VIETNAM GENERAL CONFEDERATION OF LABOUR
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HAWKES PROCESSES

by

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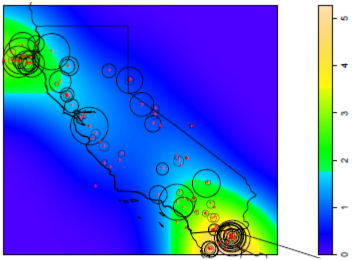
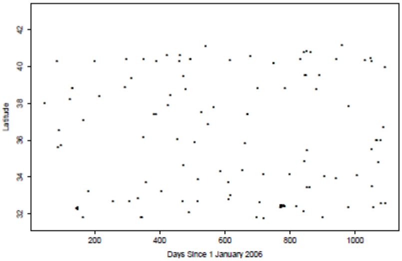
Outline

Introduction



(a)



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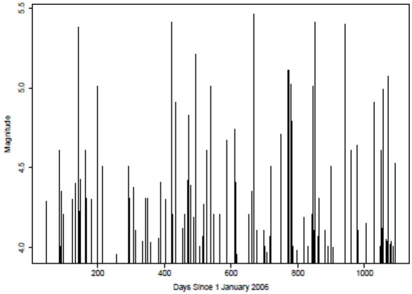

Background

Definition 1

(Point process) Let $\{T_i, i \in \mathbb{N}\}$ be a sequence of non-negative random variables such that $\forall i \in \mathbb{N}$, $T_i < T_{i+1}$. Then $\{T_i, i \in \mathbb{N}\}$ is a (simple) point process.

Background

Definition 2

(Counting process) A counting process is a stochastic process $(N(t) : t \geq 0)$ taking values in $\mathbb{N}_0$ that satisfies $N(0) = 0$, is almost surely finite, and is a right-continuous step function with increments of size $+1$. Denote by $(\mathcal{H}(u) : u \geq 0)$ the history of the arrivals up to time u.

Background

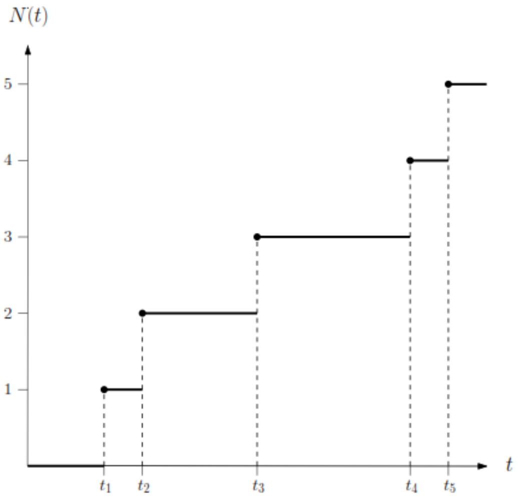

Figure 1: Point process $\{t_1, t_2, ...\}$ and corresponding counting process $N(t)$.

Background

Definition 3

(Inhomogeneous Poisson Process) Consider $(N(t) : t \geq 0)$ a counting process and that satisfies

$$\mathbb{P}(N(t+h) - N(t) = m|N(t)) = \begin{cases} \lambda(t)h, & \text{if } m = 1 \\ o(h), & \text{if } m > 1 \\ 1 - \lambda(t)h + o(h), & \text{if } m = 0 \end{cases}$$

Then $N(t)$ is called a inhomogeneous Poisson process with $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$.

Background

Definition 4

(Hawkes process) Consider $(N(t) : t \geq 0)$ a counting process, with associated history $\mathcal{H}(t) : t \geq 0$, that satisfies

$$\mathbb{P}(N(t+h) - N(t) = m|\mathcal{H}(t)) = \begin{cases} \lambda^*(t)h + o(h), & \text{if } m = 1 \\ o(h), & \text{if } m > 1 \\ 1 - \lambda^*(t)h + o(h), & \text{if } m = 0 \end{cases}$$

Suppose the process' conditional intensity function is of the form

$$\lambda^*(t) = \lambda + \int_0^t \mu(t-u)dN(u)$$

for some $\lambda > 0$ and $\mu : (0, \infty) \rightarrow [0, \infty)$ which are called the background intensity and excitation function respectively. Suppose that $\mu(.) \neq 0$, then a process $N(.)$ is a Hawkes process.

Simulation Algorithms - Inhomogeneous Poisson

Algorithm 1 Generate an inhomogeneous Poisson process by thinning.

INPUT: T is the time to simulate;
$\lambda(.)$ is the intensity function;
M is bounded value;

OUTPUT: The vector P containing the times of occurrences $\{t_1, t_2, ..., t_n\}$;

Require: $\lambda(.) \leq M$ on $[0, T]$.

Step 1: Set $P \leftarrow []$, $t \leftarrow 0$

Step 2: while $t < T$ do

a. Generate next candidate point $E \leftarrow \text{Exp}\ (M)$, $t \leftarrow t + E$
b. Keep it with some probability $U \leftarrow \text{Unif}(0, M)$
c. if $t < T$ and $U \leq \lambda(t)$ then $P \leftarrow [P, t]$

Simulation Algorithms - Inhomogeneous Poisson

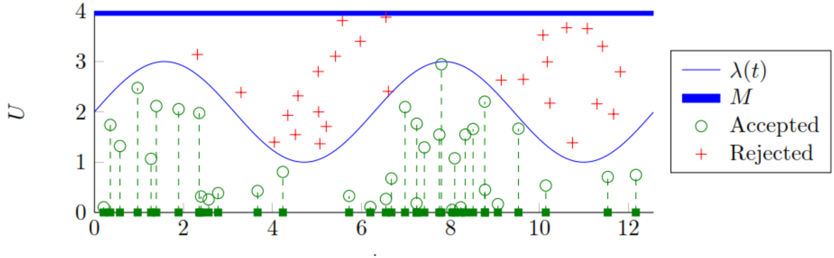

Figure 2: Inhomogeneous Poisson.

Simulation Algorithms - Intensity-based Hawkes Process

Algorithm 2 Generate a Hawkes process by thinning.

INPUT: T is the time to simulate;

$\lambda^*(.)$ is the conditional intensity function;

OUTPUT: The vector P containing the times of occurrences $\{t_1, t_2, ..., t_n\}$;

Require: $\lambda^*(.)$ non-increasing in periods without any arrivals.

Step 1: Set $\varepsilon \leftarrow 10^{-10}$, $P \leftarrow []$, $t \leftarrow 0$

Step 2: while $t < T$ do

a. Find new upper bound: $M \leftarrow \lambda^*(t + \varepsilon)$

b. Generate next candidate point $E \leftarrow \text{Exp}(M)$, $t \leftarrow t + E$

c. Keep it with some probability $U \leftarrow \text{Unif}(0, M)$

d. if $t < T$ and $U \leq \lambda^*(t)$ then $P \leftarrow [P, t]$

Simulation Algorithms - Intensity-based Hawkes Process

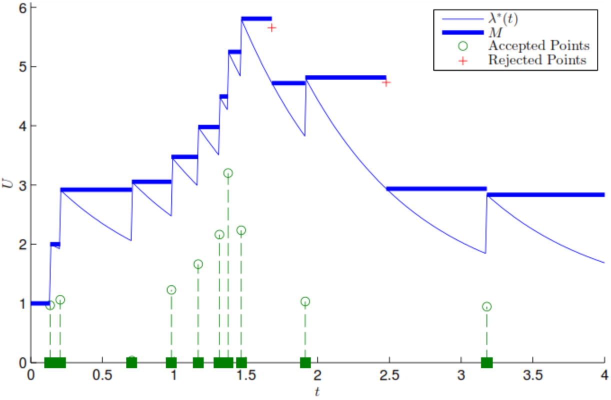

Figure 3: Intensity-based Hawkes Process.

Simulation Algorithms - Cluster-based Hawkes Process

Algorithm 3 Generate a Hawkes process by clusters.

INPUT: T is the time to simulate;
(λ, α, β) are parameters of the conditional intensity function;
OUTPUT: P is the union of all the clusters $\{C_1, C_2, ..., C_k\}$;
Step 1: Set $P \leftarrow \{\}$
Step 2: Generate the immigrants:
a. $k \leftarrow \text{Poi}(\lambda T)$
b. $C_1, C_2, ..., C_k \xleftarrow{\text{independent and identically distributed}} \text{Unif}(0, T)$
Step 3: Generate the descendants:
a. $D_1, D_2, ..., D_k \xleftarrow{\text{independent and identically distributed}} \text{Poi}(\alpha/\beta)$
Step 4: for $i \leftarrow 1$ to k do
a. if $D_i > 0$ then
a.1 $E_1, E_2, ..., E_{D_i} \xleftarrow{\text{independent and identically distributed}} \text{Exp}(\beta)$
a.2 $P \leftarrow P \cup \{C_i + E_1, C_i + E_2, ..., C_i + E_{D_i}\}$

Simulation Algorithms - Cluster-based Hawkes Process

Step 5: Remove descendant outside $[0, T]$

$$P \leftarrow \{P_i : P_i \in P, P_i \leq T\}$$

Step 6: Add in immigrants and sort: $P \leftarrow \text{Sort}\ (P \cup \{C_1, C_2, ..., C_k\})$

Simulation Algorithms - Cluster-based Hawkes Process

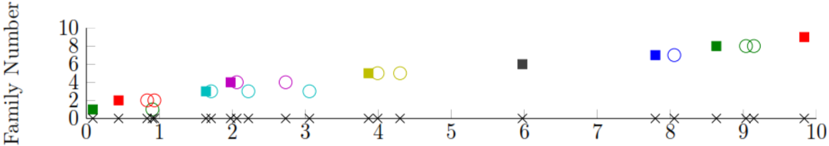

Figure 4: Cluster-based Hawkes Process.

THANK YOU!