10/17/2022

→ PRACTICE

PRACTICE

Show that for all sets A, B, C

$$B \not\equiv C \Rightarrow A \cap B \not\equiv A \cap C \vee A \cup B \not\equiv A \cup C$$

We go about this proof by proving a logically equivalent statement. Note that...

$$p \Rightarrow q \vee r \equiv \neg p \vee q \vee r \equiv \neg(p \wedge \neg r) \vee q \equiv (p \wedge \neg r) \Rightarrow q$$

So we will prove the following statement:

$$B \not\equiv C \wedge A \cup B \equiv A \cup C \Rightarrow A \cap B \not\equiv A \cap C$$

Note that $B \not\equiv C \Leftrightarrow B \not\subseteq C \vee C \not\subseteq B$. WLOG, assume $B \not\subseteq C$ i.e $(\exists x \in B)[x \notin C]$. Note if $A \cup B \equiv A \cup C$, then $x \in A$. We show this in a lemma:

LMA: $B \not\subseteq C \wedge A \cup B \equiv A \cup C \Rightarrow (\exists x \in U)[x \in A \cap x \in B \cap x \notin C]$:
We'll prove this via contradiction. We know that $B \not\subseteq C$, so there $(\exists x \in U)[x \in B \cap x \notin C]$. Assume that $x \notin A$. Then,

$$x \in B \Rightarrow x \in A \cup B \quad \wedge \quad x \notin A \wedge x \notin C \Rightarrow x \notin A \cup C$$

$$x \in A \cup B \wedge x \notin A \cup C \Rightarrow A \cup B \not\equiv A \cup C$$

which contradicts our premise. $\therefore (\exists x \in U)[x \in A \cap x \in B \cap x \notin C]$.

So, we can say of x: $x \in A \wedge x \in B \wedge x \notin C$

$$\Rightarrow x \in A \cap B \wedge x \notin A \cap C$$

$$\Rightarrow A \cap B \not\equiv A \cap C$$

This arguement applies WLOG if instead $C \not\subseteq B$, so we have proven $B \not\equiv C \Rightarrow A \cap B \not\equiv A \cap C \vee A \cup B \not\equiv A \cup C$.

QED

Show that for all sets A, B

$$P(A \cap B) \equiv P(A) \cap P(B)$$

We prove this via a direct proof, by showing that $P(A \cap B) \subseteq P(A) \cap P(B)$ and $P(A) \cap P(B) \subseteq P(A \cap B)$. Recall that $S \subseteq T \equiv (\forall x \in S)[x \in T]$.

PART 1: $P(A \cap B) \subseteq P(A) \cap P(B)$: We can say that for all elements $X \in P(A \cap B)$ ...

$$X \subseteq A \cap B \Rightarrow X \subseteq A \wedge X \subseteq B \Rightarrow X \in P(A) \wedge X \in P(B)$$

$$\Rightarrow X \in P(A) \cap P(B)$$

PART 2: $P(A) \cap P(B) \subseteq P(A \cap B)$: We can say that for all elements $X \in P(A) \cap P(B)$ ...

$$X \in P(A) \wedge X \in P(B) \Rightarrow X \subseteq A \wedge X \subseteq B \Rightarrow X \subseteq A \cap B$$

$$\Rightarrow X \in P(A \cap B)$$

Thus, we have shown that $P(A \cap B) \subseteq P(A) \cap P(B)$ and $P(A) \cap P(B) \subseteq P(A \cap B)$, allowing us to conclude that $P(A \cap B) \equiv P(A) \cap P(B)$.

QED