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A New Outer Bound on the Capacity Region of a Class of Z-interference Channels

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***Abstract*—Following the work of Liu and Goldsmith in 2009, we study a class of Z-interference channels that satisfy the shift-invariant condition but not the maximum entropy condition. We provide a new capacity region outer bound for this class of Z-interference channels. We show the tightness of the proposed outer bound by finding the capacity region of certain Z-interference channels which were not previously known.**

I. Introduction

The interference channel, introduced in [1], is a simple network consisting of two pairs of transmitters and receivers. Each pair wishes to communicate at a certain rate with negligible probability of error. However, the two communications interfere with each other. The problem of finding the capacity region of the interference channel is difficult and therefore remains essentially open except in some special cases [2].

The Z-interference channel is an interference channel where one transmitter-receiver pair is interference-free. Though finding the capacity region of the Z-interference channel is a simpler problem than that of the interference channel, capacity results are still limited, with the following exceptions: the capacity region of the Z-interference channel is known when the interference is deterministic [3, Section IV] or when the channel satisfies the shift-invariant condition and the maximum entropy condition [4]. The sum capacity of the Z-interference channel is known when the Z-interference channel is Gaussian [5], or when the interference-free link is also noise-free [6], although in both cases, the full capacity region has not been characterized.

In this paper, we study a class of Z-interference channels that satisfy the shift-invariant condition but *not* the maximum entropy condition. For these Z-interference channels, we develop a new capacity region outer bound. The converse techniques that we use include the introduction of imaginary channels [4], the single-letterization technique [7, page 314], and the technique of replacing two auxiliary random variables with one [8]. We show that for the class of Z-interference channels considered in this paper, the new outer bound is strictly tighter than the outer bound provided in [4] for Z-interference channels that satisfy the shift-invariant condition. Furthermore, we show the usefulness of this new outer bound by finding the capacity region of certain Z-interference channels which were not previously known. For these Z-interference channels, we show that superposition encoding and partially decoding the interference [9] is optimal.

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II. System Model

Consider a Z-interference channel, see Fig. 1, characterized by $p(y_1|x_1)$ and $p(y_2|x_1,x_2)$ with input alphabets $\mathcal{X}_1$, $\mathcal{X}_2$ and output alphabets $\mathcal{Y}_1$, $\mathcal{Y}_2$. There are two independent messages W_1 and W_2 which are uniform on sets $\{1,2,\cdots,N_1\}$ and $\{1,2,\cdots,N_2\}$, respectively. W_i is known at Transmitter i and is intended for Receiver i, $i=1,2$. An (N_1,N_2,n,ϵ_n) code for this channel consists of a sequence of two encoding functions:

$$f_i^n : \{1,2,\cdots,N_i\} \rightarrow \mathcal{X}_i^n, \quad i=1,2$$

and two decoding functions:

$$g_i^n : \mathcal{Y}_i^n \rightarrow \{1,2,\cdots,N_i\}$$

with probability of error ϵ_n defined as

$$\max_{i=1,2} \frac{1}{N_1 N_2} \sum_{w_1,w_2} \Pr\left[g_i^n(Y_i^n) \neq w_i | W_1=w_1, W_2=w_2\right]$$

A rate pair (R_1,R_2) is said to be achievable if there exists a sequence of $\left(2^{nR_1},2^{nR_2},n,\epsilon_n\right)$ codes such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. The capacity region of the Z-interference channel is the closure of the set of all achievable rate pairs.

The class of Z-interference channels that we focus on in this paper satisfy the following two conditions. Both conditions are placed on the multiple access link between the two transmitters and Receiver 2, i.e., $p(y_2|x_1,x_2)$.

Condition 1: The multiple access link of the Z-interference channel, i.e., $p(y_2|x_1,x_2)$, has the structure shown in Fig. 2. More specifically, there exist a random variable T taking values in $\mathcal{T}$ and a deterministic function f such that

1) $p(y_2|x_1,x_2)$ can be expressed as

$$p(y_2|x_1,x_2) = \sum_{t \in \mathcal{T}} V(y_2|t,x_2) p(t|x_1)$$

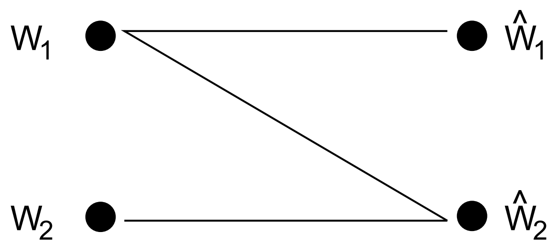

Fig. 1. The Z-interference Channel.

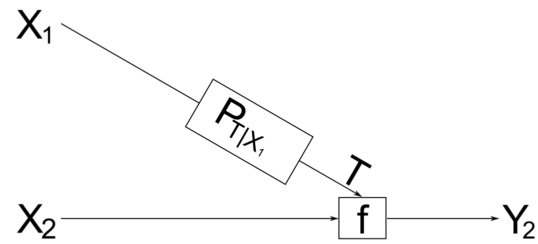

Fig. 2. The multiple access link $p(y_2|x_1, x_2)$.

where $V(y_2|t, x_2)$ is equal to 1 when $y_2 = f(x_2, t)$ and 0 otherwise.

2) The function $Y_2 = f(X_2, T)$ satisfies that there exists a function h such that $T = h(Y_2, X_2)$.

Condition 2: For any $n = 1, 2, \cdots$, the optimal distribution that solves the following optimization problem

$$\max_{p(x_2^n)} \quad H(Y_2^n) \tag{1}$$

is $p^*(x_2^n) = \prod_{i=1}^n p^*(x_{2i})$, i.e., the optimizing distribution is an independent and identically distributed (i.i.d.) distribution according to a single-letter distribution $p^*(x_2)$, irrespective of the distribution $p(t^n)$.

Remark 1: Using functional representation lemma [2], random variable T and deterministic function f that satisfies 1) of Condition 1 always exist.

Remark 2: Condition 1 is similar to the conditions on f_1 and f_2 in [3]. While in [3] the channel from X_1 to T is deterministic, here, we allow for random $p(t|x_1)$. However, the channels studied in this paper also have to satisfy Condition 2.

The shift-invariant condition and the maximum entropy condition defined in [4] are highly relevant to Conditions 1 and 2 defined in this paper. Thus, we repeat the definitions of the conditions in [4] here.

Shift-invariant condition: for any $n = 1, 2, \cdots$, $H(Y_2^n|X_2^n = x_2^n)$, when evaluated with the distribution $\sum_{x_1^n} p(x_1^n)p(y_2^n|x_1^n, x_2^n)$, is independent of x_2^n for any $p(x_1^n)$.

Maximum entropy condition: define τ as

$$\tau = \max_{p(x_1)p(x_2)} H(Y_2),$$

there exists a $p^*(x_2)$ such that $H(Y_2)$, when evaluated with the distribution $\sum_{x_1,x_2} p(x_1)p^*(x_2)p(y_2|x_1, x_2)$, is equal to τ for any $p(x_1)$.

We now show that the class of Z-interference channels that satisfy Condition 1 also satisfy the shift-invariant condition. For Z-interference channels that satisfy Condition 1, we have

$$\begin{aligned}
&H(Y_2^n|X_2^n = x_2^n) \\
&= H(Y_2^n, T^n|X_2^n = x_2^n) && (2) \\
&= H(T^n|X_2^n = x_2^n) + H(Y_2^n|X_2^n = x_2^n, T^n) \\
&= H(T^n) && (3)
\end{aligned}$$

which is not a function of x_2^n, where (2) follows because the Z-interference channel satisfies Condition 1, i.e., once we know (X_2, Y_2), we know T, and (3) follows from the structure of Fig. 2. Thus, Z-interference channels that satisfy Condition 1 also satisfy the shift-invariant condition.

III. Examples

In this section, we provide an example of the multiple access link $p(y_2|x_1, x_2)$ that satisfies Conditions 1 and 2. We call this example the modulo-sum erasure channel.

For the modulo-sum erasure channel, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{T} = \mathcal{Y}_2 = \{0, e, 1\}$, and $p(t|x_1)$ and $\mathcal{X}_1$ are both arbitrary. The deterministic channel between (X_2, T) and Y_2 is such that

$$Y_2 = \begin{cases} X_2 \oplus T, & \text{if } T \neq e \\ e, & \text{if } T = e. \end{cases}$$

where $\oplus$ is the modulo-2 sum.

It is easy to see that the modulo-sum erasure channel satisfies Condition 1. We now argue that the i.i.d. distribution uniform on $\mathcal{X}_2 = \{0, 1\}$ achieves the maximization in (1), which means that the modulo-sum erasure channel also satisfies Condition 2. Let us define the random variable $\bar{T}$, which is a deterministic function of T, as

$$\bar{T} = \begin{cases} 0, & \text{if } T \neq e \\ 1, & \text{if } T = e \end{cases}$$

$\bar{T}$ is the indicator whether T is equal to e, and $\bar{T}^n$ contains the information about the erasure positions in T^n. Denote $p(\bar{t}|t)$ as the deterministic channel from T to $\bar{T}$. Note that once we know Y_2^n, though T^n is not completely known, $\bar{T}^n$ is known. Thus, we have

$$\begin{aligned}
H(Y_2^n) &= H(Y_2^n, \bar{T}^n) \\
&= H(\bar{T}^n) + H(Y_2^n|\bar{T}^n) \\
&= H(\bar{T}^n) + \sum_{\bar{t}^n \in \{0,1\}^n} H(Y_2^n|\bar{T}^n = \bar{t}^n)\Pr[\bar{T}^n = \bar{t}^n]. && (4)
\end{aligned}$$

We will next show that the i.i.d. distribution uniform on $\mathcal{X}_2 = \{0, 1\}$ maximizes $H(Y_2^n|\bar{T}^n = \bar{t}^n)$ for each $\bar{t}^n \in \{0, 1\}^n$. Let k be the number of 1's contained in $\bar{t}^n$, $0 \leq k \leq n$. Then, given $\bar{T}^n = \bar{t}^n$, Y_2^n can take at most 2^{n-k} many values, thus,

$$H(Y_2^n|\bar{T}^n = \bar{t}^n) \leq n - k \tag{5}$$

with equality when X_2^n is the i.i.d. distribution uniform on $\mathcal{X}_2 = \{0, 1\}$. Hence, when X_2^n takes the i.i.d. distribution uniform on $\{0, 1\}$, $H(Y_2^n|\bar{T}^n = \bar{t}^n)$ is maximized for each $\bar{t}^n \in \{0, 1\}^n$ and therefore, according to (4), $H(Y_2^n)$ is maximized. Thus, the modulo-sum erasure channel satisfies Conditions 1 and 2 in this paper.

We will now show that the modulo-sum erasure channel does not satisfy the maximum entropy condition. For any $p(x_1)p(x_2)$, we have

$$H(Y_2) = H(\bar{T}) + \sum_{\bar{t}\in\{0,1\}} H(Y_2|\bar{T}=\bar{t})\Pr[\bar{T}=\bar{t}] \tag{6}$$

$$\begin{aligned} &= H(\bar{T}) + \Pr[\bar{T}=0]\cdot H(X_2\oplus T|T\neq e) \\ &\quad + \Pr[\bar{T}=1]\cdot 0 \end{aligned} \tag{7}$$

where (6) follows from the same reasoning as (4). Hence, from (7) we see that the maximum value of $H(Y_2)$ over $p(x_2)$ is $H(\bar{T})+\Pr[\bar{T}=0]$, which is in fact a function of $p(x_1)$ and is not equal to its maximum value over all $p(x_1)p(x_2)$. Hence, the modulo-sum erasure channel does not satisfy the maximum entropy condition.

The modulo-sum erasure channel is an example to show that there are Z-interference channels that satisfy Conditions 1 and 2, but do not satisfy the maximum entropy condition. For these Z-interference channels, the capacity region is not found in [4] and is generally unknown.

IV. Converse

In this section, we provide an outer bound on the capacity region of a Z-interference channel that satisfies Conditions 1 and 2.

First define $\mathcal{P}$ to be the set of distributions that satisfy

$$\begin{aligned} &p(u,v,x_1,\tilde{x}_2,y_1,t,\tilde{y}_2) \\ &= p(u,v)p(x_1|u)p^*(\tilde{x}_2)p(t|x_1)p(y_1|x_1)V(\tilde{y}_2|t,\tilde{x}_2) \end{aligned} \tag{8}$$

Define $\mathcal{P}_1$ to be a subset of $\mathcal{P}$ where the distributions satisfy $I(U;Y_1|V) < I(U;\tilde{Y}_2|V)$, and define $\mathcal{P}_2$ to be a subset of $\mathcal{P}$ where the distributions satisfy $I(U;Y_1|V) \geq I(U;\tilde{Y}_2|V)$. From the definitions, we see that $\mathcal{P}_1$ and $\mathcal{P}_2$ form a partition of $\mathcal{P}$. For $p\in\mathcal{P}$, define the two-dimensional region $\mathcal{R}_1(p)$ to be

$$\begin{aligned} \mathcal{R}_1(p) = \big\{&(R_1,R_2)| \\ &R_1 \leq \min\Big(I(X_1;Y_1), I(X_1;Y_1|V)+I(V;\tilde{Y}_2), \\ &\qquad I(X_1;Y_1|U)+I(U;T)\Big) \\ &R_2 \leq H(\tilde{Y}_2|V)-H(T|U)-I(U;Y_1|V) \end{aligned}$$

and the two-dimensional region $\mathcal{R}_2(p)$ to be

$$\begin{aligned} \mathcal{R}_2(p) = \big\{&(R_1,R_2)| \\ &R_1 \leq \min\left(I(X_1;Y_1), I(X_1;Y_1|U)+I(U;T)\right) \\ &R_2 \leq H(\tilde{Y}_2|U)-H(T|U) \\ R_1+R_2 &\leq H(\tilde{Y}_2)+I(X_1;Y_1|U)-H(T|U)\big\}, \end{aligned}$$

The main result of the paper is the following theorem which provides an outer bound on the capacity region of a Z-interference channel that satisfies Conditions 1 and 2.

Theorem 1: For a Z-interference channel that satisfies Conditions 1 and 2, an outer bound to the capacity region is

$$\mathcal{R}_O \triangleq \left(\bigcup_{p\in\mathcal{P}_1}\mathcal{R}_1(p)\right)\bigcup\left(\bigcup_{p\in\mathcal{P}_2}\mathcal{R}_2(p)\right) \tag{9}$$

Proof: The proof is provided in Section VII. ■

Remark 3: The reason why we used $\tilde{X}_2$ and $\tilde{Y}_2$, rather than X_2 and Y_2, in (8) is because the distribution of X_2 is taken to be the $p^*(x_2)$ distribution. The corresponding output at Receiver 2 is denoted as $\tilde{Y}_2$ to emphasize that it is the output when X_2 distributed as $p^*(x_2)$. Similar notations are used in the proof of Theorem 1 in Section VII.

A capacity region outer bound was given in [4, Theorem 4] for Z-interference channels that satisfy the shift-invariant condition. Z-interference channels that satisfy Condition 1 also satisfy the shift-invariant condition. Thus, applying the outer bound in [4, Theorem 4] to Z-interference channels that satisfy Condition 1, we obtain

$$R_1 \leq \min\left(I(X_1;Y_1|Q), I(X_1;Y_1|U,Q)+I(U;T|Q)\right) \tag{10}$$

$$R_2 \leq H(Y_2|Q)-H(T|U,Q) \tag{11}$$

$$R_1+R_2 \leq H(Y_2|Q)+I(X_1;Y_1|U,Q)-H(T|U,Q) \tag{12}$$

for some $p(q)p(x_1,u|q)p(x_2|q)$ where the distributions satisfy

$$\begin{aligned} &p(q,u,x_1,x_2,y_1,t,y_2) \\ &= p(q)p(x_1,u|q)p(x_2|q)p(y_1|x_1)p(t|x_1)V(y_2|t,x_2) \end{aligned} \tag{13}$$

Taking $Q=\phi$ and $p(x_2)=p^*(x_2)$, the distribution in (13) becomes

$$\begin{aligned} &p(u,x_1,\tilde{x}_2,y_1,t,\tilde{y}_2) \\ &= p(u,x_1)p^*(\tilde{x}_2)p(y_1|x_1)p(t|x_1)V(\tilde{y}_2|t,\tilde{x}_2), \end{aligned} \tag{14}$$

and we obtain an inner bound to the outer bound (10)-(12) as

$$\mathcal{R}_{IO} \triangleq \bigcup_{p\in\mathcal{P}_1\bigcup\mathcal{P}_2}\mathcal{R}_3(p) \tag{15}$$

where the two-dimensional region $\mathcal{R}_3(p)$ is defined as

$$\begin{aligned} \mathcal{R}_3(p) = \big\{&(R_1,R_2)| \\ &R_1 \leq \min\left(I(X_1;Y_1), I(X_1;Y_1|U)+I(U,T)\right) \\ &R_2 \leq H(\tilde{Y}_2)-H(T|U) \\ R_1+R_2 &\leq H(\tilde{Y}_2)+I(X_1;Y_1|U)-H(T|U)\big\}, \end{aligned}$$

Notice that in evaluating $\mathcal{R}_{IO}$, taking the union over distributions of the form (14) is sufficient. In (15), we have added the random variable V in the distribution, i.e., $p\in\mathcal{P}_1\bigcup\mathcal{P}_2$, for comparison purposes with $\mathcal{R}_O$.

Due to conditioning reduces entropy, we have $\mathcal{R}_2(p)\subseteq\mathcal{R}_3(p)$ for $p\in\mathcal{P}_2$ and $\mathcal{R}_1(p)\subseteq\mathcal{R}_3(p)$ for $p\in\mathcal{P}_1$. Hence, we have

$$\mathcal{R}_O \subseteq \mathcal{R}_{IO}$$

Thus, we conclude that the derived outer bound $\mathcal{R}_O$ is tighter than the outer bound in [4] applied to Z-interference channels that satisfy Condition 1.

V. ACHIEVABILITY

The following is an achievable region for the Z-interference channel that satisfies Conditions 1 and 2.

Theorem 2: An inner bound to the capacity region of a Z-interference channel that satisfies Condition 1 and 2 is

$$\mathcal{R}_I \triangleq \bigcup_{p \in \mathcal{P}_1 \bigcup \mathcal{P}_2} \mathcal{R}_2(p) \tag{16}$$

Proof: The proof is omitted due to space limitations. ■

The region in Theorem 2 is the Han/Kobayashi achievable region [9] specialized to the Z-interference channel that satisfies Conditions 1 and 2. Again, in evaluating $\mathcal{R}_I$, taking the union over distributions of the form (14) is sufficient. The reason why we added a random variable V into the description of the inner bound in (16), i.e., $p \in \mathcal{P}_1 \bigcup \mathcal{P}_2$, is for comparison purposes with the outer bound in (9).

VI. SOME CAPACITY RESULTS

In this section, we demonstrate the usefulness of our outer bound by finding the capacity region of certain Z-interference channels which were not previously known.

For Z-interference channels that satisfy Conditions 1 and 2, comparing the inner and outer bounds of the capacity region in (16) and (9), we see for distribution $p \in \mathcal{P}_2$, the pentagon $\mathcal{R}_2(p)$ is achievable. So if a corner point in $\mathcal{R}_2(p)$ for some $p \in \mathcal{P}_2$ happens to be on the boundary of the outer bound $\mathcal{R}_O$, then, it is a capacity point, i.e., it is a point on the capacity region.

More specifically, if the channel is such that the distribution set $\mathcal{P}_1$ is empty, then the inner and outer bounds coincide, yielding the capacity region. Thus, we have the following theorem.

Theorem 3: For Z-interference channels that satisfy Conditions 1 and 2, if the channel further satisfies that $I(U;Y_1) \geq I(U;\tilde{Y}_2)$ for all $p(u,x_1)$, where the mutual informations are evaluated using (14), then its capacity region is

$$\mathcal{C} = \bigcup_{p \in \mathcal{P}_2} \mathcal{R}_2(p),$$

and superposition encoding and partially decoding the interference is optimal.

Remark 4: In Theorem 3, $I(U;Y_1) \geq I(U;\tilde{Y}_2)$ for all $p(u,x_1)$ is equivalent to saying that the channel $p(y_1|x_1)$ is less noisy than the channel $p(\tilde{y}_2|x_1) = \sum_{x_2} p^*(x_2)p(\tilde{y}_2|x_1,x_2)$. Channel $p(\tilde{y}_2|x_1)$ is the channel from X_1 to Y_2 where X_2 is regarded as noise with distribution $p^*(x_2)$.

VII. PROOF OF THEOREM 1

For any rate pair (R_1,R_2) that is achievable, there exist two sequences of codebooks 1 and 2, denoted by $\mathcal{C}_1^n$ and $\mathcal{C}_2^n$, of rates R_1 and R_2, and probability of error less than ϵ_n, where $\epsilon_n \to 0$ as $n \to \infty$. Let X_1^n and X_2^n be uniformly distributed on codebooks 1 and 2, respectively. Let Y_1^n be the output of the discrete memoryless channel (DMC) $p(y_1|x_1)$ with the input X_1^n, T^n be the output of the DMC $p(t|x_1)$ with the input X_1^n, and Y_2^n be the output of the DMC $p(y_2|x_1,x_2)$ with the inputs X_1^n and X_2^n. We have

$$\begin{aligned} nR_1 &= H(W_1) \\ &\leq I(W_1;Y_1^n) + n\epsilon_n && (17) \\ &= H(Y_1^n) - \sum_{i=1}^n H(Y_{1i}|X_{1i}) + n\epsilon_n && (18) \\ &\leq \sum_{i=1}^n H(Y_{1i}) - \sum_{i=1}^n H(Y_{1i}|X_{1i}) + n\epsilon_n && (19) \\ &= \sum_{i=1}^n I(X_{1i};Y_{1i}) && (20) \end{aligned}$$

where (17) follows from Fano's inequality and (19) follows from conditioning reduces entropy. We also have

$$\begin{aligned} nR_2 &= H(W_2) \\ &\leq I(W_2;Y_2^n) + n\epsilon_n && (21) \\ &= H(Y_2^n) - H(Y_2^n|X_2^n) + n\epsilon_n && (22) \\ &= H(Y_2^n) - H(T^n) + n\epsilon_n && (23) \end{aligned}$$

where (21) follows from Fano's inequality, (18) and (22) follow from the fact that without loss of generality, we may consider deterministic encoders f_1^n and f_2^n, and (23) follows from (3).

Next, we define imaginary channels and random variables. The purpose of doing this is to utilize Conditions 1 and 2 and define auxiliary random variables during single-letterization that are independent to X_2^n. First, define $\tilde{X}_2^n$ to be i.i.d. according to the distribution $p^*(x_2)$. Further define $\tilde{Y}_2^n$ to be the output of T^n and $\tilde{X}_2^n$ into the deterministic channel $V(y_2|t,x_2)$. Note that $\tilde{Y}_2^n$ is independent to X_2^n. Due to Condition 2, we have

$$H(Y_2^n) \leq H(\tilde{Y}_2^n). \tag{24}$$

Next, we proceed with the single-letterization of n-letter entropies using [7, page 314, eqn (3.34)]:

$$H(\tilde{Y}_2^n) - H(Y_1^n) = \sum_{i=1}^n H(\tilde{Y}_{2i}|V_i) - H(Y_{1i}|V_i) \tag{25}$$

$$H(T^n) - H(Y_1^n) = \sum_{i=1}^n H(T_i|U_i) - H(Y_{1i}|U_i) \tag{26}$$

where auxiliary random variables V_i and U_i, $i = 1, 2, \cdots, n$ are defined as

$$V_i = \left(Y_1^{i-1}, \tilde{Y}_{2(i+1)}^n\right), \quad U_i = \left(Y_1^{i-1}, T_{(i+1)}^n\right) \tag{27}$$

Note that since $\tilde{X}_2^n$ is i.i.d., $\tilde{X}_{2i}$ is independent to (U_i,V_i).

From (25) and (26), similar to [7, page 315, (3.44)], we may deduce that there exists two real numbers γ_1 and γ_2 such that

$$H(\tilde{Y}_2^n) = \sum_{i=1}^{n} H(\tilde{Y}_{2i}|V_i) + n\gamma_1 \tag{28}$$

$$H(Y_1^n) = \sum_{i=1}^{n} H(Y_{1i}|V_i) + n\gamma_1 \tag{29}$$

$$H(T^n) = \sum_{i=1}^{n} H(T_i|U_i) + n\gamma_2 \tag{30}$$

$$H(Y_1^n) = \sum_{i=1}^{n} H(Y_{1i}|U_i) + n\gamma_2 \tag{31}$$

$$0 \leq \gamma_1 \leq \min\left(\frac{1}{n}\sum_{i=1}^{n} I(V_i;\tilde{Y}_{2i}), \frac{1}{n}\sum_{i=1}^{n} I(V_i;Y_{1i})\right) \tag{32}$$

$$0 \leq \gamma_2 \leq \min\left(\frac{1}{n}\sum_{i=1}^{n} I(U_i;T_i), \frac{1}{n}\sum_{i=1}^{n} I(U_i;Y_{1i})\right) \tag{33}$$

Define time sharing random variable Q which is independent to everything else and uniform on $\{1, 2, \cdots, n\}$. Further define

$$V = (V_Q, Q), U = (U_Q, Q), X_1 = X_{1Q},$$
$$\tilde{X}_2 = \tilde{X}_{2Q}, Y_1 = Y_{1Q}, T = T_Q, \tilde{Y}_2 = \tilde{Y}_{2Q}.$$

The following Markov chain holds

$$V \to U \to (X_1, Y_1, T, \tilde{Y}_2) \tag{34}$$

because given U_i, the randomness in V_i comes from $\tilde{X}_{2(i+1)}^n$ and is independent to $(X_{1i}, Y_{1i}, T_i, \tilde{Y}_{2i})$. Thus, the auxiliary random variables defined satisfy (8). From (28)-(33), we have

$$\frac{1}{n}H(Y_1^n) = H(Y_1|V) + \gamma_1 = H(Y_1|U) + \gamma_2 \tag{35}$$

$$\frac{1}{n}H(T^n) = H(T|U) + \gamma_2 \tag{36}$$

$$\frac{1}{n}H(\tilde{Y}_2^n) = H(\tilde{Y}_2|V) + \gamma_1 \tag{37}$$

$$= H(\tilde{Y}_2|U) + \gamma_2 + I(U;\tilde{Y}_2|V) - I(U;Y_1|V) \tag{38}$$

$$0 \leq \gamma_1 \leq \min\left(I(V;\tilde{Y}_2), I(V;Y_1)\right) \tag{39}$$

$$0 \leq \gamma_2 \leq \min\left(I(U;T), I(U;Y_1)\right) \tag{40}$$

where (38) follows from (35) and the Markov chain in (34). Depending on the sign of $I(U;Y_1|V) - I(U;\tilde{Y}_2|V)$, we have the following 2 cases:

1) $I(U;Y_1|V) < I(U;\tilde{Y}_2|V)$;
Based on (20), (23), (24), (35)-(37), (39), (40) and letting $n \to \infty$, we have

$$R_1 \leq I(X_1;Y_1|Q) \leq I(X_1;Y_1) \tag{41}$$

$$\begin{aligned} R_1 &\leq H(Y_1|V) + \gamma_1 - H(Y_1|X_1) \\ &\leq I(X_1;Y_1|V) + I(V;\tilde{Y}_2) \end{aligned} \tag{42}$$

$$\begin{aligned} R_1 &\leq H(Y_1|U) + \gamma_2 - H(Y_1|X_1) \\ &\leq I(X_1;Y_1|U) + I(U;T) \end{aligned} \tag{43}$$

$$\begin{aligned} R_2 &\leq H(\tilde{Y}_2|V) + \gamma_1 - H(T|U) - \gamma_2 \\ &= H(\tilde{Y}_2|V) - H(T|U) - I(U;Y_1|V) \end{aligned} \tag{44}$$

2) $I(U;Y_1|V) \geq I(U;\tilde{Y}_2|V)$;
Based on (20), (23), (24), (35)-(40) and letting $n \to \infty$, we have

$$R_1 \leq I(X_1;Y_1) \tag{45}$$

$$\begin{aligned} R_1 &\leq H(Y_1|U) - H(Y_1|X_1) + \gamma_2 \\ &\leq I(X_1;Y_1|U) + I(U;T) \end{aligned} \tag{46}$$

$$\begin{aligned} R_2 &\leq H(\tilde{Y}_2|U) + \gamma_2 - H(T|U) - \gamma_2 \\ &= H(\tilde{Y}_2|U) - H(T|U) \end{aligned} \tag{47}$$

$$\begin{aligned} R_1 + R_2 &\leq H(\tilde{Y}_2|V) + \gamma_1 - H(T|U) - \gamma_2 \\ &\quad + H(Y_1|U) + \gamma_2 - H(Y_1|X_1) \end{aligned} \tag{48}$$

$$\leq H(\tilde{Y}_2) + I(X_1;Y_1|U) - H(T|U) \tag{49}$$

From (41)-(44) and (45)-(49), Theorem 1 is proved.

VIII. Conclusions

We studied a class of Z-interference channels that satisfy the shift-invariant condition but not the maximum entropy condition. For these Z-interference channels, we provided an outer bound on the capacity region. The outer bound is shown to be tight for certain Z-interference channels whose capacity regions were not previously known. The techniques employed in this paper may be useful in providing converses to other problems in multi-user information theory.

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