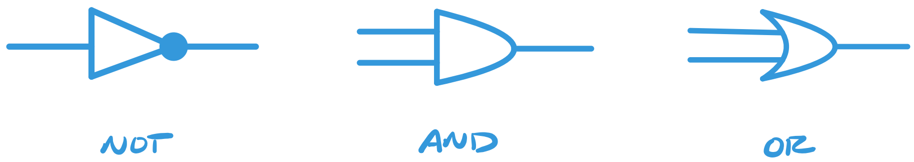
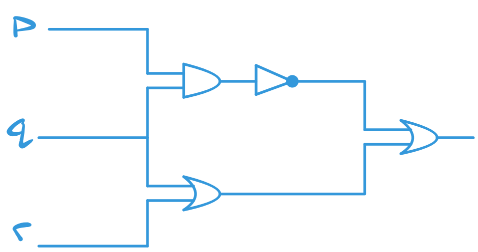
CIRCUITS

Circuits or **Logic Gates** are elements which recieve a number of bit inputs and return a number of bit outputs. For this course, we will consider 3 types of gates:



Note that these gates directly map to the three basic operators in propositional logic. This allows us an alternative / useful / intuitive representation of compound statements as we know them.

$$\neg(p \wedge q) \vee (r \vee q)$$



You should be able to create and interpret these circuits i.e go from statement to circuit and vice versa.

However, we have a third way to represent compound statements: truth-tables. By now you should be able to fill out a truth-table for a given statement, but can we go the other way? That is, given the outputs of a truth-table, can we determine the statement/circuit that generates it?.

The answer is yes, and the methodology is either **Disjunctive Normal Form (DNF)** or **Conjunctive Normal Form (CNF)**.

A compound statement is in **DNF** if it is a disjunction of conjunctions of atomic propositions and their negations:

$(p \wedge q \wedge \neg r) \vee (p \wedge q)$ ✓ $\qquad$ $p \wedge (q \vee r)$ ✗

Similarly, a statement is in CNF if it is a conjunction of disjunctions of atomic propositions and their negations:

$(p \vee \neg q) \wedge (p \vee s)$ ✓ $\qquad$ $(p \wedge q) \vee r$ ✗

These two forms will allow us to derive a statement from just the outputs of a truth-table:

p	q	r	?
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

We'll first discuss how to get an answer in DNF: simply create a disjunction of all the inputs which generate a "true" output:

$$(\bar p \wedge \bar q \wedge r) \vee (\bar p \wedge q \wedge \bar r) \vee (\bar p \wedge q \wedge r) \vee (p \wedge \bar q \wedge r)$$

To get an answer in CNF, we'll simply get the DNF form for the inputs that generate a "false" output, and then negate the statement:

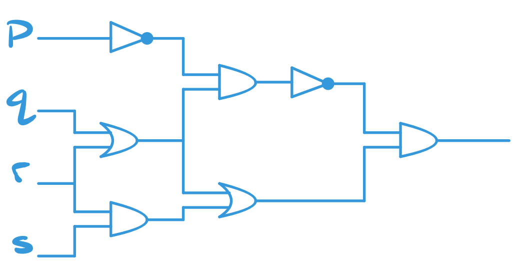
$$\neg\left[(\bar p \wedge \bar q \wedge \bar r) \vee (p \wedge \bar q \wedge \bar r) \vee (p \wedge q \wedge \bar r) \vee (p \wedge q \wedge r)\right]$$

$$\equiv (p \vee q \vee r) \wedge (\bar p \vee q \vee r) \wedge (\bar p \vee \bar q \vee r) \wedge (\bar p \vee \bar q \vee \bar r)$$

From here, you can use laws of equivalence to to get a logically equivalent but symbolically simpler statement.

PRACTICE

Give the corresponding statement for the circuit below:



$$\neg(\neg p \wedge (q \vee r)) \wedge ((q \vee r) \vee (r \wedge s))$$

For the following problem, represent the truth-table below as a circuit using at most 3 gates. Show your simplification process:

p	q	r	?
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Laws	$p \wedge 1 \equiv p$	$p \vee 0 \equiv p$
Negation Laws	$p \vee \sim p \equiv 1$	$p \wedge \sim p \equiv 0$
Double Negation Law	$\sim(\sim p) \equiv p$	
Idempotent Laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound Laws	$p \vee 1 \equiv 1$	$p \wedge 0 \equiv 0$
DeMorgan's Laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation Laws of t and c	$\sim 1 \equiv 0$	$\sim 0 \equiv 1$
Definition of Implication	$p \Rightarrow q \equiv \sim p \vee q$	
Definition of Biconditional	$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \wedge q) \vee \sim(p \vee q)$	
Contrapositive	$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$	

DNF: $(\bar{p} \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge q \wedge \bar{r}) \vee (\bar{p} \wedge q \wedge r)$

$\equiv \bar{p} \wedge ((\bar{q} \wedge r) \vee (q \wedge \bar{r}) \vee (q \wedge r))$ DISTRIBUTIVE

$\equiv \bar{p} \wedge ((\bar{q} \wedge r) \vee (q \wedge (\bar{r} \vee r)))$ DISTRIBUTIVE

$\equiv \bar{p} \wedge ((\bar{q} \wedge r) \vee (q \wedge 1))$ NEGATION

$\equiv \bar{p} \wedge ((\bar{q} \wedge r) \vee q)$ IDENTITY

$\equiv \bar{p} \wedge ((\bar{q} \vee q) \wedge (r \vee q))$ DISTRIBUTIVE

$\equiv \bar{p} \wedge (1 \wedge (r \vee q))$ NEGATION

$\equiv \bar{p} \wedge (r \vee q)$ IDENTITY