Noise-Shaped Quantization for Nonuniform Sampling

Adam Mashiach
Dept. Electrical Engineering-Systems
Tel-Aviv University
Tel-Aviv, Israel
Email: adam.mashiach@gmail.com

Ram Zamir
Dept. Electrical Engineering-Systems
Tel-Aviv University
Tel-Aviv, Israel
Email: zamir@eng.tau.ac.il

***Abstract*—The Nyquist theorem (for perfect reconstruction of a band-limited signal from its noiseless samples) depends, essentially, only on the average sampling rate. In contrast, reconstruction from imperfect samples strongly depends also on the sampling pattern. Specifically, when the samples are corrupted with independent noise, the reconstruction distortion is generally higher for nonuniform sampling than for uniform sampling at the same average rate - a phenomenon known as "noise amplification". We show that this degradation in performance can be avoided if the noise spectrum can be controlled; for any periodic nonuniform sampling pattern, there exists a quantization noise-shaping scheme that mitigates the noise amplification. Moreover, a scheme that combines noise shaping, Wiener filtering and entropy-coded dithered quantization (ECDQ) achieves the rate-distortion function of a (white or colored) Gaussian source, up to the granular loss of the lattice quantizer. This loss tends to zero, for a sequence of good latices, as the lattice dimension tends to infinity.**

I. Introduction

Sampling and quantization are the basic operations which convert a continuous physical signal into a digital representation, that can be stored, transferred and processed by digital systems. The basic sampling theorem (Nyquist-Shannon) deals with samples that are taken in constant time intervals, which is referred to as uniform sampling. In practice, sometimes the sampling times are not uniformly spaced, whether as a consequence of physical constraints (inaccurate sampling times, sensor networks etc.) or a deliberate design (anti-aliasing sampling, ambiguous sparse spectrum resolution [1], multiple-description coding [2] ect.).

We concentrate on *periodic* nonuniform sampling, which was widely studied. In this case, the sampling grid has a periodic pattern, so the samples can be divided into a finite number M of uniform sampling subsequences. It has been shown that a band-limited signal can be perfectly reconstructed from its periodic nonuniform samples if the average sampling rate is at least the Nyquist rate. This sampling scheme was first studied by Yen in [3], who has also provided an interpolation formula. Periodic sampling can be viewed as a special case of Papoulis generalized sampling expansion [4].

While sampling (at high enough rate) of bandlimited signals is reversible (preserves all the information about the original signal), quantization of continuous-valued signals is not. Thus the quantization process can be regarded as adding noise to the samples. It is well known that uniform sampling is less sensitive to additive white noise (quantization or other types of noise) than other forms of sampling (at the same average sampling rate). In [5] Seidner and Feder derive a closed-form expression for the reconstruction error in periodic nonuniform sampling in the presence of additive white noise. They have concluded that in any periodic nonuniform sampling there is an inherent amplification of the reconstruction error, relative to the uniform sampling case.

Another important factor in signal digitization, beyond the reconstruction error (distortion), is the coding rate. Since quantization is generally a nonlinear operation, it is hard to analyze the performance of a given quantization scheme. One way to circumvent this difficulty is to linearize the quantization operation using random dither. This idea is used by entropy coded dithered (lattice) quantization (ECDQ). It was shown in [6] that ECDQ (with pre and post filters) can asymptotically achieve the rate-distortion function (RDF) of a Gaussian source sampled uniformly. Whenever we state that ECDQ can asymptotically achieve the RDF, we mean asymptotically as the lattice quantizer dimension tends to infinity.

In this work, we consider a compression scheme for analog bandlimited signals, which is based on nonuniform sampling and ECDQ. In previous work [7] we concentrated on the over-sampling case, and proved that it is possible to avoid noise amplification in periodic nonuniform sampling, if the oversampling ratio (above Nyquist) is equal to the period length. Here we concentrate on the case in which the average sampling rate equals the Nyquist rate (no oversampling), referred to as *periodic Nyquist sampling*. We show that while nonuniform sampling amplifies the reconstruction noise, it lowers the ECDQ rate, though not enough to achieve the corresponding Gaussian RDF. Nevertheless, we show that shaping the spectrum of the quantization noise (e.g. by adding a feedback loop to the ECDQ scheme) can eliminate the noise amplification. Moreover, an ECDQ-based scheme combined with noise shaping, which is tailored to the specific sampling pattern, can asymptotically (in the lattice dimension) achieve the RDF of a Gaussian source.

The paper is organized as follows. Section II presents the problem of noise amplification in nonuniform sampling. In Section III we discuss the performance of ECDQ for nonuniform sampling. The noise shaping solution for the noise amplification problem is introduced and studied in section IV. Section VI concludes the paper.

II. Noise Amplification in Nonuniform Sampling

Throughout this paper, we will use upper case letters for stochastic variables and bold math font for Deterministic matrices, as $\mathbf{H}$. Stochastic vectors will be indicated by underline, while sequence of N stochastic vectors will be indicated by lower and upper indices, thus $[\underline{X}_1, ..., \underline{X}_N] = \underline{X}_1^N$.

We consider a continuous-time bandlimited stationary Gaussian source $X(t)$, with zero mean and flat power spectrum $S_{XX}(f)$, where $S_{XX}(f) = 0$ for $f > B$. Since the RDF of a bandlimited signal equals the RDF of its Nyquist rate samples (multiplied by the sampling rate), the RDF of $X(t)$ equals $B \log_2 \left(\frac{\sigma_X^2}{D}\right)$ bits per second. The source is sampled at time instants t_n, and quantized to yield the digital samples A_n. We are modeling the quantization operation as an additive white noise channel. This model is only approximate for deterministic quantization in high resolution, while accurate for subtractive dithered quantization, as will be discussed in section III. This model can also include noise that is added to the samples before the quantization process (like sensor noise). Thus the samples can be represented as

$$A_n = X(t_n) + V_n, \tag{1}$$

where V_n is the quantization noise (or any other source of noise) with variance σ_V^2, and is usually assumed to be white. We are interested in the optimal (in MMSE sense) linear reconstruction of $X(t)$ given the samples $\{A_n\}_{n=-\infty}^{\infty}$. Denote this reconstruction by $\hat{X}(t)$, and its average MSE by D.

Since $X(t)$ is bandlimited, it can be replaced (in mean square sense) by its Nyquist interpolation

$$\mathbb{E}\left[X(t) - \sum_{k=-\infty}^{\infty} X_k \mathrm{sinc}\left(2Bt - k\right)\right]^2 = 0, \quad \forall t \tag{2}$$

where $X_n = X(n\frac{1}{2B})$ is a sequence of uniform Nyquist samples. Since Fourier transform is unitary (Parseval's theorem), estimating $X(t)$ is equivalent (and leads to the same MMSE) to estimating the uniform samples X_n. Thus, the estimation problem can be reformulated in a discrete-time space. By inserting (1) into (2) (with $t = t_n$) we have

$$A_n \overset{\text{M.S.}}{=} \sum_{k=-\infty}^{\infty} X_k \mathrm{sinc}\left(2Bt_n - k\right) + V_n. \tag{3}$$

Equation (3) can be rewritten in a matrix form as

$$\underline{A} = \mathbf{H}\underline{X} + \underline{V}, \tag{4}$$

where $\mathbf{H}_{n,k} = \mathrm{sinc}\left(2Bt_n - k\right)$. These vectors and matrix are infinite in size, but for simplicity we will look at (large) finite N-sample vectors (finite time interval). This approximation introduces edge effects that are negligible for large N.

We concentrate on a special case of nonuniform sampling: periodic sampling. In periodic sampling, with an M-samples per-period, the sampling grid consists of M uniform sampling subsequences, where the i^{th} subsequence is delayed with some time shift τ_i. Thus, the sampling times t_n have the form $t_{Mk+i} = \tau_i + kT$ for any integer k and $i = 0, 1, ..., M-1$, where T is the period length in time. In the periodic Nyquist sampling case, the average sampling rate $\frac{M}{T}$ equals the Nyquist rate $2B$. In this case, a perfect reconstruction is possible in the absence of quantization noise [3]. When adding noise to the samples, the reconstruction error depends on the time delays $\tau_0, \tau_1, ..., \tau_{M-1}$.

It is known that nonuniform sampling results in higher distortion than uniform sampling. We define the noise amplification factor of a specific nonuniform sampling pattern as the ratio between the MMSE of the reconstruction from the nonuniform samples and the one from a uniform sampling pattern with the same average sampling rate (in the presence of additive white noise). A simple way to observe that uniform sampling is the optimal sampling pattern (at Nyquist average sampling rate), is by looking at the approximate finite discrete-time problem (4) at high SNR regime. In the case of Nyquist average sampling rate, $\mathbf{H}$ is square and non singular, thus the reconstruction in the high SNR case is simply $\underline{\hat{X}} = \mathbf{H^{-1}}\underline{A} = \underline{X} + \mathbf{H^{-1}}\underline{V}$. The resulting MMSE is $D_N = \frac{\sigma_V^2}{N}\mathrm{trace}\left(\mathbf{H^{-1}H^{-T}}\right) = \frac{\sigma_V^2}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i}$, where λ_i are the eigenvalues of the matrix $\mathbf{H}^T\mathbf{H}$. It can be shown that the norm of each row of $\mathbf{H}$ tends to 1 as $N \to \infty$, thus $\lim_{N\to\infty}\frac{1}{N}\mathrm{trace}\left(\mathbf{H}\mathbf{H}^T\right) = 1$, and since $\mathbf{H}\mathbf{H}^T$ and $\mathbf{H}^T\mathbf{H}$ share the same eigenvalues we have $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\lambda_i = 1$. Using the harmonic-arithmetic mean inequality we have that $\lim_{N\to\infty} D_N \geq \sigma_V^2$, with equality iff all the eigenvalues of $\mathbf{H}^T\mathbf{H}$ are 1, which corresponds to the uniform sampling case.

To illustrate the impact of noise amplification on the distortion let us look at the special case of periodic Nyquist sampling with $M = 2$, under the restriction of perfect reconstruction in the absence of noise. Using the closed-form expression of Seidner and Feder [5] the noise amplification in this case is $\frac{1}{sin^2(\pi B \Delta\tau)}$, where $\Delta\tau \triangleq \tau_2 - \tau_1$ is the time difference between two samples in a period. For $\Delta\tau = \frac{1}{2B}$, which corresponds to uniform sampling, there is no noise amplification (as desired by its definition). As the two samples in a period get closer to one another, the noise amplification increases and tends to ∞ when $\Delta\tau \to 0$. Thus noise amplification can be unbounded in the high SNR regime.

III. ECDQ of Nonuniform Samples

In this section we consider a simple sampling and quantization scheme, which is based on lattice ECDQ and Wiener filtering. Before diving into the main discussion of the section, we will present L-dimensional lattice ECDQ (Fig. 1) and summarize some of its properties. ECDQ is based on subtractive dithered quantization, in which a random dither vector $\underline{Z}$ is being added to the source vector $\underline{X}$ before the quantization process at the encoder, and subtracted from the received lattice vector at the decoder. The dither vector, which is known to both the encoder and decoder, is independent of

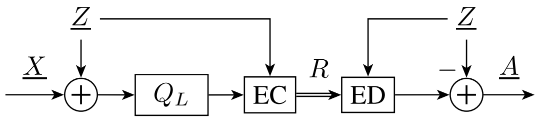

Fig. 1. Source coding by an ECDQ.

the input signal and previous realizations of the dither, and is uniformly distributed over the basic Voronoi cell of the L-dimensional lattice quantizer ($\mathcal{V}_0$). The resulting vector at the decoder $\underline{A} = Q_L\left(\underline{X}+\underline{Z}\right)-\underline{Z}$ can be written as $\underline{A}=\underline{X}+\underline{V}$, where $\underline{V}$ is the quantization noise and is independent of the input and uniformly distributed over $\mathcal{V}_0$. Moreover, $\underline{V}$ has white components (for a good lattice [8]), thus, for ECDQ the additive noise model is accurate in any resolution and for any lattice dimension.

If N successive quantizer outputs $Q_L\left(\underline{X}_1^N+\underline{Z}_1^N\right)$ are entropy-coded jointly, conditioned on the dither values $\underline{Z}_1^N$, then the coding rate (in bits per second) is given by [6]

$$\begin{aligned} R_Q &= \frac{2B}{NL} H\left(Q_L\left(\underline{X}_1^N+\underline{Z}_1^N\right)|\underline{Z}_1^N\right) \\ &= \frac{2B}{NL} I\left(\underline{X}_1^N;\underline{X}_1^N+\underline{V}_1^N\right). \end{aligned} \tag{5}$$

Joint entropy coding takes care of the memory inside the oversampled source. Since $\underline{V}$ is uniformly distributed over $\mathcal{V}_0$, its entropy rate is

$$\frac{1}{L}h\left(\underline{V}\right) = \frac{1}{2}\log_2\left(2\pi e\sigma_V^2\right) - \frac{1}{2}\log_2\left(2\pi e G_L\right), \tag{6}$$

where G_L is the normalized second moment of the L-dimensional lattice quantizer [6]. It can be shown that

$$R_Q \le \frac{2B}{NL} I\left(\underline{X}_1^N;\underline{X}_1^N+\underline{V^*}_1^N\right) + B\log_2\left(2\pi e G_L\right), \tag{7}$$

where V_n^* is a Gaussian i.i.d process with variance σ_V^2. Thus, the ECDQ rate achieves the mutual information rate over a Gaussian test channel, denoted by R_Q^*, up to a loss of $B\log_2\left(2\pi e G_L\right)$. It is known that there exists a sequence of optimal lattice quantizers (one for each dimension) with $\lim_{L\to\infty}\log_2\left(2\pi e G_L\right)=0$, thus $\lim_{L\to\infty} R_Q = R_Q^*$. We note that for scalar quantization ($L=1$), we have $G_1=\frac{1}{12}$, thus we loose only about $\frac{1}{4}$bit relative to the RDF. For a further background on ECDQ, the reader is referred to [6] and [8].

The benefits of using (high dimensional) lattice quantizer are apparent from (7). There are several ways to apply a vector ECDQ on a sequence of source samples, and we use the same approach as in [8]. The source samples are divided into blocks of N successive samples, and each L blocks are encoded in parallel using L dimensional ECDQ. The i^{th} L dimensional vector in the input to the lattice quantizer consists of the i^{th} sample in each of the L blocks. This interleaving mechanism will enable us in section IV to insert a causal feedback loop into the ECDQ scheme.

Let us now study the ECDQ performance for nonuniform sampling (at Nyquist average sampling rate) of a Gaussian source. For simplicity of exposition we will use finite time approximation of the nonuniform sampling process that is given by $\underline{\tilde{X}} = \mathbf{H}\underline{X}$. In this simple model, the nonuniform sampling with dithered quantization process is given by (4), where $\underline{V}$ is the quantization noise. For simplicity, we restrict ourselves to a linear reconstruction (which is asymptotically optimal for $L\to\infty$), thus the minimum distortion is lower bounded by

$$D_N = \frac{1}{N}\text{trace}\left(\left[\mathbf{R}_{XX}^{-1}+\mathbf{H}^T\mathbf{R}_{VV}^{-1}\mathbf{H}\right]^{-1}\right) \tag{8}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_X^2\sigma_V^2}{\lambda_i\sigma_X^2+\sigma_V^2} \overset{\text{(a)}}{\ge} \frac{\sigma_X^2\sigma_V^2}{\sigma_X^2\frac{1}{N}\sum_{i=1}^N\lambda_i+\sigma_V^2}, \tag{9}$$

where λ_i are the eigenvalues of the matrix $\mathbf{H}^T\mathbf{H}$. Inequality (a) is due to the convexity of $1/x$, and is achieved with equality iff all the eigenvalues of $\mathbf{H}^T\mathbf{H}$ are the same, which corresponds to uniform sampling. In the limit of $N\to\infty$ the lower bound tends to the uniform sampling MMSE $\frac{\sigma_X^2\sigma_V^2}{\sigma_X^2+\sigma_V^2}$. Thus the ECDQ distortion for nonuniform sampling is higher then for uniform sampling, which is the well known noise amplification phenomenon.

Now we consider the scheme's coding rate. In the limit of $L\to\infty$, the ECDQ coding rate (7) tends to the mutual information over a Gaussian test channel, which is given by

$$\begin{aligned} R_Q^* &= \frac{2B}{N}I\left(\underline{\tilde{X}};\underline{\tilde{X}}+\underline{V}^*\right) = \frac{2B}{N}h\left(\underline{\tilde{X}}+\underline{V}^*\right) - \frac{2B}{N}h\left(\underline{V}^*\right) \\ &= \frac{B}{N}\log_2\left(\frac{\det\left(\mathbf{H}\mathbf{R}_{XX}\mathbf{H}^T+\mathbf{R}_{VV}\right)}{\det\left(\mathbf{R}_{VV}\right)}\right) \\ &= \frac{B}{N}\sum_{i=1}^{N}\log_2\left(\frac{\lambda_i\sigma_X^2+\sigma_V^2}{\sigma_V^2}\right). \end{aligned} \tag{10}$$

Due to the concavity of $\log(x)$, in the limit of $N\to\infty$ the rate is upper bounder by $B\log_2\left(\frac{\sigma_X^2+\sigma_V^2}{\sigma_V^2}\right)$, which is the rate achieved by ECDQ (with the same parameters) for uniform sampling. Again, equality is achieved only for uniform sampling. We observe an interesting result: while nonuniform sampling increases the distortion, it *decreases* the coding rate.

The observation above naturally leads to the question whether the rate reduction is enough to compensate for the increase in distortion. Using the concavity of $\log(x)$ in (10) and inserting (9) we have

$$\begin{aligned} R_Q^* &= -\frac{B}{N}\sum_{i=1}^{N}\log_2\left(\frac{\sigma_X^2\sigma_V^2}{\sigma_X^2\left(\lambda_i\sigma_X^2+\sigma_V^2\right)}\right) \\ &\ge -B\log_2\left(\frac{1}{\sigma_X^2}\frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_X^2\sigma_V^2}{\lambda_i\sigma_X^2+\sigma_V^2}\right) = B\log_2\left(\frac{\sigma_X^2}{D_N}\right), \end{aligned} \tag{11}$$

where equality is achieved only for uniform sampling. Thus, the rate reduction is not enough to achieve the Gaussian RDF. The exact gap of the ECDQ performance for nonuniform sampling from the RDF is a subject for further research.

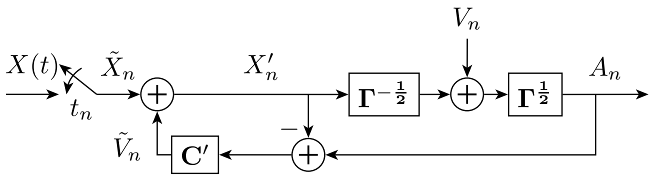

Fig. 2. Effective channel for ECDQ with a noise shaping loop. The noise shaping matrix $\mathbf{C}'$ is strictly lower triangular and $\mathbf{\Gamma}$ is diagonal.

IV. Mitigating Noise Amplification by Noise Shaping

In the previous section we discussed the ECDQ loss due to noise amplification in nonuniform sampling. We observe that the noise amplification stems from the fact that the noise is white at the original (nonuniform) sampling times, rather than at the uniform sampling times. Thus, trying to reconstruct the original source by reversing the nonuniform sampling operation (given by (4)) amplifies the white noise. If the quantization noise was not white but shaped by the matrix $\mathbf{H}$, then we would have the quantization model $\underline{A} = \mathbf{H}(\underline{X} + \underline{V})$, in which it is easy to verify that there is no noise amplification (since $\mathbf{H}$ is invertible). It is important to note that in order to cancel the noise amplification, we can shape $\underline{V}$ by any matrix $\tilde{\mathbf{H}}$, such that $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^T = \mathbf{H}\mathbf{H}^T$. If we can find such $\tilde{H}$ that is also lower triangular, then the shaping operation will be causal. In order to achieve this causal noise shaping, we insert a feedback loop into the ECDQ scheme.

We begin by introducing a simple *time invariant* ECDQ with feedback loop. In the decoder the source is being quantized sample-by-sample using a scalar quantizer. Before being quantized, each source sample x_n is combined with noise feedback $\tilde{v}_n$ which is created by feeding the quantization error v_n back through a causal filter $C'(z)$. The resulting signal $x'_n = x_n + \tilde{v}_n$ is sequentially quantized using ECDQ, to yield the quantized series a_n. Using the definition of v_n as the quantization noise, we have $a_n = x'_n + v_n = x_n + u_n$, where we define the "equivalent noise" $u_n \triangleq \tilde{v}_n + v_n$. Thus the overall scheme is equivalent to an additive noise channel with u_n as the additive noise. We notice that u_n is not white and is obtained by passing the quantization error v_n through the equivalent noise-shaping filter $C(z) = C'(z) + 1$.

In the nonuniform sampling case, the feedback scheme is not time invariant, but periodically time-varying. First we study a finite block length version of the scheme and state our main result. Then we explain how to extend it to the infinite block case. At the encoder (Fig. 2), we divide the process of source nonuniform samples $\tilde{X}$ into vectors of size N. This vector of nonuniform samples is quantized using ECDQ with feedback loop as described before, but with the following changes. The LTI filter $C'(z)$ is replaced by a strictly lower triangular matrix $\mathbf{C}'$. Thus each "equivalent noise" sample u_n is a different finite linear combination of previous quantization noise samples. Moreover, since we look at a finite block length, we should add some scaling factors before and after the quantization to compensate for edge effects, as will be clear in the following discussion. The vector $\underline{X}'$ in the input of the quantizer is scaled by a diagonal matrix $\mathbf{\Gamma}^{-\frac{1}{2}}$, quantized sample-by-sample using ECDQ and scaled back by $\mathbf{\Gamma}^{\frac{1}{2}}$. The resulting scheme is shown in Fig. 2, where we replace the ECDQ with the effective noise channel. We are now ready to state the main result of this paper.

Theorem 1: Let $X(t)$ be a continuous-time bandlimited stationary Gaussian source, sampled periodically nonuniformly at Nyquist rate (or higher). Then the ECDQ with the noise shaping feedback loop scheme in Fig. 2 (where the matrices $\mathbf{C}'$ and Γ are defined in the proof below through Cholesky decomposition of the matrix $\mathbf{H}\mathbf{H}^T$, with $\mathbf{H}$ as defined in (4)), asymptotically achieves the source RDF, as the block length N and lattice dimension L tend to infinity.

Proof: For simplicity of exposition we will use scalar quantization in the derivation, and then extend to general lattice dimension. We will also use the finite time approximation as before, which is asymptotically exact as $N \to \infty$.

The output of the ECDQ with a feedback loop in Fig. 2 can be written as

$$\begin{aligned} \underline{A} &= \mathbf{\Gamma}^{\frac{1}{2}} \left(\mathbf{\Gamma}^{-\frac{1}{2}} \left(\mathbf{C}'\mathbf{\Gamma}^{\frac{1}{2}}\underline{V} + \underline{\tilde{X}} \right) + \underline{V} \right) \\ &= \underline{\tilde{X}} + \mathbf{C}\mathbf{\Gamma}^{\frac{1}{2}}\underline{V} = \mathbf{H}\underline{X} + \underline{U}, \end{aligned} \tag{12}$$

where the effective noise is $\underline{U} = \mathbf{C}\mathbf{\Gamma}^{\frac{1}{2}}\underline{V}$, and $\mathbf{C} = \mathbf{C}' + \mathbf{I}$ is a lower triangular matrix with ones on the diagonal. Thus, the dithered-quantization feedback scheme, is equivalent to a simple additive noise channel with an effective noise $\underline{U}$. This effective noise is independent of the source and is generally not white, since

$$\mathbf{R}_{UU} = \sigma_V^2 \mathbf{C}\mathbf{\Gamma}\mathbf{C}^T. \tag{13}$$

As discussed before, our goal is to cancel the noise amplification by shaping the effective noise so that $\mathbf{R}_{UU} = \sigma_V^2\mathbf{H}\mathbf{H}^T$. This is possible (for any finite N) by choosing the matrices $\mathbf{\Gamma}$ and $\mathbf{C}$ according to the Cholesky decomposition of $\mathbf{H}\mathbf{H}^T$

$$\mathbf{C}\mathbf{\Gamma}\mathbf{C}^T = \mathbf{H}\mathbf{H}^T. \tag{14}$$

The distortion (for linear reconstruction) is given by replacing $\mathbf{R}_{VV}$ in (8) by $\mathbf{R}_{UU}$

$$D_{NS} = \frac{\sigma_X^2\sigma_V^2}{\sigma_X^2 + \sigma_V^2}. \tag{15}$$

Now, let us consider the scheme's coding rate. It is important to note that due to the feedback, the quantizer input x'_n is not independent of the past quantization noise. In this case (5) no longer holds, and it should be replaced by directed information, as been showed in [8, Theorem 2], defined as

$$I(X' \to A) = \sum_{n=1}^{N} I\left(X'^n_1; A_n | A_1^{n-1}\right). \tag{16}$$

Using the fact that the channel from X'_n to A_n is memoryless we have

$$I(X' \to A) = \sum_{n=1}^{N} h\left(A_n | A_1^{n-1}\right) - \sum_{n=1}^{N} h\left(A_n | X'^n_1, A_1^{n-1}\right) \tag{17}$$

Since the feedback is causal, given the input to the ECDQ X'_n, its output A_n is independent of the previous inputs and outputs, thus

$$\begin{aligned} I(X' \to A) &= h\left(\underline{A}\right) - \sum_{n=1}^{N} h\left(A_n | X'_n\right) \\ &\overset{\text{(a)}}{=} h\left(\underline{A}\right) - h\left(\mathbf{\Gamma}^{\frac{1}{2}} \underline{V}\right). \end{aligned} \quad (18)$$

In (a) we used the additive relation $A_n = X'_n + \mathbf{\Gamma}^{\frac{1}{2}}_{n,n} V_n$ and the fact that the components of $\underline{V}$ are i.i.d. Using (18) the scheme's rate, in bits per second, can be upper-bounded by

$$\begin{aligned} R_{NS} &= \frac{2B}{N} h\left(\underline{A}\right) - \frac{2B}{N} h\left(\mathbf{\Gamma}^{\frac{1}{2}} \underline{V}\right) \\ &= \frac{2B}{N} h\left(\mathbf{H}\underline{X} + \underline{U}\right) - \frac{2B}{N} h\left(\underline{V}\right) - \frac{2B}{N} \log_2\left(\det \mathbf{\Gamma}^{\frac{1}{2}}\right) \\ &\overset{\text{(a)}}{=} \frac{2B}{N} h\left(\mathbf{H}\underline{X} + \underline{U}\right) - B\log_2\left(2\pi e \sigma_V^2\right) \\ &\quad + B\log_2\left(2\pi e G_1\right) - \frac{2B}{N} \log_2\left(\det \mathbf{\Gamma}^{\frac{1}{2}}\right) \\ &\overset{\text{(b)}}{\leq} \frac{B}{N} \log_2\left(\frac{\det\left(\mathbf{H}\mathbf{R}_{XX}\mathbf{H}^T + \mathbf{R}_{UU}\right)}{\det\left(\sigma_V^2 \mathbf{\Gamma}\right)}\right) + B\log_2\left(2\pi e G_1\right) \\ &\overset{\text{(c)}}{=} \frac{B}{N} \log_2\left(\frac{\det\left((\sigma_X^2 + \sigma_V^2)\mathbf{H}\mathbf{H}^T\right)}{\det\left(\sigma_V^2 \mathbf{H}\mathbf{H}^T\right)}\right) + B\log_2\left(2\pi e G_1\right) \\ &= B\log_2\left(\frac{\sigma_X^2 + \sigma_V^2}{\sigma_V^2}\right) + B\log_2\left(2\pi e G_1\right) \\ &= B\log_2\left(\frac{\sigma_X^2}{D_{NS}}\right) + B\log_2\left(2\pi e G_1\right). \end{aligned} \quad (19)$$

In (a) we used (6) and in (b) we upper-bound the differential entropy $h\left(\mathbf{H}\underline{X} + \underline{U}\right)$ by that of a Gaussian vector with the same covariance matrix. In (c) we used (13), (14) and the fact that $\det \mathbf{C} = 1$.

Now, we notice that the finite time representation of the nonuniform sampling is not accurate at the edges of the N samples block. Thus, in order to get the distortion and rates as in (15) and (19) we have to take $N \to \infty$. Up to now we didn't use the periodic structure in the sampling times. In the periodic sampling case, the input process is cyclostationary (with period M) and as $N \to \infty$ the scheme converges to a periodically time varying scheme, where the matrix $\mathbf{C}$ is replaced with M parallel casual filters.

As described in section III, we can use L dimensional lattice quantization by parallel encoding of L blocks of N samples. Each such parallel encoding scheme has its own noise shaping loop in order to maintain the causality. In this case (19) still holds when replacing G_1 with G_L. For a sequence of good lattice quantizers we have $\lim_{M\to\infty} \log_2 2\pi e G_M = 0$, thus $\lim_{M\to\infty} R_{NS}(D) = R_X(D)$, concluding the proof. ∎

Remark 1: We notice (from (15)) that if we consider only linear reconstruction, there is no noise amplification. When considering optimal (nonlinear) reconstruction, it is possible that the noise amplification is not entirely canceled, but as the lattice dimension tends to ∞ the noise amplification vanishes.

Remark 2: It is important to note that the noise shaping filter is tailored to the specific sampling pattern through the matrix $\mathbf{H}$. Moreover, also the entropy coding operation depends on the sampling pattern, through the distribution it induces on the quantizer output.

Remark 3: For a colored Gaussian source, we should add an analog pre-filter to the noise shaping scheme as in [6] to realize the water puring solution for the RDF. It is interesting to note that we get some kind of separation solution: the pre and post filters depend only on the source spectrum but not on the sampling times, while the noise shaping filter depends only on the sampling times but not on the source spectrum.

Remark 4: The noise shaping technique is commonly used in delta-sigma quantization, where the source is oversampled and the feedback is used to shape the noise away from the source bandwidth. Here we use noise shaping for a different purpose: to whiten the noise in the original (continuous-time) signal space. As opposed to delta-sigma quantization, here there is no need for oversampling.

V. Conclusion

In this paper we suggest a novel method to cancel quantization-noise amplification in nonuniform sampling by noise shaping. The noise shaping results in an effective noise that is white at uniform spaced times rather than at the original nonuniform sampling times. We show that such a noise shaping can be achieved in a causal manner by inserting a feedback loop into the ECDQ scheme. We note that the method we suggest is applicable only when the noise spectrum can be controlled, for example in time-interleaved A/D converters or in multiple-description coding [2]. Unfortunately, in some cases, as in sensor networks, this is not possible.

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