# General Relativity

Trakatus

April 20, 2021

## 1 Lorentz Transformation

We use natural unit: $c = 1$.

**Definition 1.1** (Spacetime)**.** The **spacetime** is a 4 dimentional differentiable manifold $X$ with Levi-Chivita connection. We say every element in spacetime an **event**. And a curve $\gamma$ in spacetime is called **worldline**.

**Definition 1.2** (Minkowski Space)**.** In special relativity, we consider a **Minkowski space** that spacetime with metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$ globally: $(X, \eta)$

**Definition 1.3** (Inertial Frame)**.** An **intertial frame** $\mathcal{O}$ is a basis for Minkowski space $X$ as vector space sence.

In special relativity we only consider stop one or moving one with constant velocity. In physics we need to make meaningfull basis change between inertial frames.

Consider two inertial frames $\mathcal{O}(t, x, y, z)$ and $\bar{\mathcal{O}}(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ with same origin. If $\bar{\mathcal{O}}$ is moving away on origin with speed $v$ in $x$-axis. Then those frame must satisfies following axioms:

**Axiom of frame transformation**

**A.1** The speed of light $c = 1$ is universal.

**A.2** A linear transformation between two inetial frames exists

**A.3** $(t, vt, 0, 0)$ corrensponded to $(\bar{t}, 0, 0, 0)$.

These axioms implies following two reults, and those are compatiable.

**Theorem 1.4** (Lorentz)**.** Algebraically or Drawing spacetime diagram, we get (same) result, **Lorentz Transform** on $\mathcal{O}$ to $\bar{\mathcal{O}}$. For $\gamma = (1 - v^2)^{-1/2}$,

$$\begin{bmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

On the other hand, the transform of vector of basis vector $(e_t, e_x, e_y, e_z)$ in $X^*$ is exactly inverse of event coordinate. i.e. Let $\Lambda^\nu_\mu(\mathbf{v})$ be a coordinate transform with inertial frame between relative velocity $\mathbf{v}$: For vector field $x \in X$, if $(e_0, e_1, e_2, e_3)$ be transformed by mapping $\Lambda^\mu_\nu(\mathbf{v})$ then

$$x = x^\mu e_\mu = x^\nu e_\mu = x^\nu(\Lambda^\nu_\mu(\mathbf{v})e_\mu) = x^\nu(\Lambda^\nu_\mu(\mathbf{v})\Lambda^\mu_\nu(\mathbf{v}))e_\nu$$

So $\Lambda^\nu_\mu(\mathbf{v})\Lambda^\mu_\nu(\mathbf{v}) = \mathrm{id}$, So $\Lambda^\mu_\nu(\mathbf{v}) = [\Lambda^\nu_\mu(\mathbf{v})]^{-1} = \Lambda^\nu_\mu(-\mathbf{v})$.

**Theorem 1.5** (Spacetime Distance)**.** For each frame $\mathcal{O}(t, x, y, z)$, $\Delta s^2$ is invariant, where

$$\begin{aligned}\Delta s^2 &= -(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (t_2 - t_1)^2 \\ &= -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2.\end{aligned}$$

i.e. In inertial frames, metric $g$ on spacetime $X$ is $\eta$: For vector $v = (v^0, v^1, v^2, v^3)$ in $X$

$$g\langle v, v\rangle = \eta\langle v, v\rangle = v^\intercal \cdot \eta \cdot v, \quad \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case, the distance between event $E$ and origin is **time-like** if $\Delta s^2 < 0$, **space-like** if $\Delta s^2 > 0$, or **null-like** if $\Delta s^2 = 0$.[3]

**Corollary 1.6** (Speed Addition Law)**.** Consider three inertial frames $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, with relative velocity $v_{12}$ and $v_{23}$. Then

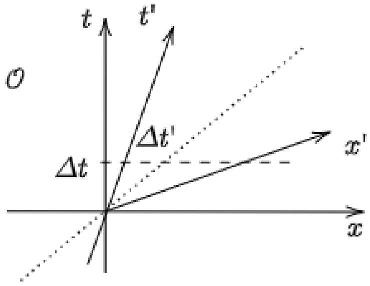
$$v_{13} = \frac{v_{12} + v_{23}}{1 + v_{12}v_{23}}$$

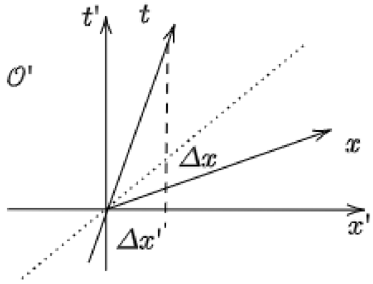
**Corollary 1.7.** Lorentz transformation preserve form of Maxwell equations.

We embedding such transform in 2-dimension, 1 dimensional time and 1 dimensional space. Using hyperbolic fuctions we can easily show symmetricity of Lorentz transform and spacetime distance.

**Definition 1.8** (Rapidity)**.** The **rapidity** $\zeta$ of speed $v$ is given by $\zeta = \tanh^{-1}(v)$. Then $\gamma = (1 - v^2)^{-1/2} = \cosh\xi$, and $v\gamma = \sinh\zeta$. Then a Lorentz transform $\Lambda$ is given by compactly:

$$\Lambda^\nu_\mu(v) = \begin{bmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{bmatrix}, \quad \Lambda^\mu_\nu(v) = \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix}.$$




**Theorem 1.9.** Consider two spacetime diagram between $\mathcal{O}$ and $\mathcal{O}'$ with relative speed $v$:

$$\frac{\Delta t'}{\Delta t} = \cosh\zeta = \gamma, \quad \frac{\Delta x}{\Delta x'} = \cosh\zeta = \gamma$$

It means $\Delta t' = \gamma\Delta t > \Delta t$, the **time dilation** and $\Delta x' = \gamma^{-1}\Delta x < \Delta x$, the **length contraction**.

It can be proved by phytagorian theorem for given metric.

## 2 4-vector Calculation

For Minkowski space $(X, \eta)$, the standard basis $\mathcal{B}$ and the dual basis $\mathcal{B}^*$ for $X$ is:

$$\mathcal{B} = \{\partial_0, \partial_1, \partial_2, \partial_3\}, \quad \mathcal{B}^* = \{dx^0, dx^1, dx^2, dx^3\}$$

The metric $\eta : X \times X \to \mathbb{R}$ can be represent as 2-form that is:[2]

$$\begin{aligned}\eta_{\mu\nu} =& \mathbf{e}_\mu \otimes \mathbf{e}_\nu = \langle \mathbf{e}_\mu, \mathbf{e}_\nu \rangle \\ =& \partial_\mu \otimes \partial_\nu = \langle \partial_\mu, \partial_\nu \rangle\end{aligned}$$

In vector analytic sence, Lorentz transform is isometry on $(X, \eta)$: We got $\eta\langle A, B\rangle = (A^\mu)^\intercal \eta^\mu_\nu B^\nu = \eta^\mu_\nu A_\mu B^\nu$.
For Lorentz transform by $\mu \to \mu'$ and $\nu \to \nu'$,

$$\eta\langle \mathbf{A}, \mathbf{B}\rangle = (A^\mu)^\intercal \eta_{\mu\nu} B^\nu = (\Lambda^\mu_{\mu'} A^{\mu'})^\intercal \eta^\mu_\nu (\Lambda^\nu_{\nu'} B^{\nu'}) = (\Lambda^{\mu'}_\nu \eta^\mu_\nu \Lambda^\nu_{\nu'}) A_{\mu'} B^{\nu'} = \eta^{\mu'}_{\nu'} A_{\mu'} B^{\nu'}$$

So in matrix representation, $\Lambda^\intercal \eta \Lambda = \eta$. So it means another definition of Lorentz transform;

**Definition 2.1** (Lorentz Group)**.** Consider a subgroup of $GL(\mathbb{R}^4)$, the general linear group of $\mathbb{R}^4$:

$$O(3,1) := \{\Lambda \in \mathrm{GL}(\mathbb{R}^4) : \Lambda^\intercal \eta \Lambda = \eta\}, \quad \text{Lorentz transform is element in } O(1,3).$$

In this sence $O(1,3)$ is called **Lorentz group**. c.f. $O(0,n) = O(n)$, the orthogonal group.

**Remark 2.2.** Lorentz group $O(1,3)$ is also 3 dimensional manifold. i.e. It's Lie group. c.f. $O(n)$ is $n(n-1)/2$ dimensional Lie group.

We assume the background space as 4-dimensional manifold. So we redefine physical quatities in 4-dimensional.

**Definition 2.3** (Proper Time)**.** Consider a time measured in comoving, or relatively rest frame. It calls **proper time**, denoted $\tau$. Using spacetime distance,

$$\Delta s^2 = -\Delta\tau^2 \implies \Delta\tau = \sqrt{-\Delta s^2}$$

Since $\Delta s^2$ is invariance on inertial frames, so proper time is also invariance. As infinisimal change of them defined by $\mathcal{B}^*$:

$$ds := \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}, \quad d\tau := \sqrt{-ds^2} = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

**Definition 2.4** (4-velocity, 4-acceleration)**.** Consider a derivative $\mathbf{u}$ for worldline on $\mathbf{x} =$

$(x^0, x^1, x^2, x^3)$ by proper time $\tau$. That is **4-velocity** $u^\alpha = dx^\alpha/d\tau$. For $\alpha = (0, 1, 2, 3)$. Similarly a second derivative $\mathbf{a}$ of $\mathbf{x}$ is **4-acceleration**, $a^\alpha = du^\alpha/d\tau = d^2x^\alpha/d\tau^2$.

**Proposition 2.5.** (a) Between intertial frames with relative velocity $(v^1, v^2, v^3)$,

$$\mathbf{u} = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) = \left(\gamma, \gamma\frac{dx^1}{dx^0}, \gamma\frac{dx^2}{dx^0}, \gamma\frac{dx^3}{dx^0}\right) = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3)$$

here $\gamma = (1 - v^2)^{-1/2}$, when $v = \|(v^1, v^2, v^3)\|$. So $\gamma = [1 - (v^1)^2 - (v^2)^2 - (v^3)^2]^{-1/2}$.

(b) $\eta\langle \mathbf{u}, \mathbf{u}\rangle = -1$. So $\mathbf{u}$ is time-like. And $\eta\langle \mathbf{u}, \mathbf{a}\rangle = 0$. So $\mathbf{a} \perp \mathbf{u}$. Thus $\mathbf{a}$ is space-like.

(c) The worldline with constant acceleration $\alpha^2 = \eta\langle \mathbf{a}, \mathbf{a}\rangle$ $(\alpha > 0)$ with $x^1$ direction is hyperbola:

$$\mathbf{x}(\tau) = \frac{1}{\alpha}\left(\sinh(\alpha\tau), \cosh(\alpha\tau) - 1, 0, 0\right)$$

when worldline start at origin.

**Definition 2.6** (Energy Momentum Vector)**.** A **4-momentum** is $\mathbf{p} := m\mathbf{u}$, where $m$ is (rest) mass. Recall (a) of 2.5: Between inertial frames, $\mathbf{p} = (m\gamma, m\gamma v^1, m\gamma v^2, m\gamma v^3)$. Then what is the physical meaning of $p^0$? It is energy. We can approximate $p^0$ as Taylor expansion:

$$\begin{aligned} p^0 = m\gamma = \frac{m}{\sqrt{1 - v^2}} &= m + \frac{1}{2}mv^2 + \frac{3}{8}mv^4 + \cdots \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots \qquad \text{(recover the speed of light.)} \\ &\approx \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic}} \end{aligned}$$

So $p^0$ is sum of kinetic energy and something. Since Minkowski space does not have potential, that is new kind of energy, the **rest energy**. Therefore $p^0$ is total energy. So $\mathbf{p}$ combine two quatities, energy (time component) and momentum (spatial component). So $\mathbf{p}$ is called **energy-momentum 4-vector**.

**Proposition 2.7.** Consider a length of energy-momentum $\mathbf{p} = (E, p^1, p^2, p^3)$:

$$\begin{aligned} &\eta\langle \mathbf{p}, \mathbf{p}\rangle = m^2\eta\langle \mathbf{u}, \mathbf{u}\rangle = -m^2 \\ \Longrightarrow\ & -(p^o)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -m^2 \\ \Longrightarrow\ & -E^2 + p^2 = -m^2 \qquad \text{(define } p := \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}.\text{)} \\ \Longrightarrow\ & E^2 = m^2 + p^2 \\ \Longrightarrow\ & E^2 = m^2c^4 + p^2c^2. \qquad \text{(recover the speed of light.)} \end{aligned}$$

**Corollary 2.8.** When $v = 0$, $p^0$ implies that: $E = mc^2$.

**Theorem 2.9** (SR Doppler Effect)**.** Consider a Lorentz transform of energy-momentum: $p^\beta = \Lambda^\beta_\alpha p^\alpha$, where $\beta$ means moving frame and $\alpha$ means rest frame. For frames moving along $x$-axis, we have $E' = \gamma E - v\gamma p^1$. In case of light, their rest mass is zero. So $\eta\langle \mathbf{p}, \mathbf{p}\rangle = 0$. So $E = p^1 =: p$. So $E' = \gamma(1-v)E$. i.e.

$$E' = \sqrt{\frac{1-v}{1+v}}\, E$$

Hence, in quantum mechanics, photon's energy $E$ is $E = hf$, where $h$ is plank constant, and $f$ is frequency of that photon. Therefore

$$f' = \sqrt{\frac{1-v}{1+v}}\, f$$

When $v > 0$, i.e. frame goes far, $\sqrt{(1-v)/(1+v)} < 1$. So frequency goes lower, the red-shift. Conversely $v < 0$, frame goes closer, it turns to blue – the blue shift.

**Definition 2.10** (Lagrangian, Action)**.** An action functional for Minkowski space is given by

$$S = -\int_{\tau_1}^{\tau_2} m d\tau = -m\int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}$$

where $\lambda$ is another parameter for worldline, compatiable with $\tau$. When $\lambda = \tau$ exactly, then the lagrangian is given by

$$L = -m\sqrt{-\eta_{\mu\nu}u^\mu u^\nu}$$

Apply Euler-Lagrange equation, the equation of motion in Minkowski space is $ma^\mu = 0$.

# 3 Differetial Tools

Tensor is multilinear map from vector space and its dual space. And generally metric tensor is one of them, which convert vector to covector and vice versa. i.e. It raise and fall indices of coefficient. We fix base on manifold $M$, it means converter between vector field and covector field. So in mathematics, using **musical isomorphism** $\sharp : \Gamma(T^*M) \to \Gamma(TM)$ and $\flat : \Gamma(TM) \to \Gamma(T^*M)$ instead $g^{\mu\nu}$ and $g_{\mu\nu}$. $\Gamma(TM)$, an algebra of vector fields on $M$, also denote $\mathfrak{X}(M)$, and $\Gamma(T^*M)$, an algebra of 1-forms on $M$, also denote $\Omega^1(M)$, are building blocks for tensor fields on $M$ by tensor product $\otimes$.

**Remark 3.1** (Covariant, Contravariant)**.** For spacetime $(X, g)$, A vector field $\mathbb{A} \in \mathfrak{X}(X)$ and 1-form $\mathbb{B} \in \Omega^1(X)$, for fixed local basis $\{\mathbf{e}_\alpha\}$ and $\{\boldsymbol{\phi}^\beta\}$:

$$\mathbb{A} = A^\alpha\ \mathbf{e}_\alpha, \quad \mathbb{B} = B_\beta\ \boldsymbol{\varphi}^\beta$$

Every basis change has a physical meaning, the Lorentz transform, coefficients of $\mathbb{A}$ and $\mathbb{B}$ change straightly or reversly: $\Lambda : (\alpha, \beta) \to (\mu, \nu)$ be a Lorentz transform for local coordinates, then

$$\begin{cases} A^\mu = \Lambda^\mu_\alpha\ A^\alpha \\ \mathbf{e}_\mu = \Lambda^\alpha_\mu\ \mathbf{e}_\alpha \end{cases}, \quad \begin{cases} B_\nu = \Lambda^\beta_\nu\ B_\beta \\ \boldsymbol{\varphi}^\nu = \Lambda^\nu_\beta\ \boldsymbol{\varphi}^\beta \end{cases}$$

$\Lambda$ converts lower index to upper index, thus $A$, $\boldsymbol{\varphi}$ change straightly, and $B$, $\mathbf{e}$ change revesely. Last two cases called **covariant** and first two cases called **contravariant**. i.e. Upper indices mean contravariant, and lower indices mean covariant.

**Remark 3.2.** Note that Lorentz transform $\Lambda$ is $\binom{1}{1}$-tensor, preserve rank of tensor.

**Definition 3.3** (Gradient of Scalar Fields)**.** Let $\phi : X \to \mathbb{R}$ is an scalar field on spacetime $X$. Note that proper time $\tau$ is a frame independent quatity on $X$, so we can consider derivative by $\tau$. By chain rule,

$$\frac{d\phi}{d\tau} = \frac{\partial \phi}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} = \frac{\partial \phi}{\partial x^\alpha}\ u^\alpha = \partial_\alpha \phi \cdot u^\alpha$$

Here, we can assume $\partial_\alpha \phi$ as $\binom{0}{1}$-tensor, denote $\nabla_\alpha \phi$. So $\nabla \phi = (\nabla_0 \phi, \nabla_1 \phi, \nabla_2 \phi, \nabla_3 \phi) = (\partial_0 \phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi)$ called **gradient vector** of $\phi$.

In general sence, $\nabla$ is combination of exterior derivative d, musical isomorphisms $\sharp, \flat$ and Hodge star $\star$.

**Definition 3.4** (Frame Transform)**.** Consider two local frames $x$ and $y$ for given point. Then they following formula determinates transform between of them.

$$dy^\beta = \frac{\partial y^\beta}{\partial x^\alpha} dx^\alpha, \quad \partial_{y^\beta} = \frac{\partial x^\alpha}{\partial y^\beta} \partial_{x^\alpha}$$

Here The vector of partial derivative is actually coordinate transform and the metric $g$ bridge to partial derivative and total derivative. Thus for scalar field $\phi$: $(d\phi)^\alpha = g^{\alpha\beta}\partial_\beta\phi$.

**Definition 3.5** (Covariance Derivative)**.** Let $T \in \mathcal{T}_q^p(X)$, the $\binom{p}{q}$-tensor on spacetime $(X, g)$. The **Covariance derivative** $\mathbf{\nabla}T$ of $T$ along direction $\mu$ is given by

$$\mathbf{\nabla}_\mu T^{a_1\cdots a_p}_{b_1\cdots b_q} = \partial_\mu T^{a_1\cdots a_p}_{b_1\cdots b_q} + \sum_{1\leq i\leq p} T^{a'_1\cdots a'_p}_{b_1\cdots b_q}\Gamma^{a_i}_{\alpha\mu} - \sum_{1\leq j\leq q} T^{a_1\cdots a_p}_{b'_1\cdots b'_q}\Gamma^{\beta}_{b_j\mu}$$

where $\alpha$ and $\beta$ is dummy indices accociate to modified indecies; $a'_k$ is $\alpha$ for $k = i$, otherwise $a'_k = a_k$ and $b'_l$ is $\beta$ for $l = j$, otherwise $b'_l = b_l$. Here we call $\Gamma^i_{jk}$ as **Christoffel symbol**.

In generally, this is related to **connection** of manifold, how the connect to each point with different shapes. (Manifold is locally Euclidean but not globally) But we take **Levi-Chivita connection**, unique connection on manifold intuitively compatiable to tangent bundle and tensor fields belongs to it, i.e. connection $\mathbf{\nabla}$ satisfies following two axioms,

**B.1 Metric Compatiablity**

$$\mathbf{\nabla}_\mu g_{\alpha\beta} = 0, \text{ for all direction } \mu$$

**B.2 Torsion Free**

$$\mathbf{\nabla}_i\partial_j - \mathbf{\nabla}_j\partial_i = [\partial_i, \partial_j] = 0, \text{ for each direction pairs } (i, j)$$

Those condition implies $\partial_i g_{jk} = g_{lk}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ik}$, and $\Gamma^l_{ij} = \Gamma^l_{ji}$ respectively. In Levi-Chivita connection, we get $\Gamma$ explicitly.

**Definition 3.6** (Christoffel Symbol)**.**

$$\Gamma^l_{ij} = \frac{g^{lk}}{2}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij})$$

Note that $\mathbf{\nabla}$ add one lower index on tensor and in preserve form of tensor, but $\Gamma$ is not a tensor. But it can consider as derivation of metric.

**Definition 3.7** (Geodesic)**.** A **geodisic** is a curve in $(X, g)$ that is the worldline with no exterior forces. Let $\gamma : [a, b] \to X$ is such curve then it determinated by

$$\mathbf{\nabla}_{\dot\gamma}\dot\gamma = 0$$

**Theorem 3.8** (Geodesic Equation)**.** A shortest way along the spacetime $(X, g)$ satisfies

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

This can be proved by two different way: 1. Get a variance of functional, 2. Fix the local coordinate, then get explicitly.

We finally define length and volume on $(X, g)$.

**Definition 3.9** (Length, Volume)**.** A length of curve $\gamma : [a, b] \to X$ is given by

$$L = \int_a^b dl = \int_a^b \|g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))\| d\lambda = \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

And the volume of given by **volume form** $dV := dx^1 \wedge \cdots \wedge dx^{\dim X} \in \Omega^{\dim X}(X)$. Under Coordinate transform about $f : x \to y$, it is just $f^* dV$ that[1]

$$dV = \det \left( \frac{\partial x^1, \cdots, x^{\dim X}}{\partial y^1, \cdots, y^{\dim X}} \right) dy^1 \wedge \cdots \wedge dy^{\dim X}$$

Note that $g = (df)^\intercal (df)$. So $\det g = -J^2$, where $J$ is jacobian, the determinant of pushforward. So on spacetime, $g = (df)^\intercal \eta (df)$ (By local flatness). Let $\mathcal{G} = \det g$. So

$$dV = dx^0 \wedge \cdots \wedge dx^3 =: \mathrm{d}^4 x = \sqrt{-\mathcal{G}} \; \mathrm{d}^4 x'$$

**Theorem 3.10** (Stokes)**.**

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

**Definition 3.11** (Curvature Tensor)**.** The curvature tensor is kind of second derivative of vector field along pararell transport: Let $A, B, C \in \mathfrak{X}(X)$, The **curvature tensor** $R \in \mathcal{T}_3^1(X)$ such that

$$R_{X,Y}(Z) = \boldsymbol{\nabla}_X \boldsymbol{\nabla}_Y Z - \boldsymbol{\nabla}_Y \boldsymbol{\nabla}_X Z - \boldsymbol{\nabla}_{[X,Y]} Z$$

Using Lie-bracket,

$$[\boldsymbol{\nabla}_x, \boldsymbol{\nabla}_y] V^\alpha = R^\alpha_{\mu x y} V^\mu$$

Using indices notation,

$$R^\alpha_{\mu x y} = \partial_x \Gamma^\alpha_{\mu y} - \partial_y \Gamma^\alpha_{\mu x} + \Gamma^\alpha_{\beta x} \Gamma^\beta_{\mu y} - \Gamma^\alpha_{\beta y} \Gamma^\beta_{\mu x}$$

Using term of metric tensor,

$$R^\alpha_{\mu x y} = \frac{g^{\alpha\beta}}{2} \left( \partial^2_{\mu x} g_{\beta y} - \partial^2_{\mu y} g_{\beta x} + \partial^2_{\beta y} g_{\mu x} - \partial^2_{\beta x} g_{\mu y} \right).$$

We usually use lowing the index: $R_{abxy} := g_{ak} R^k_{bxy}$,

$$R_{abxy} = \frac{\partial^2_{bx} g_{ay} - \partial^2_{by} g_{ax} - \partial^2_{ax} g_{by} + \partial^2_{ay} g_{bx}}{2}$$

**Proposition 3.12.**

$$R_{abxy} = -R_{baxy}, \quad R_{abxy} = R_{abyx}, \quad R_{abxy} = R_{xyab}$$

$$R_{a(\mathrm{cyl})} = 0 \iff R_{abxy} + R_{axyb} + R_{aybx} = 0$$

$$R_{abxy} = \sum_{\pi,\sigma \in S_2} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) R_{\pi_a \pi_b \sigma_x \sigma_y}, \quad \sum_{\pi \in S_4} \mathrm{sgn}(\pi) R_{\pi_a \pi_b \pi_x \pi_y} = 0$$

**Theorem 3.13** (Bianchi Identity)**.**

$$\partial_x R_{abyz} + \partial_y R_{abzx} + \partial_z R_{abxy} = 0,$$
$$\boldsymbol{\nabla}_x R_{abyz} + \boldsymbol{\nabla}_y R_{abzx} + \boldsymbol{\nabla}_z R_{abxy} = 0.$$

Riemann curvature tensor has many indices. So we consider contraction of it, trace in some sence.

**Definition 3.14** (Ricci Tensor, Ricci Scalar)**.**

$$R_{ab} := R^x_{axb}, \quad R := g^{ab} R_{ab} = g^{ab} g^{xy} R_{axby}$$

**Definition 3.15** (Einstein Tensor)**.**

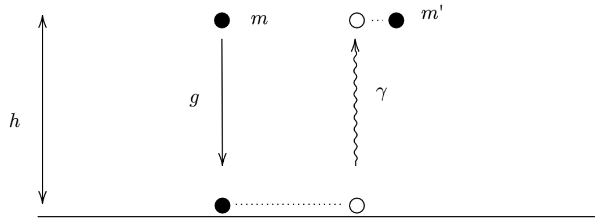
$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

This get from bianchi identify when we interchange Riemann curvature to metric and Ricci's.

$$\boldsymbol{\nabla}_\mu G^{\mu\nu} = 0$$

# 4 Slightly Curved Spacetime

**Theorem 4.1** (GR Doppler shift)**.** Consider a particle in constant gravitation field $g$. In freely falling of particle, energy conservation holds. When it truns to light and rise to top, and go back to particle. Threfore $E_{top} = m + mgh$ and $E_{bot} = m'$. By energy conservation




$m = m'$, so the rate of frequencies of photon at top and bottom is:

$$\frac{E_{bot}}{E_{top}} = \frac{f_{bot}}{f_{top}} \approx \frac{m}{m + mgh} = \frac{1}{1 + gh} \approx 1 - gh < 1$$

On other condition, consider gravitation is given as $\Phi = -GM/R$.

$$\frac{E_{bot}}{E_{top}} = \frac{f_{bot}}{f_{top}} \approx \frac{1 + \frac{GM}{R+h}}{1 + \frac{GM}{R}}$$

When $h \to \infty$, frequency rate goes to $1 - \frac{GM}{R}$, when gravitation is not very strong.

First, we give a local structure on $(X, g)$ that is compatiable to special relativity.

**Theorem 4.2** (Local-Flatness)**.** For spacetime $(X, g)$ has local diffeomorphism to Minkowski space $(U, \eta)$, when $U \hookrightarrow X$. In other word, For a event $p \in U \subseteq X$,

$$g|_p(x) = \eta(p) + \mathcal{O}(\partial^2 g)$$

**Corollary 4.3.** Since first order derivation of $g - \eta$ is zero, so we can locally identify rank 1 tensors on Minkowski space in that open set.

**Definition 4.4** (Slightly Curved Spacetime)**.** A spacetime $(X, g)$ is **slightly curved** if the metric is given by:

$$g_{\mu\nu} = \begin{bmatrix} -(1+2\Phi) & 0 & 0 & 0 \\ 0 & (1-2\Phi) & 0 & 0 \\ 0 & 0 & (1-2\Phi) & 0 \\ 0 & 0 & 0 & (1-2\Phi) \end{bmatrix}$$

where $\Phi$ is newtonian graviational potential.

## 4.1 Equation of Motion

### 4.1.1 Week Gravity

Consider a motion in curved spacetime $(X, g)$, with week gravity and non-relativistic case. Since geodesic can represent as $\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{u} = 0$, for 4-velocity $\mathbf{u}$. Since energy-momentum vector $\mathbf{p}$ is given as $m\mathbf{u}$ for rest mass $m$, $\boldsymbol{\nabla}_{\mathbf{p}}\mathbf{p} = 0$. Thus $p^\alpha \boldsymbol{\nabla}_\alpha p^\beta = 0$,

$$m\frac{dp^\alpha}{d\tau} + \Gamma^\alpha_{\beta\mu} p^\mu p^\beta = 0.$$

First term is classical motion of equation, and second term is not. On slightly curved metric $g$, on $\mu = \beta = 0$, $p^0 \simeq m$ (non-relativistic), and $|\Phi| \ll 1$ (week gravity),

$$\Gamma^0_{00} = \frac{g^{0k}}{2}(2\partial_0 g_{0k} - \partial_k g_{00}) = -\frac{g^{00}\partial_0 g_{00}}{2} = (1+2\Phi)^{-1}\partial_0\Phi \approx \partial_0\Phi$$

$$\Gamma^i_{00} = \frac{g^{ik}}{2}(2\partial_0 g_{0k} - \partial_k g_{00}) = -\frac{g^{ii}\partial_i g_{00}}{2} = (1-2\Phi)^{-1}\partial_i\Phi \approx \partial_i\Phi$$

So we get motion equations:

$$\frac{dp^0}{d\tau} = -m\partial_t\Phi, \quad \frac{dp^i}{d\tau} = -m\partial_i\Phi.$$

### 4.1.2 Light Bending

Consider a geodesic equation for light in weak gravity.

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

Here the parameter $\lambda \neq \tau$ since light's proper time goes infinite in time-like frame. Hence $\mu = \beta$

$$\Gamma^\alpha_{\gamma\beta} = \frac{g^{\alpha\nu}}{2}(2\partial_\beta g_{\nu\beta} - \partial_\nu g_{\gamma\beta}) = g^{\alpha\nu}\left(\partial_\beta g_{\nu\beta} - \frac{1}{2}\partial_\nu g_{\gamma\beta}\right) = -\frac{1}{2}g^{\alpha\nu}\partial_\nu g_{\gamma\beta}$$

So it takes down index of acceleration, we get

$$\frac{d^2x_\nu}{d\lambda^2} = \frac{1}{2}\partial_\nu g_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Take $\nu = 0$. Then $\partial_0 g_{\beta\gamma} = 0$, since light's time expand infinitely, therefore we got

$$\frac{d^2x_0}{d\lambda^2} = 0$$

here $x_0 = \lambda$, so $\lambda$ is time for observer. (in cartesian $x^i = x_i$) In other indices for $\nu$,

$$\frac{d^2 x_i}{dt^2} = -2\partial_i \Phi$$

Recall $dp^i/d\tau = -m\partial_i\Phi$. Divide both term by $m$, and approx $\tau \simeq t$,

$$\frac{d^2 x_i}{dt^2} = -\partial_i \Phi$$

So light need two times larger acceleration than massive particle.

### 4.2 Energy-Momentum Conservation

Locally $g_{\mu\nu}p^\mu p^\nu = \eta_{\mu\nu}p^\mu p^\nu = -m^2$. Thus

$$g_{\mu\nu}p^\mu p^\nu = -(1+2\Phi)(p^0)^2 + (1-2\Phi)p^2 = -m^2$$

So

$$p^0 = \sqrt{\frac{m^2 + (1-2\Phi)p^2}{1+2\Phi}} \simeq m\left(1 - \Phi + \frac{p^2}{2m^2}\right) = \underbrace{m}_{\text{Rest}} - \underbrace{m\Phi}_{\text{Potential}} + \underbrace{\frac{p^2}{2m}}_{\text{Kinetic}}$$

Classically $\partial_0\Phi = 0$, then $p_0$ is constant under $t$. But $t$ is dependent on frame, thus we conclude that mass conservation is frame dependent.

# 5 Stress-Energy- Tensor

We need to define the tensor that represent distribution of mass and energy. The conclusion of general relativity is metric and that is related. (Einstein equation: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.)

**Definition 5.1** (Stress-Energy-Tensor)**.** For mass density $\rho$,

$$T^{\alpha\beta} = \rho \mathbf{u} \otimes \mathbf{u} = \begin{bmatrix} \rho\gamma^2 & \rho\gamma^2 v^1 & \rho\gamma^2 v^2 & \rho\gamma^2 v^3 \\ \rho\gamma^2 v^1 & \rho\gamma^2 (v^1)^2 & \rho\gamma^2 v^1 v^2 & \rho\gamma^2 v^1 v^3 \\ \rho\gamma^2 v^2 & \rho\gamma^2 v^1 v^2 & \rho\gamma^2 (v^2)^2 & \rho\gamma^2 v^2 v^3 \\ \rho\gamma^2 v^3 & \rho\gamma^2 v^1 v^3 & \rho\gamma^2 v^2 v^3 & \rho\gamma^2 (v^3)^2 \end{bmatrix}$$

As view point of **flux**, $T^{\mu\nu}$ is $p^\mu$ flux of surface $x^\nu = \text{Const.}$

**Theorem 5.2** (Conservation Law)**.** Consider a conservation of flux; $\partial_\mu T^{\mu\nu} = 0$ in flat spacetime. On curved spacetime, partial derivative turns to covarience derivative:

$$\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0.$$

## 5.1 Perfect Fluid

Consider a flux of perfect fluid. Then $T^{ij} = P\delta^{ij}$, where $P$ is presure. So[3]

$$T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P\eta^{\alpha\beta}.$$

# References


[1] L. D. Landau. *The classical theory of fields*, volume 2. Elsevier, 2013.

[2] A. Marsh. *Mathematics for Physics: An Illustrated Handbook.* World Scientific, 2017.

[3] B. Schutz. *A first course in general relativity.* Cambridge university press, 2009.