1. Cálculo de momentos de distribución uniforme continua.

$$X \sim U(a,b),\ a<b \iff f_X(x) = \frac{1}{b-a} \quad \forall x \in (a,b)$$

$$F_X(x) = \frac{x-a}{b-a} \qquad M_X(t) = E[e^{tX}] = \frac{e^{tb}-e^{ta}}{t(b-a)},\ t\in\mathbb{R}$$

• Momentos No centrados de orden K.

$$m_K = E[X^K] = \int_a^b \frac{x^K}{b-a}\,dx = \left[\frac{x^{K+1}}{(K+1)(b-a)}\right]_{x=a}^{x=b} = \frac{b^{K+1}-a^{K+1}}{(K+1)(b-a)}$$

• Momentos centrados de orden K.

$$\mu_K = E[(X-m_1)^K] = \int_a^b \frac{(x-m_1)^K}{b-a}\,dx = \frac{1}{(b-a)(K+1)}\left(\left(\frac{b-a}{2}\right)^{K+1} - \left(\frac{a-b}{2}\right)^{K+1}\right)$$

$$= \begin{cases} 0 & \text{si } K\in\mathbb{N} \text{ impar} \\ \dfrac{(b-a)^K}{2^K(K+1)} & \text{si } K\in\mathbb{N} \text{ par} \end{cases}$$

2. Cálculo de las funciones generatrices de momentos de distribuciones: Unif. continua, Normal, exponencial.

$X \sim U(a,b)$

$$M_X(t) = E[e^{tX}] = \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{e^{tb}-e^{ta}}{t(b-a)},\ t\in\mathbb{R}$$

$X \sim N(0,1)$

$$M_X(t) = E[e^{tX}] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{tx-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t^2/2}\,\frac{e^{tx}\,e^{-x^2/2}}{e^{t^2/2}}\,dx$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx-\frac{x^2}{2}-\frac{t^2}{2}}\,dx = \frac{e^{t^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2}\,dx = e^{t^2/2}$$

$$X \sim N(\mu, \sigma^2) \iff Z = \frac{X-\mu}{\sigma} \sim N(0,1) \qquad X = \sigma Z + \mu$$

$$M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z+\mu)}] = e^{t\mu} E[e^{t\sigma Z}]$$

$$= e^{t\mu} M_Z(t\sigma) = e^{t\mu} e^{(t\sigma)^2/2} = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$X \sim \exp(\lambda)$, $\lambda > 0$

$$M_X(t) = E[e^{tX}] = \int_0^\infty \lambda e^{x(t-\lambda)} dx = \left[\frac{\lambda e^{x(t-\lambda)}}{t-\lambda} \right]_{x=0}^{x=\infty} = -\frac{\lambda}{t-\lambda}$$

3. Enunciar propiedades de la f. distribución de un vector aleatorio.

Sea $X = (X_1, \dots, X_n) : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^n, \mathcal{B}^n, P_X)$

se define la f. distribución

$$F_X : \mathbb{R}^n \longrightarrow [0,1]$$

$$F_X(x) = P_X((-\infty, x]) = P[X \leq x]$$

esto es, $F_X(x_1, \dots, x_n) = P[X_1 \leq x_1, \dots, X_n \leq x_n]$.

Propiedades:

•) Monotona no decreciente $\forall i = 1, \dots, n$, $\forall x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$

$$x_i < x_i' \Rightarrow F_X(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \leq F_X(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$$

•) Continua a la derecha $\forall i = 1, \dots, n$ $\forall x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$

$$\lim_{x_i' \to x_i^+} F_X(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n) = F_X(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$$

•) $\forall i = 1 \dots n \quad \forall x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$

$$\lim_{x_i \to -\infty} F_X(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots x_n) = F_X(x_1, \dots, x_{i-1}, -\infty, x_{i+1}, \dots, x_n) = 0$$

$$\lim_{x_1 \dots x_n \to +\infty} F_X(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = F_X(+\infty, \dots, +\infty) = 1$$

•) $\forall (x_1, \dots, x_n) \in \mathbb{R}^n \quad \forall (\varepsilon_1 \dots \varepsilon_n) \in \mathbb{R}_+^n$

$$F_X(x_1+\varepsilon_1, \dots, x_n+\varepsilon_n) - \sum_{i=1}^{n} F_X(x_1+\varepsilon_1, \dots, x_{i-1}+\varepsilon_{i-1}, x_i, x_{i+1}+\varepsilon_{i+1}, \dots, x_n+\varepsilon_n)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} F_X(x_1+\varepsilon_1, \dots, x_{i-1}+\varepsilon_{i-1}, x_i, x_{i+1}+\varepsilon_{i+1}, \dots, x_{j-1}\,\varepsilon_{j-1}, x_j, x_{j+1}+\varepsilon_{j+1}, \dots x_n+\varepsilon_n)$$

$$+ \dots + (-1)^n F_X(x_1 \dots x_n) \geq 0$$

4. Dem caracterización de independencia por conjuntos Borel.

Sea $X = (X_1, \dots X_n)$ vector aleatorio. $X_1 \dots X_n$ independientes

$\iff P[X_1 \in B_1, \dots, X_n \in B_n] = P[X_1 \in B_1] \dots P[X_n \in B_n] \quad \forall B_1 \dots B_n \in \mathcal{B}$

Dem

$\Leftarrow$| si la relación cierta $\forall B_1 \dots B_n \in \mathcal{B}$, tomando $B_i = (-\infty, x_i]$ $\forall i = 1 \dots n$, se tiene la def de independencia.

$\Rightarrow$| Caso discreto $X = (X_1 \dots X_n)$ discreto.

$$P[X_1 \in B_1, \dots, X_n \in B_n] = \sum_{x_i \in B_i} P[X_1 = x_1, \dots, X_n = x_n] \overset{\text{indep}}{=}$$

$$= \sum_{x_i \in B} P[X_i = x_i] \, P[X_n = x_n] = \sum_{x_i \in B_i} P[X_i = x_i] \cdots \sum_{x_i \in B_n} P[X_n = x_n]$$

$$= P[X_1 \in B_1] \dots P[X_n \in B_n]$$

caso continuo: $X=(X_1,\dots,X_n)$ continuo.

$$P[X_1\in B_1,\dots,X_n\in B_n]=\int_{B_1}\cdots\int_{B_n} f_X(x_1\dots x_n)\,dx_n\dots dx_1$$

$$\overset{\text{indep}}{=}\int_{B_1}\cdots\int_{B_n} f_1(x_1)\cdots f_n(x_n)\,dx_n\dots dx_1=\int_{B_1}f_1(x_1)\,dx_1\cdots\int_{B_n}f_n(x_n)\,dx_n$$

$$=P[X_1\in B_1]\cdots P[X_n\in B_n]$$

5. Dem de la reproductividad de la Binomial.

$$\underset{i=1\dots n}{X_i}\sim B(n_i,p)\ \text{indep}\implies\sum_{i=1}^n X_i\sim B\left(\sum_{i=1}^n n_i,p\right)$$

Dem

$X\sim B(n,p)\qquad M_X(t)=E[e^{tX}]=(pe^t+(1-p))^n,\quad t\in\mathbb{R},\ p\in[0,1]$, $n\in\mathbb{N}$.

$$M_{\sum_{i=1}^n X_i}(t)=E\left[e^{t\sum_{i=1}^n X_i}\right]=E\left[\prod_{i=1}^n e^{tX_i}\right]=\prod_{i=1}^n E[e^{tX_i}]$$

$$=\prod_{i=1}^n (pe^t+(1-p))^{n_i}=(pe^t+(1-p))^{\sum_{i=1}^n n_i}\sim B\left(\sum_{i=1}^n n_i,p\right)$$

~~6. Enunciar Tma multiplicación esperanzas~~

~~$X_1,\dots,X_n$ independientes y $\exists\,E[X_i]$, $\forall i=1\dots n$ entonces:~~

~~a) $\exists\,E[X_1\dots X_n]$~~

~~b) $E[X_1\dots X_n]=E[X_1]\cdots E[X_n]$~~

7. Dem del Tma descomposición de la varianza

Sean X,Y var. aleatorias definidas en el mismo esp. prob. tal que $\exists\,E[X^2]$, entonces.

$\exists\,E[Var[X/Y]]$ y $Var[E[X/Y]]$ y además,

$$Var[X]=E[Var[X/Y]]+Var[E[X/Y]]$$

Dem

$$Var[X] = E[(X-E[X])^2]$$

$$= E[E[(X-E[X])^2/Y]]$$

$$= E[E[(X^2+E[X]^2-2XE[X])/Y]]$$

$$= E[E[X^2/Y]+E[X]^2-2E[Y]E[X/Y]]$$

$$= E[E[X^2/Y]-E[X/Y]^2+E[X/Y]^2+E[X]^2-2E[X]E[X/Y]]$$

$$= E[E[X^2/Y]-E[X/Y]^2]+E[(E[X/Y]-E[X])^2]$$

$$= E[Var[X/Y]] + Var[E[X/Y]]$$

8. Propiedades de la esperanza condicionada.

1) $E[C/Y]=C$

2) Linealidad. Si $\exists E[X]$, $a,b\in\mathbb{R}$

$\Rightarrow \exists E[aY+b/Y] = aE[X/Y]+b$

3) $X_1 \dots X_n$ va, $\exists E[X_i]$ $\forall i=1\dots n$

$\Rightarrow \forall a_1\dots a_n \in \mathbb{R}$, $\exists E[a_1X_1+\dots+a_nX_n/Y] = a_1E[X_1/Y]+\dots+a_nE[X_n/Y$

4) Si $X\geq 0$ tal que $\exists E[X] \Rightarrow E[X/Y]\geq 0$

y $E[X/Y]=0 \Leftrightarrow P[X=0]=1$

5) Si $\exists E[X], E[Y]$, $X_1 \doteq X_2 \Rightarrow E[X_1/Y]=E[X_2/Y]$

6) X,Y, indep., $\exists E[g(X)] \Rightarrow E[g(X)/Y] = E[g(X)]$

en particular, $E[X/Y] = E[X]$

7) Si $\exists E[g(X)] \Rightarrow \exists E[E[g(X)/Y]] = E[g(X)]$

en particular, $E[E[X/Y]] = E[X]$

6. Enunciat el T^{ma} multiplicación de esperanzas

1) Si $X_1 \dots X_n$ indep $\exists E[X_i]$ $\Longrightarrow \exists E[X_1 \dots X_n] = E[X_1] \cdots E[X_n]$

2) Si $X_1 \dots X_n$ indep.
$g: \mathbb{R} \to \mathbb{R}$ med.
$\exists E[g(x)]$ $\Longrightarrow \exists E[g_1(X_1), \dots g(X_n)] = E[g_1(X_1)] \cdots E[g_n(X_n$
$g(X_1) \dots g(X_n)$ indp

3) X, Y indep $\underset{\not\Leftarrow}{\Longrightarrow}$ $Cov(X, Y) = 0$

4) Si $X_1 \dots X_n$ indep. con $MNC2 < \infty$

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_i a_i^2 Var(X_i)$$