Finding the Number of Feasible Solutions for Linear Interference Alignment Problems

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***Abstract*—In this paper, we study how many different solutions exist for a feasible interference alignment (IA) problem. We focus on linear IA schemes without symbol extensions for the K-user multiple-input multiple-output (MIMO) interference channel. When the IA problem is feasible and the number of variables matches the number of equations in the polynomial system, the number of solutions is known to be finite. Unfortunately, the exact number of solutions is only known for a few particular cases, mainly single-beam MIMO networks. In this paper, we prove that the number of IA solutions is given by an integral formula that can be numerically approximated using Monte Carlo integration methods. More precisely, the number of solutions is the scaled average over a subset of the solution variety (formed by all triplets of channels, precoders and decoders satisfying the IA polynomial equations) of the determinant of certain Hermitian matrix related to the geometry of the problem. Our results can be applied to arbitrary interference MIMO networks, with any number of users, antennas and streams per user.**

I. INTRODUCTION

Interference alignment (IA) has received a lot of attention in recent years as a key technique to achieve the maximum degrees of freedom (DoF) of wireless networks in the presence of interference. The basic idea of IA consists of designing the transmitted signals in such a way that the interference at each receiver falls within a lower-dimensional subspace, therefore leaving a subspace free of interference for the desired signal. Since its inception in [1], [2], IA schemes have been applied in different forms and adapted to various wireless networks [3]–[6].

In this paper we consider the linear IA problem (i.e., signal space alignment by means of beamforming) for the K-user multiple-input multiple-output (MIMO) interference channel (IC) with constant channel coefficients and without symbol extensions. The feasibility of linear IA in this scenario, which is closely related to the problem treated in this work, has been an active research topic during the last years [7]–[12]. Here, we study the problem of how many different IA solutions exist for a feasible system, which is relevant as a diversity metric. Some IA solutions lead to highly collinear signal and interference subspaces causing a substantial desired signal loss after decoding. In this sense, the number of solutions serve as a measure of the number of independently faded IA designs. Additionally, the number of solutions is also related to the algebraic complexity required to compute a single solution.

We specifically consider the case in which the number of variables matches the number of equations of the IA polynomial system, because in this situation the number of solutions is finite and constant, as proved in [12]. Following the nomenclature recently introduced in [13] we refer to these systems as *tightly feasible*, stressing the fact that removing a single antenna from the network turns the IA problem infeasible.

For tightly feasible single-beam (i.e., when all users wish to transmit $d = 1$ stream of data) MIMO networks, it was shown in [7] that the number of alignment solutions coincides with the mixed volume of the Newton polytopes that support each equation of the polynomial system. In practice, however, the computation of the mixed volume of a set of IA equations using the available software tools [14] can be very demanding, therefore only a few cases have been solved so far. For single-beam networks, some upper bounds on the number of solutions using Bezout's Theorem have also been proposed in [7], [15]. For multi-beam scenarios, however, the genericity of the system of polynomial equations is lost and it is not possible to resort to mixed volume calculations to find the number of solutions.

The main contribution of this paper is an integral formula for the number of IA solutions for arbitrary, tightly feasible, interference networks. Although the integral, in general, is hard to compute analytically, it can be easily estimated using Monte Carlo integration. To speed up the convergence of the Monte Carlo integration method, we specialize the general integral formula for square symmetric cases, i.e., equal number of transmit and receive antennas and equal number of streams per user.

II. SYSTEM MODEL AND BACKGROUND MATERIAL

A. System model and notation

In this paper, we consider the K-user MIMO IC where each user wishes to send $d_k \geq 0$ streams, each transmitter has $M_k \geq 1$ antennas, and each receiver is equipped with $N_k \geq 1$ antennas. We denote the set of users as $\mathcal{K} = \{1, \ldots, K\}$ and the set of interfering links as $\Phi = \{(k,l) : k,l \in \mathcal{K}, k \neq l\}$. Also, $\sharp(\Phi)$ denotes the cardinality of Φ, that is the number of elements in the finite set Φ.

We adhere to the notation used in [7] and denote this (fully connected) asymmetric interference channel as $\prod_{k\in\mathcal{K}}(M_k \times N_k, d_k) = (M_1 \times N_1, d_1)\cdots(M_K \times N_K, d_K)$. The symmetric case in which all users transmit d streams and are equipped with M transmit and N receive antennas is denoted as $(M \times N, d)^K$. In the square symmetric case all users have the same number of antennas $M = N$.

The MIMO channel from transmitter l to receiver k is denoted as H_{kl} and assumed to be flat-fading and constant over time. Each H_{kl} is an $N_k \times M_l$ complex matrix with independent entries drawn from a continuous distribution. The interference alignment (IA) problem is to find the decoders and precoders, V_l and U_k, in such a way that the interfering signals at each receiver fall into a reduced-dimensional subspace. The receivers can then extract the projection of the desired signal that lies in the interference-free subspace. To this end, it is required that the polynomial equations

$$U_k^T H_{kl} V_l = 0, \qquad (k,l) \in \Phi, \tag{1}$$

are satisfied (the superscript T denoting transpose), while the signal subspace for each user must be linearly independent of the interference subspace and must have dimension d_k, that is

$$\operatorname{rank}(U_k^T H_{kk} V_k) = d_k, \qquad \forall\, k \in \mathcal{K}. \tag{2}$$

B. Feasibility of IA: A brief review

The IA feasibility problem, which has been deeply investigated in [7]–[10], amounts to study the relationship between d_k, M_k, N_k, K such that the linear alignment problem is feasible. In the following, we make a short review of the main feasibility result presented in [11], [12], which forms the starting point of this work. Let us first describe the three main algebraic sets involved in the feasibility problem.

- Input space formed by the MIMO matrices, which is formally defined as
$$\mathcal{H} = \prod_{(k,l)\in\Phi} \mathbb{C}^{N_k \times M_l}$$
where $\prod$ holds for Cartesian product, and $\mathbb{C}^{N_k\times M_l}$ is the set of $N_k \times M_l$ complex matrices.
- Output space of precoders and decoders (i.e., the set where possible outputs exist)
$$\mathcal{S} = \left(\prod_{k\in\mathcal{K}} \mathbb{G}_{d_k,N_k}\right) \times \left(\prod_{l\in\mathcal{K}} \mathbb{G}_{d_l,M_l}\right),$$
where $\mathbb{G}_{a,b}$ is the Grassmannian formed by the linear subspaces of (complex) dimension a in $\mathbb{C}^b$.
- The solution variety, which is given by
$$\mathcal{V} = \{(H,U,V) \in \mathcal{H}\times\mathcal{S} :\ (1) \text{ holds}\}$$
where H is the collection of all H_{kl} and, similarly, U and V denote the collection of U_k and V_l, respectively. The set $\mathcal{V}$ is given by the polynomial equations in (1), linear in each of the H_{kl}, U_k, V_l and therefore is an algebraic subvariety of the product space $\mathcal{H}\times\mathcal{S}$.

Once the main algebraic sets have been defined, it is interesting to consider the following diagram

$$\begin{array}{ccc} & \mathcal{V} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{H} & & \mathcal{S} \end{array} \tag{3}$$

where the sets and the main projections involved in the feasibility problem are depicted. Note that, given $H \in \mathcal{H}$, the set $\pi_1^{-1}(H)$ is a copy of the set of U, V such that (1) holds, that is the solution set of the linear interference alignment problem.

The feasibility question can then be posed as, *is* $\pi_1^{-1}(H) \neq \emptyset$ *for a generic* H*?* The question was solved in [12], basically stating that for the problem to be feasible two conditions have to be fulfilled:

1) The algebraic dimension of $\mathcal{V}$ must be larger than or equal to the dimension of $\mathcal{H}$, i.e.,
$$s = \left(\sum_{k\in\mathcal{K}} d_k(N_k + M_k - 2d_k)\right) - \left(\sum_{(k,l)\in\Phi} d_k d_l\right) \geq 0. \tag{4}$$
In other words this condition means that, for the problem of polynomial equations to have a solution, the number of variables must be larger than or equal to the number of equations. This condition was already established in [7], hereby classifying interference channels as proper ($s \geq 0$) or improper ($s < 0$). In [8], [9] it was rigorously proved that improper systems are always infeasible.
2) For *some* element $(H,U,V) \in \mathcal{V}$, the linear mapping θ given by
$$(\dot{U}_1,\ldots,\dot{U}_K,\dot{V}_1,\ldots,\dot{V}_K) \mapsto \left\{\dot{U}_k^T H_{kl} V_l + U_k^T H_{kl} \dot{V}_l\right\} \tag{5}$$
must be surjective, i.e., it must have maximal rank equal to $\sum_{(k,l)\in\Phi} d_k d_l$. Here, $(\dot{U}_1,\ldots,\dot{U}_K,\dot{V}_1,\ldots,\dot{V}_K)$ denotes a set of complex matrices of dimensions $N_k\times d_k$ or $M_l\times d_l$ depending on whether $\dot{U}_k$ or $\dot{V}_l$ is considered, which are affine representations of the components of a vector in the tangent space of $\mathcal{V}$. This condition amounts to saying that the projection from the tangent plane at an arbitrary point of the solution variety to the tangent plane of the input space must be surjective. Moreover, in this case, the mapping (5) is surjective for *almost every* $(H,U,V) \in \mathcal{V}$.

III. The number of IA solutions

A. Preliminaries

In this paper we start by taking structured matrices given by

$$H_{kl} = \begin{pmatrix} 0_{d_k\times d_l} & A_{kl} \\ B_{kl} & C_{kl} \end{pmatrix}, \tag{6}$$

with precoders and decoders given by

$$V_l = \begin{pmatrix} I_{d_l} \\ 0_{(M_l-d_l)\times d_l} \end{pmatrix}, \qquad U_k = \begin{pmatrix} I_{d_k} \\ 0_{(N_k-d_k)\times d_k} \end{pmatrix}, \tag{7}$$

which trivially satisfy $U_k^T H_{kl} V_l = 0$ and therefore belong to the solution variety. We claim that essentially all the useful information about $\mathcal{V}$ can be obtained from the subset of $\mathcal{V}$ consisting on triples (H_{kl}, U_k, V_l) of the form (6) and (7). The reason is that given any other element $(H'_{kl}, U'_k, V'_l) \in \mathcal{V}$, one can easily find sets of orthogonal matrices P_k and Q_l satisfying

$$U_k = P_k U'_k, \qquad V_l = Q_l V'_l,$$

and

$$(U')_k^T H'_{kl} V'_l = U_k^T \left(P_k^*\right)^T H'_{kl} Q_l^* V_l = 0,$$

where the superscript $*$ denotes Hermitian. That is, the transformed channels $H_{kl} = \left(P_k^*\right)^T H'_{kl} Q_l^*$ have the form (6), and the transformed precoders V_l and decoders U_k have the form (7). Thus, we have just described an isometry which sends (H'_{kl}, U'_k, V'_l) to (H_{kl}, U_k, V_l). The situation is thus similar to that of a torus: every point can be sent to some predefined vertical circle through a rotation, thus the torus is essentially understood by "moving" a circle and keeping track of the visited places. The same way, $\mathcal{V}$ can be thought of as moving the set of triples of the form (6) and (7), and keeping track of the visited places. Technically, $\mathcal{V}$ is the orbit of the set of triples of the form (6) and (7) under the isometric action of a product of unitary groups.

In summary, the main idea is that, for the purpose of checking feasibility or counting solutions, we can replace the set of arbitrary complex matrices $\mathcal{H}$ by the set of structured matrices

$$\mathcal{H}_I = \prod_{k \neq l} \begin{pmatrix} 0_{d_k \times d_l} & A_{kl} \\ B_{kl} & C_{kl} \end{pmatrix}.$$

Similarly, we can replace the mapping θ in (5) by a new mapping Ψ of the form

$$(\dot{U}_1, \ldots, \dot{U}_K, \dot{V}_1, \ldots, \dot{V}_K) \mapsto \left(\dot{U}_k^T B_{kl} + A_{kl} \dot{V}_l\right). \tag{8}$$

We will be interested in the function $\det(\Psi\Psi^*)$, which depends on the channel realization H only through the blocks A_{kl} and B_{kl}. The vectorization of the mapping (8) reveals that Ψ is composed of two main kinds of blocks, $\Psi_{kl}^{(A)}$ and $\Psi_{kl}^{(B)}$, i.e.

$$\begin{aligned} &\mathrm{vec}(\dot{U}_k^T B_{kl} + A_{kl} \dot{V}_l) = \\ &\underbrace{(A_{kl} \otimes I_{d_k}) K_{(N_k - d_k), d_k}}_{\Psi_{kl}^{(A)}} \mathrm{vec}(\dot{U}_k) + \underbrace{(I_{d_l} \otimes B_{kl}^T)}_{\Psi_{kl}^{(B)}} \mathrm{vec}(\dot{V}_l), \end{aligned} \tag{9}$$

where $\otimes$ denotes Kronecker product and $K_{m,n}$ is the $mn \times mn$ commutation matrix which is defined as the matrix that transforms the vectorized form of an $m \times n$ matrix into the vectorized form of its transpose. Block $\Psi_{kl}^{(B)}$ has dimensions $d_l d_k \times d_l(M_l - d_l)$, whereas block $\Psi_{kl}^{(A)}$ is $d_l d_k \times d_k(N_k - d_k)$. For a given tuple (k, l), $\Psi_{kl}^{(B)}$ and $\Psi_{kl}^{(A)}$ are placed in the row partition that corresponds to the interfering link indicated by the tuple (k, l). $\Psi_{kl}^{(B)}$ is placed in the $l{+}K$-th column partition, whereas $\Psi_{kl}^{(A)}$ occupies the k-th column partition. The rest of blocks are occupied by null matrices. The dimensions of Ψ are therefore $\sum_{k \neq l} d_k d_l \times \sum_{j=1}^{K} (M_j + N_j - 2d_j) d_j$. In the particular case of $s = 0$, Ψ is a square matrix of size $\sum_{k \neq l} d_k d_l$.

Notice that Ψ has the same structure as the incidence matrix of the network connectivity graph. Taking the 3-user system as an example, Ψ is constructed as follows

$$\begin{bmatrix} \Psi_{12}^{(A)} & 0 & 0 & 0 & \Psi_{12}^{(B)} & 0 \\ \Psi_{13}^{(A)} & 0 & 0 & 0 & 0 & \Psi_{13}^{(B)} \\ 0 & \Psi_{21}^{(A)} & 0 & \Psi_{21}^{(B)} & 0 & 0 \\ 0 & \Psi_{23}^{(A)} & 0 & 0 & 0 & \Psi_{23}^{(B)} \\ 0 & 0 & \Psi_{31}^{(A)} & \Psi_{31}^{(B)} & 0 & 0 \\ 0 & 0 & \Psi_{32}^{(A)} & 0 & \Psi_{32}^{(B)} & 0 \end{bmatrix}$$

where the blocks $\Psi_{kl}^{(B)}$ and $\Psi_{kl}^{(A)}$ are given by (9).

B. Main results

Given a Riemannian manifold X with total finite volume denoted as $Vol(X)$ let

$$⨏_{x \in X} f(x)\, dx = \frac{1}{Vol(X)} \int_{x \in X} f(x)\, dx$$

be the average value of a integrable (or measurable and nonnegative) function $f : X \to \mathbb{R}$. The main results of the paper are Theorems 1 and 2 below, which give integral expressions for the number of IA solutions when the system is tightly feasible ($s = 0$): this number is denoted as $\sharp(\pi_1^{-1}(H_0))$ and is the same for all channel realizations out of some zero-measure set. Detailed proofs of the theorems can be found in [16].

Theorem 1: For a tightly feasible ($s = 0$) fully connected interference channel, and for every $H_0 \in \mathcal{H}$ out of some zero–measure set, we have:

$$\sharp(\pi_1^{-1}(H_0)) = C ⨏_{H \in \mathcal{H}_I, \|H_{kl}\|_F = 1} \det(\Psi\Psi^*)\, dH, \tag{10}$$

where Ψ is defined by (8) and

$$C = \prod_{(k,l) \in \Phi} \left(\frac{\Gamma(N_k M_l)}{\Gamma(N_k M_l - d_k d_l)} \right) \times$$

$$\prod_{k \in \mathcal{K}} \left(\frac{\Gamma(2) \cdots \Gamma(d_k) \cdot \Gamma(2) \cdots \Gamma(N_k - d_k)}{\Gamma(2) \cdots \Gamma(N_k)} \right) \times$$

$$\prod_{l \in \mathcal{K}} \left(\frac{\Gamma(2) \cdots \Gamma(d_l) \cdot \Gamma(2) \cdots \Gamma(M_l - d_l)}{\Gamma(2) \cdots \Gamma(M_l)} \right)$$

and $\Gamma(a) = (a-1)!$ denotes the Gamma function.

Remark 1: As proved in [11], [12], if the system is infeasible then $\det(\Psi\Psi^*) = 0$ for every choice of H, U, V and hence Theorem 1 still holds. Moreover, if the system is feasible and $s > 0$ then there is a continuous of solutions for almost every H_{kl} and hence it is meaningless to count them.

Theorem 1 can be used to approximate the number of solutions of a given MIMO system using Monte Carlo integration (see Section IV below). The convergence of the integral however is quite slow in general. In the square symmetric case, when $N = M \geq 2d$ or equivalently $N = M$ and $K \geq 3$, we can write another integral which has faster convergence in practice.

Theorem 2: Let us consider a tightly feasible ($s = 0$) fully connected *square* interference channel such that all users have the same number of transmit and receive antennas, $N_k = M_k = N$, and send the same number of streams $d_k = d$. Assuming additionally that $K \geq 3$, then for every $H_0 \in \mathcal{H}$ out of some zero–measure set, we have:

$$\sharp(\pi_1^{-1}(H_0)) = C' \fint_{(A_{kl}^*, B_{kl}) \in \mathcal{U}^2_{(N-d)\times d}} \det(\Psi\Psi^*)\, dH,$$

where Ψ is again defined by (8) and the input space of MIMO channels where we have to integrate are now

$$H_{kl} = \begin{pmatrix} 0_{d\times d} & A_{kl} \\ B_{kl} & 0_{(N-d)\times(N-d)} \end{pmatrix},$$

whose blocks, A_{kl} and B_{kl}, are matrices in the complex Stiefel manifold, denoted as $\mathcal{U}_{(N-d)\times d}$, and formed by all orthonormal d-dimensional vectors in $\mathbb{C}^{(N-d)}$. The constant preceding the integral in this case is

$$C' = \left(\frac{\Gamma(N-d+1)\cdots\Gamma(N)}{\Gamma(N-2d+1)\cdots\Gamma(N-d)}\right)^{K(K-1)} \left(\frac{\Gamma(2)\cdots\Gamma(d)}{\Gamma(N-d+1)\cdots\Gamma(N)}\right)^{2K}.$$

In the next section we discuss how the results in Theorems 1 and 2 can be used to get approximations to the number of IA solutions for a given interference network.

IV. Estimating the number of solutions

The integrals in Theorems 1 and 2 are too difficult to be computed analytically, but one can certainly try to compute it approximately using Monte Carlo integration. Our main reference here is [17, Sec. 5]. The *Crude Monte Carlo* method for computing the average

$$\fint_{x\in X} f(x)\, dx$$

of a function f defined on a finite-volume manifold X consists just in choosing many points at random, say $x_1, \ldots, x_n$ for $n >> 1$, uniformly distributed in X, and approximating

$$\fint_{x\in X} f(x)\, dx \approx E_n = \frac{1}{n}\sum_{j=1}^{n} f(x_j). \tag{11}$$

The most reasonable way to implement this in a computer program is to write down an iteration that computes $E_1, E_2, E_3, \ldots$. The unique point to be decided is how many such x_j we must choose to get a reasonable approximation of the integral. A usual tool for measuring that is the standard deviation, that can be approximated by

$$\Sigma_n = \left(\frac{1}{n-1}\sum_{j=1}^{n}(f(x_j) - E_n)^2\right)^{1/2}. \tag{12}$$

Algorithm 1: Computing the number of IA solutions for symmetric square scenarios $(N \times N, d)^K$.

Input: Relative error, ε; N; d; and K.
Output: Approximate number of IA solutions, E_n.
$n = 1$
repeat
 Generate a set of i.i.d. $(N-d) \times d$ matrices $\{A_{kl}^*\}$ and $\{B_{kl}\}$ in the Stiefel manifold.
 Build matrix Ψ according to (9).
 Compute $D_n = C' \det(\Psi\Psi^*)$.
 Calculate E_n and Σ_n according to (11) and (12), respectively, where $f(x_j)$ is now D_j.
 $n = n + 1$.
until $\frac{\Sigma_n}{E_n} < \varepsilon$

If we stop the iteration when $\frac{\Sigma_n}{E_n} < \varepsilon$, then, with a probability of 0.95 on the set of random sequences of n terms, the relative error satisfies

$$\frac{\left|\fint_{x\in X} f(x)\, dx - E_n\right|}{|E_n|} \leq 2\varepsilon.$$

For example, if we stop the iteration when $\frac{\Sigma_n}{E_n} < 0.05$, then, we can expect a maximum error of about 10 percent in our calculation of $\fint_{x\in X} f(x)\, dx$.

The whole procedure for the case of square symmetric systems is illustrated in Algorithm 1 which follows Theorem 2. The extension to general networks according to Theorem 1 is straightforward.

V. Numerical experiments

Table I shows the estimated number of alignment solutions for some feasible MIMO networks with $d = 2$, indicating in parentheses the approximation error in percentage. These results have been obtained using the integral formula in Theorem 1, except the square cases ($M = N$), for which we used the expression in Theorem 2. For instance, we can mention that the system $(5\times 5, 2)^4$ has, with a high confidence level, 3700 different solutions. This result was not known so far. As it can be observed, the estimate of the integral formula in Theorem 2 converges much faster than that of Theorem 1, thus allowing us to get smaller relative errors. More examples can be found in [16].

Although these results have mainly a theoretical interest, they might also have some important practical implications. For instance, knowing the rate of increase of $\sharp(\pi_1^{-1}(H))$ with K could have interest to analyze the asymptotic performance of linear IA, as discussed in [15]. Also, for moderate-size networks for which the total number of solutions is not very high, the results of this paper also open the possibility to provide a systematic way to compute all (or practically all) interference alignment solutions for a channel realization. Although all IA solutions are asymptotically equivalent, their sum-rate performance in low or moderate SNRs behavior may differ significantly [15], [18]. The main idea here is that if we are able to obtain all or almost all IA solutions for a

	$M=3$	$M=4$	$M=5$	$M=6$
	$(3\times(2K-1),2)^K$	$(4\times(2K-2),2)^K$	$(5\times(2K-3),2)^K$	$(6\times(2K-4),2)^K$
$K=2$	0	1	–	–
$K=3$	1	6	1	1
$K=4$	≈ 9 (5.8 %)	≈ 973 (7.0 %)	$\approx 3\,700$ (0.1 %)	≈ 973 (7.0 %)
$K=5$	≈ 223 (14.8 %)	$\approx 530\,725$ (11.3 %)	$\approx 72\,581\,239$ (17.8 %)	$\approx 387\,682\,648$ (0.7 %)

TABLE I

APPROXIMATE NUMBER OF IA SOLUTIONS FOR SEVERAL SYMMETRIC 2-BEAM SCENARIOS, $(M\times(2K-M+2),2)^K$.

particular channel realization, we can get all or almost all IA solutions for any other channel realization by using a homotopy-continuation based method such as that described in [19]. This idea is reflected in Figure 1, which shows in grey the sum-rate curves of 973 different solutions for the $(4\times6,2)^4$ network. The maximum sum-rate solution is plotted in a thicker solid line, while the average sum-rate of all solutions is represented with a dashed line. The relative performance improvement provided by the maximum sum-rate solution over the average is always above 10 % for SNR values below 40 dB, and is more than 20 % for SNR=20 dB.

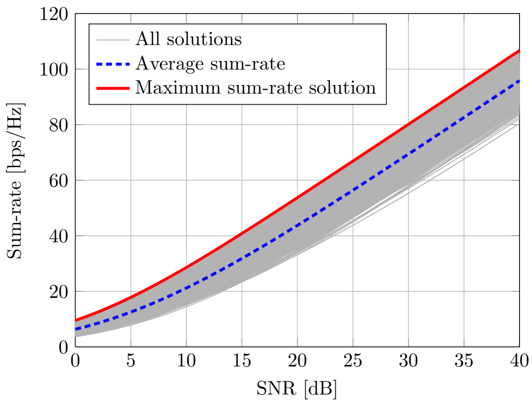

Fig. 1. Comparison of the sum rate achieved by different solutions for the system $(4\times6,2)^4$.

VI. CONCLUSION

In this paper we have provided two integral formulae to compute the finite number of IA solutions in tightly feasible problems, including multi-beam ($d_k>1$) networks. The first one can be applied to arbitrary K-user channels, whereas the second one solves the symmetric square case. Both integrals can be estimated by means of Monte Carlo methods. Using our results, we found, for instance, that the system $(5\times5,2)^4$ has, with a high confidence level, 3700 different solutions.

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