ERROR BOUNDS: AN OVERVIEW

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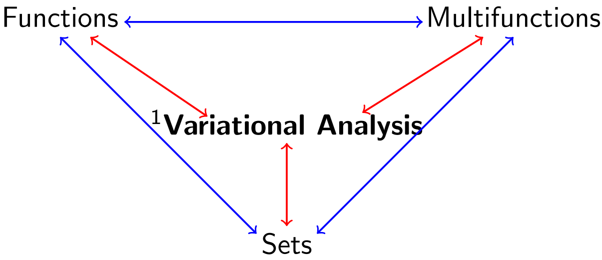
OPTIMIZATION WEBINAR

2 April 2023

Outline

- INTRODUCTION
- BASIC TOOLS
- SUFFICIENT & NECESSARY CONDITIONS
- CONNECTIONS

Introduction



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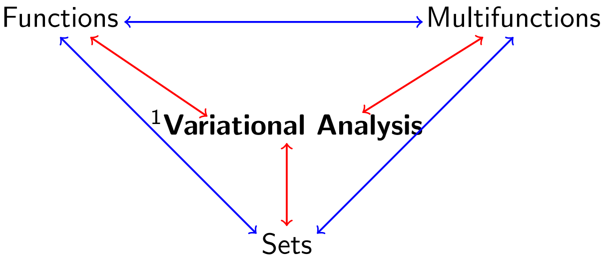
Introduction



- functions: subderivatives & subdifferentials
- sets: tangent cones & normal cones
- set-valued mappings: derivatives & coderivatives

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Notations

- $\alpha_+ := \max\{0, \alpha\}$
- Sublevel set: $[f \le 0] := \{x \in X \mid f(x) \le 0\}$
- Domain: $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$
- $B_\delta(\bar{x}) := \{x \in X \mid d(x, \bar{x}) < \delta\}$
- Dual space: X^*
- Set-valued mapping: $F : X \rightrightarrows Y$
- Distance function: $d(x, \Omega) := \inf_{\omega \in \Omega} d(x, \omega)$
- Indicator function: $i_\Omega(x) = 0$ if $x \in \Omega$ and $i_\Omega(x) = +\infty$ if $x \notin \Omega$

Historical Reviews

Journal of Research of the National Bureau of Standards Vol. 49, No. 4, October 1952 Research Paper 2362

On Approximate Solutions of Systems of Linear Inequalities*

Alan J. Hoffman

Let $Ax \leqq b$ be a consistent system of linear inequalities. The principal result is a quantitative formulation of the fact that if x "almost" satisfies the inequalities, then x is "close" to a solution. It is further shown how it is possible in certain cases to estimate the size of the vector joining x to the nearest solution from the magnitude of the positive coordinates of $Ax-b$.

1. Introduction

In many computational schemes for solving a system of linear inequalities

$$
\begin{aligned}
A_1 \cdot x &= a_{11}x_1 + \ldots + a_{1n}x_n \leqq b_1 \\
&\;\;\vdots \\
A_m \cdot x &= a_{m1}x_1 + \ldots + a_{mn}x_n \leqq b_m
\end{aligned} \qquad (1)
$$

(briefly, $Ax \leqq b$), one arrives at a vector $\bar{x}$ that "almost" satisfies (1). It is almost obvious geo-

Theorem: Let (1) *be a consistent system of inequalities and let F_n and F_m each satisfy* (3). *Then there exists a constant $c>0$ such that for any x there exists a solution x_0 of* (1) *with*

$$F_n(x-x_0) \leqq cF_m((Ax-b)^+).$$

The proof is essentially contained in two lemmas (2 and 3 below) given by Shmuel Agmon.[2]

Lemma 1. *If F_m satisfies* (3), *there exists an $e>0$ such that for every y and every subset S of the half spaces* (1)

$$F_m(\bar{y}) \leqq eF_m(y)$$

where $y=(y_1, \ldots, y_m)$, $\bar{y}=(\bar{y}_1, \ldots, \bar{y}_m)$, and

Historical Reviews

[2] **Theorem** Let A be a $m \times n$ matrix, $b \in \mathbb{R}^m$, $f(x) := \|(Ax - b)_+\|$ for $x \in \mathbb{R}^n$.

[2] Hoffman, A. J.: On approximate solutions of systems of linear inequalities. J. Res. Nat. Bur. Standards 49, 263–265 (1952)

[3] Donchev, A.: Lectures notes on variational analysis, Springer (2021)

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Applications

- [3] Linear programming
- Polyhedral optimization
- [4] Variational inequalities

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[6] Luo, X.D., Luo, Z.Q.: Extension of Hoffman's error bound to polynomial systems SIAM J. Optim. 4(2), 383–392 (1994)

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Error bounds

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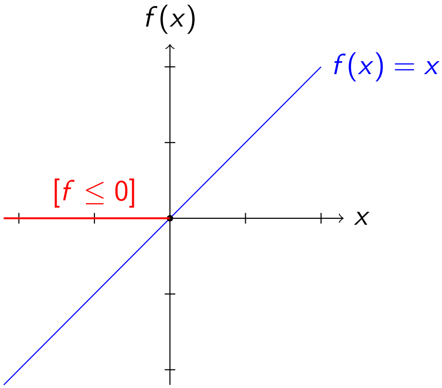
Example: linear error bounds



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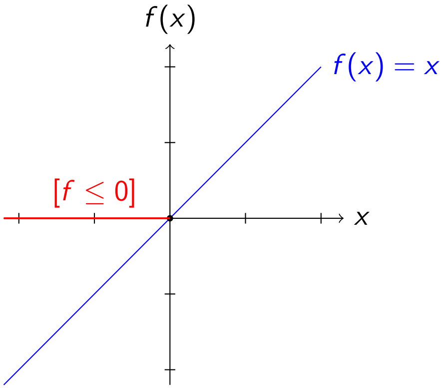
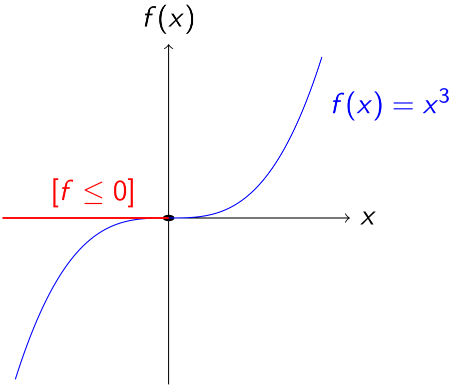


It holds that

$$d(x, [f \leq 0]) = f_+(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Example: no linear error bounds



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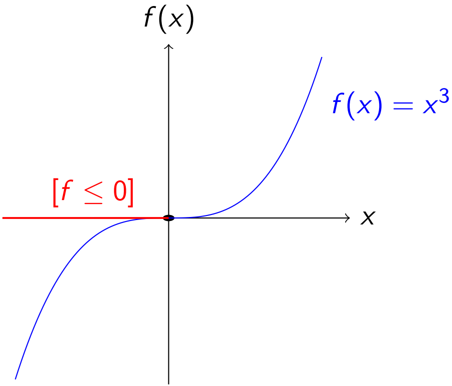

Let $\varphi(t) = t^{\frac{1}{3}}$. Then

$$d(x, [f \leq 0]) = \varphi(f_+(x)) \text{ for all } x \in \mathbb{R}.$$

Extensions

$X-$metric space, and $f(\bar{x}) \leq 0$.

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Definition

- f has a $q-$error bound at $\bar{x}$ if $\exists \tau, \delta > 0$ such that

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$$\partial^F f(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}$$

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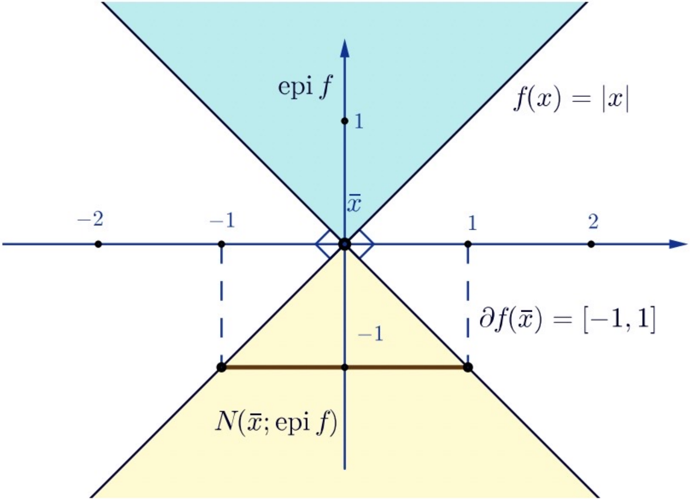
- $f-$Fréchet differentiable: $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$

Example



Sum rule

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[10] Zălinescu, C.: Convex analysis in general vector spaces . World Scientific Publishing Co. Inc., River Edge, NJ (2002)

[11] Fabian, M.: Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss. Acta Univ. Carolinae 30, 51–56 (1989)

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[10] **Theorem** Let f_1, f_2-convex, f_1 be continuous in a point in $\operatorname{dom} f_2$.

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Notes: Clarke/proximal/abstract subdifferentials

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Azé, D., Corvellec, J.N., Lucchetti, R.E.: **Variational pairs and applications to stability in nonsmooth analysis**. Nonlinear Anal. 49(5, Ser. A: Theory Methods), 643–670 (2002)

seems to be the very fact that the slope of De Giorgi-Mario-Tosques for the first time has appeared in the context of metric regularity theory."

[13] Ioffe, A.D.: Towards metric theory of metric regularity. In: Approximation, optimization and mathematical economics (Pointe-à-Pitre, 1999), pp. 165–176. Physica, Heidelberg (2001)

Ekeland variational principle

[14] **Theorem** X - complete metric space, $f-$lower semicontinuous, $x \in X$, $\varepsilon, \lambda > 0$.

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Characterizations of error bounds

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Metric subregularity & Error bounds

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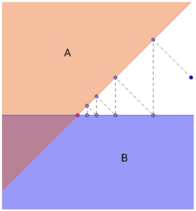

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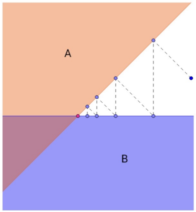

Definition $\{A, B\}$ is subtransversal at $\bar{x}$ if $\exists \tau, \delta > 0$ such that

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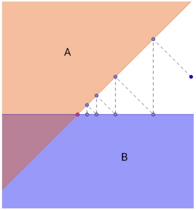

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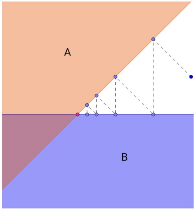

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[20] **Theorem**

Let X, Y - Banach spaces, $A \in \mathcal{L}(X, Y)$. The following statements are equivalent:

- A is surjective
- A is open
- $O \in \text{int}A(\text{int}\mathbb{B})$
- $\exists \tau > 0$ such that for all $y \in Y$, $\exists x \in X$ with $Ax = y$ and $\tau\|x\| \leq \|y\|$

The last statement can be rewritten as:

"$\exists \tau > 0$ such that $\tau d(x, A^{-1}(y)) \leq d(y, Ax)$ for all $x \in X, y \in Y$".

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THANK
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