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→ CONDITIONALS
→ LOGICAL EQUIVALENCE

Conditionals

Recall that there are three fundamental operators from which all compound-statements are built. Well, one of these statements was so useful that we gave it its own symbol, the implication:

Implication : $\Rightarrow$: if - then

We define implication as follows:

$$p \Rightarrow q \equiv \neg p \vee q$$

where p is the premise & q is the conclusion.

So...

$c \equiv$ it is cloudy $r \equiv$ it will rain

$c \Rightarrow r \equiv$ if it is cloudy, then it will rain

Note: if a then b $\not\equiv$ a if b

infact, if a then b $\equiv$ b if a

The reason implications are so important is because they map to an if - then statement. Crucial for proofs. Lets look at this truth-table:

	c	r	$\neg c$	$\neg c \vee r \equiv c \Rightarrow r$
1	0	0	1	1
2	0	1	1	1
3	1	0	0	0
4	1	1	0	1

Its worth walking through each row and verifying why it generates the truth-value it does:

ROW 1: its not cloudy and it doesn't rain
This is consistent with $c \Rightarrow r$

ROW 2: its not cloudy and it rains
This confuses students sometimes, it is consistent because when the premise is not met, we make no claims on the conclusion.

Note that when the premise is not met the implication holds because it is VACUOUSLY TRUE.

ROW 3: its cloudy and it doesn't rain
Clearly inconsistent with $c \Rightarrow r$, hence false.

ROW 4: its cloudy and it rains
Consistent, makes sense.

An implication statement $p \Rightarrow q$ generates numerous other statements which we note here:

Inverse: $\neg p \Rightarrow \neg q$

Converse: $q \Rightarrow p$

Contrapositive: $\neg q \Rightarrow \neg p$

These will be important shortly.

PRACTICE

$s \equiv$ you go swimming

$w \equiv$ you get wet

if you go swimming, you will get wet: $s \Rightarrow w$

you will get wet if you go swimming: $s \Rightarrow w$

p	q	$\bar{p}$	$\bar{q}$	$p \Rightarrow q$	$\bar{p} \Rightarrow \bar{q}$	$q \Rightarrow p$	$\bar{q} \Rightarrow \bar{p}$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

$\neg p \vee q$

$p \vee \neg q$

$\neg q \vee p$

$q \vee \neg p$

you will die if you jump in a volcano

INVERSE: if you don't jump in a volcano, then you will not die

CONVERSE: if you die, then you jumped into a volcano

CONTRAPOSITIVE: if you do not die, then you did not jump in a volcano

LOGICAL EQUIVALENCE

We have seen that sometimes two compound statements, which we can think of as boolean functions, are logically equivalent i.e they produce the same output for any given input in the domain.

We are interested in proving logical equivalence. This can be done via truth-tables or via laws of equivalence:

* Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
* Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Laws	$p \wedge 1 \equiv p$	$p \vee 0 \equiv p$
Negation Laws	$p \vee \sim p \equiv 1$	$p \wedge \sim p \equiv 0$
Double Negation Law	$\sim(\sim p) \equiv p$	
Idempotent Laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound Laws	$p \vee 1 \equiv 1$	$p \wedge 0 \equiv 0$
DeMorgan's Laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation Laws of *t* and *c*	$\sim 1 \equiv 0$	$\sim 0 \equiv 1$
Definition of Implication	$p \Rightarrow q \equiv \sim p \vee q$	
Definition of Biconditional	$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \wedge q) \vee \sim(p \vee q)$	
Contrapositive	$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$	

* You don't need to state these in simplification problems

The idea here is to manipulate a statement via the laws above until it is symbolically identical to the other. An example:

$$p \Rightarrow (q \vee r) \equiv (p \wedge \bar{q}) \Rightarrow r$$

$p \Rightarrow (q \vee r)$

$\equiv \bar{p} \vee (q \vee r)$ DEFINITION OF IMPLICATION

$\equiv \bar{p} \vee q \vee r$ DEMORGANS

$\equiv \neg(\bar{p} \vee q) \Rightarrow r$ DEFINITION OF IMPLICATION

$\equiv (p \wedge \bar{q}) \rightarrow r$ DEMORGANS

PRACTICE

$$((p \vee q) \wedge p) \vee r \equiv \neg(\bar{p} \wedge \bar{r})$$

$$((p \vee q) \wedge p) \vee r$$

$\equiv p \vee r$ ABSORPTION

$\equiv \neg\neg(p \vee r)$ DOUBLE NEGATION

$\equiv \neg(\bar{p} \wedge \bar{r})$ DEMORGANS

$$p \wedge \neg(q \wedge \neg q) \equiv p$$

$$p \wedge \neg(q \wedge \neg q)$$

$\equiv p \wedge \neg(0)$ NEGATION

$\equiv p \wedge 1$ NEGATION LAW OF T : C

$\equiv p$ IDENTITY

$$(p \to q) \wedge (q \to p) \equiv (p \wedge q) \vee (\bar{p} \wedge \bar{q})$$

$$(p \to q) \wedge (q \to p)$$

$\equiv (\bar{p} \vee q) \wedge (\bar{q} \vee p)$ DEF. IMPLICATION

$\equiv (\bar{p} \wedge (\bar{q} \vee p)) \vee (q \wedge (\bar{q} \vee p))$ DISTRIBUTIVE

$\equiv (\bar{p} \wedge \bar{q}) \vee (\bar{p} \wedge p) \vee (q \wedge \bar{q}) \vee (q \wedge p)$ DISTRIBUTIVE

$\equiv (\bar{p} \wedge \bar{q}) \vee 0 \vee 0 \vee (q \wedge p)$ NEGATION

$\equiv (\bar{p} \wedge \bar{q}) \vee (p \wedge q)$ IDETITY