Amplify-and-Compute: Function Computation over Layered Networks

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***Abstract*—We study layered wireless networks in which destinations decode linear combinations of transmitted messages over a finite field. We propose *amplify-and-compute*, a simple but effective scheme in which sources encode their messages with lattice codes, relays employ amplify-and-forward relaying, and destinations decode incoming signals to integer combinations of lattice codewords. We focus on the two-user setting and show that, by carefully choosing relay amplification weights, it is possible to align the equivalent end-to-end channel to an arbitrary matrix of non-zero integers (up to a set of channel matrices of zero measure). Such a choice achieves a computation rate to within a gap to capacity that is independent of the SNR. Amplify-and-compute therefore achieves the maximum degrees of freedom while providing good performance at moderate SNR. Finally, we show that amplify-and-compute offers similar performance when applied to layered two-user interference networks.**

I. Introduction

Building on the success of interference alignment for single-hop interference networks [1], [2], researchers recently have begun to characterize the capacity limits of multi-hop interference networks. As in the single-hop case, initial work has focused on determining the degrees of freedom (DoF) of canonical network configurations. Most of the proposed coding schemes can be viewed as variations on the Cadambe-Jafar asymptotic alignment scheme [2], in which symbol extensions are generated either from many independent channel realizations (across time- or frequency-varying channels) or from many coding layers (for static channels). For example, in the $2 \times 2 \times 2$ network, comprising two sources communicating to two destinations via two dedicated relays, it was shown by aligned interference neutralization that two DoFs are achievable [3]. Very recent work has generalized this result. It has been proven via network diagonalization that the $K \times K \times K$ network has K degrees of freedom [4], and the ergodic capacity of the $2\times2\times2$ network has been characterized to within a constant gap under i.i.d. Rayleigh fading [5].

Owing to the need to code across many layers or channel realizations, the above coding strategies require either very high SNR or very long delays, and it is unclear whether these performance limits are accessible by practical systems. One possibility is to employ the compute-and-forward strategy [6], in which sources transmit lattice codewords, relays decode and retransmit integer combinations of codewords, and destinations decode enough integer combinations to recover the desired codewords. As shown in [7], however, a direct application of compute-and-forward achieves only a single DoF under the critical assumption that sources do not make use of channel state information (CSI). This suboptimal DoFs performance is mainly due to the Diophantine penalty associated with approximating real-valued channel coefficients with rational numbers. Similar to the case with interference channels, this penalty can be overcome either by employing many layers of lattice codewords [7] or by coding across many channel realizations [8]; again, these approaches either require very high SNR or incur long delays.

Aiming to overcome these difficulties, we propose a simple strategy, *amplify-and-compute*, which requires only a single coding layer and a single channel realization. Amplify-and-compute combines compute-and-forward with amplify-and-forward relaying: Sources transmit lattice codewords, relays scale and retransmit incoming signals, and destinations recover integer combinations of lattice points. This permits scalar precoding at both sources and relays, which affords a surprising amount of flexibility.

We investigate the performance of amplify-and-compute in two topologies. First, we study the $2 \times 2 \times 2$ network where destinations seek linear combinations of messages. We show that, for almost every pair of channel matrices, it is possible to choose the precoding weights to align the equivalent end-to-end channel matrix with any matrix of non-zero integers. Such a choice eliminates the Diophantine penalty and achieves the full DoFs. Furthermore, the simplicity of amplify-and-compute enables us to give a simple expression for the achievable rate for which the gap to capacity is independent of the SNR. Second, we study the $2 \times 2 \times 2 \times 2 \times 2$ *interference* network, showing that cascaded use of amplify-and-compute again achieves rates near capacity.

II. System Model

We study a network composed of two sources, two relays, and two destinations, as depicted in Figure 1. Each source (indexed by $\ell \in \{1, 2\}$) has a message $\mathbf{b}_\ell \in \mathbb{F}_{p^2}^{k_\ell}$, where p is a large prime. Source ℓ employs an encoder $\mathcal{E}_{s,\ell} : \mathbb{F}_{p^2}^{k_\ell} \to \mathbb{C}^n$ that maps its finite-field message to a complex-valued codeword $\mathbf{x}_{s,\ell} = \mathcal{E}_{s,\ell}(\mathbf{b}_\ell)$ which must satisfy the usual power constraint, $\mathbb{E}\,\|\mathbf{x}_{s,\ell}\|^2 \leq nP$.

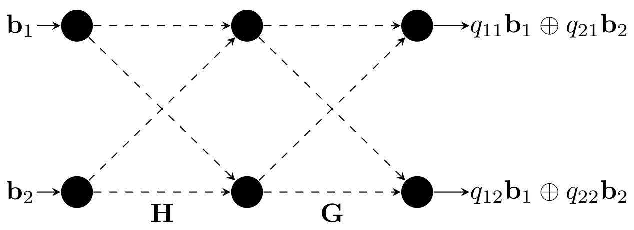

Fig. 1. The $2 \times 2 \times 2$ layered computation channel.

The relays (indexed by $m \in \{1, 2\}$) receive noisy linear combinations of source codewords. Let $\mathbf{X}_s = \begin{bmatrix}\mathbf{x}_{s,1} \ \mathbf{x}_{s,2}\end{bmatrix}^T$ be the $2 \times n$ matrix of source transmissions. Let $\mathbf{y}_{r,m}$ be the received signal at relay m and let $\mathbf{Y}_r = \begin{bmatrix}\mathbf{y}_{r,1} \ \mathbf{y}_{r,2}\end{bmatrix}^T$, which can be written as

$$\mathbf{Y}_r = \mathbf{H}\mathbf{X}_s + \mathbf{Z}_r, \tag{1}$$

where $\mathbf{H} = \{h_{m\ell}\}$ is a complex-valued channel matrix assumed to be fixed and known globally, and the noise $\mathbf{Z}_r$ is elementwise i.i.d. $\mathcal{CN}(0, 1)$. Relay m employs an encoder $\mathcal{E}_{r,m} : \mathbb{C}^n \to \mathbb{C}^n$ which maps its received signals to transmit signals $\mathbf{x}_{r,m}$. Again, we impose the power constraint $\mathbb{E}\,\|\mathbf{x}_{r,m}\|^2 \leq nP$.

The destinations (indexed by $i \in \{1, 2\}$) receive noisy linear combinations of the relays' signals. Let $\mathbf{X}_r = \begin{bmatrix}\mathbf{x}_{r,1} \ \mathbf{x}_{r,2}\end{bmatrix}^T$ denote the $2 \times n$ matrix of relay transmissions. Let $\mathbf{y}_{d,i}$ be the received signal at destination i, and let $\mathbf{Y}_d = \begin{bmatrix}\mathbf{y}_{d,1} \ \mathbf{y}_{d,2}\end{bmatrix}^T$ be the $2 \times n$ matrix of received signals. Then,

$$\mathbf{Y}_d = \mathbf{G}\mathbf{X}_r + \mathbf{Z}_d, \tag{2}$$

where $\mathbf{G} = \{g_{im}\}$ is a complex-valued channel matrix, and the noise $\mathbf{Z}_d$ is elementwise i.i.d. $\mathcal{CN}(0, 1)$. Each destination i intends to recover a linear combination of the messages:

$$\mathbf{u}_i = q_{i1}\mathbf{b}_1 \oplus q_{i2}\mathbf{b}_2, \tag{3}$$

where $\oplus$ denotes addition over $\mathbb{F}_{p^2}$ and the shorter vector is zero-padded to length $k = \max\{k_1, k_2\}$. The coefficients $q_{im} \in \mathbb{F}_{p^2}$ are chosen such that the resulting equations are linearly independent, but are otherwise arbitrary. Each destination i employs a decoder $\mathcal{D}_i : \mathbb{C}^n \to \mathbb{F}_{p^2}^k$ that maps its observation to an estimate $\hat{\mathbf{u}}_i$ of its desired linear combination.

The rate of each encoder i, in bits per channel use, is

$$R_\ell = \frac{2k_\ell \log_2(p)}{n}. \tag{4}$$

A computation rate pair (R_1, R_2) is said to be *achievable* if, for any $\epsilon > 0$ and n and p large enough, there exist encoders and decoders such that two linearly independent combinations $\mathbf{u}_1$, $\mathbf{u}_2$ can be recovered at the destinations with total probability of error at most ϵ,

$$\Pr\big\{\{\hat{\mathbf{u}}_1 \neq \mathbf{u}_1\} \cup \{\hat{\mathbf{u}}_2 \neq \mathbf{u}_2\}\big\} < \epsilon\ .$$

The *computation capacity region*, denoted $\mathcal{C}$, is the closure of the set of all achievable rate pairs.

The *computation degrees of freedom* (CDoFs) of the network is the maximum sum computation capacity, normalized by the capacity of a single AWGN link, in the limit of high SNR:

$$\text{CDoF} = \lim_{P\to\infty} \max_{(C_1,C_2)\in\mathcal{C}} \frac{C_1 + C_2}{\log_2(P)}. \tag{5}$$

While computation capacity of this channel is unknown, the CDoFs is known to be 2 up to a set of channel matrices of measure zero. The full CDoFs can be achieved by one of several instantiations of real interference alignment [3], [4], [7]; however, these approaches are effective only at very high SNR. Finally, we provide a simple outer bound on $\mathcal{C}$.

Theorem 1: Define $\bar{h}$ and $\mathcal{C}^+$ as

$$\bar{h} = \min\left\{\max_{\ell,m} |h_{m\ell}|,\ \max_{m,i} |g_{im}|\right\}$$
$$\mathcal{C}^+ = \left\{(R_1, R_2) : R_1, R_2 \leq \log_2(1 + |\bar{h}|^2 P)\right\}\ .$$

Then, $\mathcal{C} \subset \mathcal{C}^+$.

Owing to space limitations, we omit the proof. It follows from cut-set arguments and the genie-aided bound of [6, Theorem 14].

III. Amplify-and-compute

Here we introduce *amplify-and-compute*, an encoding strategy that combines compute-and-forward lattice-based coding at the sources, amplify-and-forward at the relays, and the decoding of linear combinations at the destinations.

The following theorem encapsulates key results on compute-and-forward from [6], [9] in a form suitable for amplify-and-forward.

Theorem 2: Assume that each destination has access to an observation $\mathbf{y}_{\text{eq},i} \in \mathbb{C}^n$ that be written in the following form:

$$\begin{bmatrix}\mathbf{y}_{\text{eq},1}^T \\ \mathbf{y}_{\text{eq},2}^T\end{bmatrix} = \begin{bmatrix}\mathbf{h}_{\text{eq},1}^T \\ \mathbf{h}_{\text{eq},2}^T\end{bmatrix} \mathbf{X}_s + \mathbf{Z}_{\text{eq}} \tag{6}$$

where $\mathbf{h}_{\text{eq},1}, \mathbf{h}_{\text{eq},2} \in \mathbb{C}^2$ are effective channel vectors and the noise $\mathbf{Z}_{\text{eq}}$ is elementwise i.i.d. $\mathcal{CN}(0, 1)$. Then, any computation rate pair (R_1, R_2) satisfying

$$R_\ell < \min_{i: a_{i\ell}\neq 0} \log\left(\frac{1 + P\,\|\mathbf{h}_{\text{eq},i}\|^2}{\|\mathbf{a}_i\|^2 + P\big(\|\mathbf{h}_{\text{eq},i}\|^2\,\|\mathbf{a}_i\|^2 - |\mathbf{a}_i^H \mathbf{h}_{\text{eq},i}|^2\big)}\right)$$

for some linearly independent integer vectors $\mathbf{a}_1, \mathbf{a}_2 \in \{\mathbb{Z} + j\mathbb{Z}\}^2$ is achievable.

Remark 1: The Gaussian integers $a_{i\ell} \in \mathbb{Z} + j\mathbb{Z}$ correspond to the coefficients of the linear combinations via $q_{i\ell} = f\big([a_{i\ell}] \mod p\big)$ where f is the natural mapping between $\mathbb{Z}_p + j\mathbb{Z}_p$ and $\mathbb{F}_{p^2}$. See [6], [9] for more details.

These rates are achieved using lattice codes, which were proven to achieve capacity for AWGN channels in [10]. In amplify-and-compute, each source also transmits lattice codewords subject to scalar precoding. This is expressed as

$$\mathbf{X}_s = \sqrt{P}\mathbf{V}\mathbf{C}, \tag{7}$$

where $\mathbf{V} \in \mathbb{C}^{2\times 2}$ is a diagonal matrix having nonzero entries v_1, v_2, and $\mathbf{C} \in \mathbb{C}^{2\times n}$ contains the lattice codewords $\mathbf{c}_1$ and

$\mathbf{c}_2$, each of which has unit average power. Each relay simply amplifies its incoming signal by a scalar, resulting in the following transmit signal:

$$\mathbf{X}_r = \mathbf{W}\mathbf{Y}_r, \tag{8}$$

where $\mathbf{W} \in \mathbb{C}^{2\times 2}$ is also a diagonal matrix with nonzero entries w_1, w_2 representing the amplification weights. The receivers employ lattice decoding, trying to match the received signals to the linear combinations $\mathbf{a}_i\mathbf{C}$ corresponding to the desired linear combinations $\mathbf{u}_i$.

In the following theorem, we characterize the rates achieved by this scheme.

Theorem 3: Amplify-and-compute achieves any rate pair (R_1, R_2) satisfying

$$R_\ell \le \min_{i:a_{i\ell}\neq 0} \log_2\left(\frac{1+P\left\|\mathbf{h}_{\mathrm{eq},i}\right\|^2/N_i}{\|\mathbf{a}_i\|^2 + P(\left\|\mathbf{h}_{\mathrm{eq},i}\right\|^2\|\mathbf{a}_i\|^2 - |\mathbf{a}_i^H\mathbf{h}_{\mathrm{eq},i}|^2)/N_i}\right), \tag{9}$$

for any linearly independent integer vectors $\mathbf{a}_1, \mathbf{a}_2$, where

$$h_{\mathrm{eq},i\ell} = v_\ell(g_{i1}h_{1\ell}w_1 + g_{i2}h_{2\ell}w_2) \tag{10}$$

$$\mathbf{h}_{\mathrm{eq},i}^T = [h_{\mathrm{eq},i1}\ h_{\mathrm{eq},i2}] \tag{11}$$

$$P_m' = \frac{P}{1+\|\mathbf{h}_m\|^2 P} \tag{12}$$

$$N_i = |g_{i1}|^2P_1' + |g_{i2}|^2P_2' + 1, \tag{13}$$

and the weighting coefficients must satisfy

$$|v_i|^2 \le 1,\ \forall i \tag{14}$$

$$|w_m|^2 \le P_m'\ . \tag{15}$$

Proof: First, we derive the constraints on the weighting coefficients. Since the source power must be less than P, v_i must satisfy

$$\left\|\sqrt{P}v_i\mathbf{c}_i\right\|^2 \le nP \implies |v_i|^2 \le 1\ .$$

The amplification weights at the relays must satisfy

$$|w_m|^2\left\|\mathbf{y}_{rm}\right\|^2 < nP$$

$$|w_m|^2 < \frac{P}{1+P(|h_{m1}v_1|^2+|h_{m2}v_2|^2)}.$$

Since $|v_i|^2 \le 1$, any coefficients satisfying (15) satisfy the power constraint.

The signals received at the destinations can be written as

$$\begin{aligned}\mathbf{Y}_d &= \mathbf{G}\mathbf{X}_r + \mathbf{Z}_d\\ &= \mathbf{G}(\mathbf{W}(\mathbf{H}\mathbf{V}\mathbf{C}+\mathbf{Z}_r)+\mathbf{Z}_d\\ &= \mathbf{G}\mathbf{W}\mathbf{H}\mathbf{V}\mathbf{C}+\mathbf{G}\mathbf{W}\mathbf{Z}_r+\mathbf{Z}_d\\ &= \mathbf{H}_{\mathrm{eq}}\mathbf{C}+\mathbf{Z}_{\mathrm{eq}},\end{aligned}$$

where the entries of $\mathbf{H}_{\mathrm{eq}}$ follow (10), and where $\mathbf{Z}_{\mathrm{eq}} = \mathbf{G}\mathbf{W}\mathbf{Z}_r + \mathbf{Z}_d$. Then, the receiver obtains a noisy linear combination of dithered lattice codewords. The equivalent noise power at destination m obeys

$$\begin{aligned}\frac{1}{n}\left\|\mathbf{z}_{\mathrm{eq},i}\right\|^2 &= |g_{i1}w_1|^2 + |g_{i2}w_2|^2 + 1\\ &\le |g_{i1}|^2P_1' + |g_{i2}|^2P_2' + 1 = N_i.\end{aligned}$$

With this representation of the received signal and noise, we can invoke Theorem 2, which yields the desired result. ■

As in the original compute-and-forward approach of [6], the obstacle to near-capacity performance is the Diophantine penalty in (9). When the equivalent channels are nearly co-linear with suitable function coefficients, the penalty is small, and the achievable rate is approximately $\log_2(P)$. However, when the equivalent channels cannot be matched up, the penalty is large. In the following section we will show that careful selection of the amplification weights permits perfect alignment between equivalent channel gains and integer function coefficients.

IV. Integer Co-linearization

While the lattice amplify-and-forward strategy proposed in the previous section is relatively simple—consisting of standard lattice codes and scalar precoding—it achieves performance previously obtained only by rather complicated interference alignment schemes. Indeed, the key observation of this work is that, in a layered network, amplify-and-compute is sufficient to align perfectly the equivalent channels to suitable linear combinations of sources' messages, which eliminates the Diophantine penalty.

In particular, we can align $\mathbf{H}_{\mathrm{eq}}$ with a matrix of non-zero Gaussian integers. Let $\mathbf{A} \in \{\mathbb{Z}+j\mathbb{Z}\}^{2\times 2}$ be a matrix whose entries are all non-zero. In order to make each row of $\mathbf{H}_{\mathrm{eq}}$ co-linear with each row of $\mathbf{A}$, the following system of equations must be satisfied:

$$h_{\mathrm{eq},i1}a_{i2} - h_{\mathrm{eq},i2}a_{i1} = 0, \tag{16}$$

for $i = 1, 2$. Suppose momentarily that $v_2 = w_2 = 1$. Then (16) becomes

$$v_1(g_{11}h_{11}w_1 + g_{12}h_{21})a_{12} - (g_{11}h_{12}w_1 + g_{12}h_{22})a_{11} = 0 \tag{17}$$

$$v_1(g_{21}h_{11}w_1 + g_{22}h_{21})a_{22} - (g_{21}h_{12}w_1 + g_{22}h_{22})a_{21} = 0. \tag{18}$$

This is a system of two equations having two variables and order two, and it can be solved for almost every choice of channel gains. Solving (18) for v_1 yields

$$v_1 = \frac{(g_{21}h_{12}w_1 + g_{22}h_{22})a_{21}}{(g_{21}h_{11}w_1 + g_{22}h_{21})a_{22}}. \tag{19}$$

Substituting (19) back into (17) yields a quadratic in w_1. Its solutions are too unwieldy to print, but we point out a few of their features. In particular, we want to characterize the conditions under which co-linearization fails. Examining the quadratic resulting from (17), we see that if

$$h_{11}h_{12}g_{11}g_{21}(a_{12}a_{21} - a_{11}a_{22}) = 0, \tag{20}$$

and

$$h_{11}h_{22}(g_{11}g_{22}a_{12}a_{21} - g_{12}g_{21}a_{11}a_{22}) + \\ h_{12}h_{21}(g_{12}g_{21}a_{12}a_{21} - g_{11}g_{22}a_{11}a_{22}) = 0, \quad (21)$$

then there may be no solution for w_1. Similarly, if the solution for w_1 is such that

$$(g_{21}h_{11}w_1 + g_{22}h_{21})a_{2,2} = 0, \quad (22)$$

then there is no solution for v_1. Note that, by contrast to interference alignment, which fails on the (dense) set of rational channel coefficients, co-linearization fails only on a single algebraic curve.

Recall that we supposed $w_2 = v_2 = 1$, which means that these choices of precoding weights may violate the power constraints outlined in Theorem 3. However, for any fixed choices, one can always scale $\mathbf{V}$ and $\mathbf{W}$ down by constants in order to meet the constraints without affecting the co-linearity of the equivalent channels.

Co-linearization turns out to achieve near-capacity performance, as we show in the next theorem.

Theorem 4: For $P \geq 1$ and for almost every $\mathbf{H}, \mathbf{G}$, amplify-and-compute achieves rates (R_1, R_2) that are within $c(\mathbf{H}, \mathbf{G})$ bits of the boundary of the capacity region, where $c(\mathbf{H}, \mathbf{G})$ depends on the channel matrices but not on the SNR. *A fortiori*, amplify-and-compute achieves the maximum CDoFs of two.

Proof: Select a full-rank matrix $\mathbf{A}$ with non-zero entries. As discussed above, for almost every $\mathbf{H}, \mathbf{G}$ we can choose $\mathbf{V}$ and $\mathbf{W}$ to co-linearize $\mathbf{H}_{\text{eq}}$ with $\mathbf{A}$. Since the Diophantine penalty is eliminated, the rates achieved are at least

$$\begin{aligned} R_1, R_2 &\geq \min_i \log_2 \left(1 + P\frac{\|\mathbf{h}_{\text{eq},i}\|^2}{N_i}\right) - \log_2(\|\mathbf{a}_i\|^2) \\ &\geq \log_2(P) - c_1, \end{aligned}$$

where

$$c_1 = \max_i \log_2 \left(\frac{\|\mathbf{a}_i\|^2 \left(1 + |g_{i1}^2| / \|\mathbf{h}_1\|^2 + |g_{i2}^2| / \|\mathbf{h}_2\|^2\right)}{\|\mathbf{h}_{\text{eq},i}\|^2} \right).$$

Similarly, the upper bound from Theorem 1 stipulates that achievable rates are at most

$$\begin{aligned} R_1, R_2 &\leq \log_2(1 + P|\bar{h}|^2) \\ &\leq \log_2(P(1 + |\bar{h}|^2)) \\ &\leq \log_2(P) + c_2, \end{aligned}$$

where

$$c_2 = \log_2(1 + |\bar{h}|^2).$$

Choosing $c(\mathbf{H}, \mathbf{G}) = c_1 + c_2$ yields the claim. ■

Remark 2: As mentioned before, the maximum CDoFs can also be achieved by interference alignment techniques such as those presented in [4], [7]. However, our approach offers a guarantee of near-capacity performance at moderate SNRs, whereas existing alignment techniques are effective only at very high SNR. Furthermore, since our scheme is relatively simple, it is amenable to practical implementation.

Remark 3: We have not characterized the constant $c(\mathbf{H}, \mathbf{G})$ other than to show that it does not depend on the SNR. Since the performance of co-linearization depends on the choice of $\mathbf{A}$, one could tighten the gap by judicious choice of the function coefficients.

In Figure 2 we plot the achievable rate for amplify-and-compute, averaged over 1000 i.i.d. Rayleigh-distributed realizations of channel matrices $\mathbf{H}, \mathbf{G}$, as a function of SNR. We also plot the upper bound of Theorem 1 and the rates achieved when the relays perform ordinary compute-and-forward. Note that the slope of the amplify-and-compute rates approaches that of the upper bound, indicating optimality in the CDoFs.

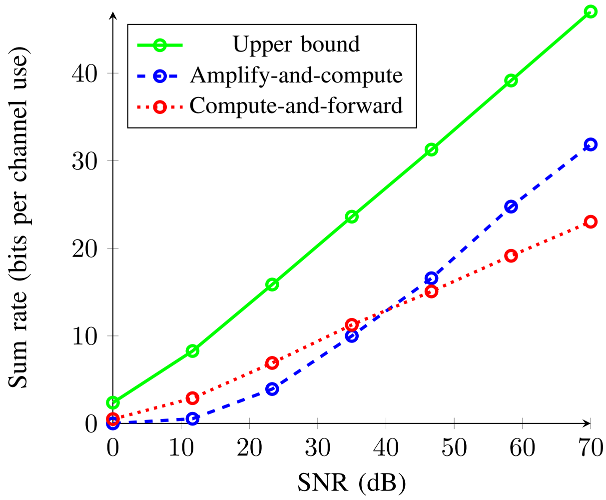

Fig. 2. Average achievable rates for ordinary compute-and-forward and amplify-and-compute, plotted against the upper bound.

V. Applications to Layered Interference Channels

Amplify-and-compute also is effective for a class of layered *interference* channels. Consider a network that structurally resembles the cascade of two $2 \times 2 \times 2$ networks, as depicted in Figure 3. Two source nodes, having messages $\mathbf{b}_1, \mathbf{b}_2$, transmit messages through three layers of relays to a pair of destinations. Instead of recovering arbitrary full-rank linear combinations of messages, destinations recover individual messages $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. In lieu of a full system description, we briefly describe the signals transmitted and received. Let $\mathbf{X}_s, \mathbf{X}_{r1}, \mathbf{X}_{r2}, \mathbf{X}_{r3} \in \mathbb{C}^{2 \times n}$ be the signals transmitted by the source and relays, respectively. Then, the received signals at the relays and destination, respectively, are given by

$$\begin{aligned} \mathbf{Y}_{r1} &= \mathbf{H}_1\mathbf{X}_s + \mathbf{Z}_{r1} \\ \mathbf{Y}_{r2} &= \mathbf{G}_1\mathbf{X}_{r1} + \mathbf{Z}_{r2} \\ \mathbf{Y}_{r3} &= \mathbf{H}_2\mathbf{X}_{r2} + \mathbf{Z}_{r3} \\ \mathbf{Y}_d &= \mathbf{G}_2\mathbf{X}_{r3} + \mathbf{Z}_d, \end{aligned}$$

where the channel matrices are fixed and known globally, and the noise is unit-variance complex Gaussian. We again impose a common power constraint P.

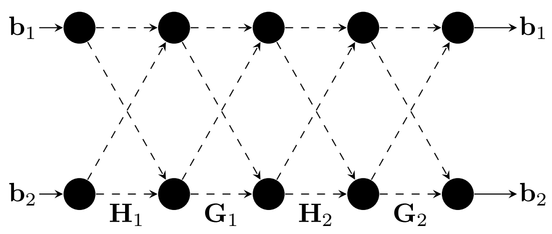

Fig. 3. Layered interference channel.

For this channel, repeated use of amplify-and-compute achieves capacity to within a gap not depending on the SNR. Our approach, as suggested by our choice of notation, is to divide the network into two stages, each of which performs amplify-and-compute. Similar to before, let $\mathbf{C}_1 \in \mathbb{C}^{2\times n}$ contain the lattice codewords corresponding to $\mathbf{b}_1$ and $\mathbf{b}_2$. Then, the sources transmit scaled lattice codewords, and the first layer of relays amplify their incoming signals:

$$\mathbf{X}_s = \mathbf{V}_1\mathbf{C}_1 \tag{23}$$

$$\mathbf{X}_{r1} = \mathbf{W}_1\mathbf{Y}_{r1}, \tag{24}$$

where $\mathbf{V}_1$ and $\mathbf{W}_1$ are diagonal matrices, similar to Section III. Then, the second layer of relays obtains

$$\mathbf{Y}_{r2} = \mathbf{G}_1\mathbf{W}_1\mathbf{H}_1\mathbf{V}_1\mathbf{C}_1 + \mathbf{G}_1\mathbf{W}_1\mathbf{Z}_{r1} + \mathbf{Z}_{r2} \tag{25}$$

$$= \mathbf{H}_{\mathrm{eq},1}\mathbf{C}_1 + \mathbf{Z}_{\mathrm{eq},1}, \tag{26}$$

where $\mathbf{H}_{\mathrm{eq},1}$ and $\mathbf{Z}_{\mathrm{eq},1}$ are defined as expected. Using the procedure outlined in Section IV, we can choose $\mathbf{V}_1$ and $\mathbf{W}_1$ to align the rows of $\mathbf{H}_{\mathrm{eq},1}$ with any non-zero matrix $\mathbf{A}_1$. The second layer of relays decodes $\mathbf{Y}_{r2}$ to the integer combination of lattice codewords corresponding to $\mathbf{A}_1$.

Next, we perform another stage of amplify-and-compute. Let $\mathbf{C}_2 \in \mathbb{C}^{2\times n}$ be the dithered lattice codewords corresponding to $\mathbf{f}_1$ and $\mathbf{f}_2$. The relays transmit

$$\mathbf{X}_{r2} = \mathbf{V}_2\mathbf{C}_2 \tag{27}$$

$$\mathbf{X}_{r3} = \mathbf{W}_2\mathbf{Y}_{r3}, \tag{28}$$

where, yet again, $\mathbf{V}_2$ and $\mathbf{W}_2$ are diagonal. The destinations obtain

$$\mathbf{Y}_d = \mathbf{G}_2\mathbf{W}_2\mathbf{H}_2\mathbf{V}_2\mathbf{C}_2 + \mathbf{G}_1\mathbf{W}_1\mathbf{Z}_{r3} + \mathbf{Z}_d \tag{29}$$

$$= \mathbf{H}_{\mathrm{eq},2}\mathbf{C}_2 + \mathbf{Z}_{\mathrm{eq},2}, \tag{30}$$

where again $\mathbf{H}_{\mathrm{eq},2}$ and $\mathbf{Z}_{\mathrm{eq},2}$ are defined as expected. At this stage, we choose $\mathbf{V}_2$ and $\mathbf{W}_2$ to align $\mathbf{H}_{\mathrm{eq},2}$ to a non-zero matrix, but this time we choose $\mathbf{A}_2 = \det(\mathbf{A}_1)\mathbf{A}_1^{-1}$, which is clearly non-zero and Gaussian integer valued. Finally, the destinations decode $\mathbf{Y}_d$ to the integer combination of lattice points corresponding to $\mathbf{A}_2$. Supposing successful decoding at the second layer of relays and the destinations, the destinations obtain a linear combinations of messages described by $\mathbf{A}_2\mathbf{A}_1 = \det(\mathbf{A}_1)\mathbf{I}$. Since the matrix is diagonal, the destinations can recover the individual messages as desired.

Using arguments similar to Theorem 4, one can show that repeated amplify-and-compute achieves rates to within a gap of capacity that does not depend on the SNR, and therefore it achieves the maximum degrees of freedom. Again our approach offers advantages over existing techniques: guaranteed finite-SNR performance, simple implementation, etc.

VI. Conclusion

We have proposed and studied *amplify-and-compute*, a simple scheme involving lattice coding and scalar precoding. For $2 \times 2 \times 2$ computation networks, amplify-and-compute is sufficient to achieve computation rates within a gap to capacity that is independent of the SNR. For interference networks with three layers of relays, amplify-and-forward can be leveraged to achieve similar near-capacity performance. These results are somewhat surprising, since previous (C)Dof-optimal approaches to layered networks involve impractically high SNRs or long delays.

Of interest is the generalization of these results to larger networks. One can imagine a network with K sources, K destinations, and multiple dedicated relays arranged into multiple layers. The feasibility of co-linearization in such a network is a difficult problem. Whereas the two-user case required the solution to a small system of equations, the general case entails solving many simultaneous polynomial equations. The result is a challenging algebraic geometry problem, and it is the topic of ongoing investigation.

References

[1] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis," *IEEE Transactions on Information Theory*, vol. 54, no. 8, pp. 3457–3470, August 2008.

[2] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the K-user interference channel," *IEEE Transactions on Information Theory*, vol. 54, no. 8, pp. 3425–3441, August 2008.

[3] T. Gou, S. A. Jafar, S.-W. Jeon, and S.-Y. Chung, "Aligned interference neutralization and the degrees of freedom of the 2x2x2 interference channel," *IEEE Transactions on Information Theory*, vol. 58, no. 7, pp. 4381–4395, July 2012.

[4] I. Shomorony and A. Avestimehr, "Degrees of freedom of two-hop wireless networks: 'Everyone gets the entire cake'," in *Proc. Allerton*, 2012.

[5] S. W. Jeon, C. Wang, and M. Gastpar, "Approximate ergodic capacity of a class of fading 2-user 2-hop networks," *IEEE Transactions on Information Theory*, Submitted October 2012, available online: http://arxiv.org/abs/1210.2182.

[6] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *IEEE Trans. Inform. Theory*, vol. 57, no. 10, pp. 6463–6486, Oct. 2011.

[7] U. Niesen and P. Whiting, "The degrees of freedom of compute-and-forward," *IEEE Trans. Info. Theory*, vol. 58, no. 8, pp. 5214 –5232, Aug. 2012.

[8] U. Niesen, B. Nazer, and P. Whiting, "Computation alignment: Capacity approximation without noise accumulation," *IEEE Transactions on Information Theory*, to appear.

[9] S. N. Hong and G. Caire, "Lattice coding strategies for cooperative distributed antenna systems," *IEEE Transactions on Information Theory*, to appear, available online: http://arxiv.org/abs/1210.0160.

[10] U. Erez and R. Zamir, "Achieving $\frac{1}{2}\log(1 + \mathrm{SNR})$ on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inform. Theory*, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.