Strong Coordination over a Line Network

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Abstract—We study the problem of strong coordination in a three-terminal line network, in which agents use common randomness and communicate over a line network to ensure that their actions follow a prescribed behavior, modeled by a target joint distribution of actions. We provide inner and outer bounds to the coordination capacity region, and show that these bounds are partially optimal. We leverage this characterization to develop insight into the interplay between communication and coordination. Specifically, we show that common randomness helps achieve optimal communication rates between agents, and that matching the network topology to the behavior structure may reduce inter-agent communication rates.

I. Introduction

One fundamental problem in decentralized networked systems is to coordinate activities of different agents so that they reach a state of agreement. In this paper, we measure coordination by the ability to achieve a prescribed joint probability distribution of actions at all agents in the network. The information-theoretic limits of such a coordination have been partly characterized [1] using the notions of *empirical coordination*, which only requires the normalized histogram of induced joint actions to approach a desired target distribution, and *strong coordination*, where the sequence of induced joint actions must be statistically indistinguishable from the target distribution. If the actions to coordinate are dependent, it has been shown that the communication rate among agents can be significantly reduced compared to a basic approach of communicating explicit messages describing the actions. Further study of the fundamental limits of networked coordination could help develop insight into the interplay between communication and coordination, which could in turn guide the design of many applications, for example in distributed control or multi-agent based exploration and surveillance.

The information-theoretic limits of empirical coordination for small and large networks have been the subject of several investigations. For instance, [2] studies the rate required to reconstruct the empirical distribution of a source at the output of a communication channel rather than reconstructing the source itself. Further, [3], [4] analyze the rate of noise-free communication required to mimic a noisy memoryless communication channel. The work in [5] considers a distributed multi-agent control problem, in which each agent generates actions based on its own observations of a source of randomness. Recently, [6], [7] have proposed coordination schemes based on polar codes achieving empirical coordination and strong coordination for specific distributions of actions. The
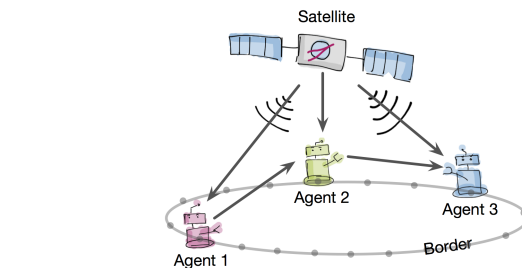

Fig. 1. Coordination of three agents in a perimeter defense scenario.

generation of dependent random variables in networks under a strong coordination constraint is considered in [8], [9], [10], by considering bidirectional transmissions in several rounds; however, these works only address the coordination of two nodes.

In this paper, we attempt to develop further insight into the relation between network communication topology and coordination by extending the work in [1] about point-to-point strong coordination to a three-terminal line network. We provide inner and outer bounds to the coordination capacity region and characterize the optimal communication rates between agents. We also analyze the impact of the underlying network topology on the minimization of the communication requirements for coordination.

II. Problem Statement and Main Results

A. Motivating scenario

As a motivation, consider the perimeter defense scenario illustrated in Fig. 1, in which three agents patrol a border to avoid intrusions. Each agent is able to take sequences of binary actions ("move left", "move right") described by the random variables $X_i \sim \text{Bern}(p_i)$, and the objective is to communicate to achieve a desired behavior, captured by a prescribed joint distribution $q_{X_1 X_2 X_3}$ of the actions. The agents also have access to common randomness, which is pictorially illustrated by an overhead satellite. In this scenario, it is possible to deploy several communication network topologies to coordinate the actions of the agents, and we are interested in determining the optimal topology that minimizes the inter-agent communication rate. As a first step towards this goal, we study the problem of coordination along a *line network* with unidirectional communication, in which Agent 1 can only communicate with Agent 2, and Agent 2 can only
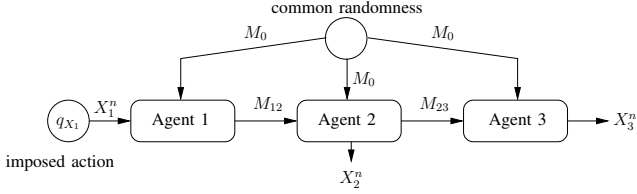
Fig. 2. Coordination over a line network.

communicate with Agent 3, possibly assisted by common randomness.

B. Problem setting and main result

Formally, we consider the setting illustrated in Fig. 2, in which three agents wish to coordinate their actions. The actions taken by Agent $i \in \{1, 2, 3\}$ is described by a sequence of discrete actions $x_i^n \in \mathcal{X}_i^n$, and the behavior is captured by the joint probability distribution of the actions. The network has a line structure in the sense that there are only two communication links:

- a noiseless link between Agent 1 and Agent 2, over which Agent 1 transmit messages in the set $\mathcal{M}_{12} \triangleq [\![1, 2^{nR_{12}}]\!]$;
- a noiseless link between Agent 2 and Agent 3, over which Agent 2 transmit messages in the set $\mathcal{M}_{23} \triangleq [\![1, 2^{nR_{23}}]\!]$.

In addition, we assume that all agents have access to a common source of randomness, which produces uniform random numbers in the set $\mathcal{M}_0 \triangleq [\![1, 2^{nR_0}]\!]$, and that Agent 1 independently determines his own sequence of actions according to the prescribed distribution $q_{X_1^n}$; the objective is then to control the actions of Agent 2 and Agent 3 by means of a line coordination code defined as follows.

Definition 1: A $(2^{nR_0}, 2^{nR_{12}}, 2^{nR_{23}}, n)$ line coordination code consists of:

- an encoder $f_{12} : \mathcal{M}_0 \times \mathcal{X}_1^n \to \mathcal{M}_{12}$ to send messages from Agent 1 to Agent 2;
- an actuator $g_2 : \mathcal{M}_0 \times \mathcal{M}_{12} \to \mathcal{X}_2^n$ to generate the actions of Agent 2;
- an encoder $f_{23} : \mathcal{M}_0 \times \mathcal{M}_{12} \to \mathcal{M}_{23}$ to send messages from Agent 2 to Agent 3;
- an actuator $g_3 : \mathcal{M}_0 \times \mathcal{M}_{23} \to \mathcal{X}_3^n$ to generate the actions of Agent 3.

The communication and processing at each agent induces joint actions characterized by the joint distribution of actions $p(x_1^n, x_2^n, x_3^n)$. The goal is to design a code so that $p(x_1^n, x_2^n, x_3^n)$ is arbitrarily close to the target joint distribution $q(x_1^n, x_2^n, x_3^x) \triangleq \prod_{i=1}^n q_{X_1 X_2 X_3}(x_{1,i}, x_{2,i}, x_{3,i})$. Formally, a rate triplet (R_0, R_{12}, R_{23}) is achievable if there exists a sequence of $(2^{nR_0}, 2^{nR_{12}}, 2^{nR_{23}}, n)$ line coordination codes with increasing length n such that

$$\lim_{n \to \infty} \mathbb{V}\big(p_{X_1^n X_2^n X_3^n}, q_{X_1^n, X_2^n, X_3^n}\big) = 0,$$

where $\mathbb{V}$ denotes the, L_1-distance between two distributions. The set of all achievable rate triplets is called the coordination capacity region $\mathcal{C}(q_{X_1 X_2 X_3})$, and the central result of this paper is a partial characterization of $\mathcal{C}(q_{X_1 X_2 X_3})$.

As a baseline, let us start by considering a simple line coordination code, in which Agent 1 takes a sequence of actions x_1^n drawn according to $q_{X_1^n}$ and sends an *explicit* description of x_1^n to Agent 2; this requires a rate of $R_{12} \geqslant H(X_1)$ bits/action. Agent 2 then takes a sequence of actions x_2^n drawn according to $q_{X_2^n | X_1^n = x_1^n}$ and sends an explicit description of both x_1^n and x_2^n to Agent 3; this requires a rate of $R_{23} \geqslant H(X_1 X_2)$ bits/action. Agent 3 finally takes a sequence of actions x_3^n generated according to $q_{X_3^n | X_2^n = x_2^n, X_1^n = x_1^n}$. Note that this code does not exploit common randomness. Consequently, the rates achievable by this code are

$$\left\{ (R_0, R_{12}, R_{23}) : \begin{array}{l} R_0 \geqslant 0 \\ R_{12} \geqslant H(X_1) \\ R_{23} \geqslant H(X_1 X_2) \end{array} \right\}. \tag{1}$$

We show next that a much improved rate region can be achieved by exploiting common randomness.

Our result is expressed in terms of the following short-hand notation. Let $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $V \in \mathcal{W}$ be auxiliary random variables. We define the sets $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ of joint distributions on $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ as

$$\mathcal{S}_{\text{out}} \triangleq \left\{ p_{UVWX_1X_2X_3} : \begin{array}{l} p_{X_1X_2X_3} = q_{X_1X_2X_3} \\ X_1 \to UW \to VX_2X_3 \\ X_1X_2U \to VW \to X_3 \end{array} \right\}, \tag{2}$$

$$\mathcal{S}_{\text{in}} \triangleq \left\{ p_{UVWX_1X_2X_3} : \begin{array}{l} p_{UVWX_1X_2X_3} \in \mathcal{S}_{\text{out}} \\ U \to W \to V \end{array} \right\}. \tag{3}$$

For a fixed distribution $p_{UVWX_1X_2X_3}$, we also define the rate region

$$\mathcal{R}(p_{UVWX_1X_2X_3}) \triangleq$$
$$\left\{ \begin{array}{l} (R_0, R_{12}, R_{23}) : \\ R_0 + R_{12} + R_{23} \geqslant \mathbb{I}(UVW; X_1X_2X_3) \\ R_0 + R_{12} \geqslant \mathbb{I}(UW; X_1X_2X_3) \\ R_0 + R_{23} \geqslant \mathbb{I}(VW; X_1X_2X_3) \\ R_0 \geqslant \mathbb{I}(W; X_2X_2X_3) \\ R_{12} + R_{23} \geqslant \mathbb{I}(UVW; X_1) \\ R_{12} \geqslant \mathbb{I}(UW; X_1) \\ R_{23} \geqslant \mathbb{I}(VW; X_1) \end{array} \right\}. \tag{4}$$

Theorem 1: The coordination capacity region $\mathcal{C}(q_{X_1 X_2 X_3})$ of the line network satisfies

$$\bigcup_{p \in \mathcal{S}_{\text{in}}} \mathcal{R}(p) \subseteq \mathcal{C}(q_{X_1 X_2 X_3}) \subseteq \bigcup_{p \in \mathcal{S}_{\text{out}}} \mathcal{R}(p).$$

Proof: For clarity, the proofs are relegated to Section III and Section IV, and we do not provide the cardinality bounds on the auxiliary random variables. ∎

Note that the inner and outer bounds for $\mathcal{C}(q_{X_1 X_2 X_3})$ may not match because $\mathcal{S}_{\text{in}}$ is a strict subset of $\mathcal{S}_{\text{out}}$, in general. The constraint $U \to W \to V$ prevents us from choosing W independently of $X_1 X_2 X_3$, so that $R_0 \geqslant \mathbb{I}(W; X_2 X_2 X_3) > 0$ in general, as $X_1 X_2 X_3$ depend on UV (see Section III). Consequently, the proposed coding scheme does not specialize to the baseline scheme whose rates are given in Eq. (1).

C. Applications

The multiple auxiliary random variables involved in Theorem 1 make it rather difficult to parse the bounds obtained for $\mathcal{C}(q_{X_1X_2X_3})$. To obtain additional insight, we specialize Theorem 1 and characterize the region $\mathcal{C}^*(q_{X_1X_2X_3})$, defined as the projection of $\mathcal{C}(q_{X_1X_2X_3})$ onto the plane $R_0 = 0$. In other words, the region $\mathcal{C}^*(q_{X_1X_2X_3})$ characterizes the optimal communication rates between the agents for coordination, assuming the rate of common randomness can be chosen arbitrarily.

Corollary 1: The coordination capacity region $\mathcal{C}^*(q_{X_1X_2X_3})$ is characterized by

$$\mathcal{C}^*(q_{X_1X_2X_3}) = \left\{ (R_{12}, R_{23}) : \begin{array}{l} R_{12} \geqslant \mathbb{I}(X_2X_3; X_1) \\ R_{23} \geqslant \mathbb{I}(X_3; X_1) \end{array} \right\}.$$

Proof: We first simplify the outer bound of Theorem 1 by noting the following. Because of the Markov chains $X_1 - UW - VX_2X_3$ and $X_1X_2U - VW - X_3$, the constraint on the sum rate $R_{12} + R_{23}$ in Eq. (4) is ineffective and subsumed by the constraint on R_{12}. In addition, the data processing inequality ensures that

$$\mathbb{I}(UW; X_1) \geqslant \mathbb{I}(X_2X_3; X_1) \text{ and } \mathbb{I}(VW; X_1) \geqslant \mathbb{I}(X_3; X_1).$$

To show that the region is achievable, note that the choice $U \triangleq X_2$, $W \triangleq X_3$, $V \triangleq X_3$ satisfies the constraints of the set $\mathcal{S}_{\text{in}}$, so that it may be substituted into the inner bound. This choice directly yields the desired result. ∎

We now develop several insights regarding the interplay between coordination and communication topology by leveraging the simple expression of Corollary 1, which does not involve any auxiliary random variables.

Insight 1: Common randomness helps. Comparing the result of Corollary 1 with the baseline performance in Eq. (1), we observe that $\mathbb{I}(X_2X_3; X_1) \leqslant \mathbb{H}(X_1)$ and $\mathbb{I}(X_3; X_1) \leqslant \mathbb{H}(X_1X_2)$, so that the baseline scheme is suboptimal, in general.

Insight 2: The communication topology should match the coordination structure. Assume now that the desired behavior $q_{X_1X_2X_3}$ is such that $X_1 - X_2 - X_3$ forms a Markov chain. In other words, the actions of Agent 3 should be conditionally independent of the actions of Agent 1 given the actions of Agent 2. Note that the *communication* topology (a line network here) *matches* this *coordination* structure, since Agent 3 only communicates with Agent 1 through Agent 2. Specializing Corollary 1, we obtain

$$\mathcal{C}^*(q_{X_1X_2X_3}) = \left\{ (R_{12}, R_{23}) : \begin{array}{l} R_{12} \geqslant \mathbb{I}(X_2; X_1) \\ R_{23} \geqslant \mathbb{I}(X_3; X_1) \end{array} \right\}. \quad (5)$$

Let us consider a modified communication topology as shown in Fig. 3, such that the roles of Agent 3 and Agent 2 are swapped. There is now a *mismatch* between the communication topology and the desired behavior, since Agent 3 becomes the communication bottleneck. This effect is captured by specializing again Corollary 1, which yields the region

$$\mathcal{C}^*(q_{X_1X_2X_3}) = \left\{ (R_{13}, R_{32}) : \begin{array}{l} R_{13} \geqslant \mathbb{I}(X_2; X_1) \\ R_{32} \geqslant \mathbb{I}(X_2; X_1) \end{array} \right\}. \quad (6)$$
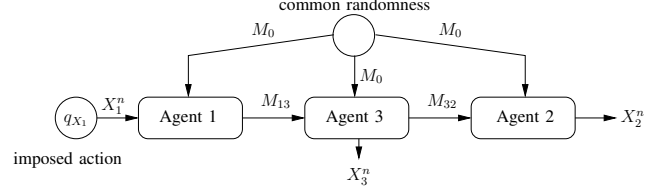

Fig. 3. Modified line network topology.

Since $\mathbb{I}(X_2; X_1) \geqslant \mathbb{I}(X_3; X_1)$ by the data-processing inequality, we see that the mismatch translates into a penalty in terms of the communication rates between agents.

III. ACHIEVABILITY PROOF

Because of space constraints, we do not detail some of the more technical steps of the proof. Let U, V, W, X_1, X_2, X_3 be discrete random variables with joint distribution

$$p(u, v, w, x_1, x_2, x_3) \triangleq W(x_1|u, w)W(x_2|u, v, w)$$
$$W(x_3|v, w)p(u|w)p(v|w)p(w)$$

such that the marginal $p_{X_1X_2X_3}$ satisfies $p(x_1, x_2, x_3) = q(x_1, x_2, x_3)$. Note that at least one such distribution exists (with $U \triangleq X_2$, $W \triangleq X_3$, $V \triangleq X_3$) and that it belongs to the set $\mathcal{S}_{\text{in}}$.

Following the idea in [4], we solve the coordination problem by constructing a code for an intermediate problem, which is illustrated in Fig. 4. Three uniformly distributed messages $M_0 \in [\![1, 2^{nR_0}]\!]$, $M_{12} \in [\![1, 2^{nR_{12}}]\!]$ and $M_{23} \in [\![1, 2^{nR_{23}}]\!]$ are encoded into codewords $U^n \in \mathcal{U}^n$, $V^n \in \mathcal{V}^n$ and $W^n \in \mathcal{W}^n$ using the following encoding functions:

- $f_U : [\![1, 2^{nR_0}]\!] \times [\![1, 2^{nR_{12}}]\!] \to \mathcal{U}^n$;
- $f_W : [\![1, 2^{nR_0}]\!] \to \mathcal{W}^n$;
- $f_V : [\![1, 2^{nR_0}]\!] \times [\![1, 2^{nR_{23}}]\!] \to \mathcal{V}^n$.

The codewords U^n, W^n, V^n, are then transmitted into channels with transition probabilities $W_{X_1|UW}$, $W_{X_2|UVW}$, $W_{X_3|VW}$, respectively. The encoders induce a joint distribution between messages, and channel outputs, which we denote by $\hat{p}_{X_1^nX_2^nX_3^nM_0M_{12}M_{23}}$. The problem is to identify triplets (R_0, R_{12}, R_{23}) that are achievable, in the sense that there
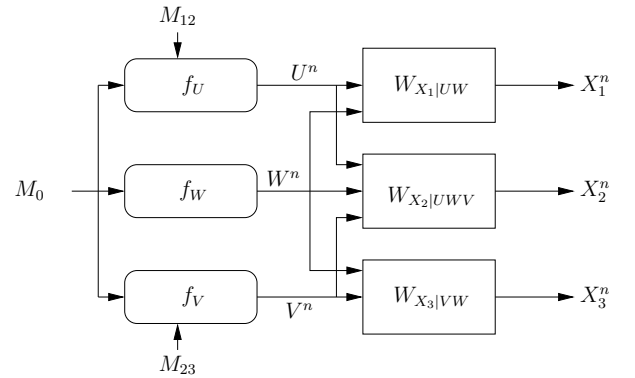

Fig. 4. Intermediate problem used in achievability proof.

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exists a sequence of encoders with increasing length n such that

$$\lim_{n\to\infty} \mathbb{V}\big(\hat{p}_{X_1^n, X_2^n, X_3^n}, q_{X_1^n X_2^n X_3^n}\big) = 0 \tag{7}$$

$$\text{and } \lim_{n\to\infty} \mathbb{V}\big(\hat{p}_{X_1^n, M_0}, \hat{p}_{X_1^n} \hat{p}_{M_0}\big) = 0 \tag{8}$$

The constraint in Eq. (7) is a *channel resolvability* constraint, which requires the encoders to simulate the distribution $q_{X_1^n X_2^n X_3^n}$, while the constraint in Eq. (8) is a *secrecy* constraint, which requires the message M_0 to be independent of the output X_1^n. This ensures the compatibility of a code for the intermediate problem in Fig. 4 with a code for the original problem in Fig. 2.

Since all channels considered here have multiple inputs, the key conceptual tool we rely on is a variation of multiple-access channel resolvability [11], [12], [13].

To simplify notation in the sequel, we denote $Z^n \triangleq X_1^n X_2^n X_3^n$, which we use when the analysis does not require us to treat X_1^n, X_2^n, and X_3^n separately.

We start by generating three codebooks randomly.

- We generate 2^{nR_0} sequences, labeled $w_i^n = (w_{i,1}, \cdots, w_{i,n})$, independently according to $\prod_{\ell=1}^{n} p_W(w_{i,\ell})$;
- For each w_i^n, we generate $2^{R_{12}}$ sequences, labeled u_{ij}^n, independently according to $\prod_{\ell=1}^{n} p_{U|W}(u_{ij,\ell}|w_{i,\ell})$;
- For each w_i^n, we generate $2^{R_{23}}$ sequences, labeled v_{jk}^n, independently according to $\prod_{i=1}^{n} p_{V|W}(v_{jk,\ell}|w_{i,\ell})$.

The indices of sequences in the codewords define the encoding functions f_U, f_W, f_V of the code.

Next, we analyze $\mathbb{E}\big(\mathbb{V}\big(\hat{P}_{Z^n}, q_{Z^n}\big)\big)$, where the expectation is over the randomly generated code and $\hat{P}$ denotes the probability $\hat{p}$ for a random code. We denote by $T_\delta^n(Z)$ the δ-typical set for the distribution p_Z and by $T_\delta^n(UVWZ|z^n)$ the δ-conditional typical set for the distribution p_{UVWZ} and $z^n \in T_\delta^n(Z)$. Upon defining

$$\hat{p}_{Z^n}^{(1)}(z^n) \triangleq \sum_{i,j,k} W(z^n|u_{ij}^n, v_{ik}^n, w_i^n) 2^{-n(R_0+R_{12}+R_{23})}$$
$$\mathbf{1}\left\{(u_{ij}^n, v_{ik}^n, w_i^n) \in T_\delta^n(UVWZ|z^n)\right\}$$

and

$$\hat{p}_{Z^n}^{(2)}(z^n) \triangleq \sum_{i,j,k} W(z^n|u_{ij}^n, v_{ik}^n, w_i^n) 2^{-n(R_0+R_{12}+R_{23})}$$
$$\mathbf{1}\left\{(u_{ij}^n, v_{ik}^n, w_i^n) \notin T_\delta^n(UVWZ|z^n)\right\},$$

so that $\hat{p}_{Z^n}(z^n) = \hat{p}_{Z^n}^{(1)}(z^n) + \hat{p}_{Z^n}^{(2)}(z^n)$, and using the triangle inequality repeatedly, one obtains the following upper bound

$$\mathbb{V}\big(\hat{P}_{Z^n}, q_{Z^n}\big) \leqslant \sum_{z^n \in T_\delta^n(Z)} \left|\hat{P}_{Z^n}^{(1)}(z^n) - \mathbb{E}\big(\hat{P}_{Z^n}^{(1)}(z^n)\big)\right|$$
$$+ \sum_{z^n \in T_\delta^n(Z)} \left|\hat{P}_{Z^n}^{(2)}(z^n) - \mathbb{E}\big(\hat{P}_{Z^n}^{(2)}(z^n)\big)\right|$$
$$+ \sum_{z^n \notin T_\delta^n(Z)} \left|\hat{P}_{Z^n}(z^n) - q_{Z^n}(z^n)\right|.$$

One can then show that

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{z^n \notin T_\delta^n(Z)} \left|\hat{P}_{Z^n}(z^n) - q_{Z^n}(z^n)\right|\right) = 0,$$

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{z^n \in T_\delta^n(Z)} \left|\hat{P}_{Z^n}^{(2)}(z^n) - \mathbb{E}\big(\hat{P}_{Z^n}^{(2)}(z^n)\big)\right|\right) = 0.$$

The last sum is upper bounded using Jensen's inequality as

$$\mathbb{E}\left(\sum_{z^n \in T_\delta^n(Z)} \left|\hat{P}_{Z^n}^{(1)}(z^n) - \mathbb{E}\big(\hat{P}_{Z^n}^{(1)}(z^n)\big)\right|\right)$$
$$\leqslant \sum_{z^n \in T_\delta^n(Z)} \sqrt{\mathbb{E}\big((\hat{P}_{Z^n}^{(1)}(z^n))^2\big) - \mathbb{E}\big(\hat{P}_{Z^n}^{(1)}(z^n)\big)^2}. \tag{9}$$

Note that $\mathbb{E}\big((\hat{P}_{Z^n}^{(1)}(z^n))^2\big)$ is written explicitly as

$$\mathbb{E}\big((\hat{P}_{Z^n}^{(1)}(z^n))^2\big) = \sum_{ijk} \sum_{i'j'k'} A_{ijki'j'k'} \text{ with}$$

$$A_{ijki'j'k'} \triangleq \mathbb{E}\big(W(z^n|U_{ij}^n, V_{ik}^n, W_i^n) W(z^n|U_{i'j'}^n, V_{i'k'}^n, W_{i'}^n)$$
$$\mathbf{1}\left\{(U_{ij}^n, V_{ik}^n, W_i^n) \in T_\delta^n(UVWZ|z^n)\right\}$$
$$\mathbf{1}\left\{(U_{i'j'}^n, V_{i'k'}^n, W_{i'}^n) \in T_\delta^n(UVWZ|z^n)\right\}\big) \tag{10}$$

Because of the properties of the random code generation procedure, the analysis of the sum can be split into 5 parts.

If $i \neq i'$, and for any j, k, j', k', we obtain

$$\sum_{i,j,k,i'\neq i,j',k'} A_{ijki'j'k'} \leqslant \mathbb{E}\big(\hat{p}_{Z^n}^{(1)}(z^n)\big)^2 \tag{11}$$

If $i = i'$, $j = j'$, $k = k'$, we obtain

$$\sum_{i,j,k} A_{ijkijk} \leqslant 2^{-n(R_0+R_{12}+R_{23}+\mathbb{H}(Z|UVW)+\mathbb{H}(Z)-\mathcal{O}(\delta))} \tag{12}$$

If $i = i'$, $j = j'$, $k \neq k'$, we obtain

$$\sum_{i,j,k,k'\neq k} A_{ijkijk'} \leqslant 2^{-n(R_0+R_{12}+\mathbb{H}(Z|UW)+\mathbb{H}(Z)-\mathcal{O}(\delta))} \tag{13}$$

If $i = i'$, $j \neq j'$, $k = k'$, we obtain

$$\sum_{i,j,k,j'\neq j} A_{ijkij'k} \leqslant 2^{-n(R_0+R_{23}+\mathbb{H}(Z|VW)+\mathbb{H}(Z)-\mathcal{O}(\delta))} \tag{14}$$

If $i = i'$, $j \neq j'$, $k \neq k'$, we obtain

$$\sum_{i,j,k,j'\neq j,k'\neq k} A_{ijkij'k'} \leqslant 2^{-n(R_0+\mathbb{H}(Z|W)+\mathbb{H}(Z)-\mathcal{O}(\delta))} \tag{15}$$

Substituting Eq. (11)-(15) into Eq. (9) and using the bound $|T_\delta^n(Z)| \leqslant 2^{n(\mathbb{H}(Z)+\mathcal{O}(\delta))}$, one can finally show that if

$$R_0 + R_{12} + R_{23} \geqslant \mathbb{I}(UVW; Z),$$
$$R_0 + R_{12} \geqslant \mathbb{I}(UW; Z),$$
$$R_0 + R_{23} \geqslant \mathbb{I}(VW; Z),$$
$$R_0 \geqslant \mathbb{I}(W; Z), \tag{16}$$

then

$$\lim_{n \to \infty} \mathbb{E}\left(\sum_{z^n \in T^n_\delta(Z)} \left| \hat{p}^{(1)}_{Z^n}(z^n) - \mathbb{E}\left(\hat{p}^{(1)}_{Z^n}(z^n) \right) \right| \right) = 0.$$

The analysis of $\mathbb{E}\left(\mathbb{V}\left(\hat{p}_{X_1^n, M_0}, \hat{p}_{X_1^n} \hat{p}_{M_0} \right) \right)$ follows similar steps once using the triangle inequality repeatedly to show

$$\mathbb{V}\left(\hat{p}_{X_1^n, M_0}, \hat{p}_{X_1^n} \hat{p}_{M_0} \right) \leqslant 2 \mathbb{E}_{M_0}\left(\mathbb{V}\left(\hat{p}_{X_1^n | M_0}, q_{X_1^n} \right) \right).$$

The secrecy constraint then reduces to another multiple-access channel resolvability constraint, and we can show that if

$$\begin{align}
R_{12} + R_{23} &\geqslant \mathbb{I}(UVW; X_1), \\
R_{12} &\geqslant \mathbb{I}(UW; X_1), \tag{17} \\
R_{23} &\geqslant \mathbb{I}(VW; X_1),
\end{align}$$

then $\lim_{n \to \infty} \mathbb{E}\left(\mathbb{V}\left(\hat{p}_{X_1^n, M_0}, \hat{p}_{X_1^n} \hat{p}_{M_0} \right) \right) = 0$. Using Markov's inequality, one then shows the existence of a sequence of codes achieving the rates (R_0, R_{12}, R_{23}) satisfying the constraints in Eq. (16) and Eq. (17).

All that remains to show now is how the code for the intermediate problem can be used as a code for the original coordination problem. We choose the encoding and decoding for the coordination problem to operate as follows.

- Agent 1 generates message m_{12} from his actions x_1^n and the common randomness m_0 according to $\hat{p}(m_{12}|x_1^n, m_0)$; Agent 1 then sends m_{12} to Agent 2;
- Agent 2 generates a message m_{23} uniformly at random and receives m_{12}, m_0; Agent 2 then simulates the transmission of the codewords $u^n_{m_0, m_{12}}$, $v^n_{m_0, m_{23}}$, $w^n_{m_0}$ through the channel with transition probabilities $W_{X_2|UVW}$ to obtain his actions x_2^n; Agent 2 then sends m_{23} to Agent 3;
- Agent 3 receives m_{23}, m_0 and simulates the transmission of the codewords $v^n_{m_0, m_{23}}$ and $w^n_{m_0}$ through the channel with transition probabilities $W_{X_3|VW}$ to obtain his actions x_3^n.

We denote the joint distribution induced by this scheme by $\tilde{p}_{X_1^n X_2^n X_3^n M_0 M_{12} M_{23}}$. Using the triangle inequality,

$$\begin{align}
\mathbb{V}\left(\tilde{p}_{X_1^n X_2^n X_3^n}, q_{X_1^n X_2^n X_3^n} \right) &\leqslant \mathbb{V}\left(\tilde{p}_{X_1^n X_2^n X_3^n}, \hat{p}_{X_1^n X_2^n X_3^n} \right) \\
&\quad + \mathbb{V}\left(\hat{p}_{X_1^n X_2^n X_3^n}, q_{X_1^n X_2^n X_3^n} \right).
\end{align}$$

The first term on the right-hand side vanishes since the code satisfies Eq. (8) after some intermediate steps, while the second term vanishes since the code satisfies Eq. (7). Hence, the achievable coordination rates are exactly the achievable rates for the intermediate problem. Combining all rate constraints and recalling the construction of UVW, one obtains that $\bigcup_{p \in \mathcal{S}_{\text{in}}} \mathcal{R}(p) \subset \mathcal{C}$.

IV. Converse Proof

Let (R_0, R_{12}, R_{23}) be achievable coordination rates, so that for all $\epsilon > 0$, there exists a $(2^{nR_0}, 2^{nR_{12}}, 2^{nR_{23}}, n)$ line coordination code for which $\mathbb{V}\left(p_{X_1^n X_2^n X_3^n}, q_{X_1^n, X_2^n, X_3^n} \right) \leqslant \epsilon$. For brevity, we only sketch the calculation of some converse constraints. Following the steps as in [4], we obtain

$$R_0 + R_{12} + R_{23} \geqslant \mathbb{I}(X_{1,Q} X_{2,Q} X_{3,Q}; M_0 M_{12} M_{23} Q) - \delta(\epsilon),$$

where the random variable Q is uniformly distributed in $[\![1, n]\!]$ and independent of all others, and $\delta(\epsilon)$ denotes a function such that $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$. Upon introducing $X_i \triangleq X_{i,Q}$ for $i \in \{1, 2, 3\}$, $U \triangleq M_{12}$, $V \triangleq M_{23}$ and $W \triangleq M_0$, we obtain

$$R_0 + R_{12} + R_{23} \geqslant \mathbb{I}(X_1 X_2 X_3; UVW) - \delta(\epsilon).$$

Similar bounds are obtained for $R_0 + R_{12}$, $R_0 + R_{23}$ and R_0. Next, using the independence of M_0 and X_1^n, we can also show

$$R_{12} + R_{23} \geqslant \mathbb{I}(X_1; UVW).$$

Similar bounds are obtained for R_{12} and R_{23}. Finally, using a functional dependence graph, one can check that the following Markov chains hold:

$$X_1 - UW - VX_2X_3 \quad \text{and} \quad X_1X_2U - VW - X_3.$$

Note that the constraint $U - W - V$ from the achievability proof does *not* hold here, which puts an additional restriction on the coding scheme used in the achievability proof and shows that this scheme is sub-optimal in general. Combining all constraints and taking the limit as ϵ goes to zero yields the desired converse (see [14, Lemma VI] for a careful justification).

V. Acknowledgement

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