Gaussian HDA Coding with Bandwidth Expansion and Side Information at the Decoder

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***Abstract*—Lossy transmission of a Gaussian source over an additive white Gaussian (AWGN) channel with side information at the decoder is tackled under the regime of bandwidth expansion. A previously known scheme, hybrid digital/analog Wyner-Ziv (HDA-WZ) coding is shown to remain optimal when extended from the bandwidth matched case to the bandwidth expansion case. The extended scheme also exhibits similar robustness properties under mismatched side information and/or channel quality. Finally, under the criterion of *min-max distortion loss*, the extended HDA-WZ scheme is shown to outperform a purely-digital scheme known as the common description scheme (CDS).**

I. Introduction

We consider the lossy transmission of a unit-variance i.i.d. Gaussian source X^{nN} over an AWGN channel with bandwidth expansion, where the receiver has access to side information Y^{nN} correlated with the source. More specifically, $X^{nN} = Y^{nN} + Z^{nN}$ with $Z^{nN} \sim \mathcal{N}\left(0, \sigma_Z^2 \mathbf{I}\right)$ and $Y^{nN} \sim \mathcal{N}\left(0, \left(1-\sigma_Z^2\right) \mathbf{I}\right)$ being independent. The transmitter maps X^{nN} to the channel input U^{mN} with $m \geq n$, which is then corrupted by the channel according to $V^{mN} = U^{mN} + W^{mN}$, where $W^{mN} \sim \mathcal{N}\left(0, \sigma_W^2 \mathbf{I}\right)$ is independent of U^{mN}. The receiver then estimates the source as $\hat{X}^{nN}$ using both V^{mN} and Y^{nN}. This scenario is depicted in Figure 1.

As usual, the channel input is power constrained, i.e.,

$$\frac{1}{mN} \sum_{i=1}^{mN} \mathbb{E}[U_i^2] \leq P \tag{1}$$

and the quality of reconstruction is measured by the average expected square-error distortion

$$D = \frac{1}{nN} \sum_{i=1}^{nN} \mathbb{E}[(X_i - \hat{X}_i)^2] \, .$$

We take m, n, and N as integers, where N is the length of blocks consisting of super-symbols X^n, Y^n, $\hat{X}^n$, U^m, V^m, and W^m, while m and n determine the bandwidth expansion ratio as $\kappa = \frac{m}{n}$. In the subsequent asymptotic analysis, N will approach infinity while m and n will be finite and fixed.

While this problem is known to be separable [8], i.e., a combination of digital source and channel codes can achieve the optimal (D, P) tradeoff, digital codes are vulnerable to changes in channel and/or side information quality: After the system is built for a target (σ_Z^2, σ_W^2), if σ_Z^2 and/or σ_W^2 increases, the decoding of the digital information breaks down (also known as the thresholding effect). On the other hand, decreasing σ_Z^2 and σ_W^2 does not translate to any reduction in distortion (also known as the leveling-off effect). It is well-known that both thresholding and leveling-off can be mitigated

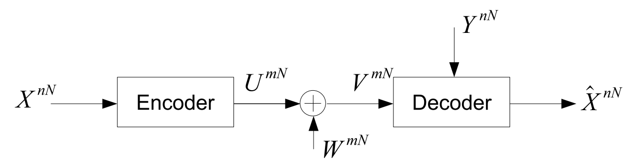

Figure 1. The block diagram of the scenario.

by hybrid digital/analog (HDA) coding (for example, see [1], [2], [7], [9] and references therein). However, to the best of our knowledge, no HDA scheme was explicitly proposed before for bandwidth expansion with side information at the decoder.

In this paper, we propose an extension of Wilson et al.'s HDA-WZ scheme in [9] for $\kappa > 1$, and show that it achieves the optimal (D, P) tradeoff. We then analyze the robustness of our scheme to a mismatch of σ_Z^2 and/or σ_W^2, and in particular show that it remains optimal whenever

$$\frac{(1 + \frac{P}{\sigma_W^2})^\kappa - 1}{\sigma_Z^2 (1 + \frac{P}{\sigma_W^2})}$$

stays constant, thereby generalizing the robustness condition already known for the $\kappa = 1$ case [3], [4]. A similar (but different) robustness behavior was previously demonstrated in [6] for a purely digital (joint source-channel coding) method termed CDS.

Our second contribution is a "distortion loss" comparison of the HDA-WZ scheme, CDS, and uncoded transmission under bandwidth expansion and side information and/or channel quality mismatch. The distortion loss is defined as the ratio

$$L\left(\sigma_Z^2, \sigma_W^2\right) = \frac{D\left(\sigma_Z^2, \sigma_W^2\right)}{D_{\text{opt}}\left(\sigma_Z^2, \sigma_W^2\right)} \tag{2}$$

where $D(\sigma_Z^2, \sigma_W^2)$ is the distortion a given fixed scheme achieves at a particular pair (σ_Z^2, σ_W^2), and $D_{\text{opt}}\left(\sigma_Z^2, \sigma_W^2\right)$ is the minimum distortion achievable when (σ_Z^2, σ_W^2) is known beforehand, i.e.,

$$D_{\text{opt}}\left(\sigma_Z^2, \sigma_W^2\right) = \frac{\sigma_Z^2}{(1 + \frac{P}{\sigma_W^2})^\kappa} \, .$$

We observe that each scheme outperforms the others in certain regions on the (σ_Z^2, σ_W^2)-plane.

Finally, we introduce a robustness criterion which measures the *maximum distortion loss* in a rectangular region on the (σ_Z^2, σ_W^2)-plane, and show that according to this criterion, the HDA-WZ scheme is always superior to CDS.

II. Existing Schemes and the Proposed Scheme for Bandwidth Expansion

We first discuss two existing schemes, namely, uncoded transmission and CDS, and then introduce the extension of the HDA-WZ scheme to the case of bandwidth expansion. We will use the notation $\rho = \sqrt{1-\sigma_Z^2}$ and $\gamma = \frac{P}{\sigma_W^2}$ whenever they simplify the formulae. When ρ or γ has a subscript, it is to be understood that σ_Z^2 or σ_W^2 has the same subscript.

A. Uncoded Transmission

The simplest option is uncoded transmission, i.e., setting $N = 1$ and sending $U^m = \mathbf{M}X^n$ where $\mathbf{M}$ is an $m \times n$ matrix. This problem was studied by Lee and Petersen [5] for a more general case of correlated source and channel noise vectors, and their result can be specialized to our case as

$$\mathbf{M}_{ij} = \begin{cases} \kappa P & i = j \\ 0 & i \neq j \end{cases} \tag{3}$$

with which the distortion becomes

$$D_{\mathrm{UNC}}(\sigma_Z^2, \sigma_W^2) = \frac{\sigma_Z^2}{1 + \kappa \sigma_Z^2 \gamma} . \tag{4}$$

Uncoded transmission cannot achieve the optimum distortion (unless $\kappa = 1$ and $\rho = 0$). However, we include it because of its robustness to changes in the channel and/or the side information quality.

B. CDS

CDS is a purely digital scheme that was proposed in [6], and it can be utilized for general bandwidth expansion ratios. In this scheme, the source is quantized to one of $2^{nN[I(X;S)+\epsilon]}$ codewords indexed as $S^{nN}(j)$, and then this quantized version is mapped into an independently and randomly generated channel word $U^{mN}(j)$. At the receiver, channel output V^{mN} will be jointly typical with $2^{nN[I(X;S)-\kappa I(U;V)+2\epsilon]}$ channel codewords. Tracking back these codewords to the source codebook, one sees a *virtual bin* from which the unique source codeword $S^{nN}(j)$ is decoded successfully with the aid of the side information Y^{nN} if and only if

$$I\left(X; S|Y\right) \leq \kappa I\left(U; V\right) .$$

Thus, by choosing S and U to achieve the Wyner-Ziv rate distortion function and channel capacity, respectively, it is possible to achieve $D_{\mathrm{opt}}(\sigma_Z^2, \sigma_W^2)$. Moreover, it was shown in [6] that after setting S and U as described above, CDS still achieves $D_{\mathrm{opt}}(\sigma_Z^2, \sigma_W^2)$ when (σ_Z^2, σ_W^2) changes as long as

$$\frac{(1+\gamma)^\kappa - 1}{\sigma_Z^2}$$

stays constant.

C. HDA-WZ Coding

In [9], the HDA-WZ scheme was proposed along with several other schemes, and was proven to be optimal in the bandwidth-matched case, which corresponds to $m = n = 1$ in our notation. It operates as follows. A codebook of size $2^{N[I(X;T)+\epsilon]}$ is created using codewords generated according to an auxiliary random variable defined by

$$T = U + kX$$

where $U \sim \mathcal{N}(0, P)$ is independent of X. The encoder then finds a codeword $T^N(j)$ that is jointly typical with X^N and transmits $U^N = T^N(j) - kX^N$. Given the channel output $V^N = U^N + W^N$, the receiver can find the unique $T^N(j)$ which is jointly typical with V^N and Y^N if and only if

$$I(X; T) \leq I(T; Y, V) .$$

The receiver then computes the MMSE estimate $\hat{X}^N = aT^N + bV^N + dY^N$. In [9], it was shown that by choosing

$$k^2 = \frac{P^2}{\sigma_Z^2 \left(P + \sigma_W^2\right)}$$

one can achieve the distortion $D_{\mathrm{opt}}(\sigma_Z^2, \sigma_W^2)$.

We now generalize this scheme and its optimality to include bandwidth expansion, i.e., $m > n$. As mentioned before, we can think of the source X^{nN} as a length-N collection of super symbols X^n, and map it into a length-N collection of super symbols U^m. For this purpose, we generate $2^{N[I(X^n;T^m)+\epsilon]}$ auxiliary super-codewords T^{mN} according to

$$T^m = U^m + \mathbf{K}X^n$$

where $\mathbf{K}$ is an $m \times n$ matrix and $U^m \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_U)$ is independent of X^n, and after finding $T^{mN}(j)$ jointly typical with X^{nN}, transmit $U^{mN} = T^{mN}(j) - (\mathbf{I}_N \otimes \mathbf{K})X^{nN}$ with $\otimes$ denoting the Kronecker product. Upon receiving V^{mN}, the decoder then searches for T^{mN} that is jointly typical with V^{mN} and Y^{nN}, and produces the MMSE estimate

$$\hat{X}^n = \mathbf{A}T^m + \mathbf{B}V^m + \mathbf{D}Y^n . \tag{5}$$

For $T^{mN}(j)$ to be the unique such sequence, we need

$$I(X^n; T^m) \leq I(T^m; Y^n, V^m) . \tag{6}$$

Now we are ready to prove the main result of this section.

Theorem 1: The optimum distortion $D_{\mathrm{opt}}(\sigma_Z^2, \sigma_W^2)$ can always be achieved by choosing $\mathbf{C}_U$ and $\mathbf{K}$ appropriately.

Proof: Using the fact that V^m and Y^n are independent, we can write (6) as

$$h\left(T^m, V^m, Y^n\right) \leq h\left(U^m\right) + h\left(V^m\right) + h\left(Y^n\right) \tag{7}$$

which, in turn, is the same as

$$\left| \mathbf{C}_U + \sigma_Z^2 \mathbf{K}\mathbf{K}^T - \mathbf{C}_U \left(\mathbf{C}_U + \sigma_W^2 \mathbf{I}\right)^{-1} \mathbf{C}_U \right| \leq \left|\mathbf{C}_U\right| . \tag{8}$$

From the principle of orthogonality, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ must satisfy

$$\mathbf{A}\mathbf{C}_T + \mathbf{B}\mathbf{C}_{VT} + \mathbf{D}\mathbf{C}_{YT} = \mathbf{C}_{XT} \tag{9a}$$

$$\mathbf{A}\mathbf{C}_{TV} + \mathbf{B}\mathbf{C}_V + \mathbf{D}\mathbf{C}_{YV} = \mathbf{C}_{XV} \tag{9b}$$

$$\mathbf{A}\mathbf{C}_{TY} + \mathbf{B}\mathbf{C}_{VY} + \mathbf{D}\mathbf{C}_Y = \mathbf{C}_{XY} \tag{9c}$$

and the corresponding minimum distortion becomes

$$\begin{aligned} D &= \frac{1}{n}\mathbb{E}\|X^n - \hat{X}^n\|^2 \\ &= \frac{1}{n}\mathbb{E}\left[(X^n - \hat{X}^n)^t X^n\right] \\ &= 1 - \frac{1}{n}\mathbb{E}\left[\left(X^n\right)^t \left(\mathbf{A}T^m + \mathbf{B}V^m + \mathbf{D}Y^n\right)\right] \\ &= 1 - \frac{1}{n}\left[\mathrm{tr}\left(\mathbf{A}\mathbf{K}\right) + \mathrm{tr}\left(\rho^2 \mathbf{D}\right)\right] \\ &= \sigma_Z^2 - \frac{1}{n}\mathrm{tr}\left(\sigma_Z^2 \mathbf{A}\mathbf{K}\right) \end{aligned} \tag{10}$$

where in the last step, we used (9c).

Now, if $\mathbf{C}_U = P\mathbf{I}$, the power constraint (1) is automatically satisfied, and (8) becomes

$$\left|\sigma_Z^2 \mathbf{K}\mathbf{K}^t + \frac{P\sigma_W^2}{P+\sigma_W^2}\mathbf{I}\right| \le P^m \ . \tag{11}$$

Moreover, solving (9) for $\mathbf{A}$ yields

$$\mathbf{A} = \sigma_Z^2 \mathbf{K}^t \left(\frac{P\sigma_W^2}{P+\sigma_W^2}\mathbf{I} + \sigma_Z^2 \mathbf{K}\mathbf{K}^t\right)^{-1} \tag{12}$$

and reduces (10) to

$$D = \sigma_Z^2 - \frac{1}{n}\mathrm{tr}\left(\sigma_Z^4 \mathbf{K}\mathbf{K}^t \left(\frac{P\sigma_W^2}{P+\sigma_W^2}\mathbf{I} + \sigma_Z^2 \mathbf{K}\mathbf{K}^t\right)^{-1}\right) \tag{13}$$

Observing that both (11) and (13) depend only on $\mathbf{K}\mathbf{K}^t$, which is a symmetric positive semi-definite matrix with a rank$\le n$, we can write $\mathbf{K}\mathbf{K}^t = \Phi\Lambda\Phi^t$ where Φ is an $m \times m$ orthogonal matrix, and Λ is an $m \times m$ diagonal matrix with $\Lambda_{ii} = \lambda_i \ge 0$ such that $\lambda_i = 0$ for $i > n$.

With this simplification, we can write (11) as

$$\xi^{m-n} \prod_{i=1}^{n} \left(\xi + \sigma_Z^2 \lambda_i\right) \le P^m \tag{14}$$

where $\xi = \frac{P}{1+\gamma}$, and (10) becomes

$$\begin{aligned} D &= \sigma_Z^2 - \frac{1}{n}\mathrm{tr}\left(\sigma_Z^4 \Phi\Lambda\Phi^t \left(\xi\mathbf{I} + \sigma_Z^2 \Phi\Lambda\Phi^t\right)^{-1}\right) \\ &= \sigma_Z^2 - \frac{1}{n}\sum_{i=1}^{n} \sigma_Z^2 \frac{\sigma_Z^2 \lambda_i}{\xi + \sigma_Z^2 \lambda_i} \\ &= \frac{\sigma_Z^2}{n}\sum_{i=1}^{n} \frac{\xi}{\xi + \sigma_Z^2 \lambda_i} \ . \end{aligned} \tag{15}$$

It is easy to show that the minimization of (15) subject to (14) yields

$$\lambda_i = \frac{P\left((1+\gamma)^\kappa - 1\right)}{\sigma_Z^2 (1+\gamma)} \tag{16}$$

for $1 \le i \le n$, and the corresponding distortion is

$$D = \frac{\sigma_Z^2 \xi}{\xi + \frac{P((1+\gamma)^\kappa - 1)}{1+\gamma}} = \frac{\sigma_Z^2}{(1+\gamma)^\kappa}$$

completing the proof. ■

III. Performance of the Schemes With Mismatched SNR and/or Side Information Quality

In this section we will consider the case when the side information and/or channel quality are not the same as what the encoder targets for, say, $(\sigma_{Z_0}^2, \sigma_{W_0}^2)$, but could be any (σ_Z^2, σ_W^2).

A. Uncoded Transmission

Since the optimal $m \times n$ matrix $\mathbf{M}$ given in (3) does not depend on either σ_Z^2 or σ_W^2, the expression (4) is valid for any (σ_Z^2, σ_W^2) pair.

B. Separate Source-Channel Coding

When source and channel coding are performed separately, the optimum source coder will use the test channel $S = X + E$ to quantize X^{nN} so that

$$I(X; S|Y_0) = \kappa I(U; V_0) \tag{17}$$

where Y_0 with $X = Y_0 + Z_0$ and $V_0 = U + W_0$ are the random variables corresponding to the target variances $(\sigma_{Z_0}^2, \sigma_{W_0}^2)$. The relation (17) is the same as

$$\frac{1}{\sigma_E^2} = \frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2} \tag{18}$$

with $\gamma_0 = \frac{P}{\sigma_{W_0}^2}$.

As σ_Z^2 and/or σ_W^2 deviate from the target values, for S^{nN} still to be decodable, one needs

$$\begin{aligned} I(X; S|Y_0) &\le \kappa I(U; V) \qquad (19) \\ I(Y_0; S) &\le I(Y; S) \qquad (20) \end{aligned}$$

where (19) ensures that the bin index is reliably decoded, and (20) is needed to uniquely decode S^{nN} from the given bin. It should be clear that (19) and (20) are the same as $\sigma_W^2 \le \sigma_{W_0}^2$ and $\sigma_Z^2 \le \sigma_{Z_0}^2$, respectively.

For any (σ_Z^2, σ_W^2) so that S^{nN} is decodable, optimizing the estimator $\hat{X}^{nN} = aY^{nN} + bS^{nN}$ yields

$$\begin{aligned} D_{\mathrm{SEP}}(\sigma_Z^2, \sigma_W^2) &= \left(\frac{1}{\sigma_E^2} + \frac{1}{\sigma_Z^2}\right)^{-1} \\ &= \left(\frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2} + \frac{1}{\sigma_Z^2}\right)^{-1} . \end{aligned} \tag{21}$$

As expected, when $\sigma_Z^2 = \sigma_{Z_0}^2$ and $\sigma_W^2 = \sigma_{W_0}^2$, (21) yields $D_{\mathrm{SEP}}(\sigma_Z^2, \sigma_W^2) = D_{\mathrm{opt}}(\sigma_Z^2, \sigma_W^2)$.

On the other hand, if either $\sigma_Z^2 > \sigma_{Z_0}^2$ or $\sigma_W^2 > \sigma_{W_0}^2$, since the decoder can use nothing other than the side information to estimate X^{nN}, we have $D_{\mathrm{SEP}}(\sigma_Z^2, \sigma_W^2) = \sigma_Z^2$.

C. CDS

Similar to separate coding, the encoder uses a test channel $S = X + E$ to quantize X^{nN} such that (17), and equivalently (18), is satisfied for some target pair $(\sigma_{Z_0}^2, \sigma_{W_0}^2)$. However, since no explicit binning is involved, the decodability condition for any pair (σ_Z^2, σ_W^2) is now given only by [6]

$$I(X; S|Y) \le \kappa I(U; V) \tag{22}$$

which, combined with (18), translates into

$$\frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2} \le \frac{(1+\gamma)^\kappa - 1}{\sigma_Z^2} \ . \tag{23}$$

Observing that once S^{nN} is decoded, the optimal estimator has the same form as in separate coding, and therefore yields the same distortion

$$D_{\mathrm{CDS}}(\sigma_Z^2, \sigma_W^2) = \left(\frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2} + \frac{1}{\sigma_Z^2}\right)^{-1} \tag{24}$$

and also that if S^{nN} is not decodable, $D_{\mathrm{CDS}}(\sigma_Z^2, \sigma_W^2) = \sigma_Z^2$. As noted in [6], we not only have $D_{\mathrm{CDS}}(\sigma_{Z_0}^2, \sigma_{W_0}^2) =$

$D_{\rm opt}(\sigma^2_{Z_0}, \sigma^2_{W_0})$, but $D_{\rm CDS}(\sigma^2_Z, \sigma^2_W) = D_{\rm opt}(\sigma^2_Z, \sigma^2_W)$ also for any (σ^2_Z, σ^2_W) satisfying (23) with equality.

Finally, although expressions in (21) and (24) are the same, since the decodability region of separate coding is a subset of that of CDS, it is always true that

$$D_{\rm CDS}(\sigma^2_Z, \sigma^2_W) \le D_{\rm SEP}(\sigma^2_Z, \sigma^2_W) \ .$$

Therefore, in the sequel, we exclude separate coding in performance comparisons.

D. HDA-WZ

In the proof of Theorem 1, the expressions for decodability (14) and distortion (15) are valid for *any* (σ^2_Z, σ^2_W) and $\lambda_i \ge 0$, $1 \le i \le n$. If we set in particular

$$\lambda_i = \frac{P\left((1+\gamma_0)^\kappa - 1\right)}{\sigma^2_{Z_0}(1+\gamma_0)}$$

as in (16) for a "target" $(\sigma^2_{Z_0}, \sigma^2_{W_0})$ pair, we guarantee that $D_{\rm HDA}(\sigma^2_{Z_0}, \sigma^2_{W_0}) = D_{\rm opt}(\sigma^2_{Z_0}, \sigma^2_{W_0})$. Also, (14) and (15) become

$$\frac{(1+\gamma_0)^\kappa - 1}{\sigma^2_{Z_0}(1+\gamma_0)} \le \frac{(1+\gamma)^\kappa - 1}{\sigma^2_Z(1+\gamma)} \tag{25}$$

and

$$D_{\rm HDA}(\sigma^2_Z, \sigma^2_W) = \left(\frac{(1+\gamma_0)^\kappa - 1}{\sigma^2_{Z_0}} \cdot \frac{1+\gamma}{1+\gamma_0} + \frac{1}{\sigma^2_Z}\right)^{-1} \tag{26}$$

respectively. As before, $D_{\rm HDA}(\sigma^2_Z, \sigma^2_W) = \sigma^2_Z$ if the auxiliary codeword T^{mN} is not decodable, i.e., if (25) is violated.

E. Performance Comparison

We now compare the performances of CDS, HDA-WZ, and uncoded transmission methods in terms of the *distortion loss* given in (2) that they incur. We likewise define the *distortion gain* as $G\left(\sigma^2_Z, \sigma^2_W\right) = \frac{1}{L\left(\sigma^2_Z, \sigma^2_W\right)}$. Note that $L\left(\sigma^2_Z, \sigma^2_W\right)$ is larger than or equal to 1 while $G\left(\sigma^2_Z, \sigma^2_W\right)$ is between 0 and 1 for any $\left(\sigma^2_Z, \sigma^2_W\right)$ pair with any scheme. The distortion loss expressions for HDA-WZ, CDS, and uncoded transmission are given in Table I.

	If(23) or (25) satisfied	Otherwise
$L_{\rm HDA}$	$\left(\frac{(1+\gamma_0)^\kappa-1}{\sigma^2_{Z_0}} \cdot \frac{1+\gamma}{1+\gamma_0} + \frac{1}{\sigma^2_Z}\right)^{-1} \frac{(1+\gamma)^\kappa}{\sigma^2_Z}$	$(1+\gamma)^\kappa$
$L_{\rm CDS}$	$\left(\frac{(1+\gamma_0)^\kappa-1}{\sigma^2_{Z_0}} + \frac{1}{\sigma^2_Z}\right)^{-1} \frac{(1+\gamma)^\kappa}{\sigma^2_Z}$	$(1+\gamma)^\kappa$
$L_{\rm UNC}$	$\frac{(1+\gamma)^\kappa}{1+\sigma^2_Z\gamma\kappa}$	

Table I
THE DISTORTION LOSS EXPRESSIONS FOR THE THREE SCHEMES

In Figure 2, the distortion gains are depicted in an RGB-coded diagram, where "red", "green", and "blue" colors account for $G_{\rm HDA}$, $G_{\rm CDS}$, and $G_{\rm UNC}$ respectively. In this sense a true "yellow" color at any point means $G_{\rm HDA} = G_{\rm CDS} = 1$ and $G_{\rm UNC} = 0$ at that $\left(\sigma^2_Z, \sigma^2_W\right)$ pair. Likewise, a true "white" color corresponds to all the schemes being optimal whereas a true "black" color means distortion loss for all of the schemes are very high. In the figure, $\kappa = 1.2$ and the target is set to $\rho_0 = \sqrt{1-\sigma^2_{Z_0}} \approx 0.7071$ and $\gamma_0 = \frac{P}{\sigma^2_{W_0}} = 5$.

Table II shows numerical values for $G_{\rm HDA}$, $G_{\rm CDS}$, and $G_{\rm UNC}$ at several points indicated on Figure 2. On the curve indicated by points **f-a-g**, $G_{\rm HDA} = 1$ since (25) is satisfied with equality. For all points to the upper right of that curve, the HDA-WZ scheme successfully decodes T^{mN}. Likewise, on the curve **b-a-c**, $G_{\rm CDS} = 1$ since (23) is satisfied with equality and to the upper-right of that curve, CDS decodes S^{nN} successfully. On the point **a**, which corresponds to the target, both CDS and HDA-WZ achieves a gain of 1, as expected. On points **d** and **e**, the uncoded transmission scheme outperforms the other two.

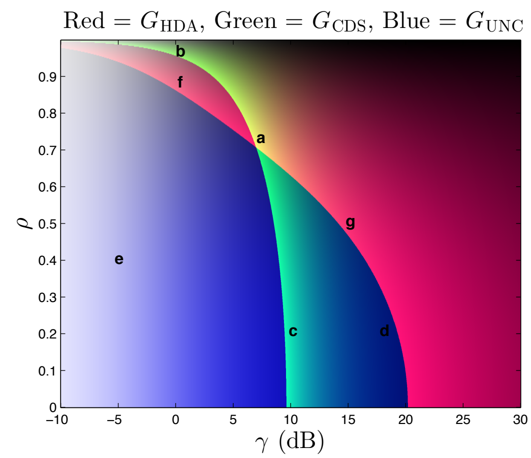

Figure 2. An RGB-coded representation of $G_{\rm HDA}$, $G_{\rm CDS}$, and $G_{\rm UNC}$.

Point	R($G_{\rm HDA}$)	G($G_{\rm CDS}$)	B($G_{\rm UNC}$)	γ (dB)	ρ
a	1.000	1.000	0.465	6.99	0.707
b	0.627	1.000	0.487	-0.16	0.958
c	0.064	1.000	0.721	9.48	0.194
d	0.005	0.080	0.478	19.00	0.200
e	0.719	0.719	0.948	-5.00	0.400
f	1.000	0.428	0.566	0.12	0.860
g	1.000	0.217	0.467	14.54	0.486

Table II
THE POINTS ON FIGURE 2 AND CORRESPONDING VALUES.

IV. CODING WITH A MIN-MAX DISTORTION LOSS CRITERION

In this section we focus on coding aimed for a predefined rectangular region $\mathcal{R}$ of parameters, i.e., $\sigma^2_{Z_1} \le \sigma^2_Z \le \sigma^2_{Z_2}$ and $\sigma^2_{W_1} \le \sigma^2_W \le \sigma^2_{W_2}$ for some $\sigma^2_{Z_1} < \sigma^2_{Z_2}$ and $\sigma^2_{W_1} < \sigma^2_{W_2}$. We use the notation $\gamma_i = \frac{P}{\sigma^2_{W_i}}$ for $i = 1, 2$. For each scheme, we aim to minimize the *maximum distortion loss* over $\mathcal{R}$:

$$\mathcal{L}(\mathcal{R}) = \min \max_{(\sigma^2_Z, \sigma^2_W) \in \mathcal{R}} L\left(\sigma^2_Z, \sigma^2_W\right) \ .$$

For uncoded transmission, since $L_{\rm UNC}$ does not depend on any preset $(\sigma^2_{Z_0}, \sigma^2_{W_0})$ and is a monotonically decreasing function of σ^2_Z and σ^2_W, the min-max of the distortion loss in uncoded transmission is

$$\mathcal{L}_{\rm UNC}(\mathcal{R}) = \frac{(1+\gamma_1)^\kappa}{1+\sigma^2_{Z_1}\gamma_1\kappa} \ . \tag{27}$$

For CDS and the HDA-WZ scheme, the main point is to search for the optimum values for $\sigma^2_{Z_0}$ and $\sigma^2_{W_0}$. However, the

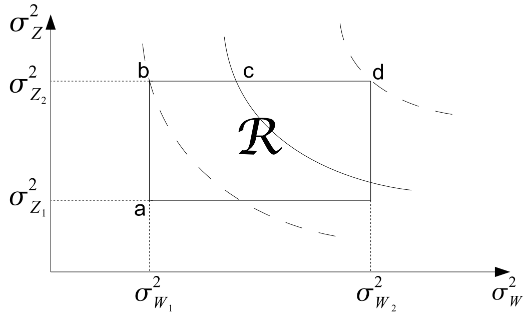

Figure 3. The predefined region $\mathcal{R}$ and some critical points. Depending on the scheme, either (23) or (25) is satisfied on or below the curves, and therefore the codeword S^{nN} or T^{mN} can be decoded. The point **c** represents the target pair $(\sigma_{Z_0}^2, \sigma_{W_0}^2)$.

search only has one degree of freedom. To see that, observe that neither condition (23) nor the expressions for $L_{\rm CDS}$ given in Table I change with $(\sigma_{Z_0}^2, \sigma_{W_0}^2)$ when $\frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2}$ is fixed. The same is true for $L_{\rm HDA}$ when $\frac{(1+\gamma_0)^\kappa - 1}{\sigma_{Z_0}^2(1+\gamma_0)}$ is fixed. Therefore, without loss of generality, we will assume that the target side information parameter $\sigma_{Z_0}^2$ coincides with $\sigma_{Z_2}^2$, which corresponds to point **c** in Figure 3. For a fixed target point **c**, we will use the notation $L_{\rm CDS}^i(\mathbf{x}|\mathbf{c})$ and $L_{\rm HDA}^i(\mathbf{x}|\mathbf{c})$ for the distortion loss given in the ith column of Table I evaluated at point $\mathbf{x}$. We will be especially interested in the cases where **x** is one of points **a**, **b**, **c**, and **d** shown in Figure 3.

Lemma 1: If

$$(1+\gamma_2)^\kappa \le \frac{(1+\gamma_1)^\kappa}{1+\zeta\left([1+\gamma_2]^\kappa - 1\right)} \tag{28}$$

with $\zeta = \frac{\sigma_{Z1}^2}{\sigma_{Z2}^2}$, then

$$\mathcal{L}_{\rm CDS}(\mathcal{R}) = (1+\gamma_0)^\kappa \tag{29}$$

where γ_0 is the solution of

$$(1+\gamma_0)^\kappa = \frac{(1+\gamma_1)^\kappa}{1+\zeta\left([1+\gamma_0]^\kappa - 1\right)} .$$

Otherwise,

$$\mathcal{L}_{\rm CDS}(\mathcal{R}) = \frac{(1+\gamma_1)^\kappa}{1+\zeta\left([1+\gamma_2]^\kappa - 1\right)}. \tag{30}$$

Proof: We begin by observing that when (23) holds, $L_{\rm CDS}$ is a decreasing function of σ_Z^2 and σ_W^2, and when (23) does not hold, $L_{\rm CDS}$ is a decreasing function of σ_W^2. Therefore, for a fixed target point **c**, the maximum loss in $\mathcal{R}$ is always equal to either $L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$ or $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c})$.

Now, when point **c** is exactly at **b**, $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c}) > L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$. As we move **c** from **b** toward **d**, $L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$ increases and $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c})$ decreases. Therefore, $\mathcal{L}_{\rm CDS}(\mathcal{R})$ is achieved either (i) by setting **c** between **b** and **d** such that $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c}) = L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$, or (ii) if that is not possible, then by setting $\mathbf{c} = \mathbf{d}$, thereby ensuring that (23) is satisfied in the entire region $\mathcal{R}$ and resulting in $\mathcal{L}_{\rm CDS}(\mathcal{R}) = L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$.

For condition (i) to occur, we must necessarily have $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c}) \le L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$ when $\mathbf{c} = \mathbf{d}$, which is the same as (28). In this case, solving for the point **c** such that $L_{\rm CDS}^2(\mathbf{c}|\mathbf{c}) = L_{\rm CDS}^1(\mathbf{a}|\mathbf{c})$ yields (29). Otherwise, setting $\mathbf{c} = \mathbf{d}$ yields (30). ■

Lemma 2: If

$$(1+\gamma_2)^\kappa \le \frac{(1+\gamma_2)(1+\gamma_1)^{\kappa-1}}{\frac{(1+\gamma_2)}{(1+\gamma_1)} + \zeta\left([1+\gamma_2]^\kappa - 1\right)} \tag{31}$$

then

$$\mathcal{L}_{\rm HDA}(\mathcal{R}) = (1+\gamma_0)^\kappa$$

where γ_0 is the solution of

$$(1+\gamma_0)^\kappa = \frac{(1+\gamma_0)(1+\gamma_1)^{\kappa-1}}{\frac{(1+\gamma_0)}{(1+\gamma_1)} + \zeta\left([1+\gamma_0]^\kappa - 1\right)} .$$

Otherwise,

$$\mathcal{L}_{\rm HDA}(\mathcal{R}) = \frac{(1+\gamma_2)(1+\gamma_1)^{\kappa-1}}{\frac{(1+\gamma_2)}{(1+\gamma_1)} + \zeta\left([1+\gamma_2]^\kappa - 1\right)} .$$

Proof: The proof follows the exact same steps as in the proof of Lemma 1, except expressions for $L_{\rm HDA}$ are used. ■

We now prove the main result of this section.

Theorem 2: For all rectangular regions $\mathcal{R}$,

$$\mathcal{L}_{\rm HDA}(\mathcal{R}) < \mathcal{L}_{\rm CDS}(\mathcal{R}) .$$

Proof: We analyze two possibilities separately as in the proof of Lemma 1: (i) $\mathcal{L}_{\rm CDS}(\mathcal{R})$ is achieved at a point **c** is between **b** and **d**, or (ii) when **c** coincides with **d**.

For case (i), take $\mathbf{c}' = \mathbf{c}$ as the target point for the HDA-WZ scheme. Then comparing the expressions of $L_{\rm HDA}$ and $L_{\rm CDS}$ in Table I, we observe

$$L_{\rm HDA}^1(\mathbf{a}|\mathbf{c}') < L_{\rm CDS}^1(\mathbf{a}|\mathbf{c}) = L_{\rm CDS}^2(\mathbf{c}|\mathbf{c}) = L_{\rm HDA}^2(\mathbf{c}'|\mathbf{c}') .$$

Now, moving $\mathbf{c}'$ from **c** toward **d** even slightly will increase $L_{\rm HDA}^1(\mathbf{a}|\mathbf{c}')$ and decrease $L_{\rm HDA}^2(\mathbf{c}'|\mathbf{c}')$, thereby guaranteeing

$$\mathcal{L}_{\rm HDA}(\mathcal{R}) \le \max\{L_{\rm HDA}^1(\mathbf{a}|\mathbf{c}'), L_{\rm HDA}^2(\mathbf{c}'|\mathbf{c}')\} < \mathcal{L}_{\rm CDS}(\mathcal{R}) .$$

For case (ii), taking $\mathbf{c}' = \mathbf{d}$ yields

$$\mathcal{L}_{\rm HDA}(\mathcal{R}) \le L_{\rm HDA}^1(\mathbf{a}|\mathbf{c}') < L_{\rm CDS}^1(\mathbf{a}|\mathbf{c}) = \mathcal{L}_{\rm CDS}(\mathcal{R}) .$$

■

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