Performance Bounds for Spatially-Coupled LDPC Codes over the Block Erasure Channel

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***Abstract*—The paper provides simple lower and upper bounds on block P_B and bit P_b performances of spatially-coupled LDPC (SC-LDPC) codes over a particular model of the block erasure channel. As expected, the spatial coupling structure helps in the correction of bursty erasures, and the decoding performance of SC-LDPC codes improves if the coupling parameter $\mathtt{w}$ increases.**

I. Introduction

In this paper, we would like to investigate the behavior of spatially-coupled LDPC (SC-LDPC) codes over a particular example of the block erasure channel (BLEC) channel. Our BLEC model is relevant in the context of data transmission or data storage in networks, when big blocks of data can be lost during the transmission or simply unavailable because of a storage system failure. Also in general, the BLEC seems to be a good model for a the block-fading channel in the high-SNR regime. Block-fading channels are very relevant in wireless communication, and designing a good error-correcting code for a block-fading channel is still a quite challenging task [3]. For example, standard LDPC codes are not at all adapted for the transmission over the BLEC, and for our transmission model they would have $P_b = P_B = \epsilon$, where ϵ is the block erasure probability of the channel.

SC-LDPC codes have been first introduced in [1], and their analysis over the binary erasure channel (BEC) has been given in [2]. Note that SC-LDPC codes have very good asymptotic iterative performance over the BEC. Actually, their iterative threshold approaches the MAP threshold of the ensemble. The motivation to use SC-LDPC codes over the BLEC is justified by the fact that those codes have a convolutional (or a spatial coupling) structure, while to use such structures is one of the best code strategies, adapted for use over non-ergodic transmission channels [4]. Therefore, one would expect that SC-LDPC codes have a good performance over the BLEC.

The interesting question is how much the spatial coupling structure helps in correcting the bursty erasures. In this paper, this question is addressed by deriving some simple lower and upper bounds on both bit and block erasure probabilities after decoding, and by showing that all of them improve with the coupling parameter $\mathtt{w}$ of SC-LDPC codes. The bounds are based on evaluation of the decoding performance over dominant erasure patterns, that happen for small and large values of the block erasure probability ϵ of the channel.

II. Preliminaries

A. Spatially-Coupled LDPC (SC-LDPC) codes

Let the $(n, \mathtt{l}, \mathtt{r})$ ensemble be the ensemble of LDPC codes with corresponding variable node degree $\mathtt{l}$, check node degree $\mathtt{r}$ and codelength n. Then, a $(n, \mathtt{l}, \mathtt{r}, L, \mathtt{w})$ SC-LDPC code with variable node degree $\mathtt{l}$, check node degree $\mathtt{r}$, codelength nL and coupling parameter $\mathtt{w}$ can be constructed with the following procedure:

1) Choose at random $L+2$w codes from the $(n, \mathtt{l}, \mathtt{r})$ LDPC ensemble, and enumerate them from -w to $L + \mathrm{w} - 1$.
2) Put $p_0 = 1/(2\mathtt{w} + 1)$.
3) For each LDPC code i ($-\mathtt{w} \le i < L + \mathtt{w}$), partition the edges of its Tanner graph into $2\mathtt{w}+1$ disjoint subsets of equal size $p_0 n$. Denote these subsets as $E^i_{-\mathtt{w}}, \dots, E^i_{\mathtt{w}}$.
4) Set $i = -\mathtt{w}$.
   a) For given i and for all w from 1 to $\mathtt{w}$ do[1]: Take $E^i_w = \{(v^{i,w}_k, c^{i,w}_k)\}^{p_0 n}_{k=1}$ and $E^{i+w}_{-w} = \{(v^{i+w,-w}_k, c^{i+w,-w}_k)\}^{p_0 n}_{k=1}$. Exchange check nodes connections between E^i_w and E^{i+w}_{-w}, so that new E^i_w becomes $E^i_w = \{(v^{i,w}_k, c^{i+w,-w}_k)\}^{p_0 n}_{k=1}$ and new E^{i+w}_{-w} is $E^{i+w}_{-w} = \{(v^{i+w,-w}_k, c^{i,w}_k)\}^{p_0 n}_{k=1}$. Such an exchange is illustrated by bold lines in Fig. 1.
   b) Increment i. If $i \ge L$, then finish the procedure. Otherwise, go to 4a).

By the procedure above, one obtains a SC-LDPC code of length Ln of rate $r_{SC} < 1 - \mathtt{l}/\mathtt{r}$, for which the coupling is made on both left and right sides. Note that the SC-LDPC code is defined as the set of all binary assignments in L LDPC codes, indexed from 0 to $L-1$, so that they (assignments) form valid codewords at local parity codes in the spatially-coupled structure. Binary values for LDPC codes, indexed from $-\mathtt{w}$ to -1 and from L to $L + \mathtt{w} - 1$, are fixed and set to 0.

The $(n, \mathtt{l}, \mathtt{r}, L, \mathtt{w})$ SC-LDPC ensemble is defined over all possible permutations of underlying LDPC codes and choices of edge subsets for spatial coupling.

[1] In what follows an edge is given by its variable and check node connection, i.e. $e = (v, c)$

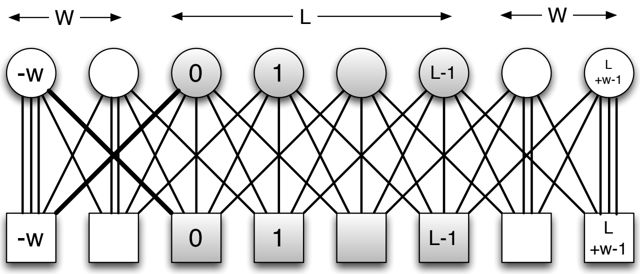

Fig. 1. Coupling structure for $\mathtt{w} = 2$ and $L = 4$. Each set of variable nodes (circles) or parity nodes (boxes) is coupled to its neighbours via edge subsets E_w^i (straight lines). Bold lines represent subsets $E_{\mathtt{w}}^{-\mathtt{w}}$ and $E_{-\mathtt{w}}^{0}$, to illustrate the point 4a).

B. BLEC Model and Density Evolution of SC-LDPC Codes for the BLEC(ϵ)

We consider the transmission over the BLEC channel with erasure probability ϵ, where *each transmitted packet is of length n and carries one LDPC codeword part from the SC-LDPC codeword*. The bit erasure rate ε per received packet is therefore 1 with probability ϵ and 0 otherwise.

It has been observed in [3] that the decoding performance of LDPC codes does not depend much on the codelength n, but rather on the ratio n/N, N being the size of transmitted packets. If, for instance, $n/N = 1$, that the received packet will be either perfectly known or completely erased, and the bit erasure rate is the same for any n, and equals to ϵ, as the coding cannot give an improvement. For our transmission model, $n = N$. It is therefore reasonable to assume that the SC-LDPC decoding performance is independent from n, for n sufficiently large:

Assumption 2.1: The decoding performance of the $(\mathtt{l}, \mathtt{r}, L, \mathtt{w})$ ensemble over the BLEC(ϵ) is a good approximation of the decoding performance of a $(n, \mathtt{l}, \mathtt{r}, L, \mathtt{w})$ code over the BLEC(ϵ) with $n = N$, under condition that in both cases one LDPC codeword corresponds to one transmission packet.

This assumption simplifies the analysis, as the average performance of the $(\mathtt{l}, \mathtt{r}, L, \mathtt{w})$ ensemble can be obtained by density evolution, that can be defined similar to the binary erasure channel case [2].

Let $x_i(y_i)$ be the erasure probability of messages going out (going into) variable nodes of the i-th LDPC block, then:

$$\begin{cases} x_i^{(l)} = & \varepsilon_i \left(\sum_{j=-\mathtt{w}}^{\mathtt{w}} p_0 y_{i-j}^{(l)} \right)^{\mathtt{l}-1}; \\ y_i^{(l)} = & 1 - \left(1 - \sum_{k=-\mathtt{w}}^{\mathtt{w}} p_0 x_{i+k}^{(l-1)} \right)^{\mathtt{r}-1}; \\ & \text{for } i = -\mathtt{w}, \ldots, L + \mathtt{w} - 1. \end{cases} \tag{1}$$

Here $\varepsilon_i = 0$ for $-\mathtt{w} \le i < 0$ and $L \le i < L + \mathtt{w}$, and

$$\varepsilon_i = \begin{cases} 0, & w.p.\ 1 - \epsilon \\ 1, & w.p.\ \epsilon \end{cases} \quad \text{if } 0 \le i < L.$$

(1) can be rewritten in another form, which is more convenient for our purposes. Let $\alpha_i = (y_{i-\mathtt{w}}, \ldots, y_{i-1})$, $\gamma_i = (y_{i+1}, \ldots, y_{i+\mathtt{w}})$, $\delta_i = (x_{i-\mathtt{w}}, \ldots, x_{i-1})$ and $\beta_i = (x_{i+1}, \ldots, x_{i+\mathtt{w}})$, and denote

$$\bar{\alpha}_i^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} \alpha_{i,j}^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} y_{i-j}^{(l)}, \qquad \bar{\gamma}_i^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} y_{i+j}^{(l)},$$

$$\bar{\beta}_i^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} \beta_{i,j}^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} x_{i+j}^{(l)}, \qquad \bar{\delta}_i^{(l)} = p_0 \sum_{j=1}^{\mathtt{w}} x_{i-j}^{(l)}.$$

Then, for $0 \le i \le L$,

$$\begin{cases} x_i^{(l)} = & \varepsilon_i \left(\bar{\alpha}_i^{(l)} + \bar{\gamma}_i^{(l)} + p_0 y_i^{(l)} \right)^{\mathtt{l}-1}; \\ y_i^{(l)} = & 1 - \left(1 - \bar{\delta}_i^{(l-1)} - \bar{\beta}_i^{(l-1)} - p_0 x_i^{(l-1)} \right)^{\mathtt{r}-1}. \end{cases} \tag{2}$$

III. Erasure Pattern and Its Decoding

Assume the all-zero SC-LDPC codeword was transmitted. Given the transmission model of Section II-B, the received vector is a sequence of blocks of n erasures and of blocks of n zeros. If t consecutive blocks were erased, the received vector contains tn successive erasures. Let us call it the *erasure pattern* of length t. Note that the received vector is completely described by the list of couples (t, i), where t is the length of an erasure pattern and i is the index of its first block.

In this section, the decoding of one disjoint erasure pattern is considered.

Lemma 3.1: Consider the erasure pattern of length t and of starting index i. Denote by $P_B(e|t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1})$ and $P_b(e|t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1})$ its corresponding block and bit erasure probabilities after decoding, assuming fixed vectors $\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}$. Then, there exists some value $T = T(\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1})$ so that

$$P_B(e|t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) = \begin{cases} 1, & t \ge T, \\ 0, & t < T, \end{cases} \tag{3}$$

and

$$P_b(e|t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) = \begin{cases} p_b, & t \ge T, \\ 0, & t < T, \end{cases} \tag{4}$$

with $p_b = p_b(t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) > 0$.

The proof of lemma is given below. Note that, unfortunately, one cannot simplify the expressions by using scalars $\bar{\delta}_i, \bar{\alpha}_i, \bar{\beta}_{t+i-1}, \bar{\gamma}_{t+i-1}$ instead of vectors $\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}$: patterns with different $\delta, \alpha, \beta, \gamma$ (and therefore different P_b) may have the same average values $\bar{\delta}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}$.

Proof: In order to find P_b and P_B, one defines the density evolution equations for $x_i, \ldots, x_{i+t-1}$ and $y_i, \ldots, y_{l+t-1}$, given fixed initial parameters $\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}$. Note that they are similar to (2), except that now $\varepsilon_{i+j} = 1$ for all j, $0 \le j < t$. Therefore, the solution of density equation $x_i^{(\infty)}, \ldots, x_{t+i-1}^{(\infty)}$ and, therefore, P_b and P_B will only depend on the length t. The monotonicity of P_b and P_B with t comes from the monotonicity of the iterative decoder. If the decoding fails for some erasure length t, then it also fails for any $t' > t$ as $x_{i+k}^{(l)}(t) \le x_{i+k}^{(l)}(t')$ for any $l > 0$ and any $i \le k \le t + i$. Inversely, if the decoding succeeds for some t, then it also succeeds for any $t' < t$. ■

Unfortunately, T and p_b depend on too many parameters, and they are difficult to evaluate. In the following lemma some simpler bounds on T are developed:

Lemma 3.2: We have

$$\begin{aligned} T(\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) &\geq T(\delta_i, \mathbf{0}, \beta_{t+i-1}, \mathbf{0}) \\ &= T_{min}(\delta_i, \beta_{t+i-1}); \quad (5) \\ T(\delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) &\leq T(\delta_i, \mathbf{a}, \beta_{t+i-1}, \mathbf{b}) \\ &= T_{max}(\delta_i, \beta_{t+i-1}), \quad (6) \end{aligned}$$

with $\mathbf{0} = (0, ..., 0)$, $\mathbf{a} = (1, ..., 1, 0)$ and $\mathbf{b} = (0, 1, ..., 1)$.

The proof of lemma follows directly from (2) and is omitted. Moreover, the bounds on p_b can be found:

Lemma 3.3: For some $t \geq T$,

$$p_{min}(t) \leq p_b(t, \delta_i, \alpha_i, \beta_{t+i-1}, \gamma_{t+i-1}) \leq p_{max}(t),$$

with

$$\begin{aligned} p_{min}(t) &= p_b(t, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \frac{1}{t} \sum_{k=0}^{t-1} \left(p_0 \sum_{i=-\mathtt{w}}^{\mathtt{w}} \hat{y}_{k+i}^{(\infty)} \right)^{\mathtt{l}}, \\ p_{max}(t) &= p_b(t, \mathbf{a}, \mathbf{a}, \mathbf{b}, \mathbf{b}) = \frac{1}{t} \sum_{k=0}^{t-1} \left(p_0 \sum_{i=-\mathtt{w}}^{\mathtt{w}} \tilde{y}_{k+i}^{(\infty)} \right)^{\mathtt{l}}. \end{aligned}$$

If one denotes $q = \min(k, \mathtt{w})$, $s = \min(t-k, \mathtt{w})$ and $v = 2\mathtt{w} - 2 - q_k - s_k$, then $\hat{y}_{k+i}^{(\infty)}$'s are the solution of

$$\begin{cases} x_{i+k} = (\sum_{j=-s}^{q} p_0 y_{i+k-j})^{\mathtt{l}-1}; \\ y_{i+k} = 1 - (1 - \sum_{j=-q}^{s} p_0 x_{i+k+j})^{\mathtt{r}-1}; \end{cases} \quad 0 \leq k < t,$$

and $\tilde{y}_{k+i}^{(\infty)}$'s are the solution of

$$\begin{cases} x_{i+k} = (p_0 v + \sum_{j=-s}^{q} p_0 y_{i+k-j})^{\mathtt{l}-1}; \\ y_{i+k} = 1 - (1 - p_0 v - \sum_{j=-q}^{s} p_0 x_{i+k+j})^{\mathtt{r}-1}; \end{cases} \quad 0 \leq k < t.$$

Lemma 3.3 is obtained by developing (1) for aforementioned initial conditions.

IV. Performance Bounds over the BLEC(ϵ)

Using facts from Section III, it is possible to develop lower and upper bounds on the bit erasure probability P_b and the block erasure probability P_B after iterative decoding of SC-LDPC codes over the BLEC(ϵ):

Theorem 4.1: For the $(\mathtt{l}, \mathtt{r}, L, \mathtt{w})$ ensemble described in Section II-A and the block erasure probability ϵ,
$P_B^{LB} \leq P_B \leq P_B^{UB}$ and $P_b^{LB} \leq P_b \leq P_b^{UB}$, with

$$P_B^{LB} = 1 - \frac{1 - \epsilon z_0}{(1-\epsilon) z_0} \cdot \frac{z_0^{-L}}{1 + T_{max}(\mathbf{0}, \mathbf{0})(1 - z_0)} \quad (7)$$

$$P_b^{LB} = \frac{p_{min}(T_{min}(\mathbf{0}, \mathbf{0})) T_{max}(\mathbf{0}, \mathbf{0})}{L} P_B^{LB} \quad (8)$$

$$P_B^{UB} = \sum_{\delta=\mathbf{0}}^{\mathbf{a}} \sum_{\beta=\mathbf{0}}^{\mathbf{b}} P_B(\delta, \beta) \quad (9)$$

$$P_b^{UB} = \sum_{\delta=\mathbf{0}}^{\mathbf{a}} \sum_{\beta=\mathbf{0}}^{\mathbf{b}} p_{max}(T_{min}(\delta, \beta)) P_B(\delta, \beta) \quad (10)$$

where z_0 is the smallest positive root of (16), the expressions for T_{max}, T_{min}, p_{max}, p_{min} are given by Lemmas 3.2 and 3.3, and

$$P_B(\delta, \beta) = \epsilon^{wt(\delta) + wt(\beta) + T_{min}(\delta, \beta)} (1-\epsilon)^{2\mathtt{w} - wt(\delta) - wt(\beta)}. \quad (11)$$

In the rest of the section, the derivation of these upper and lower bounds is presented.

A. Lower Bounds

1) P_B^{LB}: Let us start with:

$$P_B = \sum_S \mathbf{P}(S) P_B(S) \quad (12)$$

where the summation is performed over all 0-? sequences of length L, where ? occurs independently with probability ϵ. Note that each 0-? sequence can be seen as the list of k erasure patterns with known starting positions, each pattern being of length t_i, $1 \leq i \leq k$. Note that, as the starting positions of erasure patterns are known, the distances between two neighbouring erasure patterns are also known. Unless this distance is large, the erasure probabilities within erasure patterns, computed during the decoding process, are dependent from each other.

By using the pattern representation, one obtains that

$$P_B = \sum_{k=1}^{\lfloor L/2 \rfloor} \sum_{\{t_1, \ldots, t_k\}} \mathbf{P}(\{t_1, \ldots, t_k\}) P_B(\{t_1, \ldots, t_k\}). \quad (13)$$

By Lemma 3.1, each t_i has the corresponding threshold value $T_i = T_i(\delta_{i_k}^{(\infty)}, \alpha_{i_k}^{(\infty)}, \beta_{t_k+i_k-1}^{(\infty)}, \gamma_{t_k+i_k-1}^{(\infty)})$. Hence

$$P_B(\{t_1, \ldots, t_k\}) \geq \mathbf{P}(\exists \text{ at least one pattern s.t. } t_j \geq T_j);$$

$$P_B(\{t_1, \ldots, t_k\}) \geq 1 - \prod_i (1 - \mathbf{P}(t_i \geq T_i))$$

$$= 1 - \sum_{\delta, \alpha, \beta, \gamma} \mathbf{P}(\delta, \alpha, \beta, \gamma) \prod_i (1 - \mathbf{P}(t_i \geq T_i(\delta, \alpha, \beta, \gamma)))$$

Now note that

$$\mathbf{P}(t_i \geq t) = 1 - \sum_{t_i=0}^{t-1} \mathbf{P}(t = i) \propto 1 - \sum_{i=0}^{t-1} \epsilon^i (1-\epsilon) = \epsilon^t. \quad (14)$$

Therefore, if $T_i(\delta, \alpha, \beta, \gamma) \leq T_{max}(\delta, \alpha))$,
then $\mathbf{P}(t_i \geq T_i(\delta, \alpha, \beta, \gamma)) \geq \mathbf{P}(t_i \geq T_{max}(\delta, \alpha))$, and

$$P_B(\{t_1, ..., t_k\}) \geq 1 - \sum_{\delta, \beta} \mathbf{P}(\delta, \beta) \prod_i (1 - \mathbf{P}(t_i \geq T_{max}(\delta, \beta)))$$

We also have $\mathbf{P}(\{t_1, \ldots, t_k\}) \geq \prod_i \mathbf{P}(\{t_i\})$, where $\mathbf{P}(\{t_i\})$ is the probability to have an erasure pattern of length t_i alone, independently of others. So,

$$P_B \geq \sum_{\delta, \beta} \mathbf{P}(\delta, \beta) \mathbf{P}(\exists \text{ pattern of length } t \geq T_{max}(\delta, \beta)).$$

As $T_{max}(\delta, \beta) \leq T_{max}(\mathbf{0}, \mathbf{0})$,

$$P_B \geq \mathbf{P}(\exists \text{ pattern of length } t \geq T_{max}(\mathbf{0}, \mathbf{0})). \quad (15)$$

Let us use the rephrased version of the theorem from [5]:

Theorem 4.2: Given the BLEC(ϵ), the probability to get an erased pattern of length at least T within L blocks is asymptotically

$$\mathbf{P}(\exists t \geq T) = 1 - \frac{1-\epsilon z_0}{(1-\epsilon)z_0} \cdot \frac{z_0^{-L}}{T+1-Tz_0} + o(z_0^{-L}),$$

where z_0 is the smallest positive root of

$$1 - z + (1-\epsilon)\epsilon^T z^{T+1} = 0. \tag{16}$$

By applying Theorem 4.2 one directly obtains (7).

2) P_b^{LB}: (13) can be also adapted for the P_b case. Then,

$$P_b(e|S) = P_b(e|\{t_1 \dots t_k\}) \geq \sum_i \frac{t_i}{L} P_b(e|\{t_i\}). \tag{17}$$

where $P_b(e|\{t_i\})$ denotes the bit erasure probability after decoding the independent erasure pattern of length t_i, and $\frac{t_i}{L}$ is a renormalisation coefficient.

One obtains

$$P_b \geq \sum_{\delta,\beta} \mathbf{P}(\delta,\beta) \frac{tp_b(T)}{L} \mathbf{P}(\exists \text{ pattern of length } t \geq T_{max}(\delta,\beta))$$

and, by bounding the expression on the right,

$$P_b \geq \frac{p_{min}T_{min}}{L} \mathbf{P}(\exists \text{ pattern of length } t \geq T_{max}(\mathbf{0},\mathbf{0})).$$

B. Upper Bounds

1) P_B^{UB}: Start from equation (13). By the union bound,

$$P_B(\{t_1,\dots t_k\}) \leq \sum_i P(t_i \geq T_i), \tag{18}$$

where $T_i = T(\delta_{i_k}^{(\infty)}, \alpha_{i_k}^{(\infty)}, \beta_{t_k+i_k}^{(\infty)}, \gamma_{t_k+i_k}^{(\infty)})$. Note that the interdependence of erasure patterns, located close to each other, is expressed via initial conditions for T_i.

By Lemma 3.2 and (14),

$$\begin{aligned} P_B(\{t_1,\dots t_k\}) \leq \sum_k \sum_{\delta,\beta} &\mathbf{P}(\delta_{i_k}^{(\infty)}, \beta_{t_k+i_k}^{(\infty)}) \\ &\cdot P(t_i \geq T_{min}(\delta_{i_k}^{(\infty)}, \beta_{t_k+i_k}^{(\infty)})). \end{aligned}$$

One can further upper bound $P(t_i \geq T_{min}(\delta_{i_k}^{(\infty)}, \beta_{t_k+i_k}^{(\infty)}))$ by $P(t_i \geq T_{min}(\delta_{i_k}^{(0)}, \beta_{t_k+i_k}^{(0)}))$, i.e, one upper bounds the erasure probabilities by channel erasure probabilities. By taking this into account, one obtains that

$$P_B \leq \sum_{\delta=\mathbf{0}}^{\mathbf{a}} \sum_{\beta=\mathbf{0}}^{\mathbf{b}} \mathbf{P}(\delta)\mathbf{P}(\beta)\mathbf{P}(\exists \text{ pattern of length } T_{min}(\delta,\beta)), \tag{19}$$

The summation in δ/β is done over all 0-1 vectors, whose support is included into the support of **a**/**b**. These vectors represent all possible channel input parameters for a given erasure pattern. For some value $\delta = (b_{-\mathtt{w}}, \dots, b_2, b_1)$, $P(\delta)$ is computed as:

$$P(\delta) = \epsilon^{wt(\delta)}(1-\epsilon)^{\mathtt{w}-wt(\delta)-1}, \tag{20}$$

here $wt(\delta)$ represents the weight of vector δ. $P(\beta)$ is evaluated in the same way. (19-20) give (9).

2) P_b^{UB}: One has

$$P_b(\{t_1,\dots t_k\}) = \sum_i p_b(t_i)\mathbf{P}(t_i \geq T_i), \tag{21}$$

and similarly as for P_B^{UB}, one obtains

$$\begin{aligned} P_b \leq \sum_{\delta=\mathbf{0}}^{\mathbf{a}} \sum_{\beta=\mathbf{0}}^{\mathbf{b}} &\mathbf{P}(\delta,\beta)p_{max}(T_{min}(\delta,\beta)) \\ &\cdot \mathbf{P}(\exists \text{ pattern of length } T_{min}(\delta,\beta)), \end{aligned} \tag{22}$$

Then (10) and (11) follow.

V. Estimation of T_{min} and T_{max}

The coupling parameter $\mathtt{w}$ impacts P_B and P_b via the minimum length T of the erasure pattern which cannot be corrected by iterative decoding: the larger is $\mathtt{w}$, the larger is T. In general, T should be estimated numerically, as well as its bounds T_{min} and T_{max}. In this section, we present how T_{min} and T_{max} can be estimated analytically, with the help of the so called forward and backward evaluations of x's and y's within one erasure pattern of some length .

A. Upper Bounding $T_{max}(\mathbf{0},\mathbf{0})$

Remind that

$$x_i = \left(\bar{\alpha}_i + \bar{\gamma}_i + p_0\left(1 - \left(1 - p_0 x_i - \bar{\beta}_i - \bar{\delta}_i\right)^{\mathtt{r}-1}\right)\right)^{1-1}$$

Consider the evaluation of x_i's in the backward manner[2], assuming that the values δ and α are close to 1 (i.e. the erasure pattern is long enough so that the decoder does not succed to correct erasures in the middle). Then one can write

$$x_i \geq \left(1 - p_0\left(p_0 - p_0(1-p_0)^{1-1} - \bar{\beta}_i\right)^{\mathtt{r}-1}\right)^{1-1} = \tilde{x}_i,$$

one can bound the values $\bar{\beta}_{T-1}, \bar{\beta}_{T-2}, \dots, \bar{\beta}_1$ by

$$\bar{\beta}_{T-j} = \sum_{k=1}^{\mathtt{w}} p_0 x_{T-j+k} \quad \geq \quad \tilde{\bar{\beta}}_{T-j} = \sum_{k=1}^{\mathtt{w}} p_0 \tilde{x}_{T-j+k}. \tag{23}$$

Similarly, for the evaluation of x's in the forward manner[3], assuming that the values β and γ are close to 1:

$$x_i \geq \left(1 - p_0\left(p_0 - p_0(1-p_0)^{1-1} - \bar{\delta}_i\right)^{\mathtt{r}-1}\right)^{1-1} = \hat{x}_i,$$

and one bounds the values $\bar{\delta}_2, \dots, \bar{\delta}_T$ by

$$\bar{\delta}_j = \sum_{k=1}^{\mathtt{w}} p_0 x_{j-k} \quad \geq \quad \tilde{\bar{\delta}}_j = \sum_{k=1}^{\mathtt{w}} p_0 \hat{x}_{j-k}, \tag{24}$$

where the initial conditions are $\bar{\delta}_0 = \tilde{\bar{\delta}}_0 = 0$.

The expressions of $\tilde{\bar{\delta}}_j$ and $\tilde{\bar{\beta}}_{T-j}$ are given by nonlinear recursions not depending on x's. Moreover, they are similar as the coupling is symmetric. Hence, if $\tilde{T}_{\bar{\delta}}$ and $\tilde{T}_{\bar{\beta}}$ are corresponding hitting times of $\tilde{\bar{\delta}}_j$ and $\tilde{\bar{\beta}}_{T-j}$, then $\tilde{T}_{\bar{\delta}} = \tilde{T}_{\bar{\beta}}$, and one can bound $T_{max}(\mathbf{0},\mathbf{0}) \leq 2\tilde{T}_{\bar{\delta}}$.

[2] i.e. sequentially starting from the largest value of i to the smallest value

[3] i.e. sequentially starting from the smallest value of i to the largest value

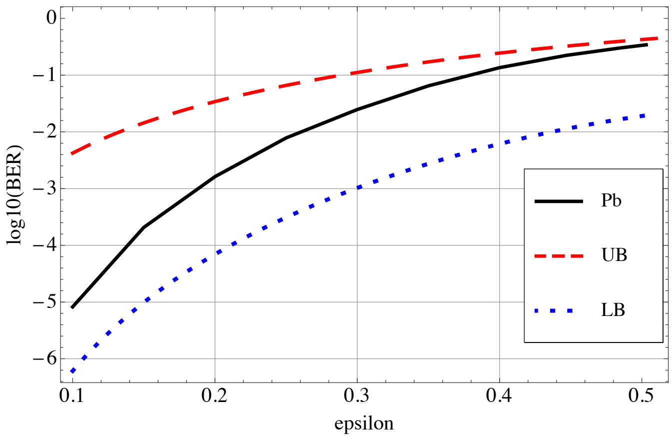

Fig. 2. Upper and lower bounds on P_b of a $(2000, 3, 6, 5, 200)$ code.

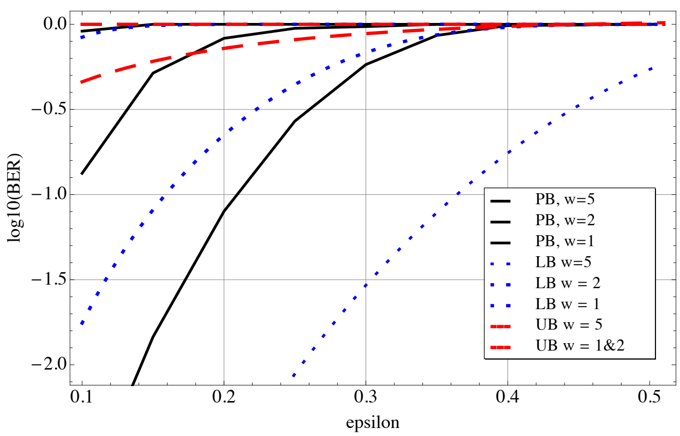

Fig. 3. Upper and lower bounds on P_B of $(2000, 3, 6, \mathtt{w}, 200)$ codes, $\mathtt{w} = 1, 2, 5$.

B. Lower Bounding $T_{min}(\delta, \beta)$

In order to lower bound $T_{min}(\delta, \beta)$, we use the procedure, described above. The backward recursion can be bounded as

$$\bar{\beta}_{T-k} \quad \leq \quad \hat{\bar{\beta}}_{T-k} = \sum_{t=1}^{\mathtt{w}} p_0 \left(1 - p_0 \left(-\hat{\bar{\beta}}_i\right)^{\mathtt{r}-1}\right)^{1-1}, \quad (25)$$

and the expression is similar for the forward recursion (and they have the same hitting time $\hat{T}_{\bar{\delta}}(\delta, \beta)$). Finally, one can obtain that $T_{min}(\delta, \beta) \leq 2\hat{T}_{\bar{\delta}}(\delta, \beta)$.

VI. Numerical Results

Let us illustrate Theorem 4.1 by several numerical results.

Fig.2 presents a lower and an upper bound on P_b for a $(2000, 3, 6, 5, 200)$ code. The bounds were calculated by (8) and (10), where the values of $T_{max} = 7$ and T_{min} varying from 0 to 4, for different values of δ and β, were obtained numerically. For the lower bound, $p_{min}(t = 7)$ was estimated to be 0.4 (the actual erasure probability is around 0.65). One can see that the upper bound goes close to the P_b curve for large values of ϵ. Moreover, the lower bound has the same slope as the P_b curve.

$\mathtt{w}$	1	2	5
$T_{max}(\mathbf{0}, \mathbf{0})$	2	3	7
UB on T_{max}	4	6	10

TABLE I
Values of T_{max} for different values of $\mathtt{w}$.

Fig.3 presents lower and upper bounds on the P_B for $(2000, 3, 6, \mathtt{w}, 200)$ codes with $\mathtt{w} = 1, 2, 5$ respectively. Note that, once again, the lower bounds captures well the slope of the corresponding P_B curve. Upper (union) bounds, unfortunately, become tight only for large values of ϵ.

What concerns the estimations in Section V, Table I presents the actual values of $T_{max}(\mathbf{0}, \mathbf{0})$ and estimated values.

VII. Discussion

Our results show that, indeed, spatially-coupled codes behave well over the BLEC(ϵ), compared to standard LDPC codes, which are not at all suited for such channels. In the asymptotic case, when $L \to \infty$ and $\mathtt{w} \to \infty$, the BLEC performance of SC-LDPC codes is expected to approach the BEC performance.

This paper also gives simple upper and lower bounds on erasure probabilities P_b and P_B as functions of ϵ and lengths of erasure patterns T, the patterns in their turn depending on boundary conditions δ, α, β, γ. Obtained bounds are not very tight, although the lower bound becomes tighter at small values of ϵ, and the upper (union) bound – at large values of ϵ

Our future work is focused on improvement of these results, namely on closing the constant gap between the lower bound and the performance curve, proposing a better upper bound than the union bound, and deriving a better analytical estimation of lengths T of erasure patterns. A comparison with other existing coding solutions for the BLEC(ϵ) (e.g. convolutional codes) should also be done.

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