09/19/2022

→ INTRO TO SETS
→ PREDICATES / QUANTIFIERS

We took off a fair amount of points on the recent homework and quiz for incorrect style (truth-tables with wrong row order, etc). The material we're teaching right now will add even more style rules you need to follow. They are all laid out in the style guide, it is your responsibility to ensure you're consistent with it.

INTRO TO SETS

In the coming weeks we're going to cover topics such as sets, number theory, predicates, and quantifiers in preparation for proofs. Many of these topics are dependent on one another which makes teaching them complex. So, we're going to take a minute to cover some general fundamentals so we can better teach the topics above.

We begin with sets: These are fundamental mathematical objects defined to be an unordered collection of objects with no sense of repeat elements. So,

✓ {NAV, CLIFF, PAUL} ≡ {CLIFF, PAUL, NAV}

✗ {NAV, CLIFF, NAV, PAUL}

A set can be empty, known as the empty or null set ($\varnothing$, {}) or have everything which is known as the Universal Set which is usually context-dependent and needs a domain to be defined.

There's a few ways to define sets which you need to be aware of, namely ① listing / ellipses, ② set-builder notation, ③ interval notation, and ④ set modification:

① listing / ellipses : This one is very intuitive, simply list all the elements in the set:

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

If the set is infinite or just large, ellipses are used to imply that a established pattern holds. So,

$$A = \{0, 1, \ldots, 9\}$$

Note, we do not recommend ellipses as they are technically ambigious. I will generally try to avoid using them.

② set-builder : The GOAT. In this notation, the form of the element is given as well as a defining property which all the elements satisfy. So, numbers greater than 24 or less than 18 would be...

$$S = \{x \mid x < 18 \lor x > 24\}$$

This is the recommended notation. Wonderful for proofs since it defines the set with a mathematical property you can utilize.

③ interval : You have probably seen this used to define the domains of continuous functions. You provide an infimum and supremum and the set is all the numbers between them:

$$S = (18, 24) \qquad \text{NOTE, } 18 \notin S \land 24 \notin S$$

You can include the infimum/supremum by using a square-bracket:

$$S' = [18, 24] \qquad \text{NOW, } 18 \in S \land 24 \in S$$

The last notation is set modification to a base set, but before we can get to that we need to introduce the base sets you need to know:

$\mathbb{R}$ REAL NUMBERS every number on an axis

$\mathbb{Q}$ RATIONALS $\equiv \{a/b \mid a, b \in \mathbb{Z} \land b \neq 0\}$

$\mathbb{Z}$ INTEGERS you know what these are

$\mathbb{N}^*$ NATURALS $\equiv \{0, 1, 2, 3, \ldots\}$

$\mathbb{R}-\mathbb{Q}$** IRRATIONALS $\equiv \{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$

$\mathbb{P}$ PRIMES 1 is not a prime

* ZERO being a natural number or not is a source of contention. In this class $0 \in \mathbb{N}$

** The symbol used here is a set minus, we'll define it formally later.

(4) set modification : This notation is used when making a small change to a base set :

naturals not including 0 : $\mathbb{N}^{\neq 0}$

reals greater than 4 : $\mathbb{R}^{>4}$

even integers : $\mathbb{Z}^{\text{EVENS}}$

PRACTICE

Give the truth-values of the following statements:

$1 \in \mathbb{Z}$	TRUE
$-4.5 \in \mathbb{Z}$	FALSE
$\pi \in \mathbb{R}$	TRUE
$4/5 \in \mathbb{Q}$	TRUE
$928 \in \mathbb{P}$	FALSE
$1234 \in \mathbb{Z}^{odd}$	FALSE

Define the following sets using set-builder notation:

integers between a & b exclusive $\equiv \{x \in \mathbb{Z} \mid x > a \wedge x < b\}$

real numbers that aren't integers $\equiv \{x \in \mathbb{R} \mid x \notin \mathbb{Z}\}$

positive rationals $\equiv \{x \in \mathbb{Q} \mid x > 0\}$

PREDICATES / QUANTIFIERS

Up until this point, we've talked about manipulating atomic statements. We would like to reason about other types of inputs such as $\mathbb{N}$. The question is how to square large domains with true/false framework we've developed.

Our first tool is propositional functions or predicates:

We can think of these as functions that take one or more inputs from any domain we desire and outputs a boolean:

$T \equiv$ CMSC250 Instructors $\quad U(t) \equiv t$ is an undergrad

$U(\text{nav}) = \text{True} \qquad U(\text{cliff}) = \text{False}$

Predicates also allow us to reason about multiple non-boolean inputs:

$$E(x, y, z) := x + y = z$$

$E(2, 6, 8) = \text{True} \qquad E(8, 2, 6) = \text{False}$

So we can now handle non-boolean inputs, but what if we wanted to make statements about domains themselves? To do this, we use quantifiers. Quantifiers express the extent to which a predicate holds over a range of elements. The two most important by far are universal $(\forall)$ and existential $(\exists)$:

$$\text{DOMAIN} \equiv \{x_1, x_2, \dots, x_n\} \equiv D$$

$$\text{UNIVERSAL} : \forall x \in D\, P(x) \equiv \bigwedge_{i=1}^{n} P(x_i)$$

$$\text{EXISTENTIAL} : \exists x \in D\, P(x) \equiv \bigvee_{i=1}^{n} P(x_i)$$

In terms of quantifiers, we can verify via DeMorgan's that the following holds:

$$\neg (\forall x)[P(x)] = (\exists x)[\neg P(x)]$$

Note, another quantifier worth knowing is uniqueness quantifier ($\exists!$)

Often times we need to quantify over multiple domains. For example, the statement

"For every integer, there exists a real that is smaller than it"

$$\equiv (\forall x \in \mathbb{Z}, \exists y \in \mathbb{R})[y < x]$$

This statement is true of course, but note that the following is not:

$$(\exists y \in \mathbb{R}, \forall x \in \mathbb{Z})[y < x] \equiv$$

"there is a real such that every integer is greater than it"

The point is that when we have multiple qualifiers, <u>ORDER</u> <u>MATTERS</u>.

PRACTICE

Give the truth values of the following statements:

$(\forall x \in \{\text{ppl in this room}\})[AGE(x) > 40]$ FALSE

$(\forall x \in \{32, 1/8, 4, 16\})[x \text{ is a power of } 2]$ TRUE

$(\exists x \in \mathbb{P})[0 \equiv x \pmod 2]$ TRUE

$(\exists x \in \mathbb{Z}^{\neq 0})[(x \cdot x = x) \wedge (x + x = x)]$ FALSE

$(\forall x \in \mathbb{R})[4 | x \Rightarrow 2 | x]$ TRUE

$(\exists x \in \mathbb{P}, \forall y \in \mathbb{P})[x \cdot y = x]$ FALSE

$(\forall x \in \mathbb{R}, \exists y \in \mathbb{R})[xy \in \mathbb{Z}]$ TRUE

$(\forall \varepsilon > 0, \exists x \in \mathbb{Q})[1 < x < 1 + \varepsilon]$ TRUE

$(\forall x, \exists y)[P(x,y)] \Rightarrow (\exists y, \forall x)[P(x,y)]$ FALSE

$(\exists x \forall y)[P(x,y)] \Rightarrow (\forall y, \exists x)[P(x,y)]$ TRUE