View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Matrices with Permanent Equal to One

Victor A. Nicholson
Department of Mathematics
Kent State University
Kent, Ohio 44242

Submitted by Richard S. Varga

ABSTRACT

We show that a nonnegative square matrix M is nilpotent if and only if the permanent of $M+I$ is one. We also show that a 2-complex obtained by sewing disks to a wedge of circles is collapsible if and only if its incidence matrix has permanent one.

1. INTRODUCTION

We show in Theorem 1 that a nonnegative square matrix M is nilpotent if and only if the permanent of $M+I$ is one. We consider the geometry underlying this result in Corollary 1. Corollary 2 characterizes the square matrices with integer entries that have permanent equal to one. We use this result to characterize the collapsible 2-complexes obtained by sewing disks to a wedge of circles (Theorem 2).

2. MAIN RESULTS

Let M be a nonnegative square matrix and r a positive integer. If r positive elements m_{ij} of M can be arranged to have the form $m_{t_1t_2}, m_{t_2t_3}, \ldots, m_{t_rt_1}$, they will be called a *positive cycle* (of length r) of elements in M. The permanent of an $n\times n$ matrix M with entries m_{ij} is defined by

$$\operatorname{per}(M)\equiv\sum_{\sigma}\prod_{j=1}^{n} m_j\sigma(j),$$

© American Elsevier Publishing Company, Inc., 1975

where the sum extends over all $n!$ permutations σ of the first n positive integers. We use I to denote the $n\times n$ identity matrix. We say an $n\times n$ matrix M is *upper triangular* if $m_{ij}=0$ for all $i>j$, and *strictly upper triangular* if $m_{ij}=0$ for all $i\geqslant j$.

THEOREM 1. *Let M be a nonnegative $n\times n$ matrix. Then the following are equivalent*:

(1) *there exists a permutation matrix P such that PMP^T is strictly upper triangular,*

(2) *there is no positive cycle of elements in M,*

(3) $\operatorname{per}(M+I)=1$,

(4) *M is nilpotent.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3). This is immediate.

(3)$\Rightarrow$(1). Since M is nonnegative, $\prod_{i=1}^{n}(m_{ii}+1)\geqslant 1$. Thus, the permanent of $M+I$ is equal to the product of its diagonal elements. By Lemma 2 of [1], there is a permutation matrix P such that $P(M+I)P^T=PMP^T+I$ is upper triangular. The diagonal elements of PMP^T+I are all ones because each diagonal element is $\geqslant 1$, $\operatorname{per}(PMP^T+I)$ is the product of the diagonal elements, and $\operatorname{per}(PMP^T+I)=1$. Thus PMP^T is strictly upper triangular.

(2)$\Leftrightarrow$(4). The matrix M is nilpotent if and only if all of the eigenvalues of M are zero. A nonnegative square matrix has a real eigenvalue equal to its spectral radius [5, Theorem 2.7]. Thus M is nilpotent if and only if M has no positive eigenvalues. By Theorem 1 of [4], M has a positive eigenvalue if and only if there is a positive cycle of elements in M. ∎

The following corollary and Fig. 1 make clear the geometry underlying Theorem 1. If G is a loopless directed graph (we allow G to have multiple lines) with vertices $v_1,\ldots,v_n$, then the *adjacency* matrix $M=(m_{ij})$ of G is given by $m_{ij}=$ the number of arrows from v_i to v_j. The *bipartite graph of a nonnegative square matrix M* is the bipartite graph $G(M)$ whose points

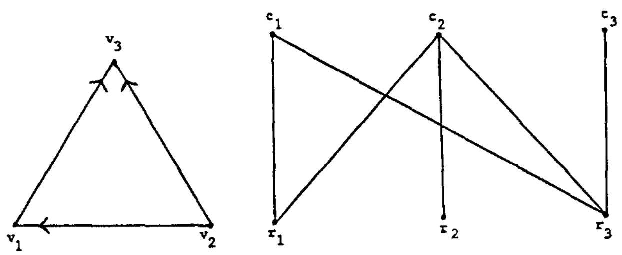

FIG. 1.

consist of the two sets $R=$(the rows of M) and $C=$(the columns of M), and whose lines are the ordered pairs (r_i, c_j), where $m_{ij} \neq 0$. The bipartite graph $G(M)$ does not have multiple lines. A *1-factor* of a graph is a family F of lines of the graph such that every point of the graph is incident with exactly one line in F.

COROLLARY 1. *Let G be a loopless directed graph and M its adjacency matrix. Then G is acyclic if and only if the bipartite graph $G(M+I)$ has a unique 1-factor.*

Proof. The graph G is acyclic if and only if M has no positive cycles. By Theorem 1, M has no positive cycles if and only if the permanent of $M+I$ is one. It is easy to see that the permanent of $M+I$ is one if and only if $G(M+I)$ has a unique 1-factor. ∎

COROLLARY 2. *Suppose M is an $n \times n$ matrix with nonnegative integer entries. Then the permanent of M is one if and only if there exist $n \times n$ permutation matrices P and Q such that PMQ is upper triangular with all ones on the main diagonal.*

Proof. Suppose the permanent of M is one. Since M is nonnegative, there is a permutation σ such that $\prod_{i=1}^{n} M_{i\sigma(i)}=1$. Since each entry is an integer, $M_{i\sigma(i)}=1$ for each $i=1,\ldots,n$. Let $R=(r_{ij})$ be the $n \times n$ permutation matrix with $r_{\sigma(j)j}=1$ for each $j=1,\ldots,n$. Then $RM=N+I$ for some nonnegative matrix N. Since $\operatorname{per}(N+I)=1$, Theorem 1 implies that there exists a permutation matrix S such that SNS^T is strictly upper triangular. Let $P=RS$ and $Q=S^T$. Then $PMQ=SRMS^T=S(N+I)S^T=SNS^T+I$, which is upper triangular with ones down the main diagonal. The converse is immediate. ∎

3. APPLICATION

A 2-complex K obtained by sewing disks to a wedge of circles is *collapsible* if it is possible to order the disks $D_1, D_2, \ldots, D_n$ so that for each $i=1,\ldots,n$ there is a circle S_i that D_i is sewn onto exactly once but that D_j is not sewn onto for all $j>i$. Intuitively, we are able to grasp D_1 at the part of its edge that is sewn to S_1, pluck D_1 from K as one plucks a petal from a flower, and continue to pluck the remaining disks. If K is not collapsible, there will be a stage at which no disk has an edge that we can grasp. For a discussion of collapsing see [3, p. 42].

THEOREM 2. *Let* $\langle a_1, a_2, \ldots, a_n : w_1 = w_2 = \cdots = w_n = 1\rangle$ *be a free group with n generators and n relations. Let K be the 2-complex formed by sewing n disks to a wedge of n simple closed curves by the words* $w_1, \ldots, w_n$. *Let* $M = (m_{ij})$ *be the n-square nonnegative matrix formed by* m_{ij} = *the sum of the absolute values of the exponents on* a_i *in* w_j. *Then K collapses to a point if and only if the permanent of M is one.*

Proof. Let D_i denote the disk corresponding to the word w_i, and let S_i be the curve corresponding to generator a_i, for each $i = 1, 2, \ldots, n$. The complex K collapses to a point if and only if there is an ordering of the disks $D_{\alpha(1)}, \ldots, D_{\alpha(n)}$ and a permutation β such that, for each $i = 1, 2, \ldots, n$, $S_{\beta(i)}$ is sewn to $D_{\alpha(i)}$ exactly once and is not sewn to $D_{\alpha(j)}$ for any $j > i$. Suppose the permanent of M is one. Then, by Corollary 2, there exist permutation matrices P and Q such that $PMQ = (x_{ij})$ is upper triangular with ones down the main diagonal. Since P interchanges the rows of M and Q interchanges the columns, there exist permutations α and β such that $x_{ij} = M_{\alpha(i)\beta(j)}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Since $x_{ii} = 1$, $S_{\beta(i)}$ is sewn to $D_{\alpha(i)}$ exactly once for each $i = 1, \ldots, n$. Since $x_{ij} = 0$ whenever $i > j$, $S_{\beta(i)}$ is not sewn to $D_{\alpha(j)}$ whenever $j > i$ for each $i = 1, \ldots, n$. Thus K collapses to a point. Conversely, if K collapses to a point, then the two permutations α and β give rise to permutation matrices P and Q, so that PMQ is upper triangular with ones on the diagonal. By Corollary 2, the permanent of M is one. ■

The author wishes to thank R. S. Varga for his very helpful comments during the preparation of this work.

REFERENCES

1 G. M. Engel and H. Schneider, Inequalities for determinants and permanents, *Linear and Multilinear Algebra* 1 (1973), 187–201.
2 F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.
3 J. F. P. Hudson, *Piecewise Linear Topology*, Mathematics Lecture Note Series, Benjamin, New York, 1969.
4 J. L. Ullman, On a theorem of Frobenius, *Mich. Math. J.* 1 (1953), 189–193.
5 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.