Concentration of measure and superdense coding

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Concentration of measure effect on quantum communication

In short, this presentation is showing the fact that:
Given a bipartite system $A \otimes B$, assume $\dim(B) \geq \dim(A) \geq 3$, as the dimension of the smaller system A increases, with a very high probability, a random pure state $\sigma = |\psi\rangle\langle\psi|$ selected from $A \otimes B$ is almost as good as the maximally entangled state.

Why we cannot sent unlimited amount of information via quantum states

The fact that Hilbert space contains infinitely many different state vectors does not aid us in transmitting an unlimited amount of information. The more states are used for transmission, the closer they are to each other and hence they become less and less "distinguishable".

Recall the Definition of measurement

Definition of measurement

A measurement (observation) of a system prepared in a given state produces an outcome x, x is a physical event that is a subset of the set of all possible outcomes. For each x, we associate a measurement operator M_x on $\mathscr{H}$.
Given the initial state (pure state, unit vector) u, the probability of measurement outcome x is given by:

$$p(x) = \|M_x u\|^2$$

Note that to make sense of this definition, the collection of measurement operators $\{M_x\}$ must satisfy the completeness requirement:

$$1 = \sum_{x \in X} p(x) = \sum_{x \in X} \|M_x u\|^2 = \sum_{x \in X} \langle M_x u, M_x u \rangle = \langle u, (\sum_{x \in X} M_x^* M_x) u \rangle$$

So $\sum_{x \in X} M_x^* M_x = I$.

Example of distinguishable states

Proposition of indistinguishability

Suppose that we have two system $u_1, u_2 \in \mathscr{H}_1$, the two states are distinguishable if and only if they are orthogonal.

Ways to distinguish the two states:

1. set $X = \{0, 1, 2\}$ and $M_i = |u_i\rangle\langle u_i|$, $M_0 = I - M_1 - M_2$
2. then $\{M_0, M_1, M_2\}$ is a complete collection of measurement operators on $\mathscr{H}$.
3. suppose the prepared state is u_1, then $p(1) = \|M_1 u_1\|^2 = \|u_1\|^2 = 1$, $p(2) = \|M_2 u_1\|^2 = 0$, $p(0) = \|M_0 u_1\|^2 = 0$.

If they are not orthogonal, then there are no choice of measurement operators to perfectly distinguish the two states.
intuitively, if the two states are not orthogonal, then for any measurement (projection) there exists non-zero probability of getting the same outcome for both states.

Theoretical bound on the amount of information that can be transmitted by a quantum system

Holevo bound

The maximal amount of classical information that can be transmitted by a quantum system is given by the Holevo bound. $\log_2(d)$ is the maximum amount of classical information that can be transmitted by a quantum system with d levels (that is basically the number of qubits).

Superdense coding

Bell state

The Bell states are the following four states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

These are basis of the 2-qubit Hilbert space.

It is a procedure defined as follows:

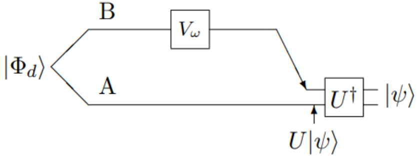
Superdense coding

Suppose A and B share a Bell state (or other maximally entangled state) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where A holds the first part and B holds the second part.
A wish to send 2 **classical bits** to B.

Superdense coding (continued)

Superdense coding (continued)

A performs one of four Pauli unitaries (Some fancy Quantum Gates named X, Y, Z, I) on the combined state of entangled qubits $\otimes$ one qubit. Then A sends the resulting one qubit to B.
This operation extends the initial one entangled qubit to a system of one of four orthogonal Bell states.
B performs a measurement on the combined state of the one qubit and the entangled qubits he holds.
B decodes the result and obtains the 2 classical bits sent by A.



Common misconceptions about superdense coding

Misconception 1

Superdense coding is a way to send 2 classical bits of information by sending 1 qubit with 1 entangled qubit. **The role of the entangled qubit** is to help them to distinguish the 4 possible states of the total 3 qubits system where 2 of them (the pair of entangled qubits) are mathematically the same.

Misconception 2

No information can be gained by measuring a pair of entangled qubits. To send information from A to B, we need physically send the qubits from A to B. That means, we cannot send information faster than the speed of light.

Levy's concentration inequality

Levy's concentration inequality

Given an η-Lipschitz function $f : S^n \to \mathbb{R}$ with median M, the probability that a random $x \in_R S^n$ is further than ϵ from M is bounded above by $\exp(-\frac{C(n-1)\epsilon^2}{\eta^2})$, for some constant $C > 0$.

$$\Pr[|f(x) - M| > \epsilon] \leq \exp(-\frac{C(n-1)\epsilon^2}{\eta^2})$$

That basically means that the probability of the random variable $f(x)$ being further than ϵ from the median M is bounded above by $\exp(-\frac{C(n-1)\epsilon^2}{\eta^2})$.

Devil in the details 1

What is the exact value of C? (How it relates to the Lipschitz constant, dimension of the bipartite system, etc.)

Levy's concentration inequality on random subspaces

Choose a *random* pure state $\sigma = |\psi\rangle\langle\psi|$ from $A \otimes B$.
The expected value of the entropy of entanglement is known and satisfies a concentration inequality. (Page's formula)

$$\mathbb{E}[H(\psi_A)] \geq \log_2(d_A) - \frac{1}{2\ln(2)}\frac{d_A}{d_B}$$

Recall that the von Neumann entropy of a mixed state ρ is given by

$$H(\rho) = -\operatorname{Tr}(\rho \log_2(\rho))$$

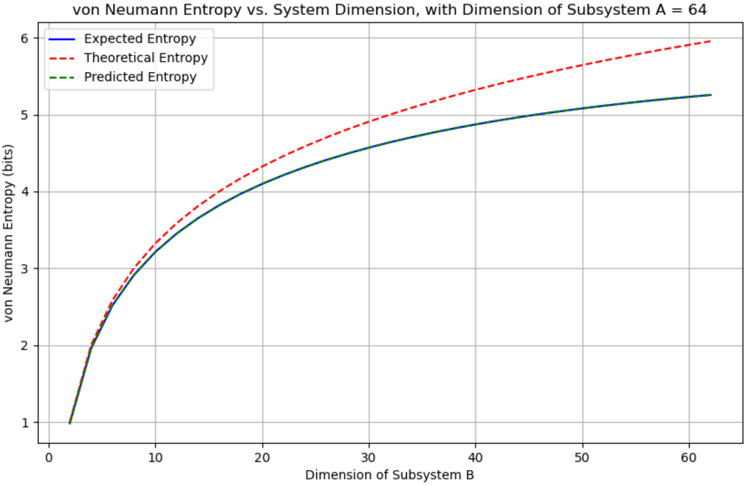
Devil in the details 2

How the *random sampling* is defined in the space of pure states?

Devil in the details 3

How to prove the Page's formula?

Page's formula, experimental verification



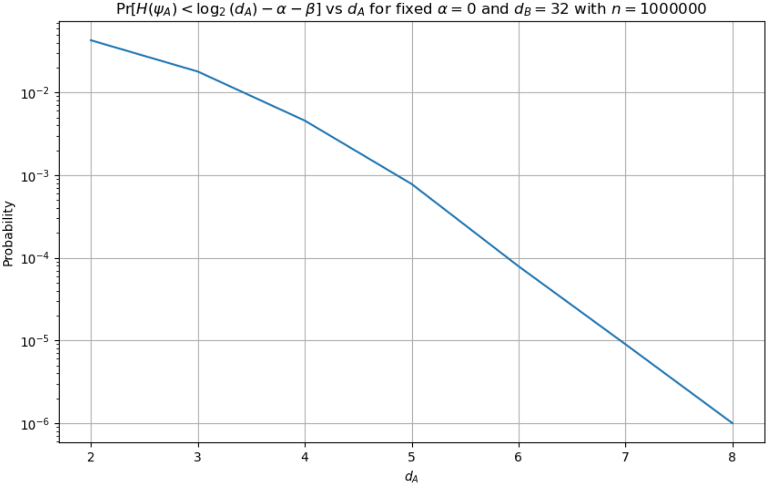
Levy's concentration inequality on random subspaces (continued)

From the Levy's lemma, we have
If we define $\beta = \frac{1}{\ln(2)}\frac{d_A}{d_B}$, then we have, for appropriate α,

$$\Pr[H(\psi_A) < \log_2(d_A) - \alpha - \beta] \leq \exp\left(-\frac{(d_A d_B - 1)C\alpha^2}{(\log_2(d_A))^2}\right)$$

where C is a small constant and $d_B \geq d_A \geq 3$.
That basically means that the probability of a random pure state $\sigma = |\psi\rangle\langle\psi|$ from $A \otimes B$ is almost as good as the maximally entangled state.

Levy's concentration inequality on random subspaces (continued)



Future Plans

- Understand the detailed proof of Levy's concentration inequality on random subspaces in a more generalized settings. (Section $3\frac{1}{2}.19$ on [Gromov, 1981]) from the perspective of
  - Observable diameter
  - Lipschitz functions
  - The Maxwell-Boltzmann distribution
- Check the detailed proof of the Page's formula.
- Check different ways to sample the random pure state (via Unitary transformations, or other measurements)
- Continue researching on the applications of Quantum error correction codes.
  - Gottesman-Kitaev-Preskill (GKP) code
  - Surface code

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Q&A