

On a Multiple-Access in a Vector Disjunctive Channel

Alexey Frolov and Victor Zyablov
 Inst. for Information Transmission Problems
 Russian Academy of Sciences
 Moscow, Russia
 Email: {alexey.frolov, zyablov}@iitp.ru

Vladimir Sidorenko and Robert Fischer
 Institute of Communications Engineering
 University of Ulm
 Ulm, Germany
 Email: {vladimir.sidorenko, robert.fischer}@uni-ulm.de

Abstract—We address the problem of increasing the sum rate in a multiple-access system from [1] for small number of users. We suggest an improved signal-code construction in which in case of a small number of users we give more resources to them. For the resulting multiple-access system a lower bound on the relative sum rate is derived. It is shown to be very close to the maximal value of relative sum rate in [1] even for small number of users. The bound is obtained for the case of decoding by exhaustive search. We also suggest reduced-complexity decoding and compare the maximal number of users in this case and in case of decoding by exhaustive search.

I. INTRODUCTION

In this paper we consider a noiseless multiuser vector disjunctive (logical OR) channel which is also called Z-channel. This means that “1” is always transmitted correctly, and “0” may be replaced by “1”. Let us denote the number of active users by S , $S \geq 2$. So for some time τ the channel inputs are binary vectors $\mathbf{x}_i^{(\tau)}$, $i = 1, 2, \dots, S$, and the channel output at time τ is an elementwise disjunction of vectors at input

$$\mathbf{y}^{(\tau)} = \bigvee_{i=1}^S \mathbf{x}_i^{(\tau)}.$$

The motivation to consider the channel model is as follows. Consider a communication scheme which uses M -ary pulse position modulation (PPM). In this case the channel consists of M subchannels which correspond to either frequencies [2], [3] or time slots. To transmit the i^{th} element of an M -ary alphabet the user needs to transmit energy (e.g., a short pulse) in the i^{th} subchannel. We can say that the M -ary symbol is transmitted as a binary vector of length M and weight one. The detector at the receiver measures the energy in the i^{th} subchannel and decides if “1” or “0” was transmitted by comparing the energy with the threshold. In this case the probability of receiving “1” as “0” is much smaller than the probability of receiving “0” as “1” as we need to suppress the energy in the first case. Thus we use a vector disjunctive channel model as an idealized model for this or similar communication schemes.

The channel model is similar to the A channel [4], [5], [6], [7], but in the present paper we remove the restriction on the weight of the vectors transmitted by users (in the A channel

model users are only allowed to transmit vectors of weight one).

In [1], a signal-code construction for the multiple-access system using the A channel was introduced. The construction was based on Kautz–Singleton (KS) codes [8]. For the resulting multiple-access system a lower bound on the relative sum rate was derived and it was shown that the bound coincides with an upper bound asymptotically. The signal-code construction requires neither block synchronization nor feedback which are significant advantages of it. All the results were obtained in case of decoding by exhaustive search. Sure, the decoding algorithm is not applicable in case of codes with large dimensions. In [9], a reduced-complexity decoding of KS codes based on Reed–Solomon (RS) codes was suggested. It was done by a modification of the Guruswami–Sudan list decoder [10] or the soft-input Koetter–Vardy [11] decoder of RS codes. Unfortunately the maximal number of active users was much smaller in comparison to [1].

The major disadvantage of the multiple-access system from [1] is as follows. In case of small (in comparison to the number of subchannels) number of users the relative sum rate (or sum rate per subchannel) is very close to zero. Our main goal in the paper is to increase the relative sum rate for a small number of users. To achieve this we propose a new signal-code construction in which in case of small number of users we will give more resources to them. Let the channel consist of Q subchannels. We divide all the range of subchannels into nonoverlapping subranges of q subchannels and give $m = m(S)$ subranges to each user.

Our contribution is as follows. We propose a new signal-code construction, which is an improvement of signal-code construction from [1]. A lower bound on the relative sum rate is obtained and shown to be very close to the maximal value of sum rate in [1] even for small number of users. We find the maximal number of active users in the system. Finally, reduced-complexity decoding is suggested and the maximal number of users is found in this case.

II. BASIC SIGNAL-CODE CONSTRUCTION

A. Kautz-Singleton Codes

The Kautz-Singleton code is a concatenation of an outer (n, k) -code over \mathbb{F}_q and the following inner code. Let us

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enumerate the elements of the field \mathbb{F}_q in some order as follows

$$\mathbb{F}_q = \{ \alpha_1, \alpha_2, \dots, \alpha_q \}.$$

The inner code one-to-one maps every field element $\alpha_i \in \mathbb{F}_q$ to a binary column vector of length q having a single nonzero element at the i^{th} position. The vector positions are counted from 1 to q .

Example 1. Let $\mathbf{c} = [\alpha_2, \alpha_4, \alpha_6, \alpha_1, \alpha_2, \alpha_5]$ be a codeword of $RS(6, 2)$ over \mathbb{F}_7 . The RS codeword \mathbf{c} will be encoded into a KS codeword \mathbf{C} as follows

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

B. Transmission

Let us recall that the channel consists of Q subchannels. A time interval during which one vector of length Q is transmitted will be called a tact. Assume that all the users use the same alphabet: symbols of \mathbb{F}_q . Assume also that each user is given m subranges of q subchannels and the transmission duration (in tacts) is t .

Each user encodes information to be transmitted with the help of a $(n = mt, k, d)$ code \mathcal{C} (all users use the same code). Consider the process of sending a message by the i^{th} user. We denote the codeword to be transmitted by c_i . Let \mathbf{C}_i be a KS codeword of size $q \times mt$ corresponding to c_i . Then \mathbf{C}_i is splitted into m disjoint parts $\mathbf{C}_i^{(j)}$, $j = 1, 2, \dots, m$, of size $q \times t$ and each of the parts is sent in the corresponding subrange.

The subranges are allocated dynamically. For this purpose permutations of length Q are used. Permutations used by a current user are known to nobody except for a “transmitter-receiver” pair.

So the transmission can be seen in such a way. The word \mathbf{C}_i is splitted into m parts

$$\mathbf{C}_i = [\mathbf{C}_i^{(1)} \mathbf{C}_i^{(2)} \dots \mathbf{C}_i^{(m)}]_{q \times mt}.$$

Then a matrix \mathbf{T}_i is formed

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{C}_i^{(1)} \\ \mathbf{C}_i^{(2)} \\ \vdots \\ \mathbf{C}_i^{(m)} \\ \mathbf{0} \end{bmatrix}_{Q \times t}.$$

Before sending a binary vector, a permutation of its elements is made (a new permutation is used for each vector). In what follows we assume the permutations to be chosen equiprobably and independently from the set of all the $Q!$ possible permutations.

C. Reception

The base station sequentially receives messages from all the users. Let us consider the process of receiving a message from the i^{th} user. We assume that the base station is synchronized with transmitters of all the users. This means that t columns that correspond to a codeword sent by i^{th} user are known at the receiver. At receiving of each column the inverse permutation is performed. Thus, we obtain a matrix

$$\mathbf{Y}_i = \mathbf{T}_i \vee \left(\bigvee_{\substack{j=1, \dots, S \\ j \neq i}} \mathbf{X}_j \right),$$

where \mathbf{T}_i is a matrix corresponding to a KS codeword (\mathbf{C}_i) transmitted by the i^{th} user and matrices \mathbf{X}_j , $j = 1, \dots, S$, $j \neq i$, are the results of another users activity. Note that matrices \mathbf{X}_j may not contain whole codewords sent by another users.

Example 2. Let $m = 1$, $Q = q = 7$. If, for example, the code matrix \mathbf{C} from Example 1 was transmitted then we can receive the following matrix \mathbf{Y}_i

$$\mathbf{Y}_i = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the channel does not touch the “1”s of the matrix \mathbf{C} but replaces 8 zeros by “1”.

The decoding problem in this case is equivalent to decoding problem for KS code. For our basic signal-code construction we use the decoding by exhaustive search. Consider the codeword $c_l \in \mathcal{C}$. We need to construct a matrix \mathbf{T}_l corresponding to c_l in the manner described above. Since the described multiple-access system uses a disjunctive channel all the elements of the matrix at the channel output corresponding to the codeword sent by the i^{th} user will be non-zero. Therefore, the assumption that the codeword $c_l \in \mathcal{C}$ was transmitted by the i^{th} user is true only if the condition follows

$$\mathbf{T}_l \wedge \mathbf{Y}_i = \mathbf{T}_l, \quad (1)$$

where \wedge is an element-wise conjunction of matrices.

To decode we need to check the condition (1) for all the words of \mathcal{C} . If the list of codewords satisfying the decoding condition consists of only one word, then the decoder outputs the word; if the list of codewords consists of several words then the decoder outputs a decoding failure (decoding error is not possible in this case).

Remark 1. Note, that the presence of a block synchronization is not needed here like for the system from [1].

Remark 2. In a real system for permutations to be known both on the transmitter and the receiver it is advisable to use pseudorandom number generators, which are a part of any system based on frequency hopping [12].

D. Probability of failure

Let us estimate the probability p_* of decoding failure for the i^{th} user. Let $\beta = 1 - \left(1 - \frac{m}{Q}\right)^{S-1}$.

Theorem 1.

$$p_* \leq \sum_{W=d}^n [A(W) \beta^W] < q^k \beta^d,$$

where $A(W)$ is the number of codewords of weight W in the code C .

Proof: Let the i^{th} user send a codeword c_i , let the matrix \mathbf{T}_i correspond to it. The existence of at least one codeword $c^* \neq c_i$ such that the decoding condition follows for a matrix \mathbf{T}^* corresponding to it is sufficient for the decoder to output a failure.

Consider some codeword $c' \neq c_i$ and let $D = d(c_i, c')$. The codeword c' will be included in the list of codewords satisfying the decoding condition if and only if a matrix \mathbf{T}' is covered by “1”s in \mathbf{Y}_i .

We can state that at least $n - D$ of n positions to be checked are non-zero as codewords c' and c_i coincide on these positions. Thus a codeword c' will be included in the list if the remaining D positions are non-zero.

Let us consider one column and let the column contain l of D positions (which should be checked) of a codeword c' . Let us enumerate these positions. We are interested in the probability $P\left(\bigcup_{j=1}^l A_j\right)$, where A_j is an event consisting in the fact that the position with number j is covered. The probability can be calculated as follows

$$P\left(\bigcup_{j=1}^l A_j\right) = P(A_1) P(A_2|A_1) \cdot \dots \cdot P(A_l|A_1 \dots A_{l-1}).$$

Note that as random independent equiprobable permutations are used

$$P(A_1) = \beta,$$

and

$$P(A_b|A_1 \dots A_{b-1}) \leq \beta, \quad b = 2 \dots l.$$

Thus, we obtain

$$P\left(\bigcup_{j=1}^l A_j\right) \leq \beta^l.$$

As a result the probability of a codeword c' to be included in a list is less or equal to β^D . After getting a sum over all the codewords (except c_i) we obtain the needed result. ■

E. Minimal number of tacts

In this paragraph we estimate the minimal number of tacts t needed to transmit k information symbols with given probability of failure (p_r). Let Q, q, S, m, k and p_r be fixed.

From Theorem 1 we see that if

$$d \geq \frac{k \log_2 q - \log_2 p_r}{-\log_2 \beta}, \quad (2)$$

then $p_* < p_r$. We choose the smallest d satisfying (2), i.e.

$$d = \left\lceil \frac{k \log_2 q - \log_2 p_r}{-\log_2 \beta} \right\rceil.$$

In accordance to the Gilbert–Varshamov bound if n, d and k satisfy the inequality

$$n \geq k + \log_q \left[\sum_{i=0}^{d-2} \binom{n-1}{i} (q-1)^i \right],$$

then there exist a code with such parameters. Thus, we choose n in such a way

$$n = \left\lceil \frac{\log_2 q}{\log_2 q - 1} (k + d - 1) \right\rceil.$$

Now we obtain the estimate on the minimal number of tacts needed in this case:

$$\begin{aligned} t &= \left\lceil \frac{n}{m} \right\rceil \\ &= \left\lceil \frac{\log_2 q}{\log_2 q - 1} \left(\frac{k (\log_2 q - \log_2 \beta) - \log_2 p_r}{m (-\log_2 \beta)} \right) \right\rceil. \end{aligned}$$

F. Maximal number of users

Let Q, q, m, t, k, d and p_r be fixed. In this paragraph we estimate the maximal number of users for which $p_* < p_r$. Directly from Theorem 1 we obtain

$$S \leq \frac{-\ln \left(1 - \frac{\sqrt[t]{p_r}}{q^{R/\delta}}\right)}{-\ln \left(1 - \frac{m}{Q}\right)} + 1,$$

where $\delta = d/n, R = k/n$.

And thus

$$S_{\max} \geq \left\lceil \frac{-\ln \left(1 - \frac{\sqrt[t]{p_r}}{q^{R/\delta}}\right)}{-\ln \left(1 - \frac{m}{Q}\right)} \right\rceil + 1. \quad (3)$$

Remark 3. Using the inequalities for logarithm

$$|x| \leq -\ln(1 - |x|) \leq \frac{|x|}{1 - |x|},$$

we obtain

$$S_{\max} \geq \left\lceil \left(\frac{Q}{m} - 1 \right) \frac{\sqrt[t]{p_r}}{q^{R/\delta}} \right\rceil + 1.$$

G. Sum rate

Let us introduce some notions. The *rate for one user* (in bits per tact)

$$R_i(Q, q, S, m, k, p_r) = \frac{k}{t} \log_2 q.$$

By analogy with the rate of a single user, we define the *sum rate* of all active users as the amount of information (in bits) transmitted in a system during one tact. Since users transmit information independently, this value can be computed as follows:

$$R_\Sigma(Q, q, S, m, k, p_r) = S \frac{k}{t} \log_2 q.$$

Relative sum rate (the rate per subchannel)

$$\rho(Q, q, S, m, k, p_r) = \frac{R_\Sigma}{Q}.$$

In Fig. 1 the dependency of ρ on m is shown. The parameters are chosen as follows: $Q = 4096$, $q = 64$, $p_r = 10^{-10}$, $k = 120$. We see that there is a maximum of relative sum rate at some m .

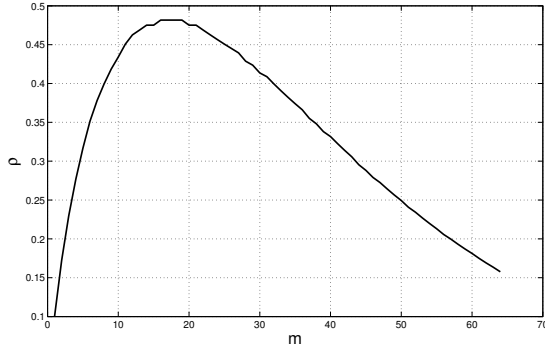


Fig. 1. Dependency of ρ on m

Now define

$$\rho^*(Q, q, S, k, p_r) = \max_{1 \leq m \leq Q/q} [\rho(Q, q, S, m, k, p_r)].$$

In Fig. 2 the dependency of ρ^* on S is shown. The parameters are chosen as follows: $Q = 4096$, $p_r = 10^{-10}$, k is chosen so that $k \log_2 q = 720$ (bits).

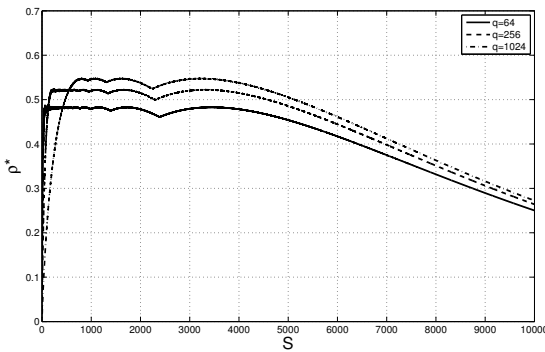


Fig. 2. Dependency of ρ^* on S

Now we derive an asymptotic estimate of a sum rate.

Requiring p_r to decrease exponentially with n (in other words, assuming $p_r = 2^{-cn}$, $c > 0$) and assuming that k is chosen such that $\frac{k}{m} \rightarrow \infty$, $\frac{k}{m} = o(Q)$, we obtain

$$t \sim \left(\frac{\log_2 q}{\log_2 q - 1} \right) \frac{k}{m} \left(\frac{\log_2 q - \log_2 \beta}{-\log_2 \beta - c'} \right), \quad (4)$$

where $c' = c \frac{\log_2 q}{\log_2(q-1)}$.

Remark 4. Note, that the transmission time t here is much better than in [1]. Also note, that unlike [1] k should be chosen large.

Let $\mu = m/Q$, $1/Q \leq \mu \leq 1/q$. Let us introduce an asymptotic quantity

$$\rho_\infty(q, S, \mu, k, c) = \lim_{Q \rightarrow \infty} \rho(Q, q, S, m, k, p_r). \quad (5)$$

After substituting (4) into (5) we obtain

$$\begin{aligned} \rho_\infty(q, S, \mu, k, c) &\geq \underline{\rho}_\infty(q, S, \mu, c) \\ &= S \mu (-\log_2 \beta - c') \frac{\log_2 q - 1}{\log_2 q - \log_2 \beta}. \end{aligned}$$

Let us introduce one more quantity

$$\underline{\rho}_\infty^*(q, S, c) = \max_{0 < \mu \leq 1/q} \{ \underline{\rho}_\infty(q, S, \mu, c) \}.$$

Let ε be an arbitrarily small positive value, the dependency of $\underline{\rho}_\infty^*(q, S, \varepsilon)$ is shown in Fig. 3.

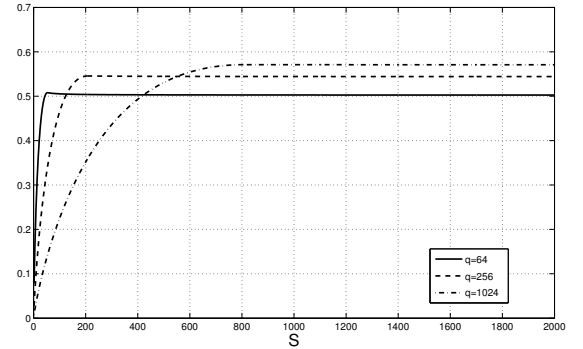


Fig. 3. Dependency of $\underline{\rho}_\infty^*$ on S

Let $\hat{\mu} = 1 - S^{-1} \sqrt{\frac{1}{2}}$, then

$$\underline{\rho}_\infty^*(q, S, c) \geq \begin{cases} \underline{\rho}_\infty(q, S, \hat{\mu}, c), & \hat{\mu} < \frac{1}{q} \\ \underline{\rho}_\infty(q, S, 1/q, c), & \text{otherwise} \end{cases}$$

Remark 5. Note, that the condition $\hat{\mu} < \frac{1}{q}$ holds when

$$S > \frac{1}{-\log_2(1 - 1/q)} + 1,$$

so after strengthening it we obtain

$$S > q \ln 2 + 1.$$

Remark 6. Note, that

$$\underline{\rho}_\infty(q, S, \hat{\mu}, \varepsilon) \geq (1 - \varepsilon') \frac{\log_2 q - 1}{\log_2 q + 1} \ln 2,$$

where $\varepsilon' = S\varepsilon$.

III. REDUCED-COMPLEXITY DECODING

As in the proposed scheme k should be chosen large, then the exhaustive search over q^k codewords is not applicable in practice. In this section we propose a reduced-complexity decoding for the signal-code construction.

Let us choose the code \mathcal{C} of length $n = mt$ to be concatenated, i.e., $\mathcal{C} = \mathcal{C}_O \diamond \mathcal{C}_I$, where \mathcal{C}_O is an outer (m, k_O, d_O) -code over $GF(q^{k_I})$, \mathcal{C}_I is an inner (t, k_I, d_I) -code over $GF(q)$.

We will decode the code in such a way. First we will decode all the inner codes independently by exhaustive search (as described above). We choose k_I to be small to have small number of words. Then we will correct the erasures by the outer code. In what follows we assume that the erasure correction is done by means of gaussian elimination (the complexity is $O(m^3)$) to get theoretical results, but in practise it is better to use codes with simple decoding algorithms (e.g., low-density parity-check (LDPC) codes).

Let us denote by $p_*^{(I)}$ the probability of failure for one inner code. Then in accordance to the Chernoff bound

$$p_* \leq \min_{s>0} \left\{ e^{-sd_O} \left[1 + p_*^{(I)} (e^s - 1) \right]^m \right\}.$$

Thus the largest value of $p_*^{(I)}$ for which the requirement $p_* < p_r$ is satisfied can be calculated as follows

$$\hat{p}_*^{(I)} = \max_{s>0} \left\{ \frac{\sqrt[m]{p_r} e^{s\delta_O} - 1}{e^s - 1} \right\},$$

where δ_O is the relative minimum distance of \mathcal{C}_O , i.e., $\delta_O = d_O/m$.

After substituting of $\hat{p}_*^{(I)}$ to (3) we obtain the lower bound on the maximal number of users in case of reduced-complexity decoding

$$S_{\max} \geq \left\lceil \frac{-\ln \left(1 - \frac{d_I \sqrt[m]{\hat{p}_*^{(I)}}}{q^{k_I/\delta_I}} \right)}{-\ln \left(1 - \frac{m}{Q} \right)} \right\rceil + 1.$$

In Fig. 4 the dependency of the maximal number of users on the rate of concatenated code is shown for $Q = 2^{18}$, $p_r = 10^{-10}$, $m = 200$, $t = 50$, $q = 64$, $k_I = \{1, 2, 3, 4\}$. The comparison with the decoding by exhaustive search is made.

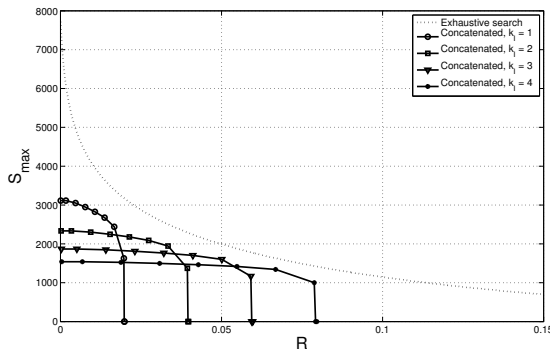


Fig. 4. Comparison with the decoding by exhaustive search

We see that the number of users in case of concatenated construction is smaller in comparison with the decoding by exhaustive search. But at the same time we significantly gain in the complexity of decoding.

IV. CONCLUSION

In the present paper, a novel signal-code construction for a multiple-access system using a disjunctive vector channel is proposed. The construction is an improvement of signal-code construction from [1] and it also requires neither block synchronization nor feedback. The main advantage of the signal-code construction in comparison to the construction from [1] is that the sum rate in case of small (in comparison to the number of subchannels) number of users is increased. To achieve this we divide all the range of subchannels into nonoverlapping subranges of subchannels and give $m = m(S)$ subranges to each user, where S is the number of users. A lower bound on the relative sum rate for the resulting multiple-access system is obtained and shown to be very close to the maximal value of sum rate in [1] even for small number of users. A lower bound on the maximal number of active users in the system is derived. These two bounds are obtained for the case of decoding by exhaustive search. Reduced-complexity decoding is suggested. The maximal number of users for reduced-complexity decoding is calculated and compared with the maximal number of users in the case of decoding by exhaustive search.

REFERENCES

- [1] D. Osipov, A. Frolov, V. Ziyabov. Multiple Access System for a Vector Disjunctive Channel. *Problems of Information Transmission*, vol. 48, no. 3 pp. 243–249, 2012.
- [2] T. Tsang, M. El-Gamal. Ultra-Wideband (UWB) Communications Systems: An Overview. In *3rd International IEEE-NEWCAS Conference*, pp. 381–386, Jun. 2005.
- [3] K. Witrals, G. Leus, G. Janssen, M. Pausini, F. Troesch, T. Zasowski, J. Romme. Noncoherent Ultra-Wideband Systems. *IEEE Signal Processing Magazine*, pp. 48–66, July 2009.
- [4] S.-C. Chang, J. Wolf. On the t -User m -Frequency Noiseless Multiple-Access Channel with and without Intensity Information. *IEEE Transactions on Information Theory*, pp. 41–48, Jan. 1981.
- [5] L. Wilhelmsson, K. S. Zigangirov. On the Asymptotic Capacity of a Multiple-Access Channel. *Problems of Information Transmission*, pp. 12–20, 1997.
- [6] L. A. Bassalygo, M. S. Pinsker. Evaluation of the Asymptotics of the Summarized Capacity of an m -Frequency t -User Noiseless Multiple-Access Channel. *Problems of Information Transmission*, pp. 3–9, 2000.
- [7] A. Han Vinck, K. Keuning. On the Capacity of the Asynchronous t -User m -Frequency Noiseless Multiple-Access Channel without Intensity Information. *IEEE Transactions on Information Theory*, pp. 2235–2238, Nov. 1996.
- [8] W. Kautz, R. Singleton. Nonrandom Binary Superimposed Codes. *IEEE Transactions on Information Theory*, pp. 363–377, 1964.
- [9] V. Sidorenko, R. Fischer. Low-Complexity List Decoding of Reed-Solomon Coded Pulse Position Modulation. In *9th Int. ITG Conference on Systems, Communication and Coding*, Jan. 21–24, 2013, Munich, Germany.
- [10] V. Guruswami, M. Sudan. Improved Decoding of Reed-Solomon and Algebraic-Geometry Codes. *IEEE Transactions on Information Theory*, pp. 1757–1767, 1757–1767, Sept. 1999.
- [11] R. Koetter, A. Vardy. Algebraic Soft-Decision Decoding of Reed-Solomon Codes. *IEEE Transactions on Information Theory*, pp. 2809–2825, Nov. 2003.
- [12] K. Zigangirov. *Theory of Code Division Multiple Access Communication*. IEEE Press, 2004.