IV.

THE FINITE INTEGRAL $\Sigma^n \varphi x$ EXPRESSED BY A SIMPLE DEFINITE INTEGRAL.

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It is known that one can use the theorem of *Parseval* to express the finite integral $\Sigma^n \varphi x$ as a definite double integral. However, if I am not mistaken, no one has expressed the same integral as a single definite integral. That is the objective of this memoire.

Letting φx denote an arbitrary function of x, it is easy to see that we can always assume

(1) $$\varphi(x) = \int e^{vx} f(v).dv,$$

where the integral is taken between two arbitrary limits of v, independent of x. The function $f(v)$ is a function of v, whose form depends on that of $\varphi(x)$. Assuming $A(x) = 1$, by taking the finite integral of both sides of equation (1), we have

(2) $$\Sigma\varphi(x) = \int e^{vx} \frac{f(v)}{e^v - 1} dv,$$

where an arbitrary constant must be added. By taking the finite integral again, we obtain

$$\Sigma^2\varphi(x) = \int e^{vx} \frac{fv}{(e^v - 1)^2} dv.$$

In general, we have

(3) $$\Sigma^n\varphi(x) = \int e^{vx} \frac{fv}{(e^v - 1)^n} dv.$$

To complete this integral, we need to add to the right-hand side a function of the form

$$C + C_1 x + C_2 x^2 + \ldots + C_{n-1} x^{n-1},$$

where C, C_1, C_2, etc. are arbitrary constants.

Now let us find the value of the definite integral $\int e^{vx} \frac{fv}{(e^v-1)^n} dv$. For this purpose, I use a theorem due to M. *Legendre* (Exerc. de calc. int. t. II, p. 189), which states that

$$\frac{1}{4}\frac{e^v+1}{e^v-1} - \frac{1}{2v} = \int_0^{\frac{1}{0}} \frac{dt.\sin vt}{e^{2\pi t}-1}.$$

We obtain from this equation

$$\frac{1}{e^v-1} = \frac{1}{v} - \frac{1}{2} + 2\int_0^{\frac{1}{0}} \frac{dt.\sin vt}{e^{2\pi t}-1}. \tag{4}$$

By substituting this value of $\frac{1}{e^v-1}$ in equation (2), we have

$$\Sigma\varphi x = \int e^{vx}\frac{fv}{v}dv - \frac{1}{2}\int e^{vx} fv.dv + 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\int e^{vx} fv.\sin vt.dv.$$

The integral $\int e^{vx} fv.\sin vt.dv$ can be found as follows. By replacing x successively by $x+t\sqrt{-1}$ and $x-t\sqrt{-1}$ in equation (1), we obtain

$$\varphi(x+t\sqrt{-1}) = \int e^{vx}e^{vt\sqrt{-1}} fv.dv,$$

$$\varphi(x-t\sqrt{-1}) = \int e^{vx}e^{-vt\sqrt{-1}} fv.dv,$$

from which we obtain, by subtracting and dividing by $2\sqrt{-1}$,

$$\int e^{vx}\sin vt.fv.dv = \frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}}.$$

Thus, the expression for $\Sigma\varphi x$ becomes

$$\Sigma\varphi x = \int \varphi x.dx - \frac{1}{2}\varphi x + 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}}.$$

Now to find the value of the general integral

$$\Sigma^n\varphi x = \int e^{vx} fv\frac{dv}{(e^v-1)^n},$$

let us set

$$\frac{1}{(e^v-1)^n} = (-1)^{n-1}\left(A_{0,n}p + A_{1,n}\frac{dp}{dv} + A_{2,n}\frac{d^2p}{dv^2} + \ldots + A_{n-1,n}\frac{d^{n-1}p}{dv^{n-1}}\right)$$

where p is equal to $\frac{1}{e^v-1}$, and $A_{0,n}$, $A_{1,n}\ldots$ are numerical coefficients that must be determined. If we differentiate the previous equation, we have

$$\frac{ne^v}{(e^v-1)^{n+1}} = (-1)^n\left(A_{0,n}\frac{dp}{dv} + A_{1,n}\frac{d^2p}{dv^2} + \ldots + A_{n-1,n}\frac{d^np}{dv^n}\right)$$

Now

$$\frac{ne^v}{(e^v-1)^{n+1}} = \frac{n}{(e^v-1)^n} + \frac{n}{(e^v-1)^{n+1}},$$

so
$$\frac{ne^v}{(e^v-1)^{n+1}} = n(-1)^{n-1}\left(A_{0,\,n}p + A_{1,\,n}\frac{dp}{dv} + \ldots + A_{n-1,\,n}\frac{d^{n-1}p}{dv^{n-1}}\right)$$
$$+\, n(-1)^n\left(A_{0,\,n+1}p + A_{1,\,n+1}\frac{dp}{dv} + \cdots + A_{n,\,n+1}\frac{d^n p}{dv^n}\right).$$
Comparing these two expressions for $\frac{ne^v}{(e^v-1)^{n+1}}$, we obtain the following equations:
$$\begin{array}{lcl}
A_{0,\,n+1} - A_{0,\,n} = 0 & ɔ: & \Delta A_{0,\,n} = 0, \\
A_{1,\,n+1} - A_{1,\,n} = \frac{1}{n}A_{0,\,n} & ɔ: & \Delta A_{1,\,n} = \frac{1}{n}A_{0,\,n}, \\
A_{2,\,n+1} - A_{2,\,n} = \frac{1}{n}A_{1,\,n} & ɔ: & \Delta A_{2,\,n} = \frac{1}{n}A_{1,\,n}, \\
\ldots\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots \\
A_{n-1,\,n+1} - A_{n-1,\,n} = \frac{1}{n}A_{n-2,\,n} & ɔ: & \Delta A_{n-1,\,n} = \frac{1}{n}A_{n-2,\,n}, \\
\qquad A_{n,\,n+1} = \frac{1}{n}A_{n-1,\,n}, & &
\end{array}$$
from which we deduce
$$A_{0,\,n} = 1,\ A_{1,\,n} = \Sigma\frac{1}{n},\ A_{2,\,n} = \Sigma\left(\frac{1}{n}\Sigma\frac{1}{n}\right),\ A_{3,\,n} = \Sigma\left[\frac{1}{n}\Sigma\left(\frac{1}{n}\Sigma\frac{1}{n}\right)\right] \quad \text{etc.}$$
$$A_{n,\,n+1} = \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}\cdots\frac{1}{2}.\frac{1}{1}.A_{0,\,1} = \frac{1}{\Gamma(n+1)}.$$
This last equation can be used to determine the constants that appear in the expressions of $A_{1,\,n}$, $A_{2,\,n}$, $A_{3,\,n}$ etc.

Having thus determined the coefficients $A_{0,\,n}$, $A_{1,\,n}$, $A_{2,\,n}$, etc., we will have, upon substituting the resulting value of $\frac{1}{(e^v-1)^n}$ in equation (3),
$$\Sigma^n\varphi x = (-1)^{n-1}\int e^{vx} fv.dv\left(A_{0,\,n}p + A_{1,\,n}\frac{dp}{dv} + \ldots + A_{n-1,\,n}\frac{d^{n-1}p}{dv^{n-1}}\right);$$
now we have
$$p = \frac{1}{v} - \frac{1}{2} + 2\int_0^{\frac{1}{0}} \frac{dt.\sin vt}{e^{2\pi t}-1},$$
from which we obtain, by differentiating,
$$\frac{dp}{dv} = -\frac{1}{v^2} + 2\int_0^{\frac{1}{0}} \frac{tdt.\cos vt}{e^{2\pi t}-1},$$
$$\frac{d^2p}{dv^2} = \quad \frac{2}{v^3} - 2\int_0^{\frac{1}{0}} \frac{t^2dt.\sin vt}{e^{2\pi t}-1},$$
$$\frac{d^3p}{dv^3} = -\frac{2.3}{v^4} - 2\int_0^{\frac{1}{0}} \frac{t^3dt.\cos vt}{e^{2\pi t}-1} \quad \text{etc.};$$
therefore, by substituting
$$\Sigma^n\varphi x = \int\left(A_{n-1,\,n}\frac{\Gamma n}{v^n} - A_{n-2,\,n}\frac{\Gamma(n-1)}{v^{n-1}} + \ldots + (-1)^{n-1}A_{0,\,n}\frac{1}{v} + (-1)^n.\frac{1}{2}\right)e^{vx}fv.dv$$
$$+\, 2(-1)^{n-1}\iint_0^{\frac{1}{0}} \frac{P\sin vt.dt}{e^{2\pi t}-1}e^{vx}fv.dv + 2(-1)^{n-1}\iint_0^{\frac{1}{0}} \frac{Q\cos vt.dt}{e^{2.Tt}-1}e^{vx}fv.dv.$$

From the equation $\varphi x = \int e^{vx} fv.dv$ we obtain by integrating:

$$\int \varphi x.dx = \int e^{vx} fv \frac{dv}{v},$$
$$\int^2 \varphi x.dx^2 = \int e^{vx} fv \frac{dv}{v^2},$$
$$\int^3 \varphi x.dx^3 = \int e^{vx} fv \frac{dv}{v^3} \quad \text{etc.;}$$

in addition, we have

$$\int \sin vt.e^{vx} fv.dv = \frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}},$$
$$\int \cos vt.e^{vx} fv.dv = \frac{\varphi(x+t\sqrt{-1}) + \varphi(x-t\sqrt{-1})}{2},$$

therefore we will have, by substituting

$$\Sigma^n \varphi x = A_{n-1,\,n}\Gamma n \int^n \varphi x.dx^n - A_{n-2,\,n}\Gamma(n-1)\int^{n-1} \varphi x.dx^{n-1} + \ldots + (-1)^{n-1}\int \varphi x.dx$$
$$+ (-1)^n.\frac{1}{2}\varphi x + 2(-1)^{n-1}\int_0^{\frac{1}{0}} \frac{Pdt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$+ 2(-1)^{n-1}\int_0^{\frac{1}{0}} \frac{Qdt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) + \varphi(x-t\sqrt{-1})}{2}$$

where

$$P = A_{0,\,n} - A_{2,\,n}t^2 + A_{4,\,n}t^4 - \ldots,$$
$$Q = A_{1,\,n}t - A_{3,\,n}t^3 + A_{5,\,n}t^5 - \ldots.$$

By letting e.g. $n = 2$, we will have

$$\Sigma^2 \varphi x = \iint \varphi x.dx^2 - \int \varphi x.dx + \frac{1}{2}\varphi x - 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$-2\int_0^{\frac{1}{0}} \frac{tdt}{e^{2.\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) + \varphi(x-t\sqrt{-1})}{2}.$$

Setting e.g. $\varphi x = e^{ax}$, we have

$$\varphi(x \pm t\sqrt{-1}) = e^{ax}e^{\pm at\sqrt{-1}}, \quad \int e^{ax}dx = \frac{1}{a}e^{ax}, \quad \iint e^{ax}dx^2 = \frac{1}{a^2}e^{ax},$$

therefore, by substituting and dividing by e^{ax},

$$\frac{1}{(e^a-1)^2} = \frac{1}{2} - \frac{1}{a} + \frac{1}{a^2} - 2\int_0^{\frac{1}{0}} \frac{dt.\sin at}{e^{2\pi t}-1} - 2\int_0^{\frac{1}{0}} \frac{tdt.\cos at}{e^{2\pi t}-1}.$$

The most remarkable case is when $n = 1$. In that case, as we have seen previously:

$$\Sigma \varphi x = C + \int \varphi x\, dx - \frac{1}{2}\varphi x + 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}}.$$

Assuming that both integrals $\Sigma\varphi x$ and $\int\varphi x\,dx$ vanish for $x=a$, it is clear that we will have:

$$C=\frac{1}{2}\varphi a-2\int_0^{\frac{1}{0}}\frac{dt}{e^{2\pi t}-1}\frac{\varphi(a+t\sqrt{-1})-\varphi(a-t\sqrt{-1})}{2\sqrt{-1}};$$

therefore

$$\Sigma\varphi x=\int\varphi x\,dx-\frac{1}{2}(\varphi x-\varphi a)+2\int_0^{\frac{1}{0}}\frac{dt}{e^{2\pi t}-1}\frac{\varphi(x+t\sqrt{-1})-\varphi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$-2\int_0^{\frac{1}{0}}\frac{dt}{e^{2\pi t}-1}\frac{\varphi(a+t\sqrt{-1})-\varphi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

If we let $x=\infty$, assuming that φx and $\int\varphi x.dx$ vanish for this value of x, we will have:

$$\varphi a+\varphi(a+1)+\varphi(a+2)+\varphi(a+3)+\dots\ \text{ in inf.}$$
$$=\int_a^{\frac{1}{0}}\varphi x.dx+\frac{1}{2}\varphi a-2\int_0^{\frac{1}{0}}\frac{dt}{e^{2\pi t}-1}\frac{\varphi(a+t\sqrt{-1})-\varphi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

Setting e.g. $\varphi x=\frac{1}{x^2}$, then we have

$$\frac{\varphi(a+t\sqrt{-1})-\varphi(a-t\sqrt{-1})}{2\sqrt{-1}}=\frac{-2at}{(a^2+t^2)^2},$$

so

$$\frac{1}{a^2}+\frac{1}{(a+1)^2}+\frac{1}{(a+2)^2}+\dots=\frac{1}{2a^2}+\frac{1}{a}+4a\int_0^{\frac{1}{0}}\frac{tdt}{(e^{2\pi t}-1)\,(a^2+t^2)^2},$$

and by letting $a=1$

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots=\frac{\pi^2}{6}=\frac{3}{2}+4\int_0^{\frac{1}{0}}\frac{tdt}{(e^{2\pi t}-1)\,(1+t^2)^2}.$$