Multi-Dimensional Spatially-Coupled Codes

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***Abstract*—Spatially-coupled (SC) codes are constructed by coupling many regular low-density parity-check codes in a chain. The decoding chain of SC codes aborts when facing burst erasures. This problem cannot be overcome by increasing the chain length. In this paper, we introduce multi-dimensional (MD) SC codes to circumvent it. Numerical results show that two-dimensional SC codes are more robust against the burst erasures than one-dimensional SC codes. Furthermore, we consider designing multi-dimensional SC codes with smaller rateloss.**

I. Introduction

Spatially-coupled (SC) low-density parity-check (LDPC) codes have attracted much attention due to their capacity-achieving performance [1]–[6] and a memory-efficient sliding-window decoding algorithm [7]. The studies on SC-LDPC codes date back to the invention of convolutional LDPC codes by Felström and Zigangirov [8]. Lentmaier *et al.* observed that (4,8)-regular convolutional LDPC codes exhibit the decoding performance surpassing the belief propagation (BP) threshold of (4,8)-regular block LDPC codes [4]. Further, the BP threshold of the convolutional LDPC codes coincides with the maximum a posterior (MAP) threshold of the underlying block LDPC codes with a lot of accuracy. Constructing convolutional LDPC codes from a block LDPC code improves the BP threshold up to the MAP threshold of the underlying codes.

Kudekar *et al.* named this phenomenon "threshold saturation" and proved rigorously for the binary-input erasure channel (BEC) [1], [3] and the binary-input memoryless output-symmetric (BMS) channels [5]. In the limit of large d_l, d_r, L and w, the SC-LDPC code ensemble (d_l, d_r, L, w) [5] was shown to *universally* achieve the Shannon limit of the BMS channels under BP decoding.

In this paper, we deal with a serious problem of SC-LDPC codes. SC-LDPC codes are constructed by coupling L regular LDPC codes in a chain. BP is employed to decode the chain of codes from the end points of the chain. The BP decoding of SC codes aborts when facing burst erasures. In other words, the decoding error probability remains strictly positive from the section at which the burst erasures are received. This problem cannot be solved by increasing the coupling number, i.e. the number of sections. In this paper, we introduce multi-dimensional (MD) SC codes to overcome this problem. Numerical results show that two-dimensional (2D) SC codes are more robust against the burst erasures than one-dimensional (1D) SC codes. Furthermore, we consider designing MD-SC codes with small rateloss as $O(1/L^D)$, where D is the dimension of coupling.

II. Multi-Dimensional Coupled Codes

A. Definition: $(d_l, d_r, L, \omega, \mathcal{Z})$ codes

Definition 1: Define $\mathbb{Z}_L := \mathbb{Z}/L\mathbb{Z} = \{0, 1, \dots, L-1\}$. Consider L^D sections on D-dimensional discrete torus $\mathbb{Z}_L^D$. For coupling window size w, throughout this paper, we fix $\omega_{\underline{j}}$ as

$$\omega_{\underline{j}} = \begin{cases} 1/w^D & \underline{j} \in [0, w-1]^D \\ 0 & \text{otherwise,} \end{cases}$$

where we denoted $[a, b] := \{a, a+1, \dots, b-1, b\}$. For bit node degree $d_l \geq 3$, check node degree $d_r > d_l$, the coupling number $L > w$, the set of connecting rates $\omega = \{\omega_{\underline{j}} : \underline{j} \in \mathbb{Z}_L^D\}$, and a shortened domain $\mathcal{Z} \subset \mathbb{Z}_L^D$, we define MD-SC $(d_l, d_r, L, \omega, \mathcal{Z})$ codes as follows: Each section $\underline{i} \in [0, L-1]^D$ has M bit nodes of degree d_l and $\frac{d_l}{d_r}M$ check nodes of degree d_r. Connect edges between the bit nodes and the check nodes uniformly at random so that bit nodes in section $\underline{i}$ are connected to check nodes in section $\underline{i} + \underline{j}$ ($\underline{j} \in \mathbb{Z}_L^D$) with $\omega_{\underline{j}} d_l M$ edges, respectively. Shorten the bit nodes in section $\underline{i} \in \mathcal{Z} \subset \mathbb{Z}_L^D$. Namely, the shortened bit nodes are set to 0 and not transmitted through the channel.

Discussion 1: In [3], SC codes of the coupling number $2L+1$ were defined over section $[-L, +L]$, and the bit nodes outside $[-L, +L]$ were shortened. One might regard it is more natural to shorten the bit nodes outside $[0, L-1]^D$ instead of introducing a shorten domain $\mathcal{Z}$. If one defined so, the MD codes might be regarded as a bundle of 1D-SC codes. We have introduced a shorten domain $\mathcal{Z}$ to reveal the effect of the MD extension more explicitly. This is why we employ the codes in Definition 1.

Lemma 1: The coding rate $R(d_l, d_r, L, \omega, \mathcal{Z})$ of $(d_l, d_r, L, \omega, \mathcal{Z})$ codes is given by

$$1 - \frac{d_l}{d_r} \frac{1}{L^D - \#\mathcal{Z}} \sum_{\underline{i} \in \mathbb{Z}_L^D} \Big(1 - \big(\sum_{\underline{j}: \underline{i}+\underline{j} \in \mathcal{Z}} \omega_{\underline{j}}\big)^{d_r}\Big). \qquad (1)$$

Proof: We shall count the numbers of transmitted bit nodes and *valid* check nodes. Let V and C denote these numbers, respectively. Since the check nodes adjacent only to shortened bit nodes give no constraint on the code, it is sufficient to count the check nodes adjacent to unshortened bit nodes. Since the degree of check nodes is d_r, a check node in section $\underline{i}$ has d_r edges connecting to shortened bit nodes with probability

$$\Big(\sum_{\underline{j}:\underline{i}+\underline{j}\in\mathcal{Z}} \omega_{\underline{j}}\Big)^{d_r}.$$

Therefore, the average number of check nodes which are adjacent to at least one unshortened bit nodes is given by

$$C = \frac{d_l}{d_r} M \sum_{\underline{i}\in\mathbb{Z}_L^D} \Big(1 - \big(\sum_{\underline{j}:\underline{i}+\underline{j}\in\mathcal{Z}} \omega_{\underline{j}}\big)^{d_r}\Big).$$

There are $V = M(L^D - \#\mathcal{Z})$ unshortened bit nodes. We calculate the coding rate as $1 - C/V$, which concludes (1). □

III. Density Evolution Analysis

We consider the transmission takes place over the BEC(ϵ) with erasure probability ϵ. The performance of the BP decoding is analyzed in the limit $M \to \infty$. Let $p_{\underline{i}}^{(\ell)}$ denote the erasure probability of BP messages from each bit node to check nodes at the ℓ-th BP iteration round. Let $q_{\underline{i}}^{(\ell)}$ denote the erasure probability of BP messages from each check node to bit nodes at the ℓ-th BP iteration round. Since the bit nodes in $\mathcal{Z}$ are shortened, $p_{\underline{i}}^{(0)}$ are given as

$$p_{\underline{i}}^{(0)} = \epsilon_{\underline{i}} := \begin{cases} 0 & (\underline{i}\in\mathcal{Z}), \\ \epsilon & (\underline{i}\notin\mathcal{Z}). \end{cases}$$

For $\ell \geq 1$, $p_{\underline{i}}^{(\ell)} = 0$ for shortened section $\underline{i}\in\mathcal{Z}$ and

$$p_{\underline{i}}^{(\ell)} = \epsilon_{\underline{i}} \Big(\sum_{\underline{j}} \omega_{\underline{j}} q_{\underline{i}+\underline{j}}^{(\ell)}\Big)^{d_l-1}, \tag{2}$$

$$q_{\underline{i}}^{(\ell)} = 1 - \Big(1 - \sum_{\underline{j}} \omega_{\underline{j}} p_{\underline{i}-\underline{j}}^{(\ell-1)}\Big)^{d_r-1},$$

for $\underline{i}\notin\mathcal{Z}$. The decoding erasure probability $\mathbb{P}_{\mathrm{b}}^{(\ell)}$ is given by

$$\mathbb{P}_{\mathrm{b}}^{(\ell)} = \frac{1}{L^D - \#\mathcal{Z}} \sum_{\underline{i}\in\mathbb{Z}_L^D} \epsilon_{\underline{i}} \Big(\sum_{\underline{j}} \omega_{\underline{j}} q_{\underline{i}+\underline{j}}^{(\ell)}\Big)^{d_l}.$$

We define the BP threshold ϵ^* as

$$\epsilon^* = \sup\{\epsilon > 0 \mid \lim_{\ell\to\infty} \mathbb{P}_{\mathrm{b}}^{(\ell)} = 0\}.$$

Namely, for the erasure probability below the threshold ϵ^*, the asymptotic decoding erasure probability goes to zero after infinite iterations.

A. Shortening w hyperplanes

Choose w axis-aligned hyperplanes in D-dimensional space as the shortened domain $\mathcal{Z}$. To be precise,

$$\mathcal{Z} := \{\underline{i} = (i_1,\dots,i_D) \in \mathbb{Z}_L^D \mid i_1 \in [0, w-1]\}.$$

We simply refer to $\mathcal{Z}$ as w hyperplanes. For $D = 1$, we have $\mathcal{Z} = [0, w-1] =: \tilde{\mathcal{Z}}$. We use $\tilde{\cdot}$ to represent variables for the 1D system throughout this paper for the sake of readability. The following proposition asserts that the asymptotic properties of MD systems are equivalent to those of the 1D system. See [9] for the proof of a general case.

Proposition 1:

$$\epsilon^*(d_l, d_r, L, \omega, \mathcal{Z}) = \tilde{\epsilon}^*(d_l, d_r, L, \tilde{\omega}, \tilde{\mathcal{Z}}), \tag{3}$$

$$R(d_l, d_r, L, \omega, \mathcal{Z}) = \tilde{R}(d_l, d_r, L, \tilde{\omega}, \tilde{\mathcal{Z}}) \tag{4}$$

$$= 1 - \frac{d_l}{d_r} - O(1/L). \tag{5}$$

Proof: We give a proof for $D = 2$. The proof for $D > 2$ follows similarly. It is sufficient to show $p_{\underline{i}}^{(\ell)} = \tilde{p}_{i_1}^{(\ell)}$ for any $\ell \geq 0$ by induction. It is obvious that

$$p_{(i_1,i_2)}^{(0)} = \tilde{p}_{i_1}^{(0)} = \begin{cases} 0 & i_1 \in [0, w-1] \\ \epsilon & \text{otherwise}, \end{cases}$$

for any $i_2 \in \mathbb{Z}_L$. Assume $p_{(i_1,i_2)}^{(\ell)} = \tilde{p}_{i_1}^{(\ell)}$ for ℓ. From (2) and the definition of $\tilde{\omega}$ and ω, it follows that

$$\begin{aligned} q_{(i_1,i_2)}^{(\ell+1)} &= 1 - \Big(1 - \frac{1}{w^2} \sum_{j_1=0}^{w-1} \sum_{j_2=0}^{w-1} p_{(i_1-j_1, i_2-j_2)}^{(\ell)}\Big)^{d_l-1} \\ &= 1 - \Big(1 - \frac{1}{w} \sum_{j_1=0}^{w-1} \tilde{p}_{i_1-j_1}^{(\ell)}\Big)^{d_l-1} = \tilde{q}_{i_1}^{(\ell+1)}, \\ p_{(i_1,i_2)}^{(\ell+1)} &= \epsilon \Big(\frac{1}{w^2} \sum_{j_1=0}^{w-1} \sum_{j_2=0}^{w-1} q_{(i_1+j_1, i_2+j_2)}^{(\ell+1)}\Big)^{d_l-1} \\ &= \epsilon \Big(\frac{1}{w} \sum_{j_1=0}^{w-1} \tilde{q}_{(i_1+j_1)}^{(\ell+1)}\Big)^{d_l-1} = \tilde{p}_{i_1}^{(\ell+1)}, \end{aligned}$$

for $(i_1, i_2) \notin \mathcal{Z}$. Thus we have $p_{(i_1,i_2)}^{(\ell)} = \tilde{p}_{i_1}^{(\ell)}$ for any $\ell \geq 0$, which concludes (3).

We derive (4) as follows.

$$\begin{aligned} &R(d_l, d_r, L, \omega, \mathcal{Z}) \\ &= 1 - \frac{d_l/d_r}{L^2 - wL} \sum_{(i_1,i_2)\in\mathbb{Z}_L^2} \Big(1 - \big(\sum_{(j_1,j_2):(i_1+j_1, i_2+j_2)\in\mathcal{Z}} \omega_{(j_1,j_2)}\big)^{d_r}\Big) \\ &= 1 - \frac{d_l/d_r}{L^2 - wL} L \sum_{(i_1,0)\in\mathbb{Z}_L^2} \Big(1 - \big(\sum_{(j_1,j_2):(i_1+j_1, j_2)\in\mathcal{Z}} \omega_{(j_1,j_2)}\big)^{d_r}\Big) \\ &= 1 - \frac{d_l/d_r}{L - w} \sum_{(i_1,0)\in\mathbb{Z}_L^2} \Big(1 - \big(\sum_{j_2=0}^{w-1} \sum_{j_1:0\leq i_1+j_1\leq w-1} 1/w^2\big)^{d_r}\Big) \\ &= \tilde{R}(d_l, d_r, L, \tilde{\omega}, \tilde{\mathcal{Z}}). \end{aligned}$$

Equation (5) follows from

$$\tilde{R}(d_l, d_r, L, \tilde{\omega}, \tilde{\mathcal{Z}}) = \Big(1 - \frac{d_l}{d_r}\Big) - \frac{d_l}{d_r} \frac{1 - w - 2\sum_{i=0}^{w} (\frac{i}{w})^{d_r}}{L - w}$$

of which proof appears in [3] for coupled codes defined on $[-L, L]$. □

In the next section, we will see these MD-SC codes with hyperplanes as the shortened domain behave differently from the 1D-SC codes.

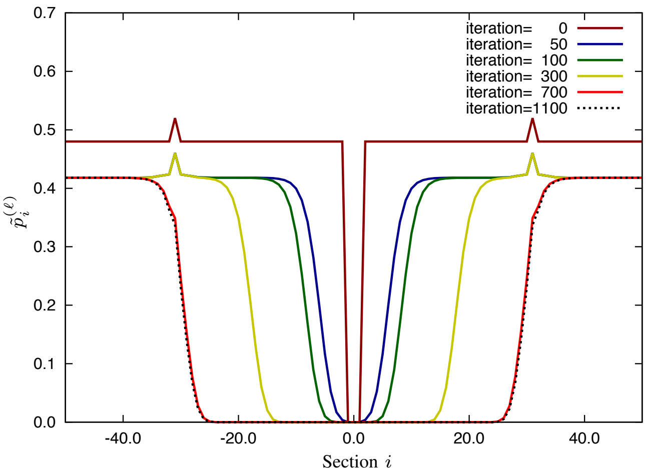

Fig. 1. Transition of the message error probability $\tilde{p}_i^{(\ell)}$ at each section i and at iteration ℓ of 1D-SC $(d_l = 3, d_r = 6, L = 101, \tilde{\omega}, \tilde{\mathcal{Z}} = \{0, \pm 1\})$ codes with $w = 4$. The channel is BEC($\epsilon = 0.48$) with 2 burst section erasures injected. Decoding aborts around at sections $i = \pm 31$ where burst erasures are injected as $\tilde{p}_i^{(0)} = 0.52$.

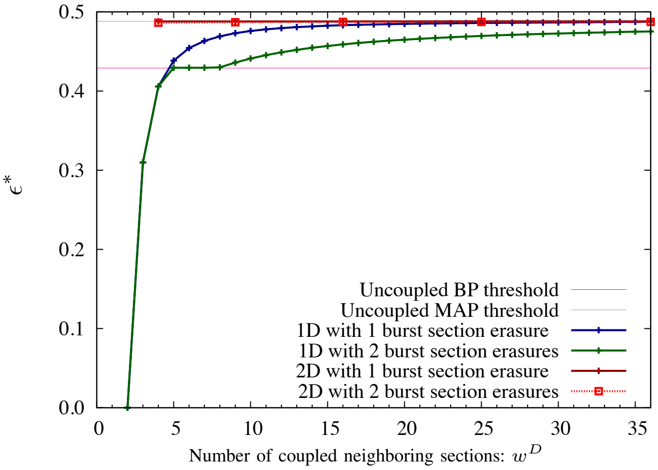

Fig. 2. The BP threshold of 1D-SC and 2D-SC $(d_l = 3, d_r = 6, L, \omega, \mathcal{Z})$ with w hyperplanes as $\mathcal{Z}$. The BP threshold of 1D-SC codes are seriously degraded when burst section erasures exist. The 2D-SC codes are more robust than the 1D-SC codes.

IV. Robustness for Burst Erasures

In this section, we consider burst erasures and demonstrate robustness of 2D coupled codes. 1D-SC codes are constructed by coupling L regular LDPC codes of length M. Assume that we are transmitting bits coded by 1D-SC codes and burst erasures of length M occur at some section i. We call such burst erasures a *burst section erasure*. Such a burst section erasure is described as $\epsilon_{\underline{i}} = \epsilon_i = 1$.

Can 1D-SC codes correct such burst erasures? Figure 1 shows the transition of the message error probability of 1D-SC $(d_l = 3, d_r = 6, L = 101, \tilde{\omega}, \tilde{\mathcal{Z}} = \{0, \pm 1\})$ codes with $w = 4$. The channel is BEC($\epsilon = 0.48$) with 2 burst section erasures injected. Decoding aborts around at section $i = \pm 31$ where burst erasures are injected as $\tilde{p}_i^{(0)} = 0.52$. The 1D-SC codes cannot recover such burst erasures.

Figure 4 in the last page shows the transition of the decoding error probability of 2D-SC $(d_l = 3, d_r = 6, L = 101, \omega, \mathcal{Z})$ codes with $w^2 = 4$ and 2 lines as $\mathcal{Z}$. The channel is BEC($\epsilon = 0.48$) with 20 burst section erasures injected. Each burst section erasure is described as $p_{\underline{i}}^{(0)} = 1$. The 2D-SC codes are capable of recovering such burst section erasures.

Figure 2 compares the BP threshold of 1D-SC codes to that of 2D-SC codes with w shortened lines. The degrees d_l and d_r are set to 3 and 6, respectively. The horizontal axis indicates w^D. This is intended to be dealt fairly with respect to the number of coupled neighboring sections both at 1D and 2D. We injected one or two burst section erasures. The coupling number L for each plotted point is chosen sufficiently large so that the BP threshold does not increase due to the rateloss, and that the burst error sections do not affected each other. Note that the BP threshold is approximately 0.4882 when there are no burst section erasures. For small coupling window size $w \geq 4$, the BP threshold of 1D-SC codes is 0. This is seriously degraded from 0.4882. This degradation cannot be mitigated by increasing L. When $w = 2$ the BP threshold is 0, namely the burst section erasure is not recovered even if all other sections were recovered. This can be explained by the theorem in the next section.

On the other hand, 2D-SC codes are not degraded from the case of no burst section erasures even for small w. From this observation, 2D-SC codes are more robust against burst section erasures than 1D-SC codes.

A. Bound on Performance

In the previous section we observed that 1D-SC codes do not recover a single burst section erasure when $w = 2$. This can be explained by the following theorem.

Theorem 1: The MD $(d_l, d_r, L, \omega, \mathcal{Z})$ code of dimension D cannot recover a burst section erasure at section $\underline{i}$ if $\epsilon_{\underline{i}} > \epsilon^{\mathrm{BP}}(d_l, d_r) w^D$ for $\underline{i} \notin \mathcal{Z}$, where $\epsilon^{\mathrm{BP}}(d_l, d_r)$ is the BP threshold of uncoupled (d_l, d_r) codes.

Proof: Let us consider the best case, namely other sections have no erasures. To be precise, $\epsilon_{\underline{j}} = 0$ for $\underline{j} \neq \underline{i}$. The density evolution equations can be written as

$$p_{\underline{j}}^{(\ell)} = 0 \ (\underline{j} \neq \underline{i})$$
$$p_{\underline{i}}^{(\ell)} = \epsilon_{\underline{i}} \Big(1 - \sum_{\underline{j}} \omega_{\underline{j}} \big(1 - \sum_{\underline{k}} \omega_{\underline{k}} p_{\underline{i}+\underline{j}-\underline{k}}^{(\ell-1)}\big)^{d_r - 1}\Big)^{d_l - 1}$$
$$= \epsilon_{\underline{i}} \Big(1 - \big(1 - \frac{1}{w^D} p_{\underline{i}}^{(\ell-1)}\big)^{d_r - 1}\Big)^{d_l - 1}.$$

Denoting $\hat{p}_{\underline{i}}^{(\ell)} := p_{\underline{i}}^{(\ell)} / w^D$, we have

$$\hat{p}_{\underline{i}}^{(\ell)} = \frac{\epsilon_{\underline{i}}}{w^D} (1 - (1 - \hat{p}_{\underline{i}}^{(\ell-1)})^{d_r - 1})^{d_l - 1}.$$

This can be viewed as the density evolution of uncoupled (d_l, d_r) codes over BEC($\epsilon_{\underline{i}}/w^D$). Hence $p_{\underline{i}}^{(\infty)} > 0$ if $\frac{\epsilon_{\underline{i}}}{w^D} > \epsilon^{\mathrm{BP}}(d_l, d_r)$, which concludes the theorem. ■

From Theorem 1, one can see that $(d_l = 3, d_r = 6, L, \tilde{\omega}, \tilde{\mathcal{Z}})$ with $w = 2$ cannot recover a single burst section erasure since $\epsilon_{\underline{i}} = 1 > 0.4294 \times 2 = 0.8588 = \epsilon^{\mathrm{BP}}(d_l, d_r) w^D$.

V. Rateloss Problem of Multi-Dimensional SC Codes and Its Mitigation

As one can see in (1), the rate of SC codes is less than that of the uncoupled codes $1 - \frac{d_l}{d_r}$. The 1D-SC codes have a rateloss of $O(1/L)$. The 1D-SC codes could have a rateloss of $O(1/L^D)$ by coupling L^D sections as 1D-SC codes. The D-dimensional SC codes with shortened hyperplanes have only $O(1/L)$ while there are L^D sections. This is a crucial drawback of selecting hyperplanes as the shortened domain. Is it possible to design MD-SC codes with a rateloss of $O(1/L^D)$ without changing the BP threshold?

Define the shortened domain as a hypercube of size z

$$\mathcal{Z} = [0, z-1]^D.$$

We claim that the rateloss of the codes with this $\mathcal{Z}$ has a rateloss of $O(1/L^D)$. The number C of check nodes that are adjacent to unshortened bit nodes is not greater than the number of all check nodes.

$$C \le \frac{d_l}{d_r} M L^D.$$

There are $V = M(L^D - z^D)$ unshortened bit nodes. Thus we have the coding rate as

$$\begin{aligned} R = 1 - C/V &\ge 1 - \frac{d_l}{d_r}\frac{L^D}{L^D - z^D} \\ &= \left(1 - \frac{d_l}{d_r}\right) - O\left(\frac{z^D}{L^D}\right). \end{aligned}$$

Note that we are not claiming that this rate is better than the coding rate of 1D-SC codes. It is fair to compare the coding rate for the same number of sections L_C. From this point of view, the rateloss of both 1D-SC codes and the MD-SC codes scales with $O(1/L_C)$, where $L_C = L$ for 1D-SC codes and $L_C = L^D$ for MD-SC codes of dimension D.

Does the BP threshold coincide with the MAP threshold of the uncoupled codes? Figure 3 shows the BP threshold of 2D-SC $(d_l = 3, d_r = 6, L, w = 2)$ codes with a square of size z as the shortening domain $\mathcal{Z}$. We take sufficiently large coupling number L so that each plotted point converges. We observe that the BP threshold approaches the MAP threshold of uncoupled codes as z gets large. Figure 5 shows the transition of the decoding error probability of 2D-SC $(d_l = 3, d_r = 6, L = 101, \omega, \mathcal{Z})$ codes with $w = 2$ and a square shortened domain of size 15×15. The channel is BEC($\epsilon = 0.48$) with 20 burst section erasures injected. These burst section erasures are recovered by 2D-SC codes. It is observed that 2D-SC codes can recover more burst section erasures as L gets large.

VI. Conclusion

We proposed MD-SC codes. MD-SC codes achieve the same BP threshold as 1D-SC codes when axis-aligned hyperplanes are used as the shortened domain. 2D-SC codes are more robust against burst section erasures than 1D-SC codes.

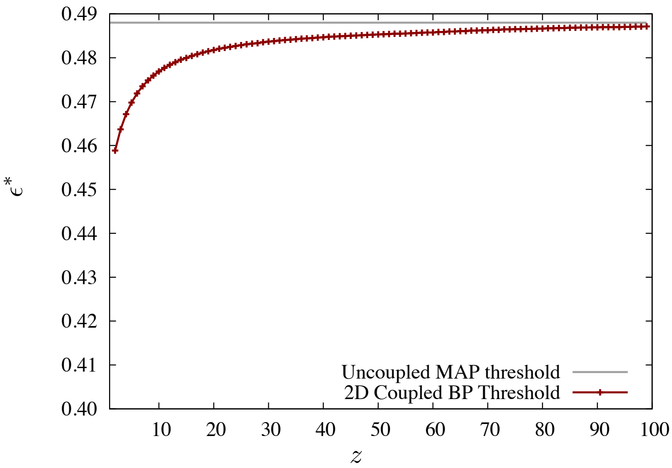

Fig. 3. The BP threshold of 2D-SC $(d_l = 3, d_r = 6, L, w = 2)$ codes with a square of size z as the shortened domain $\mathcal{Z}$.

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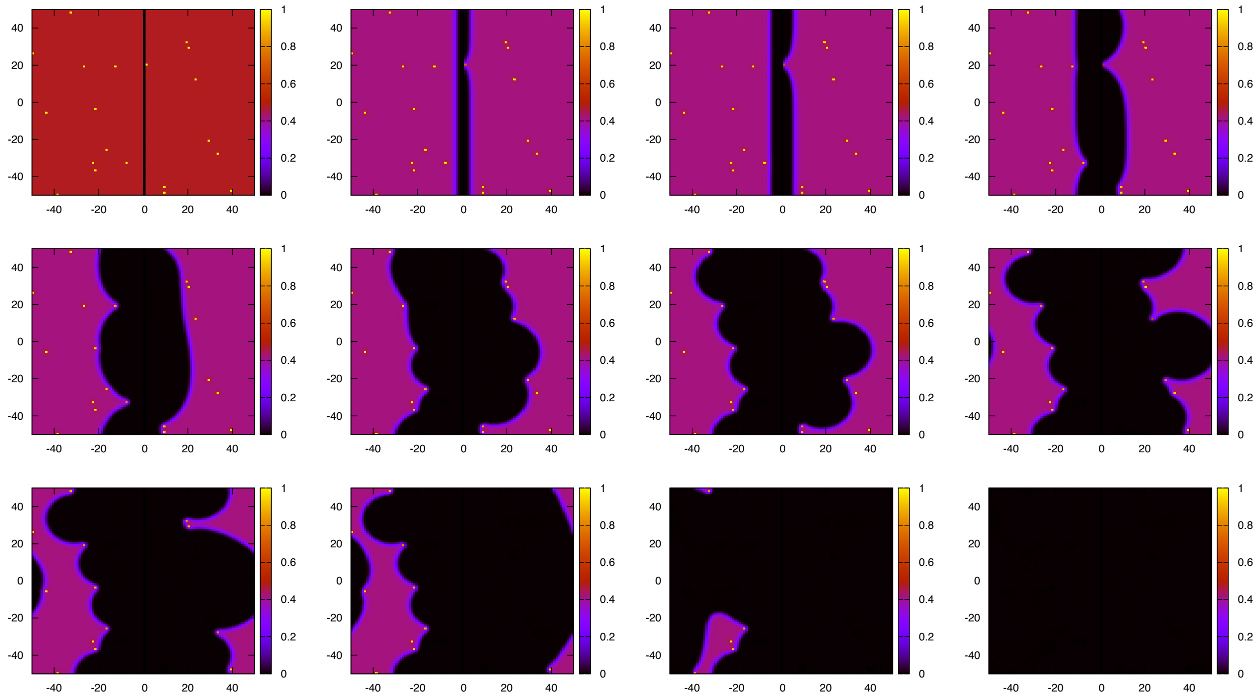

Fig. 4. Transition of the decoding error probability of 2D-SC $(d_l = 3, d_r = 6, L = 101, \omega, \mathcal{Z})$ codes with $w = 2$ and 2 shortened lines. The channel is BEC($\epsilon = 0.48$) with 20 burst section erasures injected.

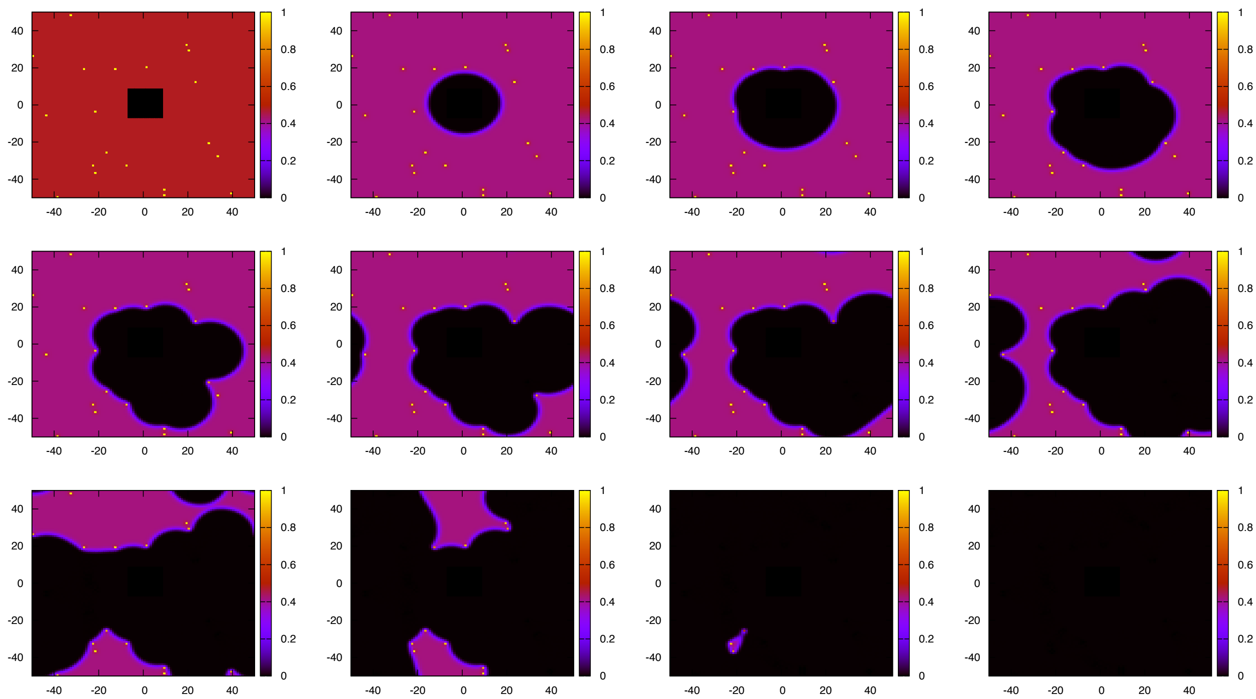

Fig. 5. Transition of the decoding error probability of 2D-SC $(d_l = 3, d_r = 6, L = 101, \omega, \mathcal{Z})$ codes with $w = 2$ and a shortened square of size 15. The channel is BEC($\epsilon = 0.48$) with 20 burst section erasures injected.