A Perturbation Proof of the Vector Gaussian One-Help-One Problem

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***Abstract*—In this paper, we give a perturbation proof to vector Gaussian one-help-one problem for characterizing the rate distortion region, in which the challenge is that the conventional entropy power inequality used in scalar Gaussian case is not necessarily tight in vector case. Different from *enhancement* technique, we take the Fisher information matrix to present the entropy, and then derive a new extremal inequality based on the method of integration along a path of a continuous Gaussian perturbation. This new extremal inequality enables us to give a *perturbation* proof of Rahman and Wagner's theorem.**

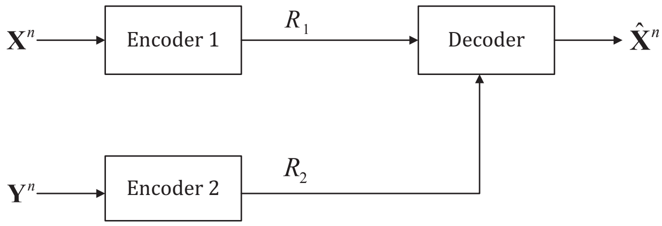

Fig. 1. The vector Gaussian one-help-one problem.

I. Introduction

In their seminal work [1], Liu and Viswanath showed a problem of maximization of the difference between two entropy terms as below

$$\max_{p(\mathbf{X}):\mathrm{cov}(\mathbf{X})\preccurlyeq\mathbf{S}} h(\mathbf{X}+\mathbf{Z}_1)-\mu h(\mathbf{X}+\mathbf{Z}_2), \tag{1}$$

in which $\mathbf{X}$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are independent random vectors taking value in m dimensional real number space $\mathcal{R}^m$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are vector Gaussian random variables with positive definite covariance matrix, and $\mu > 1$ is a positive real mumber. They demonstrated that the above optimization problem cannot be solved by using the entropy power inequality, and then provided two different approaches [1]. The first one is based on the so-called *enhancement* technique, which is introduced by Weingarten *et al* for solving the capacity of multiple-input multiple-output (MIMO) Gaussian broadcast channel [2]. Another one is based on the *perturbation* technique, from which a monotone integration path from random vector $\mathbf{X}$ to the optimal Gaussian vector was established [3].

Motivated by the work on the capacity of MIMO Gaussian broadcast channel in [2], the optimization problem (1) is led to an extremal inequality [1], a new version of entropy power inequality, which was further applied by Liu and Viswanath to establish an outer bound for the rate region of vector Gaussian one-help-one problem [1] .

The one-help-one problem is one of the simplest version of multi-terminal distributed source coding problems with distortion. For the scalar Gaussian source, Oohama completely characterized the rate-distortion region by using Shannon entropy power inequality [4]. Recently, several works are devoted to extend Oohama's result to vector Gaussian source [1], [5], [6]. Among these works, Rahman and Wagner [6] applied *"souse enhancement"* which is first proposed in [5], combined with their precious work on the so called *"distortion projection"* [7], from which the entire rate distortion region of the vector Gaussian one-help-one problem was eventually obtained.

Therefore, one question arises naturally. Is it possible to solve this problem by the *perturbation* technique? Different from the *enhancement* technique, the *perturbation* method is more direct from information theoretic perspective, henceforth it might be more suitable to solve the rate distortion region of more generalized multi-terminal source coding problem. The main contribution of this paper is to give a affirmative answer to this question. We strengthen Liu and Viswanath's extremal inequality to a more appropriate form, and then use it to bound the weighted sum of the rate, which completes the perturbation proof of Rahman-Wagner theorem.

This paper is organized as follows. In section II we will revisit Rahman and Wagner's formulation of the vector Gaussian one-help-one problem and their theorem established in [6]. Section III is devoted to go through some mathematical preliminaries which will be used to prove the new extremal inequality. In section IV, we will prove the extremal inequality in details and then use it to characterize the rate region of the vector Gaussian one-help-one problem.

II. Rahman and Wagner's Theorem

We now introduce Rahman and Wagner's theorem and their formulation. The vector Gaussian one-help-one problem is defined as two-terminal distributed source coding problem with a single quadratic distortion. As depicted in Fig. 1, let $\{\mathbf{X}_i,\mathbf{Y}_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random vectors, and $\mathbf{X}_i,\mathbf{Y}_i$, $i = 1,2,\ldots,$ be two generic zero-mean jointly $m\times 1$ dimensional Gaussian random vectors with covariance matrices $\mathbf{K_X}$ and $\mathbf{K_Y}$ respectively. We assume $\mathbf{K_X}$ and $\mathbf{K_Y}$ are both positive definite for the sake of simplicity. Throughout this paper, we denote n-length sequence by a superscript n for simplicity, *e.g.* $\mathbf{X}^n = \{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ and $\mathbf{Y}^n = \{\mathbf{Y}_1, \ldots, \mathbf{Y}_n\}$.

In Fig. 1, the encoder 1 observes sequence $\mathbf{X}^n$ and separately uses itself encoding function:

$$\phi_1^n : \mathcal{R}^{m\times n} \mapsto \mathcal{M}_1^n = \{1, \cdots, 2^{nR_1}\},$$

i.e., $C_1 = \phi_1^n(\mathbf{X}^n)$. Analogously, the encoder 2 observes $\mathbf{Y}^n$ using another encoding function:

$$\phi_2^n : \mathcal{R}^{m\times n} \mapsto \mathcal{M}_2^n = \{1, \cdots, 2^{nR_2}\},$$

i.e., $C_2 = \phi_2^n(\mathbf{Y}^n)$. The decoder collects the values of the two encoders and estimate $\mathbf{X}^n$ using the decoding function:

$$\varphi^n : \mathcal{M}_1^n \times \mathcal{M}_2^n \mapsto \mathcal{R}^{m\times n},$$

i.e., $\hat{\mathbf{X}}^n = \varphi^n(C_1, C_2)$.

The rate-distortion tuple $(R_1, R_2, \mathbf{D})$ is said to be achievable, if there exists a sequence of $(\phi_1^n, \phi_2^n, \varphi^n, n)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E\left[(\mathbf{X}_i - \hat{\mathbf{X}}_i)(\mathbf{X}_i - \hat{\mathbf{X}}_i)^T\right] \preceq \mathbf{D}.$$

The rate-distortion region $\mathcal{R}(\mathbf{D})$ is the closure of all achievable rate tuples (R_1, R_2) subject to distortion $\mathbf{D}$.

For a fixed distortion constraint $\mathbf{D}$, the rate region $\mathcal{R}(\mathbf{D})$ is convex and it can be characterized by its supporting hyperplanes. Equivalently, it can be expressed by solving the following optimization problem

$$R(\mathbf{D}, \mu) = \inf_{(R_1, R_2) \in \mathcal{R}(\mathbf{D})} \mu R_1 + R_2,$$

in which $\mu \geq 0$. Without loss of generality, it can be assumed that

$$\mathbf{X} = \mathbf{Y} + \mathbf{N},$$

where the zero-mean Gaussian random vector $\mathbf{N}$ is independent of $\mathbf{Y}$, and its covariance $\mathbf{K_N}$ satisfies $\mathbf{K_X} = \mathbf{K_Y} + \mathbf{K_N}$.

Using above notations, the rate distortion region of the vector Gaussian one-help-one problem can be characterized by the following optimization problem [1] in [6],

$$R^*(\mathbf{D}, \mu) \triangleq \min_{\mathbf{B}_1, \mathbf{B}_2} \frac{\mu}{2} \log \frac{|\mathbf{K_X} - \mathbf{B}_2|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2|}$$
$$\text{subject to } \mathbf{B}_1,\ \mathbf{B}_2 \succcurlyeq \mathbf{0}, \text{ and}$$
$$\mathbf{D} \succcurlyeq \mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2.$$

Theorem 1 ([6, Theorem 1]): Given the distortion matrix $\mathbf{D}$ and the weight $\mu \geq 0$, the rate distortion region of the vector Gaussian one-help-one problem is given by the solution of the above optimization problem:

$$R(\mathbf{D}, \mu) = R^*(\mathbf{D}, \mu).$$

[1]In paper [6], the authors considered two cases of $0 \leq \mu < 1$ and $\mu \geq 1$, separately. And they show the case of $0 \leq \mu < 1$ is one point-to-point problem. In this paper, we consider the two cases using the same converse technology. Actually, the case of $0 \leq \mu < 1$ can also be included in the case of $\mu \geq 1$. This will be explained in Appendix.

As pointed out in [6], the converse part of this theorem requires to show

$$\mu R_1 + R_2 \geq R^*(\mathbf{D}, \mu),$$

for any $(R_1, R_2) \in \mathcal{R}(\mathbf{D})$.

III. Mathematical Preliminaries

Our perturbation proof is based on some properties of conditional Fisher information matrix.

Definition 1: Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors with differentiable conditional probability density function $f(\mathbf{x}|u)$. The vector-valued score function is defined as

$$\nabla \log f(\mathbf{x}|u) = \left[\frac{\partial \log f(\mathbf{x}|u)}{\partial x_1}, \cdots, \frac{\partial \log f(\mathbf{x}|u)}{\partial x_m}\right]^T.$$

The conditional Fisher Information of $\mathbf{X}$ respect to U is given by

$$J(\mathbf{X}|U) = \mathbb{E}\left[(\nabla \log f(\mathbf{x}|u)) \cdot (\nabla \log f(\mathbf{x}|u))^T\right].$$

Lemma 1 (Cramér–Rao Inequality): Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors. Assuming that the conditional covariance matrix $\mathrm{cov}(\mathbf{X}|U) \succ \mathbf{0}$, then

$$J(\mathbf{X}|U)^{-1} \preceq \mathrm{cov}(\mathbf{X}|U).$$

One can refer to the proof of unconditional version in [3, Theorem 20].

Lemma 2 (Fisher Information Matrix Inequality): Let $(\mathbf{X}, \mathbf{Y}, U)$ be a tuple of jointly distributed random vectors. Assume that $\mathbf{X}$ and $\mathbf{Y}$ being conditionally independent given U, then for any square matrix $\mathbf{A}$,

$$J(\mathbf{X} + \mathbf{Y}|U) \preceq \mathbf{A} J(\mathbf{X}|U) \mathbf{A}^T + (\mathbf{I} - \mathbf{A}) J(\mathbf{Y}|U) (\mathbf{I} - \mathbf{A})^T.$$

The proof is similar to the unconditional version in [1, Appendix II].

Lemma 3 (Data Processing Inequality): Let $(\mathbf{X}, U, V)$ be a tuple of jointly distributed random vectors, and $U, V, \mathbf{X}$ form a Markov chain. *i.e.* $U \to V \to \mathbf{X}$, then

$$J(\mathbf{X}|U) \preceq J(\mathbf{X}|V).$$

The proof is easily followed by the chain rule of Fisher information matrix [8, Lemma 1], and it is analogous to another form of data processing inequality [8, Lemma 3].

Lemma 4 (de Brujin's Identity): Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors, and $\mathbf{N} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ be a standard Gaussian random vector, which is independent of $(\mathbf{X}, U)$, then

$$\frac{d}{d\gamma} h(\mathbf{X} + \sqrt{\gamma}\mathbf{N}|U) = \frac{1}{2} \mathrm{tr}\left\{J(\mathbf{X} + \sqrt{\gamma}\mathbf{N}|U)\right\}.$$

This lemma is the conditional version of [3, Theorem 14]. The de Brujin's identity recovers the links between entropy and Fisher information matrix.

IV. The perturbation proof

In this section, we first exam the KKT (Karush–Kuhn–Tucker) conditions for the optimization problem $R^*(\mathbf{D},\mu)$, then derive a new extremal inequality via *perturbation* method. Finally, the rate distortion region is bounded by this extremal inequality.

A. KKT Conditions

The Lagrangian of $R^*(\mathbf{D},\mu)$ is

$$\frac{\mu}{2}\log\frac{|\mathbf{K_X}-\mathbf{B}_2|}{|\mathbf{K_X}-\mathbf{B}_1-\mathbf{B}_2|}+\frac{1}{2}\log\frac{|\mathbf{K_Y}|}{|\mathbf{K_Y}-\mathbf{B}_2|}$$
$$-\operatorname{tr}\left\{\mathbf{B}_1\mathbf{\Psi}_1+\mathbf{B}_2\mathbf{\Psi}_2+(\mathbf{D}-\mathbf{K_X}+\mathbf{B}_1+\mathbf{B}_2)\mathbf{\Lambda}\right\},$$

where $\mathbf{\Psi}_1,\mathbf{\Psi}_2$ and $\mathbf{\Lambda}$ are Lagrange multipliers. Let $\mathbf{B}_1^*,\mathbf{B}_2^*$ (non-unique) be one optimal solution of $R^*(\mathbf{D},\mu)$. The KKT conditions for this optimization problem are given by

$$\frac{\mu}{2}(\mathbf{K_X}-\mathbf{B}_1^*-\mathbf{B}_2^*)^{-1}-\mathbf{\Psi}_1-\mathbf{\Lambda}=\mathbf{0}, \tag{2}$$

$$\frac{\mu}{2}(\mathbf{K_X}-\mathbf{B}_1^*-\mathbf{B}_2^*)^{-1}-\frac{\mu}{2}(\mathbf{K_X}-\mathbf{B}_2^*)^{-1}+\frac{1}{2}(\mathbf{K_Y}-\mathbf{B}_2^*)^{-1}-\mathbf{\Psi}_2-\mathbf{\Lambda}=\mathbf{0}, \tag{3}$$

$$\mathbf{B}_1^*\mathbf{\Psi}_1=\mathbf{0}, \tag{4}$$

$$\mathbf{B}_2^*\mathbf{\Psi}_2=\mathbf{0}, \tag{5}$$

$$(\mathbf{D}-\mathbf{K_X}+\mathbf{B}_1^*+\mathbf{B}_2^*)\mathbf{\Lambda}=\mathbf{0}, \tag{6}$$

$$\mathbf{\Psi}_1,\mathbf{\Psi}_2,\mathbf{\Lambda}\succeq\mathbf{0}.$$

Notice that the optimization problem $R^*(\mathbf{D},\mu)$ is not convex, thus in order to show that the optimal solution $(\mathbf{B_1^*},\mathbf{B_2^*})$ fulfills the KKT conditions, the constraint qualifications are examined in [6, Lemma 2].

B. The Extremal Inequality

Theorem 2: Let $(\mathbf{B_1^*},\mathbf{B_2^*})$ be the optimal solution for the optimization problem $R^*(\mathbf{D},\mu)$. Let U,V be two random variables such that $\mathbf{X}\to\mathbf{Y}\to V$ forms a Markov chain. Then for every distribution of (U,V) satisfying constraints:

$$E\left[(\mathbf{X}-E\left[\mathbf{X}|U,V\right])\left(\mathbf{X}-E\left[\mathbf{X}|U,V\right]\right)^T\right]\preceq\mathbf{D},$$

we have

$$\begin{aligned}&\mu h(\mathbf{X}|V)-h(\mathbf{Y}|V)-\mu h(\mathbf{X}|U,V)\\ \geq&\frac{\mu}{2}\log|(2\pi e)(\mathbf{K_X}-\mathbf{B}_2^*)|-\frac{1}{2}\log|(2\pi e)(\mathbf{K_Y}-\mathbf{B}_2^*)|\\ &-\frac{\mu}{2}\log|(2\pi e)(\mathbf{K_X}-\mathbf{B}_2^*-\mathbf{B}_1^*)|.\end{aligned}$$

Proof: Define three zero mean Gaussian random vectors $\mathbf{X}_1^G$, $\mathbf{Y}_2^G$ and $\mathbf{X}_2^G$ independent of $(\mathbf{X},\mathbf{Y},U,V)$. Furthermore, they are mutually independent. Their covariance matrices are: $\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*},\mathbf{K_Y}-\mathbf{B_2^*},\mathbf{K_X}-\mathbf{B_2^*}$.

We can construct the "covariance preserving transform" proposed by Dembo *et al.* in [3]: for any $\gamma\in(0,1)$,

$$\mathbf{X}_{1,\gamma}=\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\mathbf{X}_1^G, \tag{7}$$

$$\mathbf{Y}_{2,\gamma}=\sqrt{1-\gamma}\mathbf{Y}+\sqrt{\gamma}\mathbf{Y}_2^G, \tag{8}$$

$$\mathbf{X}_{2,\gamma}=\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\mathbf{X}_2^G, \tag{9}$$

Let's consider the following function:

$$g(\gamma)=\mu h(\mathbf{X}_{2,\gamma}|V)-h(\mathbf{Y}_{2,\gamma}|V)-\mu h(\mathbf{X}_{1,\gamma}|U,V). \tag{10}$$

Following from de Brujin's identity in lemma 4, we have

$$\frac{d}{d\gamma}h(\mathbf{X}_{2,\gamma}|V)=\frac{1}{2(1-\gamma)}\operatorname{tr}\left\{(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)-\mathbf{I}\right\} \tag{11}$$

$$\frac{d}{d\gamma}h(\mathbf{Y}_{2,\gamma}|V)=\frac{1}{2(1-\gamma)}\operatorname{tr}\left\{(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)-\mathbf{I}\right\} \tag{12}$$

$$\frac{d}{d\gamma}h(\mathbf{X}_{1,\gamma}|U,V)=\frac{1}{2(1-\gamma)}\operatorname{tr}\left\{(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\cdot J(\mathbf{X}_{1,\gamma}|U,V)-\mathbf{I}\right\} \tag{13}$$

Substituting (11), (12) and (13) into (10), we see

$$\begin{aligned}&2(1-\gamma)g'(\gamma)\\ &=\operatorname{tr}\left\{\mu\left[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)-\mathbf{I}\right]\right.\\ &\quad-\left[(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)-\mathbf{I}\right]\\ &\quad\left.-\mu\left[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)-\mathbf{I}\right]\right\}\end{aligned} \tag{14}$$

If we define

$$\tilde{\mathbf{X}}_2^G=\mathbf{Y}_2^G+\tilde{\mathbf{N}}^G,$$

where $\tilde{\mathbf{N}}^G\sim\mathcal{N}(\mathbf{0},\mathbf{K_N})$ and independent of $\mathbf{Y}_2^G$. Define

$$\begin{aligned}\tilde{\mathbf{X}}_{2,\gamma}&=\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\tilde{\mathbf{X}}_2^G\\ &=\sqrt{1-\gamma}(\mathbf{Y}+\mathbf{N})+\sqrt{\gamma}(\mathbf{Y}_2^G+\tilde{\mathbf{N}}^G)\\ &=\mathbf{Y}_{2,\gamma}+\tilde{\mathbf{N}},\end{aligned} \tag{15}$$

where $\tilde{\mathbf{N}}\sim\mathcal{N}(0,\mathbf{K_N})$. Note that $(\tilde{\mathbf{X}}_{2,\gamma},V)$ and $(\mathbf{X}_{2,\gamma},V)$ have the same distribution so we have $J(\mathbf{X}_{2,\gamma}|V)=J(\tilde{\mathbf{X}}_{2,\gamma}|V)$. Apply Fisher information matrix inequality in Lemma 2 to obtain the following matrix inequality:

$$\begin{aligned}&J(\mathbf{X}_{2,\gamma}|V)=J(\tilde{\mathbf{X}}_{2,\gamma}|V)\\ &\preceq(\mathbf{K_X}-\mathbf{B}_2^*)^{-1}(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)(\mathbf{K_Y}-\mathbf{B_2^*})\\ &\cdot(\mathbf{K_X}-\mathbf{B_2^*})^{-1}+(\mathbf{K_X}-\mathbf{B_2^*})^{-1}\mathbf{K}_NJ(\tilde{\mathbf{N}})\mathbf{K}_N(\mathbf{K_X}-\mathbf{B_2^*})^{-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)(\mathbf{K_Y}-\mathbf{B_2^*})\succeq(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)\\ &\quad(\mathbf{K_X}-\mathbf{B_2^*})-\mathbf{K_N}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\operatorname{tr}\{\mu\left[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)-\mathbf{I}\right]-\\ &\quad\left[(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)-\mathbf{I}\right]\}\\ &=\operatorname{tr}\{(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})[\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}]\\ &\quad-(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)(\mathbf{K_Y}-\mathbf{B_2^*})[(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}]\\ &\quad-(\mu-1)\mathbf{I}\}\\ &\leq\operatorname{tr}\{(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})[\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}\\ &\quad-(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}]+\mathbf{K_N}(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}-(\mu-1)\mathbf{I}\}\}\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr}\{(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})[\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1} \\
&\quad -(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}]-[\mu\mathbf{I}-(\mathbf{K_X}-\mathbf{B_2^*})(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}]\} \\
&= \operatorname{tr}\{[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*})] \\
&\quad \cdot[\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}-(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}]\}. \qquad (16)
\end{aligned}
$$

Referring to the KKT conditions of $R^*(\mathbf{D},\mu)$, (2) may be subtracted from (3). Then the next identity is obtained,

$$
\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}-(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}=2(\mathbf{\Psi_1}-\mathbf{\Psi_2}). \qquad (17)
$$

On the other hand,

$$
\begin{aligned}
&\mu[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)-\mathbf{I}] \\
=&[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
&-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})][\mu(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})^{-1}]. \qquad (18)
\end{aligned}
$$

According to the equation (2) in the KKT conditions,

$$
\mu(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})^{-1}=2(\mathbf{\Psi_1}+\mathbf{\Lambda}). \qquad (19)
$$

Substituting (16) – (19) into formula (14), it can be simplified to the following form:

$$
\begin{aligned}
&(1-\gamma)g'(\gamma) \\
\le& \operatorname{tr}\{[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
&\cdot J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})+\mathbf{B}_1^*]\mathbf{\Psi}_1\} \\
&-\operatorname{tr}\{[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*})]\mathbf{\Psi_2}\} \\
&-\operatorname{tr}\{[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
&-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})]\mathbf{\Lambda}\} \qquad (20)
\end{aligned}
$$

To bound the first term of r.h.s. of (20), we first define

$$
\check{\mathbf{X}}_2^G=\mathbf{X}_1^G+\check{\mathbf{W}}^G,
$$

where $\check{\mathbf{W}}^G\sim\mathcal{N}(\mathbf{0},\mathbf{B_1^*})$ and independent of $\mathbf{X}_1^G$. Define

$$
\begin{aligned}
\check{\mathbf{X}}_{2,\gamma}&=\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\check{\mathbf{X}}_2^G \\
&=\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}(\mathbf{X}_1^G+\check{\mathbf{W}}^G) \\
&=\mathbf{X}_{1,\gamma}+\sqrt{\gamma}\check{\mathbf{W}}^G
\end{aligned}
$$

Note that $(\check{\mathbf{X}}_{2,\gamma},V)$ and $(\mathbf{X}_{2,\gamma},V)$ have the same distribution so we have $J(\mathbf{X}_{2,\gamma}|V)=J(\check{\mathbf{X}}_{2,\gamma}|V)$. Making use of Fisher information matrix inequality in Lemma 2, we have the following inequality:

$$
\begin{aligned}
&J(\mathbf{X}_{2,\gamma}|V)=J(\check{\mathbf{X}}_{2,\gamma}|V) \\
\preceq&(\mathbf{K_X}-\mathbf{B_2^*})^{-1}(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|V) \\
&\cdot(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})(\mathbf{K_X}-\mathbf{B_2^*})^{-1}+(\mathbf{K_X}-\mathbf{B_2^*})^{-1}\mathbf{B}_1^* \\
&\cdot J(\sqrt{\gamma}\check{\mathbf{W}}^G)\mathbf{B}_1^*(\mathbf{K_X}-\mathbf{B_2^*})^{-1}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*}) \\
\preceq&(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})+\frac{1}{\gamma}\mathbf{B}_1^* \\
\preceq&(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})+\frac{1}{\gamma}\mathbf{B}_1^*,
\end{aligned}
$$

where the last inequality comes from that $V\to(U,V)\to\mathbf{X}$ forms a Markov chain and the data processing inequality in Lemma 3.

Thus the first term of r.h.s. of (20) can be bounded as below:

$$
\begin{aligned}
&[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
&\cdot J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})+\mathbf{B}_1^*]\mathbf{\Psi}_1 \\
\preceq&\frac{1+\gamma}{\gamma}\mathbf{B}_1^*\mathbf{\Psi}_1=\mathbf{0}.
\end{aligned}
$$

The equality is because of the equation (4) in KKT conditions for $R^*(\mathbf{D},\mu)$.

To bound the second term of (20), we first give an upper bound to the inverse of the Fisher information matrix $J(\mathbf{X}_{2,\gamma}|V)^{-1}$:

$$
\begin{aligned}
&J(\mathbf{X}_{2,\gamma}|V)^{-1} \\
\overset{(a)}{\preceq}&\operatorname{cov}(\mathbf{X}_{2,\gamma}|V)\preceq\operatorname{cov}(\mathbf{X}_{2,\gamma})\overset{(b)}{=}\operatorname{cov}(\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\mathbf{X}_2^G) \\
\overset{(c)}{=}&(1-\gamma)\operatorname{cov}(\mathbf{X})+\gamma\operatorname{cov}(\mathbf{X}_2^G) \\
=&(1-\gamma)\mathbf{K_X}+\gamma(\mathbf{K_X}-\mathbf{B_2^*})=\mathbf{K_X}-\gamma\mathbf{B_2^*}, \qquad (21)
\end{aligned}
$$

where (a) follows from the Cramér–Rao inequality in Lemma 1, (b) from (9), (c) from the independence between $\mathbf{X}_2^G$ and $\mathbf{X}$.

Thus the second term of (20) can be bounded as follows:

$$
\begin{aligned}
&[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*})]\mathbf{\Psi_2} \\
\succeq&[(\mathbf{K_X}-\mathbf{B_2^*})(\mathbf{K_X}-\gamma\mathbf{B_2^*})^{-1}(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*})]\mathbf{\Psi_2} \\
=&(\mathbf{K_X}-\mathbf{B_2^*})(\mathbf{K_X}-\gamma\mathbf{B_2^*})^{-1}[(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\gamma\mathbf{B_2^*})]\mathbf{\Psi_2} \\
=&(\mathbf{K_X}-\mathbf{B_2^*})(\mathbf{K_X}-\gamma\mathbf{B_2^*})^{-1}(\gamma-1)\mathbf{B_2^*}\mathbf{\Psi_2}=\mathbf{0}.
\end{aligned}
$$

The last equality is because of the equation (5) in KKT conditions for $R^*(\mathbf{D},\mu)$.

To bound the last term of (20), we give an upper bound to the inverse of the Fisher information matrix $J(\mathbf{X}_{1,\gamma}|U,V)^{-1}$, analogously:

$$
\begin{aligned}
&J(\mathbf{X}_{1,\gamma}|U,V)^{-1} \\
\overset{(a)}{\preceq}&\operatorname{cov}(\mathbf{X}_{1,\gamma}|U,V)\overset{(b)}{=}\operatorname{cov}(\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\mathbf{X}_1^G|U,V) \\
\overset{(c)}{=}&(1-\gamma)\operatorname{cov}(\mathbf{X}|U,V)+\gamma\operatorname{cov}(\mathbf{X}_1^G) \\
=&(1-\gamma)\operatorname{cov}(\mathbf{X}|U,V)+\gamma(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
\overset{(d)}{\preceq}&(1-\gamma)\mathbf{D}+\gamma\mathbf{D}=\mathbf{D},
\end{aligned}
$$

where (a) follows from the Cramér–Rao inequality in Lemma 1, (b) from (7) , (c) from the definition of $\mathbf{X}_1^G$ and the fact that random vector $\mathbf{X}_1^G$ is independent of U, V, and (d) from the fact that $\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*}\preceq\mathbf{D}$ and the conditions of the theorem that $\operatorname{cov}(\mathbf{X}|U,V)\preceq\mathbf{D}$.

Thus the last term of (20) can be bounded as follows:

$$
\begin{aligned}
&[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}) \\
&-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})]\mathbf{\Lambda} \\
\succeq&[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\mathbf{D}^{-1}(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})-(\mathbf{K_X}-\mathbf{B_2^*} \\
&-\mathbf{B_1^*})]\mathbf{\Lambda}
\end{aligned}
$$

$$\succeq (\mathbf{K_X} - \mathbf{B_2^*} - \mathbf{B_1^*})\mathbf{D}^{-1}[\mathbf{K_X} - \mathbf{B_2^*} - \mathbf{B_1^*} - \mathbf{D}]\mathbf{\Lambda} = \mathbf{0}.$$

Finally, the last equality comes from (6) in KKT conditions for $R^*(\mathbf{D}, \mu)$.

According to the derived bounds of the three terms in (20), we obtain the inequality

$$(1-\gamma)g'(\gamma) \leq 0$$

for any $\gamma \in (0,1)$, which means that we have found a decreasing path from any $((\mathbf{X}|V), (\mathbf{Y}|V), (\mathbf{X}|U,V))$ to the optimal Gaussian distributed random vectors $(\mathbf{X}_2^G, \mathbf{Y}_2^G, \mathbf{X}_1^G)$. This completes the *perturbation* proof of the extremal inequality in Theorem 2. ■

C. Rate Region

Lemma 5: For any $(R_1, R_2) \in \mathcal{R}(\mathbf{D})$, then there exist random variables U and V such that

$$\begin{aligned} &R_1 \geq I(\mathbf{X}; U|V); \\ &R_2 \geq I(\mathbf{Y}; V); \\ &\mathrm{cov}(\mathbf{X}|U,V) \preceq \mathbf{D}, \\ &\mathbf{X} \rightarrow \mathbf{Y} \rightarrow V. \end{aligned}$$

The proof of this lemma can be found in paper [7, Lemma 2]. Notice that the result of Lemma 5 is the conditions of Theorem 2, so for any $(R_1, R_2) \in \mathcal{R}(\mathbf{D})$, we have

$$\begin{aligned} \mu R_1 + R_2 \geq& \mu I(\mathbf{X}; U|V) + I(\mathbf{Y}; V) \\ =& \mu h(\mathbf{X}|V) - \mu h(\mathbf{X}|U,V) + h(\mathbf{Y}) - h(\mathbf{Y}|V) \\ \geq& \frac{\mu}{2} \log \frac{|\mathbf{K_X} - \mathbf{B}_2^*|}{|\mathbf{K_X} - \mathbf{B}_1^* - \mathbf{B}_2^*|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2^*|} \\ =& R^*(\mathbf{D}, \mu). \end{aligned}$$

In this way, we obtain the same rate distortion region of vector Gaussian one-help-one problem as in [6].

Appendix

If we set $\mathbf{B}_2 = \mathbf{0}$ in the optimization problem $R^*(\mathbf{D}, \mu)$, we will see

$$\begin{aligned} R^*(\mathbf{D}, \mu) =& \min_{\mathbf{B}_1, \mathbf{B}_2} \frac{\mu}{2} \log \frac{|\mathbf{K_X} - \mathbf{B}_2|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2|} \\ & \text{subject to } \mathbf{B}_1,\ \mathbf{B}_2 \succcurlyeq \mathbf{0}, \text{ and} \\ & \qquad \mathbf{D} \succcurlyeq \mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2. \\ \leq& \min_{\mathbf{B}_1} \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y}|} \\ & \text{subject to } \mathbf{B}_1 \succcurlyeq \mathbf{0}, \text{ and } \mathbf{D} \succcurlyeq \mathbf{K_X} - \mathbf{B}_1. \\ \triangleq& R^{'}(\mathbf{D}, \mu), \end{aligned} \tag{22}$$

where $R^{'}(\mathbf{D}, \mu)$ is the optimization problem defined by Rahman and Wagner for the case $0 \leq \mu < 1$. Morever $R^{'}(\mathbf{D}, \mu)$ is lower bounded by $R^*(\mathbf{D}, \mu)$ for any $\mu \in [0, \infty)$.

Furthermore, when $0 \leq \mu < 1$, for any nonnegative definite matrices $\mathbf{B}_1, \mathbf{B}_2 \succcurlyeq \mathbf{0}$, we have

$$\begin{aligned} & \frac{\mu}{2} \log \frac{|\mathbf{K_X} - \mathbf{B}_2|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2|} \\ \geq& \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} \\ & - \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_2|} + \frac{\mu}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2|} \\ =& \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} \\ & + \frac{\mu}{2} \left(\log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2|} - \log \frac{|\mathbf{K_Y} + \mathbf{K_N}|}{|\mathbf{K_Y} + \mathbf{K_N} - \mathbf{B}_2|} \right) \\ \geq& \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|}. \end{aligned}$$

The last inequality comes from the fact that the function

$$f(\mathbf{B}) = \log \frac{|\mathbf{B}|}{|\mathbf{B} - \mathbf{C}|}$$

is monotonically decreasing with respect to the positive definite matrix $\mathbf{B}$ satisfying $\mathbf{B} \succ \mathbf{C} \succcurlyeq \mathbf{0}$. Therefore,

$$\begin{aligned} R^*(\mathbf{D}, \mu) \geq& \frac{\mu}{2} \min_{\mathbf{B}_1, \mathbf{B}_2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2|} \\ & \text{subject to } \mathbf{B}_1,\ \mathbf{B}_2 \succcurlyeq \mathbf{0}, \text{ and} \\ & \qquad \mathbf{D} \succcurlyeq \mathbf{K_X} - \mathbf{B}_1 - \mathbf{B}_2. \\ =& \min_{\mathbf{B}_1} \frac{\mu}{2} \log \frac{|\mathbf{K_X}|}{|\mathbf{K_X} - \mathbf{B}_1|} \\ & \text{subject to } \mathbf{B}_1 \succcurlyeq \mathbf{0}, \text{ and } \mathbf{D} \succcurlyeq \mathbf{K_X} - \mathbf{B}_1 \\ \triangleq& R^{'}(\mathbf{D}, \mu). \end{aligned}$$

Combining this with (22) will lead to

$$R^*(\mathbf{D}, \mu) = R^{'}(\mathbf{D}, \mu), \quad 0 \leq \mu < 1.$$

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