NATURAL OPERATION ON MANIFOLDS

TRAKATUS

1. Basic Definition

First, let define a space at which our philosophy something smooth going on.

Definition 1.1 (Manifolds, Charts, Atlas, Structure)**.** A Second countable Hausdorff space M is said to be an n**-dimensional manifold** if it is locally homeomorphic to $\mathbb{R}^n$. i.e. a **chart** (U, h) given, where U is an open set in M and $h : U \to h(U) \subseteq \mathbb{R}^n$ is homeomorphism that $h(U)$ is open set in $\mathbb{R}^n$. The family of charts $\{U_\alpha, h_\alpha\}_{\alpha\in A}$ is said to be **atlas**, when $\{U_\alpha\}_{\alpha\in A}$ cover M.

We determine smoothness of manifold by smoothness of chart changing: For $U_\alpha \cap U_\beta \neq \varnothing$, $\tau_{\alpha\beta} := h_\beta \circ h_\alpha^{-1}$ is C^k-class, then that atlas $\mathcal{A} \ni (U_\alpha, h_\alpha), (U_\beta, h_\beta)$ is said to be C^k-atlas.

$$\tau_{\alpha\beta} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta),$$

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ h_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\tau_{\alpha\beta}"'] & & h_\beta(U_\alpha \cap U_\beta) \end{tikzcd}$$

We say the maximal atlas, i.e. when if union of two C^k-atlas is also another atlas then we equivalent those atlases; then such class is said to be C^k**-structure**.

Definition 1.2 (Manifold with Boundary)**.** A Second countable Hausdorff space M is said to be an n**-dimensional manifold with boundary** if it is locally homeomorphic to $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n \geq 0\}$. This definition well works for other conepts, atlas, structure, etc. A **boundary** ∂M is mapped on $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\}$.

So naturally it is locally homeomorphic to $\mathbb{R}^{n-1}$. The complement of boundary is called interior, M°. Let (U_α, h_α) be a chart on M Then the chart for ∂M is $(U_\alpha \cap \partial M, h_\alpha|_{U_\alpha\cap\partial M})$ that $h_\alpha|_{U_\alpha\cap\partial M}$ maps $U_\alpha \cap \partial M$ to open subset in $\mathbb{R}^{n-1}$, where we include $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n_+$.

If $\partial M = \varnothing$, then we shall say that M is **closed**.

Example 1.3 (Unit sphere)**.** Let $M = \mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Consider $2n+2$ subsets $U_i^+ := \{x \in \mathbb{S}^n : x_i > 0\}$ and $U_i^- := \{x \in \mathbb{S}^n : x_i < 0\}$. And let $h_i^\pm : U_i^\pm \to \mathbb{R}^n$ be given by leaving out the i-th coordinate. This atlas defines the standard differential structure on $\mathbb{S}^n$. There is another way: Consider two poles $a_\pm = (0, \cdots, 0, \pm 1) \in \mathbb{R}^{n+1}$.

Let $U^\pm = \mathbb{S}^n \setminus \{a_\pm\}$ and consider the projection $h_\pm$ from $a_\pm$;

$$h_\pm : (x_1, \cdots, x_{n+1}) = \frac{1}{1 \mp x_{n+1}}(x_1, \cdots, x_n, 0).$$

Example 1.4 (Real and Complex Projective space)**.** The real projective $n-$space $\mathbb{RP}^n$ is obtained by identifying antipodal points in $\mathbb{S}^n$; Let $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ be such identification. Note that $\pi|_{U_i^+}$ is homeomorphism, and its image covers $\mathbb{RP}^n$.

$$\begin{tikzcd} U_i^+ \arrow[r, hook] \arrow[d, "h_i^+"'] & \mathbb{S}^n \arrow[r, "\pi"] & \mathbb{RP}^n \\ \mathbb{R}^n \arrow[urr, "\pi\circ(h_i^+)^{-1}"'] & & \end{tikzcd}$$

Thus $\{\pi(U_i^+, h_i^+ \circ \pi^{-1})\}_{i=1}^{n+1}$ is an atlas on $\mathbb{RP}^n$.

Another way, **the homogeneous coordinates**: Consider an equivalent relation $\sim$ on $\mathbb{R}^{n+1}$:

$$x \sim y \iff \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ s.t. } x = \lambda y$$

then

$$\mathbb{RP}^n \cong (\mathbb{R}^{n+1} \setminus \{0\})/\sim .$$

Similarly, we suppose complex $\mathbb{C}$ on here,

$$\mathbb{CP}^n \cong (\mathbb{C}^{n+1} \setminus \{0\})/\sim, \quad x \sim y \iff \exists \lambda \in \mathbb{C} \setminus \{0\} \text{ s.t. } x = \lambda y.$$

Example 1.5 (Grassmannian Manifold)**.** Projective manifold starts the idea that points of it are line contained origin of Euclidean. More generally, when we take points as n-dimensional subspace of Euclidean, this provide **Grassmannian Manifold**.

$$\mathrm{Gr}_k(\mathbb{R}^n) := \{V \subseteq \mathbb{R}^n : V \text{ is a linear subspace, } \dim V = k\}$$

It is clear that

$$\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n.$$

Definition 1.6 (Product Manifold)**.** If $\{U_\alpha, h_\alpha\}_\alpha$ is an atlas on M, $\{V_\beta, g_\beta\}_\beta$ is an atlas on N, and at least one of M, N is a closed, then $\{U_\alpha \times V_\beta, h_\alpha \times g_\beta\}_{(\alpha,\beta)}$ is an atlas on $M \times N$ and defines the product structure on it. But if both M and N are not closed, this is not an atlas.

Example 1.7 (Stiefel Manifold)**.** Consider a set of $n\times k$ matrices with real entries $\mathrm{Mat}_{n\times k}(\mathbb{R})$ that can identify $\mathbb{R}^{nk}$, and a subset $V_{n,k}$ of $\mathrm{Mat}_{n\times k}(\mathbb{R})$ that matrices of full rank can identify open subset of $\mathbb{R}^{nk}$. An $n \times k$ matrices of rank k can be viewed as k-frames, that is called **Stiefel manifold** of k-frames in $\mathbb{R}^n$. Of course $V_{n,n}$ is $\mathrm{GL}(n)$. By the foregoing, $V_{n,k}$ is nk-dimensional differentiable manifold.

Next, define a smooth maps between manifold.

Definition 1.8 (Smooth map, Diffeomorphism)**.** A map $f : M \to N$ between manifolds is said to be C^k if for each $x \in M$ and each chart (V, h_V) of $f(x)$, there is a chart (U, h_U) of x that $f(U) \subseteq V$ and related map $h_V \circ f \circ h_U^{-1} : h_U(U) \to h_V(V)$ is C^k.

A C^k map $f : M \to N$ is said to be C^k-**diffeomorphism** if it has C^k inverse $f^{-1} : N \to M$.

A map $f : M \to N$ between manifolds of the same dimension is called a **local diffeomorphism** if each $x \in M$ has an open neighborhood U such that $f|_U : U \to f(U)$ is diffeomorphism. Note that a local diffeomorphism need not to be surjective and injective.

Note that the relation of diffeomorphism is an equivalence relation between smooth structures.

Example 1.9 (Lie Group)**.** Consider $\mathrm{GL}(n)$. Just matrix multiplication provide smooth map $\mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathrm{GL}(n)$ since each entries are polynomial of entries of domain. We say a group is a smooth manifold and the group operation a smooth map, called **Lie group**. Note that set of orthogonal matrices $\mathrm{O}(n)$ is also Lie group.

Note that if L is a Lie group and $a \in L$, then the map $R_a : L \to L$ that $x \mapsto ax$ is diffeomorphism.

Definition 1.10 (Orientation)**.** Let $\mathcal{A} = \{u_\alpha, h_\alpha\}_\alpha$ be an atlas in M. Consider jacobian determinant of transition maps $\tau_{\alpha\beta} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$. When it is positive, then such $\tau_{\alpha\beta}$ is **orientation compatible**. An **oriented atlas** on M is an atlas for which all transition map are orientation preserving. M is **orientable** if it admits an oriented atlas, and **orientation** of M is an oriented structure.

If $\mathcal{A}_M = \{U_\alpha, h_\alpha\}_\alpha$ and $\mathcal{A}_N = \{V_\beta, g_\beta\}_\beta$ are oriented atlases on M and N respectively, then a diffeomorphism $f : M \to N$ is said to be **orientation preserving** if the Jacobians of all maps $g_\beta \circ f \circ h_\alpha^{-1}$ have positive determinant.

If M is not closed, and $\mathcal{A} = \{U_\alpha, h_\alpha\}_\alpha$ is an oriented structure on M, then it provide also oriented structure on ∂M. For if, take $p \in \partial M \cap U_{\alpha_1} \cap U_{\alpha_2}$ and let $h_{\alpha_2} \circ h_{\alpha_1}^{-1} := (f_1, \cdots, f_n)$, then

$$df_{h_{\alpha_1}^{-1}(p)} = J = \begin{bmatrix} J|_{\partial M} & * \\ 0 & \partial f_n / \partial x_n \end{bmatrix}$$

since, ∂M has a local coordinates with $x_n = 0$, and definitely $\det J, \det J|_{\partial M} > 0$, so $\frac{\partial f_n}{\partial x_n} > 0$.

2. Partition of Unity

Definition 2.1 (Locally Finite)**.** A collection of subsets of M is **locally finite** if each point of M is contained in some open neighborhood intesecting at most finite number of them.

Definition 2.2 (Relatively Compactness, Locally Compactness, Paracompactness)**.**

(a) A subset U of a space X is **relatively compact** if closure of U is compact in X.
(b) A space X is said to be **locally compact** if every open neighborhood of each point admits a relatively compact neighborhood. i.e. For every $x \in X$, and every open neighborhood U_x of x, there exists another smaller open neighborhood V_x such that $\overline{V_x}$ is compact and $\overline{V_x} \subseteq U_x$.

(c) A space X is said to be **paracompact** if every open covering admits a locally finite refinement. i.e. For every open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of X there exists open subcovering $\mathcal{V} = \{V_\beta\}_\beta$ that and every $x \in X$, there exists open neighborhood U of x such that $|\{\beta : V_\beta \cap U \neq \varnothing\}| < \infty$.

Locally homeomorphic to $\mathbb{R}^n$ directly implies following fact.

Lemma 2.3. Every manifold M is locally compact.

Proof. For each $x \in M$, there exists open neighborhood U of x that homeomorphic to $\mathbb{R}^n$ by some homeomorphism h. Consider a closed ball $\overline{D}$ with centering at $h(x)$ and finite radius. It is obviously compact, and h preserve compactness of $\overline{D}$. Thus $h^{-1}(\overline{D})$ be a compact neighborhood of x. □

And Following lemma is the key to show every manifold is paracompact.

Lemma 2.4. If X is locally compact and second countable space, then X can be expressed as the union of at most countably many compact spaces.

Proof. We construct countable covering of X by compact subspaces. By second countability, there exists countable base $\beta = \{B_i\}_{i=1}^\infty$. By local compactness, each $x \in X$ has a open neighborhood V_x on which $\overline{V_x}$ is compact. Then for each $x \in X$, there exists $B_{i_x} \in \beta$ such that $x \in B_{i_x} \subseteq V_x$. By assumption $\overline{V_x}$ is compact, $\overline{B_{i_x}}$ is compact. (Since closed set of compact space is also compact.) Note that $\{\overline{B_{i_x}}\}_{x\in X}$ is covering of X, but it is at most countable. So there exists at most countable set $J \subseteq X$ such that

$$X = \bigcup_{x\in J} \overline{B_{i_x}}.$$

□

Conversely, If a space X can be expressed by at most countable many compact subspaces, then it is locally compact. (Since each $x \in X$ is contained in some such compact subspace C. Then every open neighborhood U of x admits $V := U \cap C$, relativity compact.)

Lemma 2.5. X can be expressed as the union of at most countably many compact subspaces, then X can be also represented as

$$X = \bigcup_{i=1}^\infty U_i,$$

where each U_i is relatively compact, and $\overline{U_i} \subseteq U_{i+1}$ for all $i = 1, 2, 3, \cdots$.

Proof. We have $X = \bigcup_{i=1}^\infty C_i$, where C_i is compact subspace. By local compactness of X □

Definition 2.6 (Adequateness)**.** An atlas $\{U_\alpha, h_\alpha\}_\alpha$ on M is said to be **adequate** if it is locally finite, $h_\alpha(U_\alpha) = \mathbb{R}^n$ or $\mathbb{R}^n_+$ for each chart, and $\bigcup_\alpha h_\alpha^{-1}(\mathring{\mathbb{D}}^n) = M$, where $\mathring{\mathbb{D}}^n$ is an interior of unit ball; i.e. $\mathring{\mathbb{D}}^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$.

Theorem 2.7. Let $\mathcal{V} = \{V_\beta\}_\beta$ be an open covering of M. Then there exists an adequate atlas $\{U_\alpha, h_\alpha\}_\alpha$ such that $\mathcal{U} = \{U_\alpha\}_\alpha$ is a refinement of $\mathcal{V}$.

Proof. Since M is locally compact, Hausdorff and second countable, by 2.4 there exists a sequence $\{K_i\}_{i=1}^{\infty}$ of open subspaces of M, with compact closures and such that $\overline{K_i} \subseteq K_{i+1}$ and $\bigcup_i K_i = M$. □

Corollary 2.8. Smoooth manifold is paracompact.

To define partition of unity, first we construct smooth bump on $\mathbb{R}^n$.

Let smooth $f : \mathbb{R} \to \mathbb{R}$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

And define a smooth $g(x) := f(x-a)f(b-x)$. It is only positive on (a, b) and zero elsewhere. (Where $a < b$.)

Then

$$h(x) = \frac{\displaystyle\int_{-\infty}^{x} g(t)dt}{\displaystyle\int_{-\infty}^{+\infty} g(t)dt}$$

is a smooth function such that $h = 0$ for $x < a$ and $h = 1$ for $x > b$. And smoothly $0 < h < 1$ on (a, b). Now $H := 1 - h(\|x\|) : \mathbb{R}^n \to \mathbb{R}$ is a function that 1 in $\mathbb{D}^n(a)$ and 0 on outside $\mathbb{D}^n(b)$, when $a, b > 0$.

Definition 2.9 (Partition of Unity)**.** Let $\{U_\alpha, h_\alpha\}_\alpha$ be an adequate atlas on M. Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump that $\lambda|_{\mathbb{D}^n} = 1$ and zero outside of $\mathbb{D}^n(2)$. Let $\lambda_\alpha : M \to \mathbb{R}$ defined as

$$\lambda_\alpha = \begin{cases} \lambda \circ h_\alpha & \text{inside } U_\alpha \\ 0 & \text{outside } U_\alpha \end{cases}$$

Appendices

A. Calculus on $\mathbb{R}^n$

The main reference of this appendix section is spivak's book

A.1. Differentiation.

Definition A.1 (Differentiablity)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $a \in \mathbb{R}^n$ if there exists a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

The linear map λ is denoted df_a and called the **differential** of f at a.

Theorem A.2 (Uniqueness of Differential)**.** If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there exists unique linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ that $\lambda = df_a$.

Proof. Suppose $\lambda_1, \lambda_2 : \mathbb{R}^n \to \mathbb{R}^m$ are linear and

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda_1(h)\|}{\|h\|} = 0, \quad \lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda_2(h)\|}{\|h\|} = 0.$$

If $d(h) := f(a+h) - f(a)$, then

$$\begin{aligned} \lim_{h \to 0} \frac{\|\lambda_1(h) - \lambda_2(h)\|}{\|h\|} &= \lim_{h \to 0} \frac{\|\lambda_1(h) - d(h) + d(h) - \lambda_2(h)\|}{\|h\|} \\ &\leq \lim_{h \to 0} \frac{\|d(h) - \lambda_1(h)\|}{\|h\|} + \frac{\|d(h) - \lambda_2(h)\|}{\|h\|} \\ &= 0 \end{aligned}$$

Now take an approach $h \to 0$ by $tx \to 0$ as $t \to 0$. Hence for non-zero $x \in \mathbb{R}^n$, we have

$$0 = \lim_{t \to 0} \frac{\|\lambda_1(tx) - \lambda_2(tx)\|}{\|tx\|} = \frac{\|\lambda_1(x) - \lambda_2(x)\|}{\|x\|}.$$

Therefore $\lambda_1(x) = \lambda_2(x)$. ☐

adfsasdf $\varphi(x) =$

A.2. Integration.

asdfadsf $I = [a,b] \times [c,d]$

B. Vector Bundle Theory

The main reference of this appendix section is Milnor and stasheff's book.