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→ SET PROOFS

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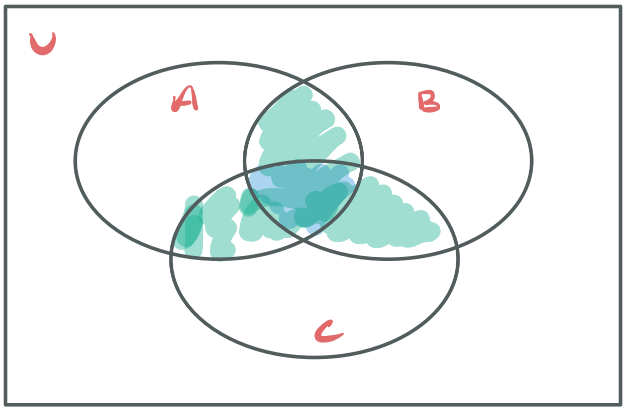
At this point we've introduced a lot of terminology, notation, and ideas around sets. Now we want to go about proving things about them using proofs. Consider this as applying the idea of proofs to another field, sets as opposed to number theory.

Here's some properties of sets we may ask you to prove. If you're ever looking for some practice, this is a good source.

Commutative	$A \cup B = B \cup A$ and $A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity	$A \cup \emptyset = A$ and $A \cap U = A$
Complement	$A \cup A^c = U$ and $A \cap A^c = \emptyset$
Double Complement	$(A^c)^c = A$
Idempotent	$A \cup A = A$ and $A \cap A = A$
Universal Bound	$A \cup U = U$ and $A \cap \emptyset = \emptyset$
DeMorgan's	$(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
Absorption	$A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
Complement of $\emptyset$ and U	$U^c = \emptyset$ and $\emptyset^c = U$
Set Difference	$A - B = A \cap B^c$

An example: Prove or disprove that for all sets A, B, C $(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq A \cap B \cap C$

💡 A note on Venn-Diagrams, they are very helpful to build intution about a problem. They are <u>NOT</u> proofs.



We see now that this statement is not true. Since we want to disprove a universal statement, we can simply construct a valid counter-example.

We disprove this statement via the following example:

$A \equiv \{1, 2, 4, 5\}$ $\quad B \equiv \{2, 3, 5, 6\}$ $\quad C \equiv \{4, 5, 6, 7\}$

Then, $A \cap B \cap C \equiv \{5\}$ and...

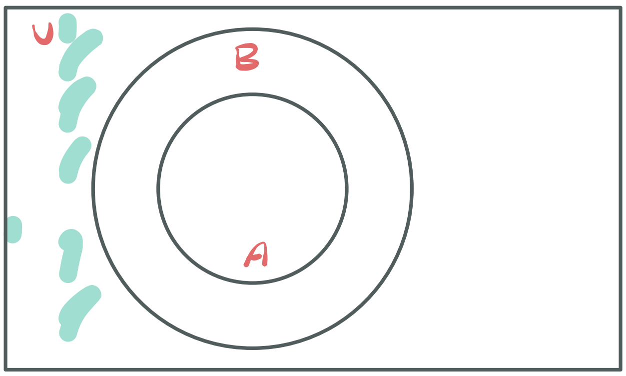
$$(A \cap B) \cup (A \cap C) \cup (B \cap C) \equiv \{2, 4, 5, 6\}$$

Clearly, $\{2, 4, 5, 6\} \not\subseteq \{5\}$ so this statement is false.

QED

PRACTICE

Prove or disprove that for all sets A, B
$A \subseteq B \Rightarrow A \cap B^c \equiv \emptyset$



We prove this using a proof by contradiction, that is we show that there exist sets A, B such that $A \subseteq B \wedge A \cap B^c \neq \emptyset$ and show this leads to a contradiction.

$$A \cap B^c \neq \emptyset \Leftrightarrow (\exists x \in U)[x \in A \cap B^c]$$

$$x \in A \cap B^c \Rightarrow x \in A \wedge x \in B^c$$

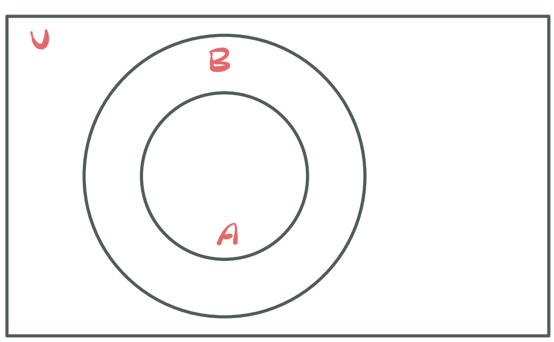
$$\Rightarrow x \in A \wedge x \notin B$$

But, $A \subseteq B \Rightarrow (\forall x \in U)[x \in A \Rightarrow x \in B]$

This is a contradiction. Thus we have proven that for all sets A, B
$A \subseteq B \Rightarrow A \cap B^c = \emptyset$.

QED

Prove or disprove that for all sets A, B
$A \equiv A \cap B \Leftrightarrow A \in P(B)$



We prove this using a direct proof. Since we are proving a bi-implication, we need to show that definition goes both ways:

LMA: $A \equiv A \cap B \Rightarrow A \subseteq B$

$$(A \equiv A \cap B) \Leftrightarrow (A \subseteq A \cap B \wedge A \cap B \subseteq A)$$

$$A \subseteq A \cap B \equiv (\forall x \in U)[x \in A \Rightarrow (x \in A \wedge x \in B)]$$

$$x \in A \Rightarrow (x \in A \wedge x \in B) \equiv x \notin A \vee (x \in A \wedge x \in B)$$

$$\equiv (x \notin A \vee x \in A) \wedge (x \notin A \vee x \in B)$$

$$\Rightarrow x \notin A \vee x \in B$$

$$\equiv x \in A \Rightarrow x \in B$$

$$\equiv A \subseteq B$$

We now go about proving the other way.

LMA: $A \subseteq B \Rightarrow A \equiv A \cap B$

$$A \subseteq B \equiv (\forall x \in U)[x \in A \Rightarrow x \in B]$$

To show that $A \equiv A \cap B$, we need to show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

$$\therefore A \cap B \subseteq A$$

$$x \in A \Rightarrow x \in B \Rightarrow x \in A \cap B$$

$$\therefore A \subseteq A \cap B$$

Thus we can conclude that $A \equiv A \cap B$.

We have shown that the implication goes both ways. Therefore, $A \equiv A \cap B \Leftrightarrow A \in \mathcal{P}(B)$.

QED