Data Structures and Algorithms

Connecting People

Acknowledgement

- The contents of these slides have origin from School of Computing, National University of Singapore.
- We greatly appreciate support from Dr. Steven Halim for kindly sharing these materials.

Policies for students

- These contents are only used for students PERSONALLY.
- Students are NOT allowed to modify or deliver these contents to anywhere or anyone for any purpose.

Recording of modifications

- Currently, there are no modification on these contents.

Outline

Minimum Spanning Tree (MST), CP3 Section 4.3

- Motivating Example & Some Definitions

Two Algorithms to solve MST (you have a choice!)

- Prim's (greedy algorithm with PriorityQueue)
  - PriorityQueue is discussed in Lecture 03-04 (not just Lecture 02)
- Kruskal's (greedy algorithm, uses sorting and UFDS)
  - UFDS is discussed in Lecture 05

Review

Definitions that we have learned before

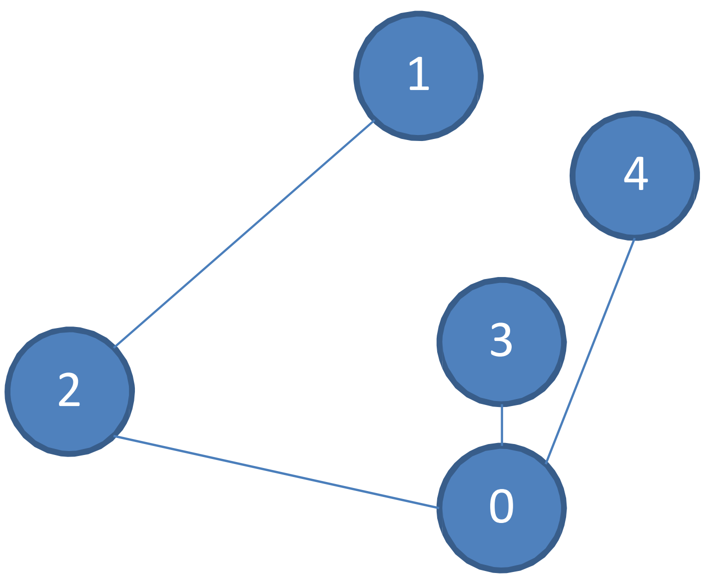
- **Tree T**
  - **T** is a **connected graph** that has **V** vertices and **V-1** edges
  - Important: One unique path between any two pair of vertices in **T**
- **Spanning** Tree **ST** of connected graph **G**
  - **ST** is a tree that spans (covers) every vertices in **G**
  - Recall the **BFS and DFS Spanning Tree**

Sorting problem & several sorting algorithms

- Rearrange set of objects so that every pair of objects **(a, b; a < b)** in the final arrangement satisfies that **a** is before **b**

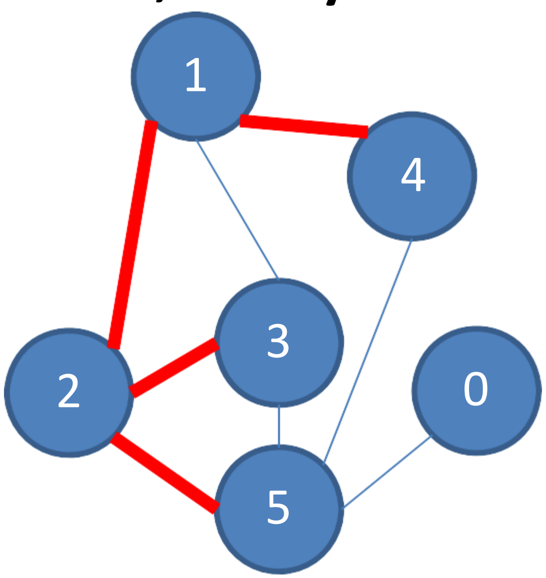
Is This A Tree?

1. Yes, why ______________
2. No, why ______________



Are the edges highlighted in red part of a spanning tree of the original graph?

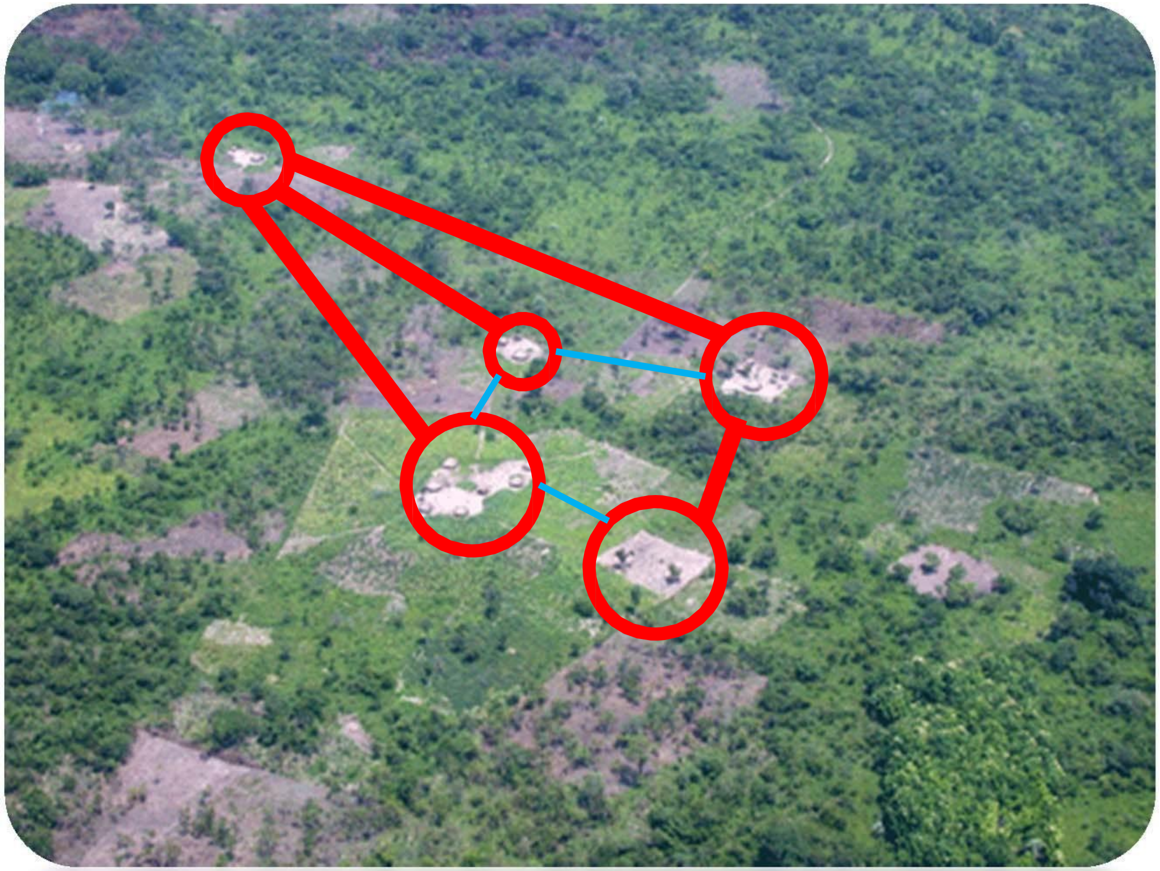
1. Yes, why ____________
2. No, why ____________



Motivating Example

Government Project

- Want to link rural villages with roads
- The cost to build a road depends on the terrain, etc
- You only have limited budget
- How are you going to build the roads?



More Definitions (1)

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - Generally undirected (e.g. bidirectional road, etc)
  - Weighted (e.g. distance, time, toll, etc)
- Weight function **w(a, b): E→ R**
  - Sets the weight of edge from **a** to **b**
- **Weighted** Graph: **G(V, E), w(a, b): E→ R**
- **Connected** undirected graph **G**
  - There is a path from any vertex **a** to any other vertex **b** in **G**

More Definitions (2)

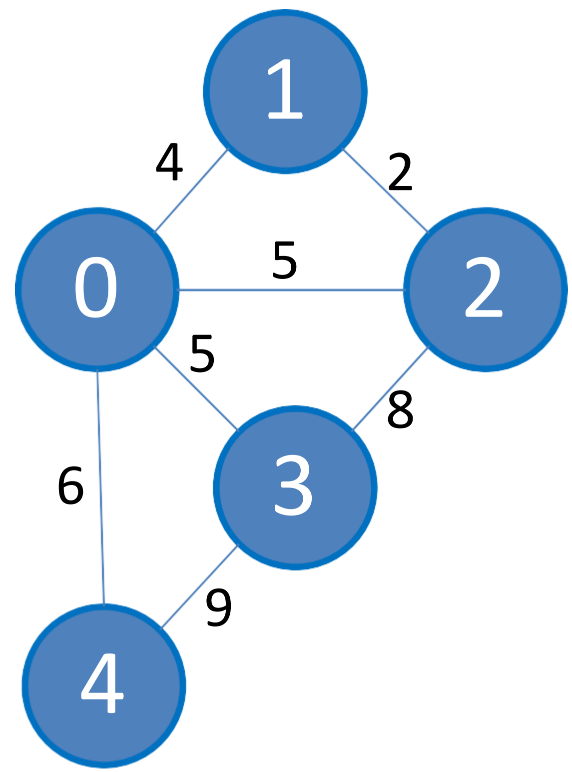
- Spanning Tree **ST** of **G**
  - Let **w(ST)** denotes the total weight of edges in **ST**

$$w(ST) = \sum_{(a,b)\in ST} w(a, b)$$

- **Minimum Spanning Tree (MST)** of connected undirected weighted graph **G**
  - **MST** of **G** is an **ST** of **G** with the minimum possible **w(ST)**

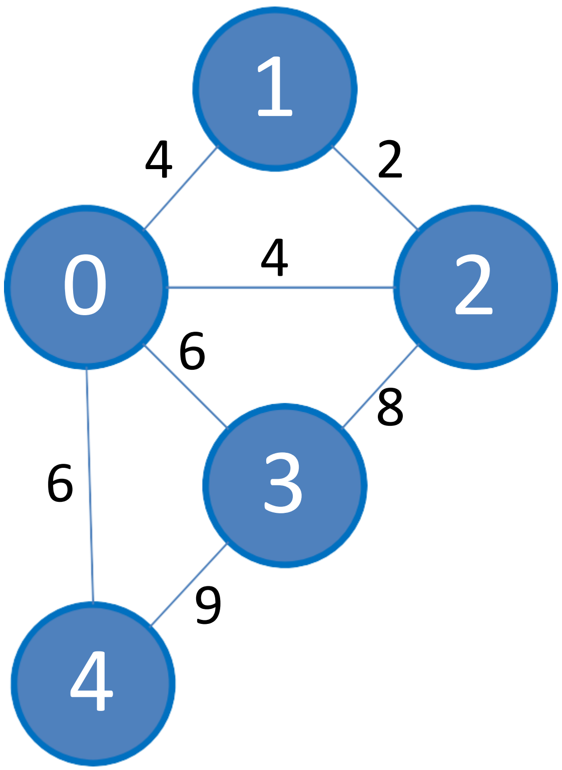
More Definitions (3)

- **The (standard) MST Problem**
  - Input: A connected undirected weighted graph **G(V, E)**
  - Select some edges of **G** such that the graph is still connected, but with minimum total weight
  - Output: Minimum Spanning Tree **(MST)** of **G**

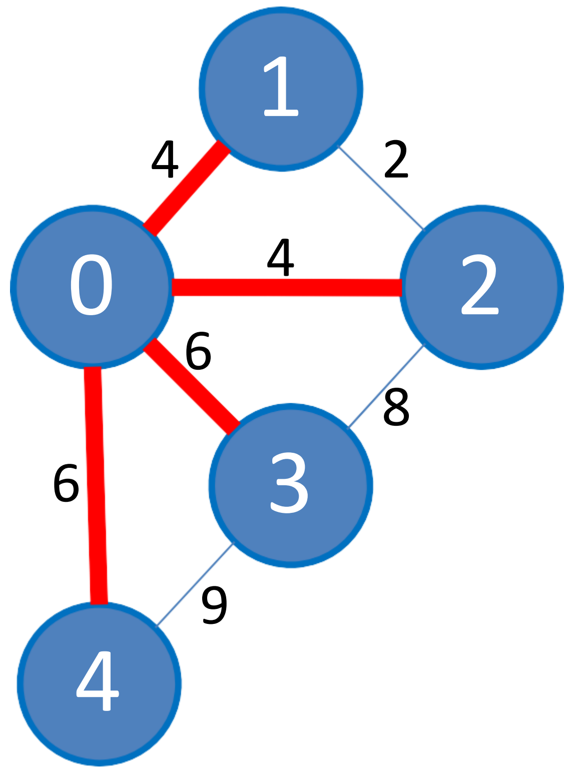


Example

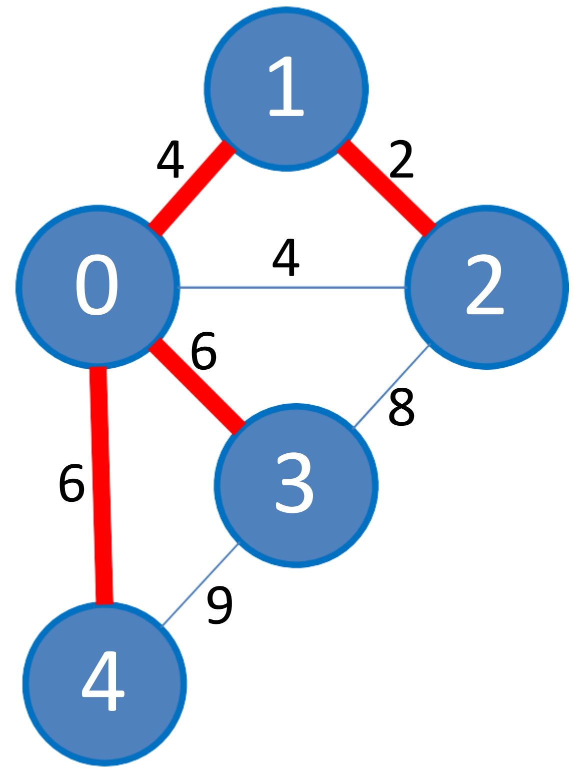
The Original Graph



A Spanning Tree
Cost: 4+4+6+6 = 20

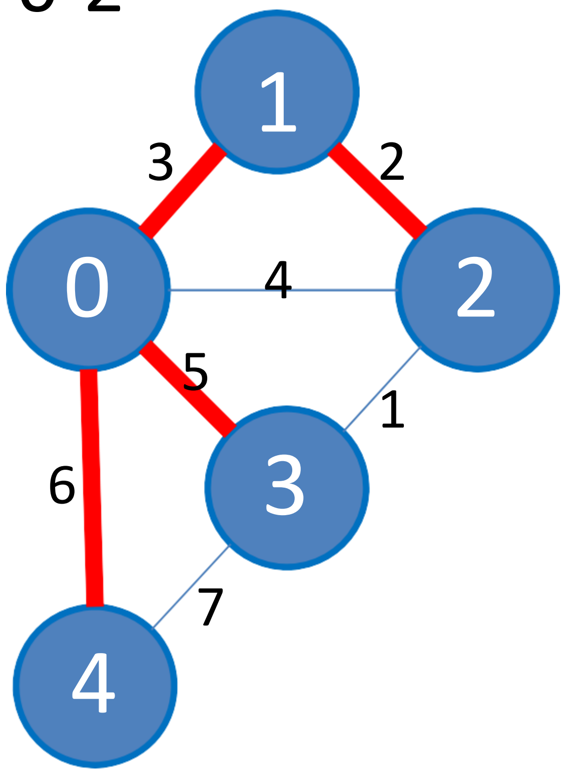


An MST
Cost: 4+6+6+2 = 18



Are the edges highlighted in red part of an MST of the original graph?

1. No, we must replace edge 0-3 with edge 2-3
2. No, we must replace edge 1-2 with 0-2
3. Yes



MST Algorithms

MST is a well-known Computer Science problem

Several efficient (polynomial) algorithms:

- Jarnik's/Prim's greedy algorithm
  - Uses PriorityQueue Data Structure taught in Lecture 02-04
- Kruskal's greedy algorithm
  - Uses Union-Find Data Structure taught in Lecture 05
- Boruvka's greedy algorithm (not discussed here)
- And a few more advanced variants/special cases…

Do you still remember Prim’s/Kruskal’s algorithms from CS1231?

1. Yes and I also know how to ***implement*** them
2. Yes, but I have not try implementing them yet
3. I forgot that particular CS1231 material… but I know it exists
4. Eh?? These two algorithms were covered before in CS1231??

Grid MST, ICPC SG Prelim 2015

https://open.kattis.com/problems/gridmst/

https://open.kattis.com/problems/gridmst/statistics

If you know basic MST algorithm…,

you still can**NOT** solve this problem

But you can solve the simplified form when **N** is small ($1 \le$ **N** ≤ 1000)

Prim's Algorithm

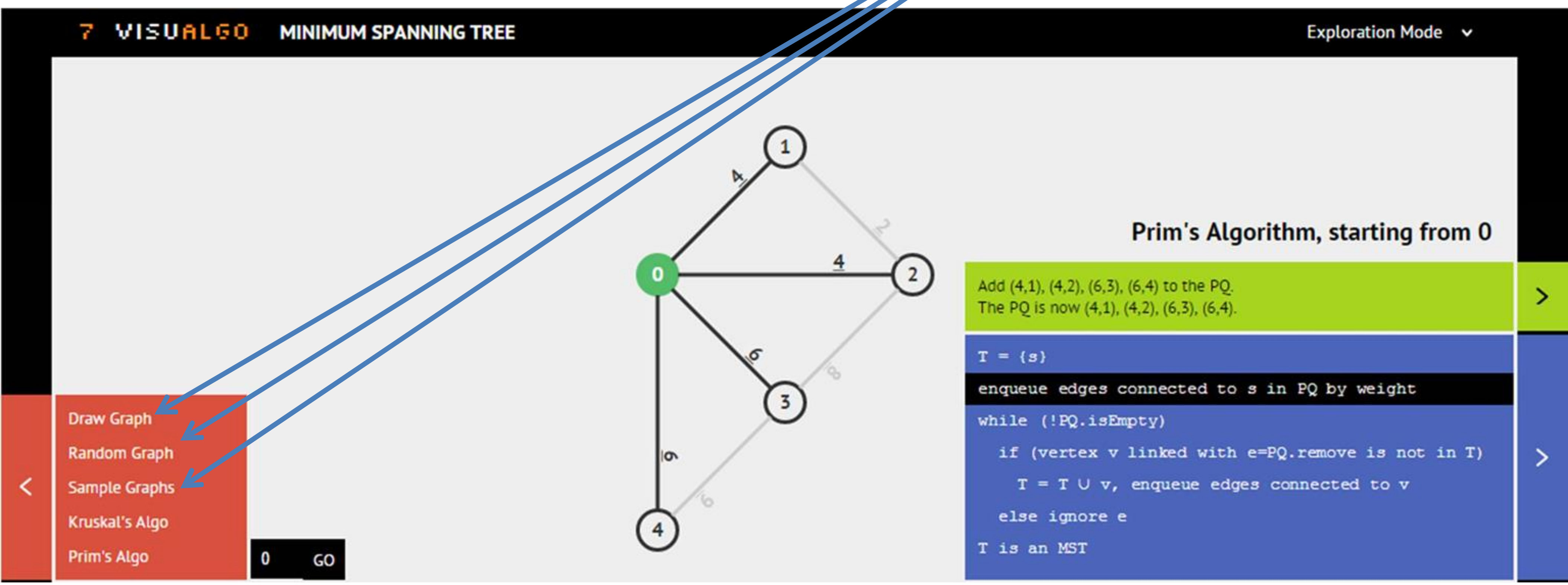
Very simple pseudo code

```
T ← {s}, a starting vertex s (usually vertex 0)
enqueue edges connected to s (only the other ending vertex and edge weight) into a priority queue PQ
  that orders elements based on increasing weight

while there are unprocessed edges left in PQ
  take out front most edge e
  if vertex v linked with this edge e is not taken yet
    T ← T U v (including this edge e)
    enqueue edges connected to v (as above)

T is an MST
```

MST Algorithm: Prim's

Ask VisuAlgo to perform Prim's *from various sources* on the sample Graph (CP3 4.12), *then try other graphs*

In the screen shot below, we show the start of **Prim(0)**



Easy Java Implementation

You just need to use two known Data Structures to be able to implement Prim's algorithm:

1. A priority queue (we can use Java PriorityQueue), and
2. A Boolean array (to decide if a vertex has been taken or not)

With these DSes, we can run Prim's in O(**E** log **V**)

- We process each edge once, O(E)
  - Each time, we Insert/ExtractMax from a PQ in O(log **E**)
  - As $\mathbf{E} = O(\mathbf{V}^2)$, we have $O(\log \mathbf{E}) = O(\log \mathbf{V}^2) = O(2 \log \mathbf{V}) = O(\log \mathbf{V})$

Let's have a quick look at PrimDemo.java

Why Prim’s Works? (1)

First, we have to realize that **Prim’s algorithm** is a **greedy algorithm**

This is because **at each step**, it always try to select the next valid edge e with **minimal weight** (greedy!)

Greedy algorithm is usually simple to implement

- However, it usually requires “proof of correctness”
- You will see such proof like this again in CS3230
- Here, we will just see a quick proof

Why Prim's Works? (2)

see visual explanation in the next two slides

Let T be the spanning tree of graph G generated by Prim's algorithm and T* be the spanning tree of G that is known to have minimal cost

- If T == T*, we are done
- If T != T*
  - Let $e_k = (u, v)$ be the first edge chosen by Prim's algorithm at the *k-th* iteration that **is not** in T*
  - Let *P* be the path from *u* to *v* in *T**, and let *e** be an edge in *P* such that one endpoint is in the tree generated at the (*k−1*)-th iteration of Prim's algorithm and the other is not
    - i.e. one endpoint of *e** is *u* **or** one endpoint is *v*, but the endpoints are not *u* **and** *v*

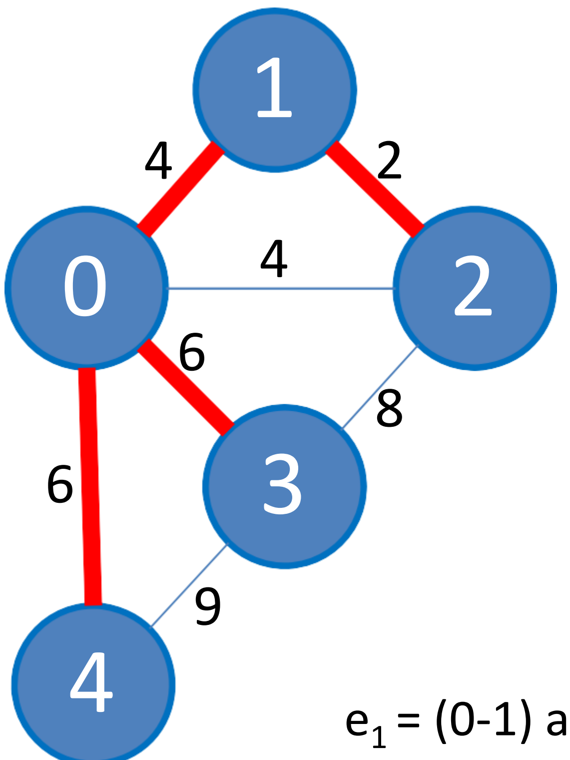
Why Prim’s Works? (3)

see visual explanation in the next slide

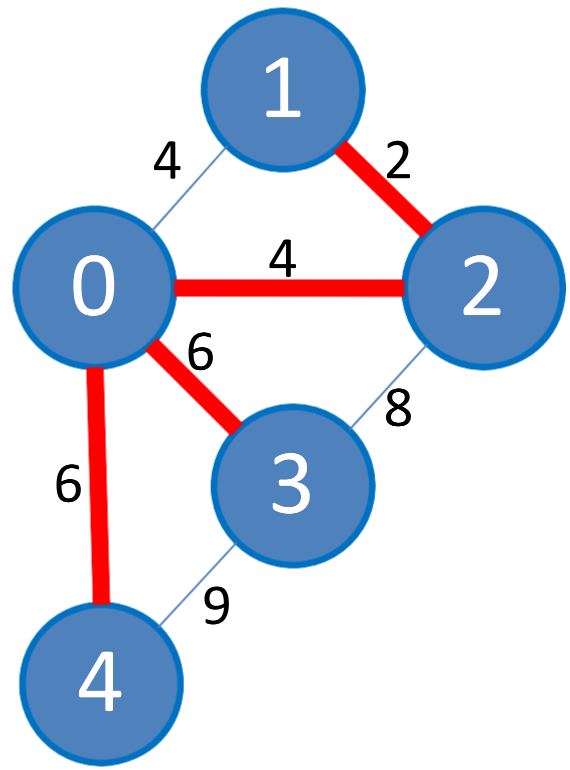
- If T != T* (continued)
  - If the weight of e^* is less than the weight of e_k, then Prim's algorithm would have chosen e^* on its k-th iteration
    - So, it is certain that $w(e^*) \geq w(e_k)$
    - When e^* has weight equal to that of e_k, the choice between the e^* or e_k is arbitrary
    - Whether the weight of e^* is greater than or equal to e_k, e^* can be substituted with e_k while preserving minimal total weight of T^*
  - This process can be repeated until T^* is equal to T
    - Thus we can show that the spanning tree generated by any instance of Prim's algorithm is a minimal spanning tree

Visual Explanation

Our Prim's algorithm reports this MST T



Suppose that this is the optimal MST T*



e_1 = (0-1) at iteration 1

P = **0-2**-1 in T*

e* is (0-2)

If we substitute e_1 with e*, we can transform T to T*

Coming up next: Kruskal’s algorithm

5 MINUTES BREAK

Kruskal’s Algorithm

Very simple pseudo code

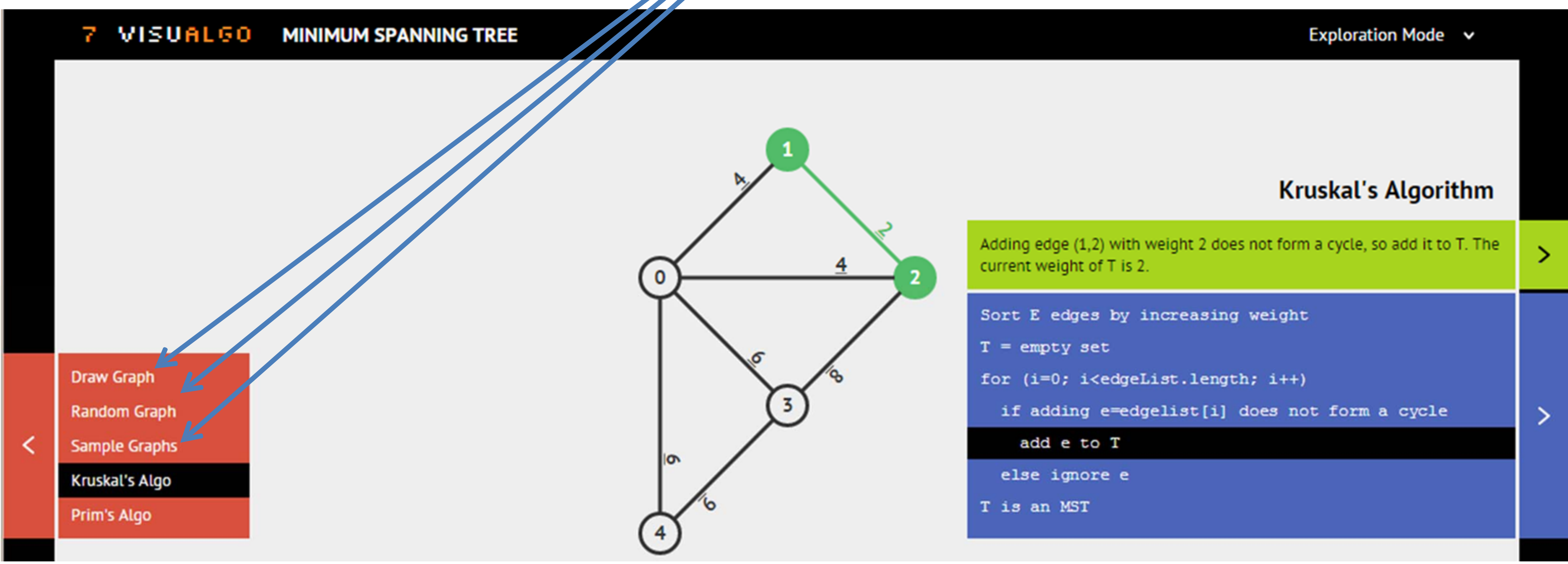
```
sort the set of E edges by increasing
weight  T ?     {}
while there are unprocessed       left
  edges  pick an unprocessed      min cost
  edge with
  if adding e to T does not form a  cycle
    add e to T
T is an MST
```

MST Algorithm: Kruskal's

Ask VisuAlgo to perform Kruskal's on the sample Graph (CP3 4.11), *then try other graphs*

In the screen shot below, we show the start of **Kruskal**

(there is no parameter for this algorithm)



Why Kruskal’s Works? (1)

Kruskal’s algorithm is also a **greedy algorithm**

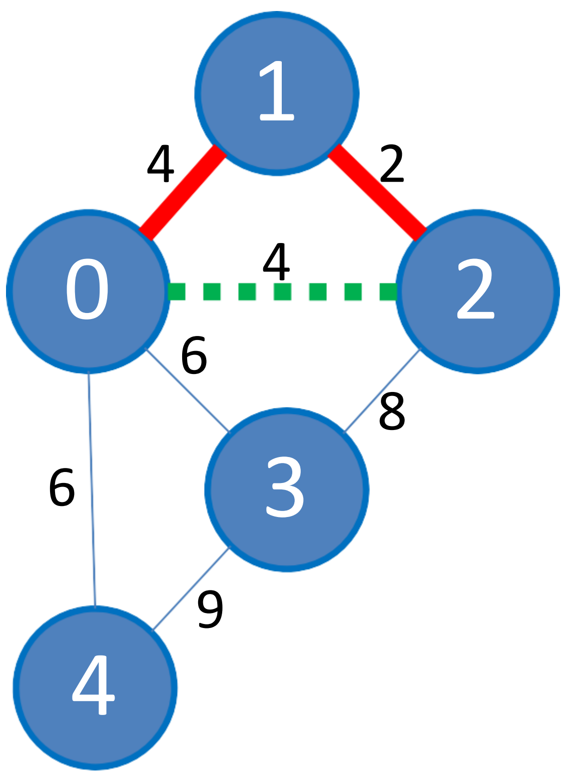
Because **at each step**, it always try to select the next unprocessed edge **e** with **minimal weight** (greedy!)

Simple proof on how this greedy strategy works

- Let’s define a loop invariant: Every edge **e** that is added into **T** by Kruskal’s algorithm is part of the MST

Why Kruskal's Works? (2)

Cannot connect 0 and 2
As it will form a cycle



Loop invariant: Every edge **e** that is added into **T** by Kruskal's algorithm is part of the MST.

```
sort E edges by increasing weight
T ⍰ {}
while there are unprocessed edges left
  pick an unprocessed edge e with min cost
  if adding e to T does not form a cycle
    add e to T
T is an MST
```

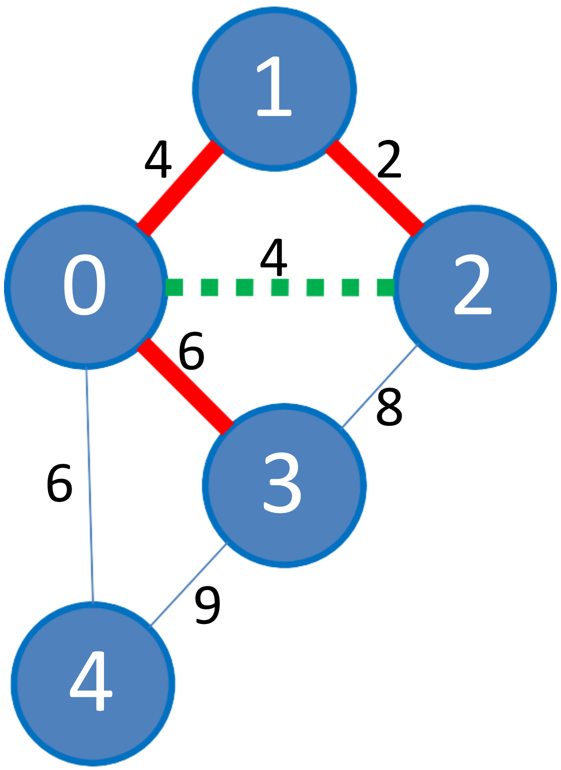
Kruskal's algorithm has a special **cycle check** before adding an edge **e** into **T**. Edge **e** will never form a cycle.

At the start of every loop, **T** is always part of **MST**.

At the end of the loop, we have selected **V-1** edges from a connected weighted graph **G** without having any cycle. This implies that we have a **Spanning Tree**.

Why Kruskal's Works? (3)

Connect 0 and 3
The next smallest edge



Loop invariant: Every edge **e** that is added into **T** by Kruskal's algorithm is part of the MST.

```
sort E edges by increasing weight
T  ?   {}
while there are unprocessed edges left
  pick an unprocessed edge e with min cost
  if adding e to T does not form a cycle
    add e to T
T is an MST
```

By keep adding the next unprocessed edge **e** with min cost, **w(T U e) ≤ w(T U any other unprocessed edge that does not form cycle)**.

At the start of every loop, **T** is always part of **MST**.
At the end of the loop, the Spanning Tree **T** must have minimal weight **w(T)**, so **T** is the final **MST**.

Kruskal's Implementation (1)

```
sort E edges by increasing weight // O(E log E)
T ?
while {there are unprocessed edges left // O(E)
} pick an unprocessed edge e with min cost O(1)
  if adding e to T      does not  a cycle // O(?)
    add e to the form O(T1)
T is an MST
```

To sort the edges:

- We use **EdgeList** to store graph information
- Then use "any" sorting algorithm that we have seen before

To test for cycles:

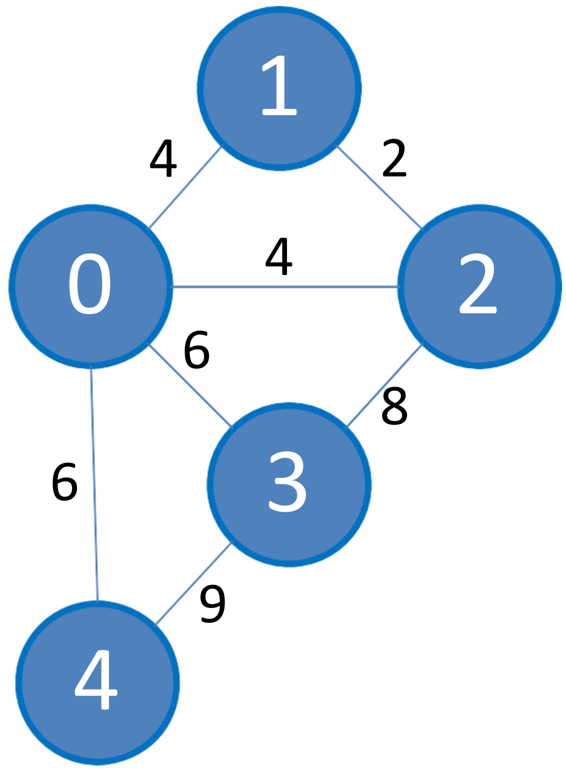
- We use **Union-Find Disjoint Sets**

Sorting Edges in Edge List

Adjacency Matrix/List that we have learned previously are *not suitable* for edge-sorting task!

To sort **EdgeList**, we use ***one liner* Java Collections.sort :O**

- Yeah, you don't have to use merge/quick sort in CS1020… :O



i	w	u	v
0	2	1	2
1	4	0	1
2	4	0	2
3	6	0	3
4	6	0	4
5	8	2	3
6	9	3	4

Kruskal's Implementation (2)

```
sort E edges by increasing weight // O(E log E)
T ⍰
while {there are unprocessed edges left // O(E)
} pick an unprocessed edge e with min cost // O(1)
  if adding e to T does not form a cycle // O(α(V)) = O(1)
    add e to the T // O(1)
T is an MST
```

To sort the edges, we need O(**E** log **E**)

To test for cycles, we need O(**α(V)**) – small, assume constant O(**1**)

In overall

- Kruskal's runs in O(**E** log **E** + ~~**E α(V)**~~) // E log E dominates!
- As **E** = O(**V**²), thus Kruskal's runs in O(**E** log **V**²) = O(**E log V**)

Let's have a quick look at KruskalDemo.java

If given an MST problem, I will…

1. Use/code Kruskal’s algorithm
2. Use/code Prim’s algorithm
3. No preference…

Summary

Re-introducing the MST problem (covered in CS1231)

Discussing the implementation of Prim's algorithm

- Revisiting the PriorityQueue ADT

Discussing the implementation of Kruskal's algorithm

- Revisiting the EdgeList and showing technique to sort edges
- Revisiting the Union-Find Disjoint Sets DS

You *may* learn MST/Prim's/Kruskal's again in CS3230

PS4 should now be doable 😊

It will not due until Sat, 17 Oct 2015, 8am, because I want to give you time off from CS2010 in Week 07

But it is always better to attempt it earlier than later

Reminder to self (re-discuss TopoSort if still have time)