On the Sum-Rate Capacity of the Phase Fading Z-Interference Channel with a Relay in the Weak Interference Regime

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Abstract—**We study the sum-rate capacity of the Z-interference channel with a relay (Z-ICR) in the weak interference regime, when the channel coefficients are subject to phase fading. This fading model represents many practical communication systems in which phase noise is a major concern, such as OFDM communications and line-of-sight microwave communications. We consider the case in which the relay receives transmissions from only one of the two transmitters, but the transmission of the relay is received at *both destinations*. We present conditions on the channel coefficients under which the sum-rate capacity of the phase fading Z-ICR is achieved by treating interference as noise at each receiver, and derive the corresponding sum-rate capacity. Our results show the benefits of relaying in the presence of interference when interference is weak and relay power is finite.**

I. Introduction

The interference channel (IC) was introduced by Shannon in 1961. The capacity region of the IC is unknown except for some special cases. Specifically, when interference is strong, then decoding both messages at each receiver is optimal. At the opposite extreme, it was recently shown in [1], [2], and [3] that for ICs with additive white Gaussian noise (AWGN), when interference is weak, then treating interference as noise is sum-rate optimal.

When one of the interfering links in the IC is missing, e.g., due to shadowing, then the IC specializes to the Z-IC model in which only one of the receivers is interfered, while the other receiver observes an interference-free signal. The sum-rate capacity of the Gaussian Z-IC was derived in [4], where it was shown that for a Z-IC with weak to moderate interference, the sum-rate capacity is achieved by letting the pair which is not interfered to communicate at a rate equal to its interference-free capacity, and letting the second pair communicate at the maximal rate possible when the interfering signal is treated as AWGN.

In this work we consider the Z-IC with an additional relay node, abbreviated as the Z-ICR. Here, the optimal relaying strategy is not known in the general case, but when the channel is fading, interference is very strong, and reception at the relay is good, it was shown in [5] that forwarding desired information to the destinations is the optimal relaying strategy. In the weak interference (WI) regime, however, as the interference at each receiver is not strong enough to be decoded without constraining the rates of desired information, then this strategy may not be optimal. The sum-rate capacity of the Gaussian IC with a *potent* relay in the WI regime was characterized in [6], where it was shown that in such a scenario, compress-and-forward (CF) at the relay is sum-rate optimal, see [6, Thm. 4]. The generalized degrees of freedom (GDoF) of the IC with a relay was studied in [7] which showed that adding a relay to the IC increases the GDoF when the source-relay links and the interfering links are weak.

To date, there has been no work that studied the sum-rate capacity of the Z-IC with a relay (Z-ICR) in the WI regime for scenarios in which the power of the relay is finite. In this work, we partially fill this gap by considering the phase fading (PF) Z-ICR with a relay that receives transmissions from only *one of the two sources*, but is received at both destinations. Specifically, we shall focus on the sum-rate capacity of the PF Z-ICR in the WI regime. Contrary to [6] which showed optimality of CF for a potent relay, we will study the optimality of decode-and-forward (DF).

Main Contributions: This paper characterizes the sum-rate capacity of the PF Z-ICR in the WI regime. In this model each node has channel state information (CSI) only on its incoming links (only Rx-CSI), the relay operation is causal and full-duplex, and all SNRs are finite. It is also shown that adding a relay to the Z-ICR strictly increases the capacity region relative to the Z-IC without a relay even in the scenarios in which interference is treated as additive noise at each receiver.

The rest of this paper is organized as follows: in Section II we define the system model, introduce the notations, and present a lemma needed to derive our results. In section III we characterize the sum-rate capacity of the PF Z-ICR in the WI regime, and discuss the implications of this result. Finally, in Section IV we give concluding remarks.

II. System Model and Preliminaries

We denote RVs with upper-case letters, e.g., X, Y, their realizations with lower-case letters, e.g., x, y, and the p.d.f. of a continuous RV X with $f_X(x)$. Capital double-stroke letters are used to denote matrices, e.g., $\mathbb{A}$, with the exception that $\mathbb{E}\{X\}$ denotes the stochastic expectation of X. Bold-face letters, e.g., $\mathbf{x}$ denote column vectors, and the i'th element of a vector $\mathbf{x}$ is denoted with x_i. We use x^j to denote the vector $(x_1, x_2, ..., x_j)$. X^* denotes the conjugate of X, $\mathbf{X}^T$ denotes the transpose of $\mathbf{X}$, $\mathbb{A}^H$ denotes the Hermitian

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transpose of $\mathbb{A}$, and $|\mathbb{A}|$ denotes the determinant of $\mathbb{A}$. Given two $n \times n$ Hermitian matrices, $\mathbb{A}, \mathbb{B}$, we write $\mathbb{B} \preceq \mathbb{A}$ if $\mathbb{A} - \mathbb{B}$ is positive semidefinite (p.s.d.) and $\mathbb{B} \prec \mathbb{A}$ if $\mathbb{A} - \mathbb{B}$ is positive definite (p.d.). We denote the circularly symmetric, complex Normal distribution with mean μ and variance σ^2 with $\mathcal{CN}(\mu, \sigma^2)$. cov($\mathbf{X}$) denotes the covariance matrix of $\mathbf{X}$. Finally, subscript "G", e.g., X_G, denotes an RV which is distributed according to a circularly symmetric, complex Normal distribution with the same mean and variance as the indicated RV, e.g., $X_G \sim \mathcal{CN}\big(\mathbb{E}\{X\}, \text{var}\{X\}\big)$. Similarly, X_G^n denotes a vector of jointly Gaussian RVs of length $n \sim \mathcal{CN}\big(\mathbb{E}\{X^n\}, \text{cov}(X^n)\big)$.

The Z-ICR consists of two transmitters, two receivers and a full-duplex relay node. Tx_k sends messages to Rx_k, $k \in \{1, 2\}$. The relay node assists the communication from Tx_1 to Rx_1. This channel is depicted in Fig. 1. The received signals at Rx_1, Rx_2 and the relay at time i are denoted by $Y_{1,i}$, $Y_{2,i}$, and $Y_{3,i}$ respectively. The channel inputs from Tx_1, Tx_2 and the relay at time i are denoted by $X_{1,i}$, $X_{2,i}$ and $X_{3,i}$, respectively. Let $H_{kl,i}$ denote the channel coefficient from node k to node l at time i. The relationship between the channel inputs and its outputs is given by:

$$Y_{1,i} = H_{11,i}X_{1,i} + H_{21,i}X_{2,i} + H_{31,i}X_{3,i} + Z_{1,i} \quad \text{(1a)}$$
$$Y_{2,i} = H_{22,i}X_{2,i} + H_{32,i}X_{3,i} + Z_{2,i} \quad \text{(1b)}$$
$$Y_{3,i} = H_{13,i}X_{1,i} + Z_{3,i}, \quad \text{(1c)}$$

$i = 1, 2, ..., n$, where Z_1, Z_2 and Z_3 are mutually independent RVs, distributed $\mathcal{CN}(0, 1)$, independent in time and independent of the channel inputs and the channel coefficients. The channel input signals are subject to per-symbol average power constraints: $\mathbb{E}\{|X_k|^2\} \leq P_k$, $k \in \{1, 2, 3\}$. The destinations and the relay node have instantaneous *causal* Rx-CSI. Under the phase fading model, the channel coefficients are given by $H_{lk,i} = a_{lk}e^{j\Theta_{lk,i}}$, where $a_{lk} \in \mathfrak{R}_+$ are non-negative constants corresponding to the attenuation of the signal power from node l to node k, and $\Theta_{lk,i}$ are uniformly distributed over $[0, 2\pi)$, independent in time and independent of each other and of the additive noises Z_k, $k \in \{1, 2, 3\}$. Note that since the receivers have CSI, we can set $a_{31} = a_{22} = 1$ by absorbing the channel coefficients into the power constraints and normalizing the remaining channel coefficients accordingly. Also note that in the phase fading model, as the phase of the channel coefficients are uniformly distributed over $[0, 2\pi)$, then we can assume that the phase of H_{31} and H_{22} are zero. This follows since we can multiply both sides of (1a) and (1b) by $e^{-j\Theta_{31,i}}$ and $e^{-j\Theta_{22,i}}$, respectively, without changing the distribution of the resulting channel coefficients or the additive noises at the receivers. The independence of the channel phases implies that $H_{lk,i}$'s are also mutually independent, independent in time, and independent of the other parameters of the scenario. We represent the Rx-CSI at Rx_1 with $\tilde{H}_1 = \big(H_{11}, H_{21}, H_{31}\big) \in \mathfrak{C}^3 \triangleq \tilde{\mathfrak{H}}_1$, the Rx-CSI at Rx_2 with $\tilde{H}_2 = \big(H_{22}, H_{32}\big) \in \mathfrak{C}^2 \triangleq \tilde{\mathfrak{H}}_2$, and the Rx-CSI at the relay with $\tilde{H}_3 = H_{13} \in \mathfrak{C} \triangleq \mathfrak{H}_3$.

Definition 1. An (R_1, R_2, n) code for the Z-ICR consists of two message sets $\mathcal{M}_k \triangleq \{1, 2, ..., 2^{nR_k}\}$, $k \in \{1, 2\}$, two encoders at the sources; $e_k : \mathcal{M}_k \mapsto \mathfrak{C}^n, k \in \{1, 2\}$, and two decoders at the destinations; $g_k : \tilde{\mathfrak{H}}_k^n \times \mathfrak{C}^n \mapsto \mathcal{M}_k$, $k \in \{1, 2\}$. Since the relay receives transmissions only from Tx_1, the transmitted signal at the relay at time i is $x_{3,i} = t_i\big(y_3^{i-1}, h_{13}^{i-1}\big) \in \mathfrak{C}$, $i = 1, 2, ..., n$.

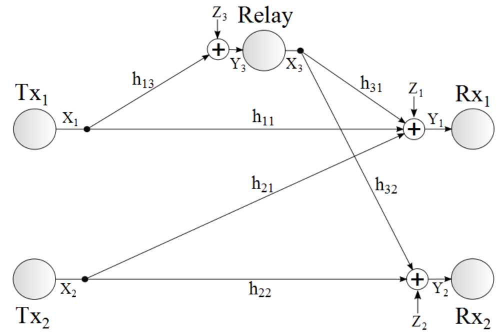

Fig. 1: The Z-ICR. The relay node receives the transmissions from Tx_1 only, but is received at both destinations.

Comment 1. Note that since the message sets at the sources are independent and there is no feedback, then the signals transmitted from Tx_1 and from Tx_2 are also independent. Additionally, as the relay receives transmissions only from Tx_1, then its transmitted signal is independent of the signal from Tx_2, i.e., $f_{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f_{\mathbf{X}_1,\mathbf{X}_3}(\mathbf{x}_1, \mathbf{x}_3) \cdot f_{\mathbf{X}_2}(\mathbf{x}_2)$.

Definition 2. The average probability of error is defined as $\text{P}_e^{(n)} \triangleq \Pr\big(g_1(\tilde{H}_1^n, Y_1^n) \neq M_1 \text{ or } g_2(\tilde{H}_2^n, Y_2^n) \neq M_2\big)$, where each source message is selected independently and uniformly from its message set.

Definition 3. A rate pair (R_1, R_2) is called achievable if for any $\epsilon > 0$ and $\delta > 0$ there exists some blocklength $n_0(\epsilon, \delta)$ such that (s.t.) for every $n > n_0(\epsilon, \delta)$ there exists an $(R_1 - \delta, R_2 - \delta, n)$ code with $\text{P}_e^{(n)} < \epsilon$. The capacity region is defined as the convex hull of all achievable rate pairs.

In the sequel, we shall use the following lemma:

Lemma 1. Let $\mathbf{Z}_{11}$ and $\mathbf{Z}_{22}$ be two n-dimensional complex Normal vectors with i.i.d. entries s.t. $Z_{kk,i} \sim \mathcal{CN}(0, 1), k \in \{1, 2\}, i \in \{1, 2, 3, ..., n\}$. Let $\mathbb{V}_{11}$ and $\mathbb{V}_{22}$ be two $2n \times n$ deterministic complex matrices s.t. $\mathbb{V}_{kk}^H \mathbb{V}_{kk} = \mathbb{D}_{kk}$, where $\mathbb{D}_{kk}, k \in \{1, 2\}$ is a $n \times n$ diagonal matrix with real entries. Let $\mathbb{S}$ be a $2n \times 2n$ p.s.d. Hermitian matrix and let $\mathbf{X}$ be a $2n$-dimensional complex vector, independent of $(\mathbf{Z}_{11}, \mathbf{Z}_{22})$. Consider the following optimization problem:

$$\max_{f(\mathbf{x})} \; h(\mathbb{V}_{11}^H \cdot \mathbf{X} + \mathbf{Z}_{11}) - h(\mathbb{V}_{22}^H \cdot \mathbf{X} + \mathbf{Z}_{22}) \quad (2)$$
$$\text{subject to } \text{cov}(\mathbf{X}) \preceq \mathbb{S},$$

Then, a circularly symmetric complex Normal $\mathbf{X}$ is an optimal solution of (2).

Proof: We follow steps similar to those used in the proof of [9, Corollary 6]. For details see [10, Lemma 6]. ∎

III. Sum-Rate Capacity in the WI Regime

Let $\mathcal{C}_{WI}$ denote the capacity region of the PF Z-ICR in the WI regime. The sum-rate capacity of the PF Z-ICR in the WI regime is characterized in the following theorem:

Theorem 1. *Consider the PF Z-ICR with Rx-CSI defined in Section II. If it holds that*

$$\frac{a_{11}^2 P_1 + P_3}{1 + a_{21}^2 P_2} \leq a_{13}^2 P_1, \tag{3}$$

and there exist two complex scalars ρ_1 and ρ_2, $0 \leq |\rho_1|, |\rho_2| \leq 1$, such that

$$|a_{32}|(1 + a_{21}^2 P_2) \leq \\ |\rho_1| \sqrt{\big(1 - |\rho_2|^2\big) - \big(a_{32}^2\big)\big(a_{11}^2 P_1 + 2a_{11}\sqrt{P_1 P_3}\big)} \tag{4a}$$

$$|a_{21}|(1 + a_{32}^2 P_3) \leq |\rho_2| \sqrt{1 - |\rho_1|^2}, \tag{4b}$$

then, the sum-rate capacity of the PF Z-ICR is given by

$$\sup\{R_1 + R_2\} = \log\Big(1 + \frac{a_{11}^2 P_1 + P_3}{1 + a_{21}^2 P_2}\Big) + \log\Big(1 + \frac{P_2}{1 + a_{32}^2 P_3}\Big), \tag{5}$$

where the supremum is taken over all achievable $(R_1, R_2) \in \mathcal{C}_{WI}$, and it is achieved by $X_k \sim \mathcal{CN}(0, P_k), k \in \{1, 2, 3\}$, mutually independent.

Comment 2. Observe that the conditions in (4) are satisfied if a_{32} and a_{21} are small. As a_{32} and a_{21} correspond to the strengths of the interfering links, the conditions in (4) correspond to the WI regime.

Proof: The proof consists of the following steps:

1) We derive an upper bound on the sum-rate of the PF Z-ICR be letting each receiver observe an appropriate genie signal.
2) We show that this upper bound is maximized by mutually independent, zero-mean, circularly symmetric complex Normal channel inputs.
3) We characterize an achievable rate region for the PF Z-ICR by using codebooks generated according to an i.i.d. in time, mutually independent circularly symmetric complex Normal distribution, by using the DF strategy at the relay, and by treating the interfering signal as noise at each receiver.
4) We derive conditions on the channel coefficients that guarantee that the upper bound on the sum-rate coincides with the achievable sum-rate.

A. An Upper Bound on the Sum-Rate

Consider the PF Z-ICR defined in (1). Assume that at time i, a genie provides the signals $S_{1,i}$ and $S_{2,i}$ and the associated Rx-CSI, $\tilde{H}_{1,i}$ and $\tilde{H}_{2,i}$, to Rx_1 and Rx_2, respectively, where $S_{1,i} \triangleq H_{11,i} X_{1,i} + X_{3,i} + \eta_1 W_{1,i}$, and $S_{2,i} \triangleq X_{2,i} + \eta_2 W_{2,i}$, $i \in \{1, 2, ..., n\}$. W_1 and W_2 are two mutually independent circularly symmetric complex Normal RVs, $\mathcal{CN}(0, 1)$, i.i.d. in time and independent of (X_1^n, X_2^n, X_3^n). Let $\mathbb{E}\{W_{k,i} Z_{k,i}^*\} = \rho_k, k \in \{1, 2\}$. Since $\text{var}(W_k) = \text{var}(Z_k) = 1, k = 1, 2$, then $|\rho_k| \leq 1$. η_1 and η_2 are two complex valued constants chosen by the genie. Therefore, the genie provides to each destination a noisy version of its desired signal. Next, following Comment 1, consider the general n-letter input distribution $f_{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f_{\mathbf{X}_1, \mathbf{X}_3}(\mathbf{x}_1, \mathbf{x}_3) \cdot f_{\mathbf{X}_2}(\mathbf{x}_2)$. Define $n\epsilon_{1n} \triangleq 1 + P_e^{(n)} n R_1$. Observe that as $P_e^{(n)} \to 0$ for $n \to \infty$, then $\epsilon_{1n} \to 0$ when $n \to \infty$. Hence, we obtain

$$\begin{aligned} n(R_1 - \epsilon_{1n}) &\overset{(a)}{\leq} I(X_1^n, X_3^n; Y_1^n, S_1^n | \tilde{H}_1^n) \\ &\overset{(b)}{=} \mathbb{E}_{\tilde{H}_1}\Big\{ h(S_1^n | \tilde{h}_1^n) - n h(S_{1G} | X_{1G}, X_{3G}, \tilde{h}_1) \\ &\quad + h(Y_1^n | S_1^n, \tilde{h}_1^n) - h(Y_1^n | X_1^n, X_3^n, S_1^n, \tilde{h}_1^n) \Big\} \\ &\overset{(c)}{\leq} \mathbb{E}_{\tilde{H}_1}\Big\{ h(S_1^n | \tilde{h}_1^n) - n h(S_{1G} | X_{1G}, X_{3G}, \tilde{h}_1) \\ &\quad + n h(Y_{1G} | S_{1G}, \tilde{h}_1) - h(Y_1^n | X_1^n, X_3^n, S_1^n, \tilde{h}_1^n) \Big\}. \end{aligned} \tag{6}$$

Here, (a) follows from Fano's inequality and the data processing inequality, (b) follows from the fact that W_1^n is generated independent of $(X_1^n, X_3^n, \tilde{H}_2^n)$, the fact that both W_1^n and H_{32}^n are i.i.d. in time and from [8, Lemma 2], and (c) follows from arguments similar to [2, Lemma 1]. See details in [10, Lemma 2]. Following the same approach for upper bounding R_2 and combining with (6), we can upper bound the sum-rate via

$$\begin{aligned} &n(R_1 + R_2 - \epsilon_{1n} - \epsilon_{2n}) \leq \\ &\quad \mathbb{E}_{\tilde{H}_1, \tilde{H}_2}\Big\{ h(S_1^n | \tilde{h}_1^n) - n h(S_{1G} | X_{1G}, X_{3G}, \tilde{h}_1) \\ &\qquad + n h(Y_{1G} | S_{1G}, \tilde{h}_1) - h(Y_1^n | X_1^n, X_3^n, S_1^n, \tilde{h}_1^n) \\ &\qquad + h(S_2^n | \tilde{h}_2^n) - n h(S_{2G} | X_{2G}, \tilde{h}_2) \\ &\qquad + n h(Y_{2G} | S_{2G}, \tilde{h}_2) - h(Y_2^n | X_2^n, S_2^n, \tilde{h}_2^n) \Big\}. \end{aligned} \tag{7}$$

B. The Maximizing Input Distribution

We now show that the upper bound in (7) is maximized by mutually independent, zero-mean circularly symmetric complex Normal channel inputs, i.i.d. in time. First note that by [8, Lemma 1] it follows that for any complex random vector $\mathbf{X}$, the random vector $\mathbf{X}' \triangleq \mathbf{X} - \mathbb{E}\{\mathbf{X}\}$ has the same entropy as $\mathbf{X}$. Then, the most energy efficient strategy would be to transmit $\mathbf{X}'$ rather than $\mathbf{X}$. Hence, it only remains to show that the differences $h(S_2^n | \tilde{H}_2^n) - h(Y_1^n | X_1^n, X_3^n, S_1^n, \tilde{H}_1^n)$ and $h(S_1^n | \tilde{H}_1^n) - h(Y_2^n | X_2^n, S_2^n, \tilde{H}_2^n)$ in (7) are maximized by circularly symmetric complex Normal channel inputs, i.i.d. in time. Let $V_1 \sim \mathcal{CN}(0, 1 - |\rho_1|^2)$, then

$$\begin{aligned} &h(S_2^n | \tilde{H}_2^n) - h(Y_1^n | X_1^n, X_3^n, S_1^n, \tilde{H}_1^n) \\ &\quad \overset{(a)}{=} \mathbb{E}_{\tilde{H}_1, \tilde{H}_2}\Big\{ h\big(X_2^n + \eta_2 \cdot W_2^n | \tilde{h}_2^n\big) \\ &\qquad\qquad - h\big((h_{21} X_2)^n + V_1^n | \tilde{h}_1^n\big) \Big\} \\ &\quad \overset{(b)}{\leq} \mathbb{E}_{\tilde{H}_1, \tilde{H}_2}\Big\{ h\big(X_{2G}^n + \eta_2 \cdot W_2^n | \tilde{h}_2^n\big) \\ &\qquad\qquad - h\big((h_{21} X_{2G})^n + V_1^n | \tilde{h}_1^n\big) \Big\} \\ &\quad \overset{(c)}{\leq} \mathbb{E}_{\tilde{H}_1, \tilde{H}_2}\Big\{ n h\big(X_{2G} + \eta_2 W_2 | \tilde{h}_2\big) \\ &\qquad\qquad - n h\big(h_{21} X_{2G} + V_1 | \tilde{h}_1\big) \Big\} \\ &\quad = n h(X_{2G} + \eta_2 W_2 | \tilde{H}_2) - n h(H_{21} X_{2G} + V_1 | \tilde{H}_1), \end{aligned} \tag{8}$$

as long as $|a_{21}\eta_2|^2 \leq 1 - |\rho_1|^2$. In the above transitions (a) follows from [10, Lemma 1] which is an extension of [1, Lemma 3], (b) follows from [9, Thm. 1] since given $\tilde{h}_1^n$ we have $h\big((h_{21} X_2)^n + V_1^n | \tilde{h}_1^n\big) = h\big(X_2^n + {\mathbb{O}_{h_{21}}^{(n)}}^{-1} V_1^n | \tilde{h}_1^n\big) - \log(a_{21}^{2n})$, where $\mathbb{O}_{h_{21}}^{(n)}$ is an $n \times n$ diagonal matrix s.t. $\mathbb{O}_{h_{21},ii}^{(n)} = h_{21,i}$.

Note that since $a_{21} \neq 0$ then $\mathbb{O}_{h_{21}}^{(n)}$ is invertible. Transition (c) follows from [10, Lemma 4], which is an extension of [3, Lemma 1], which states that if $|a_{21}\eta_2|^2 \leq 1-|\rho_1|^2$, then $h\big(X_{2G}^n+\eta_2\cdot W_2^n|\tilde{h}_2^n\big)-h\big(X_{2G}^n+{\mathbb{O}_{h_{21}}^{(n)}}^{-1}V_1^n|\tilde{h}_1^n\big)$ is maximized with $X_{2G}\sim\mathcal{CN}(0,P_2)$ i.i.d. in time.

Next, assume that $|a_{32}\eta_1|^2 \leq \big(1-|\rho_2|^2\big)-\big(a_{32}^2)(a_{11}^2P_1+2a_{11}\sqrt{P_1P_3}\big)$, and let $V_2\sim\mathcal{CN}(0,1-|\rho_2|^2)$. Thus

$$
\begin{aligned}
&h(S_1^n|\tilde{H}_1^n)-h(Y_2^n|X_2^n,S_2^n,\tilde{H}_2^n)\\
&\overset{(a)}{\leq}\ \mathbb{E}_{\tilde{H}_1,\tilde{H}_2}\Big\{h\big(X_{3G}^n+(h_{11}X_{1G})^n+\eta_1\cdot W_1^n|\tilde{h}_1^n\big)\\
&\qquad\qquad -h\big((h_{32}X_{3G})^n+V_2^n|\tilde{h}_2^n\big)\Big\}\\
&=\ \mathbb{E}_{\tilde{H}_1,\tilde{H}_2}\Big\{h\big(X_{3G}^n+(h_{11}X_{1G})^n+\eta_1\cdot W_1^n|\tilde{h}_1\big)\\
&\qquad\qquad -h\big(X_{3G}^n+\tilde{V}_2^n|\tilde{h}_2\big)\Big\}-\log(a_{32}^{2n})\\
&\overset{(b)}{\leq}\ nh(X_{3G}+H_{11}X_{1G}+\eta_1W_1|\tilde{H}_1)\\
&\qquad\qquad -nh(H_{32}X_{3G}+V_2|\tilde{H}_2),
\end{aligned}
\tag{9}
$$

where $\tilde{V}_{2,i}=\frac{V_{2,i}}{a_{32}}$. Here (a) follows from the fact that X_2^n is independent of X_3^n, from [10, Lemma 1] and from Lemma 1 by substituting

$$
\mathbf{X}=(\mathbf{X}_1^T,\mathbf{X}_3^T)^T,\mathbb{V}_{11}^H=(\mathbb{O}_{h_{11}}^{(n)},\mathbb{I}_n),\mathbb{V}_{22}^H=(\mathbb{O}_0^{(n)},\mathbb{O}_{h_{32}}^{(n)}),
$$

where $\mathbf{X}_1$ and $\mathbf{X}_3$ are two n-dimensional random vectors, $\mathbb{O}_{h_{11}}^{(n)}$ and $\mathbb{O}_{h_{32}}^{(n)}$ are two $n\times n$ diagonal matrices s.t. $\mathbb{O}_{h_{11},ii}^{(n)}=h_{11,i}$ and $\mathbb{O}_{h_{32},ii}^{(n)}=h_{32,i}$, and $\mathbb{O}_0^{(n)}$ is an $n\times n$ matrix in which all entries are equal to zero. Finally, (b) is proven in [10, Proposition 1] which follows along the lines of [3, Lemma 1]. We conclude that if we choose η_1 and η_2 s.t.

$$
|a_{32}\eta_1|^2 \leq \big(1-|\rho_2|^2\big)-a_{32}^2\big(a_{11}^2P_1+2a_{11}\sqrt{P_1P_3}\big) \tag{10a}
$$
$$
|a_{21}\eta_2|^2 \leq \big(1-|\rho_1|^2\big) \tag{10b}
$$

then, using (8) and (9) in (7) we obtain the following upper bound:

$$
\begin{aligned}
R_1+R_2 \leq &\max_{0\leq|\rho|<\sqrt{P_1P_3}} I(X_{1G},X_{3G};Y_{1G},S_{1G}|\tilde{H}_1)\\
&\qquad +I(X_{2G};Y_{2G},S_{2G}|\tilde{H}_2),
\end{aligned}
\tag{11}
$$

for some $f_{X_1,X_2,X_3}(x_1,x_2,x_3)=f_{X_1,X_3}(x_1,x_3)f_{X_2}(x_2)$, where $f_{X_2}(x_2)\sim\mathcal{CN}(0,P_2)$, and $f_{X_1,X_3}(x_1,x_3)\sim\mathcal{CN}(0,\mathbb{K}_{(X_1,X_3)})$, $\mathbb{K}_{(X_1,X_3)}=\begin{bmatrix}P_1 & \rho\\ \rho^* & P_3\end{bmatrix}$. Observe that the maximizing channel inputs are i.i.d. in time. Following the same technique as in [8, Appendix A-A], we obtain that (11) is maximized when $\rho=0$ (see a detailed derivation in [10, Appendix A]). Hence, we conclude that the upper bound on the sum-rate is maximized by mutually independent, zero-mean, circularly symmetric complex Normal channel inputs, i.i.d. in time.

C. An Achievable Rate Region

An achievable rate region for the phase fading Z-ICR is characterized in the following proposition:

Proposition 1. *Consider the PF Z-ICR with Rx-CSI defined in Section II. Let the channel inputs be distributed according to $X_k\sim\mathcal{CN}(0,P_k),k\in\{1,2,3\}$, mutually independent. If*

$$
I(X_1,X_3;Y_1|\tilde{H}_1)\leq I(X_1;Y_3|X_3,\tilde{H}_3), \tag{12}
$$

then an achievable rate region is given by all the rate pairs s.t.

$$
0<R_1 \leq I(X_1,X_3;Y_1|\tilde{H}_1) \tag{13a}
$$
$$
0<R_2 \leq I(X_2;Y_2|\tilde{H}_2). \tag{13b}
$$

Proof: The achievability is based on mutually independent codebooks, DF strategy at the relay, and backward decoding at Rx$_1$. Rx$_2$ uses the point-to-point (PtP) decoding rule. A detailed proof is provided in [10, Section IV-B]. ■

D. The Sum-Rate Capacity in the WI Regime

We now derive conditions under which the upper bound on the sum-rate in (11) coincides with the achievable sum-rate given in (13). Note that for both the maximum in (11) and the achievable sum-rate in (13) we have $f_{X_1,X_2,X_3}(x_1,x_2,x_3)=f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)$, where $X_k\sim\mathcal{CN}(0,P_k),k\in\{1,2,3\}$. The sum-rate in (13) coincides with (11) if

$$
I(X_{1G},X_{3G};S_{1G}|Y_{1G},\tilde{H}_1)=0 \tag{14a}
$$
$$
I(X_{2G};S_{2G}|Y_{2G},\tilde{H}_2)=0 \tag{14b}
$$

holds. For (14a), we obtain

$$
\begin{aligned}
&I(X_{1G},X_{3G};S_{1G}|Y_{1G},\tilde{H}_1)=0\\
&\Leftrightarrow \mathbb{E}_{\tilde{H}_1}\Big\{I(X_{1G},X_{3G};h_{11}X_{1G}+X_{3G}+\eta_1W_1|\\
&\qquad\qquad h_{11}X_{1G}+X_{3G}+h_{21}X_{2G}+Z_1,\tilde{h}_1)\Big\}=0.
\end{aligned}
\tag{15}
$$

Similar to [2, Lemma 8], we apply [10, Lemma 3] and conclude that (15) is satisfied if for all h_{lk} it holds that

$$
\begin{aligned}
&\mathbb{E}\big\{(\eta_1W_1)(h_{21}X_{2G}+Z_1)^*\big\}=\mathbb{E}\big\{|h_{21}X_{2G}+Z_1|^2\big\}\\
&\Leftrightarrow \eta_1\rho_1=1+a_{21}^2P_2.
\end{aligned}
\tag{16}
$$

Using the same arguments for (14b), we obtain that $\eta_2\rho_2=1+a_{32}^2P_3$ guarantees $I(X_{2G};S_{2G}|Y_{2G},\tilde{H}_2)=0$. Substituting this and (16) in (10), we conclude that it is possible to construct a genie signal for which (14) is satisfied, if we can find two complex scalars ρ_1 and ρ_2 s.t. $0\leq|\rho_1|,|\rho_2|\leq 1$ and (4) is satisfied. Then, the sum-rate capacity is given by

$$
\begin{aligned}
\sup_{(R_1,R_2)\in\mathcal{C}_{WI}}\{R_1+R_2\}=I(X_{1G},X_{3G};Y_{1G}|\tilde{H}_1)+\\
I(X_{2G};Y_{2G}|\tilde{H}_2),
\end{aligned}
\tag{17}
$$

and it is achieved by $X_k\sim\mathcal{CN}(0,P_k),k\in\{1,2,3\}$, mutually independent. For this input distribution, the expressions in (17) coincide with (5), and (12) coincides with (3). ■

E. Discussion

Comment 3. Note that the conditions (4) are sufficient conditions for optimality. Hence, the region described in (4) is a subset of the region in which treating interference as noise is sum-rate optimal.

Comment 4. An interesting question is whether adding a relay node to the PF Z-IC increases the sum-rate when the interfering signal is treated as noise at each receiver. Consider the achievable region of Proposition 1: when (12) holds, then by using the rate bounds (13) it can be shown that the signal from the relay increases the sum-rate at weak interference. This is demonstrated for a PF Z-ICR in Fig. 2. It can be seen

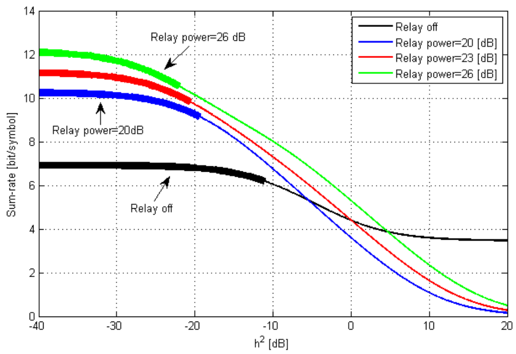

Fig. 2: The achievable sum-rate of Proposition 1 for a symmetric scenario with different relay powers where $a_{11} = a_{22} = a_{31} = 1, a_{21} = a_{32} = h$, $P_1 = P_2 = 10[dB]$, and a_{13} is set to guarantee (3).

from the figure that when the interference is weak, the relay improves the sum-rate, which follows as the rate gain for Tx_1-Rx_1 is greater than the rate loss for Tx_2-Rx_2. For values of the cross-link, h, corresponding to the bold part of each graph, treating the interfering signal as noise is sum-rate optimal, i.e, for these values the achievable sum-rate coincides with the sum-rate capacity characterized in Theorem 1.

Comment 5. Note that for the set of channel coefficients satisfying (3), the sum-rate capacity in (5) is an upper bound on the sum-rate capacity of the PF ICR in the weak interference regime when both interfering links are active and the relay node receives transmissions only for Tx_1. If the relay node additionally receives the transmission of Tx_2, then there might exist some coding strategy in the WI regime, in which the signal from the relay node can be used to facilitate the mitigation of interference at both destinations. This is currently an open issue that requires further research.

Comment 6. Fig. 3 shows the position of the relay in a 2D-plane for which DF at the relay achieves the sum-rate capacity of the PF Z-ICR. We considered a model in which the average attenuations a_{ij} are linked to the distance from node i to node j, d_{ij}, via $a_{ij} = \frac{1}{d_{ij}^2}$. The power and the channel coefficients were normalized to satisfy $a_{22} = a_{31} = 1$. This corresponds to the two-ray propagation model. Note that since the signal from the relay is desired at Rx_1 and is treated as noise at Rx_2, then for the WI conditions to hold, the relay should be closer to Rx_1 (to strengthen the desired signal at Rx_1) and farther from Rx_2 (to decrease the interference at Rx_2). However, the relay should remain relatively close to Tx_1 to allow reliable decoding of the messages from Tx_1 at the relay.

Comment 7. Consider the scenario in which $P_3 = 0$, corresponding to the case in which the relay is off. This scenario is equivalent to the case in which $a_{31} = a_{32} = 0$. In this case, since decoding at the relay does not constrain the rates, from Theorem 1 we conclude that if $|a_{21}| \leq 1$, then the sum-rate capacity of the channel is given by

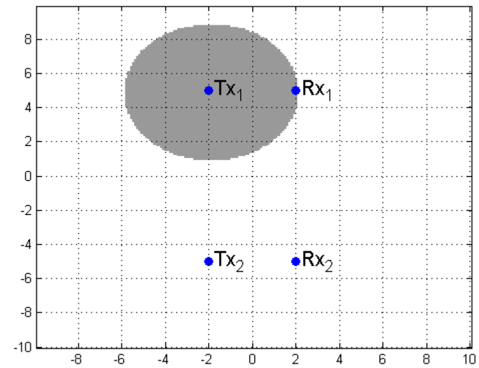

Fig. 3: The geographical position of the relay in a 2D-plane where the conditions of Theorem 1 are satisfied. The black area represents this area for $P_1 = 10, P_2 = 9$, and $P_3 = 8$.

$$\sup_{(R_1,R_2)\in\mathcal{C}_{WI}} \{R_1+R_2\} = \log\left(1+\frac{a_{11}^2 P_1}{1+a_{21}^2 P_2}\right)+\log\left(1+P_2\right).$$

which is similar to the sum-rate capacity expression for the AWGN Z-IC in the WI regime characterized in [4, Theorem 2] (although in the current work the channel is phase fading).

IV. Conclusions

The sum-rate capacity of the PF Z-ICR in the WI regime is characterized, subject to the relay receiving transmissions only from Tx_1. We identified conditions on the channel coefficients under which treating the interfering signal as additive noise is sum-rate optimal. Our results show that if interference is weak, then adding a relay to the IC increases the sum-rate. Therefore, there is a strong motivation for employing relay nodes in ICs in the WI regime as well as in the strong interference regime [5].

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