Upper Bounds on the Size of Grain-Correcting Codes

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***Abstract*—In this paper, we re-visit the combinatorial error model of Mazumdar et al. [3] that models errors in high-density magnetic recording caused by lack of knowledge of grain boundaries in the recording medium. We present new upper bounds on the cardinality/rate of binary block codes that correct errors within this model.**

I. Introduction

The combinatorial error model studied by Mazumdar et al. [3] is a highly simplified model of an error mechanism encountered in a magnetic recording medium at terabit-per-square-inch storage densites [4], [6]. In this model, a one-dimensional track on a magnetic recording medium is divided into evenly spaced bit cells, each of which can store one bit of data. Bits are written sequentially into these bit cells. The sequence of bit cells has an underlying "grain" distribution, which may be described as follows: bit cells are grouped into non-overlapping blocks called *grains*, which may consist of up to b adjacent bit cells. We focus on the case $b = 2$, so that a grain can contain at most two bit cells. We define the *length* of a grain to be the number of bit cells it contains.

Each grain can store only one bit of information, i.e., all the bit cells within a grain carry the same bit value (0 or 1), which we call the *polarity* of the grain. We assume, following [3], that in the sequential write process, the first bit to be written into a grain sets the polarity of the grain, so that all the bit cells within this grain must retain this polarity. This implies that any subsequent attempts at writing bits within this grain make no difference to the value actually stored in the bit cells in the grain. If the grain boundaries were known to the write head (encoder) and the read head (decoder), then the maximum storage capacity of one bit per grain can be achieved. However, in a more realistic scenario where the underlying grain distribution is fixed but *unknown*, the lack of knowledge of grain boundaries reduces the storage capacity. Constructions and rate/cardinality bounds for codes that correct errors caused by a fixed but unknown underlying grain distribution have been studied in the prior literature [3], [5]. In this paper, we present improved rate/cardinality upper bounds for such codes.

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The paper is organized as follows. After providing the necessary definitions and notation in Section II, we derive, in Section III, an upper bound on the cardinality of t-grain-correcting codes using the fractional covering technique from [1]. An information-theoretic upper bound on the maximum rate asymptotically achievable by codes correcting a constant fraction of grain errors is derived in Section IV. We conclude in Section V with some remarks concerning the two bounds.

II. Definitions and Notation

Let $\Sigma = \{0, 1\}$, and for a positive integer n, let $[n]$ denote the set $\{1, 2, \dots, n\}$. A track on the recording medium consists of n bit cells indexed by the integers in $[n]$. The bit cells on the track are grouped into non-overlapping grains of length at most 2. A length-2 grain consists of bit cells with indices $j-1$ and j, for some $j \in [n]$; we denote such a grain by the pair $(j-1, j)$. Let $E \subseteq \{2, \dots, n\}$ be the set of all indices j such that $(j-1, j)$ is a length-2 grain. Since grains cannot overlap, E contains no pair of consecutive integers. The set E will be called the *grain pattern*.

A binary sequence $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$ to be written on to the track can be affected by errors only at the indices $j \in E$. Indeed, what actually gets recorded on the track is the sequence $\mathbf{y} = (y_1, \dots, y_n)$, where

$$y_j = \begin{cases} x_{j-1} & \text{if } j \in E \\ x_j & \text{otherwise.} \end{cases} \tag{1}$$

For example, if $\mathbf{x} = (0001010011100010)$ and $E = \{2, 4, 7, 9, 14\}$, then $\mathbf{y} = (000001111100000)$. The effect of the grain pattern E on a sequence $\mathbf{x} \in \Sigma^n$ defines an operator $\phi_E : \Sigma^n \to \Sigma^n$, where $\mathbf{y} = \phi_E(\mathbf{x})$ is as specified by (1).

For integers $n \geq 1$ and $t \geq 0$, let $\mathcal{E}_{n,t}$ denote the set of all subsets $E \subseteq \{2, \dots, n\}$ with $|E| \leq t$, such that E contains no pair of consecutive integers. For $\mathbf{x} \in \Sigma^n$, we define

$$\Phi_t(\mathbf{x}) = \{\phi_E(\mathbf{x}) : E \in \mathcal{E}_{n,t}\}.$$

Thus, $\Phi_t(\mathbf{x})$ is the set of all possible sequences that can be obtained from $\mathbf{x}$ by the action of some grain pattern E with $|E| \leq t$. A binary code $\mathcal{C}$ of length n is said to be a *t-grain-correcting code*, if for any pair of distinct vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}$, we have $\Phi_t(\mathbf{x}_1) \cap \Phi_t(\mathbf{x}_2) \neq \emptyset$. Let $M(n, t)$ denote the maximum cardinality of a t-grain-correcting code of length n. Also, for $\tau \in [0, \frac{1}{2}]$, the maximum asymptotic rate of a $\lceil \tau n \rceil$-grain-correcting code is defined to be

$$R(\tau) = \limsup_{n \to \infty} \frac{1}{n} \log_2 M(n, \lceil \tau n \rceil). \tag{2}$$

A grain pattern E changes a sequence $\mathbf{x}$ to a different sequence $\mathbf{y}$ iff for some $j \in E$, the length-2 grain $(j-1, j)$ straddles the boundary between two successive runs in $\mathbf{x}$. Here, a *run* is a maximal substring of consecutive identical bits in $\mathbf{x}$. A run consisting of 0s (resp. 1s) is called a 0-*run* (resp. 1-*run*). The number of distinct runs in $\mathbf{x}$ is denoted by $r(\mathbf{x})$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \Sigma^n$, the *derivative sequence* $\mathbf{x}' = (x'_2, \ldots, x'_n) \in \Sigma^{n-1}$ is defined by $x'_j = x_{j-1} \oplus x_j$, $j = 2, \ldots, n$, where $\oplus$ denotes modulo-2 addition. The 1s in $\mathbf{x}'$ identify the boundaries between successive runs in $\mathbf{x}$. Thus, $\omega(\mathbf{x}') = r(\mathbf{x}) - 1$, where $\omega(\cdot)$ denotes Hamming weight.

Let $\text{supp}(\mathbf{x}') = \{j : x'_j = 1\}$ denote the support of $\mathbf{x}'$. For $\mathbf{x} \in \Sigma^n$, the sequences $\mathbf{y} \in \Phi_t(\mathbf{x})$ are in one-to-one correspondence with the different ways of selecting at most t non-consecutive integers[1] from $\text{supp}(\mathbf{x}')$ to form a grain pattern $E \in \mathcal{E}_{n,t}$. A count of the number of ways in which this can be done is obtained as follows. Let $\ell_1, \ell_2, \ldots, \ell_m$ be the lengths of the distinct 1-runs in $\mathbf{x}'$, and define the set $T = \big\{(t_1, \ldots, t_m) \in \mathbb{Z}_+^m : \sum_{j=1}^m t_j \le t\big\}$, where $\mathbb{Z}_+$ denotes the set of non-negative integers. In the above definition, t_j represents the number of integers from the support of the jth 1-run that are to be included in a grain pattern E being formed. The number of distinct ways in which t_j non-consecutive integers can be chosen from the ℓ_j consecutive integers forming the support of the jth 1-run is, by an elementary counting argument, equal to $\binom{\ell_j - t_j + 1}{t_j}$. Thus,

$$|\Phi_t(\mathbf{x})| = \sum_{(t_1,\ldots,t_m)\in T} \prod_{j=1}^{m} \binom{\ell_j - t_j + 1}{t_j}. \tag{3}$$

Simplified expressions can be obtained for small values of t.

Proposition 1. *For $\mathbf{x} \in \Sigma^n$, let $\omega = \omega(\mathbf{x}')$ denote the Hamming weight of the derivative sequence $\mathbf{x}'$. Also, let m be the number of 1-runs in $\mathbf{x}'$. We then have*
(a) $|\Phi_1(\mathbf{x})| = 1 + \omega = r(\mathbf{x})$.
(b) $|\Phi_2(\mathbf{x})| = 1 + m + \binom{\omega}{2}$.
(c) $|\Phi_3(\mathbf{x})| = 1 + m_1 + m(\omega - 3) + \binom{\omega}{3} - \binom{\omega}{2} + 2\omega$, *where m_1 denotes the number of 1-runs of length 1 in $\mathbf{x}'$.*

Proof: (a) Observe that the set $\Phi_1(\mathbf{x})$ consists of the sequence $\mathbf{x}$ itself, and the ω distinct sequences in the set $\big\{\phi_E(\mathbf{x}) : E = \{j\} \text{ for some } j \in \text{supp}(\mathbf{x}')\big\}$.

(b) For $t = 2$, the expression in (3) simplifies to

$$|\Phi_2(\mathbf{x})| = 1 + \sum_{j=1}^{m} \ell_j + \sum_{j=1}^{m} \binom{\ell_j - 1}{2} + \sum_{(i,j):i<j} \ell_i \ell_j.$$

From this, routine manipulations yield

$$|\Phi_2(\mathbf{x})| = 1 + m + \frac{1}{2}\left[\left(\sum_{j=1}^{m} \ell_j\right)^2 - \sum_{j=1}^{m} \ell_j\right],$$

which equals $1 + m + \binom{\omega}{2}$, since $\omega = \sum_{j=1}^m \ell_j$.

(c) The derivation here is analogous to that in (b) above. We omit the details due to lack of space. ∎

[1] A sequence or set of non-consecutive integers is one that does not contain a pair of consecutive integers.

III. An Upper Bound on $M(n,t)$

In this section, we explore the applicability of a technique from [1] to bound $M(n,t)$ from above.

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{X})$, where V is a finite set, called the *vertex set*, and $\mathcal{X}$ is a family of subsets of V. The members of $\mathcal{X}$ are called *hyperedges*. A *matching* of $\mathcal{H}$ is a pairwise disjoint collection of hyperedges. A *(vertex) covering* of $\mathcal{H}$ is a subset $T \subseteq V$ such that T meets every hyperedge of $\mathcal{H}$, i.e., $T \cap X \neq \emptyset$ for all $X \in \mathcal{X}$. The *matching number* $\nu(\mathcal{H})$ is the largest size of a matching of $\mathcal{H}$, while the *covering number*, $\tau(\mathcal{H})$, is the smallest size of a covering of $\mathcal{H}$.

Number the vertices and hyperedges of $\mathcal{H}$ in some arbitrary way, and define the $|V| \times |\mathcal{X}|$ vertex-hyperedge incidence matrix $A = (A_{i,j})$ by $A_{i,j} = 1$ if vertex i belongs to hyperedge j and $A_{i,j} = 0$ otherwise. It is easy to verify that

$$\begin{aligned} \nu(\mathcal{H}) &= \max\{\mathbf{1}^T \mathbf{z} : \mathbf{z} \in \{0,1\}^{|\mathcal{X}|},\ A\mathbf{z} \le \mathbf{1}\} \\ \tau(\mathcal{H}) &= \min\{\mathbf{1}^T \mathbf{w} : \mathbf{w} \in \{0,1\}^{|V|},\ A^T\mathbf{w} \ge \mathbf{1}\} \end{aligned}$$

where $\mathbf{1}$ denotes an all-ones column vector. Note that the corresponding linear programming (LP) relaxations

$$\begin{aligned} \nu_f(\mathcal{H}) &= \max\{\mathbf{1}^T \mathbf{z} : \mathbf{z} \ge \mathbf{0},\ A\mathbf{z} \le \mathbf{1}\} \\ \tau_f(\mathcal{H}) &= \min\{\mathbf{1}^T \mathbf{w} : \mathbf{w} \ge \mathbf{0},\ A^T\mathbf{w} \ge \mathbf{1}\} \end{aligned}$$

are duals of each other. By strong LP duality, we have $\nu_f(\mathcal{H}) = \tau_f(\mathcal{H})$, and hence,

$$\nu(\mathcal{H}) \le \nu_f(\mathcal{H}) = \tau_f(\mathcal{H}) \le \tau(\mathcal{H}). \tag{4}$$

The quantities $\nu_f(\mathcal{H})$ and $\tau_f(\mathcal{H})$ are called the *fractional matching number* and *fractional covering number*, respectively, of the hypergraph $\mathcal{H}$. Any non-negative vector $\mathbf{w}$ such that $A^T\mathbf{w} \ge 1$ is called a *fractional covering* of $\mathcal{H}$. To put it differently, a fractional covering is a function $\mathsf{w} : V \to \mathbb{R}_+$ such that $\sum_{v\in X} \mathsf{w}(v) \ge 1$ for all $X \in \mathcal{X}$. The *value* of a fractional covering w is defined to be $|\mathsf{w}| := \sum_{v\in V} \mathsf{w}(v)$. From the inequality $\nu(\mathcal{H}) \le \tau_f(\mathcal{H})$ in (4), we see that $\nu(\mathcal{H}) \le |\mathsf{w}|$ for any fractional covering w of $\mathcal{H}$.

Now let $V = \Sigma^n$ and $\mathcal{X} = \{\Phi_t(\mathbf{x}) : \mathbf{x} \in \Sigma^n\}$ and consider the hypergraph $\mathcal{H}_{n,t} = (V, \mathcal{X})$. Note that $\nu(\mathcal{H}_{n,t}) = M(n,t)$; thus, fractional coverings of $\mathcal{H}_{n,t}$ yield upper bounds on $M(n,t)$. Bounding the size of packings in this way has been extensively used in combinatorics, see e.g. [2]. Inspired by [1], we consider the function $\mathsf{w}_t : \Sigma^n \to \mathbb{R}_+$, defined by

$$\mathsf{w}_t(\mathbf{x}) = \frac{1}{|\Phi_t(\mathbf{x})|}. \tag{5}$$

For $t = 1, 2, 3$, we can prove that w_t is a fractional covering of $\mathcal{H}_{n,t}$, and conjecture that this is in fact the case for all $t \ge 1$.

Conjecture 1. *For all positive integers n and t, the function w_t defined in (5) is a fractional covering of $\mathcal{H}_{n,t}$: $\forall\, \mathbf{x} \in \Sigma^n$,*

$$\sum_{\mathbf{y}\in\Phi_t(\mathbf{x})} \frac{1}{|\Phi_t(\mathbf{y})|} \ge 1. \tag{6}$$

Therefore,

$$M(n,t) \le |\mathsf{w}_t| = \sum_{\mathbf{x}\in\Sigma^n} \frac{1}{|\Phi_t(\mathbf{x})|}. \tag{7}$$

Our proof of (6) for $t = 1, 2, 3$ relies on an understanding of the relationship between $|\Phi_t(\mathbf{x})|$ and $|\Phi_t(\mathbf{y})|$ for $\mathbf{y} \in \Phi_t(\mathbf{x})$. Recall, from (3), that $|\Phi_t(\mathbf{x})|$ depends only on the lengths of the 1-runs in $\mathbf{x}'$. Thus, we need to understand how the distribution of 1s changes in going from $\mathbf{x}'$ to $\mathbf{y}'$.

A. *Effect of Grains on the Derivative Sequence*

Recall that 1s in $\mathbf{x}'$ correspond to run boundaries in $\mathbf{x}$. We say that a (length-2) grain *acts on* a 1 in $\mathbf{x}'$ if it straddles the corresponding run boundary in $\mathbf{x}$. We need to distinguish between two types of 1s in the derivative sequence $\mathbf{x}'$. A *trailing* 1 is the last 1 in a 1-run, while a *non-trailing* 1 is any 1 that is not a trailing 1.

A segment of $\mathbf{x}'$ that contains a trailing 1 is of the form $*10*$, or $*1$ in case the trailing 1 is a suffix of $\mathbf{x}$. Up to complementation, the corresponding segment of $\mathbf{x}$ is of the form $*011*$ or $*01$. A grain acting on the trailing 1 in $\mathbf{x}'$ straddles the 01 run boundary in $\mathbf{x}$. In the sequence $\mathbf{y}$ obtained through the action of this grain, the segment under observation becomes $*001*$ or $*00$, and the corresponding segment of the derivative sequence $\mathbf{y}'$ is $*01*$ or $*0$.

On the other hand, a non-trailing 1 in $\mathbf{x}'$ belongs to a segment of the form $*11*$; the first 1 shown is the non-trailing 1 under consideration. Again, up to complementation, the corresponding segment in $\mathbf{x}$ is of the form $*010*$. A grain acting on the non-trailing 1 in $\mathbf{x}'$ straddles the 01 run boundary shown in $\mathbf{x}$. This grain causes the segment being observed to become $*000*$ in $\mathbf{y}$, and hence $*00*$ in $\mathbf{y}'$.

To summarize, the action of a grain on a trailing 1 converts a segment of the form $*10*$ or $*1$ in $\mathbf{x}'$ to $*01*$ or $*0$ in $\mathbf{y}'$, and a grain acting on a non-trailing 1 converts a segment of the form $*11*$ in $\mathbf{x}'$ to $*00*$ in $\mathbf{y}'$. It should be clear that the bits depicted by $*$s on either side of these segments remain unchanged by the action of the grain. Note, in particular, that Hamming weight does not increase in going from $\mathbf{x}'$ to $\mathbf{y}'$: $\omega(\mathbf{y}') \le \omega(\mathbf{x}')$. A grain acting on a trailing 1 reduces the Hamming weight by at most 1; in the case of a non-trailing 1, the Hamming weight is reduced by 2.

Finally, when dealing with a grain pattern containing $t > 1$ length-2 grains, since the grains are non-overlapping, the actions of individual grains can be considered independently.

B. *Proof of (6) for $t = 1, 2, 3$*

Consider $t = 1$ first. For any $\mathbf{y} \in \Phi_1(\mathbf{x})$, we have $\omega(\mathbf{y}') \le \omega(\mathbf{x}')$ by the discussion in Section III-A, and hence, $|\Phi_1(\mathbf{y})| \le |\Phi_1(\mathbf{x})|$ by Proposition 1. Therefore,

$$\sum_{\mathbf{y}\in\Phi_1(\mathbf{x})} \frac{1}{|\Phi_1(\mathbf{y})|} \ge \sum_{\mathbf{y}\in\Phi_1(\mathbf{x})} \frac{1}{|\Phi_1(\mathbf{x})|} = 1,$$

which proves (6) for $t = 1$.

The simple argument above does not extend directly to $t \ge 2$, the reason being that it is no longer true in general that $|\Phi_t(\mathbf{y})| \le |\Phi_t(\mathbf{x})|$ for $\mathbf{y} \in \Phi_t(\mathbf{x})$. For example, consider $\mathbf{x} = 0100$, and note that $\Phi_2(\mathbf{x}) = \{0000, 0100, 0110\}$. Take $\mathbf{y} = 0110 \in \Phi_2(\mathbf{x})$, and verify that $\Phi_2(\mathbf{y}) = \{0110, 0010, 0111, 0011\}$. Thus, $|\Phi_2(\mathbf{y})| > |\Phi_2(\mathbf{x})|$.

To prove (6) for $t = 2, 3$, we show that the sequences $\mathbf{y} \in \Phi_t(\mathbf{x})$ that violate the inequality $|\Phi_t(\mathbf{y})| \le |\Phi_t(\mathbf{x})|$ can be dealt with by suitably matching them with sequences that satisfy the inequality. To this end, for a fixed $\mathbf{x} \in \Sigma^n$, let us define $F_t(\mathbf{x}) = \{\mathbf{y} \in \Phi_t(\mathbf{x}) : |\Phi_t(\mathbf{y})| > |\Phi_t(\mathbf{x})|\}$ and $G_t(\mathbf{x}) = \{\mathbf{y} \in \Phi_t(\mathbf{x}) : |\Phi_t(\mathbf{y})| \le |\Phi_t(\mathbf{x})|\}$. We will construct a one-to-one mapping $p : F_t(\mathbf{x}) \to G_t(\mathbf{x})$ such that for all $\mathbf{y} \in F_t(\mathbf{x})$, we have

$$\frac{1}{|\Phi_t(\mathbf{y})|} + \frac{1}{|\Phi_t(p(\mathbf{y}))|} \ge \frac{2}{|\Phi_t(\mathbf{x})|}. \tag{8}$$

The mapping p will be referred to as a *pairing*. It is easy to verify that the construction of such a pairing is sufficient to prove (6), and hence, (7).

A pairing can indeed be constructed for $t = 2, 3$, and we sketch a proof of this here. Consider $\mathbf{y} \in F_t(\mathbf{x})$, with $t = 2$ or 3. Let $E \in \mathcal{E}_{n,t}$ be such that $\mathbf{y} = \phi_E(\mathbf{x})$. Let ω and $\tilde{\omega}$ be the Hamming weights of the derivative sequences $\mathbf{x}'$ and $\mathbf{y}'$, respectively. The discussion in Section III-A shows that $\tilde{\omega} \le \omega$. Using Proposition 1, it can be shown that $\tilde{\omega} \le \omega - 2$ only if $\mathbf{y} \notin F_t(\mathbf{x})$; thus, $\tilde{\omega}$ equals $\omega - 1$ or ω. In either case, the grains in E act only on trailing 1s in $\mathbf{x}$.

An *isolated* 1 in $\mathbf{x}'$ is a 1 that forms a 1-run of length 1. Let E_1 denote the subset of E consisting of grains j that act on isolated 1s of $\mathbf{x}'$, and let $E_2 = E \setminus E_1$. It can be shown (again using Proposition 1) that E_2 is non-empty. Set $E' = E_1 \cup \{j-1 : j \in E_2\}$, and consider $\mathbf{z} = \phi_{E'}(\mathbf{x})$. Clearly, $E' \in \mathcal{E}_{n,t}$, and hence, $\mathbf{z} \in \Phi_t(\mathbf{x})$. With a little effort, it can be shown that $\frac{1}{|\Phi_t(\mathbf{y})|} + \frac{1}{|\Phi_t(\mathbf{z})|} \ge \frac{2}{|\Phi_t(\mathbf{x})|}$. It follows that $\mathbf{y} \mapsto \mathbf{z}$ is the required pairing.

In summary, we have obtained the following result.

Theorem 2. *For any integer $n \ge 1$ and $t = 1, 2, 3$, we have*

$$M(n,t) \le \sum_{\mathbf{x}\in\Sigma^n} \frac{1}{|\Phi_t(\mathbf{x})|}.$$

For $t = 1$, an exact closed-form expression can be derived for $\sum_{\mathbf{x}} \frac{1}{|\Phi_t(\mathbf{x})|}$. Indeed,

$$\begin{aligned}\sum_{\mathbf{x}\in\Sigma^n} \frac{1}{|\Phi_1(\mathbf{x})|} &\overset{(a)}{=} \sum_{\mathbf{x}\in\Sigma^n} \frac{1}{|r(\mathbf{x})|} = \sum_{r=1}^{n} \sum_{\mathbf{x}:r(\mathbf{x})=r} \frac{1}{r} \\ &\overset{(b)}{=} \sum_{r=1}^{n} 2\binom{n-1}{r-1}\frac{1}{r} \overset{(c)}{=} 2\sum_{r=1}^{n} \frac{1}{n}\binom{n}{r},\end{aligned}$$

which evaluates to $\frac{2}{n}(2^n - 1)$. Equality (a) above is by virtue of Proposition 1; (b) is due to the fact that the number of $\mathbf{x} \in \Sigma^n$ with $r(\mathbf{x}) = r$ is equal to twice the number of $\mathbf{x}' \in \Sigma^{n-1}$ with $\omega(\mathbf{x}') = r - 1$; and (c) uses the identity $\frac{1}{r}\binom{n-1}{r-1} = \frac{1}{n}\binom{n}{r}$. Thus, we have

Corollary 3. $M(n,1) \le \frac{1}{n}(2^{n+1} - 2)$ *for all integers $n \ge 1$.*

For $t = 2, 3$, analogous closed-form expressions for the upper bound in Theorem 2 do not appear to exist. However, using Proposition 1, the bounds can be expressed in a form more convenient for numerical evaluation.

Corollary 4. *With the convention that* $\binom{a}{-1}$ *equals* 1 *if* $a = -1$*, and equals* 0 *otherwise, the following bounds hold:*

(a) $M(n,2) \leq 2 \cdot \sum_{\omega=0}^{n-1} \sum_{m=0}^{\omega} \binom{\omega-1}{m-1}\binom{n-\omega}{m} \frac{1}{1+m+\binom{w}{2}}$

(b) $M(n,3) \leq 2 \cdot \sum_{\omega=0}^{n-1} \sum_{m=0}^{\omega} \sum_{m_1=0}^{m} \alpha(m_1,m,\omega) \frac{1}{\phi(m_1,m,\omega)},$ *where* $\alpha(m_1,m,\omega) = \binom{m}{m_1}\binom{\omega-m-1}{m-m_1-1}\binom{n-\omega}{m}$ *and* $\phi(m_1,m,\omega) = 1 + m_1 + m(\omega-3) + \binom{\omega}{3} - \binom{\omega}{2} + 2\omega$.

The bounds above are simply alternative ways of expressing $\sum_{\mathbf{x}} \frac{1}{|\Phi_t(\mathbf{x})|}$ using Proposition 1 and elementary counting.

Table I lists, for some small values of n, the numerical values of the bounds in Corollaries 3 and 4 rounded down to the nearest even integer[2]. Two other upper bounds on $M(n,t)$ exist in the prior literature, namely Corollary 6 of [3] and Theorem 3.1 of [5]. Numerical computations for $n \leq 20$ show that our bounds above are consistently better than the bounds obtained from [5, Theorem 3.1]. On the other hand, the bound of [3, Corollary 6] may be better than our bound for small values of n: for example, the bound in [3] yields $M(10,2) \leq 92$. However, our bound is better for all n sufficiently large: for $t=1$, our bound is better for all $n \geq 8$; for $t=2$, our bound wins for $n \geq 13$.

IV. An information-theoretic upper bound on $R(\tau)$

The method of the previous section would yield a bound on the asymptotic rate $R(\tau)$, as defined in (2), were Conjecture 1 to be proved. Instead, in this section, we use an information-theoretic approach to derive an upper bound on $R(\tau)$. In fact, in Section V, we sketch an argument that indicates that our information-theoretic bound is better than any upper bound on $R(\tau)$ that can be obtained from Conjecture 1.

For every even n, by grouping together adjacent coordinates, we can view any code $C \in \{0,1\}^n$ as a code of blocklength $n/2$ over the alphabet $\{00,01,10,11\}$. Let us say that a binary n-tuple, alternatively an $n/2$-tuple over the quaternary alphabet, has *quaternary distribution* (or simply *distribution*) $(f_{00}, f_{11}, f_{01}, f_{10})$ if it has $f_{00}n/2$ symbols 00, $f_{11}n/2$ symbols 11, $f_{01}n/2$ symbols 01 and $f_{10}n/2$ symbols 10. We will say that a code has *constant distribution* if each of its codewords has the same quaternary distribution $(f_{00}, f_{11}, f_{01}, f_{10})$. Our goal is to find upper bounds on the rate of $\lceil \tau n \rceil$-grain-correcting codes of constant distribution: since the number of possible quaternary distributions for a code of length n is $O(n^3)$, the maximum of these upper bounds on constrained codes will yield an unconstrained upper bound.

Let us introduce the following notation:

$$R_f(\tau) = \limsup_{n\to\infty} \frac{1}{n} \log_2 M(n, f, \lceil \tau n \rceil)$$

where $M(n,f,t)$ denotes the maximum cardinality of a t-grain error correcting code of length n and constant quaternary distribution f.

[2]Note that $M(n,t)$ is always even, since an optimal grain-correcting code can be assumed to be closed under complementation of codewords, so that codewords that start with a 0 and those that start with a 1 are equal in number.

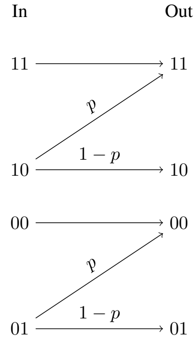

Fig. 1. A DMC whose effect can be mimicked by grain patterns

Our strategy is the following: for any given distribution $f = (f_{00}, f_{11}, f_{01}, f_{10})$, we associate to it a discrete memoryless channel (DMC) with input and output alphabets $\{00,01,10,11\}$ such that any infinite family of $\lceil \tau n \rceil$-grain-correcting codes of constant distribution f achieves vanishing error-probability when submitted through this channel. By a standard information-theoretic argument, this implies that the asymptotic rate R of any family of $\lceil \tau n \rceil$-grain-correcting codes of constant distribution f is bounded from above by half the mutual information between the channel input with probability distribution f and the channel output.

Consider the channel depicted in Figure 1. Let C be a member of a family of $\lceil \tau n \rceil$-grain-correcting codes of length n and constant distribution f. Suppose that

$$(f_{10} + f_{01})pn/2 \leq \tau n(1-\varepsilon),$$

where p is the transition probability shown in Figure 1. When a binary n-tuple, equivalently a word of length $n/2$ over the alphabet $\{00,01,10,11\}$, is transmitted over the channel, then with probability tending to 1 as n goes to infinity, the number of transitions $01 \to 00$ plus the number of transitions $10 \to 11$ is not more than $\lceil \tau n \rceil$. Since these transitions are of the kind caused by grain errors, if there are no more than $\lceil \tau n \rceil$ such transitions, then the errors they cause are correctable by any $\lceil \tau n \rceil$-grain-correcting code. Therefore, for any $\varepsilon > 0$, any family of $\lceil \tau n \rceil$-grain-correcting codes of constant distribution f can be transmitted over the above channel with vanishing error probability after decoding. By a continuity argument we conclude that

$$R_f(\tau) \leq \frac{1}{2} I(X,Y) \tag{9}$$

where X is the channel input with probability distribution $p(X) = f$, and Y is the corresponding output of the channel with parameter $p = \frac{2\tau}{f_{10}+f_{01}}$.

It remains to compute the mutual information $I(X,Y)$. Since $p = \frac{2\tau}{f_{10}+f_{01}}$ cannot exceed 1, we can write

$$f_{10} + f_{01} = 2\tau + x \quad \text{and} \quad f_{00} + f_{11} = 1 - 2\tau - x \tag{10}$$

with x non-negative. Now, for every distribution satisfying (10) we have $H(Y \mid X) = (2\tau + x)\,\mathsf{h}\left(\frac{2\tau}{2\tau+x}\right)$, where $\mathsf{h}(\cdot)$ is the binary entropy function defined by $\mathsf{h}(\xi) = -\xi \log_2 \xi -$

t \ n	2	3	4	5	6	7	8	9	10	15	20
1	2 (2)	4 (4)	6 (6)	12 (8)	20 (16)	36 (26)	62 (44)	112	204	4368	104856
2			6 (4)	10 (8)	16 (10)	26 (16)	42 (22)	70	114	1552	26418
3					16 (8)	26 (16)	40 (18)	64 (32)	100	1024	12510

TABLE I
SOME NUMERICAL VALUES OF THE UPPER BOUND OF THEOREM 2, ROUNDED DOWN TO THE NEAREST EVEN INTEGER. WITHIN PARENTHESES ARE THE CORRESPONDING LOWER BOUNDS FROM TABLE I OF [5].

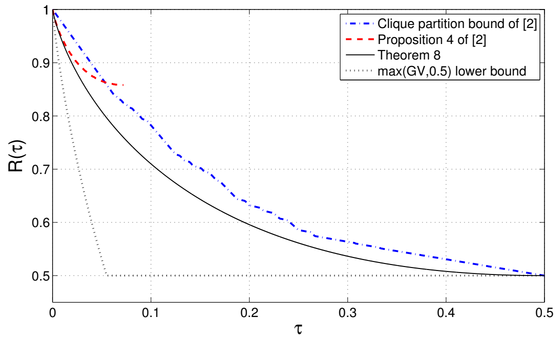

Fig. 2. The upper bound of Theorem 5 along with bounds from [3].

$(1-\xi)\log_2(1-\xi)$, for $\xi \in [0,1]$. This implies that, under the constraints in (10), $I(X,Y) = H(Y) - H(Y\,|\,X)$ is maximized when $H(Y)$ is maximized, which happens when Y is distributed as follows: $P(Y=10) = P(Y=01) = \frac{x}{2}$ and $P(Y=00) = P(Y=11) = \frac{1-x}{2}$. Therefore, we obtain

$$I(X,Y) \le 1 + \mathsf{h}(x) - (2\tau + x)\,\mathsf{h}\left(\frac{2\tau}{2\tau + x}\right), \qquad (11)$$

which together with (9) shows that $R(\tau)$ does not exceed

$$\frac{1}{2}\left[1 + \mathsf{h}(f_{10} + f_{01} - 2\tau) - (f_{10} + f_{01})\,\mathsf{h}\left(\frac{2\tau}{f_{10} + f_{01}}\right)\right].$$

The right hand side of (11) is maximized for $x = 1/2 - \tau$, thus yielding the unconstrained upper bound stated below.

Theorem 5. *For $\tau \in [0, \frac{1}{2}]$, we have*

$$R(\tau) \le \frac{1}{2}\left(1 + \mathsf{h}\left(\frac{1}{2} - \tau\right) - \left(\frac{1}{2} + \tau\right)\mathsf{h}\left(\frac{2\tau}{\frac{1}{2} + \tau}\right)\right).$$

The upper bound of Theorem 5 is plotted in Figure 2. For comparison, also plotted are the upper and lower bounds from [3, Figure 1]. The plots clearly show that the upper bound of Theorem 5 improves upon the previous upper bounds, but still remains far from the lower bound plotted. It should be pointed out that a slightly better lower bound was found by Sharov and Roth [5], but the improvement is only marginal.

V. CONCLUDING REMARKS

In this paper, we derived two upper bounds, one on the maximum cardinality, $M(n,t)$, of a binary t-grain-correcting code of blocklength n, and the other on the asymptotic rate $R(\tau)$. A natural question to ask is whether the conjectured upper bound (7) would yield a better bound on $R(\tau)$ than Theorem 5. We argue here that this would not be case, at least for $\tau \ge 0.21$.

Recall again that $|\Phi_t(\mathbf{x})|$ depends only on the lengths of the 1-runs in the derivative sequence $\mathbf{x}'$. A "typical" sequence $\mathbf{x}' \in \Sigma^{n-1}$ would contain approximately $n/2^{\ell+2}$ 1-runs of length ℓ (substrings $01^\ell 0$), $\ell = 1, 2, \ldots$. If $\mathcal{T}^{(n)}$ is the set of all $\mathbf{x} \in \Sigma^n$ with such a "typical" derivative sequence $\mathbf{x}'$, then $|\mathcal{T}^{(n)}| = 2^{n(1-o(1))}$, where $o(1)$ is a term that goes to 0 as $n \to \infty$.

Now, the number of distinct ways a grain pattern can affect a 1-run of length ℓ is equal to the number of binary sequences of length ℓ which do not contain a pair of consecutive 1s. It is well known, and indeed easy to verify, that this number is the ℓth term in the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, which is given by $q_\ell := \frac{1}{\sqrt{5}}(\varphi^\ell - \psi^\ell)$, where $\varphi = 1 - \psi = \frac{1+\sqrt{5}}{2}$. It follows from this that for $\mathbf{x} \in \mathcal{T}^{(n)}$, we have $|\Phi_t(\mathbf{x})| \lesssim \prod_{\ell \ge 1} (q_\ell)^{n/2^{\ell+2}} = 2^{\lambda n}$, where $\lambda = \sum_{\ell \ge 1} \frac{\log_2 q_\ell}{2^{\ell+2}} = 0.4124\ldots$. Therefore,

$$\sum_{\mathbf{x} \in \Sigma^n} \frac{1}{|\Phi_t(\mathbf{x})|} \ge \sum_{\mathbf{x} \in \mathcal{T}^{(n)}} \frac{1}{|\Phi_t(\mathbf{x})|} \gtrsim \frac{|\mathcal{T}^{(n)}|}{2^{\lambda n}} = 2^{n(1-\lambda)(1-o(1))}.$$

It follows from this that any upper bound on $R(\tau)$ that one could get from (7) cannot be smaller than $1 - \lambda = 0.5875\ldots$, and hence, cannot improve upon the bound of Theorem 5 for $\tau \ge 0.21$.

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REFERENCES

[1] A.A. Kulkarni and N. Kiyavash, "Non-asymptotic upper bounds for deletion correcting codes," arXiv:1211.3128, Nov. 2012.
[2] C. Berge, "Packing Problems and Hypergraph Theory: A Survey," *Annals of Discrete Mathematics*, vol. 4, pp. 3–37, 1979.
[3] A. Mazumdar, A. Barg and N. Kashyap, "Coding for high-density recording on a 1-d granular magnetic medium," *IEEE Trans. Inform. Theory*, vol. 57, no. 11, pp. 7403–7417, Nov. 2011.
[4] L. Pan, W.E. Ryan, R. Wood and B. Vasic, "Coding and detection for rectangular grain models," *IEEE Trans. Magn.*, vol. 47, no. 6, pp. 1705–1711, June 2011.
[5] A. Sharov and R.M. Roth, "Bounds and constructions for granular media coding," *Proc. 2011 IEEE Int. Symp. Inform. Theory (ISIT 2011)*, pp. 2304–2308.
[6] R. Wood, M. Williams, A. Kavcic and J. Miles, "The feasibility of magnetic recording at 10 Terabits per square inch on conventional media," *IEEE Trans. Magn.*, vol. 45, no. 2, pp. 917–923, Feb. 2009.