11 / 14 / 2022

→ COMBINATORICS

COMBINATORICS

Combinatorics is the mathematical field of counting. The utility is obvious, and although this may seem trivial, it can get rapidly complex. Consider this relatively simple example:

How many integers between 4 and 8 inclusive?

4 5 6 7 8 so five, obviously. But, this listing technique quickly becomes infeasible. For example:

How many integers between 37 and 94 inclusive?

Clearly, we don't want to list these out. Instead, we'll utilize the following formula: $|[m,n]| = m + n - 1$. So, in this case the answer is 58. What about the following example.

How many integers between 14 and 76 inclusive are divisible by 4?

Consider the following:

$$16 = 4(4) \quad 20 = 4(5) \quad \cdots \quad 72 = 4(18) \quad 76 = 4(19)$$

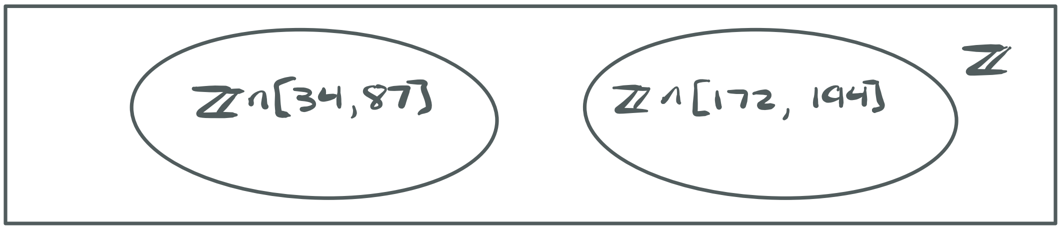
So we can apply our previous formula to compute the answer to this prompt, by calculating $|[4,19]| = 19 - 4 + 1 = 16$.

The purpose of that tedious walk through was to demonstrate how you should think about combinatorics. On your end, this section of the course will consist of picking up useful computational methods you can use to break down complex counting problems into something computable.

ADDITION RULE

Think of all the instances you want to count as a set. If you can split that set into disjoint subsets, you can compute the parts seperately and you add the results:

How many numbers between 34 and 87 or between 172 and 194?



$$87 - 34 + 1 + 194 - 172 + 1 = 77$$

MULTIPLICATION RULE

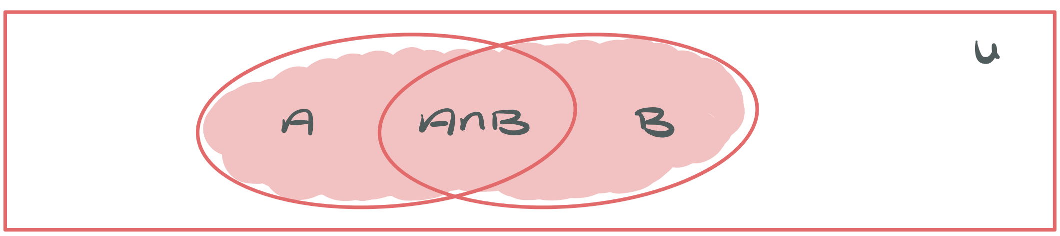
If you can think of the set you're trying to compute as a sequence of independent choices, then the combinations multiply.

You have 6 shirts, 4 pants, 2 pairs of shoes, how many unique outfits?

$$6 \cdot 4 \cdot 2 = 48$$

INCLUSION - EXCLUSION PRINCIPLE

We're familiar with splitting a complex set into to simpler, disjoint sets. What if these sets weren't disjoint sets?

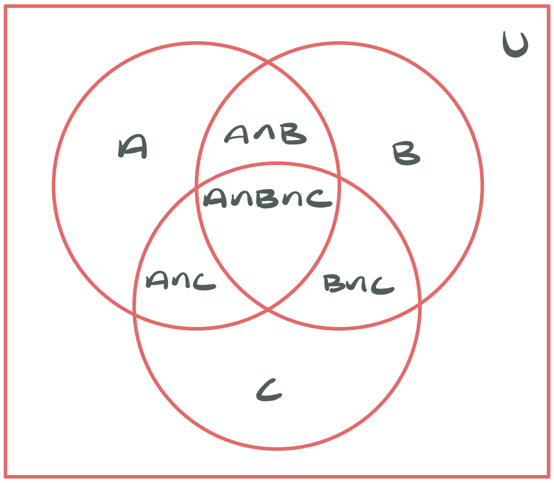


What would $|A \cup B|$ be? Convince yourself that the following is true:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can we extend this to three sets? Yes,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Convince yourself this holds by considering the overcounting that occurs when you compute $|A| + |B| + |C|$ and how the formula fixes that.

This generalizes to the inclusion - exclusion principle, that is, the cardinality of the union of any number of sets can be found by...

1) include the cardinality of the sets

2) exclude the cardinality of the pairwise intersections

3) include the cardinality of the 3-wise intersections

⋮ so on and so forth.

PERMUTATIONS

How many permutations of length k can be constructed from n unique elements? We use the following formula:

$$P(n,k) = \frac{n!}{(n-k)!}$$

This formula arises pretty directly from the multiplication rule. A common complication we should address is removing the uniqueness assumption. We can model this by asking how many permutations are there of n elements, where there are m unique elements with $a_1, a_2, \ldots, a_m$ duplicates. Then, our formula would be...

$$\frac{n!}{a_1! \, a_2! \cdots a_m!}$$

Finding permutations of length $<n$ is more complicated.

n CHOOSE k

What if we don't care about the order of the elements; we want to know that given a set of size n, how many subsets of size k there are.

$$\binom{n}{k} = \frac{P(n,k)}{P(k)} = \frac{n!}{k!\,(n-k)!}$$

PRACTICE

How many permutations of "doom"?

$$\frac{4!}{2!}$$

How many 4-letter strings from "Potsdamn"

$$P(8,4) = \frac{8}{4!}$$

How many permutations of "bookkeeper"?

$$\frac{10!}{3!\,2!\,2!}$$

How many permutations of "Mississippi"?

$$\frac{11!}{4!\,4!\,2!}$$

How many ways to choose five letters from the alphabet?

$$\binom{26}{5} = \frac{26!}{21!\ 5!}$$

We want to choose 6 ppl from a group of 10, but 3 of them refuse to be seperated

Our group either includes or doesn't include the group of 3:

Includes: $\checkmark\!\_\ \checkmark\!\_\ \checkmark\!\_\ \_\ \_\ \_$ $\binom{7}{3}$

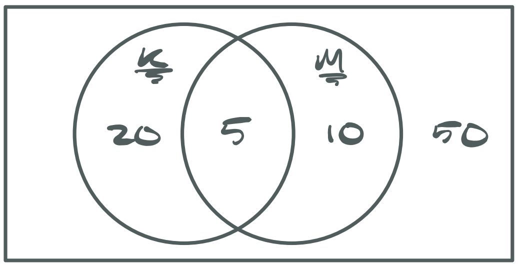
Doesn't: $\_\ \_\ \_\ \_\ \_\ \_$ $\binom{7}{6}$

$$\binom{7}{3} + \binom{7}{6}$$

We have 50 ppl in a classroom,

→ for 20, name starts with K
→ for 10, name ends with M
→ for 5, name starts with K & ends with M

How many student's have a name that doesn't start with K nor ends with M:



$$|K \cup M| = |K| + |M| - |K \cap M|$$

$$= 20 + 10 - 5 = 25$$

$$U - |K \cup M| = 50 - 25 = 25$$

How many groups of size 25 can we make such that the majority of their names start with "K"?

$$\sum_{i=13}^{20} \binom{20}{i} \binom{30}{25-i}$$