04/04/2022

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STRUCTURAL INDUCTION

Recall that performing strong/weak induction requires a countable set with some sense of ordering to iterate over. Sometimes we run into sets that lack this natural ordering. But, as long as we have 1) some sort of "base" elements and 2) recursive rules for adding new elements, we can apply an idea analogous to induction as you know it to prove properties about the set.

Examples of sets like these include...

RECURSIVE SETS

We've defined sets in many ways so far:

$\mathbb{N}$

$\{0, 1, 2, \dots\}$

$\{x \mid x \in \mathbb{Z} \wedge x \geq 0\}$

We now introduce another way; defining them recursively:

$0 \in \mathbb{N}$

$n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

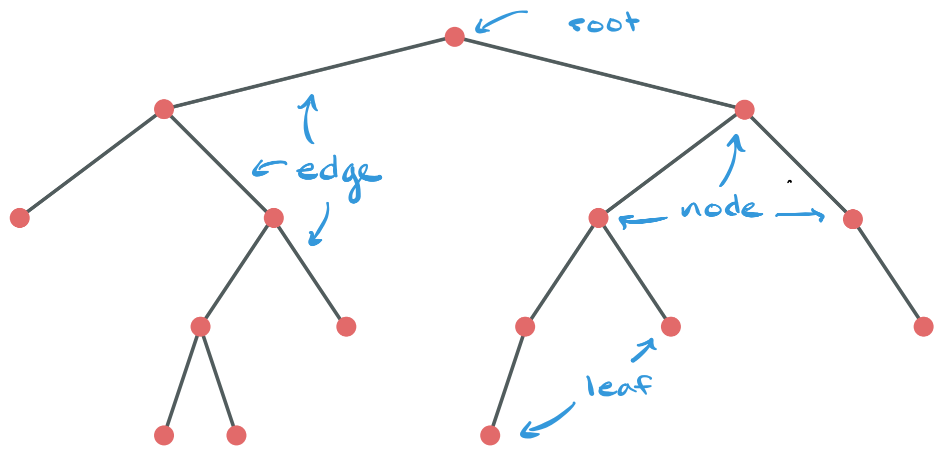
The example above has one "rule", but there can be more:

$0 \in \mathbb{Z}$

$z \in \mathbb{Z} \Rightarrow z+1 \in \mathbb{Z} \wedge z-1 \in \mathbb{Z}$

BINARY TREES

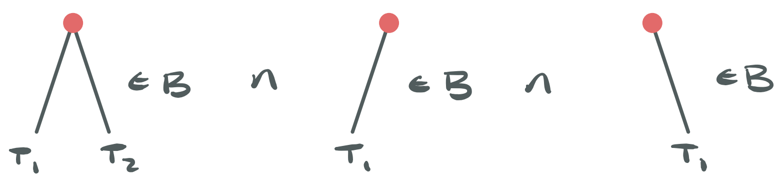
Here's an example of a binary tree:



Here's how we define the set of all possible binary trees B:

$\bullet \in B$

$T_1, T_2 \in B \Rightarrow$



So binary trees are one of those recursively defined collection of objects that we like to prove properties about. Doing so often involves the following definitions:

For a binary tree $T \in B$, we define...

$N(T) \leftarrow$ number of nodes in T

$E(T) \leftarrow$ number of edges in T

$L(T) \leftarrow$ number of leaves in T

$H(T) \leftarrow$ height of T where $H(\bullet) = 0$

Also, a tree $T \in B$ is *perfect* if every node has either 0 or 2 children and all leaves are at the same height.

PRACTICE

Consider S where ...

$(0,0) \in S \wedge (x,y) \in S \Rightarrow (x+1, y+3), (x+3, y+1) \in S$

Prove $(\forall (x,y) \in S)[\,4 \mid x+y\,]$

BC: $(0,0) \in S \wedge 4 \mid 0$ ✓

IH: Consider an arbitrary element $(x,y) \in S$ where $4 \mid x+y$.

IS: $4 \mid x+y \equiv (\exists k \in \mathbb{Z})[\,4k = x+y\,]$

$(x+1, y+3) \in S: \; x+1+y+3 \overset{\text{by IH}}{=} 4k+4 = 4(k+1) \Rightarrow 4 \mid x+1+y+3$

$(x+3, y+1) \in S: \; x+3+y+1 \overset{\text{by IH}}{=} 4k+4 = 4(k+1) \Rightarrow 4 \mid x+3+y+1$

QED.

Consider S where ...

$0 \in S \wedge x \in S \Rightarrow 2x+1 \in S$

Prove $S \subseteq \{2^n - 1 \mid n \in \mathbb{N}\} = \{0, 1, 3, 7, \dots\}$

$n = 0, 1, 2, 3$

BC: $0 \in S \wedge 0 = 2^0 - 1$ ✓

IH: Assume for an arbitrary $s \in S$, $(\exists n \in \mathbb{N})[\,s = 2^n - 1\,]$

IS: $2s+1 \in S$ by IH

$2(2^n - 1) + 1 = 2 \cdot 2^n - 2 + 1 = 2^{n+1} - 1$

$n+1 \in \mathbb{N}$ by closure

QED.

Let B be the set of non-empty binary trees.

Prove $(\forall T \in B)[\,N(T) = E(T) + 1\,]$

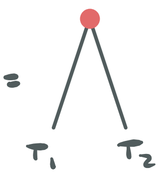
BC: $N(\bullet) = 1 \qquad E(\bullet) = 0$

$1 = 0 + 1$ ✓

IH: Assume for two arbitrary $T_1, T_2 \in B$

$$N(T_1) = E(T_1) + 1 \quad \wedge \quad N(T_2) = E(T_2) + 1$$

IS:

$T =$ 

$$N(T) = 1 + N(T_1) + N(T_2)$$
$$E(T) = 2 + E(T_1) + E(T_2)$$

by IH, $N(T) = 1 + E(T_1) + 1 + E(T_2) + 1$
$\qquad = 1 + E(T)$

$T =$ 

$$N(T) = 1 + N(T_1)$$
$$E(T) = 1 + E(T_1)$$

by IH, $N(T) = 1 + E(T_1) + 1$
$\qquad = 1 + E(T)$

$T =$ 

$$N(T) = 1 + N(T_2)$$
$$E(T) = 1 + E(T_2)$$

by IH, $N(T) = 1 + E(T_2) + 1$
$\qquad = 1 + E(T)$

QED.

Consider S where ...

$3 \in S \quad \wedge \quad x, y \in S \Rightarrow x - y + 2 \in S$

Prove $S = \mathbb{Z}$

BC: $3 \in S \quad \wedge \quad 3 \in \mathbb{Z}$ ✓

$3 - 3 + 2 \in S \quad \wedge \quad 2 \in \mathbb{Z}$

$2 - 3 + 2 \in S \quad \wedge \quad 1 \in \mathbb{Z}$

IH: Assume for an arbitrary $k \in S$, $k \in \mathbb{Z}$.

IS: $k, 1, 3 \in S$

$\Rightarrow k - 1 + 2 = k + 1 \in S$

$\Rightarrow k - 3 + 2 = k - 1 \in S$

by closure $k - 1, k + 1 \in \mathbb{Z}$. QED.