Sending a Bivariate Gaussian Source Over a Gaussian MAC with Unidirectional Conferencing Encoders

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***Abstract*—We consider the problem of transmitting a bivariate Gaussian memoryless source over a two-user additive Gaussian multiple-access channel with unidirectional conferencing encoders. Here, prior to each transmission block Encoder 1 is allowed to communicate with Encoder 2 via a unidirectional noise-free bit-pipe of given capacity. We extend the vector-quantizer scheme suggested by Lapidoth-Tinguely, for the case without conferencing, to the case with unidirectional conference and derive an achievable rate-distortion region. We compare the performance of the suggested vector-quantizer to that of the optimal scheme for lossless transmission when the capacity of the conference link is large.** [1]

I. Introduction

We consider a communication scenario where two transmitters wish to transmit a memoryless bivariate Gaussian source to a single receiver over a two-user additive white Gaussian multiple-access channel (MAC). Prior to each transmission block, Encoder 1 is allowed to communicate with Encoder 2 via a unidirectional noise-free bit-pipe of given capacity. Special cases are the classical MAC considered by Lapidoth-Tinguely in [1], where the encoders are ignorant of each others inputs (the bit-pipe is of zero capacity) and the asymmetric setting, where Encoder 2 is fully cognizant of the source input at Encoder 1 (the pipe is of infinite capacity).

The time-k output of the Gaussian MAC is given by

$$Y_k = x_{1,k} + x_{2,k} + Z_k, \tag{1}$$

where $(x_{1,k}, x_{2,k}) \in \mathbb{R}^2$ are the symbols sent by the transmitters, and Z_k is the time-k additive noise term. The sequence $\{Z_k\}$ consists of independent identically distributed (IID) zero-mean variance N Gaussian random variables that are independent of the source sequence.

The input source sequence $\{(S_{1,k}, S_{2,k})\}$ consists of zero-mean Gaussians of covariance

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix} \tag{2}$$

with $\rho \in [0, 1]$, and $0 < \sigma^2 < \infty$. The sequence $\{S_{1,k}\}$ is observed by Encoder 1 and the sequence $\{S_{2,k}\}$ is observed by Encoder 2. Prior to each block of n channel uses, the encoders may exchange information via the use of the unidirectional bit-pipe which is assumed to be:

- perfect in the sense that any input symbol is available immediately and error-free at the output of the pipe; and
- of limited capacity C_{12}, in the sense that when the input to the pipe from Encoder 1 to Encoder 2 takes values in the set $\mathcal{W}$, such that $W = f^{(n)}(\mathbf{S}_1)$ for some encoding function $f^{(n)} : \mathbb{R}^n \mapsto \mathcal{W}$, then

$$\log |\mathcal{W}| \le nC_{12}. \tag{3}$$

We define an (n, C_{12})-*conference* to be a collection of an input alphabet $\mathcal{W}$, and an encoding function $f^{(n)}(\cdot)$ as above, where n, C_{12} and the alphabet set satisfy (3).

After the conference, Encoder 2 is cognizant of the random variable W so the channel inputs $\mathbf{X}_1 = (X_{1,1}, \ldots, X_{1,n})$ and $\mathbf{X}_2 = (X_{2,1}, \ldots, X_{2,n})$ can be described via encoding functions $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$ as

$$\begin{aligned} \mathbf{X}_1 &= \varphi_1^{(n)}(\mathbf{S}_1), \\ \mathbf{X}_2 &= \varphi_2^{(n)}(\mathbf{S}_2, W) = \varphi_2^{(n)}(\mathbf{S}_2, f^{(n)}(\mathbf{S}_1)), \end{aligned}$$

where

$$\begin{aligned} \varphi_1^{(n)} &: \mathbb{R}^n \mapsto \mathbb{R}^n, \\ \varphi_2^{(n)} &: \mathbb{R}^n \times \mathcal{W} \mapsto \mathbb{R}^n. \end{aligned} \tag{4}$$

The transmitted sequences of the two encoders are average-power limited to P_1 and P_2 respectively, i.e.

$$\frac{1}{n}\mathsf{E}\left[\sum_{k=1}^{n} (X_{\nu,k})^2\right] \le P_\nu, \qquad \nu = 1, 2. \tag{5}$$

Based on the channel output $\mathbf{Y} = (Y_1, \ldots, Y_n)$ the decoder forms its estimates $\hat{\mathbf{S}}_1 = \phi_1^{(n)}(\mathbf{Y})$ and $\hat{\mathbf{S}}_2 = \phi_2^{(n)}(\mathbf{Y})$ for the source sequences respectively, where

$$\phi_\nu^{(n)} : \mathbb{R}^n \mapsto \mathbb{R}^n, \qquad \nu = 1, 2. \tag{6}$$

We are interested in the minimal expected squared-error distortions at which the receiver can reconstruct each of the source sequences.

Definition 1: Given $\sigma^2 > 0, \rho \in [0, 1], P_1, P_2, N > 0$ and $C_{12} > 0$ we say that the tuple $(D_1, D_2, \sigma^2, \rho, P_1, P_2, N, C_{12})$ is *achievable* if there exists a sequence of block-lengths n, encoding functions $f^{(n)}$ which belong to an (n, C_{12})-*conference*, encoders $(\varphi_1^{(n)}, \varphi_2^{(n)})$ as in (4) satisfying the average-power

[1]The work of S. Bross was supported by the Israel Science Foundation under Grant 497/09.

constraints (5), and reconstruction functions $(\phi_1^{(n)}, \phi_2^{(n)})$ as in (6) resulting in average distortions that fulfill

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[(S_{\nu,k} - \hat{S}_{\nu,k})^2\right] \le D_\nu, \qquad \nu = 1, 2, \quad (7)$$

whenever

$$Y_k = \varphi_{1,k}^{(n)}(\mathbf{S}_1) + \varphi_{2,k}^{(n)}(\mathbf{S}_2, f^{(n)}(\mathbf{S}_1)) + Z_k, \quad k = 1, \ldots, n$$

and $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean bivariate Gaussian vectors with covariance matrix K_{SS} as in (2) and $\{Z_k\}$ are IID zero-mean variance-N Gaussian random variables that are independent of $\{(S_{1,k}, S_{2,k})\}$.

Of special interest is the case where both encoders are subject to equal power constraints, and where $D_1 = \alpha D_2, 0 < \alpha < 1$. That is, for some given C_{12}, N and $D_1 = \alpha D, D_2 = D$ we are interested in

$$P^*(\sigma^2, \rho, \alpha, D, N, C_{12}) \triangleq \inf\{P \colon (\alpha D, D, \sigma^2, \rho, P, P, N, C_{12}) \text{ is achievable }\}.$$

II. Main Results

Our achievability result is based on an extension of the vector-quantizer scheme presented in [1] which benefits from the presence of the unidirectional conference channel. The encoding steps of our scheme are presented in Fig. 1.

The source sequence $\mathbf{S}_1$ is quantized by Encoder 1 in two steps; first it is quantized by a rate-R_1 vector-quantizer where the quantized sequence is denoted by $\mathbf{U}_1^*$, then the quantization error of the first step is quantized by a rate-R_c vector-quantizer, where $R_c + 1/2\log[1 - \rho^2 2^{-2R_1}(1 - 2^{-2R_c})] \le C_{12}$, and the quantized sequence is denoted by $\mathbf{V}^*$. The source sequence $\mathbf{S}_2$ is quantized by Encoder 2 via a rate-R_2 vector-quantizer where the quantized sequence is denoted by $\mathbf{U}_2^*$. Encoder 1 informs Encoder 2 via the conference channel on the index of $\mathbf{V}^*$, given its side-information $\mathbf{S}_2$, and consequently both encoders can cooperate in transmitting this sequence.
The channel input $\mathbf{X}_1$ is now given by

$$\mathbf{X}_1 = \alpha_{1,1}\mathbf{U}_1^* + \alpha_{1,2}\mathbf{V}^* \qquad (8)$$

where for $0 \le \beta_1 \le 1$, and $\bar{\beta}_1 \triangleq 1 - \beta_1$ the gains $\alpha_{1,1}$ and $\alpha_{1,2}$ are chosen as

$$\alpha_{1,1} = \sqrt{\frac{\bar{\beta}_1 P_1}{\sigma^2(1 - 2^{-2R_1})}}, \alpha_{1,2} = \sqrt{\frac{\beta_1 P_1}{\sigma^2 2^{-2R_1}(1 - 2^{-2R_c})}}$$

This ensures that the input $\mathbf{X}_1$ satisfies the average-power constraint P_1.
The channel input $\mathbf{X}_2$ is now given by

$$\mathbf{X}_2 = \alpha_{2,1}\mathbf{U}_2^* + \alpha_{2,2}\mathbf{V}^* \qquad (9)$$

where for $0 \le \beta_2 \le 1$, the gain $\alpha_{2,1}$ is chosen as

$$\alpha_{2,1} = \sqrt{\frac{\bar{\beta}_2 P_2}{\sigma^2(1 - 2^{-2R_2})}}$$

and the gain $\alpha_{2,2}$ is chosen as per (13). This ensures that the input $\mathbf{X}_2$ satisfies the average-power constraint P_2.

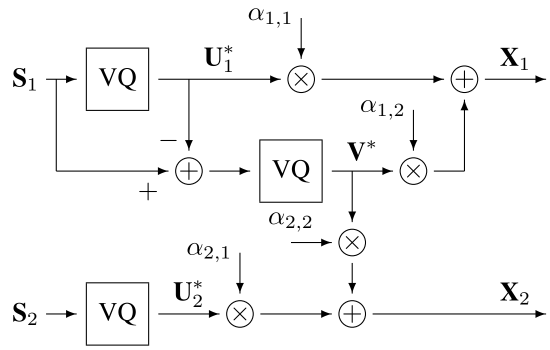

Fig. 1. The vector-quantizer flow.

Based on the channel output $\mathbf{Y}$ the decoder first estimates the triplet $(\mathbf{U}_1^*, \mathbf{V}^*, \mathbf{U}_2^*)$ by performing joint decoding which takes into account the correlation between the sequences. The resulting decoded triplet is denoted by $(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2)$. The decoder then forms its estimates of the source sequences $\mathbf{S}_\nu, \nu = 1, 2$ using approximate MMSE estimates $\hat{\mathbf{S}}_\nu$ based on $(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2)$, i.e.,

$$\begin{aligned} \hat{\mathbf{S}}_1 &= \gamma_{1,1}\hat{\mathbf{U}}_1 + \gamma_{1,2}\hat{\mathbf{U}}_2 + \gamma_{1,3}\hat{\mathbf{V}} \approx \mathsf{E}\left[\mathbf{S}_1 | \hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2\right] \\ \hat{\mathbf{S}}_2 &= \gamma_{2,1}\hat{\mathbf{U}}_1 + \gamma_{2,2}\hat{\mathbf{U}}_2 + \gamma_{2,3}\hat{\mathbf{V}} \approx \mathsf{E}\left[\mathbf{S}_2 | \hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2\right] \quad (10) \end{aligned}$$

A detailed description of the scheme is given in Section III.A.

The distortion pairs achieved by this vector-quantizer scheme are described in the next theorem.

Theorem 1: The distortions achieved by the vector-quantizer scheme are all pairs (D_1, D_2) satisfying

$$\begin{aligned} D_1 &> \sigma^2 2^{-2(R_1+R_c)} \frac{1 - \rho^2(1 - 2^{-2R_2})}{1 - \rho^2(1 - 2^{-2R_2})(1 - 2^{-2(R_1+R_c)})} \\ D_2 &> \sigma^2 2^{-2R_2} \frac{1 - \rho^2(1 - 2^{-2(R_1+R_c)})}{1 - \rho^2(1 - 2^{-2R_2})(1 - 2^{-2(R_1+R_c)})} \quad (11) \end{aligned}$$

where, for some $0 \le \beta_1, \beta_2 \le 1$, the rate-triple (R_1, R_2, R_c) satisfies

$$\begin{aligned} R_1 &< \frac{1}{2}\log\left(\frac{\bar{\beta}_1 P_1(1 - \tilde{\rho}^2 - \bar{\rho}^2) + N(1 - \bar{\rho}^2)}{N(1 - \tilde{\rho}^2 - \bar{\rho}^2)}\right) \\ R_2 &< \frac{1}{2}\log\left(\frac{\bar{\beta}_2 P_2(1 - \tilde{\rho}^2 - \bar{\rho}^2) + N}{N(1 - \tilde{\rho}^2 - \bar{\rho}^2) + \lambda_2}\right) \\ R_c &< \frac{1}{2}\log\left(\frac{\delta^2(1 - \tilde{\rho}^2 - \bar{\rho}^2) + N(1 - \tilde{\rho}^2)}{N(1 - \tilde{\rho}^2 - \bar{\rho}^2) + \lambda_c}\right) \\ R_1 + R_2 &< \frac{1}{2}\log\left(\frac{\lambda_{12} - \bar{\beta}_2 P_2 \bar{\rho}^2 + N}{(1 - \bar{\beta}_2 P_2 \bar{\rho}^2 \lambda_{12}^{-1}) N (1 - \tilde{\rho}^2)}\right) \\ R_1 + R_c &< \frac{1}{2}\log\left(\frac{(\lambda_{1c} + N)(\bar{\beta}_1 P_1 + \delta^2)}{\lambda_{1c} N}\right) \\ R_2 + R_c &< \frac{1}{2}\log\left(\frac{\lambda_{2c} - \bar{\beta}_2 P_2 \tilde{\rho}^2 + N}{(1 - \bar{\beta}_2 P_2 \tilde{\rho}^2 \lambda_{2c}^{-1}) N\,(1 - \bar{\rho}^2\ )}\right) \\ R_1 + R_2 + R_c &< \frac{1}{2}\log\left(\frac{\lambda_{12} + 2\delta\bar{\rho}\sqrt{\bar{\beta}_2 P_2} + \delta^2 + N}{N(1 - \tilde{\rho}^2)(1 - \bar{\rho}^2)}\right) \\ R_c + 1/2\log[1 - \rho^2 2^{-2R_1}&(1 - 2^{-2R_c})] < C_{12} \quad (12) \end{aligned}$$

and

$$
\begin{aligned}
\tilde{\rho} &\triangleq \rho\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})} \\
\bar{\rho} &\triangleq \rho\sqrt{2^{-2R_1}(1-2^{-2R_2})(1-2^{-2R_c})} \\
\lambda_2 &\triangleq \frac{N^2\bar{\rho}^2\tilde{\rho}^2(2+\tilde{\rho}^2)}{\beta_2 P_2(1-\tilde{\rho}^2-\bar{\rho}^2)+N} \\
\sigma_v^2 &\triangleq \sigma^2 2^{-2R_1}(1-2^{-2R_c}) \\
\alpha_{2,2} &\triangleq \sqrt{\frac{P_2}{\sigma^2}}\Bigg(\sqrt{\rho^2\bar{\beta}_2(1-2^{-2R_2})+\frac{\sigma^2\beta_2}{\sigma_v^2}} \\
&\qquad -\sqrt{\rho^2\bar{\beta}_2(1-2^{-2R_2})}\Bigg) \\
\delta &\triangleq \sigma_v(\sqrt{\beta_1 P_1}\sigma_v^{-1}+\alpha_{2,2}) \\
\lambda_c &\triangleq \frac{N^2\bar{\rho}^2(\bar{\rho}^2\bar{\beta}_1 P_1-\tilde{\rho}^2\sigma_v^2)}{\sigma_v^2(\delta^2(1-\tilde{\rho}^2-\bar{\rho}^2)+N(1-\tilde{\rho}^2))} \\
\lambda_{12} &\triangleq \bar{\beta}_1 P_1+2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2}+\bar{\beta}_2 P_2 \\
\lambda_{1c} &\triangleq \bar{\beta}_1 P_1(1-\tilde{\rho}^2)+\delta^2(1-\bar{\rho}^2) \\
&\qquad -2\delta\sigma_v^{-1}\bar{\rho}^2\sqrt{\bar{\beta}_1 P_1\sigma^2(1-2^{-2R_1})} \\
\lambda_{2c} &\triangleq \bar{\beta}_2 P_2+2\delta\bar{\rho}\sqrt{\bar{\beta}_2 P_2}+\delta^2
\end{aligned} \tag{13}
$$

Remark 1: The substitution of $C_{12}=0$ in Theorem 1 (which then implies $R_c=0$ as well as $\bar{\beta}_1=\bar{\beta}_2=1$) recovers the Lapidoth-Tinguely achievable rate-distortion region [1, Theorem IV.4].

Based on Theorem 1 we now present sufficient conditions for the achievability of $(D_1, D_2, \sigma^2, \rho, P_1, P_2, N, \infty)$.

Corollary 1: When $C_{12}=\infty$ the distortions achieved by the vector-quantizer scheme are all pairs (D_1, D_2) satisfying

$$
\begin{aligned}
D_1 &> \sigma^2 2^{-2R_c}\frac{1-\rho^2(1-2^{-2R_2})}{1-\hat{\rho}^2} \\
D_2 &> \sigma^2 2^{-2R_2}\frac{1-\rho^2(1-2^{-2R_c})}{1-\hat{\rho}^2}
\end{aligned}
$$

where, for some $0\le\beta\le 1$, the rate-pair (R_2, R_c) satisfies

$$
\begin{aligned}
R_2 &< \frac{1}{2}\log\left(\frac{\bar{\beta}P_2(1-\hat{\rho}^2)+N}{N(1-\hat{\rho}^2)}\right) \\
R_c &< \frac{1}{2}\log\left(\frac{\delta_1^2(1-\hat{\rho}^2)+N}{N(1-\hat{\rho}^2)}\right) \\
R_2+R_c &< \frac{1}{2}\log\left(\frac{\delta_2+N}{N(1-\hat{\rho}^2)}\right)
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
\hat{\rho} &\triangleq \rho\sqrt{(1-2^{-2R_2})(1-2^{-2R_c})} \\
\delta_1 &\triangleq \sqrt{P_1}+\sqrt{P_2}\left(\sqrt{\bar{\beta}\hat{\rho}^2+\beta}-\sqrt{\bar{\beta}\hat{\rho}^2}\right) \\
\delta_2 &\triangleq \bar{\beta}P_2+2\delta_1\hat{\rho}\sqrt{\bar{\beta}P_2}+\delta_1^2 \\
&= P_1+P_2+2\sqrt{(\bar{\beta}\hat{\rho}^2+\beta)P_1 P_2}
\end{aligned}
$$

Remark 2: For the achievability of the distortion pairs in Corollary 1 it suffices that $C_{12}\ge R_c+1/2\log[1-\rho^2(1-2^{-2R_c})]$.

Source-Channel Separation

We consider next the set of distortion pairs that are achieved by combining the optimal scheme for the source-coding problem without unidirectional conference link with the optimal scheme for the channel-coding problem with unidirectional conference link . The rate-distortion region associated with the source-coding problem can be found in [2], [3] and is described as follows. A normalized distortion pair (d_1, d_2) is achievable if, and only if, $(R_1, R_2)\in\mathcal{R}(d_1, d_2)$ where

$$
\begin{aligned}
\mathcal{R}(d_1,d_2) = \Bigg\{(R_1,R_2)\colon\ & R_1 \ge \frac{1}{2}\log_2^+\left[\frac{1-\rho^2(1-2^{-2R_2})}{d_1}\right] \\
& R_2 \ge \frac{1}{2}\log_2^+\left[\frac{1-\rho^2(1-2^{-2R_1})}{d_2}\right] \\
& R_1+R_2 \ge \frac{1}{2}\log_2^+\left[\frac{(1-\rho^2)\beta(d_1,d_2)}{2d_1 d_2}\right]\Bigg\}
\end{aligned}
$$

with $\beta(d_1,d_2)=1+\sqrt{1+\frac{4\rho^2 d_1 d_2}{(1-\rho^2)^2}}$.
The distortion pairs achievable by source-channel separation follow now by combining the latter set of rate pairs with the capacity region of the Gaussian MAC with unidirectional conference link reported in [4], which for $C_{12}=\infty$, is expressed by

$$
\begin{aligned}
\mathcal{C} = \bigcup_{0\le\beta\le 1}\Bigg\{(R_1,R_2)\colon\ & R_2 \le \frac{1}{2}\log_2\left(1+\bar{\beta}P_2/N\right) \\
& R_1+R_2 \le \frac{1}{2}\log_2\left[1+(P_1+P_2+2\sqrt{\beta P_1 P_2})/N\right]\Bigg\}
\end{aligned}
$$

By [5, Theorem 1], when $C_{12}=\infty$ source-channel separation is optimal for lossless transmission. Table 1 compares the performance of the vector-quantizer to that of the source-channel separation for lossy transmission.

ρ	d	α	P^*_{VQ}	$P^*_{\text{SC sep}}$	C_{12}(VQ)	C_{12}(SC sep)
0.5	0.2	0.1	47.4124	47.4124	2.6156	2.8183
0.5	0.05	0.8	97.2739	97.2739	2.1150	2.3128

TABLE I
MINIMAL POWER $P^*(1,\rho,\alpha,d,1,\infty)$ AND MINIMAL CONFERENCE RATE FOR THE VECTOR-QUANTIZER AND SOURCE-CHANNEL SEPARATION.

III. PROOFS

A. Proof of Theorem 1

1) Coding Scheme: Fix some $\epsilon>0$ and a rate tuple (R_1, R_2, R_c).

Code Construction: Three codebooks $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_c$ are generated independently. Codebook $\mathcal{C}_i, i\in 1,2$, consists of 2^{nR_i} codewords $\{\mathbf{U}_i(1), \mathbf{U}_i(2), \dots, \mathbf{U}_i(2^{nR_i})\}$. The codewords are drawn independently uniformly over the surface of the centered $\mathbb{R}^n$-sphere $\mathcal{S}_i$ of radius $r_i = \sqrt{n\sigma^2(1-2^{-2R_i})}$. Codebook $\mathcal{C}_c$, consists of 2^{nR_c} codewords $\{\mathbf{V}(1), \mathbf{V}(2), \ldots, \mathbf{V}(2^{nR_c})\}$. The codewords are drawn independently uniformly over the surface of the centered $\mathbb{R}^n$-sphere $\mathcal{S}_c$ of radius $r_c = \sqrt{n\sigma^2 2^{-2R_1}(1-2^{-2R_c})}$.

For every $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$ where neither $\mathbf{w}$ nor $\mathbf{v}$ are the zero-sequence, denote the angle between $\mathbf{w}$ and $\mathbf{v}$ by $\sphericalangle(\mathbf{w}, \mathbf{v})$. i.e.,

$$\cos\sphericalangle(\mathbf{w}, \mathbf{v}) \triangleq \frac{\langle \mathbf{w}, \mathbf{v}\rangle}{\|\mathbf{w}\|\|\mathbf{v}\|}.$$

Encoding: Given the source sequences $(\boldsymbol{s}_1, \mathbf{s}_2)$, let $\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i)$ be the set defined by

$$\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i) \triangleq \Big\{\mathbf{u}_i \in \mathcal{C}_i \colon \Big|\cos\sphericalangle(\mathbf{s}_i, \mathbf{u}_i) - \sqrt{1-2^{-2R_i}}\Big| \le \sqrt{1-2^{-2R_i}}\epsilon\Big\}.$$

Encoder 1 vector-quantizes $\mathbf{s}_1$ in two steps as follows.

1) If $\mathcal{F}(\mathbf{s}_1, \mathcal{C}_1) \neq \emptyset$ it forms the vector $\mathbf{u}_1^*$ by choosing it as the codeword $\mathbf{u}_1(j^*) \in \mathcal{F}(\mathbf{s}_1, \mathcal{C}_1)$ where j^* minimizes $|\cos\sphericalangle(\mathbf{s}_1, \mathbf{u}_1(j)) - \sqrt{1-2^{-2R_1}}|$, while if $\mathcal{F}(\mathbf{s}_1, \mathcal{C}_1) = \emptyset$ then $\mathbf{u}_1^*$ is the all-zero sequence.
2) Let

$$\mathbf{Z}_{Q_1} \triangleq \mathbf{S}_1 - \mathbf{U}_1^*. \tag{15}$$

let $\mathcal{F}(\mathbf{z}_{Q_1}, \mathcal{C}_c)$ be the set defined by

$$\mathcal{F}(\mathbf{z}_{Q_1}, \mathcal{C}_c) \triangleq \Big\{\mathbf{v} \in \mathcal{C}_c \colon \Big|\cos\sphericalangle(\mathbf{z}_{Q_1}, \mathbf{v}) - \sqrt{1-2^{-2R_c}}\Big| \le \sqrt{1-2^{-2R_c}}\epsilon\Big\}.$$

If $\mathcal{F}(\mathbf{z}_{Q_1}, \mathcal{C}_c) \neq \emptyset$ it forms the vector $\mathbf{v}^*$ by choosing it as the codeword $\mathbf{v}(k^*) \in \mathcal{F}(\mathbf{z}_{Q_1}, \mathcal{C}_c)$ where k^* minimizes $|\cos\sphericalangle(\mathbf{z}_{Q_1}, \mathbf{v}(k)) - \sqrt{1-2^{-2R_c}}|$, while if $\mathcal{F}(\mathbf{z}_{Q_1}, \mathcal{C}_c) = \emptyset$ then $\mathbf{v}^*$ is the all-zero sequence.

The channel input $\mathbf{X}_1$ is now given by (8).

Since the codebooks $\mathcal{C}_1$ and $\mathcal{C}_c$ are drawn over the centered $\mathbb{R}^n$-spheres of radii $r_1 = \sqrt{\sigma^2(1-2^{-2R_1})}$ and $r_c = \sqrt{\sigma^2 2^{-2R_1}(1-2^{-2R_c})}$, respectively, and the codewords $\mathbf{U}_1^*$ and $\mathbf{V}^*$ are uncorrelated, the channel input $\mathbf{X}_1$ satisfies the average-power constraint.

Encoder 2 vector-quantizes $\mathbf{s}_2$ as follows.

If $\mathcal{F}(\mathbf{s}_2, \mathcal{C}_2) \neq \emptyset$ it forms the vector $\mathbf{u}_2^*$ by choosing it as the codeword $\mathbf{u}_2(j^*) \in \mathcal{F}(\mathbf{s}_2, \mathcal{C}_2)$ where j^* minimizes $|\cos\sphericalangle(\boldsymbol{s}_2, \mathbf{u}_2(j)) - \sqrt{1-2^{-2R_2}}|$, while if $\mathcal{F}(\mathbf{s}_2, \mathcal{C}_2) = \emptyset$ then $\mathbf{u}_2^*$ is the all-zero sequence.

Encoder 2 acquires the index k^* via the unidirectional conference channel, given its side-information $\mathbf{S}_2$, and therefore knows $\mathbf{v}^*$ as well. The channel input $\mathbf{X}_2$ is now given by (9).

Since the codebooks $\mathcal{C}_2$ and $\mathcal{C}_c$ are drawn over the centered $\mathbb{R}^n$-spheres of radii $r_1 = \sqrt{\sigma^2(1-2^{-2R_2})}$ and $r_c = \sqrt{\sigma^2 2^{-2R_1}(1-2^{-2R_c})}$, respectively, and (as will be shown later) the codewords $\mathbf{U}_2^*$ and $\mathbf{V}^*$ are correlated, the channel input $\mathbf{X}_2$ satisfies the average-power constraint.

Reconstruction: The receiver's estimate $(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2)$ of the source pair $(\mathbf{S}_1, \mathbf{S}_2)$ is obtained via the channel output $\mathbf{Y}$ in two steps. First, the receiver makes a guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2)$ of the tuple $(\mathbf{U}_1^*, \mathbf{V}^*, \mathbf{U}_2^*)$ by choosing among all "jointly typical" tuples $(\mathbf{u}_1, \mathbf{v}, \mathbf{u}_2) \in \mathcal{C}_1 \times \mathcal{C}_c \times \mathcal{C}_2$ the tuple whose linear combination $\alpha_{1,1}\mathbf{U}_1 + \alpha_{2,1}\mathbf{U}_2 + (\alpha_{1,2} + \alpha_{2,2})\mathbf{V}$ has the smallest distance to the received sequence $\mathbf{Y}$. More formally, let $\bar{\mathcal{F}}(\mathcal{C}_1, \mathcal{C}_c, \mathcal{C}_2)$ be the set of triplets $(\mathbf{u}_1, \mathbf{v}, \mathbf{u}_2) \in \mathcal{C}_1 \times \mathcal{C}_c \times \mathcal{C}_2$ such that

$$\begin{aligned} |\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1, \mathbf{u}_2)| &\le 7\epsilon \\ |\bar{\rho} - \cos\sphericalangle(\mathbf{v}, \mathbf{u}_2)| &\le 7\epsilon \\ |\cos\sphericalangle(\mathbf{v}, \mathbf{u}_1)| &\le 3\epsilon \end{aligned} \tag{16}$$

where $(\tilde{\rho}, \bar{\rho})$ are defined in (13), and for any tuple $(\mathbf{u}_1, \mathbf{v}, \mathbf{u}_2)$ define

$$\mathbf{X}_{\mathbf{u}_1,\mathbf{v},\mathbf{u}_2} \triangleq \alpha_{1,1}\mathbf{u}_1 + \alpha_{2,1}\mathbf{u}_2 + (\alpha_{1,2} + \alpha_{2,2})\mathbf{v}.$$

Then the receiver forms its estimate by choosing

$$(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2) = \arg\min_{(\mathbf{u}_1,\mathbf{v},\mathbf{u}_2) \in \bar{\mathcal{F}}(\mathcal{C}_1, \mathcal{C}_c, \mathcal{C}_2)} \|\mathbf{Y} - \mathbf{X}_{\mathbf{u}_1,\mathbf{v},\mathbf{u}_2}\|^2. \tag{17}$$

If the channel output $\mathbf{Y}$ and the codebooks are such that there doesn't exist a member in $\bar{\mathcal{F}}(\mathcal{C}_1, \mathcal{C}_c, \mathcal{C}_2)$ that minimizes the r.h.s. in (17) then $(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2)$ are chosen to be the all-zero sequences.

In the second step, the receiver forms its estimates $(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2)$ of the source pair $(\mathbf{S}_1, \mathbf{S}_2)$ according to (10) where

$$\begin{aligned} \gamma_{1,1} = \gamma_{1,3} &= \frac{1 - \rho^2(1-2^{-2R_2})}{\big[1 - \rho^2(1-2^{-2R_2})(1-2^{-2(R_1+R_c)})\big]} \\ \gamma_{1,2} &= \frac{\rho 2^{-2(R_1+R_c)}}{\big[1 - \rho^2(1-2^{-2R_2})(1-2^{-2(R_1+R_c)})\big]} \\ \gamma_{2,1} = \gamma_{2,3} &= \frac{\rho 2^{-2R_2}}{\big[1 - \rho^2(1-2^{-2R_2})(1-2^{-2(R_1+R_c)})\big]} \\ \gamma_{2,2} &= \frac{\big[1 - \rho^2(1-2^{-2(R_1+R_c)})\big]}{\big[1 - \rho^2(1-2^{-2R_2})(1-2^{-2(R_1+R_c)})\big]} \end{aligned}$$

2) Expected Distortion: Similarly to [1], to analyze the expected distortion we first show that, when the rate constraints (12) are satisfied, the asymptotic normalized distortion of the proposed scheme remains the same as that of a genie-aided scheme in which the genie provides the decoder with the triplet $(\mathbf{U}_1^*, \mathbf{V}^*, \mathbf{U}_2^*)$. The genie-aided decoder forms its estimate $(\hat{\mathbf{S}}_1^{(g)}, \hat{\mathbf{S}}_2^{(g)})$ based on $(\mathbf{U}_1^*, \mathbf{V}^*, \mathbf{U}_2^*)$ according to (10) and ignores its guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{V}}, \hat{\mathbf{U}}_2)$ produced in the first decoding step.

Proposition 1: If (R_1, R_2, R_c) satisfy (12) then

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \mathsf{E}\left[\|\mathbf{S}_\nu - \hat{\mathbf{S}}_\nu\|^2\right] \le \overline{\lim_{n\to\infty}} \frac{1}{n} \mathsf{E}\left[\|\mathbf{S}_\nu - \hat{\mathbf{S}}_\nu^{(g)}\|^2\right], \nu = 1, 2$$

Proof: We show that for any (R_1, R_2, R_c) satisfying (12) and sufficiently large n, the probability of a decoding error, and consequently $\Pr\left[(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2) \neq (\hat{\mathbf{S}}_1^{(g)}, \hat{\mathbf{S}}_2^{(g)})\right]$ is arbitrarily small. To this end we consider the event consisting of all tuples $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ for which there exists a triplet

$(\tilde{\mathbf{u}}_1, \tilde{\mathbf{v}}, \tilde{\mathbf{u}}_2) \neq (\mathbf{u}_1^*, \mathbf{v}^*, \mathbf{u}_2^*)$ in $\mathcal{C}_1 \times \mathcal{C}_c \times \mathcal{C}_2$ that satisfies conditions (16) of the reconstructor, and for which the Euclidean distance between $\mathbf{X}_{\tilde{\mathbf{u}}_1, \tilde{\mathbf{v}}, \tilde{\mathbf{u}}_2}$ and $\mathbf{y}$ is smaller or equal to the Euclidean distance between $\mathbf{X}_{\mathbf{u}_1^*, \mathbf{v}^*, \mathbf{u}_2^*}$ and $\mathbf{y}$.

This event is split into seven sub-events. In what follows we demonstrate the analysis of one of these sub-events – the event consisting of all tuples $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ for which there exists a pair $(\tilde{\mathbf{u}}_1 \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}, \tilde{\mathbf{u}}_2 \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\})$ that satisfies (16) and

$$\|\mathbf{y} - \mathbf{X}_{\tilde{\mathbf{u}}_1, \mathbf{v}^*, \tilde{\mathbf{u}}_2}\|^2 \leq \|\mathbf{y} - \mathbf{X}_{\mathbf{u}_1^*, \mathbf{v}^*, \mathbf{u}_2^*}\|^2 \tag{18}$$

To show that the probability of this sub-event vanishes as $n \to \infty$, define

$$\mathbf{w} \triangleq \zeta_1(\mathbf{y} - (\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*) + \zeta_2(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*$$

where

$$\zeta_1 = \frac{\lambda}{\lambda + N/\sigma^2}$$
$$\zeta_2 = \frac{\alpha_{2,1}\rho(1 - 2^{-2R_2})N/\sigma^2}{(\alpha_{1,2} + \alpha_{2,2})(\lambda + N/\sigma^2)}$$

and $\lambda \triangleq \alpha_{1,1}^2(1 - 2^{-2R_1}) + 2\alpha_{1,1}\alpha_{2,1}\rho(1 - 2^{-2R_1})(1 - 2^{-2R_2}) + \alpha_{2,1}^2(1 - 2^{-2R_2})(1 - \bar{\rho}^2)$.

The proof is now based on a sequence of statements related to Condition (16) and Condition (18).

A) For every "typical" tuple $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ Condition (16) implies

$$\begin{aligned} &\left| n\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} - \langle \alpha_{1,1}\tilde{\mathbf{u}}_1, \alpha_{2,1}\tilde{\mathbf{u}}_2 \rangle \right| \\ &\qquad \leq 7n\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2}\epsilon \end{aligned} \tag{19}$$

as well as

$$\begin{aligned} &\left| \|(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*\| \bar{\rho}\sqrt{n\bar{\beta}_2 P_2} - \langle (\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*, \alpha_{2,1}\tilde{\mathbf{u}}_2 \rangle \right| \\ &\qquad \leq 7\|(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*\|\sqrt{n\bar{\beta}_2 P_2}\epsilon \end{aligned} \tag{20}$$

B) For every "typical" tuple $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ Condition (18) implies

$$\begin{aligned} &\langle \mathbf{y} - (\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*, \alpha_{1,1}\tilde{\mathbf{u}}_1 + \alpha_{2,1}\tilde{\mathbf{u}}_2 \rangle \geq \\ &\qquad n(\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2 - \xi_1\epsilon) \end{aligned} \tag{21}$$

C) Condition (16) and Condition (18) imply

$$\begin{aligned} &\frac{1}{n}\|\alpha_{1,1}\tilde{\mathbf{u}}_1 + \alpha_{2,1}\tilde{\mathbf{u}}_2 - \mathbf{w}\|^2 \leq \frac{1}{n}\|\mathbf{w}\|^2 + \xi_2\epsilon \\ &\quad + (\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2)(1 - 2\zeta_1) \\ &\quad - \frac{2}{n}\zeta_2\|(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*\|\bar{\rho}\sqrt{n\bar{\beta}_2 P_2} \end{aligned} \tag{22}$$

D) For every "typical" tuple $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$

$$\begin{aligned} &\frac{1}{n}\|\mathbf{w}\|^2 \leq \zeta_1^2(\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2 + N) \\ &\quad + \frac{2}{n}\zeta_1\zeta_2\|(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*\|\bar{\rho}\sqrt{n\bar{\beta}_2 P_2} \\ &\quad + \frac{1}{n}\zeta_2^2\|(\alpha_{1,2} + \alpha_{2,2})\mathbf{v}^*\|^2 + \xi_3\epsilon \end{aligned} \tag{23}$$

E) For every "typical" tuple $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ Condition (16) and Condition (18) together with C) and D) imply

$$\begin{aligned} &\|\alpha_{1,1}\tilde{\mathbf{u}}_1 + \alpha_{2,1}\tilde{\mathbf{u}}_2 - \mathbf{w}\|^2 \leq \\ &\quad n\frac{(\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2(1 - \bar{\rho}^2))N}{\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2(1 - \bar{\rho}^2) + N} + n\xi_4\epsilon \\ &\quad \triangleq \Upsilon(\epsilon) \end{aligned} \tag{24}$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ depend just on $P_1, P_2, N, \zeta_1, \zeta_2$.

F) For every pair $(\mathbf{u}_1 \in \mathcal{S}_1^{(n)}, \mathbf{u}_2 \in \mathcal{S}_2^{(n)})$ satisfying (16), denote by $\varphi \in [0, \pi]$ the angle between $\alpha_{1,1}\mathbf{u}_1 + \alpha_{2,1}\mathbf{u}_2$ and $\mathbf{w}$, and let

$$\begin{aligned} &\mathcal{B}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1^*, \mathbf{v}^*, \mathbf{u}_2^*, \mathbf{z}) \triangleq \Big\{(\mathbf{u}_1, \mathbf{u}_2) : \\ &\quad \cos\varphi \geq \sqrt{1 - \frac{\Upsilon(\epsilon)}{n(\bar{\beta}_1 P_1 + 2\tilde{\rho}\sqrt{\bar{\beta}_1\bar{\beta}_2 P_1 P_2} + \bar{\beta}_2 P_2)}}\Big\} \end{aligned} \tag{25}$$

where we assume that ϵ is sufficiently small such that the term inside the square-root is positive. Then, for every "typical" tuple $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z})$ Condition (16) and Condition (18) imply that $(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) \in \mathcal{B}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1^*, \mathbf{v}^*, \mathbf{u}_2^*, \mathbf{z})$.

To conclude the proof we use the following lemma.

Lemma 1: For any $\Theta \in (0, 1], \Delta \in (0, 1]$ define the set

$$\begin{aligned} \mathcal{G} \triangleq \Big\{&(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_c, \mathbf{z}) : \exists \tilde{\mathbf{u}}_1 \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}, \exists \tilde{\mathbf{u}}_2 \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\} \\ &\text{s.t. } \cos\sphericalangle(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) \geq \Theta, \cos\sphericalangle(\mathbf{w}, \alpha_{1,1}\tilde{\mathbf{u}}_1 + \alpha_{2,1}\tilde{\mathbf{u}}_2) \geq \Delta\Big\} \end{aligned}$$

If

$$R_1 + R_2 < -\frac{1}{2}\log((1 - \Theta^2)(1 - \Delta^2)) \tag{26}$$

then conditionally on the inputs being "typical" the probability of $\mathcal{G}$ vanishes as $n \to \infty$.

Proof: The proof of the lemma appears in [1, Lemma D.9]. ■

Taking Δ as the lower bound on the r.h.s. of (25) and $\Theta = \tilde{\rho} - 7\epsilon$ we arrive at the desired upper bound (13) on $R_1 + R_2$. The analysis of the other sub-events is done similarly.

Finally, we show that the distortion associated with the genie-aided decoder satisfies (11). ■

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