The capacity region of a class of two-user degraded compound broadcast channels

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***Abstract*—Weingarten et al. established the capacity region for the two-user degraded compound broadcast channel where the realizations of each user exhibit a certain degradedness order. The degradedness order is defined through an additional fictitious user whose channel is stochastically degraded with respect to each realization from one set (the stronger receiver) while each realization from the other set (the weaker receiver) is stochastically degraded with respect to him. Rather than specifying a fictitious user, we consider a two-user degraded compound broadcast channel which is pair-wise degraded, i.e., each realization from the weaker receiver is stochastically degraded with respect to each realization from the stronger receiver.**

In this paper, we first prove the capacity region for a discrete memoryless class of this broadcast channels where the weaker receiver has only two possible realizations while there is an arbitrary number of possible realizations for the stronger receiver. Next, we consider the equivalent class of degraded compound MIMO Gaussian broadcast channels. To show that Gaussian inputs attain the capacity region, we prove a new extremal entropy inequality using a technique recently introduced by Geng and Nair that was used to resolve the capacity region of the two-user MIMO Gaussian broadcast channel with common and private information.

I. Introduction

In [1], Weingarten et al. considered the two-user degraded compound broadcast channel (BC), where the first user has K_1 possible realizations Y_i, $i \in \{1, \ldots, K_1\}$, while the second user has K_2 possible realizations Z_j, $j \in \{1, \ldots, K_2\}$. The transmitter has no knowledge of the actual realization while the receivers have perfect knowledge of the actual realization. There are two private messages M_1 and M_2 which should be decoded successfully by receivers 1 and 2, respectively, regardless of the actual realization. Furthermore, there is an additional fictitious user Y^* whose channel is specified by the marginal $p(y^*|x)$. The degradedness order is then specified by the additional fictitious user such that there exist distributions $p^{'}(y^*|y_i)$ and $p^{'}(z_j|y^*)$ for each i, j, where $i \in \{1, \ldots, K_1\}$ and $j \in \{1, \ldots, K_2\}$, such that

$$p(y^*|x) = \sum_{y_i} p^{'}(y^*|y_i)\, p(y_i|x)$$
$$p(z_j|x) = \sum_{y^*} p^{'}(z_j|y^*)\, p(y^*|x).$$

Hence, the fictitious user is stochastically degraded with respect to (w.r.t.) each realization Y_i of the stronger receiver while each realization Z_j of the weaker receiver is stochastically degraded w.r.t him.

Weingarten et al. proved the capacity region of the discrete memoryless case [1, Lemma 4] as well as the equivalent two-user degraded compound MIMO Gaussian BC given by

$$\begin{aligned} \boldsymbol{Y}_i &= \boldsymbol{X} + \boldsymbol{N}_i^{(y)}, \quad i = 1, 2, \ldots, K_1; \\ \boldsymbol{Z}_j &= \boldsymbol{X} + \boldsymbol{N}_j^{(z)}, \quad j = 1, 2, \ldots, K_2, \end{aligned} \tag{1}$$

where $\boldsymbol{X}$ is a real input vector of size $t \times 1$; $\boldsymbol{Y}_i$ and $\boldsymbol{Z}_j$ are real output vectors of size $t \times 1$ corresponding to realizations i and j of user 1 and user 2, respectively; $\boldsymbol{N}_i^{(y)}$ and $\boldsymbol{N}_j^{(z)}$ are real Gaussian noise vectors with zero mean and covariance matrices $\mathbf{N}_i^{(y)}$ and $\mathbf{N}_j^{(z)}$, respectively; and where a degradedness constraint is imposed on the noise covariance matrices of the two users such that a covariance matrix, $\mathbf{N}^*$, exists where

$$\mathbf{N}_i^{(y)} \preceq \mathbf{N}^* \preceq \mathbf{N}_j^{(z)}, \quad \forall i, j. \tag{2}$$

In this paper, we consider a more general definition for the two-user degraded compound BC. Rather than specifying a fictitious user, the two-user degraded compound BC satisfies the following condition:

***Condition** 1:* For each $i \in \{1, \ldots, K_1\}$ and each $j \in \{1, \ldots, K_2\}$, there exists a distribution $p^{'}(z_j|y_i)$ such that

$$p(z_j|x) = \sum_{y_i} p^{'}(z_j|y_i)\, p(y_i|x).$$

The equivalent two-user degraded compound MIMO Gaussian BC for Condition 1 satisfies the more general constraint

$$\mathbf{N}_i^{(y)} \preceq \mathbf{N}_j^{(z)}, \quad \forall i, j, \tag{3}$$

rather than the more restrictive constraint (2).

As shown by a specific example in [1, Appendix I], where $K_1 = K_2 = 2$, the class of two-user degraded compound MIMO Gaussian BCs satisfying (3) is strictly larger than that satisfying (2).

In Section II of this paper, we first prove the capacity region for a *restricted* class of BCs satisfying Condition 1, where K_1 is arbitrary and $K_2 = 2$. Next, we consider the equivalent class of two-user degraded compound MIMO Gaussian BCs satisfying (3) and where K_1 is arbitrary and $K_2 = 2$. This class of BCs includes the specific example considered in [1, Appendix I], whose capacity region was left unresolved. We first prove a new extremal entropy inequality in Section III using the technique introduced by Geng and Nair in [2]. Finally, we employ the extremal entropy inequality to prove the capacity region of the equivalent class of two-user degraded compound MIMO Gaussian BCs in Section IV.

II. Capacity region of a class of two-user discrete memoryless degraded compound BCs

In this section, we consider the discrete memoryless degraded compound BC. Since the capacity region of a BC depends only on the marginals, all outputs can be solely defined by their conditional probability functions: $p\left(y_1|x\right), \ldots, p\left(y_{K_1}|x\right)$, $p\left(z_1|x\right)$ and $p\left(z_2|x\right)$.

***Theorem** 1:* The capacity region $\mathcal{C}$ of the discrete memoryless degraded compound BC satisfying Condition 1, where K_1 is arbitrary and $K_2 = 2$, is given by all non-negative rate pairs $\left(R_1, R_2\right)$ satisfying

$$R_1 \leq \min_{i=1,\ldots,K_1} I\left(X; Y_i|U\right) \tag{4}$$

$$R_2 \leq \min\left\{I\left(U; Z_1\right), I\left(U; Z_2\right)\right\} \tag{5}$$

for some probability distribution function (pdf) satisfying

$$p\left(u, x, y_1, \ldots, y_{K_1}, z_1, z_2\right) = p\left(u\right) p\left(x|u\right) \prod_{i=1}^{K_1} p\left(y_i|x\right) p\left(z_1|x\right) p\left(z_2|x\right). \tag{6}$$

Proof: The direct coding theorem relies on successive decoding at the stronger receiver and is standard in the literature. To prove the converse, we first note that an alternative description of the region $\mathcal{C}$ is given by all rate pairs $\left(R_1, R_2\right)$ satisfying

$$R_1 + R_2 \leq \min_{i=1,\ldots,K_1} I\left(U; Z_1\right) + I\left(X; Y_i|U\right) \tag{7}$$

$$R_1 + R_2 \leq \min_{i=1,\ldots,K_1} I\left(U; Z_2\right) + I\left(X; Y_i|U\right) \tag{8}$$

$$R_2 \leq \min\left\{I\left(U; Z_1\right), I\left(U; Z_2\right)\right\} \tag{9}$$

for some pdf satisfying (6). It is not too difficult to show that the two characterizations are equivalent.

To prove the converse, let us first consider the following:

$$\begin{aligned} I\left(M_2; Z_{1,1}^N\right) &= \sum_{n=1}^N I\left(M_2; Z_{1,n}|Z_{1,1}^{n-1}\right) \\ &\leq \sum_{n=1}^N I\left(M_2, Z_{1,1}^{n-1}, Z_{2,n+1}^N; Z_{1,n}\right) \\ &= \sum_{n=1}^N I\left(U_n; Z_{1,n}\right) \end{aligned}$$

where $U_n \triangleq \left(M_2, Z_{1,1}^{n-1}, Z_{2,n+1}^N\right)$. Likewise, we may show that the following holds:

$$\begin{aligned} I\left(M_2; Z_{2,1}^N\right) &= \sum_{n=1}^N I\left(M_2; Z_{2,n}|Z_{2,n+1}^N\right) \\ &\leq \sum_{n=1}^N I\left(M_2, Z_{1,1}^{n-1}, Z_{2,n+1}^N; Z_{2,n}\right) \\ &= \sum_{n=1}^N I\left(U_n; Z_{2,n}\right). \end{aligned}$$

Next, for each realization Y_i of the stronger receiver, we define the auxiliary outputs $\left(Z_1^{(i)}, Z_2^{(i)}\right)$ which are physically degraded w.r.t Y_i and where

$$p\left(z_1^{(i)}|x\right) = p\left(z_1|x\right), \quad \forall z_1 = z_1^{(i)} \in \mathcal{Z}_1, \tag{10}$$

$$p\left(z_2^{(i)}|x\right) = p\left(z_2|x\right), \quad \forall z_2 = z_2^{(i)} \in \mathcal{Z}_2. \tag{11}$$

We can always find such auxiliary outputs since the BC satisfies Condition 1. We then consider the following:

$$\begin{aligned} & I\left(M_2; Z_{1,1}^N\right) + I\left(M_1; Y_{i,1}^N|M_2\right) \\ & \overset{(a)}{=} I\left(M_2; Z_{1,1}^{(i)N}\right) + I\left(M_1; Y_{i,1}^N|M_2\right) \\ & \leq I\left(M_2; Z_{1,1}^{(i)N}\right) + I\left(X^N; Y_{i,1}^N|M_2\right) \\ & = I\left(M_2; Z_{1,1}^{(i)N}\right) + I\left(X^N; Z_{2,1}^{(i)N}|M_2\right) \\ & \quad + I\left(X^N; Y_{i,1}^N|M_2, Z_{2,1}^{(i)N}\right) \\ & \overset{(b)}{\leq} \sum_{n=1}^N I\left(M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}; Z_{1,n}^{(i)}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Z_{2,n}^{(i)}|M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}\right) \\ & \quad + I\left(X^N; Y_{i,1}^N|M_2, Z_{2,1}^{(i)N}\right) \\ & \leq \sum_{n=1}^N I\left(M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}; Z_{1,n}^{(i)}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Z_{2,n}^{(i)}|M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Y_{i,n}|M_2, Y_{i,1}^{n-1}, Z_{1,1}^{(i)n-1}, Z_{2,n}^{(i)}, Z_{2,n+1}^{(i)N}\right) \\ & \leq \sum_{n=1}^N I\left(M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}; Z_{1,n}^{(i)}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Z_{2,n}^{(i)}|M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Y_{i,n}|M_2, Z_{1,1}^{(i)n-1}, Z_{2,n}^{(i)}, Z_{2,n+1}^{(i)N}\right) \\ & = \sum_{n=1}^N I\left(M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}; Z_{1,n}^{(i)}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Y_{i,n}|M_2, Z_{1,1}^{(i)n-1}, Z_{2,n+1}^{(i)N}\right) \\ & \overset{(c)}{=} \sum_{n=1}^N I\left(M_2, Z_{1,1}^{n-1}, Z_{2,n+1}^N; Z_{1,n}\right) \\ & \quad + \sum_{n=1}^N I\left(X_n; Y_{i,n}|M_2, Z_{1,1}^{n-1}, Z_{2,n+1}^N\right) \end{aligned}$$

$$= \sum_{n=1}^{N} I\left(U_n; Z_{1,n}\right) + \sum_{n=1}^{N} I\left(X_n; Y_{i,n}|U_n\right)$$

where (a) follows from (10), (b) follows from the Csiszár sum lemma and (c) follows from the discrete memoryless nature of the channel as well as (10)-(11). Likewise, we may exchange the role of the index 1 & 2 to prove the following:

$$\begin{aligned} & I\left(M_2; Z_{2,1}^N\right) + I\left(M_1; Y_{i,1}^N|M_2\right) \\ & \leq \sum_{n=1}^{N} I\left(U_n; Z_{2,n}\right) + \sum_{n=1}^{N} I\left(X_n; Y_{i,n}|U_n\right). \end{aligned}$$

Finally, we obtain the result by defining a random variable Q which is uniformly distributed over the integers $1,\ldots,N$ and by setting $U \triangleq (U_Q, Q)$, $X \triangleq X_Q$, $Y_i \triangleq Y_{i,Q}$, $i \in \{1,\ldots,K_1\}$, and $Z_j \triangleq Z_{j,Q}$, $j \in \{1,2\}$. ■

III. A NEW EXTREMAL ENTROPY INEQUALITY

In this section, we consider a new extremal entropy inequality which will be used in the latter section to prove the optimality of Gaussian inputs. We consider the vector additive Gaussian channels given by (1) and satisfying (3). We assume that the inequalities in (3) are strict. Since the set of all covariance matrices are dense, by the continuity of the log det function over positive definite matrices, our result also holds for the case when the inequalities are not strict (see [2, Remark 2] as well as the proof of [3, Theorem 5]).

We wish to prove the following extremal entropy inequality:

***Theorem** 2:*

$$\begin{aligned} & \sup_{\substack{\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^T\right] \preceq \mathbf{K} \\ U \to \boldsymbol{X} \to \left(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{Z}}\right)}} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X}; \boldsymbol{Y}_i|U\right) \\ +\mu_2 \sum_{j=1}^{2} \beta_j I\left(U; \mathbf{Z}_j\right) \end{array} \\ & = \mu_1 \sum_{i=1}^{K_1} \frac{\alpha_i}{2} \log \frac{\left|\mathbf{Q}+\mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|} + \mu_2 \sum_{j=1}^{2} \frac{\beta_j}{2} \log \frac{\left|\mathbf{K}+\mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q}+\mathbf{N}_j^{(z)}\right|} \end{aligned} \tag{12}$$

for some $0 \preceq \mathbf{Q} \preceq \mathbf{K}$ and where $\mu_2 > \mu_1 > 0$, $\vec{\boldsymbol{Y}} \triangleq (\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_{K_1})$, $\vec{\boldsymbol{Z}} \triangleq (\mathbf{Z}_1,\mathbf{Z}_2)$ and $\left(\vec{\alpha},\vec{\beta}\right) \triangleq (\alpha_1,\ldots,\alpha_{K_1},\beta_1,\beta_2) \in \Gamma$. For any $\left(\vec{\alpha},\vec{\beta}\right) \in \Gamma$, we have $\alpha_i, \beta_j \geq 0$ and $\sum_{i=1}^{K_1} \alpha_i = \sum_{j=1}^{2} \beta_j = 1$.

A. Mathematical Preliminaries

As we shall rely on some properties of product channels, we shall first provide some definitions as well as some previously known results.

***Definition** 1:* The vector additive Gaussian product channel can be represented as follows:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{Y}_{i,1} \\ \boldsymbol{Y}_{i,2} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{N}_{i,1}^{(y)} \\ \boldsymbol{N}_{i,2}^{(y)} \end{bmatrix}, \quad i \in \{1,\ldots,K_1\}, \\ \begin{bmatrix} \mathbf{Z}_{j,1} \\ \mathbf{Z}_{j,2} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{N}_{j,1}^{(z)} \\ \boldsymbol{N}_{j,2}^{(z)} \end{bmatrix}, \quad j \in \{1,2\}, \end{aligned} \tag{13}$$

where the noise vectors

$$\boldsymbol{N}_{i,1}^{(y)}, \boldsymbol{N}_{i,2}^{(y)} \sim \mathcal{N}\left(0, \mathbf{N}_i^{(y)}\right), \quad i \in \{1,\ldots,K_1\},$$

$$\boldsymbol{N}_{j,1}^{(z)}, \boldsymbol{N}_{j,2}^{(z)} \sim \mathcal{N}\left(0, \mathbf{N}_j^{(z)}\right), \quad j \in \{1,2\},$$

are all independent of each other as well as $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

***Proposition** 1:* Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be zero-mean independent t-dimensional random vectors. If $\boldsymbol{X}_1 + \boldsymbol{X}_2$ and $\boldsymbol{X}_1 - \boldsymbol{X}_2$ are independent, then $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent Gaussian random vectors with identical covariance matrices.

Proof: See proof of [2, Cor. 3]. ■

***Proposition** 2:* Consider the following vector additive Gaussian product channel with identical components:

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_1 + \boldsymbol{N}_1 \\ \boldsymbol{Y}_2 &= \boldsymbol{X}_2 + \boldsymbol{N}_2 \end{aligned}$$

where $\boldsymbol{N}_1 \sim \mathcal{N}(0, \mathbf{N}_G)$ and $\boldsymbol{N}_2 \sim \mathcal{N}(0, \mathbf{N}_G)$ are independent. Let $\tilde{\boldsymbol{X}} = \frac{1}{\sqrt{2}}(\boldsymbol{X}_1 + \boldsymbol{X}_2)$, $\hat{\boldsymbol{X}} = \frac{1}{\sqrt{2}}(\boldsymbol{X}_1 - \boldsymbol{X}_2)$, $\tilde{\boldsymbol{Y}} = \frac{1}{\sqrt{2}}(\boldsymbol{Y}_1 + \boldsymbol{Y}_2)$, and $\hat{\boldsymbol{Y}} = \frac{1}{\sqrt{2}}(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)$, we must have

$$I\left(\tilde{\boldsymbol{X}}, \hat{\boldsymbol{X}}; \tilde{\boldsymbol{Y}}, \hat{\boldsymbol{Y}}\right) = I\left(\boldsymbol{X}_1, \boldsymbol{X}_2; \boldsymbol{Y}_1, \boldsymbol{Y}_2\right).$$

Proof: See proof of [2, Claim 1]. ■

***Proposition** 3:* Consider the vector additive Gaussian product channels given by (13). The random vectors $\mathbf{Z}_{1,1}$ and $\mathbf{Z}_{1,2}$, $\mathbf{Z}_{2,1}$ and $\mathbf{Z}_{2,2}$, $\mathbf{Z}_{1,1}$ and $\mathbf{Z}_{2,2}$, $\mathbf{Z}_{1,2}$ and $\mathbf{Z}_{2,1}$ are independent if and only if $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent.

Proof: See proof of [2, Claim 2]. ■

The above propositions were used to prove the capacity region of the two-user Gaussian MIMO BC with private and common messages in [2] and will also be useful later.

B. Proof of Theorem 2

Let us first show that Gaussian inputs attain the following:

$$\sup_{\substack{\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^T\right] \preceq \mathbf{K} \\ U \to \boldsymbol{X} \to \left(\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_{K_1}, \mathbf{Z}_1, \mathbf{Z}_2\right)}} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X}; \boldsymbol{Y}_i|U\right) \\ -\mu_2 \sum_{j=1}^{2} \beta_j I\left(\boldsymbol{X}; \mathbf{Z}_j|U\right) \end{array}. \tag{14}$$

We first define the following function of $P_{\boldsymbol{X}}$:

$$s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right) \triangleq \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X}; \boldsymbol{Y}_1\right) - \mu_2 \sum_{j=1}^{2} \beta_j I\left(\boldsymbol{X}; \mathbf{Z}_j\right). \tag{15}$$

We define the following function of $P_{U,\boldsymbol{X}}$:

$$s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}|U\right) \triangleq \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X}; \boldsymbol{Y}_i|U\right) - \mu_2 \sum_{j=1}^{2} \beta_j I\left(\boldsymbol{X}; \mathbf{Z}_j|U\right). \tag{16}$$

We define the following function of $P_{\boldsymbol{X}}$:

$$S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right) \triangleq \sup_{P_{U|\boldsymbol{X}}} s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}|U\right). \tag{17}$$

We also define the following function of $P_{Q,\boldsymbol{X}}$:

$$S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}|Q\right) \triangleq \mathbb{E}_Q\left[S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}|Q=q\right)\right].$$

Finally, for $\mathbf{K} \succ 0$, define

$$V_{\vec{\alpha},\vec{\beta}}\left(\mathbf{K}\right) = \sup_{\boldsymbol{X}:\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^T\right] \preceq \mathbf{K}} S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right).$$

We also denote the extensions of (15)-(17) to the vector additive Gaussian product channels by $s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1,\boldsymbol{X}_2)$, $s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1,\boldsymbol{X}_2|U)$ and $S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1,\boldsymbol{X}_2)$, respectively.

***Proposition** 4:* The following inequality holds for the vector additive Gaussian product channels:

$$S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1,\boldsymbol{X}_2) \leq S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1) + S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_2).$$

If $P_{U|\boldsymbol{X}_1,\boldsymbol{X}_2}$ attains $S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1,\boldsymbol{X}_2)$ and equality holds above, then all the following must be true: 1) $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are conditionally independent given U; 2) $(U,\boldsymbol{X}_1)$ attains $S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1)$ and 3) $(U,\boldsymbol{X}_2)$ attains $S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_2)$.

Proof: Let us first define the following auxiliary outputs:

$$\boldsymbol{Z}_{1,1}^{(i)} = \boldsymbol{Y}_{i,1} + \overline{\boldsymbol{N}}_i, \quad i \in \{1,\ldots,K_1\} \tag{18}$$

where $\overline{\boldsymbol{N}}_i \sim \mathcal{N}\left(0, \mathbf{N}_1^{(z)} - \mathbf{N}_i^{(y)}\right)$. Hence, $P_{\mathbf{Z}_{1,1}^{(i)}|\mathbf{X}_1}$, $i \in \{1,\ldots,K_1\}$, and $P_{\mathbf{Z}_{1,1}|\mathbf{X}_1}$ are equivalent. Furthermore, $\boldsymbol{Z}_{1,1}^{(i)}$ is physically degraded w.r.t $\boldsymbol{Y}_{i,1}$.

For any $P_{U|\boldsymbol{X}_1,\boldsymbol{X}_2}$, we observe the following:

$$\begin{aligned}
&\alpha_i I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}_{i,1},\boldsymbol{Y}_{i,2}|U)\\
&= \alpha_i\left[I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U) + I(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Y}_{i,1})\right]\\
&\overset{(a)}{=} \alpha_i\left[I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U,\boldsymbol{Z}_{2,2}) + I\left(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&\quad + \alpha_i\left[I(\boldsymbol{Z}_{2,2};\boldsymbol{Y}_{i,1}|U) - I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&= \alpha_i\left[I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U,\boldsymbol{Z}_{2,2}) + I\left(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&\quad + \alpha_i\left[I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Z}_{2,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right) - I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&\quad + \alpha_i I\left(\boldsymbol{Z}_{1,1}^{(i)};\boldsymbol{Z}_{2,2}|U\right)\\
&\overset{(b)}{=} \alpha_i\left[I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U,\boldsymbol{Z}_{2,2}) + I(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1})\right]\\
&\quad + \alpha_i\left[I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Z}_{2,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right) - I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&\quad + \alpha_i I(\boldsymbol{Z}_{1,1};\boldsymbol{Z}_{2,2}|U).
\end{aligned} \tag{19}$$

where (a) follows from the fact that $\boldsymbol{X}_1 \to \boldsymbol{Y}_{i,1} \to \boldsymbol{Z}_{1,1}^{(i)}$ forms a Markov chain and (b) follows from (18) and the fact that we are considering a product channel.

Let us also consider the following:

$$\begin{aligned}
&\beta_1 I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Z}_{1,1},\boldsymbol{Z}_{1,2}|U)\\
&= \beta_1\left[I(\boldsymbol{X}_1;\boldsymbol{Z}_{1,1}|U) + I(\boldsymbol{X}_2;\boldsymbol{Z}_{1,2}|U,\boldsymbol{Z}_{1,1})\right]\\
&= \beta_1\left[I(\boldsymbol{X}_1;\boldsymbol{Z}_{1,1}|U,\boldsymbol{Z}_{2,2}) + I(\boldsymbol{X}_2;\boldsymbol{Z}_{1,2}|U,\boldsymbol{Z}_{1,1})\right]\\
&\quad + \beta_1 I(\boldsymbol{Z}_{1,1};\boldsymbol{Z}_{2,2}|U)
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\beta_2 I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Z}_{2,1},\boldsymbol{Z}_{2,2}|U)\\
&= \beta_2\left[I(\boldsymbol{X}_1;\boldsymbol{Z}_{2,1}|U,\boldsymbol{Z}_{2,2}) + I(\boldsymbol{X}_2;\boldsymbol{Z}_{2,2}|U)\right]\\
&= \beta_2\left[I(\boldsymbol{X}_1;\boldsymbol{Z}_{2,1}|U,\boldsymbol{Z}_{2,2}) + I(\boldsymbol{X}_2;\boldsymbol{Z}_{2,2}|U,\boldsymbol{Z}_{1,1})\right]\\
&\quad + \beta_2 I(\boldsymbol{Z}_{1,1};\boldsymbol{Z}_{2,2}|U).
\end{aligned} \tag{21}$$

Combining (19)-(21), we obtain

$$\begin{aligned}
&\mu_1\sum_{i=1}^{K_1}\alpha_i I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}_{i,1},\boldsymbol{Y}_{i,2}|U)\\
&- \mu_2\sum_{j=1}^{2}\beta_j I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Z}_{j,1},\boldsymbol{Z}_{j,2}|U)\\
&= \mu_1\sum_{i=1}^{K_1}\alpha_i I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U,\boldsymbol{Z}_{2,2}) - \mu_2\sum_{j=1}^{2}\beta_j I(\boldsymbol{X}_1;\boldsymbol{Z}_{j,1}|U,\boldsymbol{Z}_{2,2})\\
&+ \mu_1\sum_{i=1}^{K_1}\alpha_i I(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}) - \mu_2\sum_{j=1}^{2}\beta_j I(\boldsymbol{X}_2;\boldsymbol{Z}_{j,2}|U,\boldsymbol{Z}_{1,1})\\
&+ \mu_1\sum_{i=1}^{K_1}\alpha_i\left[I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Z}_{2,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right) - I\left(\boldsymbol{Y}_{i,1};\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}^{(i)}\right)\right]\\
&+ (\mu_1-\mu_2) I(\boldsymbol{Z}_{1,1};\boldsymbol{Z}_{2,2}|U)\\
&\overset{(a)}{\leq} s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1|U,\boldsymbol{Z}_{2,2}) + s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_2|U,\boldsymbol{Z}_{1,1})\\
&\overset{(b)}{\leq} S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1) + S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_2)
\end{aligned}$$

where (a) follows from the fact that $\boldsymbol{Z}_{2,2}$ is stochastically degraded w.r.t. $\boldsymbol{Y}_{i,2}$ and the fact that $\mu_2 > \mu_1$ (assumption) and (b) follows from the definition of $S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X})$. When equality holds, we must have $I(\boldsymbol{Z}_{1,1};\boldsymbol{Z}_{2,2}|U) = 0$ and hence, from Proposition 3, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are conditionally independent given U. We also have

$$\begin{aligned}
&\mu_1\sum_{i=1}^{K_1}\alpha_i I(\boldsymbol{X}_1;\boldsymbol{Y}_{i,1}|U,\boldsymbol{Z}_{2,2}) - \mu_2\sum_{j=1}^{2}\beta_j I(\boldsymbol{X}_1;\boldsymbol{Z}_{j,1}|U,\boldsymbol{Z}_{2,2})\\
&= S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_1)
\end{aligned}$$

and

$$\begin{aligned}
&\mu_1\sum_{i=1}^{K_1}\alpha_i I(\boldsymbol{X}_2;\boldsymbol{Y}_{i,2}|U,\boldsymbol{Z}_{1,1}) - \mu_2\sum_{j=1}^{2}\beta_j I(\boldsymbol{X}_2;\boldsymbol{Z}_{j,2}|U,\boldsymbol{Z}_{1,1})\\
&= S_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_2).
\end{aligned}$$

■

The following proposition shows that the supremum is attained:

***Proposition** 5:* There is a pair of random variables $(U_*,\boldsymbol{X}_*)$ with $|U_*| \leq \frac{t(t+1)}{2} + 1$ such that

$$V_{\vec{\alpha},\vec{\beta}}(\mathbf{K}) = s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_*|U_*). \tag{22}$$

Proof: The result follows using the same arguments from the proof of [2, Claim 6]. ■

We assume that $P_{U_*,\boldsymbol{X}_*}$ attains $V_{\vec{\alpha},\vec{\beta}}(\mathbf{K})$.

***Proposition** 6:* Let $(U_1,U_2,\boldsymbol{X}_1,\boldsymbol{X}_2) \sim P_{U_1,\boldsymbol{X}_1}P_{U_2,\boldsymbol{X}_2}$ be two i.i.d copies of $P_{U_*,\boldsymbol{X}_*}$. Let $\tilde{U} = (U_1,U_2)$, $\tilde{\boldsymbol{X}}_{u_1,u_2} \sim \frac{1}{\sqrt{2}}(\boldsymbol{X}_{u_1}+\boldsymbol{X}_{u_2})$, $\hat{\boldsymbol{X}}_{u_1,u_2} \sim \frac{1}{\sqrt{2}}(\boldsymbol{X}_{u_1}-\boldsymbol{X}_{u_2})$, then the following holds: 1) $\tilde{\boldsymbol{X}}$ and $\hat{\boldsymbol{X}}$ are conditionally independent given $\tilde{U}$, 2) $\tilde{U}$, $\tilde{\boldsymbol{X}}$ attains $V_{\vec{\alpha},\vec{\beta}}(\mathbf{K})$, 3) $\tilde{U}$, $\hat{\boldsymbol{X}}$ attains $V_{\vec{\alpha},\vec{\beta}}(\mathbf{K})$.

Proof: The result follows using the same arguments from the proof of [2, Claim 7]. ■

***Proposition** 7:* There exists a unique $\boldsymbol{X}_* \sim \mathcal{N}(0,\mathbf{K}_*)$, $\mathbf{K}_* \preceq \mathbf{K}$, such that $V_{\vec{\alpha},\vec{\beta}}(\mathbf{K}) = s_{\vec{\alpha},\vec{\beta}}(\boldsymbol{X}_*)$.

Proof: The result follows using the same arguments from the proof of [2, Theorem 2]. ■

Finally, from definition and Proposition 7, we have

$$S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right) \le V_{\vec{\alpha},\vec{\beta}}\left(\mathbf{K}\right) = s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}_*\right).$$

Set $\boldsymbol{X} = \boldsymbol{X}^{'} + \boldsymbol{X}_*$, where $\boldsymbol{X}^{'} \sim \mathcal{N}\left(0, \mathbf{K} - \mathbf{K}_*\right)$. We then have

$$S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right) \ge s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}|\boldsymbol{X}^{'}\right) = s_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}_*\right) = V_{\vec{\alpha},\vec{\beta}}\left(\mathbf{K}\right).$$

Hence, there must exist a $\boldsymbol{X} \sim \mathcal{N}\left(0, \mathbf{K}\right)$ such that $S_{\vec{\alpha},\vec{\beta}}\left(\boldsymbol{X}\right) = V_{\vec{\alpha},\vec{\beta}}\left(\mathbf{K}\right)$. This complete the proof of Theorem 2.

IV. Capacity region of a class of two-user degraded compound MIMO Gaussian BCs

Theorem 3: The capacity region, $\mathcal{C}_{\mathrm{ADC}}\left(\mathbf{K}\right)$, of the channel given by (1) satisfying (3) and where $K_2 = 2$ is given by the set of all non-negative rate pairs $\left(R_1, R_2\right)$ such that

$$R_1 = \min_{i=1,\ldots,K_1} \frac{1}{2} \log \frac{\left|\mathbf{Q} + \mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|}$$

$$R_2 = \min_{j=1,2} \frac{1}{2} \log \frac{\left|\mathbf{K} + \mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q} + \mathbf{N}_j^{(z)}\right|}$$

for some $0 \preceq \mathbf{Q} \preceq \mathbf{K}$.

Proof: For $\mu_1 \ge \mu_2$, we note that

$$\max_{\left(R_1,R_2\right)\in\mathcal{C}_{\mathrm{ADC}}(\mathbf{K})} \mu_1 R_1 + \mu_2 R_2 \overset{(a)}{\le} \max_{\left(R_1+R_2,0\right)\in\mathcal{C}_{\mathrm{ADC}}(\mathbf{K})} \mu_1 \left(R_1 + R_2\right)$$

where (a) follows from the fact that we can always transmit at rate $\left(R_1 + R_2, 0\right) \in \mathcal{C}_{\mathrm{ADC}}\left(\mathbf{K}\right)$ to receiver 1. Hence, the weighted sum-rate is attained by transmitting at maximum rate to receiver 1 when $\mu_1 \ge \mu_2$. Next, we consider the case when $\mu_1 < \mu_2$.

We note the following equalities:

$$\begin{aligned}
&\sup_{\left(R_1,R_2\right)\in\mathcal{C}_{\mathrm{ADC}}(\mathbf{K})} \mu_1 R_1 + \mu_2 R_2 \\
&\overset{(a)}{=} \sup_{\substack{\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^T\right]\preceq\mathbf{K}\\ U\to\boldsymbol{X}\to\vec{\boldsymbol{Y}},\vec{\boldsymbol{Z}}}} \min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X};\boldsymbol{Y}_i|U\right) \\ +\mu_2 \sum_{j=1}^{2} \beta_j I\left(U;\mathbf{Z}_j\right) \end{array} \\
&\overset{(b)}{=} \min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} \sup_{\substack{\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^T\right]\preceq\mathbf{K}\\ U\to\boldsymbol{X}\to\vec{\boldsymbol{Y}},\vec{\boldsymbol{Z}}}} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \alpha_i I\left(\boldsymbol{X};\boldsymbol{Y}_i|U\right) \\ +\mu_2 \sum_{j=1}^{2} \beta_j I\left(U;\mathbf{Z}_j\right) \end{array} \\
&\overset{(c)}{=} \min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \frac{\alpha_i}{2} \log \frac{\left|\mathbf{Q}+\mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|} \\ +\mu_2 \sum_{j=1}^{2} \frac{\beta_j}{2} \log \frac{\left|\mathbf{K}+\mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q}+\mathbf{N}_j^{(z)}\right|} \end{array} \\
&\overset{(d)}{=} \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} \min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \frac{\alpha_i}{2} \log \frac{\left|\mathbf{Q}+\mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|} \\ +\mu_2 \sum_{j=1}^{2} \frac{\beta_j}{2} \log \frac{\left|\mathbf{K}+\mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q}+\mathbf{N}_j^{(z)}\right|} \end{array} \\
&= \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} \begin{array}{l} \mu_1 \min_{i=1,\ldots,K_1} \frac{1}{2} \log \frac{\left|\mathbf{Q}+\mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|} \\ +\mu_2 \min_{j=1,2} \frac{1}{2} \log \frac{\left|\mathbf{K}+\mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q}+\mathbf{N}_j^{(z)}\right|} \end{array}
\end{aligned}$$

where (a) follows from Theorem 1; (b) follows from the minimax theorem in [4, Theorem 4.2]; (c) follows from Theorem 2; and (d) follows from the following observations: Let us define

$$F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}, \vec{\beta}\right) \triangleq \begin{array}{l} \mu_1 \sum_{i=1}^{K_1} \frac{\alpha_i}{2} \log \frac{\left|\mathbf{Q}+\mathbf{N}_i^{(y)}\right|}{\left|\mathbf{N}_i^{(y)}\right|} \\ +\mu_2 \sum_{j=1}^{2} \frac{\beta_j}{2} \log \frac{\left|\mathbf{K}+\mathbf{N}_j^{(z)}\right|}{\left|\mathbf{Q}+\mathbf{N}_j^{(z)}\right|} \end{array}.$$

Let $\left(\vec{\alpha}^{**}, \vec{\beta}^{**}\right) \in \arg\min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}, \vec{\beta}\right)$ and let $\mathbf{Q}^{**} \in \arg\max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right)$. For any $\left(\vec{\alpha}^{'}, \vec{\beta}^{'}\right) \in \Gamma$ and $\delta \in (0,1)$, let

$$\begin{aligned}
\vec{\alpha}_\delta &\triangleq \delta\vec{\alpha}^{'} + \left(1-\delta\right)\vec{\alpha}^{**} \\
\vec{\beta}_\delta &\triangleq \delta\vec{\beta}^{'} + \left(1-\delta\right)\vec{\beta}^{**}.
\end{aligned}$$

We may readily verify that $\left(\vec{\alpha}_\delta, \vec{\beta}_\delta\right) \in \Gamma$. Next, let $\vec{\mathbf{Q}}_\delta^{**} \in \arg\max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}_\delta, \vec{\beta}_\delta\right)$. We note that

$$\begin{aligned}
F_{\mathbf{K}}\left(\mathbf{Q}_\delta^{**}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right) &\le \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right) \\
&= F_{\mathbf{K}}\left(\mathbf{Q}^{**}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right) \\
&\le F_{\mathbf{K}}\left(\mathbf{Q}_\delta^{**}, \vec{\alpha}_\delta, \vec{\beta}_\delta\right) \\
\Rightarrow F_{\mathbf{K}}\left(\mathbf{Q}_\delta^{**}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right) &\le F_{\mathbf{K}}\left(\mathbf{Q}_\delta^{**}, \vec{\alpha}^{'}, \vec{\beta}^{'}\right). \qquad (23)
\end{aligned}$$

Using the fact that $g\left(\vec{\alpha}, \vec{\beta}\right) \triangleq \max_{0\preceq\mathbf{Q}\preceq\mathbf{K}} F_{\mathbf{K}}\left(\mathbf{Q}, \vec{\alpha}, \vec{\beta}\right)$ is a continuous function of $\left(\vec{\alpha}, \vec{\beta}\right)$, the fact that the set of all $\mathbf{Q}$ such that $0 \preceq \mathbf{Q} \preceq \mathbf{K}$ is compact and the fact that $\mathbf{Q}^{**}$ is unique (see Proposition 7), there must exist a convergent subsequence $\mathbf{Q}_{\delta_n}^{**} \to \mathbf{Q}^{**}$ and $\delta_n \to 0$ as $n \to \infty$. As (23) holds also for the convergent subsequence, we must have

$$F_{\mathbf{K}}\left(\mathbf{Q}^{**}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right) \le F_{\mathbf{K}}\left(\mathbf{Q}^{**}, \vec{\alpha}^{'}, \vec{\beta}^{'}\right).$$

Since the above holds for any $\left(\vec{\alpha}^{'}, \vec{\beta}^{'}\right) \in \Gamma$, we have $\min_{\left(\vec{\alpha},\vec{\beta}\right)\in\Gamma} F_{\mathbf{K}}\left(\mathbf{Q}^{**}, \vec{\alpha}, \vec{\beta}\right) = F_{\mathbf{K}}\left(\mathbf{Q}^{**}, \vec{\alpha}^{**}, \vec{\beta}^{**}\right)$. ■

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