MATH 223 EXAM I Practice Problems Fall 2021

Contents

1. About the midterm exam I

The basic information of the first midterm is:

- (When and where) The exam is scheduled for **Sep 24 (Friday) in class** (50 min).
- (Topics of exam) The exam covers page 7-page 73 of the lecture notes (up to the chain rule).
- (Exam problems) There will be less than or equal to 10 problems and the total score is 100 pts. Some of the exam problem are based upon the practice problems and homework problems.
- (Cheat sheet) You may use **one sheet** of study paper[1] of any style (hand written, a piece of formula page from the book, etc). However, the size of the papers shoud not exceed that of A4 paper.
- (Calculator) Any type of calculator without capability of internet connection and an advanced math software such as Matlab or Maple is allowed.
- (Cheating) Zero tolerance on cheating. Any student who is involved in cheating will be punished (the minimum penality will be an F grade for this course and there could be more disciplinary actions such as an official reporting of the cheating to the university (including your department) and corresponding consequences.

The basic grading policy is the following:

- Exam grading will be different from Quiz grading, and exam grading is strict.
- You must show your work to get the full credit. A correct answer with **inconsistent work** or no work may be worth zero point.

Finally, this set of practice problems may contain typos or errors. Please, read with caution and let me know if you find any errors and discrepancies.

2. Practice Problems

1. Find the center of the sphere

$$x^2 + y^2 + z^2 - 2x + 3z = -1.$$

[1] The cheat sheet must be one sheet. For example, two sheets of paper with front 2 pages and with blank back pages are not allowed.

2. Find a real number x that makes the vectors

$$\vec{v} = x\vec{i} + 2\vec{j} - \vec{k} \quad \text{and} \quad \vec{w} = \vec{i} + 2\vec{j} + 2x\vec{k}$$

perperdicular.

3. Let θ be the angle between two vectors

$$\vec{a} = \langle 1, 1, 1 \rangle \quad \text{and} \quad \vec{b} = \langle 1, -2, 2 \rangle.$$

Compute $\cos\theta$.

4. (No explanation is necessary and No partial credit for this problem) Choose all the correct statements (If all of them are wrong, then write "None").

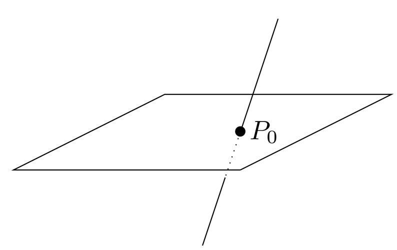
i) A line and a plane in space always meets at a point.
ii) The dot product of two vectors is a number.
iii) The cross product of $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
iv) The set of points $\{(x, y, z) : x^2 + y^2 = 1\}$ is a circle.

5. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors in space, then

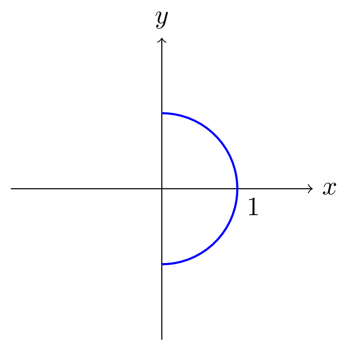
$$\frac{|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2}{|\vec{a}|^2 |\vec{b}|^2}$$

is a number. Find this number. (Of course, you must show your work. An answer without explanation gets zero point.)

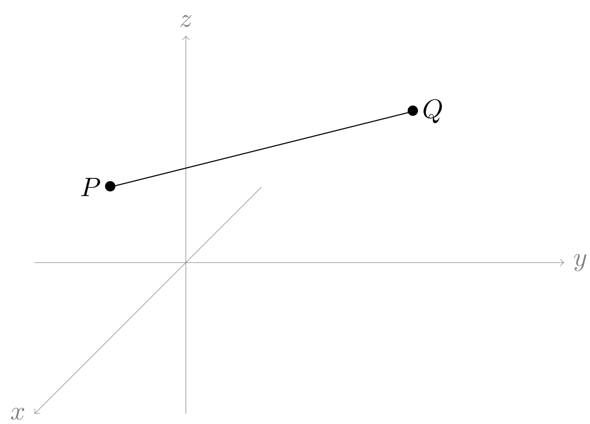
6. The parametric curve $\vec{r}(t) = \langle 1 - t, 1 + t, 2t \rangle$ meets the plane $2x + 2y + z = 5$ at a point P_0. Find this point P_0.



7. Parameterize the half circle of radius one with center at the origin as shown below. (That is, find a parametric plane curve whose image is the half circle shown below.)



8. Parameterize the line segment from $P = (1, 2, 3)$ to $Q = (1, -1, 5)$. (That is, find a parametric space curve whose image is the line segment shown below.)



9. Find the length of the parametric curve from $t = 1$ to $t = 3$

$$\vec{r}(t) = \langle t, 2\cos t, 2\sin t \rangle.$$

10. Find the arc length of the parametric curve $\vec{r}(t) = \langle t, t^2, \ln t \rangle$ from $P = (1, 1, 0)$ to $Q = (3, 9, \ln 3)$ (Find a definite integral expressing the arc length but do NOT evaluate the definite integral.)

11. Find the arc length function $s(t)$ of the parametric curve $\vec{r}(u) = \langle u, u^2, \ln u \rangle$ from $P_0 = (1, 1, 0)$ to an arbitrary point $P = (t, t^2, \ln t)$ on the curve. (Find an indefinite integral expressing the arc length but do NOT evaluate the indefinite integral.)

12. Compute $\int \vec{r}(t)dt$ where

$$\vec{r}(t) = \langle \sin 3t, t^4 \rangle.$$

13. Compute $\int_0^{2\pi} \vec{r}(t)dt$ where

$$\vec{r}(t) = \langle \sin 3t, t^4 \rangle.$$

14. For the parametric curve

$$\vec{r}(t) = t^2\vec{i} + \sin(2t)\vec{j} + \frac{1}{t}\vec{k}$$

compute the indefinite integral

$$\int \vec{r}(t)dt.$$

15. Consider the parametric curve

$$\vec{r}(t) = \langle t, t^2, e^{2t} \rangle.$$

Then

$$\vec{r'}(t) = \langle 1, 2t, 2e^{2t} \rangle \implies \overrightarrow{r''}(t) = \langle 0, 2, 4e^{2t} \rangle.$$

Find the curvature (function) $\kappa(t)$ of the curve.

16. Let $f(x)$ be a twice differentiable function. The plane curve $y = f(x)$ can be parametrized by[2]

$$\vec{r}(x) = \langle x, f(x), 0 \rangle.$$

Using Formula II, one finds that the curvature of $y = f(x)$ is

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

Show that if $\kappa(x) = 0$ for all x, then $y = f(x)$ is a line.

17. (No explanation is necessary and No partial credit for this problem) Choose all the correct statements (If all of them are wrong, then write "None").

i) The curvature function $\kappa(t)$ of a parametric curve $\vec{r}(t)$ measures how much the curve is curved at the general point $\vec{r}(t)$.

ii) For a function $f(x, y)$ of two variables,

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

iii) For a function $f(x, y)$ of two variables,

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}.$$

18. For the function $f(x, y) = x^2y^3$, find

$$\lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

19. For the function $f(x, y) = xy\sin(2x + 3y)$, find

$$\lim_{h\to 0} \frac{f(2 + h, 3) - f(2, 3)}{h}.$$

(Do not approximate your answer.)

[2]The parametrized curve still lives in the xy-plane inside the space.

20. Find an equation of the tangent plane to the surface $z = x^2 + 2y^2$ at $(1, 1, 3)$.

21. Consider the function
$$f(x, y) = x^2 + 3y^2.$$
Find the linear approximation of $f(x, y)$ at $(2, 1)$.
22. Consider the function
$$f(x, y) = x^2 + 3y^2.$$
Use the linear approximation of $f(x, y)$ at $(2, 1)$ to approximate the value $f(1.9, 1.2)$. (Do not compute the value $f(1.9, 1.2)$ by using a calulator. If you do not use linear approximation, you get no points.)

23. Find the differential of
$$f(x, y) = x^2 + 2y^2.$$
24. Consider the function
$$f(x, y) = x^2 + 2y^2.$$
Approximate the change Δf when (x, y) changes from $(1, 1)$ to $(1.1, 1.2)$ by using the differential. (Do not compute the change $\Delta f = f(1.1, 1.2) - f(1, 1)$ directly by using a calulator. If you do not use an approximation by differential, you get no points.)

25. Find the differential of
$$f(x, y, z) = \ln(x^2 + y^2 + z^2).$$

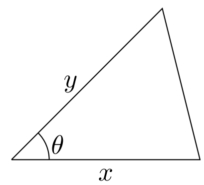
26. Use differentials to estimate the amount of metal in closed cylinderical can that is 10 cm high and 4 cm in diameter if the the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

27. If $f(x, y) = x^2y + 3xy^4$ where $x = \sin t$ and $y = \cos 2t$, find $\frac{df}{dt}$ when $t = 0$.

28. Use the chain rule to find the partial derivative $\frac{\partial z}{\partial \theta}$ where
$$z = xy, \qquad x = r\cos\theta, \qquad y = r\sin\theta.$$

29. The pressure (in kilopascals), volume V (in liters) and temperature T (in Kelvins) of a mole of an ideal gas are related by the equation
$$PV = 8.31T.$$
Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

30. The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 $radian/s$. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \frac{\pi}{6}$?



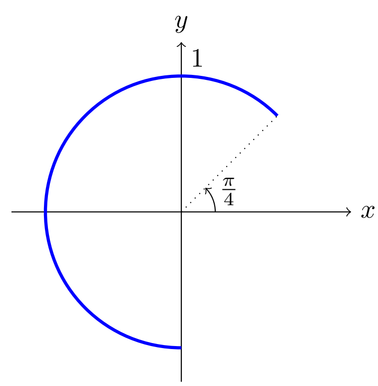
Area$=\frac{1}{2}xy\sin\theta$

3. An old exam

This was an actual midterm exam of Spring 2017 except for a few problems with (†).

1. ($\log_{10} 10000000000$-pts) What is your name ?

2. (10 pts) Parameterize the piece of the unit circle as shown below



3(†)(10 pts). (No explanation is necessary and No partial credits for this problem) Choose all the correct statements (If all of them are wrong, then write "None").

> i) The parametric curve $\vec{r}(t) = \langle t, 1-t, 0\rangle$ passes through the origin $(0,0,0)$.
> ii) The cross product of two vectors is a number.
> iii) If $\vec{u}$ and $\vec{v}$ are two vectors in space, then $|\vec{u}\cdot\vec{v}| \le |\vec{v}||\vec{u}|$.
> iv) The set of points $\{(x,y,z) : x^2+y^2+z^2=1\}$ is a sphere.

4. (10 pts) Find the length of the arc of the parametric curve

$$\vec{r}(t) = \langle t, e^t, \sin t\rangle$$

from the point $(0,1,0)$ to $(\pi, e^{\pi}, 0)$. (Do not evaluate your defnite integral.)

5. (10 pts) Consider the parametric curve

$$\vec{r}(t) = \langle t^2, t, \sin t\rangle.$$

Then

$$\vec{r'}(t) = \langle 2t, 1, \cos t\rangle \implies \overrightarrow{r''}(t) = \langle 2, 0, -\sin t\rangle.$$

Find the curvature (function) of the curve.

6. (10 pts) Consider the function

$$f(x, y) = x^3 + y^2.$$

Estimate the change in $f(x, y)$ when (x, y) changes from $(1, 2)$ to $(1.1, 1.8)$.

7. (10 pts) Let

$$z = x^2 y, \qquad x = r\cos\theta, \qquad y = r\sin\theta$$

and consider z as a function of r, θ. Use the chain rule to find the partial derivative $\frac{\partial z}{\partial r}$ at $(r, \theta) = (1, \pi)$.

8.(†-Modified)(5pts) Assume that $f(x, y)$ have partial derivative and

$$f(1, 2) = 3 \quad \text{and } f_x(1, 2) = 0 \quad \text{and } f_y(1, 2) = 0.$$

Find an equation of the tangent plane to the surface $z = f(x, y)$ at $\big(1, 2)$.

9. (†) (10 pts) Find the differential of

$$f(x, y) = \ln(x^2 + \cos(xy) + y^3).$$

10.(†) (10 pts) The length l, width w and height h of a box changes with time. At a certain instant, the dimensions are $l = 1, w = 2, h = 3$ (in meters) and l and w are increasing at a rate of 2 m/s and h is decreasing at a rate of 3 m/s. Find the rate of change of the volume of the box at that instant.