03/16/2022

→ WEAK INDUCTION

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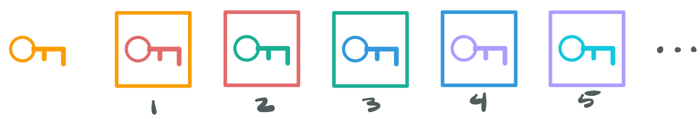
Idea behind induction:

- Key-Box Metaphor:



- Recursion

Basically trying to convey that if

2) a property holds for the k^{th} object implies it holds for the $k+1^{th}$ object
($\Rightarrow$)

1) the property holds for the first object

then, the property holds for all the objects.

Inductive proofs apply this idea to prove some property $P(n)$ holds for every element in some countable, ordered set.

why?

They have three parts:

why 1?

BASE CASE (BC): show when $n = 1$, $P(n)$

i.e show the property you're interested in holds for first element in your ordered set.

INDUCTIVE HYPOTHESIS (IH): Assume $P(k)$ for some k

together, the IH : is verify an implication. To show $p \Rightarrow q$, you first assume p. That's what we're doing here. Note, k is an arbitrary element in your set, and "for some k" is NOT $\exists k \in S$.

INDUCTIVE STEP (IS): Show $P(k) \Rightarrow P(k+1)$

Now, verify the implication. You will need to use / substitute / reference your IH at some point in this process. When you do this, write "by IH"

EXAMPLE: Verifying Gauss Summation

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{for any } n \in \mathbb{N}^{\geq 1}$$

Here, our ordered, countable set is $\mathbb{N}^{\geq 1}$ and

$$P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

BC: Show when $n=1$, $P(n)$

$$n=1 \qquad \sum_{i=1}^{1} i = 1 \qquad \frac{1(1+1)}{2} = 1 \quad \checkmark$$

IH: Assume $P(k)$ for some k

Assume for some arbitrary $k \in \mathbb{N}^{\geq 1}$,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

IS: Show $P(k) \Rightarrow P(k+1)$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 \underset{\text{by IH}}{=} \frac{k(k+1)}{2} + k+1$$

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

$$P(k+1) \equiv \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Conclusion:

Thus $(\forall n \in \mathbb{N}^{\geq 1})\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right]$

QED.

EASY TO FUCK UP: what's wrong with this proof?

All horses are the same color i.e any group of $n \in \mathbb{N}$ horses are the same color.

BC: $n=1$. 1 horse. 1 color

IH: Assume any group of $k \in \mathbb{N}$ horses are all the same color.

IS: You have $k+1$ horses. Remove any horse and you have k horses, which by our IH are all the same color. Do it again.

$k+1$ horses:

These k are the same color by IH

These k are the same color by IH

These are in both

$\therefore$ all horses are the same color.

The inductive step is not valid for $n=2$.

PRACTICE

$$a_n = \begin{cases} 3 & n=0 \\ 5a_{n-1}+8 & n \geq 1 \end{cases}$$

Show $a_n \equiv 3 \pmod 4$

BC: $n=0$

$$a_0 = 3 \equiv 3 \pmod 4$$

IH: Assume for some arbitrary $k \in \mathbb{N}$

$$a_k \equiv 3 \pmod 4$$

IS: $a_{k+1} = 5a_k + 8$

by IH, $a_k \equiv 3 \pmod 4 \iff (\exists z \in \mathbb{Z})[a_k = 3 + 4z]$

$$a_{k+1} = 5(3+4z) + 8$$
$$= 15 + 20z + 8$$
$$= 3 + 20 + 20z$$
$$a_{k+1} = 3 + 4(5+5z)$$

$5+5z \in \mathbb{Z}$ by closure $\Rightarrow a_{k+1} \equiv 3 \pmod 4$

QED.

show $2^n \geq n^2$ for $n \in \mathbb{N}^{\geq 4}$

BC: $n = 4$

$$2^4 = 16 \geq 16 = 4^2$$

IH: Assume for some arbitrary $k \in \mathbb{N}^{\geq 4}$,

$$2^k \geq k^2$$

IS: $2^{k+1} \overset{?}{\geq} (k+1)^2$

$$2^{k+1} = 2(2^k) \geq 2k^2 \quad \text{by IH}$$
$$= k^2 + k \cdot k$$
$$\geq k^2 + 4k \leftarrow \text{because } k \in \mathbb{N}^{\geq 4}$$
$$= k^2 + 2k + 2k$$
$$\geq k^2 + 2k + 1$$
$$= (k+1)^2$$

QED.

$2k^2 = k^2 + k^2 = k^2 + k \cdot k$