Sparse Phase Retrieval: Convex Algorithms and Limitations

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***Abstract*—We consider the problem of recovering signals from their power spectral densities. This is a classical problem referred to in literature as the phase retrieval problem, and is of paramount importance in many fields of applied sciences. In general, additional prior information about the signal is required to guarantee unique recovery as the mapping from signals to power spectral densities is not one-to-one. In this work, we assume that the underlying signals are sparse.**

Recently, semidefinite programming (SDP) based approaches were explored by various researchers. Simulations of these algorithms strongly suggest that signals upto $O(n^{1/2-\epsilon})$ sparsity can be recovered by this technique. In this work, we develop a tractable algorithm based on reweighted l_1-minimization that recovers a sparse signal from its power spectral density for significantly higher sparsities, which is unprecedented. We also discuss the limitations of the existing SDP algorithms and provide a combinatorial algorithm which requires significantly fewer "phaseless" measurements to guarantee recovery.

***Index Terms*—Phase Retrieval, Phaseless measurements, Semidefinite Programming (SDP), Reweighted l_1-minimization**

I. Introduction

In many practical measurement systems, the power spectral density of the signal, i.e. the magnitude square of the Fourier transform, is the measurable quantity. Phase information of the Fourier transform is completely lost, because of which signal recovery is difficult. This problem occurs in many areas of engineering and applied physics, including X-ray crystallography [3], astronomical imaging [4], microscopy, optics [5], blind channel estimation and so on.

Recovering a signal from its Fourier transform magnitude, or equivalently its autocorrelation, is known as phase retrieval. The mapping from signals to their Fourier transform magnitudes is not one-to-one, and hence unique recovery is not possible in general. Additional measurements or prior information about the signal is required in order to be able to uniquely recover the underlying signal. Constraints on the signal's values like non-negativity, bounds on the signal's support (locations where the value is non-zero), and more recently sparsity [9], [10], are commonly used as prior information.

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Considerable amount of research has been done over the last few decades ([1], [2]) and a wide range of heuristics have been proposed (see [6]), a comprehensive survey of which can be found in [7]. [8] provides a theoretical framework to understand the heuristics, which are in essence an alternating projection between a convex set and a non-convex set. Such methods often converge to a local minimum, hence drastically reducing the chances of successful signal recovery.

Recently, the phase retrieval problem was recast as a semidefinite programming problem (see [11], [12] and [13]). In [11], additional measurements with different illuminations, which is possible in an optical setup, are used to make unique recovery feasible. In [12], [13], the underlying signals are assumed to be sparse, which is a reasonable assumption in applications like X-ray crystallography, microscopy and astronomical imaging.

Numerical simulations of the existing techniques based on SDP strongly suggest that signals upto $O(n^{1/2-\epsilon})$ sparsity can be recovered with an arbitrarily high probability. This behavior for the phase retrieval problem was rigorously explained in [16], [17]. [18], [19] consider the "generalized" phase retrieval problem and observe a similar behavior. In this work, we develop an algorithm based on reweighted l_1 minimization to solve the phase retrieval problem for significantly higher sparsities. We also provide certain theoretical guarantees and discuss the limitations of the SDP based techniques, and develop a combinatorial algorithm which is significantly faster and requires far fewer measurements to guarantee recovery.

The remainder of the paper is organized as follows. In Section 2, we formulate the phase retrieval problem and recast it as an SDP problem. We discuss the limitations of the existing SDP-based techniques in Section 3 and develop an algorithm based on reweighted l_1 minimization in Section 4. In Section 5, we develop a measurement system using a combinatorial approach. Section 6 presents the results of the numerical simulations and concludes the paper.

II. Problem Formulation

Suppose $\mathbf{x} = (x_0, x_1, ...x_{n-1})$ is a real-valued discrete-time signal of length n and sparsity k, where sparsity is defined as the number of non-zero entries. Let $\mathbf{y} = (y_0, y_1, ...y_{n-1})$

be its Fourier transform, i.e.,

$$\mathbf{y} = \mathbf{F}\mathbf{x} \tag{1}$$

where $\mathbf{F}$ is the $n \times n$ DFT matrix. The phase retrieval problem can be mathematically stated as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && |\mathbf{y}| = |\mathbf{F}\mathbf{x}| \end{aligned} \tag{2}$$

Since magnitude square of Fourier transform and autocorrelation are Fourier pairs, the phase retrieval problem can be reformulated as recovering signals from their autocorrelation, i.e.,

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && a_i = \sum_j x_j x_{j+i} \qquad 0 \leq i \leq n-1 \end{aligned} \tag{3}$$

where $\mathbf{a} = (a_0, a_1, ..., a_{n-1})$ is the autocorrelation of $\mathbf{x}$.

Observe that the operations of time-shift, flipping and global sign-change do not affect the autocorrelation, because of which there is a trivial ambiguity. The signals resulting from these operations are considered equivalent, and in all the applications it is considered good enough if any equivalent signal is recovered.

The problem (2) is hard to solve due to non-convex constraints. We can relax the constraints into a set of convex constraints by embedding the problem in a higher dimensional space, a technique popularly known as lifting. Note that (2) contains n constraints of the form $|y_i| = |\mathbf{f}_i^T\mathbf{x}|$, where $\mathbf{f}_i$ is the i^{th} column of $\mathbf{F}$. Squaring both sides, the constraints can be rewritten as $|y_i|^2 = |\mathbf{x}^T\mathbf{f}_i^*\mathbf{f}_i^T\mathbf{x}|$. The problem can be recast using an $n \times n$ matrix $\mathbf{X}$ as

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{subject to} && |\mathbf{y}_i|^2 = trace(\mathbf{M}_i\mathbf{X}) \quad 0 \leq i \leq n-1 \\ &&& rank(\mathbf{X}) = 1 \quad \& \quad \mathbf{X} \succcurlyeq 0 \end{aligned} \tag{4}$$

where $\mathbf{M}_i = \mathbf{f}_i^*\mathbf{f}_i^T$.

In terms of the autocorrelation $\mathbf{a}$, the lifted problem can be formulated as

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{subject to} && \sum_j \mathbf{X}_{j,j+i} = a_i \quad 0 \leq i \leq n-1 \\ &&& rank(\mathbf{X}) = 1 \quad \& \quad \mathbf{X} \succcurlyeq 0 \end{aligned} \tag{5}$$

III. Background

The program (4) can be reformulated as a rank minimization problem as follows

$$\begin{aligned} &\text{minimize} && rank(\mathbf{X}) \\ &\text{subject to} && |\mathbf{y}_i|^2 = trace(\mathbf{M}_i\mathbf{X}) \quad 0 \leq i \leq n-1 \\ &&& X \succcurlyeq 0 \end{aligned} \tag{6}$$

This is a non-convex problem as rank is a non-convex function. It has been shown in [14] that trace minimization is the tightest convex relaxation of rank minimization for positive semidefinite matrices. This relaxation is not useful in the phase retrieval setup as $trace(\mathbf{X})$ corresponds to the energy of the signal $\mathbf{x}$, which is fixed by the magnitude of the Fourier transform. [15] proposes log-determinant function as a surrogate for rank in such problems, i.e.,

$$\text{minimize} \quad \log\det(\mathbf{X} + \epsilon\mathbf{I}) \tag{7}$$

$$\text{subject to} \quad |\mathbf{y}_i|^2 = trace(\mathbf{M}_i\mathbf{X}) \quad 0 \leq i \leq n-1 \tag{8}$$

$$X \succcurlyeq 0 \tag{9}$$

This heuristic tries to minimize a concave function in a convex domain, which can be done using gradient descent approach. This method was explored for the phase retrieval setup in [11], [12]. Simulations suggest that the algorithm converges to a rank 1 solution with high probability if the underlying signal is $O(n^{1/2-\epsilon})$ sparse.

In [13], we explored a two-stage recovery process to provably solve (2). In the first stage, we use information about the support of the autocorrelation to recover the support of the signal (see [16]). In the second stage, we solve the SDP with known support [17]. It was empirically observed and theoretically shown that signals were recovered with arbitrarily high probability if the sparsity was $O(n^{1/2-\epsilon})$. However, if the support information was available by other means, it was observed that the program recovered signals up to roughly $O(n)$ sparsity.

IV. Recovery Algorithm

In this section, we develop an iterative algorithm based on reweighted l_1-minimization to solve the phase retrieval problem outside the $O(n^{1/2-\epsilon})$ sparsity regime.

A two-stage approach like [13] wouldn't work as the support of the autocorrelation becomes full if the signal has sparsity greater than $O(n^{1/2}log(n))$. Trace minimization in the phase retrieval setup has two issues: trivial ambiguities have same objective function, trace is fixed because of which we will be solving a feasibility problem only. Weighted l_1 minimization (10) intuitively overcomes these issues and promotes sparse solutions.

$$\begin{aligned} &\text{minimize} && trace(\mathbf{V}|\mathbf{X}|) \\ &\text{subject to} && |\mathbf{y}_i|^2 = trace(\mathbf{M}_i\mathbf{X}) \quad 0 \leq i \leq n-1 \\ &&& X \succcurlyeq 0 \end{aligned} \tag{10}$$

where $\mathbf{V}$ is a weight matrix which can be designed to promote the necessary structure in the solution.

Simulations suggest that (10) has rank 1 solutions with high probability if the sparsity is $O(n^{1/2-\epsilon})$ and $\mathbf{V}$ is chosen from a random distribution like i.i.d Gaussian, but fails outside the $O(n^{1/2-\epsilon})$ region. However, the largest eigenvalue turns out to be considerably stronger than the other eigenvalues, and the eigenvector corresponding to it happens to contain a lot of information about the support of the signal even significantly outside the $O(n^{1/2-\epsilon})$ region, which is not very surprising.

This strongly suggests the possibility of an iterative algorithm, which at every iteration also knows "a lot" about where the signal's non-zero entries can be. Algorithm 1 uses this information by doing a reweighted minimization at every iteration. The weights corresponding to prospective support locations are set to zero to encourage the signal to choose those locations in the next iteration, and the weights outside this region is chosen to be some positive numbers.

Algorithm 1 Phase Retrieval Algorithm

Input: The magnitude of the Fourier transform $|\mathbf{y}|$, maximum number of iterations
Output: The underlying signal $\mathbf{x}$

1. Initialize $\mathbf{V}$ by by choosing its entries from $[0, 1]$ uniformly at random
2. Solve the optimization problem
$$\begin{aligned} &\text{minimize} && trace(\mathbf{V}|\mathbf{X}|) \\ &\text{subject to} && |\mathbf{y}_i|^2 = trace(\mathbf{M}_i\mathbf{X}) \quad 0 \leq i \leq n-1 \\ & && X \succcurlyeq 0 \end{aligned} \tag{11}$$
3. If $rank(\mathbf{X}) = 1$, return $\mathbf{X}$, else calculate $\mathbf{X}_1 = \mathbf{x}_1\mathbf{x}_1^T$, where $\mathbf{X}_1$ is the best rank-1 approximation of $\mathbf{X}$
4. Update $\mathbf{V}$ as follows: $\mathbf{V}_{ij} = 0$ if $|\mathbf{x}_i|$ and $|\mathbf{x}_j|$ are greater than a certain threshold, choose the remaining entries from $[0, 1]$ uniformly at random
5. Iterate until convergence or maximum number of iterations
6. Calculate $\mathbf{X}_1$ and return $\mathbf{x}_1$

V. Phase Retrieval as a compression problem

Theorem V.1 was proved for the phase retrieval problem in [16], [17].

Theorem V.1. *Signals can be recovered from their power spectral densities up to time-shift, reversal and global sign with probability $1-\delta$ for any $\delta, \epsilon > 0$ if*

1) $k = O(n^{1/2-\epsilon})$
2) *k entries are chosen uniformly at random*
3) *$n > n_0(\epsilon, \delta)$ (i.e., sufficiently large n)*

We can instead fix the sparsity k and consider the phase retrieval problem with partial power spectral density information (for example [20]). In particular, one might have access to only certain frequencies. We will use some classical results in the compressed sensing literature.

Theorem V.2 (Candes, Romberg, Tao, [21])**.** *Let $\mathbf{F}$ be the DFT matrix. Consider the random matrix $\mathbf{A}$ obtained by choosing m rows of $\mathbf{F}$ uniformly at random. If $\mathbf{x}_0$ is a k sparse vector, $\mathbf{x}_0$ can be recovered from observations $\mathbf{A}\mathbf{x}_0$ with arbitrarily high probability if $m \geq O(k \log n)$ via the following ℓ_1 minimization:*

$$\begin{aligned} &\min_{\mathbf{x}} && |\mathbf{x}\|_1 \\ &\textit{subject to} && \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}_0 \end{aligned} \tag{12}$$

The following theorem gives a useful result for the recovery of a sparse signal from its partial power spectral density.

Theorem V.3. *Let $\mathbf{x}_0$ be a k sparse vector whose support is chosen uniformly at random. Suppose we observe its power spectral density at m distinct frequencies chosen uniformly at random. If $m \geq O(k^2 \log n)$, $\mathbf{x}_0$ can be recovered from its partial power spectral density with arbitrarily high probability.*

Proof: If $\mathbf{x}_0$ is a k-sparse signal, its autocorrelation can have at most k^2 non-zero entries. Since power spectral density is the Fourier transform of the autocorrelation, using Theorem V.2, whenever $m \geq O(k^2 \log n)$, the autocorrelation of $\mathbf{x}_0$ can be recovered from partial power spectral density observations via ℓ_1 minimization with arbitrarily high probability. ∎

While Theorem V.3 gives a tractable algorithm for the recovery of $\mathbf{a}$ from partial power spectral density observations $\mathbf{s} = \mathbf{A}\mathbf{a}$, one can consider a more sophisticated approach . The program (12) tries to solve for a k^2-sparse solution without using the extra information that the resulting solution should be a valid autocorrelation. We propose the following method which might be of interest for future directions.

$$\min_{\mathbf{a}} \quad \|\mathbf{a}\|_1 \tag{13}$$
$$\begin{aligned} \text{subject to} \quad & \mathbf{A}\mathbf{a} = \mathbf{s} \\ & \sum_{j=1}^{n} \mathbf{X}_{j,j+i} = a_i \quad 0 \leq i \leq n-1 \end{aligned}$$
$$rank(\mathbf{X}) = 1 \quad \& \quad \mathbf{X} \succeq 0 \tag{14}$$

Observe that the only nonconvex constraint in (13) is (14). We believe that solving (13) can substantially increase the performance and instead of $O(k^2 \log n)$ measurements, just $O(k \log n)$ measurements might suffice which is similar to compressed recovery of a k sparse vector. However, further investigation is required to relax (14) in a useful manner as opposed to ignoring it.

Overall, Theorem V.3 is subject to a $O(n^{1/2-\epsilon})$-sparsity bottleneck since the full power spectral density corresponds to n measurements. While we do not provide theoretical guarantees for Algorithm 1, when full power spectral density is available, our algorithm seemingly beats the $O(n^{1/2-\epsilon})$-sparsity bottleneck. In section VI, we see that the recoverable sparsity is much higher than $O(n^{1/2-\epsilon})$.

A. Two-stage recovery

In [13], we try to solve the sparse phase retrieval problem via a two-stage approach where the first step involves finding the support of the signal from the support of the autocorrelation. We will now argue that such an approach is inherently subject to the $O(n^{1/2-\epsilon})$ bottleneck.

Lemma V.1. *Suppose $\mathbf{x}_0$ is a k-sparse signal whose support is chosen uniformly at random, and whose nonzero entries are continuous i.i.d. random variables. Then, there exists a constant c such that whenever $k \geq cn^{1/2}log(n)$, support of the autocorrelation is full with arbitrarily high probability.*

Proof: Without loss of generality, we can assume that each location belongs to the support of the signal with probability $\frac{k}{n}$ independently as the same proof will apply for k-sparse signals with standard modifications.

For a particular distance d, if no two non-zero entries in the signal are separated by a distance d, we can say that d doesn't belong to the support of the autocorrelation. This probability can be bounded by $(1-k^2/n^2)^{n/2}$ which is upper bounded by $e^{-k^2/2n}$. Union bound tells us that the probability of the support of the autocorrelation not being full is less than $ne^{-k^2/2n}$, which goes to zero if $k \geq cn^{1/2}log(n)$ for sufficiently large n. ∎

B. Relation to Gaussian Phase Retrieval

Our results on partial power spectral density can be related to the "generalized" phase retrieval problem, where the observations are of the form $|\mathbf{g}_i^T\mathbf{x}|$ for i.i.d. complex standard normal vectors $\{\mathbf{g}_i\}_{i=1}^m$. While this problem is structurally similar to phase retrieval, it is considerably simpler as there are no trivial ambiguities like time-shift and flipping.

Assuming $\mathbf{x}$ is a sparse vector, [18] and [19] analyze the following semidefinite program for the recovery of $\mathbf{x}$ up to global phase ambiguity:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_\star + \lambda\|\mathbf{X}\|_1 \\ \text{subject to} \quad & \left\langle \mathbf{g}_i\mathbf{g}_i^T, \mathbf{X}\right\rangle = \left\langle \mathbf{g}_i\mathbf{g}_i^T, \mathbf{x}\mathbf{x}^T\right\rangle \quad 1 \leq i \leq m \end{aligned} \tag{15}$$

where $||.||_\star$ is the nuclear norm and λ is any regularizer. Naturally, one would wish that the unique minimizer of (15) be $\mathbf{x}\mathbf{x}^T$ to be able to recover $\mathbf{x}$ by performing a simple decomposition. Interestingly, both [18] and [19] suggest that as long as $m \leq \min\{k^2, n\}$, recovery using (15) is not possible with very high probability for any choice of regularizer λ. To summarize, even if $k^2 \ll n$, one would still need $\Omega(k^2)$ measurements for recovery, which indicates a $O(n^{1/2-\epsilon})$ bottleneck for generic Gaussian measurements too.

Overall, we have seen that for these two important class of phaseless measurements, current theoretical guarantees are subject to a strong $O(n^{1/2-\epsilon})$-sparsity bottleneck. A natural question is whether it is possible at all to do sparse phase retrieval in a tractable way with less than $O(k^2 log(n))$ measurements.

C. Combinatorial Approach

We consider the problem of recovering $\mathbf{x}$ from phaseless measurements by making use of a specific choice of measurements. In particular, we show that one can recover a k-sparse signal from specific phaseless measurements by using only $O(k \log n)$ measurements with very high probability. This shows that phase retrieval with optimal number of measurements is in fact possible.

Theorem V.4. *Suppose* $\mathbf{x}$ *is an arbitrary* k*-sparse vector. Let* $\{\mathbf{z}_i\}_{i=1}^m$ *be i.i.d. vectors with i.i.d. entries distributed as:*

$$\begin{cases} 0 & \textit{with probability } \ 1-1/k \\ \mathcal{N}(0,1) & \textit{with probability } \ 1/k \end{cases} \tag{16}$$

Let $\{\mathbf{a}_i, \mathbf{b}_i\}_{i=1}^m$ *be i.i.d. vectors with i.i.d. entries distributed as* $\exp(\mathbf{i}\theta)$ *where* θ *is uniformly distributed in* $[0, 2\pi)$*. Denote the function* $\mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ *that returns entrywise products of two vectors by* $\cdot$*. Assume, we observe the measurements:*

$$|\left\langle \mathbf{a}_i \cdot \mathbf{z}_i, \mathbf{x}\right\rangle|^2, |\left\langle \mathbf{b}_i \cdot \mathbf{z}_i, \mathbf{x}\right\rangle|^2 \tag{17}$$

for $1 \leq i \leq m$*.*

Then, $\mathbf{x}\mathbf{x}^T$ *can be recovered with high probability whenever* $m \geq ck \log n$ *for some constant* $c > 0$*.*

Remark: We don't need the knowledge of sparsity k, if we have an extra factor of $\log n$ measurements, i.e. $m \geq klog^2(n)$ measurements.

Proof: The proof (also a recovery algorithm) consists of two main steps, details are omitted due to space constraints (see the extended version [22]).

Support recovery: Denote support of $\mathbf{x}$ by $S \subseteq [n]$ and support of $\mathbf{z}_i$ by S_i and let $\mathbf{c}_i = \mathbf{z}_i \cdot \mathbf{a}_i$. Now, consider the inner product $\mathbf{c}_i^*\mathbf{x}$. Since nonzero entries of $\mathbf{c}_i$ have continuous distribution if $S \cap S_i \neq \emptyset$, $|\mathbf{c}_i^*\mathbf{x}| \neq 0$ almost surely. Hence whenever a measurement $|\mathbf{c}_i^*\mathbf{x}|^2 = 0$ we can deduce that $S \cap S_i = \emptyset$. Let:

$$I = \{1 \leq i \leq m | S_i \cap S = \emptyset\} \tag{18}$$

Clearly, $\{S_i\}$'s are i.i.d. supports and for each i we have:

$$\mathbb{P}(S \cap S_i = \emptyset) = (1 - 1/k)^k \approx e^{-1} \tag{19}$$

By law of large numbers, w.h.p. $|I| \geq m/4$. The probability that $j \in \bar{S}$ is not contained in $\bigcup_{i \in I} S_i$ is at most $(1-1/k)^{|I|} \leq (1-1/k)^{m/4}$. Assuming $m \geq ck \log n$ and using a union bound:

$$\mathbb{P}(\bar{S} \not\subseteq \bigcup_{i \in I} S_i) \leq n^{-1} \tag{20}$$

which will approach 0, which implies that the support can be recovered. Next, with the knowledge of support, we proceed with recovery of $\mathbf{x}$ up to an overall phase ambiguity.

Signal recovery: Recovery of signal given support will be performed in two steps. We first show that magnitudes of nonzero entries of $\mathbf{x}$ can be found.

Recovering magnitudes: Assume $S_i \cap S$ is a singleton $j \in S$. Since we already have the knowledge of S from previous part, we can immediately deduce: $|x_j|^2 = \frac{|\langle \mathbf{c}_i, \mathbf{x}\rangle|^2}{|c_{i,j}|^2}$. Then, we simply need to ensure that w.h.p., for all $j \in S$ there exists $1 \leq i \leq m$ satisfying $S \cap S_i = j$. Probability of this not happening after union bounding is less than $kn^{-c/e}$ which goes to zero for $c > 2e$.

Recovering relative phases: Next, we consider the measurements satisfying $|S_i \cap S| = 2$, $1 \leq i \leq m$. For fixed i we have: $\mathbb{P}(|S_i \cap S| = 2) \geq \frac{1}{2e}$. Overall, w.h.p. there are $m/10$ measurements satisfying $|S_i \cap S| = 2$. Let:

$$I = \{1 \leq i \leq m | \bigcup |S_i \cap S| = 2\} \tag{21}$$

Next, consider the k vertex graph obtained by connecting the nodes j, l whenever $\{j, l\} = S_i \cap S$ for some $i \in I$. Observe that, each $i \in I$ picks an edge in this graph uniformly at random. This graph is connected with high probability and by fixing phase of the initial node, the rest are uniquely determined.

■

VI. Numerical Simulations

In order to demonstrate the performance of Algorithm 1, numerical simulations were performed for different values of signal length n and sparsities k. For a given n and sparsity k, the k support locations were chosen from the n locations uniformly at random. The signal values at the support locations were chosen from [0,1] uniformly at random. The Fourier transform magnitudes of the signal were computed and provided as input to Algorithm 1. If the output signal matched the input signal, it was counted as a success, else it was counted as a failure. Two other semidefinite program based algorithms (CandesPR: [11], HassibiPR: [13]) and the traditional alternating projection algorithm (GS: [6]) were used for comparison purposes.

The results of the numerical simulations are shown in Figure 1. The probability of successful recovery is plotted against various sparsities for $n = 32$ and $n = 64$. 100 simulations were performed for each sparsity to get the average success probability, which was calculated as the percentage of successful recovery. The existing SDP based algorithms start fading at around $O(n^{1/2-\epsilon})$ sparsity, the figure clearly demonstrates the superiority of using reweighted minimization outside the $O(n^{1/2-\epsilon})$ sparsity region.

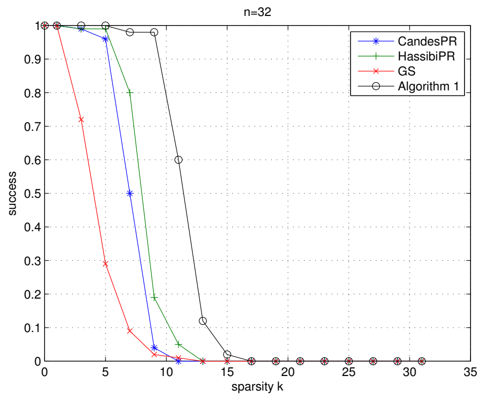

Fig. 1. Success rate of recovery for $n = 32$ and various sparsities

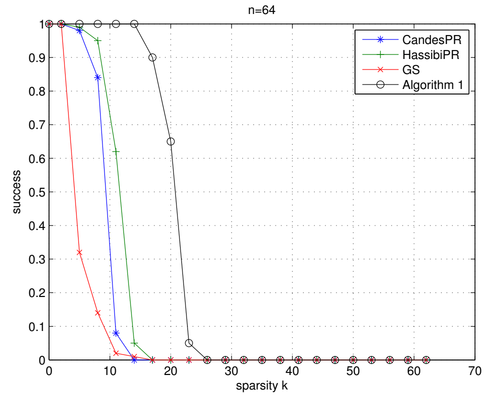

Fig. 2. Success rate of recovery for $n = 64$ and various sparsities

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