Problem 3.39

1. Problem C-3.39 page 144

   Let
   $$S = \sum_{i=1}^{n} \frac{i}{2^i}$$

   See that
   $$S = \sum_{i=1}^{n} \frac{i-1}{2^i} + \sum_{i=1}^{n} \frac{1}{2^i}$$

   Claim:

   (i) $\sum_{i=1}^{n} \frac{i-1}{2^i} < \frac{S}{2}$

   Proof.
   $$\sum_{i=1}^{n} \frac{i-1}{2^i} = \sum_{i=2}^{n} \frac{i-1}{2^i} = \sum_{i=1}^{n-1} \frac{i}{2^{i+1}} = \frac{1}{2}\sum_{i=1}^{n-1} \frac{i}{2^i} < \frac{1}{2}\sum_{i=1}^{n} \frac{i}{2^i} = \frac{S}{2}$$
   $\square$

   (ii) $\sum_{i=1}^{n} \frac{1}{2^i} < 1$

   Proof.
   $$2 * \sum_{i=1}^{n} \frac{1}{2^i} = \sum_{i=1}^{n} \frac{1}{2^{i-1}} = 1 + \sum_{i=2}^{n} \frac{1}{2^{i-1}} = 1 + \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 + \left[\sum_{i=1}^{n} \frac{1}{2^i} - \frac{1}{2^n}\right] \implies \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} < 1$$
   $\square$

   Finally, by the Claim, $S < \frac{S}{2} + 1 \implies 2S < S + 2 \implies S < 2$