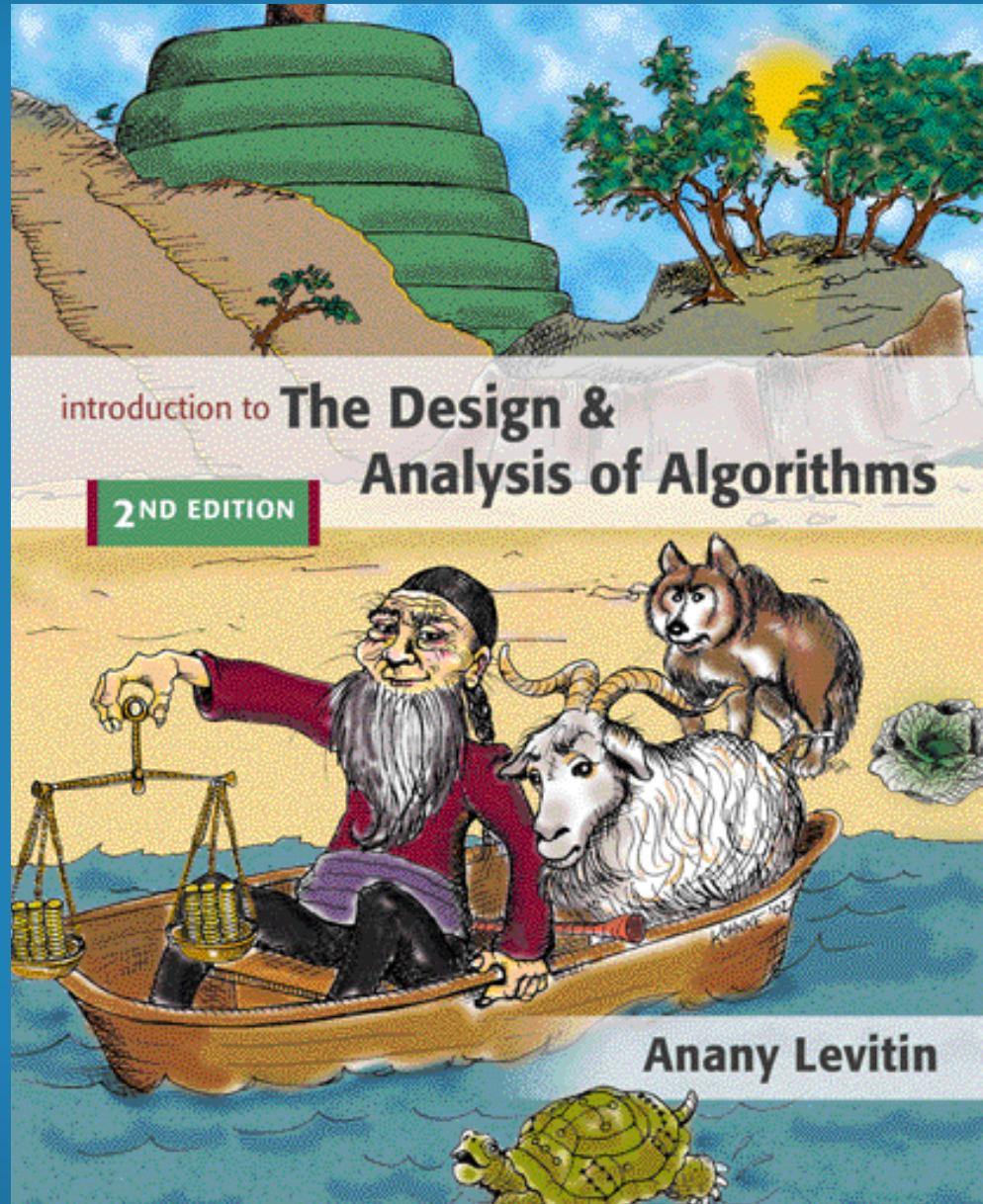


Chapter 5

Decrease-and-Conquer



Decrease-and-Conquer



1. Reduce problem instance to smaller instance of the same problem
 2. Solve smaller instance
 3. Extend solution of smaller instance to obtain solution to original instance
-
- Can be implemented either top-down or bottom-up
 - Also referred to as *inductive* or *incremental* approach

3 Types of Decrease and Conquer



- *Decrease by a constant* (usually by 1):
 - insertion sort
 - graph traversal algorithms (**DFS** and **BFS**)
 - topological sorting
 - algorithms for generating permutations, subsets
- *Decrease by a constant factor* (usually by half)
 - binary search and bisection method
 - exponentiation by squaring
 - multiplication à la russe
- *Variable-size decrease*
 - Euclid's algorithm
 - selection by partition
 - Nim-like games

This usually results in a recursive algorithm.

What's the difference?



Consider the problem of exponentiation: Compute x^n

□ **Brute Force:**

$n-1$ multiplications

□ **Divide and conquer:**

$$\begin{aligned}T(n) &= 2*T(n/2) + 1 \\&= n-1\end{aligned}$$

□ **Decrease by one:**

$$T(n) = T(n-1) + 1 = n-1$$

□ **Decrease by constant factor:**

$$\begin{aligned}T(n) &= T(n/a) + a-1 \\&= (a-1) \log_a n \\&= \log_2 n \quad \text{when } a = 2\end{aligned}$$

Insertion Sort



To sort array $A[0..n-1]$, sort $A[0..n-2]$ recursively and then insert $A[n-1]$ in its proper place among the sorted $A[0..n-2]$

- Usually implemented bottom up (nonrecursively)

Example: Sort 6, 4, 1, 8, 5

6 | 4 1 8 5
4 6 | 1 8 5
1 4 6 | 8 5
1 4 6 8 | 5
1 4 5 6 8

Pseudocode of Insertion Sort

ALGORITHM *InsertionSort($A[0..n - 1]$)*

//Sorts a given array by insertion sort
//Input: An array $A[0..n - 1]$ of n orderable elements
//Output: Array $A[0..n - 1]$ sorted in nondecreasing order

for $i \leftarrow 1$ **to** $n - 1$ **do**

$v \leftarrow A[i]$

$j \leftarrow i - 1$

while $j \geq 0$ **and** $A[j] > v$ **do**

$A[j + 1] \leftarrow A[j]$

$j \leftarrow j - 1$

$A[j + 1] \leftarrow v$

Analysis of Insertion Sort



- Time efficiency

$$C_{worst}(n) = n(n-1)/2 \in \Theta(n^2)$$

$$C_{avg}(n) \approx n^2/4 \in \Theta(n^2)$$

$$C_{best}(n) = n - 1 \in \Theta(n) \text{ (also fast on almost sorted arrays)}$$

- Space efficiency: in-place

- Stability: yes

- Best elementary sorting algorithm overall

- Binary insertion sort

Graph Traversal



Many problems require processing all graph vertices (and edges) in systematic fashion

Graph traversal algorithms:

- Depth-first search (DFS)
- Breadth-first search (BFS)

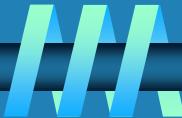
Depth-First Search (DFS)



- Visits graph's vertices by always moving away from last visited vertex to an unvisited one, backtracks if no adjacent unvisited vertex is available.
- Recurisve or it uses a stack
 - a vertex is pushed onto the stack when it's reached for the first time
 - a vertex is popped off the stack when it becomes a dead end, i.e., when there is no adjacent unvisited vertex
- “Redraws” graph in tree-like fashion (with tree edges and back edges for undirected graph)



Pseudocode of DFS

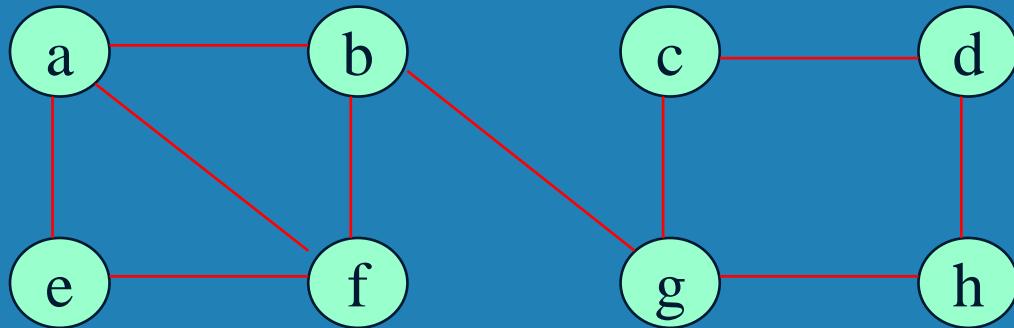


ALGORITHM $DFS(G)$

```
//Implements a depth-first search traversal of a given graph
//Input: Graph  $G = \langle V, E \rangle$ 
//Output: Graph  $G$  with its vertices marked with consecutive integers
//in the order they've been first encountered by the DFS traversal
mark each vertex in  $V$  with 0 as a mark of being "unvisited"
 $count \leftarrow 0$ 
for each vertex  $v$  in  $V$  do
    if  $v$  is marked with 0
         $dfs(v)$ 

 $dfs(v)$ 
//visits recursively all the unvisited vertices connected to vertex  $v$  by a path
//and numbers them in the order they are encountered
//via global variable  $count$ 
 $count \leftarrow count + 1$ ; mark  $v$  with  $count$ 
for each vertex  $w$  in  $V$  adjacent to  $v$  do
    if  $w$  is marked with 0
         $dfs(w)$ 
```

Example: DFS traversal of undirected graph

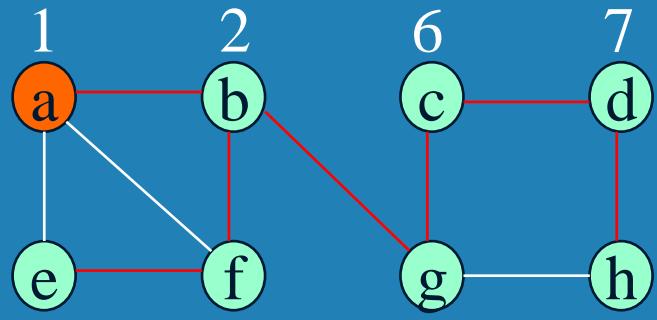


DFS traversal stack:

a
ab
abf
abfe
abf
ab
abg
abgc
abgcd
abgcdh
abgcd

:

DFS tree:



Red edges are tree edges and white edges are back edges.

Notes on DFS



- **DFS can be implemented with graphs represented as:**
 - adjacency matrices: $\Theta(|V|^2)$. Why?
 - adjacency lists: $\Theta(|V|+|E|)$. Why?
- **Yields two distinct ordering of vertices:**
 - order in which vertices are first encountered (pushed onto stack)
 - order in which vertices become dead-ends (popped off stack)
- **Applications:**
 - checking connectivity, finding connected components
 - checking acyclicity (if no back edges)
 - finding articulation points and biconnected components
 - searching the state-space of problems for solutions (in AI)

Breadth-first search (BFS)



- Visits graph vertices by moving across to all the neighbors of the last visited vertex
- Instead of a stack, BFS uses a queue
- Similar to level-by-level tree traversal
- “Redraws” graph in tree-like fashion (with tree edges and cross edges for undirected graph)

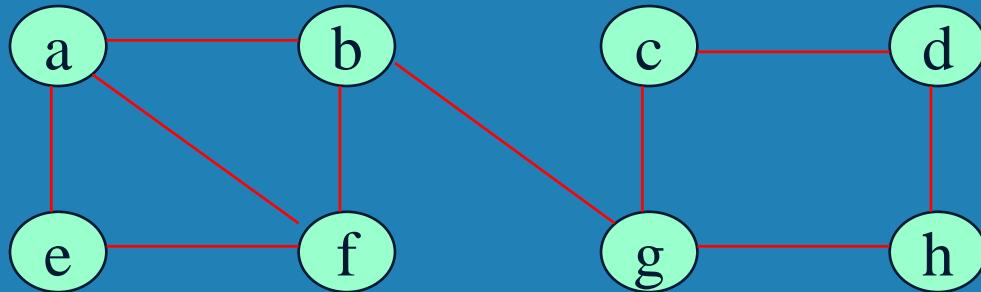
Pseudocode of BFS



ALGORITHM $BFS(G)$

```
//Implements a breadth-first search traversal of a given graph
//Input: Graph  $G = \{V, E\}$ 
//Output: Graph  $G$  with its vertices marked with consecutive integers
//in the order they have been visited by the BFS traversal
mark each vertex in  $V$  with 0 as a mark of being "unvisited"
count  $\leftarrow 0$ 
for each vertex  $v$  in  $V$  do
    if  $v$  is marked with 0
         $bfs(v)$ 
         $bfs(v)$ 
    //visits all the unvisited vertices connected to vertex  $v$  by a path
    //and assigns them the numbers in the order they are visited
    //via global variable count
    count  $\leftarrow count + 1$ ; mark  $v$  with count and initialize a queue with  $v$ 
while the queue is not empty do
    for each vertex  $w$  in  $V$  adjacent to the front vertex do
        if  $w$  is marked with 0
            count  $\leftarrow count + 1$ ; mark  $w$  with count
            add  $w$  to the queue
    remove the front vertex from the queue
```

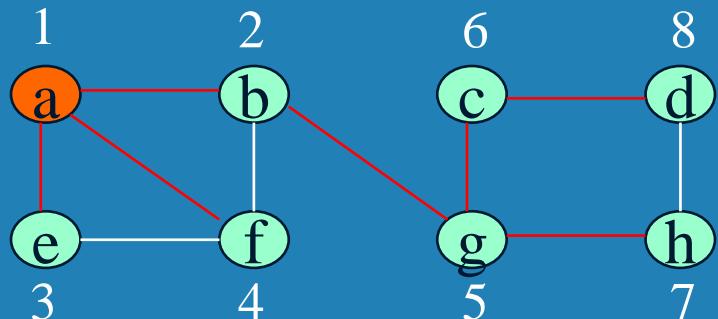
Example of BFS traversal of undirected graph



BFS traversal queue:

a bef efg₀ fg₀ g₀ ch₀ hd₀ d

BFS tree:



Red edges are tree edges and white edges are cross edges.

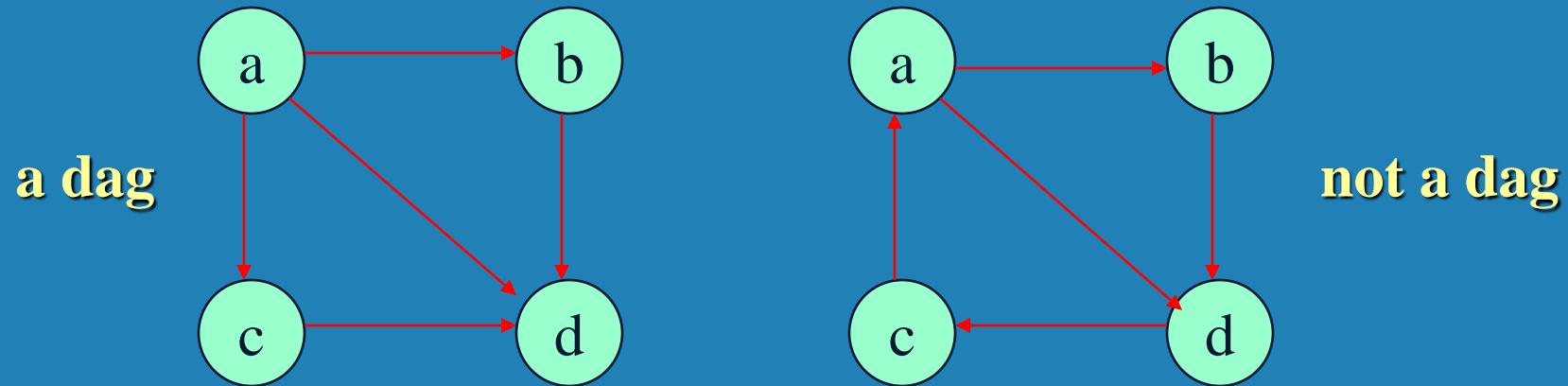
Notes on BFS



- **BFS has same efficiency as DFS and can be implemented with graphs represented as:**
 - adjacency matrices: $\Theta(|V|^2)$. Why?
 - adjacency lists: $\Theta(|V|+|E|)$. Why?
- **Yields single ordering of vertices (order added/deleted from queue is the same)**
- **Applications: same as DFS, but can also find paths from a vertex to all other vertices with the smallest number of edges**

DAGs and Topological Sorting

A dag: a directed acyclic graph, i.e. a directed graph with no (directed) cycles



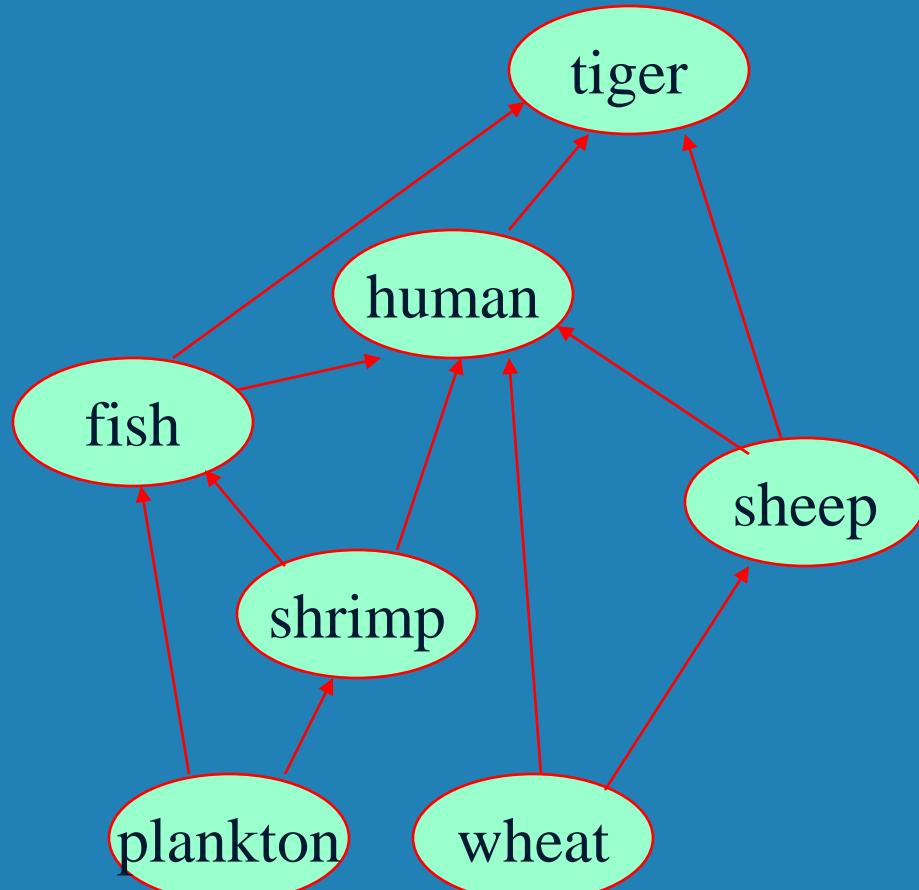
Arise in modeling many problems that involve prerequisite constraints (construction projects, document version control)

Vertices of a dag can be linearly ordered so that for every edge its starting vertex is listed before its ending vertex (topological sorting). Being a dag is also a necessary condition for topological sorting to be possible.

Topological Sorting Example



Order the following items in a food chain



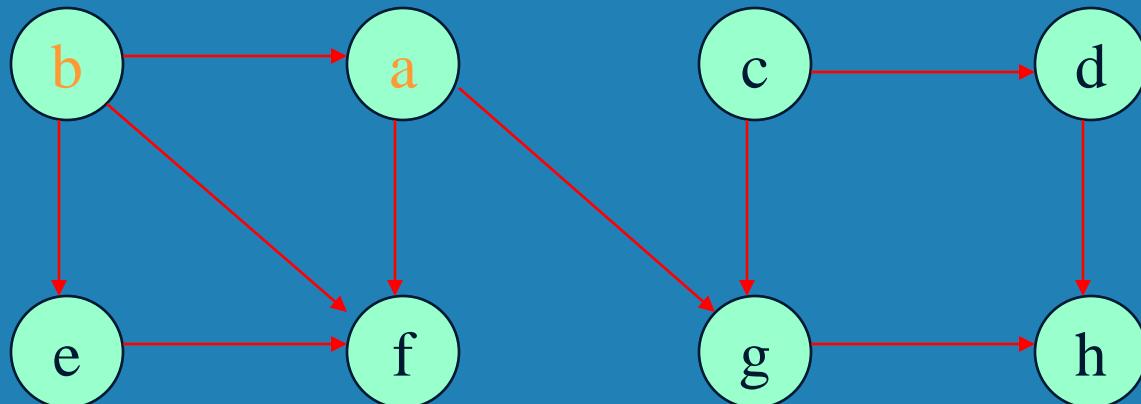
DFS-based Algorithm



DFS-based algorithm for topological sorting

- Perform DFS traversal, noting the order vertices are popped off the traversal stack
- Reverse order solves topological sorting problem
- Back edges encountered? → NOT a dag!

Example:



Efficiency: The same as that of DFS.

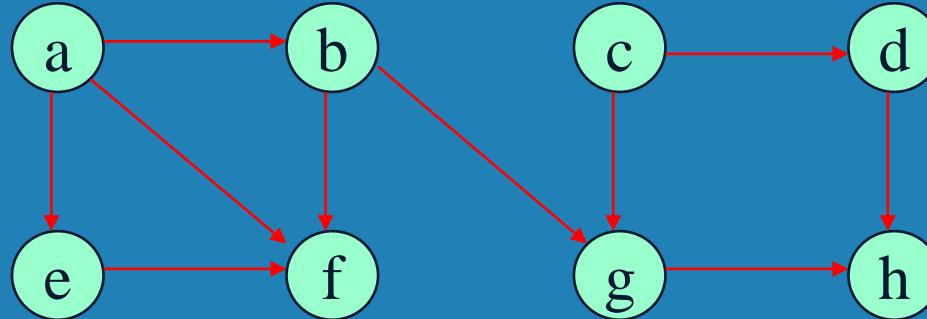
Source Removal Algorithm



Source removal algorithm

Repeatedly identify and remove a *source* (a vertex with no incoming edges) and all the edges incident to it until either no vertex is left or there is no source among the remaining vertices (not a dag)

Example:



Efficiency: same as efficiency of the DFS-based algorithm, but how would you identify a source? How do you remove a source from the dag?

“Invert” the adjacency lists for each vertex to count the number of incoming edges by going thru each adjacency list and counting the number of times that each vertex appears in these lists. To remove a source, decrement the count of each of its neighbors by one.

Decrease-by-Constant-Factor Algorithms



In this variation of decrease-and-conquer, instance size is reduced by the same factor (typically, 2)

Examples:

- **Binary search and the method of bisection**
- **Exponentiation by squaring**
- **Multiplication à la russe (Russian peasant method)**
- **Fake-coin puzzle**
- **Josephus problem**

Exponentiation by Squaring



The problem: Compute a^n where n is a nonnegative integer

The problem can be solved by applying recursively the formulas:

For even values of n

$$a^n = (a^{n/2})^2 \text{ if } n > 0 \text{ and } a^0 = 1$$

For odd values of n

$$a^n = (a^{(n-1)/2})^2 a$$

Recurrence: $M(n) = M(\lfloor n/2 \rfloor) + f(n)$, where $f(n) = 1$ or 2 ,

$$M(0) = 0$$

Master Theorem: $M(n) \in \Theta(\log n) = \Theta(b)$ where $b = \lceil \log_2(n+1) \rceil$



Russian Peasant Multiplication



The problem: Compute the product of two positive integers

Can be solved by a decrease-by-half algorithm based on the following formulas.

For even values of n :

$$n * m = \frac{n}{2} * 2m$$

For odd values of n :

$$n * m = \frac{n-1}{2} * 2m + m \quad \text{if } n > 1 \quad \text{and} \quad m \quad \text{if } n = 1$$

Example of Russian Peasant Multiplication

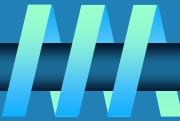


Compute $20 * 26$

n	m	
20	26	
10	52	
5	104	104
2	208	+
1	416	416
		<hr/>
		520

Note: Method reduces to adding m 's values corresponding to odd n 's.

Fake-Coin Puzzle (simpler version)



There are n identically looking coins one of which is fake. There is a balance scale but there are no weights; the scale can tell whether two sets of coins weigh the same and, if not, which of the two sets is heavier (but not by how much, i.e. 3-way comparison). Design an efficient algorithm for detecting the fake coin. Assume that the fake coin is known to be lighter than the genuine ones.

Decrease by factor 2 algorithm

$$T(n) = \log n$$

Decrease by factor 3 algorithm (Q3 on page 187 of Levitin)

$$T(n) \approx \log_3 n$$

Variable-Size-Decrease Algorithms



In the variable-size-decrease variation of decrease-and-conquer, instance size reduction varies from one iteration to another

Examples:

- Euclid's algorithm for greatest common divisor
- Partition-based algorithm for selection problem
- Interpolation search
- Some algorithms on binary search trees
- Nim and Nim-like games



Euclid's Algorithm



Euclid's algorithm is based on repeated application of equality

$$\gcd(m, n) = \gcd(n, m \bmod n)$$

Ex.: $\gcd(80, 44) = \gcd(44, 36) = \gcd(36, 8) = \gcd(8, 4) = \gcd(4, 0) = 4$

One can prove that the size, measured by the first number, decreases at least by half after two consecutive iterations.

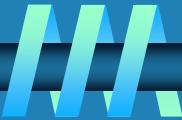
Hence, $T(n) \in O(\log n)$

Proof. Assume $m > n$, and consider m and $m \bmod n$.

Case 1: $n \leq m/2$. $m \bmod n < n \leq m/2$.

Case 2: $n > m/2$. $m \bmod n = m-n < m/2$.

Selection Problem



Find the k -th smallest element in a list of n numbers

- $k = 1$ or $k = n$

- median: $k = \lceil n/2 \rceil$

Example: 4, 1, 10, 9, 7, 12, 8, 2, 15 median = ?

The median is used in statistics as a measure of an average value of a sample. In fact, it is a better (more robust) indicator than the mean, which is used for the same purpose.

Algorithms for the Selection Problem



The sorting-based algorithm: Sort and return the k -th element

Efficiency (if sorted by mergesort): $\Theta(n \log n)$

A faster algorithm is based on using the quicksort-like partition of the list.

Let s be a split position obtained by a partition (using some pivot):



Assuming that the list is indexed from 1 to n :

If $s = k$, the problem is solved;

if $s > k$, look for the k -th smallest element in the left part;

if $s < k$, look for the $(k-s)$ -th smallest element in the right part.

Note: The algorithm can simply continue until $s = k$.

Tracing the Median / Selection Algorithm



Example: 4 1 10 9 7 12 8 2 15 Here: $n = 9, k = \lceil 9/2 \rceil = 5$

array index	1	2	3	4	5	6	7	8	9
	<u>4</u>	<u>1</u>	10	9	7	12	8	2	15
	<u>4</u>	1	2	9	7	12	8	10	15
	2	1	<u>4</u>	9	7	12	8	10	15
				<u>9</u>	7	12	8	10	15
				<u>9</u>	7	8	12	10	15
				<u>8</u>	7	<u>9</u>	12	10	15
				<u>8</u>	7				
				7	<u>8</u>				

Solution: median is 8

Efficiency of the Partition-based Algorithm



Average case (average split in the middle):

$$C(n) = C(n/2) + (n+1) \quad C(n) \in \Theta(n)$$

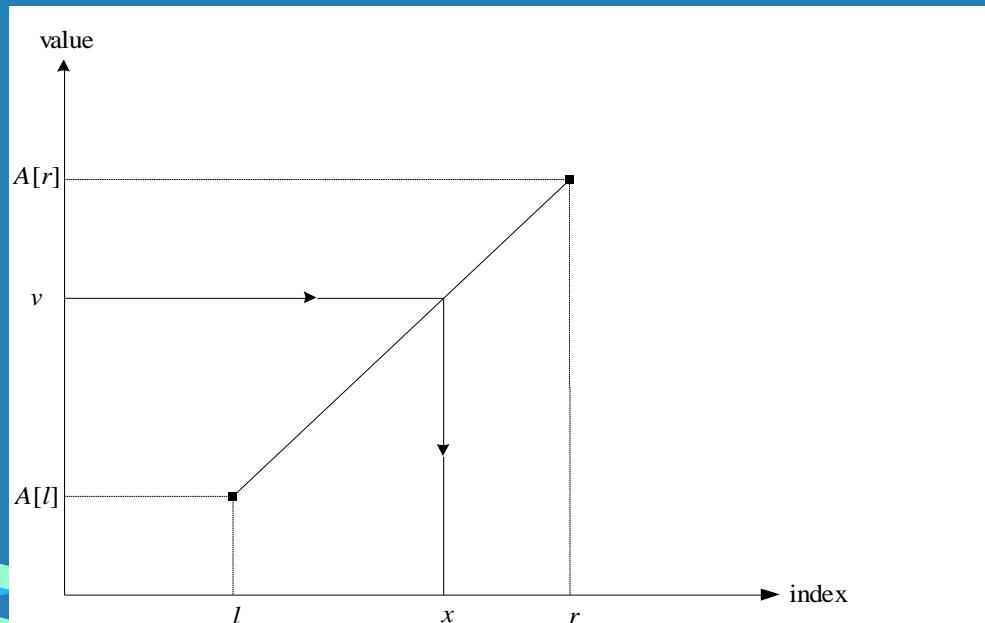
Worst case (degenerate split): $C(n) \in \Theta(n^2)$

A more sophisticated choice of the pivot leads to a complicated algorithm with $\Theta(n)$ worst-case efficiency. Details can be found in CLRS, Ch 9.3.

Interpolation Search



Searches a sorted array similar to binary search but estimates location of the search key in $A[l..r]$ by using its value v . Specifically, the values of the array's elements are assumed to grow linearly from $A[l]$ to $A[r]$ and the location of v is estimated as the x -coordinate of the point on the straight line through $(l, A[l])$ and $(r, A[r])$ whose y -coordinate is v :



$$x = l + \lfloor (v - A[l])(r - l) / (A[r] - A[l]) \rfloor$$

Analysis of Interpolation Search



- **Efficiency**

average case: $C(n) < \log_2 \log_2 n + 1$ (from “rounding errors”)

worst case: $C(n) = n$

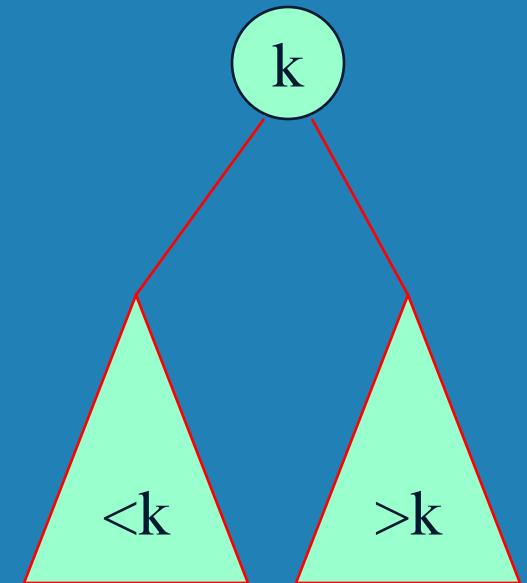
- Preferable to binary search only for VERY large arrays and/or expensive comparisons
- Has a counterpart, the method of false position (regula falsi), for solving equations in one unknown (Sec. 12.4)

Binary Search Tree Algorithms



Several algorithms on BST requires recursive processing of just one of its subtrees, e.g.,

- Searching
- Insertion of a new key
- Finding the smallest (or the largest) key



Searching in Binary Search Tree



Algorithm $BST(x, v)$

```
//Searches for node with key equal to  $v$  in BST rooted at node  $x$ 
if  $x = \text{NIL}$  return -1
else if  $v = K(x)$  return  $x$ 
else if  $v < K(x)$  return  $BST(\text{left}(x), v)$ 
else return  $BST(\text{right}(x), v)$ 
```

Efficiency

worst case: $C(n) = n$

average case: $C(n) \approx 2\ln n \approx 1.39\log_2 n$, if the BST was built from n random keys and v is chosen randomly.

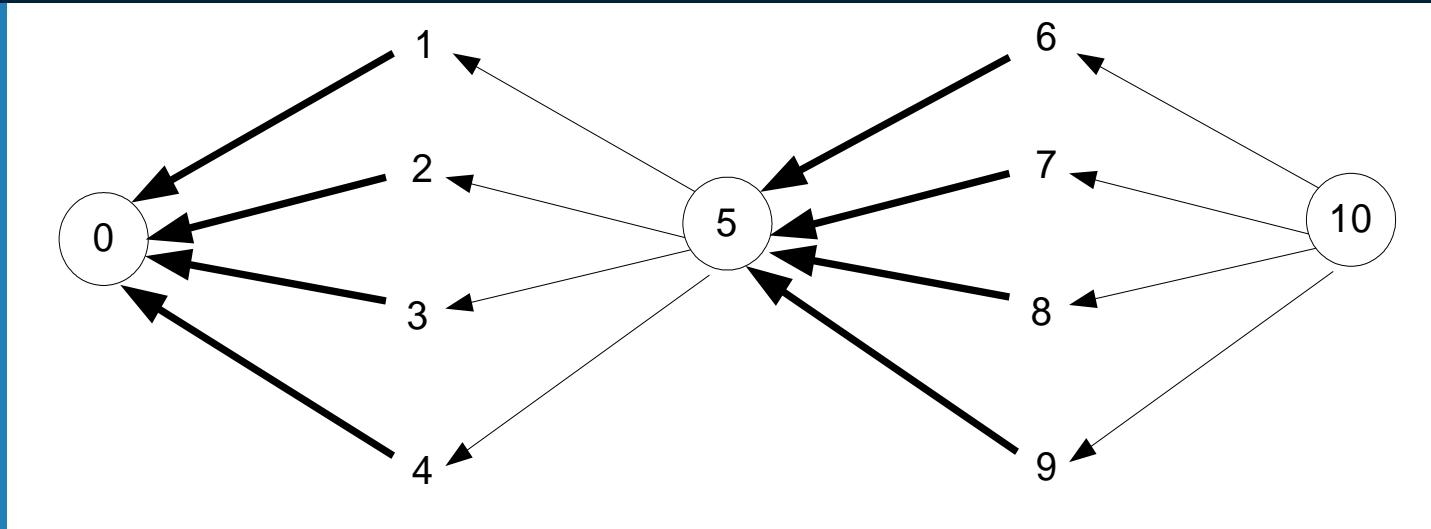
One-Pile Nim



There is a pile of n chips. Two players take turn by removing from the pile at least 1 and at most m chips. (The number of chips taken can vary from move to move.) The winner is the player that takes the last chip. Who wins the game – the player moving first or second, if both player make the best moves possible?

It's a good idea to analyze this and similar games “backwards”, i.e., starting with $n = 0, 1, 2, \dots$

Partial Graph of One-Pile Nim with $m = 4$



Vertex numbers indicate n , the number of chips in the pile. The losing positions for the player to move are circled. Only winning moves from a winning position are shown (in bold).

Generalization: The player moving first wins iff n is not a multiple of 5 (more generally, $m+1$); the winning move is to take $n \bmod 5$ ($n \bmod (m+1)$) chips on every move.