10/10/2022

→ RELATIONS
→ FUNCTIONS

RELATIONS

You are surely familiar with certain relations ($=, \geq, <$) though perhaps not in as much of a formal sense we will use them. We define a relation to be a rule that associates elements of sets together.

You can define relations on any number of sets, but we're particularly interested in binary relations i.e relations over two sets and often times those will be homogeneous binary relations i.e relations over the same set twice.

A binary relation R over sets A, B is a subset of the cartesian product $A \times B$. An example;

$$A \equiv \{1, 2, 3\} \qquad A \times B \equiv \left\{ \begin{matrix} (1,2) & , & (1,3) & , & (1,4) \\ (2,2) & , & (2,3) & , & (2,4) \\ (3,2) & , & (3,3) & , & (3,4) \end{matrix} \right\}$$

$$B \equiv \{2, 3, 4\}$$

Identify the following relations $R, S, T \subseteq A \times B$:

$$= \;:\; R \equiv \{(2,2), (3,3)\}$$

$$< \;:\; S \equiv \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

$$\geq \;:\; T \equiv \{(2,2), (3,2), (3,3)\}$$

Note that defining a relation generalizes into defining a set, though set modification and interval would be inappropriate for relations, which leaves you with listing and set builder.

When speaking of an element in a relation, we say that a is related to b under R. Notationally, we can represent this in a few ways:

$$(a,b) \in R \qquad aRb \qquad a \sim_R b$$

Homogenous binary relations, i.e relations $R \subseteq A \times A$, have certain properties we're interested in:

REFLEXIVE $\iff$ $(\forall x \in A)[x \sim_R x]$

SYMMETRIC $\iff$ $(\forall x, y \in A)[x \sim_R y \Rightarrow y \sim_R x]$

TRANSITIVE $\iff$ $(\forall x, y, z \in A)[x \sim_R y \wedge y \sim_R z \Rightarrow x \sim_R z]$

A relation that satisfies all three of these properties is known as an EQUIVALENCE relation.

Functions

Amongst binary relations $R \subseteq A \times B$, we pay special attention to those with the property that no two distinct elements of B are mapped to by the same element of A. We call these relations functions:

$$R \text{ is a function} \Rightarrow (\forall x \in A)(\forall y, z \in B)[x \sim_R y \wedge x \sim_R z \Rightarrow y = z]$$

Additionally, we don't consider R to be a function unless it maps every element in A, that is, we add the following property

$$R \text{ is a function} \Rightarrow (\forall x \in A)(\exists y \in B)[x \sim_R y]$$

Together these requirements define a function:

$$R \text{ is a function} \Leftrightarrow (\forall x \in A)(\exists! y \in B)[x \sim_R y]$$

uniqueness quantifier

We also introduce special terminology and notation for functions. If a subset of $A \times B$ is a function f, then we notate this as...

$$f : A \to B, \ f(x) = [\text{whatever algebraic function describes } f]$$

A significant subset of valid functions aren't representable as nice algebraic functions. In this case, we can use piecewise functions or fall-back on our relation notation.

Additionally, be familiar with the following terms:

Domain: the set of inputs to a function i.e A

Image: the set of actual inputs to a function which will end up being a subset of B

Codomain: a set of possible outputs of a function, B is the "standard" choice but any superset of the image is valid

Ofcourse, there's also certain properties of a function $f : A \to B$ we take interest in:

$$\text{SURJECTIVE} \Leftrightarrow (\forall y \in B)(\exists x \in A)[y = f(x)]$$

$$\text{INJECTIVE} \Leftrightarrow (\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$

A function that satisfies both of these properties is known as an BIJECTIVE function.

PRACTICE

Determine the truth-values of the following statements:

R is reflexive for...

$R \equiv \{(x,y) \mid \lfloor x \rfloor = \lceil y \rceil\} \subseteq \mathbb{R}^2$	FALSE
$R \equiv \{(x,y) \mid \lfloor x \rfloor = \lceil y \rceil\} \subseteq \mathbb{Z}^2$	TRUE
$R \equiv \{(x,y) \mid x \equiv y \pmod 4\} \subseteq \mathbb{N}^2$	TRUE

R is symmetric for...

$R \equiv \{(7,2), (\pi,\pi), (2,7)\} \subseteq \mathbb{R}^2$	TRUE
$R \equiv \{(x,y) \mid x-y > y-x\} \subseteq \mathbb{Q}^2$	FALSE
$R \equiv \{(x,y) \mid x \not\equiv y \pmod 2\} \subseteq \mathbb{Z}^2$	TRUE

R is transitive for...

$R \equiv \{(x,y) \mid \lfloor x \rfloor = \lceil y \rceil\} \subseteq \mathbb{R}^2$	FALSE
$R \equiv \{(x,y) \mid x \leq y\} \subseteq \mathbb{R}^2$	TRUE
$R \equiv \{(x,y) \mid x \not\equiv y \pmod 2\} \subseteq \mathbb{Z}^2$	FALSE

Any relation $R \subseteq A \times B$ is $\in P(A \times B)$	TRUE

The following functions are...

... surjective

$f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = \sin(x)$	FALSE
$f: \mathbb{N} \xmapsto{\geq 4} \mathbb{N}$, $f(x) = x - 4$	TRUE
$f: \mathbb{N} \mapsto \mathbb{N}$, $f(x) = x + 4$	FALSE

... injective

$f:(-\pi/2, \pi/2) \longmapsto \mathbb{R}$, $f(x) = \tan(x)$ TRUE

$f: \mathbb{N}_{\geq 4} \longmapsto \mathbb{N}$, $f(x) = x - 4$ TRUE

$f: \mathbb{N} \longmapsto \{0,1,2\}$ $f(x) = x \bmod 3$ FALSE

... bijective

$f: \mathbb{R} \longmapsto \mathbb{R}$, $f(x) = \sin(x)$ FALSE

$f:(-\pi/2, \pi/2) \longmapsto \mathbb{R}$, $f(x) = \tan(x)$ TRUE