MA1521 CALCULUS FOR COMPUTING

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Extreme Values

- One-variable: Let $y = f(x)$ be a function with domain D.
  - f has a **global** (**absolute**) **maximum** at $c \in D$

    $\Leftrightarrow f(c) \geq f(x)$ for all $x \in D$.
  - f has a **global** (**absolute**) **minimum** at $c \in D$

    $\Leftrightarrow f(c) \leq f(x)$ for all $x \in D$.
- **Maximum and Minimum of Two-Variable Function**.

  Let $z = f(x, y)$ be a function with domain $D \subseteq \mathbb{R}^2$.
  - f has a **global** (**absolute**) **maximum** at $(a, b) \in D$

    $\Leftrightarrow f(a, b) \geq f(x, y)$ for all $(x, y) \in D$.
  - f has a **global** (**absolute**) **minimum** at $(a, b) \in D$

    $\Leftrightarrow f(a, b) \leq f(x, y)$ for all $(x, y) \in D$.

Extreme Values

- **Extreme Value Theorem for Two-Variable Function**:

  Let $z = f(x, y)$ be a **continuous** function defined on a **closed**, **bounded** domain $D \subseteq \mathbb{R}^2$.
  - Then f attains the (**absolute**) **extreme** values, i.e.,

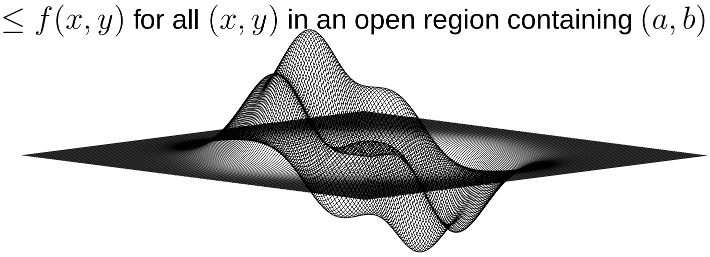
    There exist points $(a, b) \in D$ and $(c, d) \in D$ such that
    - $f(a, b) \leq f(x, y) \leq f(c, d)$ for all $(x, y) \in D$.
- **Question.** Suppose $z = f(x, y)$ is continuous on a closed, bounded domain D.
  - What are the (absolute) extreme values?
    - It may be obtained at the boundary point of the domain; or
    - It may be obtained in the interior of the domain.

Local Extreme Values

- **Local Extreme Values for Two-Variable Functions**.
  Let $z = f(x, y)$ be a function with domain $D \subseteq \mathbb{R}^2$.
  - f has a **local** (**relative**) **maximum** at $(a, b) \in D$

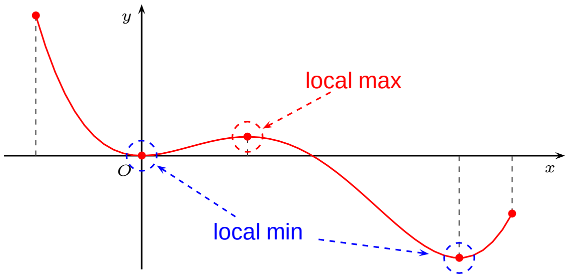
    $\Leftrightarrow f(a, b) \geq f(x, y)$ for all (x, y) in an open region containing (a, b).
  - f has a **local** (**relative**) **minimum** at $(a, b) \in D$

    $\Leftrightarrow f(a, b) \leq f(x, y)$ for all (x, y) in an open region containing (a, b).



Local Extreme Values

- Recall the **Fermat's Theorem** of one-variable function:
  If $y = f(x)$ has a local extreme value at c, and if $f'(c)$ exists, then $f'(c) = 0$.



  - If f has a local extreme value at c, then the tangent line to $y = f(x)$ at c, if exists, must be horizontal.

Local Extreme Values

- Suppose $y = f(x, y)$ has a local extreme value at (a, b).
  - It is expected that
    - The tangent plane to $z = f(x, y)$ at (a, b), if exists, must be horizontal.
  - Recall the tangent plane at (a, b):
    - $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

    It is horizontal $\Leftrightarrow$ $f_x(a, b) = f_y(a, b) = 0$.

- **First Derivative Test for Local Extreme Values**.

  Suppose $z = f(x, y)$ has a local extreme value at (a, b).
  - If $f_x(a, b)$ and $f_y(a, b)$ exist, then

$$f_x(a, b) = f_y(a, b) = 0.$$

Critical Point

- **Definition**. Let $z = f(x, y)$ be a function with domain D. Then $(a, b) \in D$ is called a **critical point** if
  - $f_x(a, b) = f_y(a, b) = 0$, or
  - at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.
- Therefore, if $z = f(x, y)$ at a local extreme value at (a, b), then (a, b) is a critical point of f.
- **Example**. $f(x, y) = x^3 - y^3 - 2xy + 6$.
  - Let $f_x = 3x^2 - 2y = 0$. Then $3x^2 = 2y$.
  - Let $f_y = -3y^2 - 2x = 0$. Then $3y^2 = -2x$.

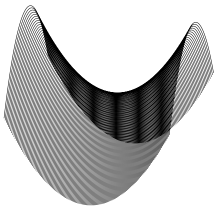
  $-2x = 3y^2 = 3\left(\frac{3}{2}x^2\right)^2 = \frac{27}{4}x^4 \Rightarrow x^4 = -\frac{8}{27}x$.
  - $x = 0 \Rightarrow y = 0$; $\quad x = -\frac{2}{3} \Rightarrow y = \frac{4}{3}$.

  Hence, f has two critical points $(0, 0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

Second Derivative Test

- **Question**. If $z = f(x, y)$ has a local extremal value at (a, b), then (a, b) is a critical point.
  - Suppose (a, b) is a critical point of $z = f(x, y)$. How can we determine whether f has a local maximum or minimum at (a, b)?

- **Definition**. Let (a, b) be a critical point of $z = f(a, b)$.
  - f is said to have a **saddle point** at (a, b) if f does not a local extremal value at (a, b).



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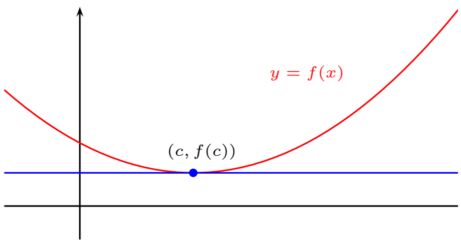
Second Derivative Test

- Consider a one-variable function $y = f(x)$:
  Suppose $f'' > 0$ on interval I. Then f is concave up.



  - If $f'(c) = 0$ at some c,
    then the tangent line of f at c is $y = f(c)$.
    - Since f is concave up,
      the graph of f lies above $y = f(c)$.
    - In other words, $f(x) > f(c)$ for all $x \neq c$.

    $\therefore$ f has the minimum at c.

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Second Derivative Test

- **Second Derivative Test for One-Variable Function**.

  Suppose $f'(c) = 0$.
  - $f''(c) > 0 \Rightarrow f$ has a local minimum at c;
  - $f''(c) < 0 \Rightarrow f$ has a local maximum at c.

- **Definition**. The **Hessian** of $z = f(x, y)$ is
  - $H(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$.

- **Second Derivative Test for Two-Variable Function**.

  Suppose $f_x(a, b) = f_y(a, b) = 0$.
  - $H(a, b) > 0$ and $f_{xx}(a, b) > 0 \Rightarrow$ local min at (a, b);
  - $H(a, b) > 0$ and $f_{xx}(a, b) < 0 \Rightarrow$ local max at (a, b);
  - $H(a, b) < 0 \Rightarrow$ saddle point at (a, b).

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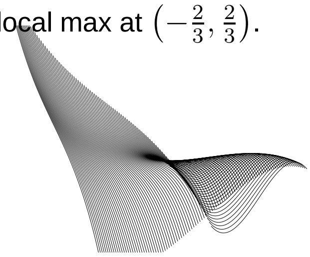
Examples

- $f(x, y) = x^3 - y^3 - 2xy + 6$.
  - It has two critical numbers $(0, 0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$.
  - $f_x = 3x^2 - 2y, \quad f_y = -3y^2 - 2x$.
    - $f_{xx} = 6x, \quad f_{xy} = -2, \quad f_{yy} = -6y$.
    - $H(x, y) = (6x)(-6y) - (-2)^2 = -36xy - 4$.
  - $H(0, 0) = -4 < 0 \Rightarrow$ saddle point at $(0, 0)$.
  - $H\left(-\frac{2}{3}, \frac{2}{3}\right) = 12 > 0$
    - $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4 < 0 \Rightarrow$ local max at $\left(-\frac{2}{3}, \frac{2}{3}\right)$.



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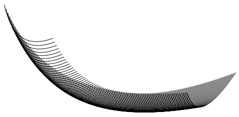
Examples

- $f(x,y) = xy + 2x - \ln x^2 y, \quad x > 0, y > 0.$
    - $f_x = y + 2 - \frac{2}{x}, \quad f_y = x - \frac{1}{y}.$
        - $f_x = f_y = 0 \Rightarrow (x,y) = \left(\frac{1}{2}, 2\right).$
    - $f_{xx} = \frac{2}{x^2}, \quad f_{xy} = 1, \quad f_{yy} = \frac{1}{y^2}.$
        - $H(x,y) = \left(\frac{2}{x^2}\right)\left(\frac{1}{y^2}\right) - 1^2 = \frac{2}{x^2y^2} - 1.$
    - $H\left(\frac{1}{2}, 2\right) = 1 > 0, \quad f_{xx}\left(\frac{1}{2}, 2\right) = 2 > 0.$

    It follows that f has a local minimum $2 + \ln 2$ at $\left(\frac{1}{2}, 2\right)$.



Examples

- $f(x,y) = 3x^2 - 2xy + y^2 - 8y + 7.$
    - $f_x = 6x - 2y, \quad f_y = -2x + 2y - 8.$

      Let $f_x = f_y = 0$. Then
        - $\begin{cases} 0 = 6x - 2y \\ 0 = -2x + 2y - 8 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 6 \end{cases}$

      $f_{xx} = 6, \quad f_{xy} = -2, \quad f_{yy} = 2.$
        - $H(x,y) = (6)(2) - (-2)^2 = 8 > 0.$
        - $f_{xx} = 6 > 0 \Rightarrow$ local minimum at $(2,6)$.
- **Remark**. Suppose $z = f(x,y)$ has a critical point (a,b).
    - $H(x,y) > 0$ & $f_{xx} > 0$ on $D \Rightarrow$ global min at (a,b);
    - $H(x,y) > 0$ & $f_{xx} < 0$ on $D \Rightarrow$ global max at (a,b).

Absolute Extreme on Closed Bounded Region

- **Absolute Extreme on Closed Bounded Region**.

  Suppose $z = f(x, y)$ is continuous on a closed and bounded region D.

Step 1. Find the critical points of f on the interior of D.

- $(a, b) \in D$ such that $f_x(a, b) = f_y(a, b) = 0$, or at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Step 2. Find the extreme values of f on the boundary of D.

- Suppose $y = y(x)$ on the boundary of D. Then
  - $f(x, y(x))$ is a function in x.

  Find its absolute extreme values.

Step 3. Compare the values of $f(x, y)$ at the points obtained in Steps 1 and 2.

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Examples

- $T(x, y) = x^2 + 2y^2 - x$ on $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

  Step 1: Find all critical points on $x^2 + y^2 < 1$.

  - $T_x = 2x - 1$ and $T_y = 4y$.
  - $T_x = T_y = 0 \Rightarrow (x, y) = \left(\frac{1}{2}, 0\right)$.

  Step 2: Find the extreme values on $x^2 + y^2 = 1$.

  - $f(x) = x^2 + 2(1 - x^2) - x = -x^2 - x + 2$, $|x| \leq 1$.
    - $f'(x) = -2x - 1$. $\quad f'(x) = 0 \Rightarrow x = -\frac{1}{2}$.
  - $f(-1) = 2$, $f(1) = 0$, $f(-\frac{1}{2}) = \frac{9}{4}$.

  Step 3: Compare to find absolute extreme values.

  - $T(\frac{1}{2}, 0) = -\frac{1}{4}, \quad f(-\frac{1}{2}) = \frac{9}{4}, \quad f(1) = 0.$

  Conclusion. $T(x, y)$ has the absolute minimum $-\frac{1}{4}$ at $(\frac{1}{2}, 0)$, and the absolute maximum $\frac{9}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

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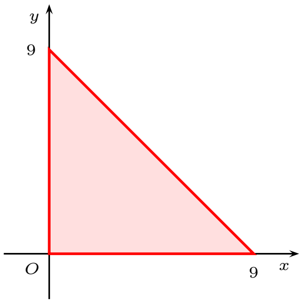
Examples

- $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the region unclosed by $x = 0$, $y = 0$ and $x + y = 9$.



i) Find critical points in the interior.

ii) Boundary points.

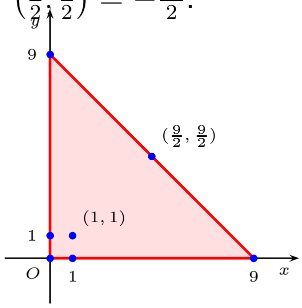
  - On x-axis;
  - On y-axis;
  - On $x + y = 9$.

Examples

- $f(x, y) = 2 + 2x + 2y - x^2 - y^2$.
  - Find critical points in the interior.
    - $f_x = 2 - 2x, \quad f_y = 2 - 2y$.
    - $f_x = f_y = 0 \Rightarrow (x, y) = (1, 1)$.
    - $f(1, 1) = 4$.
  - On the segment $(0, 0)$ to $(9, 0)$.
    - $f(x, 0) = 2 + 2x - x^2, \quad f'(x, 0) = 2 - 2x$.
    - $f'(x, 0) = 0 \Rightarrow x = 1$.
    - $f(0, 0) = 2, \quad f(9, 0) = -61, \quad f(1, 0) = 3$.
  - On the segment $(0, 0)$ to $(0, 9)$.
    - $f(0, y) = 2 + 2y - y^2, \quad f'(0, y) = 2 - 2y$.
    - $f'(0, y) = 0 \Rightarrow y = 1$.
    - $f(0, 0) = 2, \quad f(0, 9) = -61, \quad f(0, 1) = 3$.

Examples

- $f(x,y) = 2 + 2x + 2y - x^2 - y^2$.
  - On the segment $y = 9 - x, \quad 0 \le x \le 9$.
    - $f(x, 9-x) = -61 + 18x - 2x^2$.
    - $f'(x, 9-x) = 18 - 4x = 0 \Rightarrow x = \frac{9}{2}$.
    - $f(9,0) = f(0,9) = -61, \quad f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}$.



- Maximum: $f(1,1) = 4$,

  Minimum: $f(0,9) = f(9,0) = -61$.

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Examples

- Show that $\dfrac{x+y+z}{3} \ge \sqrt[3]{xyz}$ for all $x, y, z \ge 0$,

  Proof. Let $A = x + y + z$. Then $z = A - x - y$.

  Maximize $f(x,y) = xy(A - x - y)$ on the region unclosed by $x = 0$, $y = 0$ and $x + y = A$.
  - Critical points on the interior.
    - $f_x = y(A - 2x - y), \quad f_y = x(A - x - 2y)$.
    - $f_x = f_y = 0 \Rightarrow x = y = \frac{A}{3}$. $(x > 0, y > 0)$
    - $f(\frac{A}{3}, \frac{A}{3}) = \frac{A^3}{27}$.
  - Boundary points.
    - $f(x,y) = xy(A - x - y)$.
    - It is identically 0 on $x = 0$, $y = 0$, $x + y = A$.

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Examples

- Show that $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$ for all $x, y, z \geq 0$,

  Proof. Let $A = x + y + z$. Then $z = A - x - y$.

  Maximize $f(x, y) = xy(A - x - y)$ on the region unclosed by $x = 0$, $y = 0$ and $x + y = A$.

  - $f(x, y) \leq \frac{A^3}{27}$ for all x, y in the region.

  Then $xyz \leq \dfrac{(x+y+z)^3}{3^3}$, i.e., $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$.

- **Remark**. This is a special case of the **Arithmetic-Geometric Mean Inequality** for $n = 3$.
  - Let $n \in \mathbb{Z}^+$. For any $x_1, x_2, \ldots, x_n \geq 0$,
    - $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$.
  - Can you prove it?

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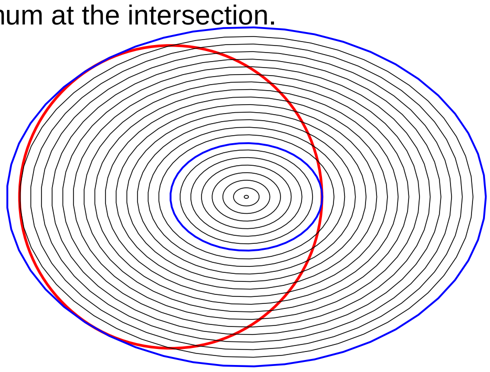
Extreme Values with Restriction

- $T(x, y) = x^2 + 2y^2 - x$ subject to $x^2 + y^2 = 1$.
  - Draw level curves $T(x, y) = c$.
    - Increase c until $T(x, y) = c$ touches $x^2 + y^2 = 1$.
      $T(x, y)$ has a minimum at the intersection.
    - Increase c until $T(x, y) = c$ leaves $x^2 + y^2 = 1$.
      $T(x, y)$ has a maximum at the intersection.



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Extreme Values with Restriction

- $T(x, y) = x^2 + 2y^2 - x$ subject to $x^2 + y^2 = 1$.
  - Suppose $T(x, y)$ has an extreme value at (x_0, y_0).
    $T(x, y) = c$ and $x^2 + y^2 - 1 = 0$:
    - are tangent to each other;
    - have the same tangent/normal line;
    - $\nabla T(x_0, y_0) \parallel \nabla g(x_0, y_0)$, $g(x, y) = x^2 + y^2 - 1$;
    - $\nabla T(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.
  - $T_x = 2x - 1$, $T_y = 4y$; $\quad g_x = 2x$, $g_y = 2y$.
    - $2x - 1 = \lambda 2x, \quad 4y = \lambda 2y, \quad x^2 + y^2 = 1.$
    - $y = 0 \Rightarrow x = \pm 1$;
      $y \neq 0 \Rightarrow \lambda = 2 \Rightarrow x = -\frac{1}{2} \Rightarrow y = \pm \frac{\sqrt{3}}{2}$.
  - $T(1, 0) = 0$, $T(-1, 0) = 2$, $T(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{9}{4}$.
    - Max: $T(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{9}{4}$; $\quad$ Min: $T(1, 0) = 0$.

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Lagrange Multiplier

- **The Method of Lagrange Multipliers**
  Find the **local maximum** and **minimum** values of $z = f(x, y)$ subject to the constraint $g(x, y) = 0$.
  - Evaluate x, y and λ that simultaneously satisfy
    - $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $g(x, y) = 0$.
- **Absolute Extreme Values with Bounded Restriction**
  Maximize/Minimize $f(x, y)$ subject to $g(x, y) = 0$,
  where $g(x, y) = 0$ is a **bounded** curve.

Step 1. Check the end points of $g(x, y) = 0$, if any.

Step 2. Use Lagrange multiplier on interior of $g(x, y) = 0$.

Step 3. Compare the values of f at points obtained in 1) & 2).

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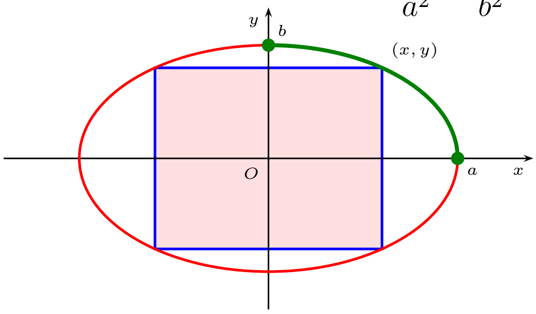
Examples

- Find the area of the largest rectangle inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a,b>0)$.



  - Maximize $f(x,y)=4xy$ subject to
    - $g(x,y)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0,\quad x,y\geq 0.$
  - End points: $(x,y)=(a,0)$, $(0,b)$.

Examples

- Find the area of the largest rectangle inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a,b>0)$.
  - Maximize $f(x,y)=4xy$ subject to
    - $g(x,y)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0,\quad x,y\geq 0.$
  - Suppose $x>0, y>0$. Apply Lagrange multipliers:

$$\left.\begin{array}{l} f_x=\lambda g_x \Rightarrow 4y=\lambda\,\frac{2x}{a^2} \\ f_y=\lambda g_y \Rightarrow 4x=\lambda\,\frac{2y}{b^2}\end{array}\right\} \Rightarrow \frac{y}{x}=\frac{x/y}{a^2/b^2} \Rightarrow \frac{x}{y}=\frac{a}{b}.$$

    - $x=ak$ & $y=bk \Rightarrow \frac{a^2k^2}{a^2}+\frac{b^2k^2}{b^2}=1 \Rightarrow k=\frac{1}{\sqrt{2}}$.
    - $(x,y)=\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$.
  - Min: $f(a,0)=f(0,b)=0$; Max: $f\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)=2ab$.

Examples

- Find the shortest distance from point $P(x_0, y_0)$ to straight line $ax + by = c$. (Assume the minimum distance exists.)

  Solution. $d(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

  - Minimize $f(x, y) = (x - x_0)^2 + (y - y_0)^2$
    - subject to $g(x, y) = ax + by - c = 0$.

    $$\left.\begin{aligned} f_x = \lambda g_x \Rightarrow 2(x - x_0) = \lambda a \\ f_y = \lambda g_y \Rightarrow 2(y - y_0) = \lambda b \end{aligned}\right\} \Rightarrow \begin{cases} x = x_0 + \frac{\lambda}{2}a \\ y = y_0 + \frac{\lambda}{2}b \end{cases}$$

    $$a(x_0 + \tfrac{\lambda}{2}a) + b(y_0 + \tfrac{\lambda}{2}b) = c \Rightarrow \frac{\lambda}{2} = \frac{c - ax_0 - by_0}{a^2 + b^2}$$

  - $x = x_0 + \dfrac{a(c - ax_0 - by_0)}{a^2 + b^2}, \quad y = y_0 + \dfrac{b(c - ax_0 - by_0)}{a^2 + b^2}$.
  - The distance is minimized at this point:

  $$\sqrt{\left(\frac{a(c-ax_0-by_0)}{a^2+b^2}\right)^2 + \left(\frac{b(c-ax_0-by_0)}{a^2+b^2}\right)^2} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

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Lagrange Multiplier of More Variables

- **The Method of Lagrange Multipliers of Three-Variables**

  Find the **local maximum** and **minimum** values of $w = f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

  - Evaluate x, y and λ that simultaneously satisfy
    - $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, $g(x, y, z) = 0$.

- **Absolute Extreme Values with Bounded Restriction**

  Maximize/Minimize $f(x, y, z)$ subject to $g(x, y, z) = 0$,

  where $g(x, y, z) = 0$ is a **bounded** surface.

Step 1. Check max/min on the boundary of $g(x, y, z) = 0$.

Step 2. Use Lagrange multiplier on interior of $g(x, y, z) = 0$.

Step 3. Compare the values of f at points obtained in 1) & 2).

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Examples

- Show that $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$ for all $x, y, z \geq 0$.

  Solution. Let $A = x + y + z$.

  - Maximize $f(x, y, z) = xyz$ subject to
    - $g(x, y, z) = x + y + z - A, \quad x, y, z \geq 0.$
  - Boundary points: $x = 0$, or $y = 0$, or $z = 0$.
    - $f(x, y, z)$ is identically zero on the boundary.
  - Suppose $x, y, z > 0$ and use Lagrange multiplier.

    $$\left.\begin{array}{l} f_x = \lambda g_x \Rightarrow yz = \lambda \\ f_y = \lambda g_y \Rightarrow zx = \lambda \\ f_z = \lambda g_z \Rightarrow xy = \lambda \end{array}\right\} \Rightarrow x = y = z = \frac{A}{3}.$$
  - $f(\frac{A}{3}, \frac{A}{3}, \frac{A}{3}) = \frac{A^3}{27}$ is the maximum.

  Therefore, $\sqrt[3]{xyz} = \sqrt[3]{f(x, y, z)} \leq \sqrt[3]{\dfrac{A^3}{27}} = \dfrac{x+y+z}{3}$.

Examples

- Find the max/min of $f(x, y, z) = ax + by + cz$ on the unit sphere $x^2 + y^2 + z^2 = 1$, where $a, b, c > 0$.

  Solution. Let $g(x, y, z) = x^2 + y^2 + z^2 - 1$.

  - The sphere $x^2 + y^2 + z^2 = 1$ has no boundary.
  - Apply the Lagrange multipliers method:

    $$\left.\begin{array}{l} f_x = \lambda g_x \Rightarrow a = \lambda 2x \\ f_y = \lambda g_y \Rightarrow b = \lambda 2y \\ f_z = \lambda g_z \Rightarrow c = \lambda 2z \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \frac{a}{2\lambda} \\ y = \frac{b}{2\lambda} \\ z = \frac{c}{2\lambda} \end{array}\right.$$

    $$1 = (\tfrac{a}{2\lambda})^2 + (\tfrac{b}{2\lambda})^2 + (\tfrac{c}{2\lambda})^2 \Rightarrow \lambda = \pm\tfrac{\sqrt{a^2+b^2+c^2}}{2}.$$
  - $(x, y, z) = \pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

Max: $f\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) = \sqrt{a^2+b^2+c^2}$

Min: $f\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right) = -\sqrt{a^2+b^2+c^2}$

Extreme Values with Two Constraints

- **Lagrange Multipliers Method with Two Constraints**

  Find the **local maximum** and **minimum** values of $w = f(x, y, z)$ subject to the constraints
  - $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

  Evaluate x, y, z and λ, μ that simultaneously satisfy
  - $f_x = \lambda g_x + \mu h_x$, $f_y = \lambda g_y + \mu h_y$, $f_z = \lambda g_z + \mu h_z$.
  - $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

- **Idea of the Lagrange multipliers method**.
  - $g(x, y, z) = h(x, y, z) = 0$ defines a curve, say C.
  - ∇g and ∇h are normal to C.
  - We seek for points at which ∇f is normal to C.
    - ∇f lies in the plane defined by ∇g and ∇h,
    - i.e., $\nabla f = \lambda \nabla g + \mu \nabla h$ for some $\lambda, \mu \in \mathbb{R}$.

Examples

- Suppose $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$.

  Find the extreme values of $f(x, y, z) = x^3 + y^3 + z^3$.

  Solution. Two restrictions $g(x, y, z) = x + y + z - 1$ and $h(x, y, z) = x^2 + y^2 + z^2 - 1$.

  $$f_x = \lambda g_x + \mu h_x \Rightarrow 3x^2 = \lambda + \mu 2x$$
  $$f_y = \lambda g_y + \mu h_y \Rightarrow 3y^2 = \lambda + \mu 2y$$
  $$f_z = \lambda g_z + \mu h_z \Rightarrow 3z^2 = \lambda + \mu 2z$$

  - The equation $3\alpha^2 = \lambda + \mu 2\alpha$ has at most 2 real roots.
    - So x, y, z cannot be all distinct.
  - Suppose $y = z$. Then

  $$\begin{cases} 1 = x + 2y \\ 1 = x^2 + 2y^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = z = 0 \end{cases} \text{ or } \begin{cases} x = -\frac{1}{3} \\ y = z = \frac{2}{3} \end{cases}$$

Examples

- Suppose $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$.

  Find the extreme values of $f(x, y, z) = x^3 + y^3 + z^3$.

  Solution. Two restrictions $g(x, y, z) = x + y + z - 1$ and $h(x, y, z) = x^2 + y^2 + z^2 - 1$.

  - Suppose $y = z$. Then
    $$\begin{cases} 1 = x + 2y \\ 1 = x^2 + 2y^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = z = 0 \end{cases} \text{ or } \begin{cases} x = -\frac{1}{3} \\ y = z = \frac{2}{3} \end{cases}$$
  - If $x = y$ then $(x, y, z) = (0, 0, 1)$, $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.
  - If $x = z$ then $(x, y, z) = (0, 1, 0)$, $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.

  Compare the values of $f(x, y, z)$ at these 6 points.

Max: $f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = 1$;
Min: $f\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \frac{5}{9}$.