# The Role of Lookahead in Estimation under Gaussian Noise

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***Abstract*—We consider mean squared estimation of a continuous-time signal corrupted by additive white Gaussian noise. We investigate the trade-off between lookahead and estimation-loss under this model. We study the class of continuous-time stationary Gauss-Markov processes (Ornstein-Uhlenbeck processes) as channel inputs, and explicitly characterize the behavior of the minimum mean squared error (MMSE) with finite lookahead and signal-to-noise ratio (SNR). The MMSE with lookahead is shown to converge exponentially rapidly to the non-causal error, with the exponent being the reciprocal of the non-causal error. We extend our results to mixtures of Ornstein-Uhlenbeck processes, and use the insight gained to present lower and upper bounds on the MMSE with lookahead for a class of stationary Gaussian input processes, whose spectrum can be expressed as a mixture of Ornstein-Uhlenbeck spectra.**

## I. Introduction

Fundamental links between information theoretic quantities and mean squared estimation loss have been established for signals corrupted by Gaussian noise. In [1], Duncan discovered the equivalence between the input-output mutual information and the integral of half the causal mean squared error in estimating the signal based on the observed process. In [2], Guo et al. present the I-MMSE relationship, which equates the derivative of the mutual information to half the average non-causal squared error.

Let $\mathbf{X} = \{X_t, t \in \mathbb{R}\}$ denote a stationary, square-integrable stochastic process. The output process $\mathbf{Y}$ of the continuous-time Gaussian channel with input $\mathbf{X}$ is given by the following equivalent model (cf. [3])

$$dY_t = \sqrt{\mathrm{snr}}\, X_t dt + dW_t, \tag{1}$$

for all $t$, where $\mathrm{snr} > 0$ is the signal-to-noise ratio (SNR) and $W_\cdot$ is a standard Brownian Motion independent of $\mathbf{X}$. Let $I(\mathrm{snr})$ denote the mutual information rate function. Let $\mathrm{mmse}(\mathrm{snr})$ and $\mathrm{cmmse}(\mathrm{snr})$, denote respectively, the smoothing and filtering squared errors which, in our setting of stationarity, can equivalently be defined[1] as

$$\mathrm{mmse}(\mathrm{snr}) = \mathbb{E}[(X_0 - \mathbb{E}[X_0|Y_{-\infty}^{+\infty}])^2], \tag{2}$$
$$\mathrm{cmmse}(\mathrm{snr}) = \mathbb{E}[(X_0 - \mathbb{E}[X_0|Y_{-\infty}^{0}])^2]. \tag{3}$$

[1]throughout, we denote, $X_a^b \doteq \{X_t, t \in [a,b]\}$, for a given process.

From [1] and [2], the above quantities are related according to

$$\frac{2I(\mathrm{snr})}{\mathrm{snr}} = \mathrm{cmmse}(\mathrm{snr}) = \frac{1}{\mathrm{snr}} \int_0^{\mathrm{snr}} \mathrm{mmse}(\gamma)\, d\gamma, \tag{4}$$

for any distribution of $\mathbf{X}$ that is square integrable.

The mean squared error with finite lookahead $d$ is defined as

$$\mathrm{lmmse}(d, \mathrm{snr}) = \mathbb{E}[(X_0 - \mathbb{E}[X_0|Y_{-\infty}^{d}])^2]. \tag{5}$$

The motivation behind the current line of work is to address the role of lookahead in the context of information and estimation in the Gaussian channel. In [4], this topic was approached from an informational perspective : where it was established that the MMSE with lookahead cannot be characterized by the mutual information rate function - unlike its causal ($d = 0$) and non-causal ($d = \infty$) counterparts. In this work, which can be viewed as a sequel to [4], we wish to characterize the role of lookahead from an estimation theoretic viewpoint. In the literature, finite lookahead estimation has been referred to as fixed-lag smoothing (cf., for example, [5] and references therein).

Motivated by the desire to understand the tradeoff between lookahead and squared error, in Section II we explicitly characterize this trade-off for the canonical family of continuous-time stationary Gauss-Markov (Ornstein-Uhlenbeck) processes. In particular, we find that the convergence of $\mathrm{lmmse}(\cdot, \mathrm{snr})$ with increasing lookahead (from causal to the non-causal error) is exponentially fast, with the exponent given by the inverse of the non-causal error, i.e. $\frac{1}{\mathrm{mmse}(\mathrm{snr})}$ itself. In Section III, we extend our results to a larger class of processes, that are expressible as a mixture of Ornstein-Uhlenbeck processes. We then consider stationary Gaussian input processes in Section IV, and characterize the MMSE with lookahead via spectral factorization methods. We summarize our findings and outline future directions in Section V.

## II. Ornstein-Uhlenbeck Process

### A. Definitions and Properties

The Ornstein-Uhlenbeck process [6], [7] is a Gaussian stochastic process characterized by the following stochastic differential equation

$$dX_t = \alpha(\mu - X_t)dt + \beta dB_t, \tag{6}$$

where $B_{(\cdot)}$ is a standard Brownian Motion and $\alpha$, $\mu$, $\beta$ are process parameters. The mean and covariance functions are given by:

$$\mathbb{E}(X_t) = \mu, \tag{7}$$

$$\mathrm{Cov}(X_t, X_s) = \frac{\beta^2}{2\alpha} e^{-\alpha|t-s|}. \tag{8}$$

The autocorrelation function and power spectral density are given by:

$$R_X(\tau) = \frac{\beta^2}{2\alpha} e^{-\alpha|\tau|}, \tag{9}$$

$$S_X(\omega) = \frac{\beta^2}{\alpha^2 + \omega^2}. \tag{10}$$

In all further analysis, we consider the process mean $\mu$ to be 0. We also note that all expressions are provided assuming $\alpha > 0$ which results in a mean-reverting evolution of the process.

### B. Mean Squared Estimation

We now consider the Gaussian channel (1) at SNR level $\gamma$, with an Ornstein-Uhlenbeck process as the channel input. In this section, we will explicitly characterize the MMSE with finite lookahead. For the rest of our discussion, we will set $\beta = 1$ in (6), as it only contributes a scaling of the channel SNR by a factor of $\beta^2$.

The filtering error for this setting can be computed explicitly via the Kalman-Bucy equations [8]. Further, by exploiting the time-reversibility and Markovity of the source, it is possible to explicitly characterize the estimation error with finite lookahead $d$ and lookback $l$, i.e. $\mathrm{Var}(X_0|Y_l^d)$. We omit details of the calculation from this presentation, and only include the main results. For detailed proofs and additional discussions beyond this manuscript, the reader is referred to [9].

The following lemma presents the filtering and smoothing error for the Ornstein-Uhlenbeck process.

*Lemma 1 (Filtering and Smoothing):* For the Ornstein-Uhlenbeck process with parameter $\alpha$, corrupted by white Gaussian noise at SNR level $\gamma$, the filtering (3) and smoothing (2) errors are given, respectively, by

$$\mathrm{cmmse}(\gamma) = \frac{\sqrt{\alpha^2 + \gamma} - \alpha}{\gamma}, \tag{11}$$

$$\mathrm{mmse}(\gamma) = \frac{1}{2\sqrt{\alpha^2 + \gamma}}. \tag{12}$$

The expressions in (11) and (12) recover the classical expressions for the optimal causal and smoothing errors (for Gaussian inputs), due to Yovits and Jackson in [10], and Wiener in [11], respectively.

Having understood causal and non-causal mean squared estimation for the Ornstein-Uhlenbeck process under Gaussian noise, we now proceed to characterize the finite lookahead mean squared loss.

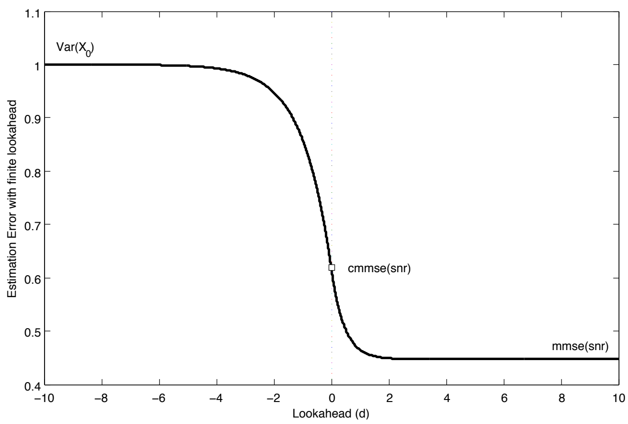


Fig. 1: $\mathrm{lmmse}(d, \mathrm{snr})$ vs. lookahead $d$ for the Ornstein-Uhlenbeck process for $\alpha = 0.5$, and $\mathrm{snr} = 1$.

*Lemma 2 (Lookahead):* For the Ornstein-Uhlenbeck process with parameter $\alpha$, the mean squared error with finite lookahead $d \in \mathbb{R}$, for observation under AWGN at SNR level $\gamma$; $\mathrm{lmmse}(d, \gamma)$ is given by

$$(1 - e^{-2d\sqrt{\alpha^2+\gamma}})\mathrm{mmse}(\gamma) + e^{-2d\sqrt{\alpha^2+\gamma}}\mathrm{cmmse}(\gamma), \tag{13}$$

when $d \geq 0$, and

$$e^{-2\alpha|d|}\mathrm{cmmse}(\gamma) + \frac{1}{2\alpha}(1 - e^{-2\alpha|d|}), \tag{14}$$

when $d < 0$;
where $\mathrm{cmmse}(\gamma)$ and $\mathrm{mmse}(\gamma)$ are as defined in (11) and (12), respectively. A plot of the estimation error with finite lookahead for the Ornstein-Uhlenbeck process is shown in Fig. 1. It is seen from the expression in Lemma 2, that the asymptotes at $d = -\infty$ and $d = +\infty$ correspond to the stationary variance $\mathrm{Var}(X_0)$ and the non-causal MMSE, respectively, for a given SNR level.

We now focus on the case where $d \geq 0$, which corresponds to estimation with finite delay. We define, for $d \geq 0$,

$$p_d \doteq \frac{\mathrm{lmmse}(d, \gamma) - \mathrm{mmse}(\gamma)}{\mathrm{cmmse}(\gamma) - \mathrm{mmse}(\gamma)}. \tag{15}$$

From Lemma 2, we observe that for the Ornstein-Uhlenbeck process,

$$p_d = e^{-2d\sqrt{\alpha^2+\gamma}} \tag{16}$$

$$= e^{-\frac{d}{\mathrm{mmse}(\gamma)}}, \tag{17}$$

where (17) follows from (12). In other words, the causal error approaches the non-causal error exponentially fast with increasing lookahead. This captures the benefit of lookahead in signal estimation with finite delay. We can state this observation as follows:

*Observation 3:* For any AWGN corrupted Ornstein-Uhlenbeck process, the mean squared error with finite positive lookahead, approaches the non-causal error exponentially fast, with decay exponent given by $\frac{1}{\mathrm{mmse}(\gamma)}$.

## III. A MIXTURE OF ORNSTEIN-UHLENBECK PROCESSES

Having presented the finite lookahead MMSE for the Ornstein-Uhlenbeck process in Lemma 2, in this section we obtain the MMSE with lookahead for the class of stochastic processes that are mixtures of Ornstein-Uhlenbeck processes. We then proceed to establish a general lower bound on the MMSE with lookahead for the class Gaussian processes whose spectra can be decomposed as a mixture of spectra of Ornstein-Uhlenbeck processes. For the same class of Gaussian processes, we also present an upper bound for the finite lookahead MMSE, in terms of a mismatched estimation loss.

Let $\mu(\alpha)$ be a probability measure defined on $[0, \infty)$. Let $P^{(\alpha)}$ be the law of the Ornstein-Uhlenbeck process with parameter $\alpha$. Note that each $P^{(\alpha)}$ is the law of a stationary ergodic stochastic process. We define the stationary distribution generated by taking a $\mu$-mixture of these processes:

$$P = \int P^{(\alpha)}\, d\mu(\alpha). \tag{18}$$

Note that $P$ need not be Gaussian.

*Lemma 4:* Let $X$ be a stationary stochastic process governed by law $P$ which is a $\mu$ mixture of Ornstein-Uhlenbeck processes, and is corrupted by AWGN at SNR $\gamma$. The MMSE with finite lookahead $d$ is given by

$$\text{lmmse}_P(d, \gamma) = \int \text{lmmse}_\alpha(d, \gamma)\, d\mu(\alpha), \tag{19}$$

where $\text{lmmse}_\alpha(d, \gamma)$ is (as defined in Lemma 2) the corresponding quantity for estimating an Ornstein-Uhlenbeck process with parameter $\alpha$.

The proof of Lemma 4 follows in one line, upon observing that the underlying 'active mode' is eventually precisely learned from the infinitely long observation of the noisy process. This relation allows us to compute the MMSE with finite lookahead for the class of processes that can be expressed as mixtures of Ornstein-Uhlenbeck processes.

As another important corollary of this discussion, consider any Gaussian process $G$ whose spectrum $S_G$ can be expressed as a mixture of spectra of Ornstein-Uhlenbeck processes, for some appropriate mixing measure $\mu(\alpha)$,i.e.

$$S_G = \int S_\alpha\, d\mu(\alpha), \tag{20}$$

where $S_\alpha$ denotes the spectrum of the Ornstein-Uhlenbeck process with parameter $\alpha$. The approach outlined above provides us with a computable lower bound on the minimum mean squared error with fixed lookahead $d$ (under AWGN) for the process $G$, which we state in the following Lemma.

*Lemma 5 (Lower Bound):* For a Gaussian process $G$ with spectrum as in (20), the finite lookahead MMSE has the following lower bound:

$$\text{lmmse}_G(d, \gamma) \geq \int \text{lmmse}_\alpha(d, \gamma)\, d\mu(\alpha), \tag{21}$$

where $\text{lmmse}_\alpha(d, \gamma)$ is characterized explicitly in Lemma 2.

To see why (21) holds note that its right hand side represents the MMSE with lookahead $d$ associated with the process whose law is expressed in (18), while the left side corresponds to this MMSE under a Gaussian source with the same spectrum.

In Fig. 2, we illustrate the bound in (21). We consider an equal mixture of two for a Gaussian process $\mathbf{G}$ whose spectrum is an equal mixture of two Ornstein-Uhlenbeck processes with parameters $\alpha_1$ and $\alpha_2$, as well as a Gaussian process with the same spectrum. Analytical expressions for this computation can be obtained, but we will omit the details in this presentation. In Section IV-A, we discuss the computation of the MMSE with lookahead for any stationary Gaussian process, of a known spectrum. This computation, based on Wiener-Kolmogorov theory, relies on the spectral factorization of the spectrum of the AWGN corrupted process. This factorization is usually difficult to perform. Thus, a simple bound on the MMSE with lookahead, such as in (21), is quite useful.

A natural question that emerges from the above discussion is, what functions can be decomposed into mixtures of spectra of Ornstein-Uhlenbeck processes? To answer this question, note that one can arrive at any spectrum $S_G$ which can be expressed (upto a multiplicative constant) as

$$S_G(\omega) = \int_0^\infty \frac{1}{\alpha^2 + \omega^2}\, d\mu(\alpha), \tag{22}$$

where we use (10) to characterize $S_\alpha(\omega)$. Equivalently, in the time domain, the desired auto-correlation function is expressible as

$$R_G(\tau) = \int_0^\infty e^{-\alpha|\tau|}\, \frac{d\mu(\alpha)}{\alpha}, \tag{23}$$

which can be viewed as a real-exponential transform of $\mu$. However, as can be seen from (22), the spectrum $S_G(\omega)$ is always constrained to be monotonically decreasing with $\omega$. This shows that the space of functions that can be candidates for the spectrum of the process G, is not exhausted by the class of functions decomposable as spectra of Ornstein-Uhlenbeck processes.

Note that Lemma 5 gives us a computable lower bound for the MMSE with lookahead for a Gaussian process $G$ whose spectrum can be expressed as a mixture of spectra of Ornstein-Uhlenbeck processes, under an appropriate mixture $\mu$.

Define, the mismatched MMSE with lookahead $d$ for a filter that assumes the underlying process has law $P^{(\beta)}$ [Ornstein-Uhlenbeck process with parameter $\beta$], whereas the true law of the process is $P^{(\alpha)}$

$$\text{lmmse}_{\alpha,\beta}\,(d, \gamma) = \mathbb{E}_\alpha[(X_0 - \mathbb{E}_\beta[X_0|Y_{-\infty}^d])^2], \tag{24}$$

where the outer expectation is with respect to the true law of the underlying process $P^{(\alpha)}$, while the inner expectation is with respect to the mismatched law $P^{(\beta)}$. Note that the mismatched filter which assumes that the underlying signal is an Ornstein-Uhlenbeck process with parameter $\beta$, is in particular, a linear filter. The process $G$ is Gaussian, and hence the optimal filter for $G$ is also linear. Thus, for any fixed $\beta$, the

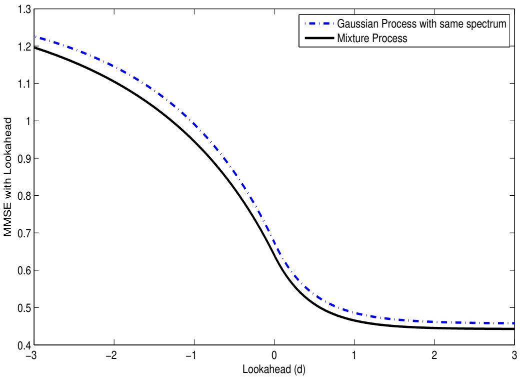


Fig. 2: Illustration of the bound in (21). Plot of MMSE with lookahead for the processes $\mathbf{G}$ and $\mathbf{X}$, for snr $= 1$; where $\mathbf{X}$ is a mixture of two Ornstein-Uhlenbeck processes with parameters $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$ respectively, and $\mathbf{G}$ is a stationary Gaussian process with the same spectrum as $\mathbf{X}$.

mismatched filter thus defined, will be suboptimal for process $G$. This yields the following natural upper bound:

*Lemma 6 (Upper Bound):* For a Gaussian process $G$ with spectrum as in (20), the finite lookahead MMSE has the following upper bound:

$$\text{lmmse}_G(d,\gamma) \leq \min_{\beta>0} \int \text{lmmse}_{\alpha,\beta}(d,\gamma)\, d\mu(\alpha). \tag{25}$$

We should note that the upper bound stated in Lemma 6 can be difficult to compute for continuous-time processes, even though the filter thus realized is linear. In summary, analysis of $\text{lmmse}(\cdot,\gamma)$ for the Ornstein-Uhlenbeck process provides us with a rich class of processes for which we can compute the finite lookahead estimation error. And for a class of Gaussian processes (23), we have a lower and upper bound on this quantity. These characterizations may be used to approximate the behavior of the MMSE with finite lookahead for a large class of Gaussian processes, instead of obtaining the optimal solution which relies on spectral factorization, and can be cumbersome.

## IV. Stationary Gaussian Processes

In this section, we focus exclusively on Gaussian processes having a known spectrum. We are interested in obtaining the MMSE with finite lookahead in estimating such a process when corrupted by AWGN at a fixed SNR level $\gamma$.

### *A. Calculation of MMSE with finite lookahead via Spectral Factorization*

Let $\{X_t, t \in \mathbb{R}\}$ stationary and Gaussian, with power spectral density $S_X(\omega)$, be the input to the continuous-time Gaussian channel at SNR $\gamma$. $Y_{(\cdot)}$, the noise corrupted version of $X_{(\cdot)}$, is stationary, Gaussian and has PSD $S_Y(\omega) = 1 + \gamma S_X(\omega)$. Let $S_Y^+(\omega)$ be the Wiener-Hopf factorization of $S_Y(\omega)$, i.e. a function satisfying $\left|S_Y^+(\omega)\right|^2 = S_Y(\omega)$ with $1/S_Y^+(\omega)$ being the transfer function of a stable causal filter. This factorization exists whenever the Paley-Wiener conditions are satisfied for $S_Y(\omega)$. Denote by $\tilde{Y}_t$ the output of the filter $1/S_Y^+(\omega)$ applied on $Y_t$, then $\tilde{Y}_t$ is a standard Brownian motion. This theory is well developed and the reader is encouraged to peruse any standard reference in continuous-time estimation for details (cf., e.g., [12] and references therein ).

Let $h(t)$ be the impulse response of the filter with transfer function

$$H(\omega) = \frac{\sqrt{\gamma} S_X(\omega)}{S_Y^-(\omega)}, \tag{26}$$

with $S_Y^-(\omega) = S_Y^+(-\omega)$. The classical result by Wiener and Hopf [13], applied to non-causal filtering of stationary Gaussian signals can be stated as,

$$\mathbb{E}\left[X_0 | Y_{-\infty}^{\infty}\right] = \mathbb{E}\left[X_0 | \tilde{Y}_{-\infty}^{\infty}\right] = \int_{-\infty}^{\infty} h(-t)\, d\tilde{Y}_t. \tag{27}$$

Moreover, an expression for the finite-lookahead MMSE estimator of $X_0$ can be immediately derived, using the fact that $\tilde{Y}_{\cdot}$ is both a standard Brownian motion and a reversible causal function of the observations:

$$\mathbb{E}\left[X_0 | Y_{-\infty}^{d}\right] = \int_{-\infty}^{d} h(-t)\, d\tilde{Y}_t. \tag{28}$$

Using again the fact that $\tilde{Y}_t$ is a Brownian motion, as well as the orthogonality property of MMSE estimators, we can obtain a simple expression for $\text{lmmse}(d,\gamma)$:

$$\begin{aligned} &\text{lmmse}(d,\gamma) \\ &\quad = \mathbb{E}\left(X_0 - \mathbb{E}\left[X_0 | Y_{-\infty}^{d}\right]\right)^2 && (29) \\ &\quad = \mathbb{E}\left(X_0 - \mathbb{E}\left[X_0 | \tilde{Y}_{-\infty}^{d}\right]\right)^2 && (30) \\ &\quad = \mathbb{E}\left(X_0 - \mathbb{E}\left[X_0 | \tilde{Y}_{-\infty}^{\infty}\right]\right)^2 \\ &\qquad + \mathbb{E}\left(\mathbb{E}\left[X_0 | \tilde{Y}_{-\infty}^{\infty}\right] - \mathbb{E}\left[X_0 | \tilde{Y}_{-\infty}^{d}\right]\right)^2 && (31) \\ &\quad = \text{mmse}(\gamma) + \mathbb{E}\left(\int_d^{\infty} h(-t)\, d\tilde{Y}_t\right)^2 && (32) \\ &\quad = \text{mmse}(\gamma) + \int_{-\infty}^{-d} h^2(t) dt, && (33) \end{aligned}$$

where (33) follows by an application of Ito's Isometry property (cf. [14]) for stochastic integrals. This classical formulation of the lookahead problem for stationary Gaussian processes shows us that the MMSE with lookahead behaves gracefully with the lookahead $d$, and is intimately connected to the impulse response $h(\cdot)$ of the filter induced by the Wiener spectral factorization process. In particular, that the solution to the lookahead problem for each value of $d$, can be obtained in this single unified manner as shown in (33), is quite satisfying.

## B. Processes with a rational spectrum

Let $S_X(\omega)$ be a rational function, i.e. of the form $\frac{P(\omega)}{Q(\omega)}$ with $P, Q$ being finite order polynomials in $\omega$. In this case $S_Y(\omega)$ is also a rational function, and the Wiener-Hopf factorization can be performed simply by factorizing the numerator and denominator of $S_Y(\omega)$ and composing $S_Y^+(\omega)$ from the stable zeros and poles.

Recall the definition of $p_d$ in (15). Clearly, $p_0 = 1$ and $\lim_{d\to\infty} p_d = 0$. The value $p_d$ goes to zero in the same rate as the lmmse converges to the non-causal MMSE. For the Ornstein-Uhlenbeck process we observed that $p_d$ converges exponentially to zero. In the following lemma, we generalize this result to include all Gaussian input processes with rational spectra.

*Lemma 7:* For any Gaussian input process with a rational spectrum, $\mathrm{lmmse}(d, \mathrm{snr})$ approaches $\mathrm{mmse}(\mathrm{snr})$ exponentially fast as $d \to \infty$. Equivalently, $p_d \to 0$ exponentially, as $d \to \infty$.

*Proof:* Since $S_Y^+(\omega)$ is a rational function, so is $H(\omega)$ defined in (26). Therefore, $h(t)$ must be a finite sum of exponentially decreasing functions, and consequently $\int_{-\infty}^{-d} h^2(t)dt$ must also decrease exponentially as $d \to \infty$. This, in conjunction with the relation in (33) concludes the proof. ■

However, it is important to investigate whether there exist Gaussian input spectra, such that the decay of $p_d$ is not exponentially fast.

The answer appears to be affirmative. As an example, consider the input power spectrum

$$S_X^{\text{triang}}(\omega) = (1 - |\omega|)\, \mathbf{1}_{\{|\omega| \le 1\}}. \tag{34}$$

In Figure 3 we plot the behaviour of $p_d$ as a function of $d$ for input power spectrum $S_X^{\text{triang}}(\omega)$ and different SNR's. This demonstrates the polynomial rate of convergence of the finite lookahead estimation error towards to non-causal MMSE. The values of $\mathrm{lmmse}(d, \mathrm{snr})$ were found by numerically approximating $h(t)$, using FFT in order to perform the factorization of $S_Y(\omega)$.

## V. Conclusions

This work can be viewed as a step towards understanding the role of finite lookahead in mean squared signal estimation under additive white Gaussian noise. We study the class of Ornstein-Uhlenbeck processes and explicitly characterize the dependence of squared estimation loss on lookahead and SNR. We extend this result to the class of processes that can be expressed as mixtures of Ornstein-Uhlenbeck processes, and use this characterization to present bounds on the finite lookahead MMSE for a class of Gaussian processes. Finally, we observe that Gaussian processes with rational spectra have the finite lookahead MMSE converging exponentially rapidly from the causal, to the non-causal MMSE.

In future work, it would be interesting to see whether the gap between the upper bound in Lemma 6, and the lower bound in Lemma 5 can be characterized explicitly, to yield tight approximations of the Gaussian MMSE with lookahead. Indeed, the plot in Fig. 2 is quite encouraging in this respect

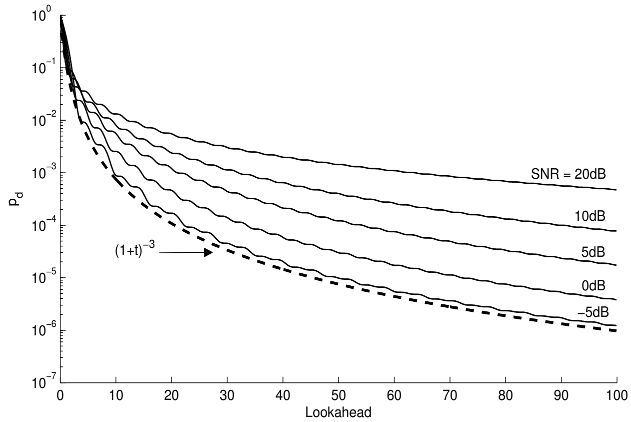


Fig. 3: Plot of $p_d$ as defined in (15), vs. lookahead in various SNRs, for input power spectrum $(1 - |\omega|)\, \mathbf{1}_{\{|\omega| \le 1\}}$. The finite lookahead MMSE is seen to converge to the non-causal MMSE at an approximately cubic rate.

- indicating that the lower bound in Lemma 5 is a good approximation of the actual finite lookahead MMSE.


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