STRONG INDUCTION

IMO, the concept of strong induction is best understood by comparing it to weak induction. It is underwhelming.

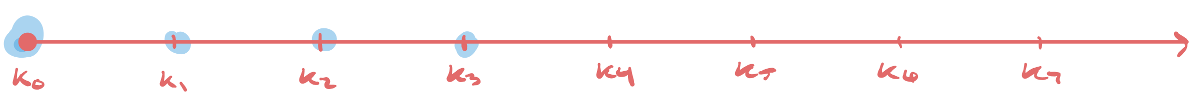
Motivation: We want to prove that some property $P(k_n)$ holds for every element k_n in some countable set $\mathbb{K}$.

WEAK INDUCTION: $P(k_0) \wedge [P(k_n) \rightarrow P(k_{n+1})]$

BC IH IS

$\therefore (\forall k \in \mathbb{K})[P(k)]$

You can see why this gets compared to dominos, right?



So to prove $P(k_3)$, for example, we utilize the fact that it holds for $P(k_2)$.

The key innovation for strong induction is realizing that by the time you want to prove $P(k_3)$, you can also rely on $P(k_0)$ and $P(k_1)$ in addition to $P(k_2)$ if doing so is useful.

So,

weak induction: $P(k_2) \Rightarrow P(k_3)$

strong induction: $[P(k_0) \wedge P(k_1) \wedge P(k_2)] \Rightarrow P(k_3)$

The strong induction outline may then look like...

STRONG INDUCTION:

$$P(k_0) \wedge \cdots \wedge P(k_i) \wedge \left[\bigwedge_{j=i}^{n} P(k_j) \Rightarrow P(k_{n+1})\right]$$

$\therefore (\forall k \in \mathbb{K})[P(k)]$

BC IH IS

Rq, we can run through how this change in logic impacts each component of our inductive proof:

BASE CASE(S): Its not uncommon to have multiple base cases when doing strong induction. How many you need is dependent on your IH; any $k \in \mathbb{K}$ to which you cannot apply your IH must be a base case.

INDUCTIVE HYPOTHESIS: We used to assume $P(k_n)$ for some arbitrary $k_n \in \mathbb{K}$. Now, we assume $P(k_i)$ for every $i \in \{0, \ldots, n\}$.

INDUCTIVE STEP: Again, the IS is the proof-y part. Everything else is pretty standard for every strong induction proof. You have more premises to work with in a strong induction proof.

PRACTICE

An airplane crashes in an island in the middle of nowhere. Each of the $n \geq 1$ passengers starts off alone (separated from the others). We call each passenger a *group*.

The passengers start wandering around. When two groups meet, everyone from one group shakes hands with everyone from the other group, and then they form a new, larger group.

Eventually, everyone meets and they form one large group of size n.

Use Strong Induction to prove that there will be a total of exactly $\frac{n(n-1)}{2}$ handshakes.

(Your proof should work no matter the order in which groups meet. In particular, do NOT make the assumption that groups can only accumulate members one by one.)

$P(n) \equiv$ a group of size n took $\frac{n(n-1)}{2}$ handshakes to form

BC: $n=1$. 0 handshakes.

$$\frac{1(1-1)}{2} = 0 \checkmark$$

IH: Assume for some arbitrary $k \in \mathbb{Z}^+$, $P(i)$ for every $i \in \{1, \ldots, k\}$

$$(\forall i \in [1,k] \cap \mathbb{N})\left[\text{a group of size } i \text{ took } \frac{i(i-1)}{2} \text{ handshakes to form}\right]$$

IS: If there's a group of size $k+1$, that means two groups of size $a,b \in \mathbb{N}$ s.t $a+b = k+1$ met.

GROUP A + GROUP B = GROUP $k+1$

Then,

total # handshakes = # of handshakes to form A

+ # of handshakes to form B

+ # handshakes when A met B.

Note this is equal to $a \cdot b$.

By IH,

# of handshakes to form A = $a(a-1)/2$

# of handshakes to form B = $b(b-1)/2$

Thus, total handshakes $= \dfrac{a(a-1)}{2} + \dfrac{b(b-1)}{2} + ab$

Recall, $a+b = k+1$

$$= \frac{a^2 - a + b^2 - b + 2ab}{2}$$

$$= \frac{(a+b)(a+b-1)}{2}$$

$P(k+1) \equiv$ total handshakes $= \dfrac{(k+1)(k)}{2}$

QED.

Let $a_n \begin{cases} 0 & n=0 \\ 4 & n=1 \\ 6a_{n-1} - 5a_{n-2} & n>1 \end{cases}$

Prove $a_n = 5^n - 1$

BC:

$n=0$, $a_0 = 0$ $5^0 - 1 = 0$

$n=1$, $a_1 = 4$ $5^1 - 1 = 4$

IH: Assume for some arbitrary $k \in \mathbb{N}$,

$$(\forall i \in [1,k] \cap \mathbb{N})[\,a_i = 5^i - 1\,]$$

IS: WTS: $a_{k+1} = 5^{k+1} - 1$

$$a_{k+1} = 6a_k - 5a_{k-1}$$

by IH $\Rightarrow = 6(5^k - 1) - 5(5^{k-1} - 1)$

$$= 6\cdot 5^k - 6 - 5\cdot 5^{k-1} + 5$$

$$= 6\cdot 5^k - 5^k - 6 + 5$$

$$= 5\cdot 5^k - 1$$

$$a_{k+1} = 5^{k+1} - 1 \equiv P(k+1)$$

QED.