Anytime Reliable LDPC Convolutional Codes for Networked Control over Wireless Channel

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***Abstract*—This paper deals with the problem of stabilizing an unstable system through networked control over the wireless medium. In such a situation a remote sensor communicates the measurements to the system controller through a noisy channel. In particular, in the AWGN scenario, we show that protograph-based LDPC convolutional codes achieve anytime reliability and we also derive a lower bound to the signal-to-noise ratio required to stabilize the system. Moreover, on the Rayleigh-fading channel, we show by simulations that resorting to multiple sensors reduces the SNR required for system stabilization.**

I. Introduction

In the last few years, networked control systems, where the observer and the controller are not physically co-located, are receiving a growing attention as a practical solution in several applications such as sensor and actuators networks. Such a scenario is suitably modelled by supposing that the remote sensor transmits its measurements to the controller through a noisy communication channel. From the information-theoretical point-of-view, such communication problem has many differences from the ordinary reliability problem on a point-to-point link. Such differences arise essentially from the fact that systems must be controlled in real-time, while the usual approach does not consider delay as a primary parameter. Moreover, past decoding errors at the receiver may have a catastrophic effect if they are not eventually corrected as time proceeds.

Based on the previous considerations, Sahai and Mitter [1] introduced the new concepts of *anytime reliability* and of *anytime capacity*. Loosely speaking, an encoding-decoding scheme is said to be anytime reliable if its bit error probability decreases exponentially with delay d, i.e., is proportional to[1] $\mathrm{e}^{-\beta d}$, where β is the *anytime exponent* of the scheme. Then, the anytime capacity $C(\beta)$ is the supremum of achievable rates for schemes with anytime exponent β. For further information, see [2] and references therein.

Anytime-reliable nonlinear tree codes were first proven to exist in [3] and then further developed in [4]. Random linear codes were first introduced in [5]. Later, Sukhavasi and Hassibi [6] showed that causal random linear codes with maximum-likelihood (ML) decoding are anytime reliable with high probability. Such schemes are characterized by a high decoder complexity, although in [6] a decoder with reasonable complexity is proposed for the erasure channel. In [7] Dossel et al. proposed a low-density parity-check (LDPC) convolutional encoding scheme on the erasure channel which is shown to be anytime reliable. In this case, a belief-propagation decoder allows anytime reliability to be achieved at an affordable complexity. For other types of coding schemes, see [8]–[10].

Since the erasure channel is more suitable for modeling the behavior of upper-layer communications, in this work, we study anytime reliable LDPC convolutional codes over the wireless channel, modeled as an AWGN channel with or without fading. The rationale behind this choice is that LDPC decoding algorithms can make efficient use of soft information from the physical layer, which is lost whenever decoding is handed over to upper layers. In addition, a physical-layer encoding-decoding scheme can better exploit the potentialities of a wireless control network where multiple remote sensors transmit simultaneously their measurements to the system controller. The contributions of the paper are as follows:

- We give a generalized description of the LDPC convolutional codes proposed by [7].
- We prove that such class of codes are anytime reliable on the AWGN channel, and we give a lower bound on their anytime exponent.
- We show by simulations that, in the presence of fading channel between the sensor and the controller, better stability margins can be achieved by using multiple sensors.

The paper is organized as follows. In Section II, we describe the (single-sensor) model of the considered system. In Section III, we generalize the LDPC convolutional codes introduced in [7]. In Section IV, we derive a lower bound on the anytime exponent for the LDPC convolutional encoding scheme on the AWGN channel. Eventually in Section V, through numerical analysis, we validate the theoretical results obtained in the previous sections and we show simulation results for the sensor network scenario.

II. System model

We consider the discrete-time dynamic time invariant system[2]

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t \tag{1}$$

where $\mathbf{x}_t \in \mathbb{R}^{n_x}$ is the state of the system at time step t (with bounded initial condition $\mathbf{x}_0$), $\mathbf{A}$ and $\mathbf{B}$ are $n_x \times n_x$ and

[1] In this paper, we use the Euler number e instead of 2 as the base of the exponential.

[2] Column vectors and matrices are denoted by lowercase and uppercase bold letters, respectively.

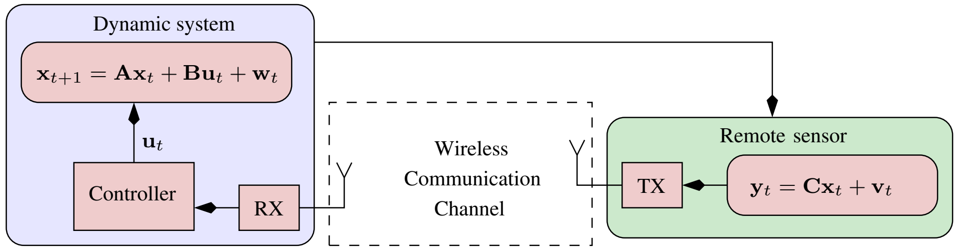

Fig. 1. Model of the discrete-time linear dynamic system: remote sensor, wireless communication channel and system controller.

$n_x \times n_u$ real matrices, respectively, $\mathbf{u}_t$ is the control input, and $\mathbf{w}_t$ is a zero-mean bounded noise process. The system in (1) is supposed to be unstable, i.e., it is characterized by $\rho(\mathbf{A}) > 1$, where $\rho(\mathbf{A})$ is the spectral radius of the matrix $\mathbf{A}$, that is the largest eigenvalue modulus of $\mathbf{A}$. The state $\mathbf{x}_t$ of the linear system in (1) is measured by a remote sensor (see Figure 1) providing the size-n_y measurement

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t \tag{2}$$

where $\mathbf{v}_t \in \mathbb{R}^{n_y}$ is a zero-mean bounded noise process independent of $\mathbf{w}_t$. Notice that the boundedness condition of the initial state and of the system and measurement noise are essential to the forthcoming theory of anytime reliability and will not be discussed further.

The remote sensor is equipped with one antenna and sends its measurement to a controller through a noisy wireless communication channel. Specifically, in the considered setting the sensor at each time step t first quantizes the measurement $\mathbf{y}_t$ into a k-bit vector $\mathbf{q}_t$. In this paper, as in [6], we consider a uniform lattice quantizer. The quantized measurements are the input of a channel encoder $\mathcal{E}_t$, whose output $\mathbf{e}_t$ is a binary vector of length n. The code rate is thus $R_c = k/n$. The encoded bits are modulated to a vector of m symbols, $\mathbf{s}_t$, belonging to a given constellation and then trasmitted over the wireless channel. The received signal is given by

$$\mathbf{r}_t = \mathbf{H}_t\mathbf{s}_t + \mathbf{z}_t \tag{3}$$

where $\mathbf{z}_t$ is a size-m vector representing additive noise with i.i.d. complex circularly symmetric Gaussian random entries with zero mean and unitary variance. The $m \times m$ diagonal channel matrix $\mathbf{H}_t = \mathrm{diag}(h_{t1}, \ldots, h_{tm})$ contains in its j-th diagonal element, $j = 1, \ldots, m$, the channel coefficient experienced at time t by the j-th transmitted symbol.

At time step t the receiver processes the received signal $\mathbf{r}_t$ and obtains the soft estimates of the quantized measurements. Specifically, $\mathbf{r}_t$ is first processed by a ML demodulator which outputs soft estimates $\hat{\mathbf{e}}_t$ of the coded bits $\mathbf{e}_t$. The soft estimates $\hat{\mathbf{e}}_t$ and the estimates computed at time steps $\tau = 1, \ldots, t-1$, i.e., $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_{t-1}$ are then sent to a decoder $\mathcal{D}_t$ which outputs estimates $\hat{\mathbf{q}}_{0|t}, \ldots, \hat{\mathbf{q}}_{t|t}$ of the quantized measurements. The notation $\hat{\mathbf{q}}_{\tau|t}$ represents the estimate of $\mathbf{q}_\tau$ obtained at the receiver at time step t.

Finally a digital-to-analog converter provides estimates $\hat{\mathbf{y}}_{1|t}, \ldots, \hat{\mathbf{y}}_{t|t}$ of the observations. Again the notation $\hat{\mathbf{y}}_{\tau|t}$ represents the estimate of $\mathbf{y}_\tau$ obtained at time step t. These estimates are sent to the system controller.

The system controller is in charge of generating suitable commands $\mathbf{u}_t$ in order to keep the system stable. In particular at time t the controller takes as input the estimates $\hat{\mathbf{y}}_{1|t}, \ldots, \hat{\mathbf{y}}_{t|t}$, and outputs the control $\mathbf{u}_t$. The controller is a chain of t filters. The τ-th filter produces the output $\hat{\mathbf{x}}_{\tau|t}$, which is an estimate of the state $\mathbf{x}_\tau$ at time step t, and has two inputs: the estimate $\hat{\mathbf{x}}_{\tau-1|t}$ and the vector $\hat{\mathbf{y}}_{\tau|t}$. The output of the t-th filter, $\hat{\mathbf{x}}_{t|t}$, provides then the estimate of the current state. The command $\mathbf{u}_t$ is finally obtained as a linear combination of the state estimate $\hat{\mathbf{x}}_{t|t}$

$$\mathbf{u}_t = \mathbf{K}\hat{\mathbf{x}}_{t|t}$$

where the $n_x \times n_u$ matrix $\mathbf{K}$ is chosen to stabilize the system, i.e. so that $\rho(\mathbf{A} + \mathbf{B}\mathbf{K}) < 1$. In this paper, hypercuboidal filters [6] have been employed in the system controller.

Let $P_e(t, d)$ be the probability that, at time t, the oldest decoding error made by the decoder $\mathcal{D}_t$ is d steps back in the past:

$$\begin{aligned} P_e(t,d) = \mathbb{P}\{\{\hat{\mathbf{q}}_{t-d+1|t} \neq \mathbf{q}_{t-d+1}\} \cap \\ \{\hat{\mathbf{q}}_{\tau|t} = \mathbf{q}_\tau, \tau < t-d+1\}\}. \end{aligned} \tag{4}$$

We say that the encoding-decoding scheme is *anytime reliable* on a given channel if it satisfies:

$$P_e(t,d) < K\mathrm{e}^{-\beta d}, \ \forall t,\, d > d_0 \tag{5}$$

where K, β and d_0 are positive constants that depend on the coding scheme and on the channel. If the code satisfies (5), then β is called its *anytime exponent* (on that channel).

Sukhavasi and Hassibi derive in [6] the conditions under which an anytime reliable encoding-decoding scheme can be used to stabilize the system of (1)-(2) in the mean-square sense, so that the expected value of $\|\mathbf{x}_t\|^2$ is bounded for all t. It is shown in [6] that, when using hypercuboidal filters, mean-square sense stability is achieved by a code with anytime exponent β satisfying $\beta > 2\log\rho(\overline{\mathbf{A}})$, where $\overline{\mathbf{A}}$ is the $n_x \times n_x$ matrix whose elements are the absolute values of the elements of $\mathbf{A}$.

III. LDPC CONVOLUTIONAL CODES

Following [7], we consider a channel coding scheme based on systematic LDPC convolutional codes. Precisely, the encoder at time t, $\mathcal{E}_t$, is the systematic encoder corresponding to the parity-check matrix

$$\mathbf{Z}_{[1:t]} = \begin{bmatrix} \mathbf{Z}_0 & & & & \\ \mathbf{Z}_1 & \mathbf{Z}_0 & & & \\ \vdots & \mathbf{Z}_1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \mathbf{Z}_{t-1} & \mathbf{Z}_{t-2} & \dots & \mathbf{Z}_1 & \mathbf{Z}_0 \end{bmatrix} \tag{6}$$

where all matrices $\mathbf{Z}_i$, $i = 0, \dots, t-1$, are $(n-k) \times n$ sparse binary matrices, and $\mathbf{Z}_0$ is full-rank (over $\mathbb{GF}(2)$) in order for $\mathbf{Z}_{[1:t]}$ to have full row rank. (Notice that we have explicitly restricted our focus to a *Toeplitz* parity-check matrix, for simplicity.) Such coding scheme is causal thanks to the lower-triangular structure of the parity-check matrix. Moreover, it can be considered as a convolutional code with boundlessly increasing memory, whose trellis at time t has a number of states equal to $2^{(t-1)k}$.

Structurally, as in [7], we have built the parity-check matrix $\mathbf{Z}_{[1:t]}$ starting from a protograph [11] matrix $\mathbf{P}_{[1:t]}$, given by

$$\mathbf{P}_{[1:t]} = \begin{bmatrix} \mathbf{P}_0 & & & & \\ \mathbf{P}_1 & \mathbf{P}_0 & & & \\ \vdots & \mathbf{P}_1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \mathbf{P}_{t-1} & \mathbf{P}_{t-2} & \dots & \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} \tag{7}$$

where all matrices $\mathbf{P}_i$, $i = 0, \dots, t-1$, are $(n_0 - k_0) \times n_0$ matrices with nonnegative integer entries and $\mathbf{P}_0$ is full-rank. $\mathbf{Z}_{[1:t]}$ is obtained by *lifting* $\mathbf{P}_{[1:t]}$ to order r, namely:

- each zero of $\mathbf{P}_{[1:t]}$ is lifted to a $r \times r$ all-zero matrix, and
- each nonzero entry of $\mathbf{P}_{[1:t]}$ equal to b is lifted to the modulo-2 sum of b permutation matrices of size $r \times r$, chosen at random between the $r!$ possible permutation matrices of that size.

As for ordinary LDPC codes [12], $\mathbf{P}_{[1:t]}$ is interpreted as the adjacency matrix of the protograph at time t, where columns represents variable nodes (VNs) and rows represent check nodes (CNs). If a given element of $\mathbf{P}_{[1:t]}$ is equal to b, there are b edges connecting the corresponding CN and VN. Moreover, if $b > 0$, the two nodes are neighbors.

Notice that $n = rn_0$ and $k = rk_0$ so that $R_c = k/n = k_0/n_0$. Since permutation matrices are chosen at random, we actually obtain an *ensemble* of codes, each one corresponding to a given choice of the permutations. The fact that $\mathbf{Z}_{[1:t]}$ is sparse even if $\mathbf{P}_{[1:t]}$ is not, allows us to choose the latter with a certain degree of freedom. In particular in our work we consider

$$\mathbf{P}_0 = \left[\mathbf{P}_{0,p} \middle| \mathbf{I}_{\overline{k}_0} \right] \tag{8}$$

and, for $\tau = 1, 2, \dots$

$$\mathbf{P}_\tau = \left[\mathbf{P}_{\tau,p} \middle| \mathbf{0}_{\overline{k}_0 \times \overline{k}_0} \right] \tag{9}$$

where $\overline{k_0} = n_0 - k_0$. Note that the description of the protograph matrices in (8) and (9) encompasses the one given in [7] where $\overline{k_0} = 1$ and $n_0 = 2$.

In our system the decoder $\mathcal{D}_t$ implements belief propagation (BP) [12] on the bipartite graph defined by $\mathbf{Z}_{[1:t]}$. In this case the advantage of deriving the code by lifting a protograph relies on the fact that the *local* structure of the code graph (i.e., the neighborhood of any given node) always looks like the protograph one ([11], [13]), while the probability of short cycles (detrimental for BP) is reduced by increasing r. Thus, convergence properties of BP decoding algorithm can be studied directly on $\mathbf{P}_{[1:t]}$, while neglecting the effects of cycles, provided that r is large enough.

Finally, notice that both the encoding and the decoding complexity scale essentially as the number of edges do, i.e., quadratically with t. Practically, we can suppose that a feedback channel is available, over which the decoder tells the encoder at regular intervals to suitably trim the memory, on the basis of the outcome of some CRC test.

IV. Anytime Reliability of LDPC Convolutional Codes on the Wireless Channel

In this section, we study the anytime reliability of the protograph-based LDPC convolutional codes. We first derive a bound on the bit error probability of the LDPC encoding/decoding scheme for a generic noisy channel. Then in Section IV-A we specialize to the AWGN case.

In order to assess the performance of the BP decoder, in what follows, we slightly modify the P-EXIT approach of [7], [13]. In particular, we suppose that the BP messages exchanged between VNs and CNs are sent through AWGN channels, and we track the evolution of the SNR of such channels with the iteration index of the BP algorithm.

Let $\mathcal{N}_c(i)$ be the set of VNs that are neighbors of i-th CN. Analogously, let $\mathcal{N}_v(j)$ be the set of CNs that are neighbors of VN j. Let us also define the following variables:

- $\rho_{ch}(j)$: the physical-channel SNR for VN j.
- $\rho^{(l)}_{C\to V,t}(i,j)$: the SNR for message travelling from CN i to VN j at the l-th iteration of the BP algorithm and at time step t (if $i \in \mathcal{N}_v(j)$).
- $\rho^{(l)}_{V\to C,t}(i,j)$: the SNR for message travelling from VN j to CN i at the l-th iteration of the BP algorithm and at time step t (if $i \in \mathcal{N}_v(j)$).

Then, the SNR evolution at time t can be approximately determined through the following set of update equations[3] [14]:

- **Initialization:** For $j = 1, \dots, n_0 t$, $i \in \mathcal{N}_v(j)$:
$$\rho^{(0)}_{V\to C,t}(i,j) = \rho_{ch}(j) \tag{10}$$
- **CN to VN update:** For $i = 1, \dots, \overline{k}_0 t$, $j \in \mathcal{N}_c(i)$:
$$\rho^{(l+1)}_{C\to V,t}(i,j) \simeq M\left(\sum_{s\in\mathcal{N}_c(i), s\neq j} M\left(\rho^{(l)}_{V\to C,t}(i,s)\right)\right) \tag{11}$$
- **VN to CN update:** For $j = 1, \dots, n_0 t$, $i \in \mathcal{N}_v(j)$:
$$\rho^{(l+1)}_{V\to C,t}(i,j) = \sum_{s\in\mathcal{N}_v(j), s\neq i} \rho^{(l+1)}_{C\to V,t}(s,j) + \rho_{ch}(j) \tag{12}$$

[3]For simplicity, the update equations are given in the hypothesis that the protograph matrix is binary. However, the results hold in general.

- **Output decision variable SNR:** For $j = 1, \ldots, n_0 t$:

$$\rho_t^{(l+1)}(j) = \sum_{s \in \mathcal{N}_v(j)} \rho_{C \to V,t}^{(l+1)}(s,j) + \rho_{ch}(j) \quad (13)$$

The function $M(\rho)$ appearing in (11) is defined as $M(\rho) = J^{-1}(1 - J(\rho))$, where

$$J(\rho) = 1 - \int_{-\infty}^{+\infty} \frac{e^{-(y-2\rho)^2/(8\rho)}}{\sqrt{8\pi\rho}} \log_2(1 + e^{-y}) dy \quad (14)$$

gives the mutual information between the input of a binary-input AWGN channel with SNR ρ and the corresponding output. Notice that $M(\rho)$ is a nonnegative, strictly decreasing function of ρ and that $M^{-1}(\rho) = M(\rho)$. The approximation involved in (11) has been observed to be tight in most cases of interest [15]. It can be easily proven [14] that the sequences $\rho_{V \to C,t}^{(l)}(i,j)$ and $\rho_{C \to V,t}^{(l)}(i,j)$ are monotonically increasing with iteration index l. Moreover, these sequences are bounded, as long as $\rho_{ch}(j)$ is bounded for all j. Thus, they converge to a limit when l goes to infinity. Let us call such limits $\rho_{C \to V,t}^{(\infty)}(i,j)$ and $\rho_{V \to C,t}^{(\infty)}(i,j)$, which are a function of the channel SNR values. The output decision variable SNR for VN j, after a large number of iterations, is then given by

$$\rho_t^{(\infty)}(j) = \sum_{s \in \mathcal{N}_v(j)} \rho_{C \to V,t}^{(\infty)}(s,j) + \rho_{ch}(j). \quad (15)$$

Next, starting from (4) we compute an upper bound to $P_e(t,d)$ after a large number of BP iterations as follows:

$$\begin{aligned} P_e(t,d) &\overset{(a)}{\leq} \mathbb{P}\{\hat{\mathbf{q}}_{t-d+1|t} \neq \mathbf{q}_{t-d+1}\} \\ &\overset{(b)}{\leq} \sum_{i=1}^{k_0} Q\left(\sqrt{\rho_t^{(\infty)}((t-d)n_0 + i)}\right) \\ &\overset{(c)}{\leq} \frac{1}{2} \sum_{i=1}^{k_0} \mathrm{e}^{-\rho_t^{(\infty)}((t-d)n_0+i)/2} \\ &\leq \frac{k_0}{2} \mathrm{e}^{-\min_{i=1}^{k_0} \rho_t^{(\infty)}((t-d)n_0+i)/2} \end{aligned} \quad (16)$$

where (a) follows from (4), (b) follows from the fact that the code is systematic and the union bound, (c) from the Chernoff bound on $Q(x)$ [16]. Thus, thanks to (5) and (16), the anytime exponent of the considered coding scheme can be lower-bounded by

$$\beta \geq \underline{\beta} = \lim_{d \to \infty} \frac{\min_{i=1}^{k_0} \rho_t^{(\infty)}((t-d)n_0 + i)}{2d}. \quad (17)$$

Thus, a sufficient condition for the stabilization of system (1)-(2) in the single-node scenario is that $\underline{\beta} > 2 \log \rho(\overline{\mathbf{A}})$. The value of $\underline{\beta}$ can be obtained numerically, for a given coding scheme, thanks to (10)-(13).

A. The AWGN case

For the AWGN case, $\rho_{ch}(j) = \rho_{ch}$ for all j. Notice that, because of (8) and (9), every systematic VN is connected to a CN that is connected to a degree-1 nonsystematic VN. Since the message coming from a degree-1 VN is set to ρ_{ch} at every iteration, thanks to the fact that $M(\rho)$ is monotonically decreasing with ρ, we can upper-bound the CN-to-VN message exchanged at iteration l as

$$M\left(\sum_{s \in \mathcal{N}_c(i), s \neq j} M(\rho_{V \to C,t}^{(l)}(i,s))\right) \leq \rho_{ch} \quad (18)$$

which clearly holds also for $l \to \infty$. By plugging the above bound into (15), we can write the following upper bound on the output SNR for systematic variable j:

$$\rho_t^{(\infty)}(j) \leq (|\mathcal{N}_v(j)| + 1)\rho_{ch}. \quad (19)$$

It is shown in [14] that this upper bound is actually reached for $\rho_{ch} \to \infty$ if some mild conditions are satisfied.

Thus, for sufficiently large ρ_{ch}, the lower bound on the anytime exponent can be approximated by

$$\underline{\beta} = \lim_{d \to \infty} \frac{\min_{i=1}^{k_0}(|\mathcal{N}_v((t-d)n_0 + i)| + 1)\rho_{ch}}{2d}. \quad (20)$$

In order to achieve anytime reliability for sufficiently large ρ_{ch}, the VN degrees must then increase linearly with d. If $\min_{i=1}^{k_0} |\mathcal{N}_v((t-d)n_0 + i)| = \gamma d + o(d)$ for all t, then

$$\underline{\beta} = \gamma \rho_{ch}/2. \quad (21)$$

The above results tells that, in order to stabilize the system in (1)-(2) over an AWGN channel with the LDPC convolutional encoding-decoding scheme and hypercuboidal filters, it is sufficient that the channel SNR satisfies $\rho_{ch} > 4 \log \rho(\overline{\mathbf{A}})/\gamma$.

Notice also that, from the analysis, it seems beneficial to increase γ, which corresponds to using a denser protograph: however, while this is true for a lifting order r going to infinity, increasing γ may affect negatively the performance for a finite r, due to the increased probability of finding short cycles.

V. Results

To validate the theoretical results, we have simulated a system characterized by

$$\mathbf{A} = \begin{bmatrix} 1.285 & 0.127 & 0. \\ 4 & 1.285 & 0.002 \\ -3.94 & -0.280 & 0.979 \end{bmatrix} \quad (22)$$

with $\rho(\overline{\mathbf{A}}) = 1.997$, and with the matrices $\mathbf{B}$ and $\mathbf{C}$ chosen as in [6, Example 1]. Moreover we used the LDPC convolutional code of [7], with $k_0 = 1$, $n_0 = 2$ and $\mathbf{P}_{\tau,p} = 1$ for all τ, so that $\gamma = 1$. In the LDPC code construction, we have used a lifting order $r = 12$, which, albeit small, already grants a good match between theoretical analysis and simulation results.

If the channel between sensor and controller can be modelled as AWGN, then, by using (21), we conclude that the system can be stabilized in the mean-square sense as long as $\rho_{ch} > 4 \log \rho(\overline{\mathbf{A}}) = 4.4$ dB. Fig. 2 shows the error probability $P_e(t,d)$ versus the delay d. The experimental curves (solid lines) obtained by simulations, are compared with the theoretical ones (dashed), obtained according to (16), in the hypothesis that the upper bound of (19) is actually achieved. As it can be seen, for $\rho_{ch} > 4$ dB, the slope of the simulated $P_e(t,d)$, which corresponds to the anytime exponent, is larger

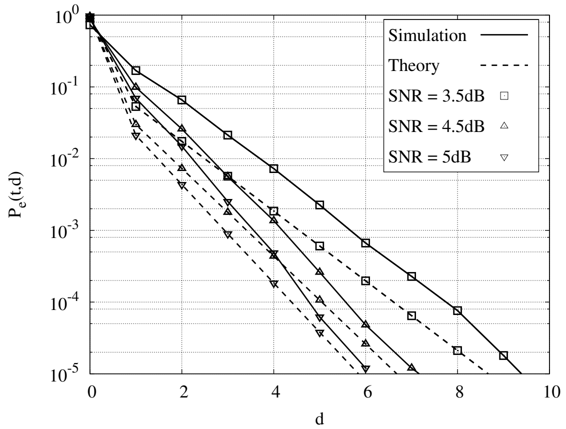

Fig. 2. Anytime exponent of the LDPC convolutional codes on the AWGN channel: theory versus simulations.

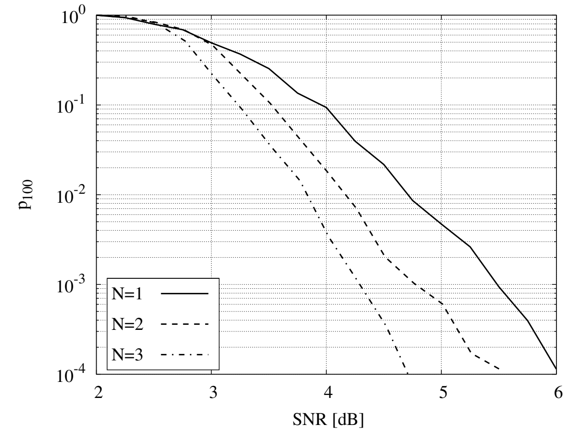

Fig. 3. Performance of multiple sensors in the fading scenario.

than the theoretical one, as predicted from the analysis. Monte Carlo simulations actually show that the system is controlled for $\rho_{ch} = 4.5$ dB.

A. Extension to the fading scenario

Through simulations, we have investigated an extension of the control system of Section II to the fading scenario in the case of a network made of multiple sensors. More precisely, we consider the case where there are N identical remote sensors, whose measurements are subject to independent bounded noise. At time t, the i-th sensor, $i = 1, \dots, N$, obtains the size-n_y measurement

$$\mathbf{y}_t^{(i)} = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t^{(i)} \tag{23}$$

where $\mathbf{v}_t^{(i)}$, $i = 1, \dots, N$, are zero-mean bounded noise processes independent of each other and of $\mathbf{w}_t$. The i-th sensor encodes the information as described in Section III and transmits the symbol vector $\mathbf{s}_t^{(i)}$ to the common receiver, which is equipped with N antennas. The signal received at the j-th receive antenna, $j = 1, \dots, N$, will then be given by

$$\mathbf{r}_t^{(j)} = \sum_{i=1}^{N} \mathbf{H}_t^{(j,i)} \mathbf{s}_t^{(i)} + \mathbf{z}_t^{(j)} \tag{24}$$

where the diagonal channel matrix $\mathbf{H}_t^{(j,i)}$ contains on its diagonal the channel coefficients from sensor i and receive antenna j. We assume Rayleigh fading, thus the instantaneous SNR at the receiver is a random variable exponentially distributed as

$$f_{\rho_{ch}}(\rho) = \mathrm{e}^{-\rho/\overline{\rho}_{ch}}/\overline{\rho}_{ch} \tag{25}$$

The receiver performs jointly optimal demodulation of the N superimposed transmitted signals. After demodulation, N decoders work in parallel to decode the information sent by the N sensors. The controller computes the feedback signal $\mathbf{u}_t$ by putting together the reconstructed measurements from all sensors. In Fig. 3, we show the performance of the simulated control network. We have measured the probability, p_{100}, that, after 100 time steps, the Euclidean distance between the system state and the estimated state is larger than 10^3, as a function of the average SNR, $\overline{\rho}_{ch}$, for $N = 1, 2, 3$. The power transmitted by each sensor has been normalized so that the total transmitted power is the same for the three cases. As it can be seen, p_{100} decreases with $\overline{\rho}_{ch}$. Notice that using multiple sensors is beneficial since it reduces the SNR required for stability.

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