

MA1521 CALCULUS FOR COMPUTING

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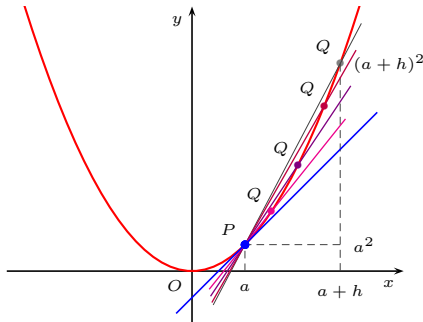
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Chapter 2: Derivatives with Applications

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The Tangent Line

- Recall that in Chapter 1 we have seen how to find the tangent line of the curve $y = x^2$ at $P(a, a^2)$:



$$\begin{aligned} m_{PQ} &= \frac{\Delta y}{\Delta x} \\ &= \frac{(a+h)^2 - a^2}{h} \end{aligned}$$

- The **slope** of the tangent line can be written as

$$m := \lim_{h \rightarrow 0} m_{PQ} = \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h}.$$

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Definition of Derivative

- The **derivative of a function f at a number a** is

$$f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

- f is **differentiable at a** if $f'(a)$ exists.
- $f'(a)$ is the **slope** of $y = f(x)$ at $x = a$.
- Let $x = a + h$. Then $h = x - a$, and $h \rightarrow 0 \Leftrightarrow x \rightarrow a$. We may use an equivalent definition:

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

- The **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)(x - a).$$

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Examples of Derivatives

- Let $f(x) = x^2 - 8x + 9$. Find $f'(3)$.

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(3+h)^2 - 8(3+h) + 9] - (3^2 - 8 \cdot 3 + 9)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(-6 - 2h + h^2) - (-6)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2h + h^2}{h} = \lim_{h \rightarrow 0} (-2 + h) = -2.
 \end{aligned}$$

- The tangent line of $y = f(x)$ passing through $(3, -6)$:

$$y - (-6) = f'(3)(x - 3) = -2(x - 3).$$

That is, $2x + y = 0$.

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Examples of Derivatives

- Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$ Find $f'(0)$.

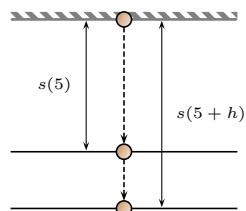
$$\begin{aligned}
 f'(0) &= \lim_{h \rightarrow 0} \frac{f(h+0) - f(0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\
 &= \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right) \\
 &= 0.
 \end{aligned}$$

$$\begin{array}{ccccc}
 \text{Note that} & -|h| & \leq & h \sin\left(\frac{1}{h}\right) & \leq & |h| & \text{for all } h \neq 0. \\
 h \rightarrow 0 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \Rightarrow & 0 & \Leftarrow & 0 & \\
 & & & \text{(Squeeze Theorem)} & & &
 \end{array}$$

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Velocity

- Let $s = s(t)$ be the position function of a particle.
 - instantaneous velocity** at time $t = a$: $s'(a)$;
 - speed** at time $t = a$: $|s'(a)|$.
- Example.** A ball is dropped from a tower 450m above the ground. Find its velocity after 5 seconds.



$$s(t) = \frac{1}{2}gt^2 = 4.9t^2.$$

$$v(5) = s'(5) = 49.$$

$$\frac{\Delta s}{\Delta t} = \frac{4.9(5+h)^2 - 4.9(5)^2}{(5+h) - 5} = \frac{49h + 4.9h^2}{h}$$

$$\text{Velocity at } t = 5: \lim_{h \rightarrow 0} \frac{49h + 4.9h^2}{h} = 49 \text{ m/s.}$$

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Example

- Let $f(x) = \frac{1}{x}$. Find $f'(a)$ at each $a \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{(a+h)a}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(a+h)a} \\ &= -\frac{1}{a^2}. \end{aligned}$$

- f' is therefore a function defined on $\mathbb{R} \setminus \{0\}$.

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Derivative as a Function

- The **derivative** of f at point $x = a$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

- The **derivative** of f as a function:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

$$\circ \quad f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D_x f(x) = \dots$$

$$\circ \quad \boxed{\frac{dy}{dx} := \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}} \quad (\text{Leibniz, 1646–1716, German})$$

$$\circ \quad f'(a) = \left. \frac{dy}{dx} \right|_{x=a}.$$

- Example.** If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$.

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Examples

- Let $f(x) = \frac{1-x}{2+x}$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-3h}{h(2+(x+h))(2+x)} \\ &= \lim_{h \rightarrow 0} \frac{-3}{(2+(x+h))(2+x)} \\ &= \frac{-3}{(2+x)^2}. \end{aligned}$$

- Domain of f : $\mathbb{R} \setminus \{-2\}$; Domain of f' : $\mathbb{R} \setminus \{-2\}$.

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Examples

- Let $f(x) = \sqrt{x}$, ($x \geq 0$). Find f' .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (x > 0). \end{aligned}$$

The domain of f' may be *smaller* than the domain of f .

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Differentiable Functions

- Definition.** (We only consider the differentiability at a point or on **open intervals**)

- f is **differentiable at** a if

$$f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

- f is **differentiable on** (a, b) if it is differentiable at **every** $c \in (a, b)$.

- Questions.**

- What's the relation between differentiability and continuity?
- What kinds of functions are differentiable?
- How to construct new differentiable functions?
- How the derivative affects the original function?

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Differentiability Implies Continuity

- **Theorem.** If f is **differentiable** at a , then f is **continuous** at a .

- **Remark.** The converse of the theorem is false.

- $f(x) = |x|$ is continuous at 0, not differentiable at 0.

- **Proof.** Suppose f is differentiable at a .

That is, $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = L$.

- $\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = L \cdot 0 = 0$.

- $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [(f(x) - f(a)) + f(a)] = 0 + f(a) = f(a)$.

Therefore, f is continuous at a .

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Differentiation Formulas

- Let c be a constant. $(c)' = 0$.

- $\frac{d}{dx}(c) = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$.

- Let f be a differentiable function, and c be a constant.

$$\begin{aligned}(cf)'(x) &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\&= \lim_{h \rightarrow 0} c \left[\frac{f(x+h) - f(x)}{h} \right] \\&= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= cf'(x).\end{aligned}$$

$$\therefore (cf)' = cf'.$$

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Differentiation Formulas

- Let f and g be differentiable functions.

$$\begin{aligned}
 (f + g)'(x) &= \lim_{h \rightarrow 0} \frac{(f + g)(x + h) - (f + g)(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[f(x + h) - f(x)] + [g(x + h) - g(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \\
 &= f'(x) + g'(x).
 \end{aligned}$$

$$\therefore (f + g)' = f' + g'.$$

$$\begin{aligned}
 (f - g)' &= [f + (-g)]' = f' + (-g)' = f' + (-g') \\
 &= f' - g'.
 \end{aligned}$$

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Differentiation Formulas

- Let f and g be differentiable functions. What is $(fg)'$?

$$\begin{aligned}
 \frac{d}{dx} [f(x)]^2 &= \lim_{h \rightarrow 0} \frac{[f(x + h)]^2 - [f(x)]^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[f(x + h) - f(x)] \cdot [f(x + h) + f(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} [f(x + h) + f(x)] \\
 &= f'(x) \cdot [f(x) + f(x)] \quad (\because f \text{ is continuous}) \\
 &= 2f'(x)f(x).
 \end{aligned}$$

$$\begin{aligned}
 (fg)' &= \frac{1}{2}[(f + g)^2 - f^2 - g^2]' \\
 &= \frac{1}{2}[2(f + g)'(f + g) - 2f'f - 2g'g] \\
 &= (f' + g')(f + g) - f'f - g'g \\
 &= f'g + fg'.
 \end{aligned}$$

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Differentiation Formulas

- Let f and g be differentiable functions. What is $(f/g)'$?
Suppose $g(x) \neq 0$.

$$\begin{aligned}
 \left(\frac{1}{g}\right)'(x) &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{g}\right)(x+h) - \left(\frac{1}{g}\right)(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{g(x) - g(x+h)}{g(x)g(x+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{g(x+h) - g(x)}{h} \cdot \frac{-1}{g(x)g(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \rightarrow 0} \frac{-1}{g(x)g(x+h)} \\
 &= g'(x) \cdot \frac{-1}{[g(x)]^2} = \left(-\frac{g'}{g^2}\right)(x).
 \end{aligned}$$

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Differentiation Formulas

- Let f and g be differentiable functions. What is $(f/g)'$?
Suppose $g(x) \neq 0$.

$$\begin{aligned}
 \left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' = f' \cdot \left(\frac{1}{g}\right) + f \cdot \left(\frac{1}{g}\right)' \\
 &= \frac{f'}{g} + \frac{f \cdot (-g')}{g^2} \\
 &= \frac{f'g - fg'}{g^2}.
 \end{aligned}$$

- $(cf)' = cf'$
- $(f \pm g)' = f' \pm g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \text{ if } g(x) \neq 0.$

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The Chain Rule

- Let $F(x) = \sqrt{x^2 + 1}$. Find F' .

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 1} - \sqrt{x^2 + 1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{(x+h)^2 + 1} - \sqrt{x^2 + 1})(\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1})}{h(\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1})} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{h(\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1})} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1})} \\
 &= \lim_{h \rightarrow 0} \frac{2x + h}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}} \\
 &= \frac{2x}{2\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}.
 \end{aligned}$$

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The Chain Rule

- Let $F(x) = \sqrt{x^2 + 1}$. Find F' .

Note that $F = f \circ g$, where

- $f(x) = \sqrt{x}$, and $g(x) = x^2 + 1$.

It is known that

- $f'(x) = \frac{1}{2\sqrt{x}}$, and $g'(x) = 2x$.

Question. Can we write F' by making use of f' and g' ?

- If $y = g(x)$, $z = f(y)$, it seems that

$$F'(x) = \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(g(x))g'(x) = \frac{2x}{2\sqrt{x^2 + 1}}.$$

Question. Can we always differentiate the composite of differentiable functions using this method?

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The Chain Rule

- **Theorem.** Let f and g be differentiable functions.
Then $F = f \circ g$ is differentiable and

$$F' = (f' \circ g)(g').$$

More precisely,

if g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and

$$F'(a) = f'(b)g'(a) = f'(g(a))g'(a).$$

In Leibniz notation, if $y = g(x)$ and $z = f(y)$, then

$$\left. \frac{dz}{dx} \right|_{x=a} = \left. \frac{dz}{dy} \right|_{y=b} \left. \frac{dy}{dx} \right|_{x=a}, \quad \text{or in short} \quad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}.$$

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Formulas of Derivatives

- $\frac{d}{dx} x^a = ax^{a-1}.$
- $\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x;$
 $\frac{d}{dx} \tan x = \sec^2 x, \quad \frac{d}{dx} \cot x = -\csc^2 x;$
 $\frac{d}{dx} \sec x = \sec x \tan x, \quad \frac{d}{dx} \csc x = -\csc x \cot x.$
- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$
 $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}.$
 $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}.$
- $\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \ln |x| = \frac{1}{x}$

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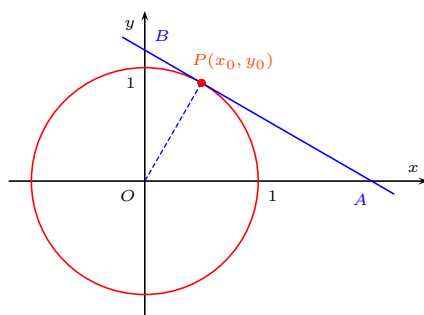
Examples

- $\frac{d}{dx} \ln \frac{\sqrt{x+1}}{e^{2x} \cos^4 5x}.$
 - $\ln \frac{\sqrt{x+1}}{e^{2x} \cos^4 5x} = \frac{1}{2} \ln(x+1) - 2x - 4 \ln |\cos 5x|.$
 - $\frac{d}{dx} \ln \frac{\sqrt{x+1}}{e^{2x} \cos^4 5x} = \frac{1}{2(x+1)} - 2 - \frac{4}{\cos 5x} \cdot (-5 \sin 5x)$
 $= \frac{1}{2(x+1)} - 2 + 20 \tan 5x.$
- $\frac{d}{dx} \left(x \sin^{-1} \frac{1}{x} \right).$
 - $\frac{d}{dx} \left(x \sin^{-1} \frac{1}{x} \right) = 1 \cdot \sin^{-1} \frac{1}{x} + x \cdot \frac{1}{\sqrt{1 - (\frac{1}{x})^2}} \cdot (-x^{-2})$
 $= \sin^{-1} \frac{1}{x} - \frac{1}{x \sqrt{1 - \frac{1}{x^2}}}.$

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Implicit Differentiation

- How to find the **slope** of the tangent line to the unit circle $x^2 + y^2 = 1$ at a point $P(x_0, y_0)$ on the circle?



- $\overline{AB} \perp \overline{OP} \Rightarrow (\text{slope of } \overline{AB})(\text{slope of } \overline{OP}) = -1$

$$y'|_P \cdot \frac{y_0}{x_0} = -1 \Rightarrow \boxed{y'|_P = -\frac{x_0}{y_0}}.$$

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Implicit Differentiation

- How to find the **slope** of the tangent line to the unit circle $x^2 + y^2 = 1$ at a point $P(x_0, y_0)$ on the circle?
 - Given that $1 = x^2 + y^2$.
Differentiate both sides with respect to x .

$$\frac{d}{dx}(1) = \frac{d}{dx}(x^2 + y^2).$$

That is,

$$\begin{aligned} 0 &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x + \frac{dy^2}{dy} \frac{dy}{dx} \\ &= 2x + 2y \frac{dy}{dx}. \end{aligned}$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \Rightarrow \left[\frac{dy}{dx} \right]_P = -\frac{x_0}{y_0}.$$

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Implicit Differentiation

- In general, we may not have a **function** $y = f(x)$.
Instead, it may be an **equation**

$$f(x, y) = 0.$$

We can still find $y' = \frac{dy}{dx}$ as follow:

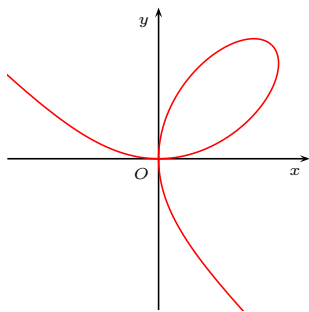
- 1). Differentiate $f(x, y)$ with respect to x ;
 - 2). Solve the equation $\frac{d}{dx}f(x, y) = 0$ to express $\frac{dy}{dx}$ in terms of x and y .
- The method introduced is called **implicit differentiation**.

Remark. In order to use the method of implicit differentiation, we shall first assume that $\frac{dy}{dx}$ exists.

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Examples

- Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$.



- Differentiate

$$x^3 + y^3 = 3xy$$

with respect to x :

$$(x^3)' = 3x^2$$

$$(y^3)' = 3y^2 y'$$

$$(xy)' = x'y + xy' = y + xy'$$

$$3x^2 + 3y^2 y' = 3(y + xy')$$

- Solve y' : $y' = \frac{x^2 - y}{x - y^2}$, $(x, y) \neq (0, 0), (\sqrt[3]{4}, \sqrt[3]{2})$.

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Find the Inverse Function

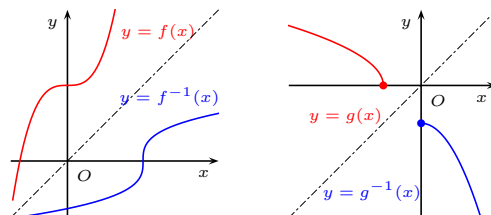
- Let f be a **one to one** function. Then it admits an inverse function f^{-1} . But how to find f^{-1} ?
- Recall that $y = f(x) \Leftrightarrow f^{-1}(y) = x$. We can thus apply the following procedure:
 - Write $y = f(x)$.
 - Solve the equation for x in terms of y : $x = f^{-1}(y)$.
 - Interchanging x and y to express f^{-1} as a function in variable x : $y = f^{-1}(x)$.
- Interchanging x and y has the same effect as the reflection with respect to $y = x$.
 - So the graphs of f and f^{-1} are **symmetric with respect to the straight line $y = x$** .

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Examples

- Find inverse of $f(x) = x^3 + 2$ and $g(x) = \sqrt{-1-x}$.

- Let $y = f(x) = x^3 + 2$.
- Solve x in terms of y : $x = \sqrt[3]{y-2}$.
- Interchange x and y : $f^{-1}(x) = y = \sqrt[3]{x-2}$.
- Let $y = g(x) = \sqrt{-1-x}$, ($x \leq -1$).
- Solve x in terms of y : $x = -y^2 - 1$, ($y \geq 0$).
- Interchange x and y : $g^{-1}(x) = -x^2 - 1$, ($x \geq 0$).



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Calculus of Inverse Functions

- Let f be a continuous function defined on an interval.
 - If f is increasing, then f is one to one.
 - $a < b \Rightarrow f(a) < f(b) \Rightarrow f(a) \neq f(b)$.
 - Similarly, if f is decreasing, then f is one to one.
 - If f is one to one, must f be increasing or decreasing?
- Theorem.** Suppose f is a one-to-one and continuous function defined on an interval. Then
 - f is either increasing or decreasing.
- Theorem.** Suppose f is a one-to-one and continuous function defined on an interval. Then
 - Its inverse function f^{-1} is continuous.

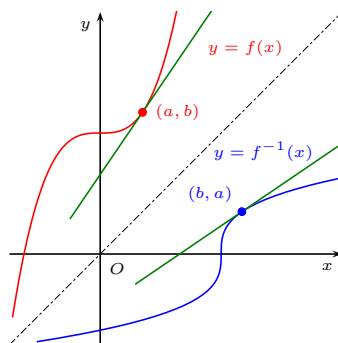
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Calculus of Inverse Functions

- **Theorem.** Let f be a one to one continuous function defined on an interval.

- If f is differentiable at a , and $f'(a) \neq 0$,
- then f^{-1} is differentiable at $b = f(a)$,

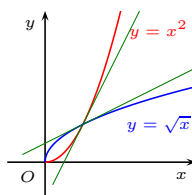
$$\text{and } (f^{-1})'(b) = \frac{1}{f'(a)}.$$



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Example

- Let $f(x) = x^2$ on $[0, 2]$. Find $(f^{-1})'(1)$.
 - Method 1: $(f^{-1})(x) = \sqrt{x}$. Then $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$.
 - $(f^{-1})'(1) = \frac{1}{2\sqrt{x}} \Big|_{x=1} = \frac{1}{2}$.
 - Method 2: $f(1) = 1$, and $f'(x) = 2x$. Then
 - $(f^{-1})'(1) = \frac{1}{f'(1)} = \frac{1}{2}$.



- The advantage of Method 2 is that we can find $(f^{-1})'$ without knowing the explicit expression of f^{-1} .

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Derivative of Trigonometric Functions

- Recall the **inverse sine function** $\sin^{-1} x : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. What is $(\sin^{-1} x)'$?
 - Let $y = \sin^{-1} x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\sin y = x$.
 $\cos y \geq 0 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.
 $\therefore (\sin^{-1} x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}, -1 < x < 1$.
- The **inverse cosine function** $\cos^{-1} x : [-1, 1] \rightarrow [0, \pi]$.
 - Let $y = \cos^{-1} x \in [0, \pi]$. Then $\cos y = x$.
 $\sin y \geq 0 \Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$.
 $\therefore (\cos^{-1} x)' = \frac{1}{(\cos y)'} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - x^2}}, -1 < x < 1$.
- Exercise.** Prove that $(\tan x)' = \frac{1}{1 + x^2}$.

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Second Derivative

- Let f be a function. We can **differentiate** it to get f' .
- f' is a function, we can **differentiate** it to get $(f')'$.
 - $f'' := (f')'$, is called the **second derivative** of f .
 - By Leibniz notation:

$$f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}.$$

- $f' = D(f) \Rightarrow f'' := D^2(f)$.
- Geometric meaning:**
 - f' measures the change of f ,
 - f'' measures the change of f' .

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Physical Meaning of Second Derivative

- Let $s = s(t)$ be the position function of an object along a straight line.
 - $s'(t) = v(t)$: the **instantaneous velocity**, it determines the change of the **position**,
 - $s''(t) = v'(t) = a(t)$: the **acceleration**, it determines the change of the **velocity**.
- Example.** Suppose the position of a particle is given by

$$s = s(t) = t^3 - 6t^2 + 9t.$$

- Velocity:** $v(t) = s'(t) = 3t^2 - 12t + 9.$
- Acceleration:** $a(t) = s''(t) = v'(t) = 6t - 12.$

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Higher Derivatives

- Let f be a function.
 - Differentiate f to get f' , the **first derivative**.
 - Differentiate f' to get f'' , the **second derivative**.
 - Differentiate f'' to get f''' , the **third derivative**.
 - Differentiate f''' to get f'''' , the **fourth derivative**.
 -
- In general, we define $f^{(0)} := f$, and for positive integer n ,

$$f^{(n)} := (f^{(n-1)})',$$

called the **n th derivative** of f .

Other notations: if $y = f(x)$, then

$$f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x).$$

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Examples

- Let $f(x) = x \cos x$. Find f' , f'' and f''' .

$$\begin{aligned}f'(x) &= (x \cos x)' = (x)' \cos x + x(\cos x)' \\&= \cos x - x \sin x.\end{aligned}$$

$$\begin{aligned}f''(x) &= (\cos x - x \sin x)' = (\cos x)' - (x \sin x)' \\&= -\sin x - [(x)' \sin x + x(\sin x)'] \\&= -\sin x - \sin x - x \cos x \\&= -2 \sin x - x \cos x.\end{aligned}$$

$$\begin{aligned}f'''(x) &= (-2 \sin x - x \cos x)' = -2(\sin x)' - (x \cos x)' \\&= -2 \cos x - [(x)' \cos x + x(\cos x)'] \\&= -2 \cos x - (\cos x - x \sin x) \\&= -3 \cos x + x \sin x.\end{aligned}$$

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Parametric Equations

- Let x and y be functions of variable t .

- $x = x(t)$ and $y = y(t)$.

This is a **parametric equation** of x and y .

- Examples.**

- $x^2 + y^2 = 1$ can be parameterized as

- $x = \cos t$ and $y = \sin t, t \in \mathbb{R}$.

- $y^2 = x^3 + x^2$ can be parameterized as

- $x = t^2 - 1$ and $y = t^3 - t, t \in \mathbb{R}$.

- Question.** Given a parametric equation, $x = x(t)$ and $y = y(t)$, how to find the derivative of y with respect to x ?

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Parametric Equations

- Suppose $x = x(t)$ and $y = y(t)$. Then
 - $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$.
- **Example.** Find an expression for $\frac{dy}{dx}$ in terms of t if the curve is defined parametrically by
 - $x = \ln t$ and $y = t^2 - e^t$.

Solution.

- $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = 2t - e^t$.
- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t(2t - e^t)$
- $\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{1}{t(2t - e^t)}.$

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Parametric Equations

- We can find the second derivative by chain rule:
 - $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dt}{dx} \cdot \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{1}{dx/dt} \cdot \frac{d}{dt} \left(\frac{dy}{dx} \right).$

Example. $x = \ln t$ and $y = t^2 - e^t$.

- $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = 2t - e^t$.
- $\frac{d^2y}{dx^2} = \frac{1}{\frac{dx}{dt}} \cdot \frac{d}{dt} \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) = t(4t - e^t - te^t).$
- $\frac{d^2x}{dy^2} = \frac{1}{\frac{dy}{dt}} \cdot \frac{d}{dt} \left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}} \right) = -\frac{4t - e^t - te^t}{t^2(2t - e^t)}.$
- **Note.** $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ but $\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} \neq 1.$

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Logarithmic Differentiation

- Find $\frac{dy}{dx}$ if $y = \frac{(x^2 + 1)\sqrt{x + 3}}{x - 1}$, $x > 1$.
 - Of course we can use product and quotient law to evaluate. But do we have a shortcut?
 - This is a product of positive functions.
 - Recall that $\ln ab = \ln a + \ln b$ for $a > 0, b > 0$.
- 1. Take logarithmic function both sides:
 - $\ln y = \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1)$.
- 2. Differentiate with respect to x :
 - $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{1}{x - 1}$.

$$\frac{dy}{dx} = \left[\frac{2x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{1}{x - 1} \right] \frac{(x^2 + 1)\sqrt{x + 3}}{x - 1}.$$

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Logarithmic Differentiation

- In general, if $y = f_1(x)f_2(x) \cdots f_n(x)$ is a product of nonzero functions, we can find the derivative as following:
 1. Take logarithmic function both sides:
 - $\ln |y| = \ln |f_1(x)| + \ln |f_2(x)| + \cdots + \ln |f_n(x)|$.
 2. Differentiate with respect to x :
 - $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \cdots + \frac{f'_n(x)}{f_n(x)}$.
 3. $\frac{dy}{dx} = \left[\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \cdots + \frac{f'_n(x)}{f_n(x)} \right] y$

$$= \left[\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \cdots + \frac{f'_n(x)}{f_n(x)} \right] f_1(x)f_2(x) \cdots f_n(x).$$
- Such method is called **logarithmic differentiation**.

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Logarithmic Differentiation

- **Example.** Find $\frac{dy}{dx}$ if $y = \frac{x \cos x}{\sqrt{\csc x}}$.

1. $\ln |y| = \ln |x| + \ln |\cos x| - \frac{1}{2} \ln |\csc x|.$
2. $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{-\sin x}{\cos x} - \frac{1}{2} \cdot \frac{-\csc x \cot x}{\csc x}.$
3. $\frac{dy}{dx} = \left[\frac{1}{x} - \tan x + \frac{1}{2} \cot x \right] \frac{x \cos x}{\sqrt{\csc x}}.$

- **Example.** Find the derivative of $y = f(x)g(x).$

1. $\ln |y| = \ln |f(x)| + \ln |g(x)|.$
2. $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}.$
3. $\frac{dy}{dx} = \left[\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right] y = f'(x)g(x) + f(x)g'(x).$

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Logarithmic Differentiation

- Evaluate $\frac{d}{dx} x^x$. Exercise: $\frac{d}{dx} (x^x)^x$ and $\frac{d}{dx} x^{(x^x)}.$

- $\frac{d}{dx} x^a = ax^{a-1}.$ Is $\frac{d}{dx} x^x = x \cdot x^{x-1} = x^x ?$
- $\frac{d}{dx} a^x = a^x \ln a.$ Is $\frac{d}{dx} x^x = x^x \ln x ?$
- Let $y = x^x$. Then $\ln y = x \ln x.$
 - $\frac{1}{y} \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$
 - $\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1).$

- In general, if $y = f(x)^{g(x)}, (f(x) > 0),$ then

- $\ln y = g(x) \ln f(x) \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} [g(x) \ln f(x)].$

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Absolute Maximum and Minimum Values

• **Definition.** Let f be a function, and D be its domain.

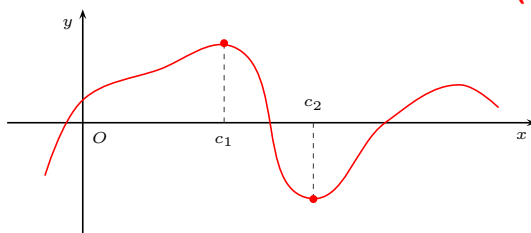
- f has an **global** (or **absolute**) **maximum** at $c \in D$

$$\iff f(c) \geq f(x) \text{ for all } x \in D.$$

- f has an **global** (or **absolute**) **minimum** at $c \in D$

$$\iff f(c) \leq f(x) \text{ for all } x \in D.$$

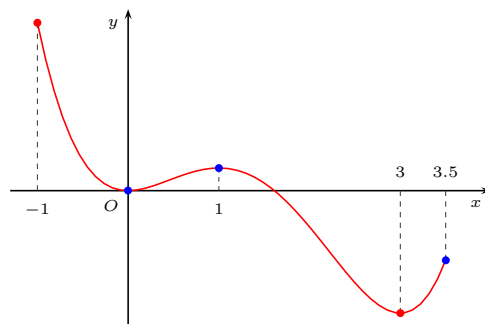
- The **absolute maximum** and **absolute minimum** are called the **(absolute) extreme values**.



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Example

- Let $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 3.5]$.

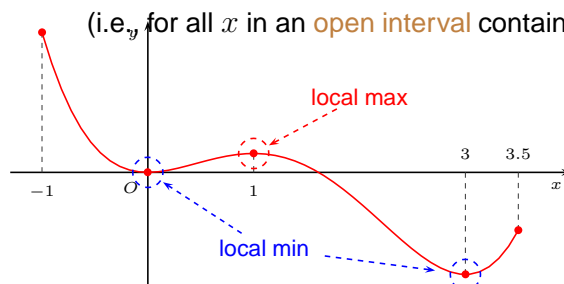


- **global max:** highest point $f(-1) = 37$.
- **global min:** lowest point $f(3) = -27$.
- What can we say about other “**turning points**” and “**end points**”?

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Local Maximum and Local Minimum

- **Definition.** Let f be a function with domain D .
 - f has a **local** (or **relative**) **maximum** at $c \in D$
 $\iff f(c) \geq f(x)$ for all x near c
 (i.e., for all x in an open interval containing c)
 - f has a **local** (or **relative**) **minimum** at $c \in D$
 $\iff f(c) \leq f(x)$ for all x near c
 (i.e., for all x in an open interval containing c)



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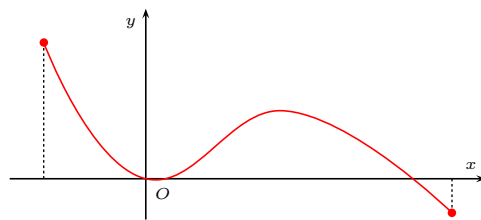
Extreme Value Theorem

- **Extreme Value Theorem.**
 If f is **continuous** on a **finite closed interval** $[a, b]$,
 - then f attains **extreme values** on $[a, b]$.
 Precisely, f attains an
 - **absolute maximum** value $f(c)$ at some $c \in [a, b]$,
 - **absolute minimum** value $f(d)$ at some $d \in [a, b]$.
 (The proof requires the “**compactness**” of finite closed interval. It is omitted in our course.)
- **Note.** Similarly as the “**Intermediate Value Theorem**”, the “**Extreme Value Theorem**” only shows the **existence** of the extreme values.
 We shall introduce a method to find out the exact value of extreme values.

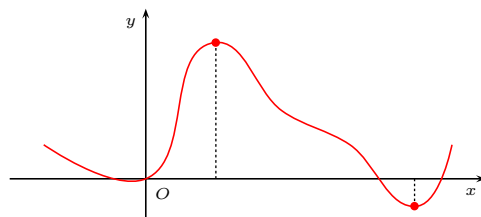
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Where are the Extreme Values?

- The extreme value may be obtained at the **endpoints**.



- If the extreme value is not obtained at the end points,



by definition it must occur as a **local max** or a **local min**.

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Finding the Extreme Values

- Let f be a **continuous** function on **closed** interval $[a, b]$.
 - Compute the values at **endpoints**: $f(a)$, $f(b)$.
 - Find **local max** and **local min** of f on (a, b) .
 - Compare the values obtained above to seek out the **extreme values**:
 - The **largest** is the **absolute maximum**,
 - The **smallest** is the **absolute minimum**.
- The 1st and the 3rd steps are easy.

How to find the **local max** and **local min** of f on (a, b) ?

- From the graphs, it seems that the local max and local min always occur at the “**turning points**”.

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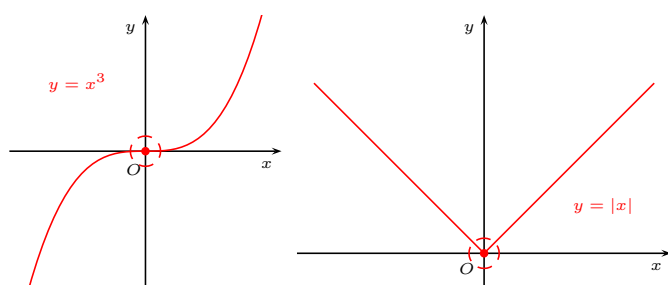
Fermat's Theorem

- **Fermat's Theorem.**
 - Suppose f has a local maximum or local minimum at c .
If $f'(c)$ exists, then $f'(c) = 0$.
- Pierre de Fermat (1601–1665), French Lawyer.
 - **Fermat's Last Theorem:** $x^n + y^n = z^n$ has no nontrivial integer solution for $n \geq 3$.
 - He wrote: "I have discovered a truly remarkable proof which this margin is too small to contain."
- **Note.** We **CANNOT** find the local maximum and local minimum by simply solving $f'(x) = 0$.
 - Even if $f'(c) = 0$, f may not have a local maximum or local minimum at c .
 - Even if f has a local maximum or a local minimum at c , $f'(c)$ may not exist, and so $f'(c)$ may not be 0.

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Examples

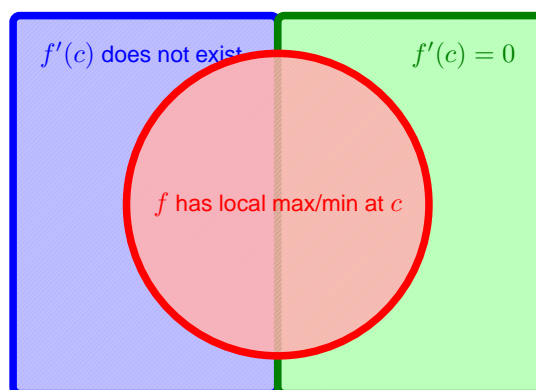
- " $f'(c) = 0 \nRightarrow f$ has local max or local min at c ".
 - Let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f'(0) = 0$.
But f has no local max or local min at 0.
- " f has local max or local min at $c \nRightarrow f'(c) = 0$ ".
 - Let $g(x) = |x|$. Then f is a local minimum at 0.
But $f'(0)$ does not exist.



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Critical Number

- Consider the following diagram:



- Definition.** Let f be a function with domain D . Then $c \in D$ is called a **critical number** of f if
 - $f'(c)$ does not exist, or $f'(c)$ exists and equals 0.

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Closed Interval Method

- Fermat's Theorem (Rephrased).**

If f has a local maximum or a local minimum at c ,

- then c is a critical number of f .

- Closed Interval Method:**

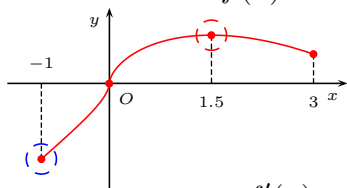
Let f be a **continuous** function on interval $[a, b]$.

- Find the values of f at **end points**: $x = a$, $x = b$,
- Find the values of f at **critical numbers** of f in (a, b) :
 - number $c \in (a, b)$ at which $f'(c)$ does not exist, or
 - number $c \in (a, b)$ at which $f'(c) = 0$.
- Compare the values of $f(x)$ evaluated in 1) and 2):
 - The largest is the **absolute maximum** value.
 - The smallest is the **absolute minimum** value.

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Examples

- Find the extreme values of $f(x) = x^{\frac{3}{5}}(4 - x)$ on $[-1, 3]$.



1) End points: $-1, 3$;

2) Critical numbers: $0, 1.5$.

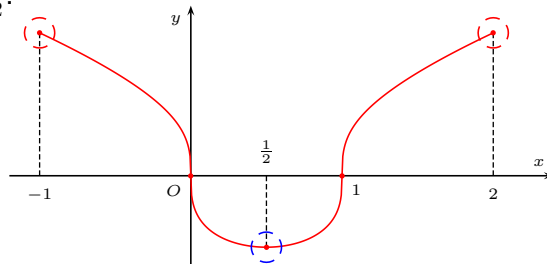
$$\begin{aligned} f'(x) &= (x^{\frac{3}{5}})'(4 - x) + x^{\frac{3}{5}}(4 - x)' \\ &= \frac{3}{5}x^{-\frac{2}{5}}(4 - x) - x^{\frac{3}{5}} = \frac{4(3 - 2x)}{5x^{\frac{2}{5}}}. \end{aligned}$$

- $f'(x)$ does not exist $\Rightarrow x = 0$,
 - $f'(x) = 0 \Rightarrow x = 1.5$.
- 3) Comparing $f(-1), f(3), f(0), f(1.5)$,
- Absolute maximum: $f(1.5) \approx 3.1886$.
 - Absolute minimum: $f(-1) = -5$.

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Examples

- Let $f(x) = \sqrt[3]{x^2 - x}$ be defined on $[-1, 2]$.
 - $f'(x) = \frac{1}{3}(x^2 - x)^{-2/3}(2x - 1) = \frac{2x - 1}{3(x^2 - x)^{2/3}}$.
 - $f'(x)$ does not exist: $x = 0, x = 1$;
 - $f'(x) = 0$: $x = \frac{1}{2}$.

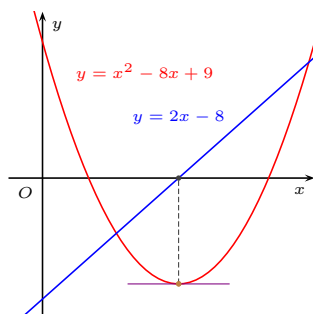


- Absolute maximum: $f(-1) = f(2) = \sqrt[3]{2}$.
- Absolute minimum: $f(\frac{1}{2}) = -\frac{1}{\sqrt[3]{4}}$.

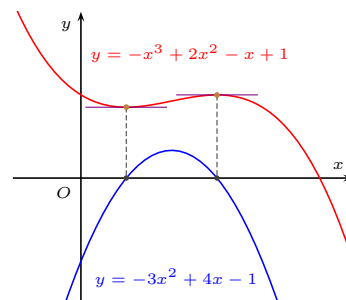
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Increasing/Decreasing Test

- Consider the following functions:



- It seems that $\begin{cases} f \text{ "turns"} & \Leftrightarrow f' = 0, \\ f \text{ is increasing} & \Leftrightarrow f' > 0, \\ f \text{ is decreasing} & \Leftrightarrow f' < 0. \end{cases}$



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Increasing/Decreasing Test

- Theorem.** Let f be a function such that
 - It is **continuous** on $[a, b]$, **differentiable** on (a, b) .

Then

- $f'(x) = 0$ on $(a, b) \Leftrightarrow f$ is **constant** on $[a, b]$;
- $f'(x) > 0$ on $(a, b) \Rightarrow f$ is **increasing** on $[a, b]$;
- $f'(x) < 0$ on $(a, b) \Rightarrow f$ is **decreasing** on $[a, b]$.

- The **converse** of Increasing/Decreasing Test **fails**:

- f is **increasing** $\nRightarrow f'(x) > 0$;
- f is **decreasing** $\nRightarrow f'(x) < 0$.

Note that f is not necessarily differentiable.

Even if f is differentiable, f' may be zero at some points.

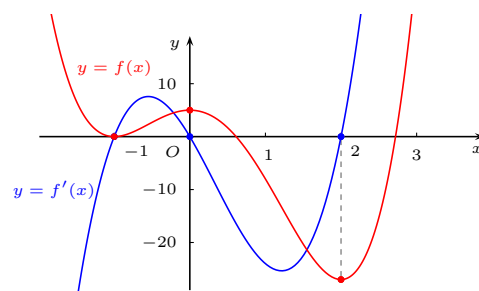
- Let $f(x) = x^3$. Then f is increasing on \mathbb{R} .
 - $f'(x) = 3x^2 \Rightarrow f'(0) = 0$.

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Example

- Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.
 - $f'(x) = 12x^3 - 12x^2 - 24x = 12(x+1)x(x-2)$.

Interval	$x+1$	x	$x-2$	$f'(x)$	$f(x)$
$(-\infty, -1)$	-	-	-	-	\searrow
$(-1, 0)$	+	-	-	+	\nearrow
$(0, 2)$	+	+	-	-	\searrow
$(2, \infty)$	+	+	+	+	\nearrow



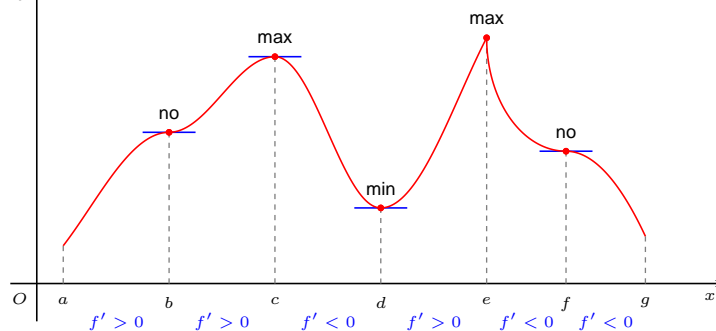
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First Derivative Test

- Let f be a continuous function. Recall that
 - if f has a **local max** or **local min** at c ,
 - then c is a **critical number** of f .

Now suppose c is a **critical number** of f .

How to check if f has a **local max** or **local min** at c ?



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First Derivative Test

- **First Derivative Test.**

Let f be **continuous** and c a **critical number** of f .
Suppose f is **differentiable** near c (except possibly at c).

- If f' changes from **positive** to **negative** at c ,
then f has a **local maximum** at c .
- If f' changes from **negative** to **positive** at c ,
then f has a **local minimum** at c .
- If f' does not change sign at c ,
then f has **no local max/min** at c .

- **Proof.** If f' changes from **positive** to **negative** at c , then

- f is **increasing** on the left of c , and
- f is **decreasing** on the right of c .

So f has a **local maximum** at c .

Other two cases can be shown similarly. (Exercise)

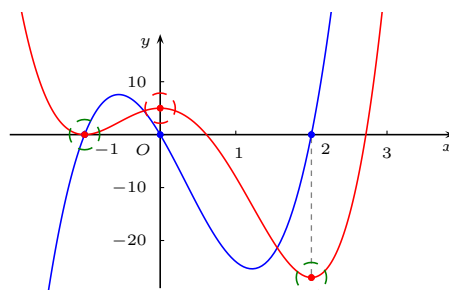
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Examples

- $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
 - $f'(x) = 12x^3 - 12x^2 - 24x = 12(x+1)x(x-2)$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	—	+	—	+
$f(x)$	↘	↗	↘	↗

- local maximum: $x = 0$;
- local minimum: $x = -1, x = 2$.



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


Examples

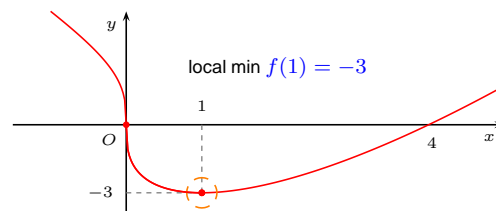
- $f(x) = x^{1/3}(x - 4)$. Find its local max and local min.

- Where are the critical numbers?

$$f'(x) = (x^{4/3} - 4x^{1/3})' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3} \frac{x - 1}{x^{2/3}}.$$

- $f'(x)$ does not exist: $x = 0$;
- $f'(x) = 0$: $x = 1$.

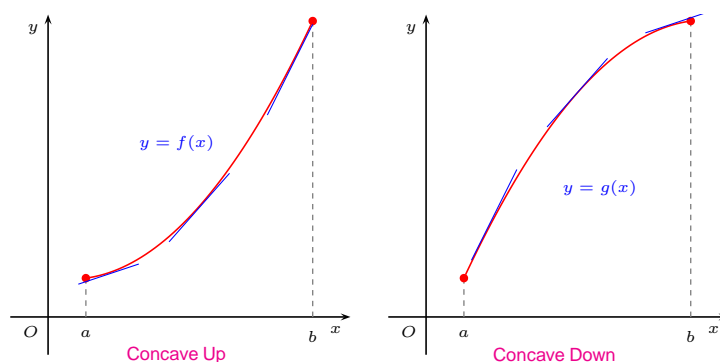
Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	—	—	+
$f(x)$			



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Concavity

- Consider two graphs with the same end points:



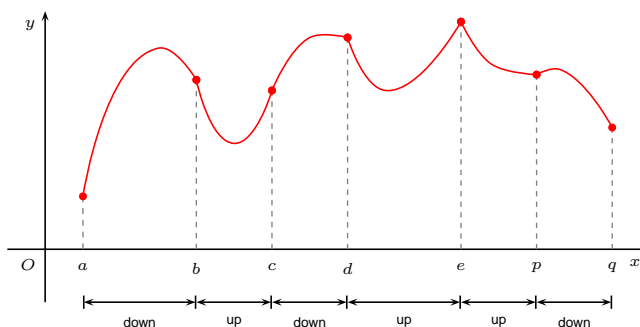
- They are both increasing functions, but look different.
- We shall define **concavity** to distinguish the two types of (differentiable) functions.

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Concavity

- **Definition.** Let f be differentiable on an open interval I .

- If the graph lies **above** all its tangent lines on I , then it is said to be **concave up**.
- If the graph lies **below** all its tangent lines on I , then it is said to be **concave down**.



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Concavity Test

- **Theorem.** Let f be **differentiable** on open interval I .
 - The graph is **concave up** $\Leftrightarrow f'$ is **increasing**.
 - The graph is **concave down** $\Leftrightarrow f'$ is **decreasing**.
- Suppose f is twice differentiable on an open interval I .
 - If $f'' > 0$ on I , by Increasing Test f' is increasing, then the graph of f is concave up.
 - If $f'' < 0$ on I , by Decreasing Test f' is decreasing, then the graph of f is concave down.
- **The Concavity Test.** Let f be a **twice differentiable** function on an open interval I .
 - If $f'' > 0$ on I , then the graph of f is **concave up** on I .
 - If $f'' < 0$ on I , then the graph of f is **concave down** on I .

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Graph f using f' and f''

- Sketch the graph of $f(x) = x^4 - 4x^3$.

- $f'(x) = 4x^2(x - 3)$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	—	—	+
$f(x)$	\searrow	\searrow	\nearrow

So $f(x)$ has local minimum at $x = 3$.

- $f''(x) = 12x(x - 2)$.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f''(x)$	+	—	+
Concavity	Up	Down	Up

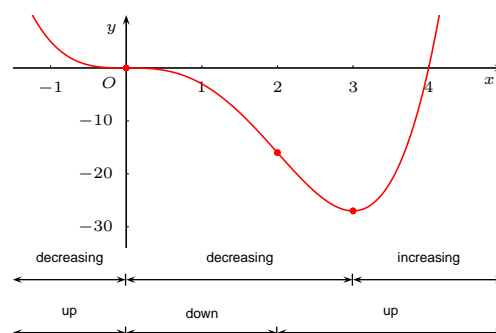
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Graph f using f' and f''

- Sketch the graph of $f(x) = x^4 - 4x^3$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f(x)$	\searrow	\searrow	\nearrow

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Concavity	Up	Down	Up



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Graph f using f' and f''

- Sketch the graph of $f(x) = x^{2/3}(6-x)^{1/3}$.

◦ $f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, \infty)$
$f'(x)$	−	+	−	−
$f(x)$	↘	↗	↘	↘

◦ $f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$.

Interval	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
$f''(x)$	−	−	+
Concavity	Down	Down	Up

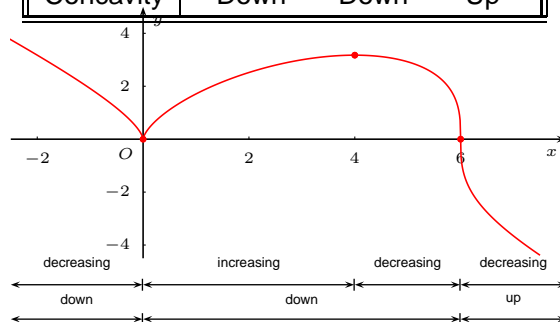
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Graph f using f' and f''

- Sketch the graph of $f(x) = x^{2/3}(6-x)^{1/3}$.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, \infty)$
$f(x)$	↘	↗	↘	↘

Interval	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
Concavity	Down	Down	Up



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Some Inequalities

- Show that for all positive $x \neq 1$, $2\sqrt{x} > 3 - \frac{1}{x}$.
 - Let $f(x) = 2\sqrt{x} - \left(3 - \frac{1}{x}\right) = 2\sqrt{x} - 3 + \frac{1}{x}$.
 - $f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{1}{x^2}(\sqrt{x^3} - 1)$.
 - $\begin{cases} f'(x) > 0, & \text{if } x > 1, \\ f'(x) < 0, & \text{if } 0 < x < 1. \end{cases}$
 - $\begin{cases} f \text{ is increasing on } [1, \infty), \\ f \text{ is decreasing on } (0, 1]. \end{cases}$
 - Then for any positive $x \neq 1$, $f(x) > f(1) = 0$.
That is,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

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Some Inequalities

- We have seen that $\sin x < x$ for all $0 < x < \frac{\pi}{2}$.
Show that $\frac{2}{\pi}x < \sin x$ when $0 < x < \frac{\pi}{2}$.
 - Let $g(x) = \frac{\sin x}{x}$ on $(0, \frac{\pi}{2}]$.
 - $g'(x) = \left(\frac{\sin x}{x}\right)' = \frac{\cos x(x - \tan x)}{x^2} < 0$.
 - By Increasing Test, g is decreasing on $(0, \frac{\pi}{2}]$.
 - For any $0 < x < \frac{\pi}{2}$, $g(x) > g(\frac{\pi}{2}) = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}$.
That is,

$$\sin x > \frac{2x}{\pi}.$$

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Some Inequalities

- Recall that $\sin x < x < \tan x$ for all $x \in (0, \frac{\pi}{2})$.
 $\tan x + \sin x$ and $2x$, which one is bigger on $(0, \frac{\pi}{2})$?
- Let $f(x) = \tan x + \sin x - 2x$.
 - $f'(x) = \sec^2 x + \cos x - 2$.
 $f''(x) = 2 \sec^2 x \tan x - \sin x > 0$
 - f' is increasing on $[0, \frac{\pi}{2})$.
 Then for any $x \in (0, \frac{\pi}{2})$, $f'(x) > f'(0) = 0$.
- f is increasing on $[0, \frac{\pi}{2})$.
 Then for any $x \in (0, \frac{\pi}{2})$, $f(x) > f(0)$.

$$\therefore \tan x + \sin x > 2x.$$

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Approximation

- Suppose f is continuous. Then $x \rightarrow a \Rightarrow f(x) \rightarrow f(a)$.
 - In other words, if $x \approx a$, then $f(x) \approx f(a)$.
 - For example, $\sqrt{1.1} \approx \sqrt{1} = 1$.
- Question.** Do we have a better approximation under some stronger assumptions?
- Suppose f' is continuous. Then $x \rightarrow a \Rightarrow f'(x) \approx f'(a)$.
 - f can be approximated by the tangent line at a .
 - $f(x) \approx f(a) + f'(a)(x - a)$.
 - For example, let $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$.
 - $f(1.1) \approx f(1) + f'(1)(1.1 - 1) = 1.05$.

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Approximation

- Suppose f'' is continuous. Then
 - $x \rightarrow a \Rightarrow f'(x) \rightarrow f'(a)$ and $f''(x) \rightarrow f''(a)$.
 - Approximate f by a quadratic function P :
 - $P(a) = f(a), P'(a) = f'(a), P''(a) = f''(a)$.

Let $P(x) = p + q(x - a) + r(x - a)^2$. Then

$$P(a) = f(a) \Rightarrow p = f(a);$$

$$P'(a) = f'(a) \Rightarrow q = f'(a);$$

$$P''(a) = f''(a) \Rightarrow 2r = f''(a).$$

$$\therefore f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

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Approximation

- Let $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$.
 - $f(x) \approx f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2$.
 - $\sqrt{1.1} \approx 1 + \frac{1}{2} \times 0.1 - \frac{1}{8} \times 0.1^2 = 1.04875$.
- In general, assume that $f^{(n)}$ is continuous, then f can be approximated by a polynomial $P(x)$ of degree n :
 - $f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$.
 - $P(a) = f(a), P'(a) = f'(a), \dots, P^{(n)}(a) = f^{(n)}(a)$.

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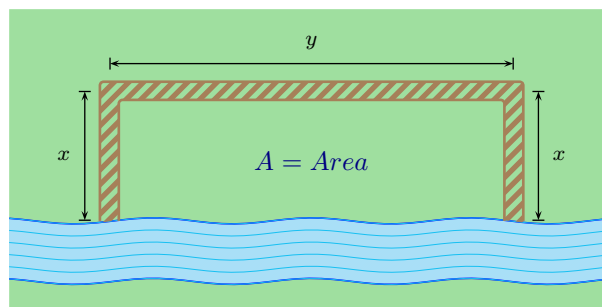
Optimization Problems

- What is **optimization problem**?
 - Finding **extreme values** in practical application.
 - **Maximize** areas, volumes, profits, . . . ,
 - **Minimize** distances, costs, times,
- How to **optimize**?
 - Understand the problem.
 - Draw a diagram.
 - Introduce notations.
 - Find relations among the variables.
 - Express the problem as finding the absolute maximum or minimum of a function $f(x)$ on a specified domain.
 - Find the absolute maximum and minimum.
 - Closed Interval Method (on finite closed interval),
 - Increasing/Decreasing Test (works for all cases).

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Example 1

- **Example.** A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the **largest** area?



- Aim: Maximize $A = xy$, where
 - $2x + y = 2400$, $x, y \geq 0$.

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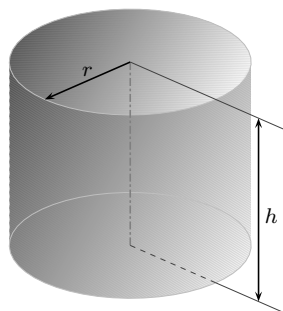
Example 1

- Maximize $A = xy$, where $2x + y = 2400$, $x, y \geq 0$.
 - $2x + y = 2400 \Rightarrow y = 2400 - 2x$.
 - $y \geq 0 \Rightarrow x \leq 1200$.
- It is equivalent to
Finding maximum of $A(x) = x(2400 - 2x)$ on $[0, 1200]$.
 - Critical number:
 - $A'(x) = 2400 - 4x$.
 - $A'(x) = 0 \Rightarrow x = 600$. $A(600) = 720\,000$.
 - Endpoints: $x = 0, 1200$. $A(0) = A(1200) = 0$.
 - $A(x)$ has maximum value 720 000 when $x = 600$.
- Conclusion: the field has the largest area 720 000 ft²,
when it has width 600 ft, and length 1200 ft.

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Example 2

- A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will **minimize** the cost of the metal to manufacture the can.



- Minimize $S = 2\pi rh + 2\pi r^2$, where
 - $V = \pi r^2 h = 1$, $r, h > 0$.

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Example 2

- Minimize $S = 2\pi r^2 + 2\pi r h$, $V = \pi r^2 h = 1$, $r, h > 0$.

- $h = \frac{1}{\pi r^2} \Rightarrow S = 2\pi r^2 + \frac{2\pi r}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$.

Find the minimum of $S(r) = 2\pi r^2 + \frac{2}{r}$ for $r > 0$.

- $S'(r) = 4\pi r - \frac{2}{r^2} = \frac{2}{r^2}(2\pi r^3 - 1)$.

- $S'(r) = 0 \Rightarrow r = \frac{1}{\sqrt[3]{2\pi}} = r_0$.

$0 < r < r_0 \Rightarrow S'(r) < 0$; $r > r_0 \Rightarrow S'(r) > 0$.

$S(r)$ is decreasing on $(0, r_0]$, is increasing on $[r_0, \infty)$.

$\therefore S(r)$ has the absolute minimum at $r = r_0$.

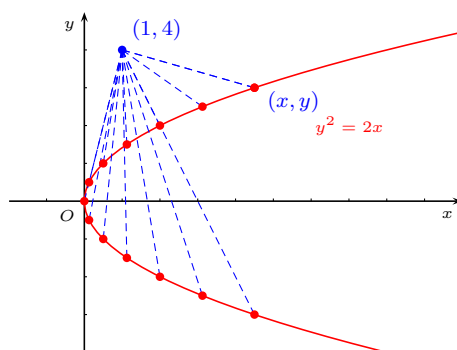
- The cost is minimized when we choose

radius $r = r_0 = \frac{1}{\sqrt[3]{2\pi}}$, and height $h = \sqrt[3]{\frac{4}{\pi}}$.

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Example 3

- Find the point on the parabola $y^2 = 2x$ that is **closest** to the point $(1, 4)$.



- Minimize $d = \sqrt{(x - 1)^2 + (y - 4)^2}$,

- where $y^2 = 2x$.

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Example 3

- Minimize $d = \sqrt{(x-1)^2 + (y-4)^2}$, with $y^2 = 2x$.

- $d(y) = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}, \quad (x = \frac{y^2}{2}).$

It is equivalent to minimizing

- $f(y) = (d(y))^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$ on \mathbb{R} .

- $f'(y) = y^3 - 8$. Then $f'(y) = 0 \Rightarrow y = 2$.
- If $y < 2$, $f'(y) < 0$; f is decreasing on $(-\infty, 2]$.
- If $y > 2$, $f'(y) > 0$; f is increasing on $[2, \infty)$.

- So $f(y)$ attains the absolute minimum at $y = 2$.

$\therefore d(y)$ attains the absolute minimum at $y = 2$. ($x = 2$)

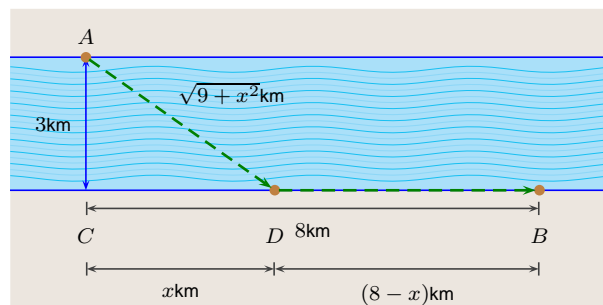
- Therefore, the point on $y^2 = 2x$ which is closest to $(4, 1)$ is $(2, 2)$. Moreover, the distance is

$$d = \sqrt{(2-4)^2 + (2-1)^2} = \sqrt{5}.$$

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Example 4

- A man launches his boat from point A on a bank of a straight river, 3km wide, and wants to reach point B , 8km downstream on the opposite bank, **as quick as possible**. If he can row 6km/h and run 8km/h, where should he land?



- Minimize $T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}, 0 \leq x \leq 8$.

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Example 4

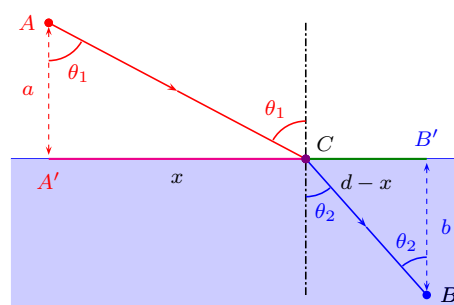
- Minimize $T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}$ on $[0, 8]$.
 - $T'(x) = \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}$.
 $T'(x) = 0 \Rightarrow 8x = 6\sqrt{9+x^2} \Rightarrow 16x^2 = 81 + 9x^2$
 $\Rightarrow 7x^2 = 81 \Rightarrow x = \frac{9}{\sqrt{7}} \ (x > 0)$.
 - Compare the values $T(0)$, $T(8)$ and $T(\frac{9}{\sqrt{7}})$:
 - $T(0) = 1.5$, $T(8) = \frac{73}{6} \approx 1.42$, and
 $T(\frac{9}{\sqrt{7}}) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$.
 - Therefore, he should land at $9/\sqrt{7}$ km away downstream from the starting point.

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Example 5: Fermat's Principle and Snell's Law

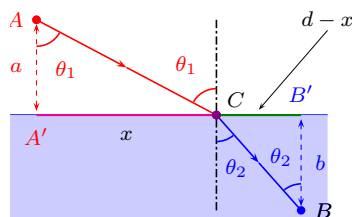
- Fermat's Principle.** The light travels along a path for which the time is minimized.
- Snell's Law.** Let v_1 and v_2 be the velocity of light in air and in water respectively. Use Fermat's Principle to show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$



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Example 5: Fermat's Principle and Snell's Law



- Minimize $T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}$ on $[0, d]$.

$$T'(x) = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{d-x}{v_2 \sqrt{b^2 + (d-x)^2}} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}.$$

- As x moves from 0 to d smoothly,

- $\theta_1 \nearrow$ and $\theta_2 \searrow \Rightarrow T'(x) \nearrow$.
- $T'(0) < 0, T'(d) > 0, T'$ is continuous on $[0, d]$.
 \Rightarrow there is a unique $x_0 \in (0, d)$ with $T'(x_0) = 0$.

$T'(x)$ increases smoothly from negative to positive.

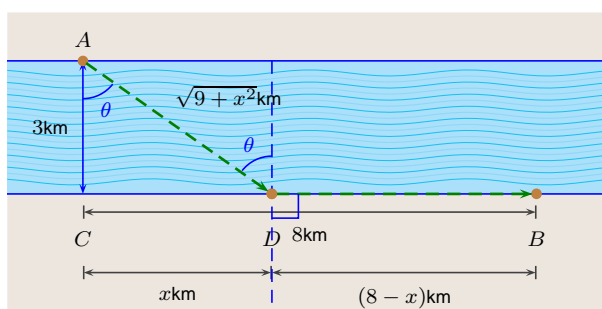
- $T'(x) < 0$ on $(0, x_0) \Rightarrow T(x) \searrow$ on $[0, x_0]$,
- $T'(x) > 0$ on $(x_0, d) \Rightarrow T(x) \nearrow$ on $[x_0, d]$.

$\therefore T(x)$ attains the min if $x = x_0$, at which $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$.

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Example 5: Fermat's Principle and Snell's Law

- Recall Example 4:



- By Snell's Law, the time is minimized when $\frac{\sin \theta}{v_1} = \frac{\sin \frac{\pi}{2}}{v_2}$.

$$\circ \frac{x/\sqrt{9+x^2}}{6} = \frac{1}{8} \Rightarrow x = \frac{9}{\sqrt{7}}.$$

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Limits of Indeterminate Forms

- How do we compute the following limits?

$$\circ \lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}, \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}, \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2}.$$

Both the numerator and denominator tend to 0 as $x \rightarrow 0$.

They have the **0/0 Indeterminate Form**.

- How to compute the following?

$$\circ \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}, \lim_{x \rightarrow \infty} \frac{x^2 + 3x}{3x^2 + 1}.$$

Both the numerator and denominator tend to $\pm\infty$ as $x \rightarrow \frac{\pi}{2}$ or $x \rightarrow \infty$.

They have the **∞/∞ Indeterminate Form**.

- Question.** Can we evaluate the limits without using ϵ, δ -definition?
 - We may use **differentiation**.

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Example

- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}.$

- Let $f(x) = 1 - \cos x$ and $g(x) = x + x^2$.

- $f(0) = g(0) = 0.$
- $f'(x) = \sin x$, and $g'(x) = 1 + 2x.$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{g(x) - g(0)} \\ &= \lim_{x \rightarrow 0} \frac{[f(x) - f(0)]/(x - 0)}{[g(x) - g(0)]/(x - 0)} \\ &= \frac{\lim_{x \rightarrow 0} [f(x) - f(0)]/(x - 0)}{\lim_{x \rightarrow 0} [g(x) - g(0)]/(x - 0)} \\ &= \frac{f'(0)}{g'(0)} = \frac{0}{1 + 2 \cdot 0} = 0. \end{aligned}$$

- However, this method does not work if $g'(0) = 0$.

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l'Hôpital's Rule

- **l'Hôpital's Rule.** Let f and g be functions such that

- $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, and
- f and g are **differentiable** near a (except at a), and
- $g'(x) \neq 0$ near a (except at a).

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists or equals $\pm\infty$.

- Guillaume François Antoine, Marquis de l'Hôpital (1661–1704) French Mathematician.
 - l'Hôpital's rule is published in his “*Analysis of the infinitely small to understand curves*”, the first book on differential calculus.
 - The rule is discovered by Johann Bernoulli (1667–1748), a Swiss Mathematician.

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Examples

- Find $\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2}$.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1-x} - 1 + \frac{x}{2})'}{(x^2)'} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{-1}{2\sqrt{1-x}} + \frac{1}{2}}{2x} = \lim_{x \rightarrow 0} \frac{\left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2}\right)'}{(2x)'} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{-1}{2} \cdot \frac{1}{2\sqrt{(1-x)^3}}}{2} \\
 &= -\frac{1}{8}.
 \end{aligned}$$

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Examples

- Find $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{(x - \sin x)'}{(x^3)'} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)'}{(3x^2)'} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \\
 &= \lim_{x \rightarrow 0} \frac{(\sin x)'}{(6x)'} \\
 &= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}.
 \end{aligned}$$

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Remarks on l'Hôpital's Rule

- Remark.**

- The condition $x \rightarrow a$ may be replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$. In other words, l'Hôpital's Rule also holds for **one sided limit**.
- l'Hôpital's Rule holds if $x \rightarrow a$ is replaced by $x \rightarrow \infty$ or $x \rightarrow -\infty$. In other words, it holds for **limit at infinity**.
 - If f and g are differentiable for large x , $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$, $g'(x) \neq 0$ for large x .
Then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$, if the limit on the right hand side exists or equals $\pm\infty$.
- l'Hôpital's Rule holds if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ is replaced by $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$.

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l'Hôpital's Rule (∞/∞)

- l'Hôpital's Rule. Suppose that

- $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$,
- f and g are differentiable near a (except at a),
- $g'(x) \neq 0$ for all x near a (except at a).

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right side exists or equals $\pm\infty$.

- Example.** Find $\lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x}$.

$$\begin{aligned} \lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x} &= \lim_{x \rightarrow \pi/2} \frac{(\sec x)'}{(1 + \tan x)'} \\ &= \lim_{x \rightarrow \pi/2} \frac{\sec x \tan x}{\sec^2 x} \\ &= \lim_{x \rightarrow \pi/2} \sin x = 1. \end{aligned}$$

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Examples

- Find $\lim_{x \rightarrow \infty} \frac{x^2 + 3x}{3x^2 + 1}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 + 3x}{3x^2 + 1} &= \lim_{x \rightarrow \infty} \frac{(x^2 + 3x)'}{(3x^2 + 1)'} = \lim_{x \rightarrow \infty} \frac{2x + 3}{6x} \\ &= \lim_{x \rightarrow \infty} \frac{(2x + 3)'}{(6x)'} = \frac{2}{6} = \frac{1}{3}. \end{aligned}$$

- Find $\lim_{x \rightarrow 1} (1 - x^2) \tan \frac{\pi x}{2}$.

$$\begin{aligned} \lim_{x \rightarrow 1} (1 - x^2) \tan \frac{\pi x}{2} &= \lim_{x \rightarrow 1} \frac{1 - x^2}{\cot \frac{\pi x}{2}} = \lim_{x \rightarrow 1} \frac{(1 - x^2)'}{(\cot \frac{\pi x}{2})'} \\ &= \lim_{x \rightarrow 1} \frac{-2x}{-\frac{\pi}{2} \csc^2 \frac{\pi x}{2}} = \frac{-2}{-\frac{\pi}{2} \cdot 1} = \frac{4}{\pi}. \end{aligned}$$

- Convert $0 \cdot \infty$ or $\infty - \infty$ indeterminate forms to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms, then apply l'Hôpital's rule.

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Correct or Wrong?

- Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + 1}{2x + 1}$.
 - $\lim_{x \rightarrow 1} \frac{x^2 + 1}{2x + 1} = \lim_{x \rightarrow 1} \frac{(x^2 + 1)'}{(2x + 1)'} = \lim_{x \rightarrow 1} \frac{2x}{2} = \lim_{x \rightarrow 1} x = 1$.
 - We **cannot** apply l'Hôpital's rule unless the limits of numerator and denominator are both 0 or both $\pm\infty$.
- Evaluate $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}$.
 - $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = \lim_{x \rightarrow \infty} \frac{(x + \sin x)'}{x'} = \lim_{x \rightarrow \infty} \frac{1 + \cos x}{1} = \lim_{x \rightarrow \infty} (1 + \cos x)$. So the limit does not exist.
 - l'Hôpital's rule is **inconclusive** if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \neq L, \pm\infty$. We shall use squeeze theorem for this question.

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Correct or Wrong?

- Evaluate $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}$.

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{1}{x/\sqrt{x^2 + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} \\
 &= \lim_{x \rightarrow \infty} \frac{x/\sqrt{x^2 + 1}}{1} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \dots
 \end{aligned}$$

The l'Hôpital's rule is useful only when the evaluation of $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ is simpler than the evaluation of

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}.$$

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Properties of the Exponential Function

- We can compute the number e numerically as a limit.

- **Theorem.** $e = \lim_{x \rightarrow 0} (1 + x)^{1/x}$.

$$\begin{aligned} \lim_{x \rightarrow 0} (1 + x)^{1/x} &= \lim_{x \rightarrow 0} \exp \left(\frac{1}{x} \ln(1 + x) \right) \\ &= \exp \left(\lim_{x \rightarrow 0} \frac{\ln(1 + x)}{x} \right) \\ &= \exp \left(\lim_{x \rightarrow 0} \frac{(\ln(1 + x))'}{(x)'} \right) \\ &= \exp \left(\lim_{x \rightarrow 0} \frac{1/(1 + x)}{1} \right) \\ &= \exp(1) = e. \end{aligned}$$

- **Remark.** Let $y = 1/x$. Then $x \rightarrow 0^+ \Leftrightarrow y \rightarrow \infty$.

- $e = \lim_{y \rightarrow \infty} (1 + 1/y)^y = \lim_{n \rightarrow \infty} (1 + 1/n)^n$.

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Example

- Evaluate $\lim_{x \rightarrow 0^+} x^x$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^x &= \lim_{x \rightarrow 0^+} e^{x \ln x} = \exp \left(\lim_{x \rightarrow 0^+} x \ln x \right) \\ &= \exp \left(\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \right) = \exp \left(\lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} \right) \\ &= \exp \left(\lim_{x \rightarrow 0^+} (-x) \right) = \exp(0) = 1. \end{aligned}$$

- In general, in order to evaluate $\lim_{x \rightarrow a} (f(x)^{g(x)})$, we use

$$\begin{aligned} \lim_{x \rightarrow a} (f(x)^{g(x)}) &= \lim_{x \rightarrow a} \exp(g(x) \ln(f(x))) \\ &= \exp \left(\lim_{x \rightarrow a} g(x) \ln f(x) \right) = \dots \end{aligned}$$

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Continuously Compounded Interest

- Initial deposit: A dollars; Interest rate (year): r .
 - Suppose the interest is credited n times per year.
 - After one year, we have $A \left(1 + \frac{r}{n}\right)^n$ dollars.
- It seems that we will get more if n gets larger.
- **Question.** What will we get after one year if $n \rightarrow \infty$, in other words, if the **interest** is **continuously compounded**?

$$\begin{aligned}
 \circ \quad \lim_{n \rightarrow \infty} A \left(1 + \frac{r}{n}\right)^n &= \lim_{x \rightarrow \infty} A \left(1 + \frac{r}{x}\right)^x \\
 &= A \lim_{x \rightarrow \infty} \exp \left(x \ln \left(1 + \frac{r}{x}\right) \right) = A \exp \left(\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{r}{x}\right)}{\frac{1}{x}} \right) \\
 &= A \exp \left(\lim_{x \rightarrow \infty} \frac{\left(1 + \frac{r}{x}\right)^{-1} \frac{-r}{x^2}}{-\frac{1}{x^2}} \right) \\
 &= A \exp \left(\lim_{x \rightarrow \infty} r \left(1 + \frac{r}{x}\right)^{-1} \right) = A \exp(r) = Ae^r.
 \end{aligned}$$

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