X.

DEMONSTRATION OF AN EXPRESSION FROM WHICH THE BINOMIAL FORMULA IS A SPECIAL CASE.

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This expression is as follows:

$$\begin{aligned}(x+\alpha)^n =& x^n + \binom{n}{1}\alpha(x+\beta)^{n-1} + \binom{n}{2}\alpha(\alpha-2\beta)(x+2\beta)^{n-2} + \dots \\ &+ \binom{n}{\mu}\alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu} + \dots \\ &+ \binom{n}{n-1}\alpha(\alpha-(n-1)\beta)^{n-2}(x+(n-1)\beta) + \alpha(\alpha-n\beta)^{n-1};\end{aligned}$$

x, α, and β are arbitrary quantities, n is a positive integer.

When $n=0$, the expression gives

$$(x+\alpha)^0 = x^0,$$

which is what was required. Now, we can prove, as follows, that if the expression holds for $n=m$, then it must also hold for $n=m+1$, i.e. it is true in general.

Let

$$\begin{aligned}(x+\alpha)^m =& x^m + \frac{m}{1}\alpha(x+\beta)^{m-1} + \frac{m(m-1)}{1.2}\alpha(\alpha-2\beta)(x+2\beta)^{m-2} + \dots \\ &+ \frac{m}{1}\alpha(\alpha-(m-1)\beta)^{m-2}(x+(m-1)\beta) + \alpha(\alpha-m\beta)^{m-1}.\end{aligned}$$

Multiplying by $(m+1)dx$ and integrating, we find that

$$\begin{aligned}(x+\alpha)^{m+1} = x^{m+1} &+ \frac{m+1}{1}\alpha(x+\beta)^m + \frac{(m+1)m}{1.2}\alpha(\alpha-2\beta)(x+2\beta)^{m-1} + \dots \\ &+ \frac{m+1}{1}\alpha(\alpha-m\beta)^{m-1}(x+m\beta) + C,\end{aligned}$$

with C being an arbitrary constant. To find its value, we let $x=-(m+1)\beta$. Then the last two equations give

$$(\alpha-(m+1)\beta)^m=(-1)^m\left[(m+1)^m\beta^m-m^m\alpha\beta^{m-1}\right.$$
$$\left.+\frac{m}{2}(m-1)^{m-1}\alpha(\alpha-2\beta)\beta^{m-2}-\frac{m(m-1)}{2.3}(m-2)^{m-2}\alpha\left(\alpha-3\beta^2\beta^{m-3}+\ldots\right]\right.,$$
$$(\alpha-(m+1)\beta)^{m+1}=(-1)^{m+1}\left[(m+1)^{m+1}\beta^{m+1}-(m+1)m^m\alpha\beta^m\right.$$
$$\left.+\frac{(m+1)m}{2}(m-1)^{m-1}\alpha(a-2\beta)\beta^{m-1}-\ldots\right]+C.$$

Multiplying the first equation by $(m+1)\beta$ and adding the product to the second equation, we find

$$C=(\alpha-(m+1)\beta)^{m+1}+(m+1)\beta(\alpha-(m+1)\beta)^m,$$

or

$$C=\alpha(\alpha-(m+1)\beta)^n.$$

It follows that the proposed equation remains valid for $n=m+1$. But it is true for $n=0$; therefore it will be true for $n=0$, 1, 2, 3, etc., that is, for any positive integer value of n.

If we set $\beta=0$, we obtain the binomial formula. If we set $\alpha=-x$, we find

$$0=x^n-\frac{n}{1}x(x+\beta)^{n-1}+\frac{n(n-1)}{1.2}x(x+2\beta)^{n-1}-\frac{n(n-1)(n-2)}{1.2.3}x(x+3\beta)^{n-1}+\ldots$$

or by dividing by x,

$$0=x^{n-1}-\frac{n}{1}(x+\beta)^{n-1}+\frac{n(n-1)}{1.2}(x+2\beta)^{n-1}-\frac{n(n-1)(n-2)}{1.2.3}(x+3\beta)^{n-1}+\ldots$$

which is also known; for the right-hand side of this equation is nothing else but

$$(-1)^n\Delta^n\left(x^{n-1}\right),$$

if one sets the constant difference equal to β.