To the Diffeology

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April 16, 2021

We both using index notation and musical isomorphisms.

1 Tensor Analysis

Definition 1.1 (Tensor Product)**.** Let V and W are vector space over $\mathbb{F}$. The **Tensor product** of V and W, $V \otimes_{\mathbb{F}} W$ with isomorphic to $V \times W$ by bilinear map

$$V \times W \ni (v, w) \mapsto v \otimes_{\mathbb{F}} w \in V \otimes_{\mathbb{F}} W$$

in universal sence. We usually skip writting $\mathbb{F}$ on $\otimes$ when we fix a field or ring.

Proposition 1.2 (Transformation of Basis)**.** Let $v \in V$, a finite dimensional vector over $\mathbb{F}$. Let $\{e_1, \cdots, e_{\dim V}\}$ and $\{\overline{e}_1, \cdots, \overline{e}_{\dim V}\}$ are basis for V. Then there uniquely exists $\{a^1, \cdots, a^{\dim V}\}$ and $\{b^1, \cdots, b^{\dim V}\}$ as subset of $\mathbb{F}$ the coefficients that

$$v = \sum_{i=1}^{\dim V} a^i e_i = \sum_{i=1}^{\dim V} b^i \overline{e_i}.$$

Covariant and Contravariant: Consider a map $\Lambda \in \mathrm{Hom}(V; V)$ that $\Lambda^{ij} : e_i \mapsto \overline{e}_j$. Then coefficients transform associatively $\overline{\Lambda}_{ij} : a_i \mapsto b_j$. Here $\overline{\Lambda}_{ij}\Lambda^{ij} = \mathrm{id}$. (Caution: It can multiplying as matrix but it cannot composite as linear map since Λ is on V and $\overline{\Lambda}$ is on V^*.)

Note (Coefficient)**.** Consider $\varphi_v : V \to \mathbb{F}$ that $x \mapsto \langle x, v\rangle$, for inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. That $\varphi_v \in V^*$, that is $\varphi_v = v^\flat$. So coefficients correspond to covector, and conversely coefficient of covector can be corresponded to vector.

Remark 1.3 (Jacobian)**.** If coordinate transform Λ is smooth and invertible; $\Lambda \in \mathrm{Diff}.\mathrm{Aut}(V)$

Definition 1.4 (Multilinear Map)**.** Let V be a **vector space over** field $\mathbb{F}$. Let $V^* := \mathrm{Hom}(V; \mathbb{F})$, be a **dual space** of V. Consider $T : \underbrace{V \times \cdots \times V}_{p-\text{copies}} \times \underbrace{V^* \times \cdots \times V^*}_{q-\text{copies}} \to \mathbb{F}$ is **multi-**

linear if it is linear for each slot. i.e. $T \in \mathrm{Hom}(\underbrace{V \times \cdots \times V}_{p} \times \underbrace{V^* \times \cdots \times V^*}_{q}; \mathbb{F})$, and denote $\mathcal{T}^p_q(V;\mathbb{F})$, called $\binom{p}{q}$-**Tensor space** over $\mathbb{F}$, and T is called $\binom{p}{q}$-tensor over $\mathbb{F}$.

Proposition 1.5 (Basis of Tensor Space)**.** Let $\mathcal{B} = \{e_i\}_{i=1}^n$ be a (hamel) **basis** for n-dimensional vector space V, and $\mathcal{B}^* = \{\varepsilon^i\}_{i=1}^n$ is dual basis of $\mathcal{B}$. For $T \in \mathcal{T}^p_q(V)$, consider

$$T^{a_1 \cdots a_p}_{b_1 \cdots b_q} = T(\varepsilon^{a_1}, \cdots, \varepsilon^{a_p}, e_{b_1}, \cdots, e_{b_q}) \in \mathbb{F}$$

the **coefficients** of tensor T.

Definition 1.6 (Tensor Fields)**.** Let M be a smooth manifold.

2 Differential Topology

Immersion Submersion.

3 Categoric Languages

4 Diffeology

References

[1] Wikipedia, `https://www.wikipedia.org/`

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[5] Patrick Iglesias-Zemmour. *Diffeology*. Vol. 185. American Mathematical Soc., 2013