# MA1521 CALCULUS FOR COMPUTING

Wang Fei

matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-16
Tel: 6516-2937

## What is a Sequence?

- Let's look at some examples of sequences:
  - Positive integers: $1, 2, 3, \ldots, n, \ldots$
  - Constant sequence: $1, 1, 1, \ldots, 1, \ldots$
  - Power sequence: $2, 4, 8, 16, \ldots, 2^n, \ldots$
  - Alternating sequence: $\dfrac{1}{\sqrt{1}}, \dfrac{-1}{\sqrt{2}}, \ldots, \dfrac{(-1)^{n+1}}{\sqrt{n}}, \ldots$
- **Definition.** A **sequence** is a list of numbers written in a definite order:

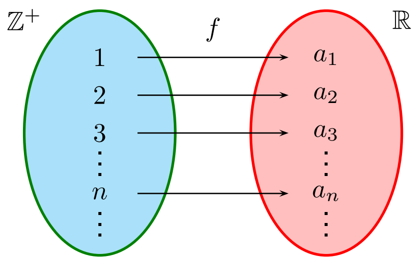
$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

  - $a_1$: the $1^{\text{st}}$ term; $a_2$: the $2^{\text{nd}}$ term; ..., $a_n$: the $n^{\text{th}}$ **term**.
- The sequence is denoted by $\{a_n\}_{n=1}^{\infty}$, or simply $\{a_n\}$.
  - $\{n\}_{n=1}^{\infty}$, $\{1\}_{n=1}^{\infty}$, $\{2^n\}_{n=1}^{\infty}$ and $\left\{\dfrac{(-1)^{n+1}}{\sqrt{n}}\right\}_{n=1}^{\infty}$.



## What is a Sequence?

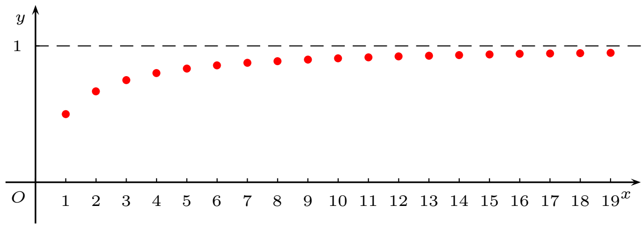
- Consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$.



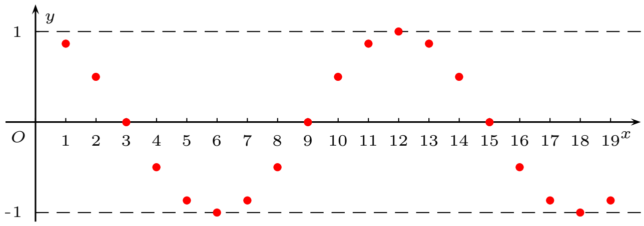

  - It defines a function $f : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = a_n$.
- Conversely, given a function $f : \mathbb{Z}^+ \to \mathbb{R}$,
  it defineds a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n = f(n)$.
- Therefore, we have an alternative definition for sequence:
  - A **sequence** is a function $\mathbb{Z}^+ \to \mathbb{R}$.

## Examples

- $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$.




- $\left\{\cos\frac{n\pi}{6}\right\}_{n=1}^{\infty}$.




5 / 78

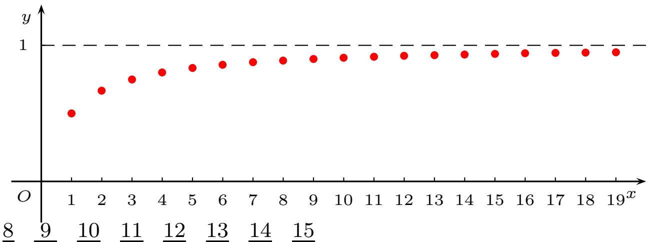
## Examples

- There are some sequences which cannot be defined by giving a simple formula for the terms, $n \mapsto a_n$.
  - $\sqrt{2}, \sqrt{\sqrt{2}+2}, \sqrt{\sqrt{\sqrt{2}+2}+2}, \ldots$.
    - $a_1 = \sqrt{2}, a_2 = \sqrt{a_1+2}, a_3 = \sqrt{a_2+2}, \ldots$.
    - $a_1 = \sqrt{2}$ and $a_n = \sqrt{a_{n-1}+2}$ for $n \geq 2$.
  - $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.
    - $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
    - It is the **Fibonacci sequence**.
      - Leonardo da Pisa, (1170s or 1180s–1250)
        Italian mathematician.
    - $F_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$.

6 / 78

## Limit of Sequence

- Since a sequence can be viewed as a function, we can similarly talk about the **limit of sequence**.
- **Example**. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$.




  - $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{10}{11}, \frac{11}{12}, \frac{12}{13}, \frac{13}{14}, \frac{14}{15}, \frac{15}{16}, \cdots$.
  - As $n$ gets larger, the term $a_n = \frac{n}{n+1}$ approaches $1$.

  We may use the similar notation as for function,

$$\lim_{n\to\infty} \frac{n}{1+n} = 1.$$

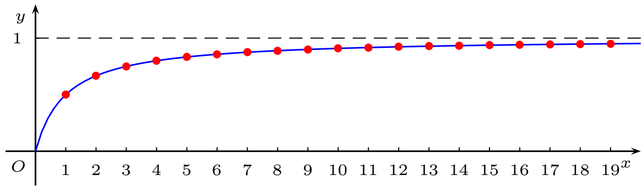


## Limit of Sequence

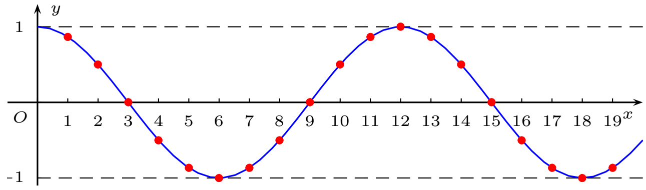
- **Definition**. Let $\{a_n\}$ be a sequence.
  - The **limit** of $\{a_n\}$ is $L$ if "$a_n$ is **arbitrarily close** to $L$ by taking $n$ **sufficiently large**".
    It is denoted by $\lim\limits_{n\to\infty} a_n = L$.
  - $\{a_n\}$ is called $\begin{cases} \textbf{convergent}, & \text{if } \lim\limits_{n\to\infty} a_n \text{ exists,} \\ \textbf{divergent}, & \text{otherwise.} \end{cases}$
- **Definition**. Let $\{a_n\}$ be a sequence.
  - The **limit** of $\{a_n\}$ is $\infty$ (resp. $-\infty$) if "$a_n$ is **arbitrarily large** (resp. **arbitrarily negatively large**) by taking $n$ **sufficiently large**".
    It is denoted by $\lim\limits_{n\to\infty} a_n = \infty$ (resp. $-\infty$).
  - **Remark**. If $\lim\limits_{n\to\infty} a_n = \pm\infty$, then $\{a_n\}$ is **divergent**.

## Examples

- We have known that $\lim_{x\to\infty} \frac{x}{x+1} = 1$.
  Can we use this fact to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$?




- $\lim_{x\to\infty} \cos\frac{\pi x}{6}$ does not exist.
  Can we conclude that $\lim_{n\to\infty} \cos\frac{\pi n}{6}$ does not exist as well?






## Limit Laws for Sequences

- **Theorem**. Let $f$ be a function and $\{a_n\}$ be the sequence such that $a_n = f(n)$ for all $n$.
  - If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.
- **Example**. Evaluate $\lim_{n\to\infty} \dfrac{\ln n}{n}$.
  - Let $f(x) = \dfrac{\ln x}{x}$, $(x > 0)$. Then $f(n) = a_n$ for all $n$.

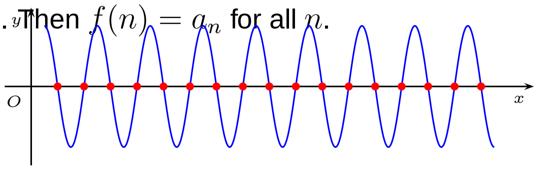
    $$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$
- **Example**. Evaluate $\lim_{n\to\infty} \sqrt[n]{n}$.
  - Let $f(x) = x^{1/x}$, $(x > 0)$. Then $f(n) = a_n$ for all $n$.

    $$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = \exp\left[\lim_{x\to\infty} \frac{\ln x}{x}\right] = e^0 = 1.$$

    $$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

## Examples

- We CANNOT use the theorem for the following cases:
  - Evaluate $\lim_{n\to\infty} \frac{n!}{n^n}$.
    - Let $f(x) = \cdots$?
    - $n!$ is only defined for natural numbers. It cannot be extended easily to a function on real numbers.
  - Evaluate $\lim_{n\to\infty} \sin n\pi$.
    - Let $f(x) = \sin x\pi$. Then $f(n) = a_n$ for all $n$.




    - $\lim_{x\to\infty} f(x)$ doesn't exist. So $\lim_{n\to\infty} a_n$ doesn't exist?
    - However, $\sin n\pi = 0$ for all $n$. $\lim_{n\to\infty} \sin n\pi = 0$.



## Limit Laws for Sequences

- **Theorem**. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences.
  - $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.
  - $\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n$.
  - $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$.
- **Theorem**. $\lim_{n\to\infty} a_n \Leftrightarrow \lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} a_{2n} = L$.
- **Theorem**. Suppose $a_n \leq b_n$ for all integer $n$.
  - If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $L \leq M$.
- **Squeeze Theorem**. Suppose $a_n \leq b_n \leq c_n$ for all $n$.
  - If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

## Limit Laws for Sequences

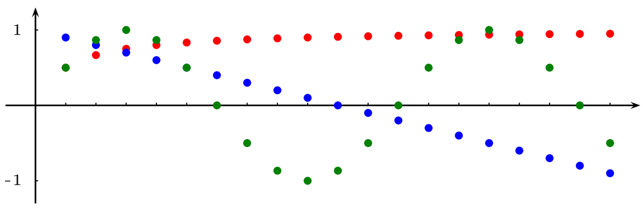
- **Example.** If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.
  - Note that $-|a_n| \le a_n \le |a_n|$ for all $n$,
    $\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0 = \lim_{n\to\infty} |a_n|$.
  - By Squeeze Theorem $\lim_{n\to\infty} a_n = 0$.
  - E.g., $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.
- **Example.** Evaluate $\lim_{n\to\infty} \dfrac{n!}{n^n}$.
  - $\dfrac{n!}{n^n} = \dfrac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{\underbrace{n \cdot n \cdot n \cdots n \cdot n}_{n \text{ times}}} = \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot \dfrac{3}{n} \cdots \dfrac{n-1}{n} \cdot \dfrac{n}{n}$.
  - $0 \le \dfrac{n!}{n^n} \le \dfrac{1}{n}$. $\left.\begin{array}{l} \lim_{n\to 0} 0 = 0 \\ \lim_{n\to 0} \frac{1}{n} = 0 \end{array}\right\} \Rightarrow \lim_{n\to\infty} \dfrac{n!}{n^n} = 0$.



## Monotonic Sequences

- Similarly as increasing/decreasing functions,
  we can talk about increasing/decreasing sequences.
- **Definition.** Let $\{a_n\}$ be a sequence.
  - $\{a_n\}$ is called $\begin{cases} \textbf{increasing} & \text{if } a_n < a_{n+1} \text{ for all } n, \\ \textbf{decreasing} & \text{if } a_n > a_{n+1} \text{ for all } n. \end{cases}$
  - $\{a_n\}$ is called **monotonic**

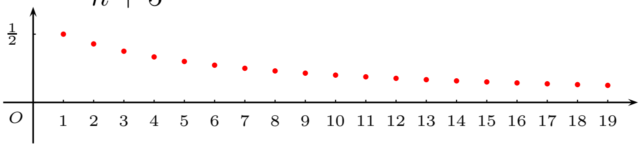
    if it is either increasing or decreasing.




$\left\{\frac{n}{n+1}\right\}$ increases; $\left\{\frac{10-n}{10}\right\}$ decreases; $\left\{\sin\frac{n\pi}{6}\right\}$ neither.

## Examples

- Show that the sequence $a_n = \dfrac{3}{n+5}$ is decreasing.




i). $n < n+1 \Rightarrow n+5 < (n+1)+5$

$\Rightarrow \frac{3}{n+5} > \frac{3}{(n+1)+5} \Rightarrow a_n > a_{n+1}$.

ii). $a_n - a_{n+1} = \frac{3}{n+5} - \frac{3}{n+6} = \frac{3}{(n+5)(n+6)} > 0$.

iii). $\frac{a_{n+1}}{a_n} = \frac{3}{n+6} \Big/ \frac{3}{n+5} = \frac{n+5}{n+6} < 1, \quad (a_n > 0)$.

iv). Let $f(x) = \frac{3}{x+5}$. $f'(x) = -\frac{3}{(x+5)^2} < 0$ for $x > 0$.

$f$ is decreasing on $\mathbb{R}^+ \Rightarrow \{a_n\}$ is decreasing.

v). $\cdots\cdots\cdots\cdots\cdots$



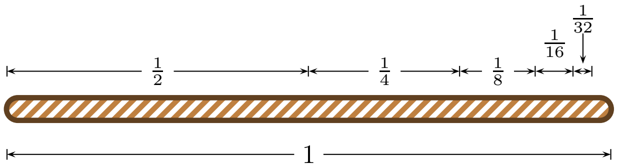
## Examples

- Determine if $a_n = \dfrac{n}{n^2+1}$ is increasing or decreasing.
  - Let $f(x) = \dfrac{x}{x^2+1}$.
    - $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$.
  - $f$ is decreasing on $[1, \infty) \Rightarrow \{a_n\}$ is decreasing.
- Determine if $a_n = \dfrac{n!}{n^n}$ is increasing or decreasing.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{(n+1)!n^n}{n!(n+1)^{n+1}}$$
$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n < 1.$$

  - Therefore, $\{a_n\}$ is decreasing.

## Examples




- Consider a segment of length $1$.
  - Cut half in the first day.
  - Cut half of the remaining in the second day.
  - In general, cut half of the remaining everyday.

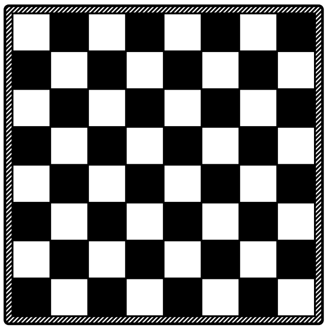
    $a_1 = \dfrac{1}{2}, a_2 = \dfrac{1}{4}, a_3 = \dfrac{1}{8}, \cdots, a_n = \dfrac{1}{2^n}, \cdots .$

  How much have we cut by the $n^{\text{th}}$ day?
  - We shall evaluate the sum of the first $n$ terms:
    - $S_n = a_1 + a_2 + \cdots + a_n = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n}.$



## Examples

- Consider an $8 \times 8$ chessboard.



  - Put $1$ gain of rice in the first square of the chessboard.
  - Doubling the number in the next square.
- How much rice do we need to fill in the chessboard?
  - $a_1 = 1, a_2 = 2, a_3 = 4, \ldots, a_n = 2^{n-1}, \ldots .$
  - $S_{64} = a_1 + a_2 + \cdots + a_{64} = 1 + 2 + 4 + 8 + \cdots + 2^{63}.$

## Series

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Then the sum of the first $n$ terms of $\{a_n\}$ forms a new sequence $\{S_n\}$.
  - $S_1 = a_1$;
  - $S_2 = a_1 + a_2$;
  - $S_3 = a_1 + a_2 + a_3$;
  - $\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$
  - $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$.

  $\{S_n\}$ is called the sequence of **partial sums** of $\{a_n\}$.
  - $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i := \sum_{n=1}^{\infty} a_n$.

  This quantity is called an **infinite series**, or simply **series**.
  - $\sum_{n=1}^{\infty} a_n$ is $\begin{cases} \textbf{convergent}, & \text{if } \{S_n\} \text{ is convergent,} \\ \textbf{divergent}, & \text{if } \{S_n\} \text{ is divergent.} \end{cases}$



## Examples

- Let us consider the examples shown at the beginning.
  - **Example 1**. $a_n = \frac{1}{2^n}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent.
    - Then $S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.
    - $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$.
  - **Example 2**. $a_n = 2^{n-1}$. Then $\sum_{n=1}^{\infty} a_n$ is divergent.
    - Then $S_n = 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.
    - $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (2^n - 1) = \infty$.
- They are special cases of **geometric series**.

## The Geometric Series

- Consider the **geometric sequence** $(a \neq 0)$.
  - $a_1 = a, a_2 = ar, a_3 = ar^2, \dots, a_n = ar^{n-1}, \dots$.
  - $a$ is the **scalar factor**, $r$ is the **common ratio**.
- $\sum\limits_{n=1}^{\infty} ar^{n-1}$ is called a **geometric series**.
  - $S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$.
  - $rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$.

  Then $(1-r)S_n = a - ar^n = a(1-r^n)$.
  - $S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r}, & \text{if } r \neq 1, \\ na, & \text{if } r = 1. \end{cases}$
  - $\sum\limits_{n=1}^{\infty} ar^{n-1} = \lim\limits_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1, \\ \text{divergent}, & \text{if } |r| \geq 1. \end{cases}$



## Examples

- Is the series $\sum\limits_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent?
  - $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{2(n+1)}3^{1-(n+1)}}{2^{2n}3^{1-n}} = \dfrac{4}{3} > 1$.
  - Then $\sum\limits_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum\limits_{n=1}^{\infty} 4\left(\dfrac{4}{3}\right)^{n-1}$ is divergent.
- Is $\sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$ convergent?
  - Geometric series of scalar factor $1$, common ratio $x$.
  - $\sum\limits_{n=0}^{\infty} x^n = \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{divergent}, & \text{if } |x| \geq 1. \end{cases}$
  - The **Taylor series** for $\dfrac{1}{1-x}$ about $0$.

## Examples

- Evaluate $\dfrac{1}{\sqrt{11}}+\dfrac{1}{\sqrt{33}}+\dfrac{1}{\sqrt{99}}+\dfrac{1}{\sqrt{297}}+\cdots$.
  - This is a geometric series with common ratio $r=\dfrac{1}{\sqrt{3}}$.
  - $\dfrac{1}{\sqrt{11}}+\dfrac{1}{\sqrt{33}}+\dfrac{1}{\sqrt{99}}+\dfrac{1}{\sqrt{297}}+\cdots=\dfrac{\frac{1}{\sqrt{11}}}{1-\frac{1}{\sqrt{3}}}$
- Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n-1}+3^{n+1}}{5^n}$.
  - $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n-1}}{5^n}=\sum_{n=1}^{\infty}\frac{1}{5}\left(\frac{3}{5}\right)^{n-1}=\frac{\frac{1}{5}}{1-\frac{3}{5}}=\frac{1}{2}$.
    $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}}{5^n}=\sum_{n=1}^{\infty}\frac{9}{5}\left(\frac{3}{5}\right)^{n-1}=\frac{\frac{9}{5}}{1-\frac{3}{5}}=\frac{9}{2}$.
  - Answer $=1/2+9/2=5$.



## Examples

- Recall that
  - Geometric series $=\dfrac{\text{leading term}}{1-\text{common ratio}}$ for $|\text{ratio}|<1$.
- Find the range of $x$ for which the series converges.
  - $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2x-1}{3}\right)^{n-2}=\frac{\frac{3}{2x-1}}{1-\frac{2x-1}{3}}=\frac{9}{(2x-1)(4-2x)}$.
    - It converges $\Leftrightarrow \left|\dfrac{2x-1}{3}\right|<1 \Leftrightarrow -1<x<2$.
  - $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}+2^n+2^{n+1}}{(x+1)^n}=\frac{\frac{7}{x+1}}{1-\frac{2}{x+1}}=\frac{7}{x-1}$.
    - It converges $\Leftrightarrow \left|\dfrac{2}{x+1}\right|<1 \Leftrightarrow x<-3$ or $x>-1$.

## Example

- Is the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ convergent?
  - $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\begin{aligned} S_n &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\ &\quad + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

  - $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$



## Telescoping Series

- The partial sum of a **telescoping series** has only a fixed number of terms after cancelation. Such evaluation is called the **method of differences**.
- **Example**. Evaluate $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

$$\frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}.$$

$$\begin{aligned} S_n &= \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} \\ &= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) \\ &= \frac{1}{1!} - \frac{1}{(n+1)!}. \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)!}\right) = 1.$$

## The Ratio Test

- Consider the series $\sum_{n=1}^{\infty} a_n$. Can we know its convergence by checking the ratio of consecutive terms?
  - If $\left|\frac{a_{n+1}}{a_n}\right| = L$ for all $n$, then $\sum_{n=1}^{\infty} |a_n|$ is a geometric series with common ratio $L$.
    - $\sum_{n=1}^{\infty} |a_n|$ is convergent $\Leftrightarrow$ if $|L| < 1$.
  - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\sum_{n=1}^{\infty} |a_n|$ is "$\textit{more or less the same}$" as the geometric series of common ratio $L$.
    - Do we have a result of convergence for $\sum_{n=1}^{\infty} a_n$ similar as that for the geometric series?



## The Ratio Test

- **Theorem**. Let $\sum_{n=1}^{\infty} a_n$ be a series.

  Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, where $0 \le L \le \infty$.
  - If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  - If $1 < L \le \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $L = 1$, the convergence of $\sum_{n=1}^{\infty} a_n$ is inconclusive.
- **Note**.
  - The ratio test does not work if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \neq L, \infty$.
  - The ratio test does not work if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

## Examples

- $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ is convergent.
  - $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^3/3^{n+1}}{(-1)^n n^3/3^n}\right| = \frac{(n+1)^3}{3n^3}$.
  - $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^3}{3} = \frac{1}{3}$.
- $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.
  - $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^n}{n^n}$.
  - $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.
    - $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.



## The Root Test

- Let $\sum_{n=1}^{\infty} a_n$ be a series.
  - If $\sqrt[n]{|a_n|} = L$, then $|a_n| = L^n$,
    - $\sum_{n=1}^{\infty} |a_n|$ is a geometric series of common ratio $L$.
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then $|a_n|$ is "*similar*" to $L^n$,
    - $\sum_{n=1}^{\infty} |a_n|$ is thus "*more or less the same*" as $\sum_{n=1}^{\infty} L^n$.
  - We can guess that the **root test** should have the same conclusion as the **ratio test**.
    - They should have the same advantage, as well as the same disadvantage.
    - However, sometimes the **root test** works better.

## The Root Test

- **Theorem**. Let $\sum_{n=1}^{\infty} a_n$ be a series.
  Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, where $0 \le L \le \infty$.
  - If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent,
  - If $1 < L \le \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $L = 1$, the convergence of $\sum_{n=1}^{\infty} a_n$ is inconclusive.
- **Example**. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.
  - $\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3}$ as $n \to \infty$.

31 / 78

## Examples

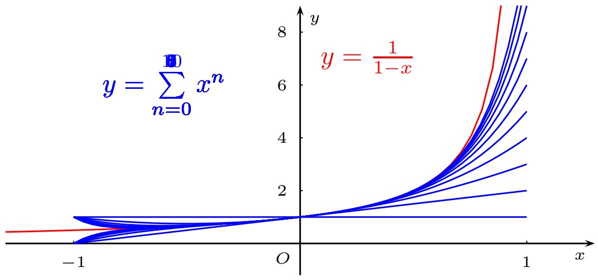
- $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is divergent.
  - $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{1+3n}}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$.
- The root test may work better than the ratio test.
  - $1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \cdots$.
  - $a_{2n-1} = a_{2n} = \frac{1}{2^{n-1}}$.
    - $\sqrt[2n-1]{a_{2n-1}} = \frac{1}{\sqrt[2n-1]{2^{n-1}}} = \frac{1}{2^{\frac{n-1}{2n-1}}} \to \frac{1}{\sqrt{2}}$,
    - $\sqrt[2n]{a_{2n}} = \frac{1}{\sqrt[2n]{2^{n-1}}} = \frac{1}{2^{\frac{n-1}{2n}}} \to \frac{1}{\sqrt{2}}$.
  - By root test the series is convergent, but the ratio test does not work.

32 / 78

## Power Series

- Consider the geometric series
  - $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots.$

  We have seen that $\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1, \\ \text{divergent}, & \text{if } |r| \geq 1. \end{cases}$
  - Viewed as function: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $-1 < x < 1$.






## Power Series

- A **power series** about $0$ is a series of the form
  - $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots,$

  $c_i$'s are constants, called **coefficients**, and $x$ is a variable.
- In general, a **power series about $a$** is a series
  - $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$
- **Remark**. By convention we write $(x-a)^0 = 1$ for all $x$.
  - In particular, $\sum_{n=0}^{\infty} c_n (a-a)^n = c_0$.
  - $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent at $x = a$ at least.
  - How to find all $x$ so that the power series is convergent?

## Examples

- Check the convergence of $\sum_{n=0}^{\infty} a_n$, where $a_n = \dfrac{x^n}{\sqrt{n}}$.
  - To check whether $\sum_{n=0}^{\infty} a_n$ is convergent, use **ratio test**.
    - $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{n+1}/\sqrt{n+1}}{x^n/\sqrt{n}}\right| = \sqrt{\dfrac{n}{n+1}}\,|x|.$
    - $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\sqrt{\dfrac{n}{n+1}}\,|x| = 1\cdot|x| = |x|.$
  - $\sum_{n=1}^{\infty}\dfrac{x^n}{\sqrt{n}}$ is $\begin{cases} \text{convergent}, & \text{if } |x|<1, \\ \text{divergent}, & \text{if } |x|>1. \end{cases}$
  - We will learn how to determine the convergence at $x = \pm 1$ soon.

35 / 78

## Examples

- $\sum_{n=0}^{\infty}\dfrac{x^n}{2^n}$ is convergent on $(-2,2)$.
  - $\sum_{n=0}^{\infty}\dfrac{x^n}{2^n} = \sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n$ is a geometric series of ratio $\dfrac{x}{2}$.

    It is convergent $\Leftrightarrow \left|\dfrac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$.
- $\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ is convergent on $\mathbb{R}$.
  - $\lim_{n\to\infty}\left|\dfrac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\dfrac{|x|}{n+1} = 0.$
- $\sum_{n=0}^{\infty} n!x^n$ is convergent at $x=0$ only.
  - $\lim_{n\to\infty}\left|\dfrac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \begin{cases} \infty, x\neq 0, \\ 0, \quad x = 0. \end{cases}$

36 / 78

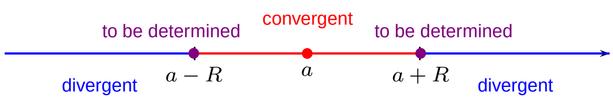
## Convergence Theorem for Power Series

- **Theorem**. Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series.
  Then its convergence is described by one of the following three possibilities:
  - (i) The series is convergent on $\mathbb{R}$;
  - (ii) The series is convergent at $x = 0$ only;
  - (iii) There is a number $R > 0$ such that
    - the series is convergent if $|x| < R$,
    - the series is divergent if $|x| > R$.
- **Remark**.
  - The convergence at $x = \pm R$ is inconclusive.
  - For case (i), we may write $R = \infty$;
  - For case (ii), we may write $R = 0$.

37 / 78

## Convergence Theorem for Power Series

- **Theorem**. Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series.
  - Then for some $0 \leq R \leq \infty$
    - the series is convergent if $|x - a| < R$;
    - the series is divergent if $|x - a| > R$.
- **Remark**. The convergence of the power series at $x = a + R$ and $x = a - R$ is inconclusive.
- **Definition**.
  - $R$ is called the **radius of convergence**.




38 / 78

## Radius of Convergence

- Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series.
  - The radius of convergence $R$ exists $(0 \le R \le \infty)$, but how to evaluate $R$?
- Consider the ratio: $\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \left|\frac{c_{n+1}}{c_n}\right| \cdot |x-a|$.
  - Suppose $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$.
    Then $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = L \cdot |x-a|$.
  - The series is $\begin{cases} \text{convergent,} & \text{if } L \cdot |x-a| < 1, \\ \text{divergent,} & \text{if } L \cdot |x-a| > 1. \end{cases}$

  $\therefore \quad R = L^{-1} = \left(\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|\right)^{-1}$.



## Radius of Convergence

- Consider the root: $\sqrt[n]{|c_n(x-a)^n|} = \sqrt[n]{|c_n|} \cdot |x-a|$.
  - Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$.
    Then $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = L \cdot |x-a|$.
  - The series is $\begin{cases} \text{convergent,} & \text{if } L \cdot |x-a| < 1, \\ \text{divergent,} & \text{if } L \cdot |x-a| > 1. \end{cases}$

  $\therefore \quad R = L^{-1} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}$.
- **Remark**.
  - If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$.
  - The formulas hold only when $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ or $\lim_{n\to\infty} \sqrt[n]{|c_n|}$ exists (or equals $\infty$).

## Examples

- $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-5)^n}{n^2}$. $\quad c_n = \dfrac{2^n}{n^2}$. $\quad R = 2^{-1} = 1/2$.
  - $\displaystyle\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/n^2} = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^2} = 2.$
- $\displaystyle\sum_{n=0}^{\infty} \frac{n^2(x-3)^{n+1}}{5^n}$. $\quad c_{n+1} = \dfrac{n^2}{5^n}$. $\quad R = (\frac{1}{5})^{-1} = 5$
  - $\displaystyle\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{n^2/5^n}{(n-1)^2/5^{n-1}} = \lim_{n\to\infty} \frac{n^2}{5(n-1)^2} = \lim_{n\to\infty} \frac{1}{5(1-\frac{1}{n})^2} = \frac{1}{5}.$



## Examples

- $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n-1}(2x+1)^n}{n!}$. $\quad c_n = \dfrac{3^{2n-1}2^n}{n!}$. $\quad R = \infty$.
  - $\displaystyle\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{3^{2n+1}2^{n+1}/(n+1)!}{3^{2n-1}2^n/n!} = \lim_{n\to\infty} \frac{18}{n+1} = 0.$
- $\displaystyle\sum_{n=0}^{\infty} \sqrt{n^n}\left(\tfrac{1}{2}x-1\right)^n$. $\quad c_n = \dfrac{\sqrt{n^n}}{2^n}$. $\quad R = 0$.
  - $\displaystyle\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{\frac{\sqrt{n^n}}{2^n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty.$

## Power Series Representation

- Recall the geometric series
  - $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ for $|x| < 1$.
  - $\frac{1}{1-x}$ is represented as power series $\sum_{n=0}^{\infty} x^n$ if $|x| < 1$.
  - $\sum_{n=0}^{\infty} x^n$ is a **power series representation** of $\frac{1}{1-x}$.
- Find a power series representation of $\frac{1}{1+x^2}$ about $0$.
  - Note that $\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$.
  - $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.
  - The identity holds $\Leftrightarrow$ $|x^2| < 1 \Leftrightarrow |x| < 1$.



## Examples

- Find a power series representation of $\frac{x^3}{x+2}$ at $0$.
  - $\frac{x^3}{x+2} = \frac{\frac{x^3}{2}}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{x^3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$
  - The identity holds $\Leftrightarrow$ $\left|-\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$.
- Find a power series representation of $\frac{1}{1-x}$ at $-1$.

$$\begin{aligned}\frac{1}{1-x} &= \frac{1}{2-(x+1)} = \frac{\frac{1}{2}}{1-\frac{x+1}{2}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}.\end{aligned}$$

  - The identity holds $\Leftrightarrow$ $\left|\frac{x+1}{2}\right| < 1 \Leftrightarrow |x+1| < 2$.

## Examples

- Find a power series representation of $\dfrac{1}{x^2+3x+2}$ at $0$.
  - $\dfrac{1}{x^2+3x+2} = \dfrac{1}{(x+1)(x+2)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$.
    - $\dfrac{1}{x+1} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$.

$$\frac{1}{x+2} = \frac{\frac{1}{2}}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-x}{2}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n.$$

  - Then $\dfrac{1}{x^2+3x+2} = \sum_{n=0}^{\infty}\left[1-\dfrac{1}{2^{n+1}}\right](-1)^n x^n$.

$\therefore$ The radius of convergence $R = \min\{1,2\} = 1$.



## Differentiation of Power Series

- $\sum_{n=0}^{\infty} c_n x^n$ is a function.
  - Is it **differentiable**? If yes, what is the **derivative**?
- Power series is a "*generalization*" of polynomial.
  Consider polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$.
  - It is continuous and differentiable,
    - $P'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$.
- **Theorem**. (Term by Term Differentiation)
  Suppose $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R > 0$.
  - Then $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is differentiable on $|x| < R$.
    - $f'(x) = \sum_{n=0}^{\infty} (c_n x^n)' = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

## Examples

- Recall that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.
  - Differentiate with respect to $x$:
    - $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$. $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.
  - Differentiate again with respect to $x$:
    - $\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{1+x}{(1-x)^3}$. $\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$.
  - They converge for $|x| < 1$. Let $x = 1/2$. We have
    - $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2 = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$.
    - $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}(1+\frac{1}{2})}{(1-\frac{1}{2})^3} = 6 = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \cdots$.



## The Coefficient of Power Series Representation

- Suppose $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R > 0$.
  - Then $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is differentiable if $|x| < R$.

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1}, \\
f''(x) &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}, \\
f'''(x) &= \sum_{n=3}^{\infty} c_n n(n-1)(n-2) x^{n-3}, \\
\cdots\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
f^{(k)}(x) &= \sum_{n=k}^{\infty} c_n n(n-1)\cdots(n-(k-1)) x^{n-k}.
\end{aligned}
$$

  - $\boxed{f^{(n)}(0) = c_n n(n-1)\cdots(n-(n-1)) = c_n n!}$

## Taylor Series and Maclaurin Series

- **Theorem**. Suppose $f$ has a power series representation $\sum_{n=0}^{\infty} c_n x^n$ of radius of convergence $R > 0$,
  - Then $c_n = \frac{f^{(n)}(0)}{n!}$, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

  Such series is called the **Maclaurin series** of $f$.
- **Theorem**. Suppose $f$ has a power series representation $\sum_{n=0}^{\infty} c_n (x-a)^n$ of radius of convergence $R > 0$,
  - Then $c_n = \frac{f^{(n)}(a)}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.

  Such series is called the **Taylor series** of $f$ at $a$.
- Power series representation, if exists, is unique $(R > 0)$.

49 / 78

## Examples

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$.
  - $c_n = 1$ for all $n$, and $c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = n!$.
- $\frac{x^3}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} = \sum_{n=3}^{\infty} \frac{(-1)^{n-3}}{2^{n-2}} x^n$.
  - $c_n = \begin{cases} 0, & n \leq 2, \\ \frac{(-1)^{n-3}}{2^{n-2}}, & n \geq 3. \end{cases}$ $f^{(n)}(0) = \begin{cases} 0, \\ \frac{(-1)^{n-3} n!}{2^{n-2}}. \end{cases}$
- **Note**. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ holds only if the power series representation of $f(x)$ exists.

  **Example**. Let $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$
  - $f^{(n)}(0) = 0$ for all $n$, but $f(x)$ is not the zero function.

50 / 78

## Examples

- Find the Maclaurin series of $f(x) = e^x$.
  - $f'(x) = e^x$, $f''(x) = e^x$, $\ldots$, $f^{(n)}(x) = e^x$, $\ldots$
  - $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$
- Find the Taylor series of $f(x) = e^{2x-1}$ at $x = 1$.
  - $\displaystyle e^{2x-1} = e^{2(x-1)+1} = e \cdot e^{2(x-1)} = \sum_{n=0}^{\infty} \frac{e\, 2^n (x-1)^n}{n!}.$
  - What is $f^{(2011)}(1)$?
    - $\displaystyle f^{(2011)}(1) = 2011!\, c_{2011} = 2011!\, \frac{e \cdot 2^{2011}}{2011!} = e \cdot 2^{2011}.$



## Examples

- Find the Maclaurin series of $f(x) = \sin x$.

| $f(x)$ | $f'(x)$ | $f''(x)$ | $f^{(3)}(x)$ |
|---|---|---|---|
| $\sin x$ | $\cos x$ | $-\sin x$ | $-\cos x$ |
| $f(0)$ | $f'(0)$ | $f''(0)$ | $f^{(3)}(0)$ |
| $0$ | $1$ | $0$ | $-1$ |
| $f^{(4)}(x)$ | $f^{(5)}(x)$ | $f^{(6)}(x)$ | $f^{(7)}(x)$ |
| $\sin x$ | $\cos x$ | $-\sin x$ | $-\cos x$ |
| $f^{(4)}(0)$ | $f^{(5)}(0)$ | $f^{(6)}(0)$ | $f^{(7)}(0)$ |
| $0$ | $1$ | $0$ | $-1$ |

  - $\displaystyle \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$
  - $\displaystyle \cos x = (\sin x)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$

## Test for Divergence

- Let $\sum_{n=1}^{\infty} a_n$ be a convergent series.
  - Suppose that $\sum_{n=1}^{\infty} a_n$ converges to $L$.
  - Let $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$.
    - $S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$ for $n \geq 2$.

    Then we have $S_n - S_{n-1} = a_n$.

  $\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0.$
- We proved: "If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$."
- **Test for Divergence**.
  - If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n$ exists but $\neq 0$,
  - then $\sum_{n=1}^{\infty} a_n$ is divergent.



## Examples

- Is the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ convergent?
  - $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0.$

  $\therefore \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is divergent.
- Consider the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, $(a \neq 0)$.
  - $$\lim_{n\to\infty} ar^{n-1} = \begin{cases} 0, & \text{if } |r| < 1, \\ a, & \text{if } r = 1, \\ \text{does not exist}, & \text{otherwise.} \end{cases}$$

  $\therefore \sum_{n=1}^{\infty} ar^{n-1}$ is divergent if $|r| \geq 1$.
- **Note**. If $\lim_{n\to\infty} a_n = 0$, test for divergence is **inconclusive**.

## Example

- Is the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4^n+1}}$ convergent?
  - $\lim_{n\to\infty} \frac{1}{\sqrt{4^n+1}} = 0 \Rightarrow$ No Conclusion.
  - We see that $\frac{1}{\sqrt{4^n+1}} < \frac{1}{\sqrt{4^n}} = \frac{1}{2^n}$.

    $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent "$\leftrightsquigarrow$" terms of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ are "small".
    - The terms of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4^n+1}}$ are "smaller".
    - It seems that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4^n+1}}$ is convergent as well.
- Is the "$comparison$" true? Does it hold in general?



## The Comparison Test

- **Theorem**. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that
  - $0 \le a_n \le b_n$ for all $n$. (Or for all $n \ge N$)

  Then $\begin{cases} \sum_{n=1}^{\infty} b_n \text{ converges} & \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \\ \sum_{n=1}^{\infty} a_n \text{ diverges} & \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges.} \end{cases}$
- **Example**. Is the series $\sum_{n=1}^{\infty} \frac{5}{2^n+4n+3}$ convergent?
  - $\frac{5}{2^n+4n+3} \le \frac{5}{2^n}$ for all $n$.
  - $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{5}{2^n} = 5 \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

    $\Rightarrow \sum_{n=1}^{\infty} \frac{5}{2^n+4n+3}$ converges.

## $p$-Series

- **Question**. For what values of $p$, is the ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?
  - Use the **test for divergence**:
    - $\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0, & \text{if } p > 0, \\ 1, & \text{if } p = 0, \\ \infty, & \text{if } p < 0. \end{cases}$

    $\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 0$.
  - However, we cannot use the test for divergence to conclude whether $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 0$.

57 / 78

## Harmonic Series

- The **Harmonic series** is the $p$-series when $p = 1$:
  - $H = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$
  - Consider the partial sum of the first $2^n$ terms:
    - $H_1 = 1;$
    - $H_2 = 1 + \frac{1}{2};$
    - $H_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$
    - $$\begin{aligned} H_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}. \end{aligned}$$

58 / 78

## Harmonic Series

- The **Harmonic series** is the $p$-series when $p = 1$:
  - $H = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$.
  - Consider the partial sum of the first $2^n$ terms:
    - $$\begin{aligned} H_{2^n} &= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + \cdots + \left(\tfrac{1}{2^{n-1}+1} + \cdots + \tfrac{1}{2^n}\right) \\ &> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + \cdots + (\tfrac{1}{2^n} + \cdots + \tfrac{1}{2^n}) \\ &= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ copies}} = 1 + \frac{n}{2}. \end{aligned}$$
  - $\lim_{n\to\infty} \left(1 + \frac{n}{2}\right) = \infty$.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_{2^n} = \infty$. So $\sum_{n=1}^{\infty}$ is **divergent**.

59 / 78

## $p$-Series

- **Theorem.** The $p$-series
  - $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is $\begin{cases} \textbf{convergent} & \text{if } p > 1, \\ \textbf{divergent} & \text{if } p \le 1. \end{cases}$
- **Remark.**
  - If $p \le 1$, $\frac{1}{n^p} \ge \frac{1}{n}$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.
  - The proof of the second statement is omitted.
- Can we use ratio test to check its convergence?
  - $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^p}{1/n^p} = \frac{n^p}{(n+1)^p}$.
  - $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^p} = 1$.
  - However, the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ depends on $p$.

60 / 78

## The Root Test

- Can the root test do better for $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$?
  - $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \frac{1}{\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^p} = \frac{1}{1^p} = 1.$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$
$$= \exp\left(\lim_{x\to\infty} \frac{\ln x}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{1/x}{1}\right) = \exp(0) = 1.$$

- In fact, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ must exist and equal $L$.
  - Hence, if one of the ratio test or root test has the limit $1$, **DO NOT** try the other test since it does not work too.

61 / 78

## Examples

- **Note**. In order to use the **Comparison Test** for a (positive) series, we shall first "$guess$"
  - whether it is **convergent** or **divergent**.
  - If we guess it is **convergent**,
    - find a (positive) **convergent** series whose terms are **bigger** than the terms of the given series.
  - If we guess it is **divergent**,
    - find a (positive) **divergent** series whose terms are **smaller** than the terms of the given series.
- **Example**. Is $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ convergent?

$\frac{\ln n}{n} \geq \frac{1}{n}$ if $n \geq 3$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

62 / 78

## Examples

- **Example.** Is $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ convergent?
  - $\frac{\ln n}{n^2} > \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ No conclusion!
  - $\frac{\ln n}{n^2} < \frac{n}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ No conclusion!

  Let's compare $\ln n$ and $\sqrt{n}$:
  - $f(x) = \ln x - \sqrt{x}$. $f'(x) = \frac{2-\sqrt{x}}{2x} < 0$ if $x > 4$.
  - For $n \geq 4$, $\ln n - \sqrt{n} \leq \ln 4 - \sqrt{4} \approx -0.6 < 0$.

  $\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ for all $n \geq 4$.
  - $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

63 / 78

## Examples

- Is the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ convergent?
  - It is "**similar**" to the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

    $\frac{1}{2^n-1} > \frac{1}{2^n} \Rightarrow$ Inconclusive by comparison test.
  - $\frac{1}{2^n-1} \leq \frac{1}{2^n - 2^{n-1}} = \frac{1}{2^{n-1}}$.

    $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent.
- Is the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ convergent?
  - It seems that $\frac{2n^2+3n}{\sqrt{5+n^5}}$ is "**similar**" to $\frac{2n^2}{\sqrt{n^5}} = \frac{2}{\sqrt{n}}$.

64 / 78

## The Limit Comparison Test

- **Theorem**. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms.

  (a) Suppose $\lim_{n=\infty} \frac{a_n}{b_n} = c$ is a positive real number.

  - $\sum_{n=1}^{\infty} b_n$ is **convergent** $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ is **convergent**.

  (b) Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

  - $\sum_{n=1}^{\infty} b_n$ is **convergent** $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is **convergent**.
  - $\sum_{n=1}^{\infty} a_n$ is **divergent** $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ is **divergent**.

  (c) Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

  - $\sum_{n=1}^{\infty} b_n$ is **divergent** $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is **divergent**.
  - $\sum_{n=1}^{\infty} a_n$ is **convergent** $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ is **convergent**.

65 / 78

## Examples

- Is the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ convergent?
  - $\lim_{n\to\infty} \frac{(2n^2+3n)/\sqrt{5+n^5}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{\frac{5}{n^5}+1}} = 2.$
  - $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is divergent.
- Is the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ convergent?
  - $\lim_{n\to\infty} \frac{1/n}{1/(\ln n)^2} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x}$
    $= \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0.$
  - $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent $\Rightarrow$ $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ is divergent.

66 / 78

## Examples

- Is the series $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n}}\sin\frac{1}{n}\right)$ convergent?
  - $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}\sin\frac{1}{n}}{\frac{1}{\sqrt{n}\,n}} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1.$
    - $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n}}$ converges $\Rightarrow \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$ converges.
- Is the series $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$ convergent?
  - $\lim_{n\to\infty}\frac{(\sin^2 n)/n^2}{1/n^2} = \lim_{n\to\infty}\sin^2 n.$ No Conclusion!
  - $\frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$. So $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$ is convergent.

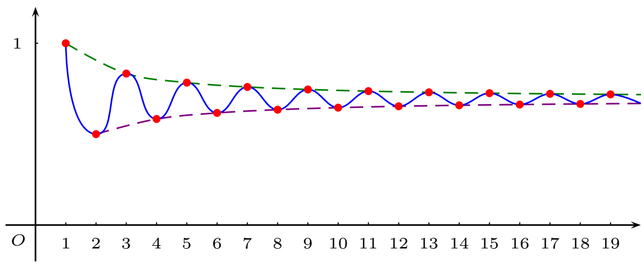


## Alternating Harmonic Series

- How about the series whose terms are not all positive?
- Is alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ convergent?

$+\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$

  - Let us check the graph of $S_n = a_1 + a_2 + \cdots + a_n$:



## The Alternating Series Test

- **Definition.** An **alternating series** is a series whose terms are alternatively *positive* and *negative*.
- **Leibniz Alternating Series Test.**
  - Let $\sum_{n=1}^{\infty} a_n$ be an **alternating series**. Suppose
    - $\lim_{n\to\infty} |a_n| = 0$, and $\{|a_n|\}$ is **decreasing**.
  - Then the series $\sum_{n=1}^{\infty} a_n$ is **convergent**.
- **Example.**
  - The **alternating Harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent;
  - although the **Harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.



## Examples

- Is the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ convergent?
  - $|a_n| = \frac{n^2}{n^3+1} \Rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$
  - Let $f(x) = \frac{x^2}{x^3+1} \Rightarrow f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$.
    - $f'(x) < 0$ if $x > \sqrt[3]{2} \Rightarrow \{|a_n|\}_{n=2}^{\infty}$ is decreasing.

  $\therefore$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is convergent.
  - However, $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ is divergent, (compare $\sum_{n=1}^{\infty} \frac{1}{n}$).
- It seems that the condition that "$\sum |a_n|$ **converges**" is "**stronger**" than the condition that "$\sum a_n$ **converges**".

## Absolute Convergence

- Let $\sum_{n=1}^{\infty} a_n$ be a series. We can consider a new series
  - $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots.$
- **Theorem**. $\sum_{n=1}^{\infty} |a_n|$ is converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
- **Examples**.
  - $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges.
  - If $\sum_{n=1}^{\infty} |a_n|$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is **inconclusive**.
    - $\sum_{n=1}^{\infty} 1$ diverges, and $\sum_{n=1}^{\infty} (-1)^n$ diverges.
    - $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.



## Absolute Convergence

- **Definition**. Let $\sum_{n=1}^{\infty} a_n$ be a series.
  - It is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.
  - It is **conditionally convergent** if $\sum_{n=1}^{\infty} |a_n|$ is divergent and $\sum_{n=1}^{\infty} a_n$ is convergent.
- **Examples**.
  - $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.
  - $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.
  - $\sum_{n=1}^{\infty} (-1)^n$ is divergent.

## Proof of Absolute Convergence Theorem

- **Proof**. Separate positive and negative terms in $\sum_{n=1}^{n} a_n$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$: | $1,$ | $1,$ | $-4,$ | $-5,$ | $1,$ | $-3,$ | $-1,$ | $2,$ | $1,$ | $7,$ |
| $a_n^+$: | $1,$ | $1,$ | $0,$ | $0,$ | $1,$ | $0,$ | $0,$ | $2,$ | $1,$ | $7,$ |
| $a_n^-$: | $0,$ | $0,$ | $4,$ | $5,$ | $0,$ | $3,$ | $1,$ | $0,$ | $0,$ | $0,$ |

  - $a_n^+ = \begin{cases} a_n, & \text{if } a_n \geq 0, \\ 0, & \text{if } a_n < 0. \end{cases}$ $a_n^- = \begin{cases} 0, & \text{if } a_n \geq 0, \\ -a_n, & \text{if } a_n < 0. \end{cases}$
    - $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$.
    - $a_n^+ + a_n^- = |a_n|$ and $a_n^+ - a_n^- = a_n$.
  - Suppose $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$0 \leq a_n^+, a_n^- \leq |a_n| \Rightarrow \sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$ is convergent.



## Examples

- **Example**. Is $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ convergent?
  - $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$.
  - $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent ($p$-series).

  $\Rightarrow \sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent by comparison test.

  $\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent by absolute convergence test.

- **Example**. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$.
  - It is $\begin{cases} \text{absolutely convergent,} & \text{if } p > 1, \\ \text{divergent,} & \text{if } p \leq 0, \\ \text{conditionally convergent,} & \text{if } 0 < p \leq 1. \end{cases}$

## Examples

- Given $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Evaluate $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$.
  - $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots$

$$=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$$
$$=\left(\frac{\pi^2}{6}-\frac{1}{4}\cdot\frac{\pi^2}{6}\right)-\left(\frac{1}{4}\cdot\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}-\frac{\pi^2}{24}=\frac{\pi^2}{12}.$$

  - $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{6}$

$$=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$$
$$=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\frac{1}{4}\cdot\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)\frac{\pi^2}{6}$$



## Example

- Can we evaluate $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ similarly?
  - $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\right)$$

$=\infty-\infty=?$ ←-- indeterminate form!

- $\lim_{n\to\infty}\frac{1/(2n-1)}{1/n}=\lim_{n\to\infty}\frac{n}{2n-1}=\frac{1}{2}$.
- $\lim_{n\to\infty}\frac{1/(2n)}{1/n}=\lim_{n\to\infty}\frac{n}{2n}=\frac{1}{2}$.

By limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$,

- $\sum_{n=1}^{\infty}\frac{1}{2n-1}=\sum_{n=1}^{\infty}\frac{1}{2n}$ are divergent.

## Conditional Convergence affects Rearrangement

- **Theorem**. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series.
  - If $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**, then every rearrangement has the same sum.
  - If $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**, then different rearrangements may have different sum.
    - Moreover, for any $L$ (a real number or $\pm\infty$), there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ whose sum is $L$.
- This theorem shows that if the series is conditionally convergent, we should not evaluate the sum by rearranging (infinitely many) terms.

77 / 78

## Example

- Find a rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ whose sum is $1$.
  - $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \cdots = \infty.$
    $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \frac{1}{18} + \cdots = \infty.$

1. If $S_n \geq 1$, add the negative terms until partial sum is $< 1$.
2. If $S_n < 1$, add the positive terms until partial sum is $\geq 1$.

- $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8}$
  $+ \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} + \frac{1}{21} - \frac{1}{12} + \frac{1}{23} + \cdots\cdots.$
- $S_1 = 1.0000. S_2 = 0.5000. S_3 = 0.8333. S_4 = 1.0333. S_5 = 0.7833. S_6 = 0.9262. S_7 = 1.0373. S_8 = 0.8706. S_9 = 0.9615. S_{10} = 1.0385. S_{11} = 0.9135. S_{12} = 0.9801. S_{13} = 1.0390. S_{14} = 0.9390. S_{15} = 0.9916. S_{16} = 1.0392. S_{17} = 0.9559. S_{18} = 0.9994.$ In fact, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2.$

78 / 78