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→ COUNTABILITY

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At this point we should be quite familiar with the idea of cardinality for finite sets. For example, if $S = \{a, b, c\}$ then $|S| = 3$.

However, what about the cardinality of infinite sets? Naively, $|\mathbb{N}| = \infty$ but this isn't very useful by itself, as it categorizes all infinite sets to be the "same" in size.

Does there exist a more meaningful method to compare the cardinalities of two infinite sets S and T in a way that isn't trivial? The answer is yes and comes in the form of bijective functions.

Assume there exists a BIJECTIVE function $f: S \to T$. Then, f is ...

$$\text{SURJECTIVE} \iff (\forall y \in T)(\exists x \in S)[y = f(x)]$$
$$\Rightarrow |S| \geq |T|$$

and

$$\text{INJECTIVE} \iff (\forall x_1, x_2 \in S)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$
$$\Rightarrow |S| \leq |T|$$

So, if this BIJECTIVE function f exists, we can conclude that $|S| = |T|$. Additionally, if this f does not exist, then $|S| \neq |T|$.

For example, consider the following function:

$$f: \mathbb{N} \to \mathbb{N}^{>0}, \quad f(x) = x + 1$$

This is a bijective function, so we can conclude that $|\mathbb{N}^{>0}| = |\mathbb{N}|$, despite the fact that $\mathbb{N}^{>0} \subset \mathbb{N}$.

We pay special attention to the cardinality of the natural numbers:

$$|\mathbb{N}| = \aleph_0$$

We say a set S is COUNTABLE if there exists a bijection $f: \mathbb{N} \to S$. If S is countable and infinite we call it COUNTABLY INFINITE. If S is not countable, we call it UNCOUNTABLE.

The naturals are also called the "counting numbers" because they are the numbers you would use to count objects. Via bijections, we make mathematically formal the idea of "counting" a set.

We now go about proving the cardinality of numerous sets. You will be expected to be able utilize all the techniques below.

Prove that the primes $\mathbb{P}$ are countable

Note that the primes $\mathbb{P}$ are a subset of the naturals $\mathbb{N}$. A subset of a countable set is countable, so $\mathbb{P}$ is countable.

Prove that the integers $\mathbb{Z}$ are countable

We establish the following bijection between the integers $\mathbb{Z}$ and naturals $\mathbb{N}$:

$$f: \mathbb{N} \to \mathbb{Z}, \quad f(x) = \begin{cases} x/2 & x \equiv 0 \pmod{2} \\ -(x+1)/2 & x \equiv 1 \pmod{2} \end{cases}$$

This corresponds to the following listing: $0, -1, 1, -2, 2, \ldots$
Therefore the integers $\mathbb{Z}$ are countable.

The bijection method utilized above is the truest-to-principal. However, it quickly becomes infeasible for complex sets, even the simple example above required careful construction. For this reason, the listing technique used above is also valid, since it is fundamentally a bijection from the naturals (why?). Be careful though, the following is an INVALID listing:

$$0, 1, 2, \ldots, -1, -2, \ldots$$

Prove that the positive rationals $\mathbb{Q}^+$ are countable

We establish a bijection $f: \mathbb{N} \longrightarrow \mathbb{Q}^+$ using the following intuition about the positive rationals $\mathbb{Q}^+$:

Let $q = a/b \in \mathbb{Q}^+$ such that $a \in \mathbb{N}$ and $b \in \mathbb{N}^{\neq 0}$

a: / b:	0	1	2	3	4	...
1	$0/1$	$1/1$	$2/1$	$3/1$	$4/1$	...
2	$0/2$	$1/2$	$2/2$	$3/2$	$4/2$	...
3	$0/3$	$1/3$	$2/3$	$3/3$	$4/3$	...
4	$0/4$	$1/4$	$2/4$	$3/4$	$4/4$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

We can create a listing of the positive rationals $\mathbb{Q}^+$ by following the snaking pattern demonstrated to the left. The construction is as follows: ① list all the positive rationals $q = a/b$ with the property that $a + b = 1$, ② list all $q = a/b$ s.t $a + b = 2$, ... ⓝ list all $q = a/b$ s.t $a + b = n$, ...

This corresponds to the listing $0/1, 0/2, 1/1, 0/3, 1/2, \ldots$ and we have therefore shown that $\mathbb{Q}^+$ is countable.

If you're really paying attention, you may have realized that this technically isn't a bijection since many values will show up multiple times in the listing (for example: $0/1 = 0/2$). However, the listing clearly induces a valid listing (remove repeats) so we'll generally accept proofs like the above.

Lets touch on the following naive, invalid listing:

$$0/1, 0/2, 0/3, \ldots, 1/1, 1/2, 1/3, \ldots, 2/1, 2/2, 2/3, \ldots$$

What are some ways to identify invalid listings? One give-away is the presence of ellipses in the middle of the list. To make this idea more mathematically definite: in a valid listing you should be able to guarantee that any given element will show by some position in the listing. For example, for our $\mathbb{Q}^+$ listing, we can guarantee that any $q = x/y \in \mathbb{Q}^+$ will show up by the following position in the list:

$$1 + 2 + \ldots + (x+y) = \frac{(x+y+1)(x+y)}{2}$$

Prove that the set of reals between 0 and 1 i.e $[0,1)$ are uncountable

We prove this via contradiction, that is we assume the set $[0,1)$ is countable and show this leads to a contradiction.

If $[0,1)$ is countable, then there exists a listing that contains every real $r \in [0,1)$. Note that any of these reals can be represented in binary as an infinite string of 0s and 1s (numbers with finite decimal representations can be padded with 0s). Then, this listing which we assumed to exist may look like this...

$$
\begin{array}{l}
0.0\,1\,1\,0\,1\,0\,0\,1\,1\,1\,0\,1\,0\,1\,1 \cdots \\
0.1\,0\,1\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0 \cdots \\
0.0\,1\,1\,1\,1\,0\,1\,0\,1\,0\,0\,0\,0\,1\,0 \cdots \\
0.1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,1\,0\,1\,0 \cdots \\
0.0\,0\,1\,0\,1\,0\,0\,0\,1\,1\,1\,0\,1\,0\,0 \cdots \\
0.1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,1\,1\,1\,0\,1 \cdots \\
0.1\,0\,1\,0\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1 \cdots \\
0.0\,0\,1\,1\,0\,1\,0\,1\,0\,1\,0\,1\,1\,0\,0 \cdots \\
0.0\,1\,1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0\,1 \cdots \\
0.0\,0\,1\,1\,1\,1\,0\,1\,0\,0\,0\,1\,0\,1\,1 \cdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots
\end{array}
$$

Again, if $[0,1)$ then every real $r \in [0,1)$ should show up somewhere in this list.

We will arrive at a contradiction by constructing a real $r^* \in [0,1)$ that is guaranteed not to show up anywhere in the listing.

We'll construct this r^* as follows: let r^i be the ith real in our listing and let r_i be the ith digit in r. Then, let $r^*_i = (r^i_i + 1) \bmod 2$. Put simply, we construct r^* by flipping the bits on the diagonal highlighted above. For the example above we would get the following $r^* = 0.1101001001\ldots$

We can confirm that $r^* \neq r^i$ for any $i \in \mathbb{N}$, because $(\forall i \in \mathbb{N})[r^*_i \neq r^i_i]$ by the definition of its construction. Therefore, r^* does not show up in our listing which is a contradiction. Thus, we have shown that $[0,1)$ is not countable.

The proof above is known as Cantor's diagonalization proof, and is utilized in an analogous manner in many uncountability proofs.

Prove that the reals $\mathbb{R}$ are uncountable

Note that the reals $\mathbb{R}$ are a superset of the uncountable set $[0,1)$. A superset of an uncountable set is uncountable, so $\mathbb{R}$ is uncountable.