How Sensitive is Compute-and-Forward to Channel Estimation Errors?

Koralia N. Pappi*, George K. Karagiannidis*, and Robert Schober†
* Department of Electrical & Computer Engineering, Aristotle University of Thessaloniki, Thessaloniki, GR-54124, Greece,
E-mails: {kpappi, geokarag}@auth.gr
† Department of Electrical & Computer Engineering, University of British Columbia, Vancouver, BC V6T1Z4, Canada,
E-mail: {rschober}@ece.ubc.ca

***Abstract*—We investigate the sensitivity of Compute-and-Forward (C&F) to channel estimation errors. More specifically, a general formula for the computation rate region of a C&F relay, suffering from imperfect channel estimation, is derived, which is then tightly approximated for Gaussian distributed channel estimation errors. Furthermore, a closed-form expression for the distribution of the C&F rate loss is presented, which can be efficiently used to compute relevant statistical parameters, such as the mean rate loss. Numerical and simulation results highlight the high sensitivity of the overall network performance to channel estimation errors.**

I. Introduction

Compute-and-Forward (C&F), first introduced in [1], is a very promising technique, which exploits interference in multiuser networks, in order to achieve high transmission rates. A C&F relay decodes an integer equation of the users' messages, which is then forwarded to the destination. Having enough equations available, the destination can decode the messages of all users. In order to achieve this, nested lattice codes are used, which are investigated in detail in [2]. The corresponding achievable computation rate region for the relay is derived in [1], while the relay is assumed to have perfect knowledge of the channel coefficients between the transmitters and the relay.

The Degrees of Freedom (DoF) of C&F were investigated in [3], where a new implementation was proposed, which achieves L DoF for a network with L transmitters and L relays. This scheme assumes that perfect Channel State Information (CSI) is available at both the transmitters and the relays. Other works on PHY-layer network coding in C&F networks, e.g. [4], also assume availability of perfect CSI, rendering it vital for their practical implementation. However, perfect CSI is not always available, as in practice, only an estimate of the actual channel coefficients is known. In order to overcome this obstacle, a blind C&F scheme was only recently introduced in [5], which avoids the need of CSI, at the cost of achieving sub-optimal performance and increasing computational complexity. However, to the best of the authors' knowledge, the actual effect of imperfect CSI on the performance of C&F has not been explored yet.

In this work, we investigate the effect of noisy channel estimation on the achievable computation rate of the C&F network presented in [1]. More specifically, our contributions are as follows.

- An analytical formula for the computation rate region of a C&F relay with channel estimation error is derived. This formula is valid for any statistical model of the channel estimation error.
- For Gaussian distributed channel estimation errors, a tight analytical approximation for the computation rate region is proposed.
- A closed-form expression for the distribution of the rate loss due to Gaussian channel estimation errors is presented. This expression can be efficiently used to provide insight into the impact of channel estimation errors on significant system performance parameters, such as the mean rate loss.

The rest of the paper is organized as follows. The system model is described in Section II. In Section III, the main results on the computation rate and the rate loss are presented. The numerical results are discussed in Section IV, while conclusions are given in Section V.

II. System Model

A. Compute and Forward

We consider a network which consists of L transmitters, M relays, and a centralized decoder. Let the channel matrix be $\mathbf{H} = [\mathbf{h_1}, \mathbf{h_2}, \dots, \mathbf{h_M}]$, where $\mathbf{h_m} = [h_{m,1}, h_{m,2}, \dots, h_{m,L}]^T$, $(\cdot)^T$ denotes transposition, $m = 1, 2, \dots, M$, and $h_{m,l}$ is the channel coefficient of the line between the l-th transmitter and the m-th relay.

Each relay observes a noisy linear combination of L transmitted signals [1]

$$\mathbf{y}_m = \sum_{l=1}^{L} h_{m,l}\mathbf{x}_l + \mathbf{z}_m, \tag{1}$$

where $\mathbf{y}_m$ is the N-dimensional received vector at the m-th relay, $\mathbf{x}_l$ is the N-dimensional transmitted vector of the l-th transmitter, and $\mathbf{z}_m$ is the N-dimensional additive noise vector. We consider a real-valued system with $\mathbf{x}_l, \mathbf{y}_m \in \mathbb{R}^N$,

The work of K. N. Pappi has been supported by the THALES Project DISCO, co-financed by the European Union (European Social Fund-ESF) and Greek National Funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALES: Reinforcement of the interdisciplinary and/or inter-institutional research and innovation.

$h_{m,l} \in \mathbb{R}$ and $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{N\times N})$, where $\mathbf{I}_{N\times N}$ is the $N \times N$ unitary matrix[1]. The N-dimensional transmitted vector $\mathbf{x}_l$ is constructed based on the k_l-dimensional message vector $\mathbf{w}_l$ using a nested lattice code. The elements of the message vector are drawn from finite field $\mathbb{F}_p$, where p is a prime. Thus, the rate of the l-th transmitter measured in bits is

$$R_l = \frac{N}{k_l} \log_2 p. \tag{2}$$

Each relay decodes an equation of the transmitted signals using integer coefficients, which form the equation coefficient vector $\mathbf{a}_m = [a_{m,1}, a_{m,2}, \ldots, a_{m,L}]^T$, $\mathbf{a}_m \in \mathbb{Z}^L$, and then forwards this equation to the centralized decoder. It was proven in [1] that, for a system with channel coefficient vector $\mathbf{h}_m$ and equation coefficient vector $\mathbf{a}_m$, the computation rate region achieved at the m-th relay is

$$\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \alpha_m) = \frac{1}{2} \log_2^+ \left(\frac{P}{\alpha_m^2 + P\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right), \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, P is the power constraint for the transmitted signals (i.e., $\|\mathbf{x}_l\|^2 \leq NP$), $\log_2^+(\cdot) = \max[0, \log_2(\cdot)]$, and α_m is a coefficient by which the relay multiplies the received signal before computing an equation.

The optimal choice for α_m in terms of maximizing the computation rate in (3) is the Minimum Mean Squared Error (MMSE) coefficient, given by [1]

$$\beta_m = \arg\max_{\alpha_m} \left[\mathcal{R}\left(\mathbf{h}_m, \mathbf{a}_m, \alpha_m\right)\right] = \frac{P\mathbf{h}_m^T \mathbf{a}_m}{1 + P\|\mathbf{h}_m\|^2}. \tag{4}$$

Thus, the maximum rate that can be achieved by the l-th transmitter is

$$R_l \leq \min_{m: a_{m,l} \neq 0} \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \beta_m). \tag{5}$$

B. Channel Estimation Error

CSI is needed at the relay for the computation of the MMSE coefficient β_m. When the CSI is not perfect, the produced error will lead to the computation of a different MMSE coefficient. So, if $\hat{\mathbf{h}}_m$ is the estimate of $\mathbf{h}_m$, the relay computes $\hat{\beta}_m$ as

$$\hat{\beta}_m = \frac{P\hat{\mathbf{h}}_m^T \mathbf{a}_m}{1 + P\|\hat{\mathbf{h}}_m\|^2}, \tag{6}$$

and the MMSE coefficient error is defined as

$$\varepsilon = \hat{\beta}_m - \beta_m. \tag{7}$$

In this paper, we only consider the effect of imperfect CSI on the computation of β_m, for a given choice of $\mathbf{a}_m$. Note that the relay also utilizes CSI in order to choose the coefficient vector $\mathbf{a}_m$. However, following the procedure of computing the optimal $\mathbf{a}_m$ as in [6], channel estimation errors may or may not lead to a sub-optimal choice of $\mathbf{a}_m$. Thus, if a sub-optimal $\mathbf{a}_m$ is chosen because of the imperfect CSI, the achievable rates will be further reduced.

[1]The analysis for real-valued systems presented in this work can be easily extended to complex-valued systems, see also [1]

III. Computation Rate Region with Channel Estimation Error

A. Computation Rate Region

Let the denominator in (3) be a function of α_m,

$$f(\alpha_m) = \alpha_m^2 + P\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2. \tag{8}$$

Then, the corresponding function for a system with channel estimation error is

$$f(\hat{\beta}_m) = \hat{\beta}_m^2 + P\|\hat{\beta}_m \mathbf{h}_m - \mathbf{a}_m\|^2. \tag{9}$$

Theorem 1: The computation rate region achieved by a C&F system with channel estimation error and MMSE coefficient error ε is

$$\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) = \frac{1}{2} \log_2^+ \left(\frac{P}{f(\beta_m) + \varepsilon^2 (1 + P\|\mathbf{h}_m\|^2)} \right). \tag{10}$$

Proof: The computation rate region for an MMSE coefficient $\hat{\beta}_m$ is given by

$$\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) = \frac{1}{2} \log_2^+ \left(\frac{P}{f(\hat{\beta}_m)} \right). \tag{11}$$

We begin the proof by rewriting $f(\hat{\beta}_m)$ as

$$\begin{aligned} f(\hat{\beta}_m) &= \hat{\beta}_m^2 + P\|\hat{\beta}_m \mathbf{h}_m - \mathbf{a}_m\|^2 \\ &= \hat{\beta}_m^2 + P\left(\hat{\beta}_m^2 \|\mathbf{h}_m\|^2 - 2\hat{\beta}_m \mathbf{h}_m^T \mathbf{a}_m + \|\mathbf{a}_m\|^2\right). \end{aligned} \tag{12}$$

Using $\hat{\beta}_m = \beta_m + \varepsilon$, (12) yields

$$\begin{aligned} f(\hat{\beta}_m) &= (\beta_m + \varepsilon)^2 + P\Big[(\beta_m + \varepsilon)^2 \|\mathbf{h}_m\|^2 + \|\mathbf{a}_m\|^2 \\ &\quad - 2\left(\beta_m + \varepsilon\right) \mathbf{h}_m^T \mathbf{a}_m\Big] \\ &= f(\beta_m) + \left(2\beta_m \varepsilon + \varepsilon^2\right)\left(1 + P\|\mathbf{h}_m\|^2\right) - 2P\varepsilon \mathbf{h}_m^T \mathbf{a}_m. \end{aligned} \tag{13}$$

From (4), it holds that

$$P\mathbf{h}_m^T \mathbf{a}_m = \beta_m \left(1 + P\|\mathbf{h}_m\|^2\right). \tag{14}$$

Using (14) in (13) yields

$$f(\hat{\beta}_m) = f(\beta_m) + \varepsilon^2 \left(1 + P\|\mathbf{h}_m\|^2\right), \tag{15}$$

which combined with (11) results in (10). This concludes the proof. ■

B. High SNR Approximation

In this section, we consider noisy estimates of the form

$$\hat{\mathbf{h}}_m = \mathbf{h}_m + \mathbf{n}_m, \tag{16}$$

where $\mathbf{n}_m$ is the channel estimation error vector of length L, assumed to be Gaussian distributed with $\mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_{L\times L})$ [7]–[9]. In order to gain more insight into the effect of channel estimation errors on the computation rate, we present an approximation of the achievable rate region for high Signal-to-Noise Ratio (SNR) values.

From (4) we have

$$|\beta_m| = \left|\frac{P\mathbf{h}_m^T\mathbf{a}_m}{1+P\|\mathbf{h}_m\|^2}\right| \leq \frac{|\mathbf{h}_m^T\mathbf{a}_m|}{\|\mathbf{h}_m\|^2}, \tag{17}$$

and for the inner product $\mathbf{h}_m^T\mathbf{a}_m$ it holds

$$|\mathbf{h}_m^T\mathbf{a}_m| \leq \|\mathbf{h}_m\|\|\mathbf{a}_m\|. \tag{18}$$

Combining (17) and (18) we get

$$|\beta_m| \leq \frac{\|\mathbf{h}_m\|\|\mathbf{a}_m\|}{\|\mathbf{h}_m\|^2} = \frac{\|\mathbf{a}_m\|}{\|\mathbf{h}_m\|} \triangleq \bar{\beta}_m. \tag{19}$$

For high values of P, (17) approaches equality. Furthermore, the achievable computation rate region at the relay depends on the choice of $\mathbf{a}_m$. It is shown in [1] that integer equation coefficients close to the channel coefficients maximize the computation rate. In this case, since $\mathbf{a}_m$ is similar to $\mathbf{h}_m$, these vectors are almost collinear, and thus (18) and (19) also approach equality. Similarly, owing to the high-SNR assumption, $\|\mathbf{h}_m\| \approx \|\hat{\mathbf{h}}_m\|$ holds and $\hat{\mathbf{h}}_m$ and $\mathbf{a}_m$ are also almost collinear, so we obtain $|\hat{\beta}_m| \approx \frac{\|\mathbf{a}_m\|}{\|\hat{\mathbf{h}}_m\|}$.

Thus, we approximate the squared MMSE channel coefficient error ε^2 by $\bar{\varepsilon}^2$, where $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{\|\mathbf{a}_m\|}{\|\hat{\mathbf{h}}_m\|} - \bar{\beta}_m, \tag{20}$$

and the approximate computation rate region becomes

$$\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m, \bar{\varepsilon}) = \frac{1}{2}\log_2^+\left(\frac{P}{f(\beta_m) + \bar{\varepsilon}^2\left(1+P\|\mathbf{h}_m\|^2\right)}\right). \tag{21}$$

Using (16), (20) can be rewritten as

$$\bar{\varepsilon} = \frac{\|\mathbf{a}_m\|}{\sqrt{\sum\limits_{l=1}^{L}\hat{h}_{m,l}^2}} - \bar{\beta}_m = \frac{\|\mathbf{a}_m\|}{\sqrt{\sum\limits_{l=1}^{L}\left(h_{m,l}+n_{m,l}\right)^2}} - \bar{\beta}_m, \tag{22}$$

where $n_{m,l}$ is the l-th element of the noise vector $\mathbf{n}_m$, which follows a zero-mean Gaussian distribution, $n_{m,l} \sim \mathcal{N}(0,\sigma^2)$. Then, $\hat{h}_{m,l}$ follows a Gaussian distribution $\mathcal{N}(h_{m,l},\sigma^2)$, and thus, if we set a random variable (RV), $c = \sqrt{\frac{1}{\sigma^2}\sum\limits_{l=1}^{L}\hat{h}_{m,l}^2}$, then c follows a noncentral chi distribution with probability density function (pdf) given by [10]

$$p_c(x) = \frac{e^{-\frac{x^2+\lambda^2}{2}}x^k\lambda}{(\lambda x)^{\frac{k}{2}}} I_{\frac{k}{2}-1}(\lambda x), \quad x \geq 0, \tag{23}$$

where $k = L$, $\lambda = \frac{1}{\sigma}\|\mathbf{h}_m\|$ and $I_\alpha(x)$ is the modified Bessel function of the first kind. Then, $\bar{\varepsilon} = \frac{\|\mathbf{a}_m\|}{\sigma c} - \bar{\beta}_m$, and the pdf of $\bar{\varepsilon}^2$ is given by (24) at the top of the next page, where

$$A(x) = -\frac{\|\mathbf{a}_m\|^2 + \left(\lambda\sigma\left(x+\bar{\beta}_m\right)\right)^2}{2\sigma^2\left(x+\bar{\beta}_m\right)^2}, \tag{25}$$

and

$$B(x) = \frac{\lambda\|\mathbf{a}_m\|}{\sigma\left(x+\bar{\beta}_m\right)}. \tag{26}$$

Now from (21), the approximate mean computation rate region is given by

$$\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m) = \mathbb{E}\left[\frac{1}{2}\log_2^+\left(\frac{P}{f(\beta_m) + \bar{\varepsilon}^2\left(1+P\|\mathbf{h}_m\|^2\right)}\right)\right], \tag{27}$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to the distribution in (24). Then, (27) becomes

$$\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m) = \int_0^\infty \frac{1}{2}\log_2^+\left(\frac{P}{f(\beta_m) + x\left(1+P\|\mathbf{h}_m\|^2\right)}\right) p_{\bar{\varepsilon}^2}(x)\mathrm{d}x. \tag{28}$$

Considering that

$$\log_2^+\left(\frac{P}{f(\beta_m) + x\left(1+P\|\mathbf{h}_m\|^2\right)}\right) = 0, \tag{29}$$

when

$$P < f(\beta_m) + x\left(1+P\|\mathbf{h}_m\|^2\right), \tag{30}$$

and also $x > 0$, we can decern the following two cases:

$$\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m) = \begin{cases} \frac{1}{2}\int\limits_0^G \log_2\left(\frac{P}{f(\beta_m)+x(1+P\|\mathbf{h}_m\|^2)}\right) p_{\bar{\varepsilon}^2}(x)\mathrm{d}x, & f(\beta_m) < P \\ 0, & f(\beta_m) \geq P, \end{cases} \tag{31}$$

where $G = \frac{P-f(\beta_m)}{1+P\|\mathbf{h}_m\|^2}$. The integral in (31) can be efficiently evaluated using the Gauss' formula, given in [11, Eq. 25.4.30].

C. Rate Loss

The rate loss for an MMSE coefficient error ε is written as

$$\begin{aligned}\Delta\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) =& \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m) - \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) \\ =& \frac{1}{2}\left[\log_2^+\left(\frac{P}{f(\beta_m)}\right)\right. \\ & \left. - \log_2^+\left(\frac{P}{f(\beta_m) + \varepsilon^2\left(1+P\|\mathbf{h}_m\|^2\right)}\right)\right].\end{aligned} \tag{32}$$

Considering that $\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m) > 0$ and $\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) > 0$ when $f(\beta_m) < P$, (32) becomes

$$\begin{aligned}\Delta\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \varepsilon) &= \frac{1}{2}\log_2\left(\frac{f(\beta_m) + \varepsilon^2\left(1+P\|\mathbf{h}_m\|^2\right)}{f(\beta_m)}\right) \\ &= \frac{1}{2}\log_2\left(1+D\varepsilon^2\right),\end{aligned} \tag{33}$$

where $D = \frac{1+P\|\mathbf{h}_m\|^2}{f(\beta_m)}$.

Using $\bar{\varepsilon}^2$ instead of ε^2, the average rate loss can be approximated by

$$\Delta\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m) = \mathbb{E}\left[\frac{1}{2}\log_2\left(1+D\bar{\varepsilon}^2\right)\right]. \tag{34}$$

Using the pdf of $\Delta\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m)$ given in (35) at the top of the next page, where $\log(\cdot)$ is the natural logarithm, (34) can be evaluated as

$$\Delta\bar{\mathcal{R}}(\mathbf{h}_m, \mathbf{a}_m) = \int_0^\infty x p_{\Delta\bar{\mathcal{R}}}(x)\mathrm{d}x. \tag{36}$$

$$p_{\bar{\varepsilon}^2}(x) = \begin{cases} \frac{\|\mathbf{a}_m\|^{\frac{k}{2}+1}}{2\sigma^{\frac{k}{2}+1}\lambda^{\frac{k}{2}-1}\sqrt{x}} \left[\frac{e^{A(\sqrt{x})}}{\left(\sqrt{x}+\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(\sqrt{x}\right)\right) + \frac{e^{A(-\sqrt{x})}}{\left(-\sqrt{x}+\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(-\sqrt{x}\right)\right) \right], & 0 \le x \le \bar{\beta}_m^2 \\ \frac{\|\mathbf{a}_m\|^{\frac{k}{2}+1} e^{A(\sqrt{x})}}{2\sigma^{\frac{k}{2}+1}\lambda^{\frac{k}{2}-1}\sqrt{x}\left(\sqrt{x}+\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(\sqrt{x}\right)\right), & \bar{\beta}_m^2 < x \end{cases} \tag{24}$$

$$p_{\Delta\bar{\mathcal{R}}}(x) = \begin{cases} \frac{\log(2) 2^{2x+1} D^{\frac{k+2}{4}} \|\mathbf{a}_m\|^{\frac{k}{2}+1}}{2\sigma^{\frac{k}{2}+1}\lambda^{\frac{k}{2}-1}\sqrt{4^x-1}} \left[\frac{e^{A\left(\sqrt{\frac{4^x-1}{D}}\right)}}{\left(\sqrt{4^x-1}+\sqrt{D}\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(\sqrt{\frac{4^x-1}{D}}\right)\right) + \frac{e^{A\left(-\sqrt{\frac{4^x-1}{D}}\right)}}{\left(-\sqrt{4^x-1}+\sqrt{D}\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(-\sqrt{\frac{4^x-1}{D}}\right)\right) \right], & 0 \le x \le \frac{1}{2}\log_2\left(1+D\bar{\beta}_m^2\right) \\ \frac{\log(2) 2^{2x+1} D^{\frac{k+2}{4}} \|\mathbf{a}_m\|^{\frac{k}{2}+1} e^{A\left(\sqrt{\frac{4^x-1}{D}}\right)}}{2\sigma^{\frac{k}{2}+1}\lambda^{\frac{k}{2}-1}\sqrt{4^x-1}\left(\sqrt{4^x-1}+\sqrt{D}\bar{\beta}_m\right)^{\frac{k}{2}+2}} I_{\frac{k}{2}-1}\left(B\left(\sqrt{\frac{4^x-1}{D}}\right)\right), & \frac{1}{2}\log_2\left(1+D\bar{\beta}_m^2\right) < x \end{cases} \tag{35}$$

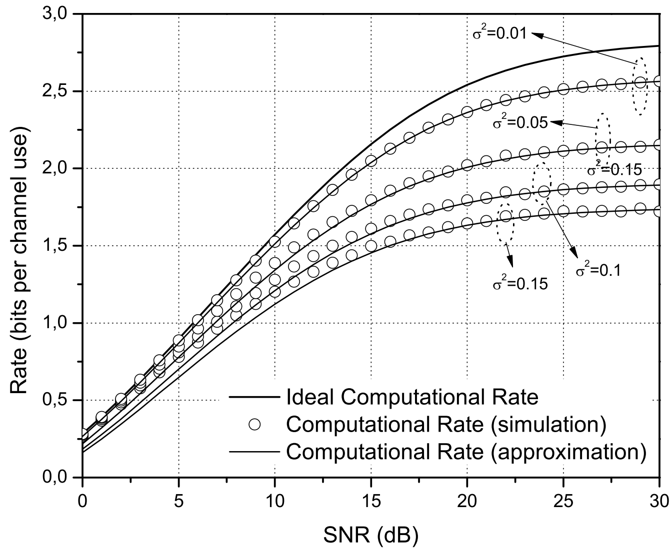

Fig. 1. Average computation rate region of a Compute-and-Forward relay in a 2-transmitter network with $\mathbf{h}_m = [0.9,\ 1.1]$ and $\mathbf{a}_m = [1,\ 1]$.

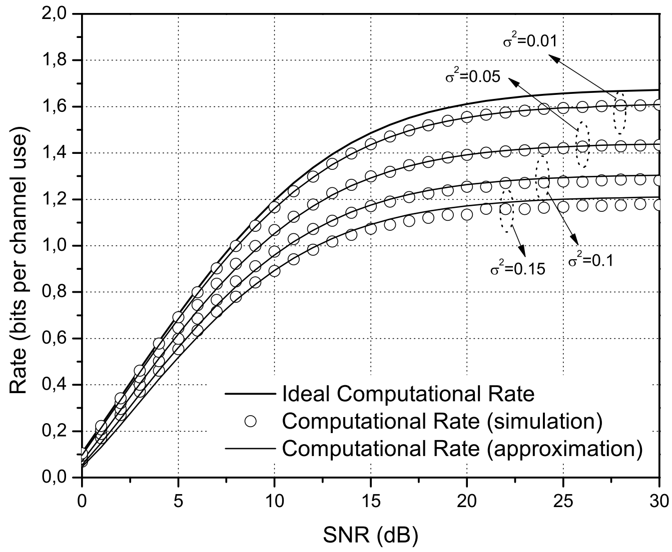

Fig. 2. Average computation rate region of a Compute-and-Forward relay in a 4-transmitter network with $\mathbf{h}_m = [0.9,\ 1.1,\ 0.8,\ 1.2]$ and $\mathbf{a}_m = [1,\ 1,\ 1,\ 1]$.

IV. Numerical Results and Discussion

In Fig. 1, the performance of a C&F relay in a system with two transmitters is examined. The channel coefficient vector is $\mathbf{h}_m = [0.9,\ 1.1]$, while the equation coefficient vector is assumed to be $\mathbf{a}_m = [1,\ 1]$. The average rate of a system with no channel estimation error is illustrated as an upper bound, and numerical results for the approximation in (21) and the simulated actual rate under channel estimation errors are given for various channel estimation error variances σ^2. The estimation error variance depends on the power of the pilot symbols used during the estimation process, which may be different from the power of the information symbols [7], and thus it is considered independent of the SNR value. It is evident that the proposed approximation is very accurate, especially for SNR values greater than 10 dB, but it can also be efficiently used for lower SNR values.

In Fig. 2, the performance of a C&F network with four transmitters is presented. The channel coefficient vector considered here is $\mathbf{h}_m = [0.9,\ 1.1,\ 0.8,\ 1.2]$, while the equation coefficient vector is chosen to be $\mathbf{a}_m = [1,\ 1,\ 1,\ 1]$. Fig. 2 shows that the proposed approximation is extremely tight even for very low values of SNR. Furthermore, one can observe that as the channel estimation error variance increases, the proposed approximation, although very tight, slightly overestimates the achievable computation rate. This is due to the choice of $\mathbf{a}_m$, which differs more from $\mathbf{h}_m$, when compared to Fig. 1. This behavior is expected, since the approximation in Section II.B assumed a "good" choice for the equation coefficient vector, which means the more $\mathbf{h}_m^T\mathbf{a}_m$ approaches $\|\mathbf{h}_m\|\|\mathbf{a}_m\|$, the better the proposed approximation is.

In Fig. 3, an ideal case of a C&F system with two transmitters is illustrated, where the channel coefficient vector and the equation coefficient vector are identical, i.e. $\mathbf{h}_m = \mathbf{a}_m =$

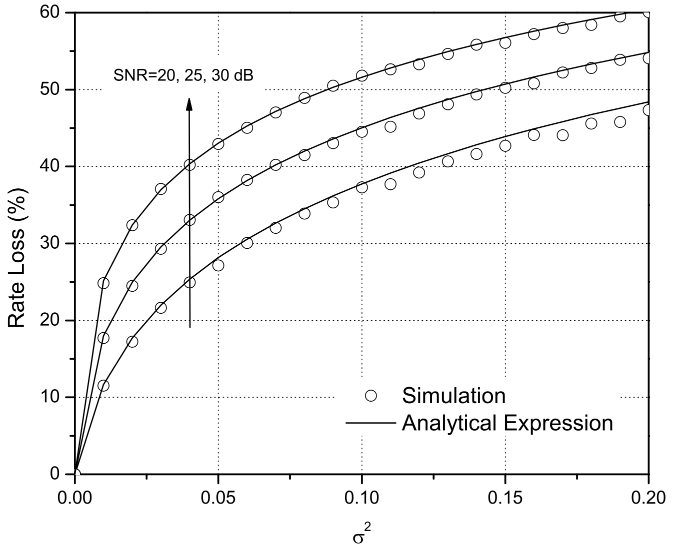

Fig. 3. Rate loss percentage of a Compute-and-Forward relay in a 2-transmitter network with $\mathbf{h}_m = [1,\, 1]$ and $\mathbf{a}_m = [1,\, 1]$.

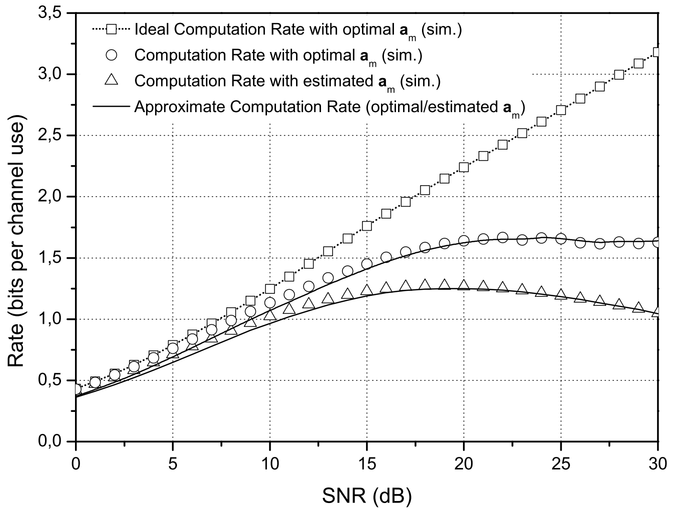

Fig. 4. Average computation rate region of a Compute-and-Forward relay in a 2-transmitter network over Rayleigh fading channels and channel estimation error with $\sigma^2 = 0.05$.

$[1,\, 1]$. In this case, the system suffers from a large rate loss, which is increasing with respect to the SNR. More specifically, the rate loss can be up to 56% for $\sigma^2 = 0.15$, and seems to be increasing for even higher values of SNR. This leads to the conclusion that even if a very beneficial - in terms of achievable rate - choice of equation coefficients can be found, the performance deterioration due to channel estimation error counterbalances this benefit.

In Fig. 4, the average computation rate region of a C&F relay in a 2-transmitter network is illustrated, when Rayleigh fading with $\Omega = \mathbb{E}[h^2] = 1$ is considered. Complex Gaussian gains before synchronization and perfect synchronization at the relay are assumed. Channel estimation error with $\sigma^2 = 0.05$ is also considered, while the computation rate is examined for two cases: a) the equation coefficient vector $\mathbf{a}_m$ is considered to be the optimal according to the criterion in [6] for each channel realization, b) $\mathbf{a}_m$ is computed using the noisy channel estimation available at the relay, which may be different from the optimal. The second case corresponds to actual relay operation. The first case is illustrated in order to highlight the performance loss when both $\mathbf{a}_m$ and β_m can be erroneously estimated, in contrast to the case when only β_m is affected. For both cases, the proposed approximation is very tight to the actual performance, as validated by the simulation results. Furthermore, a very interesting observation is that the rate decreases for high values of SNR, even for a channel estimation error as small as the one illustrated in Fig. 4. This is due to the fact that the norm of $\mathbf{a}_m$, either being the optimal or a suboptimal one, increases with increasing SNR. The error ϵ increases for higher norms of $\mathbf{a}_m$, and thus the rate is rapidly deteriorated.

V. Conclusions

In this paper, the effect of channel estimation errors on the computation rate region of a relay in a C&F network was investigated. A formula for the achievable rate region was derived. Furthermore, a tight approximation for Gaussian distributed channel estimation errors was introduced and an analytical expression for the pdf of the rate loss was proposed. The presented numerical and simulation results illustrate that C&F is quite sensitive to channel estimation errors, highlighting the need of accurate channel estimation, in order to fully exploit the achievable rates of C&F in practice.

References

[1] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6463-6486, Oct. 2011.

[2] U. Erez and R. Zamir, "Achieving $\frac{1}{2}\log(1 + SNR)$ on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inf. Theory*, vol. 50, pp. 2293-2314, Oct. 2004.

[3] U. Niesen and P. Whiting, "The Degrees of Freedom of Compute-and-Forward," *IEEE Trans. Inf. Theory*, vol. 58, pp. 5214-5232, Aug. 2012.

[4] S. Gupta and M. A. Vázquez-Castro, "Physical-layer network coding based on integer-forcing precoded compute and forward," in *Proc. IEEE 8th International Conference on Wireless and Mobile Computing, Networking and Communications (WiMob)*, Barcelona, Spain, Oct. 2012.

[5] C. Feng, D. Silva, and F. R. Kschischang, "Blind compute-and-forward," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Cambridge, MA, USA, Jul. 2012.

[6] C. Feng, D. Silva, and F. R. Kschischang. "Design criteria for lattice network coding,", in *Proc. IEEE 45th Annual Conference onInformation Sciences and Systems (CISS)*, Baltimore, MD, USA, 2011.

[7] D. S. Michalopoulos, N. D. Chatzidiamantis, R. Schober, and G. K. Karagiannidis, "The diversity potential of relay selection with practical channel estimation," *IEEE Trans. Wireless Commun.*, vol. 12, no. 2, pp. 481-493, Feb. 2013.

[8] A. Maaref and S. Aissa, "On the effects of Gaussian channel estimation errors on the capacity of adaptive transmission with space-time block coding," in *Proc. IEEE International Conference on Wireless And Mobile Computing, Networking And Communications, (WiMob)*, Montreal, Canada, Aug. 2005.

[9] M. J. Gans, "The effect of Gaussian error in maximal ratio combiners," *IEEE Trans. Commun. Technol.*, vol. COM-19, pp. 492-500, Aug. 1971.

[10] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions, Volume 2*, 2nd ed., Wiley, 1995.

[11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functionswith Formulas, Graphs, and Mathematical Tables*, 9th edition, Dover, 1970.