VIETNAM GENERAL CONFEDERATION OF LABOUR
TON DUC THANG UNIVERSITY
FACULTY OF MATHEMATICS AND STATISTICS

REPORT
HAWKES PROCESSES

by

Nguyen Le Thao Trang
Le Thi Minh Phuong
advised by
Dr. Nguyen Chi Thien

Course's Name: Stochastic Processes

Ho Chi Minh City, Jun 2019

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THE REPORT HAS BEEN ACCOMPLISHED AT TON DUC THANG UNIVERSITY

We hereby declare that this report was carried out by ourselves under the guidance and supervision of Dr. Nguyen Chi Thien; and that the work contained and the results in this report are true and have not been either submitted anywhere for any previous purpose or published in any other literature. The data and figures presented in this report are for analysis, comments, and evaluations from various resources by our own work and have been fully acknowledged in the reference part. In addition, other comments, reviews and data from other authors, and organizations used in this report have been acknowledged, and explicitly cited. We will take full responsibility for any fraud detected in our report. Ton Duc Thang University is unrelated to any copyright infringement caused on our work (if any).

Ho Chi Minh City, Jun 2019

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Chapter 1

Introduction

In reality, an earthquake can increase the geological tension in the region where it occurs. Selling a significant quantity of a stock could cause a trading flurry. These above problems are called the applications of self-exciting stochastic processes. This process is called Hawkes process which was proposed by Alan G. Hawkes [1].
The Hawkes process is a counting process that models a sequence of 'arrivals' of some types over time, for example, earthquakes, trade orders. Each arrival excites the process, i.e., the chance of a next arrival is increased for some time period after the initial arrival.
The outline of this report is organized as follows:

Chapter 2 reviews and introduces some basis knowledge. It will support us in the next chapters such as point process, counting process, conditional intensity function, inhomogeneous Poisson process.

Chapter 3 approaches two models to simulate Hawkes process: the intensity-based Hawkes process and cluster-based Hawkes process.

Chapter 4 is devoted to some simulations by using Matlab® code such as the inhomogeneous Poisson process and intensity-based Hawkes process and cluster-based Hawkes process.

Chapter 5 will introduce an application of Hawkes process in seismology.

Chapter 6 summarizes the content of the report what has been done and limited of its.

Chapter 2

Knowledge

In order to clearly understand the Hawkes process, we will introduce the definition of point process, counting process and conditional intensity function. And then we discuss the inhomogeneous Poisson process. It is necessary to discuss because one can view the Hawkes process as a generalization of the (time) inhomogeneous Poisson process.

2.1 Point process and counting process

Definition 2.1. *[1]* ***(Point process)*** *Let $\{T_i, i \in \mathbb{N}\}$ be a sequence of non-negative random variables such that $\forall i \in \mathbb{N}, T_i < T_{i+1}$. Then $\{T_i, i \in \mathbb{N}\}$ is a (simple) point process.*

Definition 2.2. *[1]* ***(Counting process)*** *A counting process is a stochastic process $(N(t) : t \geqslant 0)$ taking values in $\mathbb{N}_0$ that satisfies $N(0) = 0$, is almost surely finite, and is a right-continuous step function with increments of size +1. Denote by $(\mathcal{H}(u) : u \geqslant 0)$ the history of the arrivals up to time u.*

For example:

- If $N(t)$ is the number of persons who enter a store by time t, then $(N(t) : t \geqslant 0)$ is a counting process in which an event corresponds to a person enters the store.
- If $N(t)$ is the number of goals that a given football player scores by time t , then $(N(t) : t \geqslant 0)$ is a counting process in which an event of the process will occur whenever the football player scores a goal.

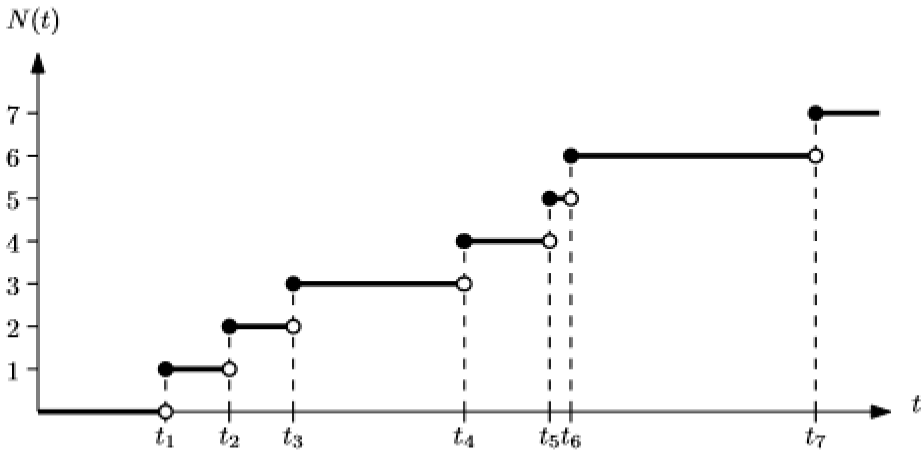

Figure 2.1: Point process $\{t_1, t_2, ...\}$ and corresponding counting process $N(t)$.

2.2 Conditional intensity function

It is difficult to work with the conditional arrival distribution $f^*(t)$, so one used another characterization of point process which was called conditional intensity function. This function is defined in the form

$$\lambda^*(t) = \frac{f^*(t)}{1 - F^*(t)}$$

where $f^*(t)$ is the conditional probability density function (p.d.f) of the next arrival given the previous arrival history and $F^*(t)$ is the cumulative distribution function (c.d.f) of the next arrival given the previous arrival history.

Definition 2.3. *[1]* ***(Conditional intensity function)*** *Consider a counting process $N(.)$ with associated histories $\mathcal{H}(.)$. If a function $\lambda^*(t)$ exists such that*

$$\lambda^*(t) = \lim_{h \downarrow 0} \frac{\mathbb{E}[N(t+h) - N(t) | \mathcal{H}(t)]}{h}$$

which only relies on information of $N(.)$ in the past (that is, $\lambda^*(t)$ is $\mathcal{H}(t)$ - measurable), then it is called the conditional intensity function of $N(.)$.

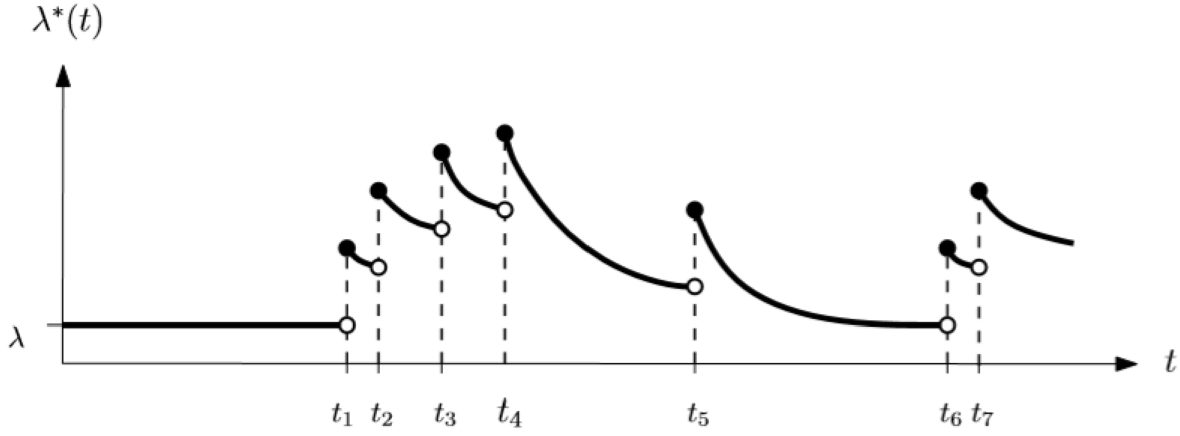

Figure 2.2: The conditional intensity function for a self-exciting process.

2.3 Inhomogeneous Poisson process

2.3.1 Description

The inhomogeneous Poisson process is a generalization of the homogeneous Poisson process that intensity function depends on the time t.

Definition 2.4. ***(Inhomogeneous Poisson Process)*** *Consider* $(N(t) : t \geqslant 0)$ *a counting process, that satisfies*

$$\mathbb{P}(N(t+h) - N(t) = m|N(t)) = \begin{cases} \lambda(t)h, & \text{if } m = 1 \\ o(h), & \text{if } m > 1 \\ 1 - \lambda(t)h + o(h), & \text{if } m = 0 \end{cases}$$

Then a counting process $N(t)$ *is called an inhomogeneous Poisson process with an intensity function* $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$.

The intensity function $\lambda(t)$ of an inhomogeneous Poisson process is a deterministic function.

Definition 2.5. *[2]* ***(Mean Value Function)*** *The function*

$$m(t) = \int_0^t \lambda(y)dy$$

is called the mean value function of the inhomogeneous Poisson process.

Theorem 2.1. *[2] If* $\{N(t), t \geqslant 0\}$ *is a non-stationary Poisson process with intensity function* $\lambda(t), t \geqslant 0$, *then* $N(t+h) - N(h)$ *is a Poisson random variable with mean* $m(t+h) - m(h) = \displaystyle\int_h^{h+t} \lambda(y)dy$.

2.3.2 The algorithm

The idea of this algorithm is to generate a homogeneous Poisson process, and then remove the points probabilistically so that the remaining points satisfy the time-varying intensity $\lambda(.)$. This algorithm requires the conditional intensity to be upper bounded, i.e., $\exists M$ such that $\lambda(.) \leqslant M$ on $[0, T]$.

Algorithm 1 Generate an inhomogeneous Poisson process by thinning.

INPUT: T is the time to simulate;

$\lambda(.)$ is the intensity function;

M is upper bounded value;

OUTPUT: Retrieve the simulated process $\{t_1, t_2, ..., t_n\}$ on $[0, T]$;

Require: $\lambda(.) \leqslant M$ on $[0, T]$.

1. $P \leftarrow []$, $t \leftarrow 0$
2. **while** $t < T$ **do**
3. Generate next candidate point: $E \leftarrow \text{Exp}\ (M)$, $t \leftarrow t + E$
4. Keep it with some probability: $U \leftarrow \text{Unif}(0, M)$
5. **if** $t < T$ **and** $U \leqslant \lambda(t)$ **then**
6. $P \leftarrow [P, t]$
7. **end if**
8. **end while**

Chapter 3

Hawkes Process

In this chapter, we discuss a self-exciting Hawkes process since the presence of past events will cause the future events more likely to appear. Firstly, we will introduce a definition of Hawkes process and then will give two algorithms to simulate them such as the Intensity-based Hawkes Process Model and the Cluster-based Hawkes Process Model.

Definition 3.1. ***(Hawkes process)*** *Consider* $(N(t) : t \geqslant 0)$ *a counting process, with associated history* $\mathcal{H}(t) : t \geqslant 0$*, that satisfies*

$$\mathbb{P}(N(t+h) - N(t) = m|\mathcal{H}(t)) = \begin{cases} \lambda^*(t)h + o(h), & \text{if } m = 1 \\ o(h), & \text{if } m > 1 \\ 1 - \lambda^*(t)h + o(h), & \text{if } m = 0 \end{cases}$$

Suppose the conditional intensity function is defined

$$\lambda^*(t) = \lambda + \int_0^t \mu(t-u)dN(u)$$

for some $\lambda > 0$ *and* $\mu : (0, \infty) \to [0, \infty)$ *which are called the background intensity and excitation function respectively. If* $\mu(.) \neq 0$ *then a process* $N(.)$ *is a Hawkes process.*

3.1 Intensity-based Hawkes Process

3.1.1 Description

Hawkes process is a Poisson process with a self-exciting intensity which is affected by the previous events. In the self-exciting point process, the conditional intensity includes the background rate of events and the memory kernel. The general intensity of Hawkes process is also defined by:

$$\lambda^*(t) = \lambda + \sum_{t_i<t} \mu(t - t_i)$$

In this case $\mu(t) = \alpha e^{-\beta t}$, we obtain

$$\lambda^*(t) = \lambda + \int_{-\infty}^{t} \alpha e^{-\beta(t-s)} dN(s) = \lambda + \sum_{t_i<t} \alpha e^{-\beta(t-t_i)},$$

where:

$\alpha > 0$ is the magnitude of self-excited jump;

$\beta > 0$ is the constant rate of decay;

$\lambda > 0$ is the background intensity.

Each arrival in the system instantaneously increases the arrival intensity by α, then over time this arrival's influence decays at rate β. We can choice $\mu(.)$ as follows:

$$\lambda^*(t) = \lambda + \int_{-\infty}^{t} \frac{k}{(c + (t-s))^p} dN(s) = \lambda + \sum_{t_i<t} \frac{k}{(c + (t-t_i))^p}$$

with some scalars $c, k, p > 0$.

On this report, we focus on the exponential form of the excitation function, one can consider as the Hawkes Process with exponentially decaying intensity.

To model a process from some time after it is started, we denote an initial condition $\lambda^*(0) = \lambda_0$. In this report, the conditional intensity process (using the exponential form of $\mu(.)$) to satisfy the stochastic differential equation.

$$d\lambda^*(t) = \beta(\lambda - \lambda^*(t))dt + \alpha dN(t), t \geqslant 0.$$

Applying stochastic calculus, we get the general solution

$$\lambda^*(t) = e^{-\beta t}(\lambda_0 - \lambda) + \lambda + \int_{0}^{t} \alpha e^{\beta(t-s)} dN(s), t \geqslant 0.$$

3.1.2 The algorithm

The idea of this algorithm is similar to inhomogeneous Poisson process, i.e., we generate a homogeneous Poisson process, and then remove the points probabilistically so that the remaining points satisfy the time-varying intensity $\lambda^*(.)$. However, the conditional intensity $\lambda^*(.)$ does not have an upper bound (the intensity to be non-increasing in periods without any arrivals). So the upper bound value M can be updated during each simulation.

Algorithm 2 Generate a Hawkes process by thinning.

INPUT: T is the time to simulate;

$\lambda^*(.)$ is the conditional intensity function;

OUTPUT: Retrieve the simulated process $\{t_1, t_2, ..., t_n\}$ on $[0, T]$;

Require: $\lambda^*(.)$ non-increasing in periods without any arrivals.

1. $\varepsilon \leftarrow 10^{-10}$, $P \leftarrow []$, $t \leftarrow 0$
2. **while** $t < T$ **do**
3. Find new upper bound: $M \leftarrow \lambda^*(t + \varepsilon)$
4. Generate next candidate point: $E \leftarrow \text{Exp}(M)$, $t \leftarrow t + E$
5. Keep it with some probability: $U \leftarrow \text{Unif}(0, M)$
6. **if** $t < T$ **and** $U \leqslant \lambda^*(t)$ **then**
7. $P \leftarrow [P, t]$
8. **end if**
9. **end while**

3.2 Cluster-based Hawkes Process

3.2.1 Description

A Hawkes process $(N(t), t \geqslant 0)$ can also be constructed as a cluster process. To count the population in a country where people arrive either via immigration or birth. Suppose that the immigrants to the country form a homogeneous Poisson process at rate λ. Each person can produce zero or more children in an independent and identically distributed, and the arrival of births form an inhomogeneous Poisson process.
An illustration of this interpretation can be represented in Figure 3.1.

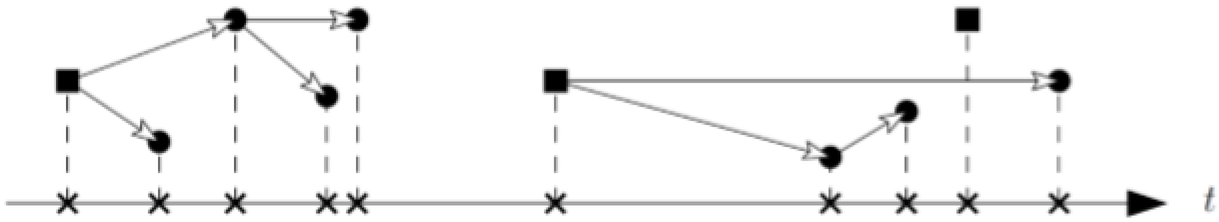

Figure 3.1: Hawkes process is represented as a collection of family trees. The squares (■) are the immigrants, the circles (•) are the descendants and the crosses (×) are the generated point process.

Theorem 3.1. *[1]* ***(Hawkes process asymptotic normality)*** *If*

$$0 < n := \int_0^t \mu(s)ds < 1 \text{ and} \int_0^\infty s\mu(s)ds < \infty$$

then the number of Hawkes process arrivals in $(0,t]$ *is asymptotically* $(t \to \infty)$ *normally distributed. More precisely, writing* $N(0,t] = N(t) - N(0)$,

$$\mathbb{P}\Big(\frac{N(0,t] - \lambda t/(1-n)}{\sqrt{\lambda t/(1-n)^3}} \leqslant y\Big) \to \phi(y),$$

where $\phi(.)$ *is the cumulative distribution function of the standard normal distribution.*

From Theorem 3.1, we have an exponentially decaying intensity

$$n = \int_0^\infty \alpha e^{-\beta s} ds = \frac{\alpha}{\beta}.$$

3.2.2 The algorithm

The idea of this algorithm is to generate the immigrants and descendants for each immigrant. The immigrants are distributed $Poi(\lambda T)$ on $[0,T]$. If k are the immigrants, then arrival times $C_1, C_2, ..., C_k$ are independent and identically distributed $Unif(0,T)$ random variables. The i^{th} descendants for each immigrant arrive at intensity $\mu(t - C_i)$ for $t > C_i$. If D_i is the number of descendants of immigrant i, then it is distributed $D_i \widetilde{\text{i.i.d}}\ Poi(n)$. So, we obtain the descendants of the i^{th} immigrant arrive at times $(C_i + E_1, C_i + E_2, ..., C_i + E_{D_i})$. Conditioned on knowing D_i, E_j are independent and identically distributed random variables distributed with the probability density function $\mu(.)/n$. For exponentially decaying intensity, one has $E_j \widetilde{\text{i.i.d}}\ Exp(\beta)$.

Algorithm 3 Generate a Hawkes process by clusters.

INPUT: T is the time to simulate;

(λ, α, β) are parameters of the conditional intensity function;

OUTPUT: P is the union of all clusters $\{C_1, C_2, ..., C_k\}$;

```
P ← {}
Generate the immigrants:
    k ← Poi(λT)
    C_1, C_2, ..., C_k ←(independent and identically distributed)− Unif(0, T)
Generate the descendants:
    D_1, D_2, ..., D_k ←(independent and identically distributed)− Poi(α/β)
for i ← 1 to k do
    if D_i > 0 then
        E_1, E_2, ..., E_{D_i} ←(independent and identically distributed)− Exp(β)
        P ← P ∪ {C_i + E_1, C_i + E_2, ..., C_i + E_{D_i}}
    end if
end for
Remove the descendant outside [0, T]: P ← {P_i : P_i ∈ P, P_i ⩽ T}
Add in immigrants and sort: P ← Sort (P ∪ {C_1, C_2, ..., C_k})
```

Chapter 4

Simulation Algorithms

In this chapter, we will give some simulations in Matlab® environment: Inhomogeneous Poisson Process, Intensity-based Hawkes Process, Cluster-based Hawkes Process.

4.1 Simulation of Inhomogeneous Poisson Process

The Inhomogeneous Poisson Process is given in Algorithm 1 that is performed Matlab® environment.

```
function [A,Ay,R,Ry] = InhomogeneousPoissonProcess(T,lambda,M)
    t = 0;
    A = []; Ay = [];
    R = []; Ry =[];
    while t < T
        t = t + exprnd(1/M);
        U = M * rand();
        if (t < T) && (U <= lambda(t))
            A = [A, t];
            Ay = [Ay, U];
        else
            R = [R, t];
            Ry = [Ry, U];
        end
    end
end
```

Example 4.1. *Simulating an inhomogeneous Poisson process with intensity function* $\lambda(t) = 2 + \sin(t)$, *bounded value* $M = 4$ *and* $T = 4\pi$.

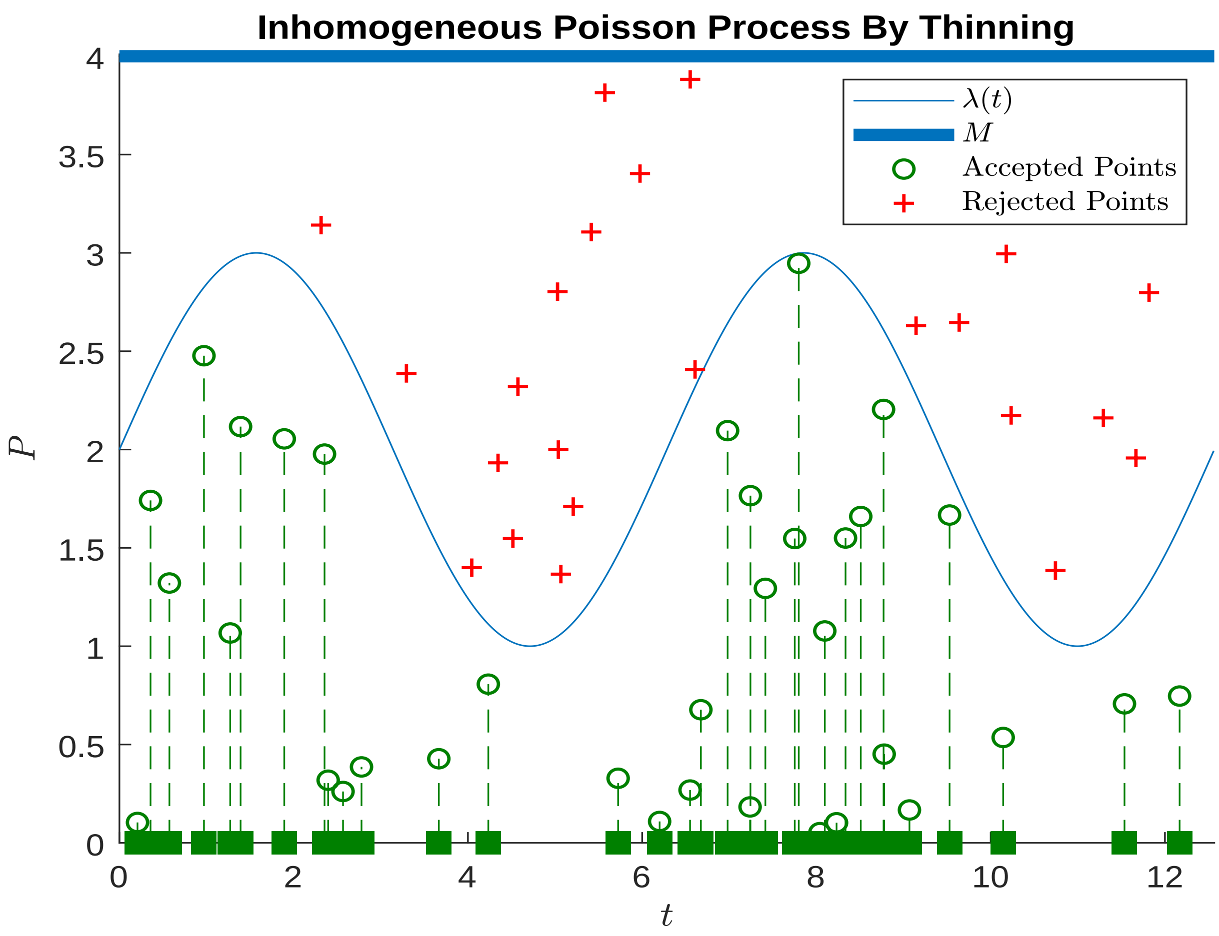

Figure 4.1: The inhomogeneous Poisson process.

Figure 4.1 illustrates the result of inhomogeneous Poisson process. Each (t, P) point describes a proposed arrival at time t whose P value. The circles are the accepted points, the plus signs are the rejected points and the squares are the point processes.

4.2 Simulation of Intensity-based Hawkes Process

The Intensity-based Hawkes Process is given in Algorithm 2 that is performed Matlab® environment.

```
function [A,Ay,R,Ry,MXs,MYs] = HawkesProcessByThinning(T,lambda,alpha,beta)
    M = lambda; t = 0;
    A = []; Ay = [];
    R = []; Ry = [];
    MXs = []; MYs = [];
    while t < T
        lastM = M; lastT = t;
```

```
        M = cif(t+1e-10,A,lambda,alpha,beta);
        t = t + exprnd(1/M);
        MXs = [MXs, [lastT; t]];
        MYs = [MYs, [M; M]];
        U = M*rand();
        if (t < T) && (U <= cif(t,A,lambda,alpha,beta))
            A = [A, t];
            Ay = [Ay, U];
        else
            R = [R, t];
            Ry = [Ry, U];
        end
    end
end
```

```
function lambda = cif(t,H,lambda_0,alpha,beta)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Consider: lambda^{*}(t) = lambda_0 + alpha*sume^{-beta(t-t_j)}
% H--the history of the process
% lambda_0--the parameter of lambda_0
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
lambda = lambda_0*ones(length(t),1);
for i = 1 : length(t)
    h = H;
    h = h(h < t(i)) ;
    if ~ isempty(h)
        lambda(i) = lambda(i) +alpha*sum(exp(- beta * (t(i) - h)));
    end
end
```

Example 4.2. *Simulating a intensity-based Hawkes Process with condition intensity function* $(\lambda, \alpha, \beta) = (1, 1, 1.2)$ *and* $T = 4$.

Figure 4.2 illustrates the result of the Intensity-based Hawkes Process in the Example 4.2. Each (t, P) point describes a proposed arrival at time t whose P value. The circles are the accepted points, the plus signs are the rejected points and the squares are the point processes.

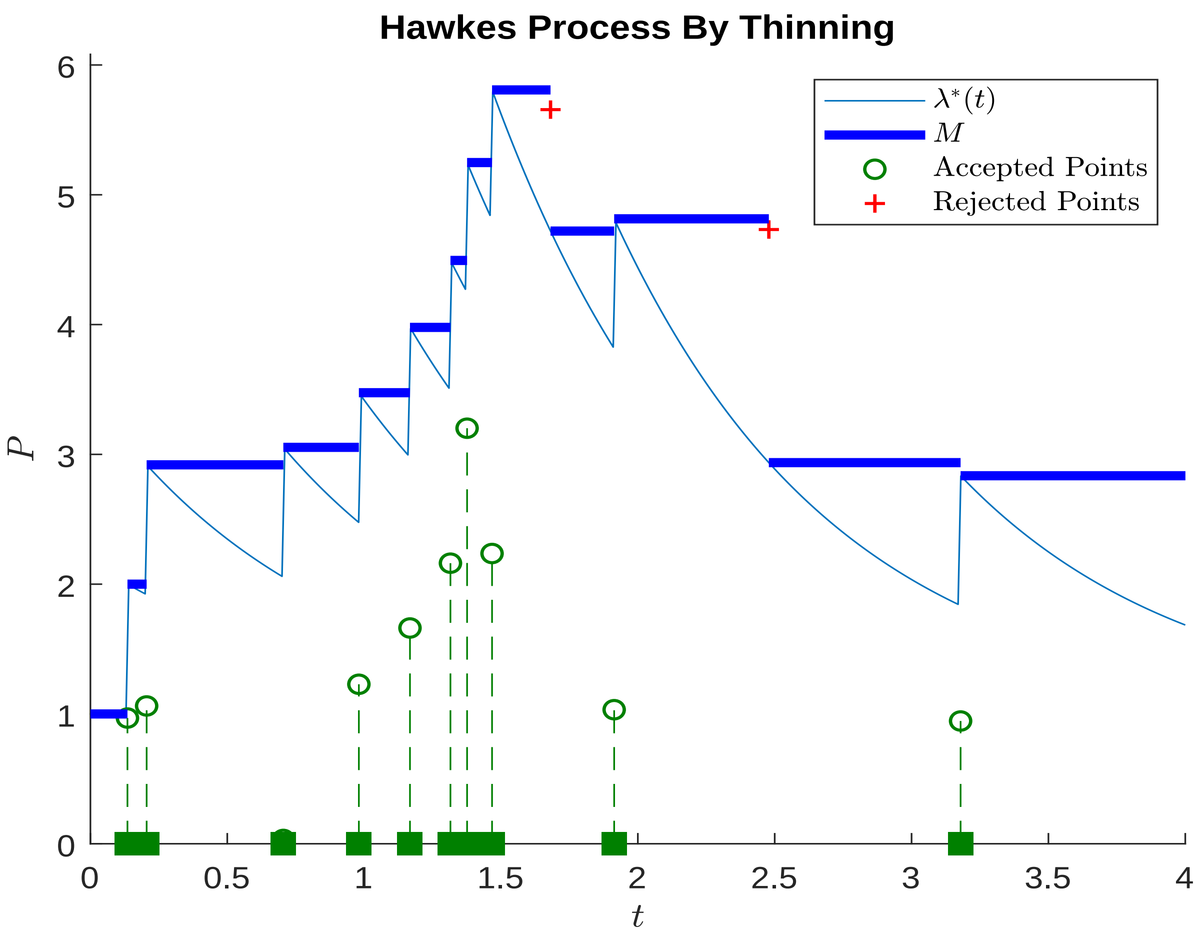

Figure 4.2: The Intensity-based Hawkes Process.

4.3 Simulation of Cluster-based Hawkes Process

The Cluster-based Hawkes Process is given in Algorithm 3 that is performed Matlab® environment.

```
function HawkesProcessByClustering(T,lambda,alpha,beta)
    k = poissrnd(lambda*T);
    C = sort(T*rand(k,1));
    D = poissrnd(alpha/beta,k,1);
    allDes = [];
    hold on;
    colorOrder = get(gca,'ColorOrder');
    for i = 1 : k
        color = colorOrder(mod(i, size(colorOrder, 1))+1,:);
        numDes = poissrnd(alpha/beta);
        Des = C(i) + exprnd(1/beta, numDes, 1);
        scatter(C(i), i, [], color,'filled','s');
```

```
        scatter(Des, i.*ones(size(Des)), [], color);
        allDes = [allDes; Des];
    end
    allpoints = sort([C; allDes]);
    title('Hawkes Process By Cluster');
    scatter(allpoints, zeros(size(allpoints)), 100, [0,0,0],'x','LineWidth'
        ,1);
    xlabel('$t$','interpreter','latex');
    ylabel('Family Number','interpreter','latex');
    a = axis(); axis([0, T, a(3), a(4)]);
end
```

Example 4.3. *Simulating a cluster-based Hawkes Process with condition intensity function* $(\lambda, \alpha, \beta) = (1, 2, 1.2)$ *and* $T = 10$.

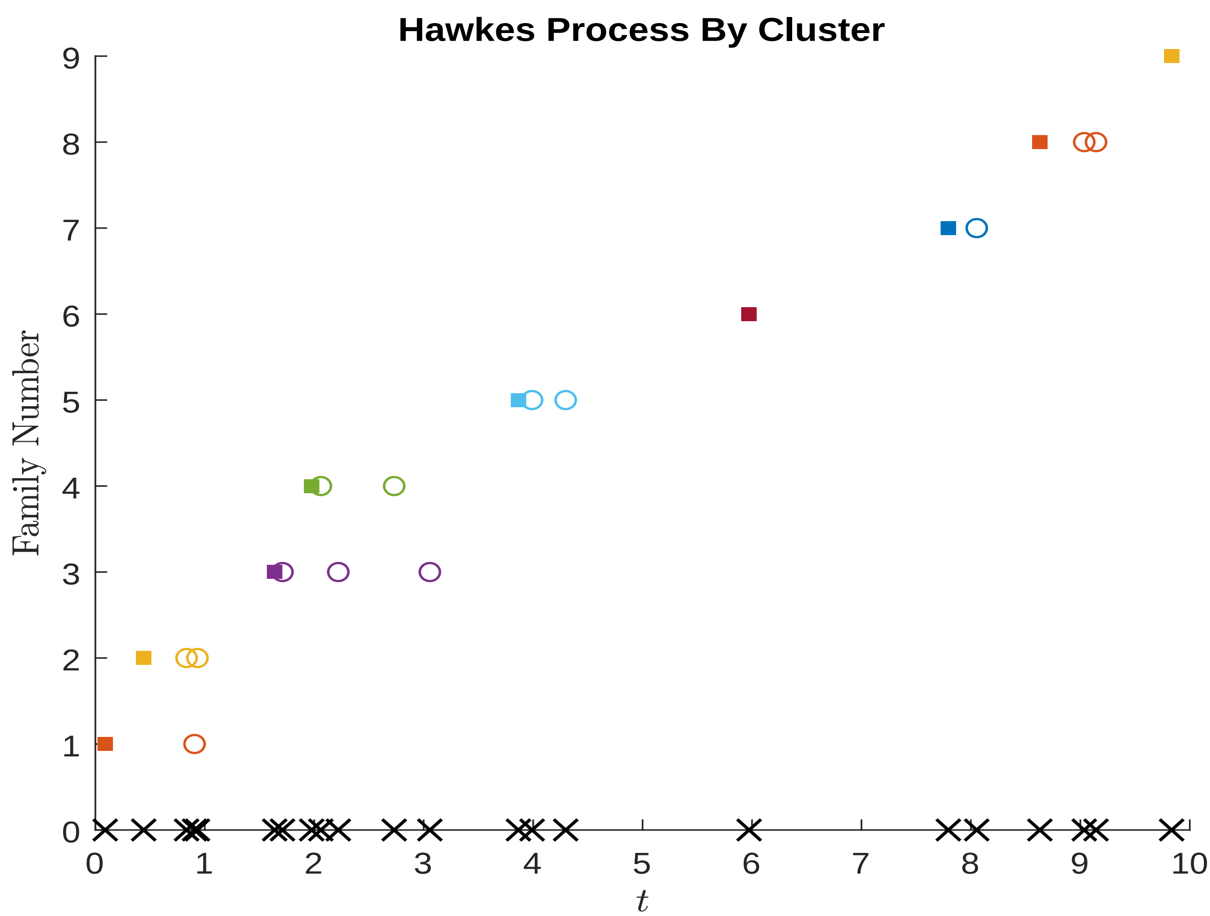

Figure 4.3: The Cluster-based Hawkes Process.

Figure 4.3 illustrates the result of the Cluster-based Hawkes Process. The squares are the immigrant points, the circles of the same height and color are descendant of immigrant points and the crosses are the point processes.

Chapter 5

An application of Hawkes process

In reality, a Hawkes processes have been applied in many areas from earthquake to financial analysis. In this chapter, we introduce an application of Hawkes process in seismology.

In the 1970s, professor Alan Hawkes introduced a mathematical model of self-exciting process in seismology, which is called Hawkes process. And then it was expanded by Y.Ogata and L.Adamopoulos. In Figure 5.1 illustrates the number of shocks in periods of three months for an area of the North Atlantic as same as the stochastic intensity function of a Hawkes process. Therefore, the ETAS model was introduced to modeling the earthquake times and magnitudes [2]. They are given as follows:

$$\lambda(t) = \lambda_0 + \alpha \sum_{T_i < t} e^{\beta \kappa_i} e^{-\delta(t-T_i)},$$

where $\kappa_i \in [0, \infty)$ is the magnitude of an earthquake occurring at time T_i and $\alpha, \beta, \delta > 0$ are parameters. In case

$$f(\kappa|t) = \gamma e^{-\gamma t}$$

Additional, we can define it by its conditional intensity function including both marks and times

$$\lambda(t, \kappa) = (\lambda_0 + \alpha \sum_{T_i < t} e^{\beta \kappa_i} e^{-\delta(t-T_i)}) \gamma e^{-\gamma t}$$

The idea of this model is that earthquakes cause aftershocks, large earthquakes increase the intensity more than small earthquakes.

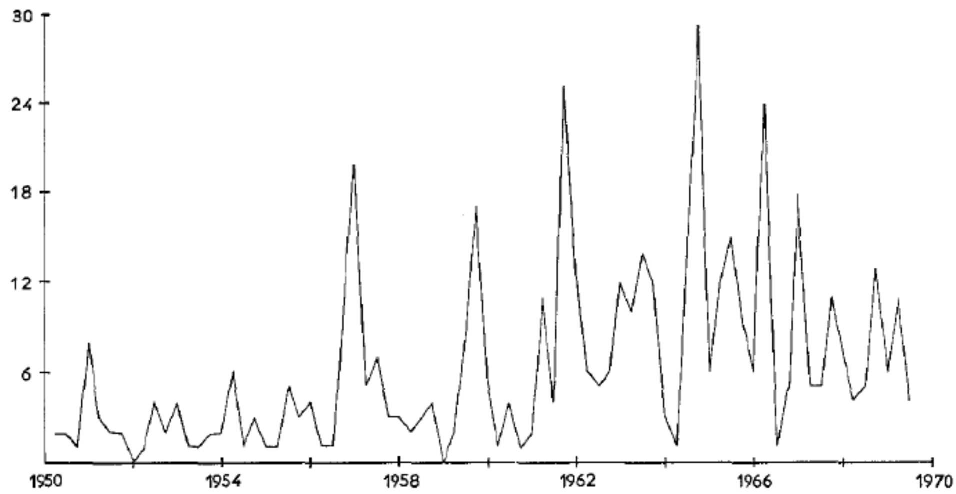

Figure 5.1: Number of shocks in periods of three months for area of North Atlantic [2].

Chapter 6

Conclusion

In this report, we have studied Hawkes process is a self-exciting stochastic process, which based on a counting process in which the intensity function depends on all previously occurred events.

In the Chapter 2, we have reviewed some basis knowledge such as point process, counting process, conditional intensity function, inhomogeneous Poisson process to apply for analysis of the Hawkes process.

In the Chapter 3, we have studied two models to simulate Hawkes process: the intensity-based Hawkes process Model and cluster-based Hawkes process Model.

In the Chapter 4, we have simulated three algorithms such as inhomogeneous Poisson process, Intensity-based Hawkes Process and Cluster-based Hawkes Process in Matlab® environment.

In the Chapter 5, we have studied an application of Hawkes process in seismology. Currently, we only simulate one dimensional Hawkes processes. However, we can extend to simulate the multidimensional Hawkes processes in the future. This is a limitation of this report.

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[1] T. T. Laub Patrick J. and P. K. Pollett, "Hawkes processes", 2015. [Online]. Available: https://arxiv.org/abs/1507.02822.

[2] K. Obral, "Simulation, estimation and applications of hawkes processes", Master's thesis, Minnesota, 2016.