u_n} > T\right], \tag{1}$$

where $0 < T < 1$ and m $(\leq N)$ is the number of combined diversity paths.

In (1), if we assume $Z = [Z_1, Z_2]$, where $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$, then (1) can be calculated in terms of the 2-dimensional joint probability density function (PDF) of Z_1 and Z_2, denoted by $p_Z(z_1, z_2)$ as

$$\mathrm{Prob}_T = \Pr\left[\frac{Z_1}{Z_1 + Z_2} > T\right] = \int_0^{\infty} \int_0^{\left(\frac{1-T}{T}\right) z_1} p_Z(z_1, z_2)\, dz_2 dz_1. \tag{2}$$

In order to find a closed-form expression of (2), we need to derive the joint PDF of Z_1 and Z_2.

III. Joint PDF of Partial Sums of Ordered I.N.D. Exponential Variates

Recently, we have introduced a unified framework to determine the joint statistics of partial sums of ordered i.i.d. RVs [2]. With this proposed approach, the joint statistics of any partial sums of ordered statistics can be obtained systematically in terms of the moment generating function (MGF) and the PDF. Since this method is valid for the identical case only, it can not be used here. Therefore, we extend the approach developed in [2] and consider in this paper the more general case in which the diversity paths are independent but not necessarily identically distributed. Similar to what was done in [2], the proposed analytical framework adopts a general two-step approach. More specifically, after obtaining the joint MGF of selected partial sum for i.n.d. RVs in a compact form, we derive the related joint PDF through an inverse Laplace transform (LT). For the cases of our interest, the joint MGF involves basic functions, for which the inverse LT can be calculated analytically. In particular, closed-form expressions for the exponential RV special case can typically be obtained.

A. General Results

Let $p_{i_l}(\cdot)$ and $P_{i_l}(\cdot)$ denote the PDF and the CDF of γ_{i_l} ($i_l = 1, 2, \ldots, N$), respectively. The N-dimensional joint PDF of the ordered RVs $\{u_i\}_{i=1}^N$ is given by [7]

$$g(u_1, u_2, \ldots, u_N) = \sum_{\substack{i_1, i_2, \ldots, i_N \\ i_1 \neq i_2 \neq \cdots \neq i_N}}^{1,2,\ldots,N} p_{i_1}(u_1)\, p_{i_2}(u_2) \cdots p_{i_N}(u_N). \quad (3)$$

Theorem 3.1: (Joint PDF of $\sum_{n=1}^{m} u_n$ and $\sum_{n=m+1}^{N} u_n$ for i.n.d. RVs)
The 2-dimensional joint PDF of $Z = [Z_1, Z_2]$ is given by

$$\begin{aligned} p_Z(z_1, z_2) = & \sum_{i_m=1}^{N} \int_0^{\infty} du_m\, p_{i_m}(u_m) \sum_{\{i_1,\ldots,i_{m-1}\} \in \mathrm{P}_{m-1}(I_N - \{i_m\})} \\ & \times L_{S_1}^{-1}\left\{ \prod_{\substack{k=1 \\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}(u_m, -S_1) \exp(-S_1 u_m) \right\} \\ & \times \sum_{\{i_{m+1},\ldots,i_N\} \in \mathrm{P}_{N-m}(I_N - \{i_m\} - \{i_1,\ldots,i_{m-1}\})} L_{S_2}^{-1}\left\{ \prod_{\substack{l=m+1 \\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m, -S_2) \right\} \\ & \text{for } z_1 \geq \frac{m}{N-m} z_2, \end{aligned} \quad (4)$$

where $c_{i_l}(\gamma, \lambda) = \int_0^{\gamma} p_{i_l}(x) \exp(\lambda x)\, dx$ is a mixture of a CDF and an MGF and $e_{i_l}(\gamma, \lambda) = \int_{\gamma}^{\infty} p_{i_l}(x) \exp(\lambda x)\, dx$ is a mixture of an exceedance distribution function (EDF) and an MGF.[1] In (4), we define the index set I_N as $I_N = \{i_1, \cdots, i_N\}$. The subset of I_N with n ($n \leq N$) elements is denoted by $\mathcal{P}_n(I_N)$. The remaining indices can be grouped in the set $I_N - \mathcal{P}_n(I_N)$.

Proof: See Appendix A. ∎

B. Special Case: Non-Identical Exponential RVs

The novel generic result in (4) applies to any set of positive RVs. We now illustrate in this subsection some results for i.n.d. exponential RV special case, where the PDF and the CDF of γ are given by $p_{i_l}(x) = \frac{1}{\bar{\gamma}_{i_l}} \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ and $P_{i_l}(x) = 1 - \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ for $\gamma \geq 0$, respectively, where $\bar{\gamma}_{i_l}$ is the average of the l-th RV. First, we need to specialize the multiple product of $c_{i_l}(\gamma, \lambda)$ and $e_{i_l}(\gamma, \lambda)$ for i.n.d. exponential RVs. More specifically, in order to apply an inverse LT for deriving final PDF closed-form expressions from MGF expressions, these multiple product expressions need to be converted to summation expressions. As shown in Appendix B, a simple summation expression function of λ from multiple product expressions can be obtained as

$$\prod_{l=n_1}^{n_2} e_{i_l}(z_a, \lambda) = \sum_{k=n_1}^{n_2} \frac{C_{k,n_1,n_2}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_k}}\right)} \exp\left(-\sum_{l=n_1}^{n_2}\left(\frac{z_a}{\bar{\gamma}_{i_l}}\right)\right) \exp\left((n_2 - n_1 + 1) z_a \lambda\right), \quad (5)$$

$$\begin{aligned} \prod_{l=n_1}^{n_2} c_{i_l}(z_a, \lambda) = & \sum_{k=n_1}^{n_2} C_{k,n_1,n_2} \\ & \times \left[\frac{1 + \left[\sum_{l=1}^{n_2 - n_1 + 1} \exp(l \cdot z_a \cdot \lambda) \left\{ (-1)^l \sum_{j_1 = j_0 + n_1}^{n_2 - l + 1} \cdots \sum_{j_l = j_{l-1} + 1}^{n_2} \exp\left(-\sum_{m=1}^{l} \frac{z_a}{\bar{\gamma}_{i_{j_m}}} \right) \right\} \right]}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_k}}\right)} \right], \end{aligned} \quad (6)$$

where $C_{l,1,n} = \frac{1}{\prod_{l=1}^{n}\left(-\bar{\gamma}_{i_l}\right) F'\left(\frac{1}{\bar{\gamma}_{i_l}}\right)}$ and $F'(x) = \left[\sum_{l=1}^{n-1} (n-l)\, x^{n-1-l} (-1)^l \sum_{j_1 = j_0 + 1}^{n-l+1} \cdots \sum_{j_l = j_{l-1}+1}^{n} \prod_{m=1}^{l} \frac{1}{\bar{\gamma}_{i_{j_m}}}\right] + (n)\, x^{n-1}$.

After substituting (6) and (5) into (4), the desired joint PDF expression for i.n.d. exponential RV assumptions can be obtained as shown in (7) at the top of the next page by applying classical inverse LT pairs and properties.

In (7), there are two integrals. The first integral easily simplifies to

$$\frac{1 - \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_1}{m}\right)}{\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)}. \quad (8)$$

For the second integral part, we need to consider two cases separately based on the valid integral region of z_1, z_2, and u_m as

$$\begin{aligned} & \int_0^{\frac{z_2}{h}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right) u_m\right) U\left(\frac{z_1}{m} - \frac{z_2}{h}\right) \\ & + \int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right) u_m\right) \left[1 - U\left(\frac{z_1}{m} - \frac{z_2}{h}\right)\right], \end{aligned} \quad (9)$$

which can easily be obtained in closed-form given the exponential dependence on u_m as

$$\begin{aligned} & \frac{1 - \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right) \frac{z_2}{h}\right)}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right)} U\left(\frac{z_1}{m} - \frac{z_2}{h}\right) \\ & + \frac{1 - \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right) \frac{z_1}{m}\right)}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right)} \left[1 - U\left(\frac{z_1}{m} - \frac{z_2}{h}\right)\right]. \end{aligned} \quad (10)$$

IV. Closed-Form Expression for the Capture Probability of GSC RAKE Receivers over i.n.d. Rayleigh Fading Channels

With the closed-form expression given in (7) at hand, we can now apply this result to obtain the closed-form expression for the capture probability. More specifically, inserting (7) into (2), a closed-form expression for i.n.d. Rayleigh fading conditions can be obtained as shown in Appendix C and is given at the bottom of the next page.

[1]Note that $c_{i_l}(\gamma, 0) = c_{i_l}(\gamma)$ is the CDF and $c_{i_l}(\infty, \lambda)$ leads to the MGF and $e_{i_l}(\gamma, 0) = e_{i_l}(\gamma)$ is the EDF while $e_{i_l}(0, \lambda)$ gives the MGF.

$$
\begin{aligned}
p_Z\left(z_1,z_2\right) = & \sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}} \sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\{i_m\}\right)} \sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1} C_{k,1,m-1} \sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\{i_m\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)} \sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N} C_{q,m+1,N} \\
& \times \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\left[\int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)u_m\right)\right. \\
& \left. +\int_0^{\infty} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)U\left(z_1-mu_m\right)U\left(z_2-hu_m\right)\right].
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
& \mathrm{Prob}_T \\
& = \sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}} \sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\{i_m\}\right)} \sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1} C_{k,1,m-1} \sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\{i_m\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)} \sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N} C_{q,m+1,N} \\
& \times\left[\frac{1}{\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}\left\{\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)dz_2dz_1-\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)\frac{z_1}{m}\right)dz_2dz_1\right\}\right. \\
& +\sum_{h=1}^{N-m}(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\left\{\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1\right. \\
& \left. -\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1\right\} \\
& +\sum_{h=1}^{N-m}(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\left\{\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\right. \\
& \left.\left. -\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\right\}\right],
\end{aligned}
\tag{11}
$$

i) The first integral:

$$
\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)dz_2dz_1=\bar{\gamma}_{i_q}\bar{\gamma}_{i_k}-\frac{\bar{\gamma}_{i_q}}{\left(\frac{1}{T\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}-\frac{1}{\bar{\gamma}_{i_q}}\right)}. \tag{12}
$$

ii) The second integral:

$$
\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)\frac{z_1}{m}\right)dz_2dz_1=\frac{\bar{\gamma}_{i_q}}{\left(\sum_{l=1}^{m}\left(\frac{1}{m\cdot\bar{\gamma}_{i_l}}\right)\right)}-\frac{\bar{\gamma}_{i_q}}{\left(\sum_{l=1}^{m}\left(\frac{1}{m\cdot\bar{\gamma}_{i_l}}\right)+\frac{1-T}{T\cdot\bar{\gamma}_{i_q}}\right)}. \tag{13}
$$

iii) The third integral:

$$
\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1=\bar{\gamma}_{i_q}\bar{\gamma}_{i_k}-\frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{\bar{\gamma}_{i_q}T}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)-\frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{\bar{\gamma}_{i_q}m}+\frac{1}{\bar{\gamma}_{i_k}}\right)}\left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right]. \tag{14}
$$

iv) The forth integral:

$$
\begin{aligned}
& \int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1 \\
& =\frac{\bar{\gamma}_{i_k}h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}-\frac{h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1-T}{T\cdot h}+\frac{1}{\bar{\gamma}_{i_k}}\right\}}U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right) \\
& -\frac{h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1}{m}+\frac{1}{\bar{\gamma}_{i_k}}\right\}}\left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right].
\end{aligned}
\tag{15}
$$

v) The fifth integral:

$$
\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1=\frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{m\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)-\frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{T\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right). \tag{16}
$$

vi) The sixth integral:

$$
\begin{aligned}
& \int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1 \\
& =\frac{m\cdot\bar{\gamma}_{i_q}}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)-\frac{m\cdot\bar{\gamma}_{i_q}}{\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)+\frac{m(1-T)}{T\cdot\bar{\gamma}_{i_q}}\right\}}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right).
\end{aligned}
\tag{17}
$$

V. Numerical Results

In this section, we present some numerical results to examine the capture probability of GSC RAKE receivers over i.n.d. Rayleigh fading channels. More specifically, we assume that the channel has an exponential MIP with $\bar{\gamma}_l = \bar{\gamma} \cdot \exp\left(-\delta\left(l-1\right)\right)$, $\left(l = 1, \cdots, N\right)$,where $\bar{\gamma}$ is the strongest average SNR (or the average SNR of the first path) and δ is the power decay factor. Fig. 1 shows the capture probability as a function of threshold, T, for both i.i.d. and i.n.d. fading scenario. As m increases, the capture probability for both i.i.d. and i.n.d. cases is increasing. As a result, we can say the system performance (or reliability) improves as m increases, as expected. Also, it can be seen that the effect of the interference of more heavily decayed MIP is smaller than it is for MIP's with smaller decay exponents. As a results, if δ decreases with fixed m, the interference level is increasing and then it leads to the system performance degradation. Fig. 2 presents the capture probability with fixed m for varying N and δ. For fixed m, if the system increases N, the system performance is degraded. Similar to Fig. 1, we can also observe the same effect of the power decay factor as expected.

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Appendices

Appendix A
Proof of Theorem 3.1

Applying an interchange of multiple integrals, the second order MGF of $Z = [Z_1, Z_2])$ can be simply re-written as

$$\begin{aligned}
&\sum_{i_m=1}^{N} \int_0^{\infty} du_m p_{i_m}\left(u_m\right) \exp\left(\lambda_1 u_m\right) \\
&\times \sum_{\{i_1,\ldots,i_{m-1}\} \in \mathrm{P}_{m-1}\left(I_N - \{i_m\}\right)} \int_{u_m}^{\infty} du_{m-1} p_{i_{m-1}}(u_{m-1}) \exp(\lambda_1 u_{m-1}) \\
&\quad \times \cdots \int_{u_2}^{\infty} du_1 p_{i_1}(u_1) \exp(\lambda_1 u_1) \\
&\times \sum_{\{i_{m+1},\ldots,i_N\} \in \mathrm{P}_{N-m}\left(I_N - \{i_m\} - \{i_1,\ldots,i_{m-1}\}\right)} \int_0^{u_m} du_{m+1} p_{i_{m+1}}(u_{m+1}) \exp(\lambda_2 u_{m+1}) \\
&\quad \times \cdots \int_0^{u_{N-1}} du_N p_{i_N}(u_N) \exp(\lambda_2 u_N).
\end{aligned} \tag{18}$$

Using the definition of $c_{i_l}\left(\gamma, \lambda\right)$ and $e_{i_l}\left(\gamma, \lambda\right)$ in (18), the simplified second order MGF can be obtained as

$$\begin{aligned}
&\sum_{i_m=1}^{N} \int_0^{\infty} du_m p_{i_m}\left(u_m\right) \exp\left(\lambda_1 u_m\right) \\
&\times \sum_{\{i_1,\ldots,i_{m-1}\} \in \mathrm{P}_{m-1}\left(I_N - \{i_m\}\right)} \prod_{\substack{k=1 \\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}\left(u_m, \lambda_1\right) \\
&\times \sum_{\{i_{m+1},\ldots,i_N\} \in \mathrm{P}_{N-m}\left(I_N - \{i_m\} - \{i_1,\ldots,i_{m-1}\}\right)} \prod_{\substack{l=m+1 \\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m, \lambda_2).
\end{aligned} \tag{19}$$

With the simplified MGF expression (19), letting $\lambda_1 = -S_1$ and $\lambda_2 = -S_2$, we can obtain the desired 2-dimensional joint PDF of Z_1 and Z_2 as given in (4) by applying an inverse LT.

Appendix B
Multiple Product of $c_{i_l}\left(\cdot,\cdot\right)$ and $e_{i_l}\left(\cdot,\cdot\right)$

In section III-B, for arbitrary n_1 to n_2, (5) and (6) have the product forms

$$\prod_{l=n_1}^{n_2} c_{i_l}(z_a, \lambda) = \frac{1}{\prod\limits_{l=n_1}^{n_2} \left(1 - \bar{\gamma}_{i_l}\lambda\right)} \prod_{l=n_1}^{n_2} \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right], \tag{20}$$

$$\prod_{l=n_1}^{n_2} e_{i_l}(z_a, \lambda) = \frac{1}{\prod\limits_{l=n_1}^{n_2} \left(1 - \bar{\gamma}_{i_l}\lambda\right)} \prod_{l=n_1}^{n_2} \left[\exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right]. \tag{21}$$

In order to convert them to sums, the following three formulas should be converted to summation expressions.

i) $\frac{1}{\prod\limits_l \left(1 - \bar{\gamma}_{i_l}\lambda\right)}$: We first derive a special case for a) the multiple product from 1 to n and then extend this result to a general case for b) the multiple product from arbitrary n_1 to n_2. For case a), after deploying the multiple product term and then rearranging and simplifying them, this multiple product expression can be converted to a summation expression of just λ as

$$\frac{1}{\prod\limits_{l=1}^{n} \left(1 - \bar{\gamma}_{i_l}\lambda\right)} = \sum_{l=1}^{n} \frac{C_{l,1,n}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right)}, \tag{22}$$

where $j_0 = 0$, $C_{l,1,n} = \frac{1}{\prod\limits_{l=1}^{n} \left(-\bar{\gamma}_{i_l}\right) F'\left(\frac{1}{\bar{\gamma}_{i_l}}\right)}$, and $F'(x) = \left[\sum\limits_{l=1}^{n-1} (n-l) x^{n-1-l} (-1)^l \sum\limits_{j_1=j_0+1}^{n-l+1} \cdots \sum\limits_{j_l=j_{l-1}+1}^{n} \prod\limits_{m=1}^{l} \frac{1}{\bar{\gamma}_{i_{j_m}}}\right] + n x^{n-1}$. For arbitrary n_1 and n_2, after applying the same steps that lead to (22), we can obtain the final result as

$$\frac{1}{\prod\limits_{l=n_1}^{n_2} \left(1 - \bar{\gamma}_{i_l}\lambda\right)} = \sum_{l=n_1}^{n_2} \frac{C_{l,n_1,n_2}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right)}. \tag{23}$$

ii) $\prod\limits_l \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right]$: For arbitrary l from n_1 to n_2, similar to B-i), after applying the same derivation steps, we can obtain the final result as

$$1 + \left[\sum_{l=1}^{n_2-n_1+1} \exp\left(l \cdot z_a \cdot \lambda\right) \left\{(-1)^l \sum_{j_1=j_0+n_1}^{n_2-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n_2} \exp\left(-\sum_{m=1}^{l} \frac{z_a}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right]. \tag{24}$$

iii) $\prod\limits_l \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)$: In this case, with the help of the property of exponential multiplication, the summation expression from the multiple product expression for arbitrary n_1 to n_2 can be obtained as

$$\exp\left(\left\{-\sum_{l=n_1}^{n_2}\left(\frac{z_a}{\bar{\gamma}_{i_l}}\right)\right\}\right)\exp\left(\left(n_2-n_1+1\right)z_a\lambda\right). \quad (25)$$

With the above results, we can now obtain the summation expressions of (5) and (6) for arbitrary n_1 to n_2.

APPENDIX C
CLOSED-FORM EXPRESSION OF (11) OVER I.N.D. RAYLEIGH FADING

Inserting the closed-form expression of (7) into (2), a closed-form expression for i.n.d. Rayleigh fading conditions can be written in (11). In (11), there are six double-integral expressions. For the first and second cases, we can directly obtain the closed-form expressions. However, for the other cases, we need to carefully consider the valid integral regions. For instance, the third and forth integral expressions, for valid integration, we need to consider two cases separately. If $\frac{h}{m}\geq\frac{1-T}{T}$, then $z_2\leq\frac{1-T}{T}z_1$ and $\frac{1}{m}\geq\frac{1-T}{T\cdot h}$. If $\frac{h}{m}<\frac{1-T}{T}$, then $z_2\leq\frac{h}{m}z_1$ and $\frac{1}{m}<\frac{1-T}{T\cdot h}$. As a result, these integral expressions can be re-written as

$$\int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\left[\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)dz_2dz_1 + \int_0^{\left(\frac{h}{m}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left\{1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right\}dz_2dz_1\right], \quad (26)$$

and

$$\int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\left[\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)dz_2dz_1 + \int_0^{\left(\frac{h}{m}\right)z_1}\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)\left\{1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right\}dz_2dz_1\right], \quad (27)$$

respectively. For the fifth and sixth integral expressions, we need to consider two cases separately for valid integration. If $\frac{1-T}{T}\geq\frac{h}{m}$, then $\frac{h}{m}z_1<z_2\leq\frac{1-T}{T}z_1$ and $\frac{1-T}{T\cdot h}\geq\frac{1}{m}$. If $\frac{1-T}{T}<\frac{h}{m}$, then there is no valid overlap integration region. As a result, these integral expressions can also be re-written as

$$\int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1, \quad (28)$$

and

$$\int_0^\infty \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right) \times \int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1, \quad (29)$$

respectively. With (26), (27), (28), and (29), we can directly derive closed-form expressions for all integrals presented in (14), (15), (16), and (17).

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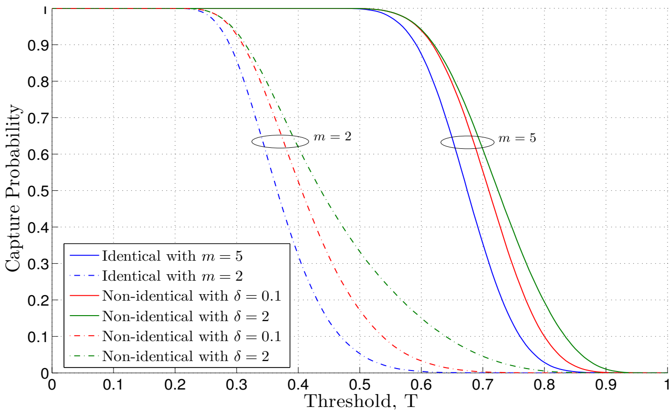

Fig. 1. Capture probability for various m and δ with $N=15$ over Rayleigh fading channel ($\bar{\gamma}=8$dB).

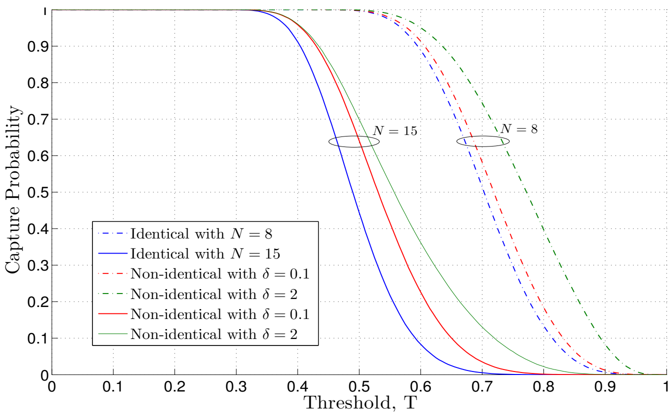

Fig. 2. Capture probability for various N and δ with fixed $m=3$ over Rayleigh fading channel ($\bar{\gamma}=8$dB).