$\boxed{R-3.2}$

$$A = 8n\log n$$

$$B = 2n^2$$

$$8n\log n = 2n^2$$

$$\Rightarrow \quad 4n\log n = n$$

$$\Rightarrow \quad \frac{n}{\log n} = 4$$

$$\Rightarrow \quad n = 16$$

$n = 16 \rightarrow$ $A = 512$
$B = 512$

Now put $n = 17$ in algorithm A and B

$$A = 8.17 \log 17 = 555.3949$$

$$B = 2.17^2 = 578$$

$\Rightarrow$ A is better than B

$\Rightarrow$ $n_0 = 16$ for $n \geq n_0$

$\boxed{R-3.3}$ $A = 40n^2$
$B = 2n^3$

$\Rightarrow \quad 40n^2 = 2n^3$

$\Rightarrow \quad 20 = n$

$n = 20 \rightarrow \quad A = 16000$
$B = 16000$

Now plugg in $n = 21$ to A & B

$A = 40 \cdot 21^2 = 17640$

$B = 2 \cdot 21^3 = 18522$

$\Rightarrow$ A is better than B for $n \geq n_0$
$\Rightarrow \quad n_0 = 20$

$\boxed{R-3.6}$ $S = 0 + 2 + 4 + 6 + \ldots + 2n$

$S = 2n + 2(n-1) + \ldots + 2 + 0$

Add both equations:

$\Rightarrow 2S = 2n(n+1)$

$\Rightarrow \quad S = n(n+1)$

So, $\sum_{i=0}^{n} 2i = n(n+1)$

- Base case $n_0 = 1$:

LHS: $\sum_{i=0}^{1} 2i = 0+2 = 2$

RHS: $1(1+1) = 2$ ✓

- Induction Hypothesis : $n = 2$

LHS: $\sum_{i=0}^{2} 2i = 0+2+4 = 6$

RHS: $2(2+1) = 6$

Assume $n = k$, $\sum_{i=0}^{k} 2i = k(k+1)$

- Inductive Step :

Assume $n = k+1$, $\sum_{i=0}^{k+1} 2i = (k+1)(k+2)$

Proof:

LHS: $\sum_{i=0}^{k+1} 2i = \sum_{i=0}^{k} 2i + 2(k+1)$

$= k(k+1) + 2(k+1)$

$= (k+1)(k+2) = RHS$ ✓

Therefore, the sum of all the even numbers from 0 to $2n$ is $n(n+1)$

R-3.9

If $d(n)$ is $O(f(n))$, then $ad(n)$ is $O(f(n))$, $a > 0$

Solution: Assume $d(n)$ is $O(f(n))$

$$\Rightarrow \quad d(n) \leq c\, f(n)$$

$$\Rightarrow \quad ad(n) \leq a.c\, f(n)$$

Let $K = a.c$, K is constant

$$\Rightarrow \quad ad(n) \leq K f(n)$$

So, $ad(n)$ is $O(f(n))$

R-3.21 Show $n \log n$ is $\Omega(n)$

Solution: $f(n)$ is $n \log n$

$g(n)$ is n

Assum $f(n)$ is $\Omega(g(n))$

$\Rightarrow \quad n\log n \geq cn$

$\Rightarrow \quad \log n \geq c$

Let $c = 1$, $n_0 = 4$

$\Rightarrow \quad \underbrace{\log 4}_{2} \geq 1$ ✓

So, $f(n) \in \Omega(g(n))$

And $n\log n$ is $\Omega(n)$

$\boxed{R-3.15}$ $f(n)$ is $O(g(n))$ if & only if $g(n)$ is $\Omega(f(n))$

Solution: Assume $f(n)$ is $O(g(n))$

$\Rightarrow \quad f(n) \leq c\, g(n)$

$\Rightarrow \quad \frac{1}{c} f(n) \leq g(n)$

let $K = \frac{1}{c}$, K is constant

$\Rightarrow \quad K f(n) \leq g(n)$

$\Rightarrow \quad g(n) \geq K f(n)$

Therefore $g(n)$ is $\Omega(f(n))$

R-3.26 , R-3.27 , C-3.36 on .py file

$\boxed{\text{C-3.38}}$

$$\sum_{i=1}^{n} i^2 = \overset{1}{1} + \overset{2}{4} + \overset{3}{9} + \overset{4}{16} + \dots + \overset{n}{n^2}$$

$$(n-1)^3 = n^3 - 3n^2 + 3n - 1$$

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

Subtituting $n = 1, 2, 3, \dots n$, we get

$$\left.\begin{aligned} 1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1 \\ 2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1 \\ 3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1 \\ &\vdots \end{aligned}\right\} +$$

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) -$$

$$3(1 + 2 + 3 + \ldots n) - (\underbrace{1 + 1 + 1 \ldots + 1}_{n \text{ times}})$$

$$\Rightarrow n^3 = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum 1$$

$$\Rightarrow 3\sum_{i=1}^{n} = n^3 + 3\frac{n(n+1)}{2} - n$$

$$= n^3 + \frac{3}{2} n(n+1) - n$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$= \frac{n(2n^2 + 2n + n + n)}{2}$$

$$= \frac{n(2n+1)(n+1)}{2}$$

$$\Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

By inductive on n

- Base case: $n_0 = 1$

$$\text{LHS: } \sum_{i=1}^{1} i^2 = 1$$

$$\text{RHS}: \frac{1(2.1+1)(1+1)}{6} = 1$$

- **IH:** $n = 2$

$$\text{LHS}: \sum_{i=0}^{2} i^2 = 1 + 4 = 5$$

$$\text{RHS}: \frac{2(2.2+1)(2+1)}{6} = 5$$

Assume $n = k$, $\sum_{i=0}^{k} i^2 = \frac{k(2k+1)(k+1)}{6}$

- **Inductive Step:** $n = k+1$

$$\text{LHS}: \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(2k+1)(k+1)}{6} + (k+1)^2$$

$$= \frac{k(2k+1)(k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2 + \overbrace{k + 6k}^{7k} + 6)}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$$\text{RHS}: \frac{(k+1)(2k+3)(k+2)}{6} \quad //$$

So, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\frac{n(n+1)(2n+1)}{6}$ is $O(n^3)$

<u>Claim</u>: $f(n)$ is $O(g(n))$ and $af(n)$ is also $O(g(n))$

$\Rightarrow$ $\underbrace{n(n+1)(2n+1)}_{2n^3+3n^2+n}$ is $O(n^3)$.

$\Rightarrow$ $2n^3+3n^2+n \leq 2n^3+3n^3+n^3 = 6n^3$

Claim: If $p \geq q$ then $n^p \geq n^q$

let $c = 6$, $n_0 = 1$

- Base Case: $n = 1 \Rightarrow 6 = 6$
- IH: Let $n = k$ assume $2k^3+3k^2+k \leq 6k^3$
- I.Step: Show that for $n = k+1$

$2(k+1)^3+3(k+1)^2+(k+1) \leq 6(k+1)^3$

RHS: $6(k+1)^3 = 6(k^2+3k^2+3k+1) = 6k^2+18k^2+18k+6$

LHS:

$$2(k^3+3k^2+3k+1)+3(k^2+2k+1)+k+1$$

$$= 2k^3+6k^2+6k+2 + 3k^2+6k+3+k+1$$

$$= \underbrace{2k^3+3k^2+k}_{\leq 6k^3} + 6k^2+12k+6 \overset{\text{by IH}}{\leq} 6k^3+6k^2+12k+6$$

$$\leq 6k^3+18k^2+18k+6 = \text{RHS}$$

Therefore: $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is $O(n^3)$