Quadratic Weight Vector for Tighter Aperiodic Levenshtein Bound

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***Abstract*—The Levenshtein bound, as a function of the weight vector, is only known to be tighter than the Welch bound on aperiodic correlation for $K \geq 4, N \geq 2$, where K and N denoting the set size and the sequence length, respectively. A quadratic weight vector is proposed in this paper which leads to a tighter Levenshtein bound for $K \geq 4, N \geq 2$ and $K = 3, N \geq 4$. The latter case was left open by Levensthein.**

Keywords: CDMA, wireless communications, correlation, lower bounds, coding theory.

I. Introduction

In asynchronous CDMA communications, the aperiodic correlations of the spreading sequence set determine the worst-case probability of detection errors and the average signal-to-noise ratio [1]. Denote by $\delta_{\max}$ the maximum aperiodic correlation magnitude over all the non-trivial auto-correlations and cross-correlations. In [2], Welch showed that

$$\delta_{\max}^2 \geq N^2 \frac{K-1}{K(2N-1)-1}, \tag{1}$$

where K and N denoting the set size and the sequence length, respectively.

In [3], Levenshtein improved the Welch bound on aperiodic correlation. It is shown that the Levenshtein bound is tighter than the Welch bound for binary sequence set when $K \geq 4$ and $N \geq 2$ [3]. As pointed out by Boztaş, the tightness of this bound is also valid for sequence sets over the *complex roots-of-unity* [4]. The idea of the Levenshtein bound is that the mean of the weighted aperiodic correlation squares for any sequence subset over the *complex roots-of-unity* is equal to or greater than that of the whole set which includes all the *complex roots-of-unity* sequences. In [5], Liu and Guan showed that the Levenshtein bound is met with equality by the weighted-correlation complementary sequences. In [6], Liu, Guan and Mow derived the generalized Levenshtein bound for quasi-complementary sequence set which is tighter than that of Welch [2]. The reader is referred to [7], [8], and [9] for more information on perfect/quasi complementary sequences.

It is noted that the tightness of the Levenshtein bound depends on the weight vector, a vector of non-negative elements whose sum is 1, in the bound equation. When the set size $K = 1$ or 2, convex optimization shows that the tightest Levenshtein bound is equal to the Welch bound. For $K \geq 3$, the Levenshtein bound becomes non-convex and hence the analytical optimization seems to be infeasible. In particular, when $K = 3$, Levenshtein surmised that the Welch bound may not be able to be improved.

A new weight vector, which takes the form of a quadratic function, is proposed in this paper. We will show that it leads to tighter Levenshtein bound for $K = 3$ and $N \geq 4$, and $K \geq 4$ and $N \geq 2$.

This paper is organized as follows. In Section II, necessary notations and a review of the Levenshtein bound are given. The proposed weight vector and the resultant Levenshtein bound are presented in Section III. Finally, the work is summarized in Section V.

II. Preliminaries

A. Notations

The following notations are used throughout this paper.

- Let $\xi = \exp\left(2\pi\sqrt{-1}/H\right)$ and $E = \left\{1, \xi^1, \cdots, \xi^{H-1}\right\}$, where H is a positive integer greater than 1;
- Denote by $T^i\mathbf{x}$ the i-cyclic shifts of any column vector $\mathbf{x} = (x_0, x_1, \cdots, x_{N-1})^{\mathrm{T}}$, e.g.,

$$T^i\mathbf{x} = (x_{N-i}, x_{N-i+1}, \cdots, x_{N-i-1})^{\mathrm{T}},$$

  where $0 \leq i \leq N-1$ and $(\cdot)^{\mathrm{T}}$ denotes the transpose of $(\cdot)$;
- For two length-N complex-valued sequences $\mathbf{a}$ (or $\{a_t\}$) and $\mathbf{b}$ (or $\{b_t\}$), denote their *aperiodic correlation func-*

tion by

$$\rho_{\mathbf{a},\mathbf{b}}(\tau)=\begin{cases}\sum\limits_{t=0}^{N-1-\tau} a_t b_{t+\tau}^*, & 0\le\tau\le(N-1);\\ \sum\limits_{t=0}^{N-1+\tau} a_{t-\tau} b_t^*, & -(N-1)\le\tau\le-1;\\ 0, & |\tau|\ge N.\end{cases} \tag{2}$$

When $\mathbf{a}\neq\mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called the aperiodic cross-correlation function; otherwise, it is called the aperiodic auto-correlation function and will be written as $\rho_{\mathbf{a}}(\tau)$ for simplicity.

Let $\mathbf{A}$ to be a sequence set with K length-N sequences.

$$\begin{aligned}\mathbf{A}&=\left\{\mathbf{a}^0,\mathbf{a}^1,\cdots,\mathbf{a}^u,\cdots,\mathbf{a}^{K-1}\right\}, \quad 0\le u\le K-1\\ \mathbf{a}^u&=\left\{a_0^u,a_1^u,\cdots,a_t^u,\cdots,a_{N-1}^u\right\}, \quad 0\le t\le N-1.\end{aligned}$$

***Definition** 1:* $\mathbf{A}$ is called a sequence set over the *complex roots-of-unity* if every entry belongs to E.

***Definition** 2:* Denote by $\delta_{\max}$ the maximum *non-trivial aperiodic correlation magnitude* of $\mathbf{A}$, e.g.,

$$\delta_{\max}=\max\left\{|\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)|:\ u\neq v,\ \text{or } u=v,\tau\neq 0\right\}. \tag{3}$$

B. Review of the Levenshtein bound

***Definition** 3:* $\mathbf{w}=(w_0,w_1,\cdots,w_{2N-2})^{\mathrm{T}}$ is called a weight vector if

$$w_i\ge 0,\quad \sum_{i=0}^{2N-2} w_i=1. \tag{4}$$

For $0\le s,t\le 2N-2$, define

$$l_{s,t,N}:=\min\Big\{|t-s|,2N-1-|t-s|\Big\}. \tag{5}$$

For $\mathbf{q}_a=(a,1,2,\cdots,N-1,N-1,\cdots,2,1)^{\mathrm{T}}$, let

$$\mathcal{Q}_a=\left[\mathbf{q}_a,T^1\mathbf{q}_a,T^2\mathbf{q}_a,\cdots,T^{2N-2}\mathbf{q}_a\right]$$

and

$$\begin{aligned}Q_{2N-1}(\mathbf{w},a):&=a\sum_{i=0}^{2N-2} w_i^2+\sum_{s,t=0}^{2N-2} l_{s,t,N}w_s w_t\\ &=\mathbf{w}^{\mathrm{T}}\mathcal{Q}_a\mathbf{w}.\end{aligned} \tag{6}$$

***Lemma** 1:* (The Levenshtein bound [3]) For sequence set $\mathbf{A}\subseteq E^N$,

$$\delta_{\max}^2\ge N-\frac{Q_{2N-1}\left(\mathbf{w},\frac{N(N-1)}{K}\right)}{1-\frac{1}{K}\sum\limits_{i=0}^{2N-2} w_i^2}. \tag{7}$$

***Remark** 1:* If the following weight vector

$$w_i=\begin{cases}\frac{1}{m}, & i\in\{0,1,\cdots,m-1\};\\ 0, & i\in\{m,m+1,\cdots,2N-2\}\end{cases} \tag{8}$$

is applied (with $1\le m\le N$), the lower bound in (7) will be reduced to

$$\delta_{\max}^2\ge\frac{NKm-N^2-\frac{K(m^2-1)}{3}}{mK-1},\ 1\le m\le N. \tag{9}$$

Note that the lower bound in (9) for $m=N$ is tighter than the Welch bound for $K\ge 4, N\ge 2$ [3].

We note that for $K=1$ and $K=2$, the bound function in terms of $\mathbf{w}$ in (7) is convex and the tightest bound is equal to the Welch bound and achieved by choosing $\mathbf{w}=\frac{1}{2N-1}(1,1,\cdots,1)^{\mathrm{T}}$; for $K\ge 3$ however, the fractional quadratic term in (7) becomes non-convex and thus the analytical optimization seems to be infeasible. In [3], Levenshtein pointed out that:

"For $K=1$ and $K=2$ the Welch bound cannot be improved by the method under consideration. This is probably also the case for $K=3$."

Thus, it is interesting to search new weight vector which leads to a tighter Levenshtein bound for $K=3$. We will present such a weight vector in the next section.

III. Proposed quadratic weight vector for tighter Levenshtein bound

A. Proposed weight vector and its resultant Levenshtein bound

The proposed quadratic weight vector $\mathbf{w}$ is defined as follows,

$$w_i=\begin{cases}\frac{6i(m-i)}{m(m^2-1)}, & i\in\{0,1,\cdots,m-1\};\\ 0, & i\in\{m,m+1,\cdots,2N-2\}\end{cases} \tag{10}$$

where $2\le m\le N+1$. It is easy to check that $\sum_{i=0}^{2N-2} w_i=1$ and

$$\sum_{i=0}^{2N-2} w_i^2=\frac{6(m^2+1)}{5m(m^2-1)}. \tag{11}$$

Substituting the proposed quadratic weight vector $\mathbf{w}$ into (7), we have:

***Theorem** 1:* For $2\le m\le N+1$,

$$\delta_{\max}^2\ge N-\frac{42N(N-1)(m^2+1)+3K(m^2-4)(3m^2+1)}{35Km(m^2-1)-42(m^2+1)}. \tag{12}$$

Furthermore, (12) can be simplified as follows,

$$\delta_{\max}^2\ge N-\frac{42N(N-1)+9K(m^2+1)}{35Km-42}. \tag{13}$$

Proof: To prove ***Theorem** 1*, we need to calculate $Q_{2N-1}(\mathbf{w},a)$. To this end, we carry out the discussion into the following two cases: $(1):\ 2\le m\le N$; $(2): m=N+1$.

Note that for $0\le s,t\le m-1$,

$$l_{s,t,N}=\min\Big\{|t-s|,2N-1-|t-s|\Big\}=|t-s|. \tag{14}$$

If $2 \le m \le N$, we have

$$\begin{aligned} Q_{2N-1}(\mathbf{w}, a) &= a \sum_{i=0}^{2N-2} w_i^2 + \sum_{s,t=0}^{2N-2} l_{s,t,N} w_s w_t \\ &= a \sum_{i=0}^{m-1} w_i^2 + \sum_{s,t=0}^{m-1} l_{s,t,N} w_s w_t \\ &= a \sum_{i=0}^{m-1} w_i^2 + \sum_{s,t=0}^{m-1} |s-t| w_s w_t \end{aligned} \tag{15}$$

In [10], it is shown that

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} \\ \sum_{i=1}^{n} i^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ \sum_{i=1}^{n} i^5 &= \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \\ \sum_{i=1}^{n} i^6 &= \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}. \end{aligned} \tag{16}$$

By (16), we have

$$\begin{aligned} \sum_{s,t=0}^{m-1} |s-t| w_s w_t &= 2 \sum_{t=1}^{m-1} t \left[\sum_{i=0}^{m-t-1} w_i w_{i+t} \right] \\ &= \frac{3(m^2-4)(3m^2+1)}{35m(m^2-1)}. \end{aligned} \tag{17}$$

Now, by (11), (17), and (15), we have

$$Q_{2N-1}(\mathbf{w}, a) = \frac{6a(m^2+1)}{5m(m^2-1)} + \frac{3(m^2-4)(3m^2+1)}{35m(m^2-1)}. \tag{18}$$

Letting $a = N(N-1)/K$, by (11) and (18), we arrive at the lower bound in (12).

If $m = N+1$, note that $w_0 w_N = 0$. Therefore,

$$\begin{aligned} \sum_{s,t=0}^{2N-2} l_{s,t,N} w_s w_t &= \sum_{s,t=0}^{m-1} l_{s,t,N} w_s w_t \\ &= 2 \sum_{t=1}^{m-1} t \left[\sum_{i=0}^{m-t-1} w_i w_{i+t} \right] \\ &\qquad + 2(N-1) w_0 w_N \\ &= 2 \sum_{t=1}^{m-1} t \left[\sum_{i=0}^{m-t-1} w_i w_{i+t} \right] \\ &= \frac{3(m^2-4)(3m^2+1)}{35m(m^2-1)}. \end{aligned} \tag{19}$$

By (19), we assert that the lower bound in (12) also holds for $m = N+1$.

Given (12), the proof of (13) is easy and thus omitted. ■

B. Tightness analysis

Denote by $\mathbf{B}_W$ the right-hand term of the Welch bound in (1). Also, denote by $\mathbf{B}_m^{(0)}$ and $\mathbf{B}_m^{(1)}$ the right-hand terms of the Levenshtein bounds in (12) and (13), respectively. Note that

$$\mathbf{B}_m^{(0)} \ge \mathbf{B}_m^{(1)}. \tag{20}$$

Let $\epsilon^{(0)} = \mathbf{B}_m^{(0)} - \mathbf{B}_W$ and $\epsilon^{(1)} = \mathbf{B}_m^{(1)} - \mathbf{B}_W$. In particular, we call $\epsilon^{(0)}$ the *bound improvement*, which is a function of m, K, N.

To perform the tightness analysis, we claim that

***Corollary** 1:* Let $m = N+1$ in ***Theorem** 1*. The resultant lower bound in (12) is tighter than the Welch bound for $K \ge 4, N \ge 2$, and $K = 3, N \ge 4$.

Proof: Our goal is to show $\epsilon^{(0)}\big|_{m=N+1} > 0$ for $K \ge 4, N \ge 2$, and $K = 3, N \ge 4$. First, we remark that:

For given N, it is easy to show that

$$\begin{aligned} \min_{K\ge 3} \epsilon^{(0)}\Big|_{m=N+1} &= \epsilon^{(0)}\Big|_{K=3,m=N+1}, \\ \min_{K\ge 3} \epsilon^{(0)}\Big|_{m=N+1} &= \epsilon^{(1)}\Big|_{K=3,m=N+1}. \end{aligned} \tag{21}$$

Meanwhile, for $K = 3$ and $N \ge 7$,

$$\begin{aligned} \epsilon^{(1)}\Big|_{m=N+1} &= \frac{2N^2-2N}{3N-2} - \frac{23N^2+4N+18}{35N+21} \\ &= \frac{N^3+6N^2-88N+36}{(3N-2)(35N+21)} \\ &> 0. \end{aligned}$$

By (20), $\epsilon^{(0)}\big|_{m=N+1} > 0$ for $K = 3$ and $N \ge 7$. Also, note the values of $\epsilon^{(0)}\big|_{m=N+1}$ for the following cases:

N	2	3	4	5	6	7	8
$K=3$	0	0	0.0131	0.0289	0.0448	0.0600	0.0746

Therefore, $\epsilon^{(0)}\big|_{m=N+1} > 0$ for $K \ge 3$ and $N \ge 4$. For $K \ge 4$, by noting that

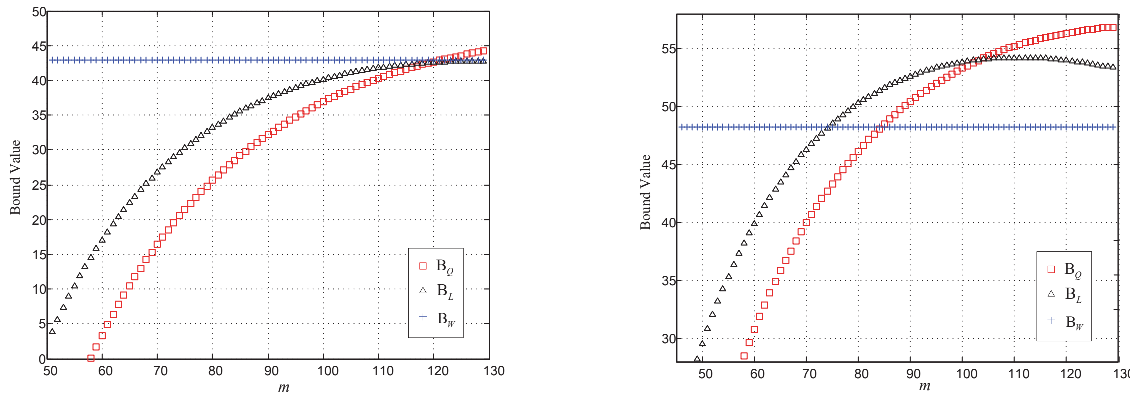
$$\begin{aligned} \epsilon^{(0)}\Big|_{N=2,m=3} &= \frac{K(K-3)}{(3K-1)(2K-1)} > 0 \\ \epsilon^{(0)}\Big|_{N=3,m=4} &= \frac{90K(K-3)}{(5K-1)(50K-17)} > 0 \end{aligned}$$

completes the proof. ■

In the above proof, some values of $\epsilon^{(0)}\big|_{m=N+1}$ for $K = 3$ are shown. For completeness, the numerical values of $\epsilon^{(0)}\big|_{m=N+1}$ for $K \in \{3, 4, \cdots, 8\}$ and $N \in \{2, 3, \cdots, 10\}$ are shown in TABLE I, with which ***Corollary** 1* can be verified.

TABLE I: The values of $\epsilon^{(0)}\big|_{m=N+1}$ for $K \in \{3, 4, \cdots, 8\}$ and $N \in \{2, 3, \cdots, 10\}$

N	2	3	4	5	6	7	8	9	10
$K=3$	0	0	0.0131	0.0289	0.0448	0.0600	0.0746	0.0886	0.1020
$K=4$	0.0519	0.1035	0.1687	0.2379	0.3081	0.3786	0.4490	0.5191	0.5891
$K=5$	0.0794	0.1609	0.2569	0.3577	0.4603	0.5637	0.6673	0.7711	0.8748
$K=6$	0.0963	0.1974	0.3136	0.4353	0.5594	0.6846	0.8104	0.9365	1.0627
$K=7$	0.1077	0.2226	0.3531	0.4897	0.6291	0.7699	0.9114	1.0534	1.1956
$K=8$	0.1159	0.2410	0.3822	0.5300	0.6807	0.8331	0.9865	1.1404	1.2946



(a) $K = 3$ and $N = 128$

(b) $K = 4$ and $N = 128$

Fig. 1: The Levenshtein bounds from different weight vectors, where $\mathbf{B}_Q$ (from the proposed quadratic weight vector) and $\mathbf{B}_L$ are the right-hand terms of the Levenshtein bounds in (12) and (9), respectively, $\mathbf{B}_W$ is the Welch bound.

For large N, we show the following corollary:

***Corollary** 2:* The Welch bound in (1) can be simplified as

$$\delta^2_{\max}\big|_{N\to+\infty} \gtrsim N\left[\frac{1}{2}-\frac{1}{2K}\right]. \tag{22}$$

In comparison, the Levenshtein bounds in (12) can be simplified as

$$\delta^2_{\max}\big|_{N\to+\infty} \gtrsim \max\left\{N\left[\frac{26}{35}-\frac{6}{5K}\right], N\frac{K-N}{K-1}\right\}, \tag{23}$$

which is tighter than that in (22) for $K \geq 3$.

Proof: The proof of (22) is simple and omitted. For (23), setting $m = 2$ and $m = N+1$ into (12), we have

$$\delta^2_{\max}\big|_{N\to+\infty} \gtrsim N\frac{K-N}{K-1} \tag{24}$$

and

$$\delta^2_{\max}\big|_{N\to+\infty} \gtrsim N\left[\frac{26}{35}-\frac{6}{5K}\right], \tag{25}$$

respectively, and thus the proof follows. ■

Next, we show the numerical comparison of the lower bounds in (12) and (9) in Fig. 1 for $K \in \{3, 4\}$ and $N = 128$. To this end, we denote by $\mathbf{B}_Q$ and $\mathbf{B}_L$ the right-hand terms of the Levenshtein bounds in (12) and (9), respectively. For a fair comparison, the range of m has been extended to $1 \leq m \leq N+1$. In particular, the proposed weight vector is defined as $\mathbf{w} = \frac{1}{2N-1}(1, 0, 0, \cdots, 0)^{\mathrm{T}}$ if $m = 1$. One can see that in either case, the maximum lower bound in (12) in tighter than that in (9).

We close this section by noting that

***Remark** 2:* When $K = 3$ and $N = 2$, the tightest Levenshtein bound is also equal to the Welch bound and is achieved by any weight vector.

Proof: When $K = 3$ and $N = 2$, for any weight vector $\mathbf{w}$, we have

$$Q_{2N-1}\left(\mathbf{w}, \frac{N(N-1)}{K}\right) = 1 - \frac{1}{3}\sum_{i=0}^{2} w_i^2.$$

Hence the Levenshtein bound in (7) in this case can be reduced to

$$\delta^2_{\max} \geq N - \frac{Q_{2N-1}\left(\mathbf{w}, \frac{N(N-1)}{K}\right)}{1 - \frac{1}{K}\sum\limits_{i=0}^{2N-2} w_i^2} = 1.$$

We complete the proof by computing the corresponding Welch bound in (1). ■

IV. Conclusion

This paper is devoted to the tightening of the Levenshtein bound for set size $K = 3$, which was left open by Levenshtein in [3].

Our main contribution is that we have proposed a quadratic weight vector in (10) which leads to the tighter Levenshtein bound (shown in ***Theorem*** *1*) for $K = 3, N \geq 4$, and $K \geq 4, N \geq 2$. This work was initially motivated by the numerical optimization results, in this paper however, we are concerned with the analytical result only. We think it is interesting that a simple weight vector such as the proposed quadratic one can give such simple and closed form of the Levenshtein bound.

As shown in ***Remark*** *2*, the Welch bound for $K = 3$ and $N = 2$ cannot be improved. Therefore, it is interesting to ask if there exist other weight vectors which lead to tighter Levenshtein bound for $K = 3, N \geq 3$, and $K \geq 4, N \geq 2$. We will show such a weight vector in our forthcoming journal paper.

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