II.

ON THE DEFINITE INTEGRAL $\int_0^1 x^{\alpha-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx.$

In Exercices de calcul intégral by Mr. Legendre, we find the following expression:

$$\int_0^1 x^{a-1}(1-x)^{c-1}dx = \frac{\Gamma(a)\cdot\Gamma(c)}{\Gamma(a+c)} \tag{1}$$

so

$$\log\int_0^1 x^{a-1}(1-x)^{c-1}dx = \log\Gamma(a)+\log\Gamma(c)-\log\Gamma(a+c).$$

By differentiating with respect to a and c, and noting that

$$\frac{dl\Gamma(a)}{da} = La - C,$$

we have

$$\frac{\int_0^1 x^{a-1}(1-x)^{c-1}lx.dx}{\int_0^1 x^{a-1}(1-x)^{c-1}dx} = La - L(a+c),$$

$$\frac{\int_0^1 x^{a-1}(1-x)^{c-1}l(1-x).dx}{\int_0^1 x^{a-1}(1-x)^{c-1}dx} = Lc - L(a+c).$$

These two equations, combined with equation (1), yield

$$\int_0^1 x^{a-1}(1-x)^{c-1}lx.dx = [La-L(a+c)]\frac{\Gamma a.\Gamma c}{\Gamma(a+c)},$$

$$\int_0^1 x^{a-1}(1-x)^{c-1}l(1-x)dx = [Lc-L(a+c)]\frac{\Gamma a.\Gamma c}{\Gamma(a+c)}.$$

The last equation can also be deduced from the penultimate equation by exchanging a and c, and replacing x with $1-x$.

When $c=1$, we have, because $L(1+a) = \frac{1}{a}+L(a)$ and $\Gamma(a+1) = a\Gamma(a)$,

$$\int_0^1 x^{a-1}\ell x\,dx = -\frac{1}{a^2},$$

a known result, and

$$\int_0^1 x^{a-1}\ell(1-x)\,dx = -\frac{L(1+a)}{a},$$

so

$$L(1+a)=-a\int_0^1 x^{a-1}\ell(1-x)\,dx.$$

By expanding $(1-x)^{c-1}$ in a series, we find

$$\int_0^1 x^{a-1}(1-x)^{c-1}l\left(\frac{1}{x}\right)dx$$
$$=\int_0^1 x^{a-1}l\left(\frac{1}{x}\right)dx-(c-1)\int_0^1 x^{a}l\left(\frac{1}{x}\right)dx+\frac{(c-1)(c-2)}{2}\int_0^1 x^{a+1}l\left(\frac{1}{x}\right)dx-\ldots;$$

But $\int_0^1 x^k l\left(\frac{1}{x}\right)dx=\frac{1}{(k+1)^2}$, so

$$\int_0^1 x^{a-1}(1-x)^{c-1}l\left(\frac{1}{x}\right)dx$$
$$=\frac{1}{a^2}-(c-1)\frac{1}{(a+1)^2}+\frac{(c-1)(c-2)}{2}.\frac{1}{(a+2)^2}-\frac{(c-1)(c-2)(c-3)}{2.3}.\frac{1}{(a+3)^2}+\ldots;$$

yet we know that $\int_0^1 x^{a-1}(1-x)^{c-1}l\left(\frac{1}{x}\right)dx=[L(a+c)-L(a)]\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}$. Therefore,

$$[L(a+c)-L(a)]\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}$$
$$=\frac{1}{a^2}-(c-1)\frac{1}{(a+1)^2}+\frac{(c-1)(c-2)}{2}.\frac{1}{(a+2)^2}-\frac{(c-1)(c-2)(c-3)}{2.3}.\frac{1}{(a+3)^2}+\ldots. \tag{2}$$

For example, taking $c=1-a$, we have

$$L(a+c)-La=-La,\ \Gamma(a+c)=1,$$
$$\Gamma(a)\Gamma(c)=\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(a\pi)};$$

so

$$-La\cdot\frac{\pi}{\sin(a\pi)}=\frac{1}{a^2}+\frac{a}{(a+1)^2}+\frac{a(a+1)}{2(a+2)^2}+\frac{a(a+1)(a+2)}{2\cdot 3\cdot(a+3)^2}+\ldots.$$

Letting $a=\frac{1}{2}$, we have $-La=2\log 2$, $\sin\frac{\pi}{2}=1$, so

$$2\pi\log 2=2^2+\frac{2}{3^2}+\frac{3}{2\cdot 5^2}+\frac{3\cdot 5}{2^2\cdot 3\cdot 7^2}+\frac{3\cdot 5\cdot 7}{2^3\cdot 3\cdot 4\cdot 9^2}+\ldots.$$

Letting $a=1-x$, $c=2x-1$, and noting that $L(1-x)-Lx=\pi\cot(\pi x)$, we will have

$$-\pi\cot(\pi x)\cdot\frac{\Gamma(1-x)\Gamma(2x-1)}{\Gamma(x)}$$
$$=\frac{1}{(1-x)^2}-\frac{2x-2}{(2-x)^2}+\frac{(2x-2)(2x-3)}{2(3-x)^2}-\frac{(2x-2)(2x-3)(2x-4)}{2\cdot 3\cdot(4-x)^2}+\ldots.$$

By exchanging a and c in equation (2), we obtain

$$[L(a+c)-Lc]\frac{\Gamma a.\Gamma c}{\Gamma(a+c)}=\frac{1}{c^2}-(a-1)\frac{1}{(c+1)^2}+\frac{(a-1)(a-2)}{2(c+2)^2}-\dots.$$

By dividing equation (2) by this one, member by member, we have

$$\frac{L(a+c)-L(a)}{L(a+c)-L(c)}=\frac{\frac{1}{a^2}-\frac{c-1}{(a+1)^2}+\frac{(c-1)(c-2)}{2(a+2)^2}-\dots}{\frac{1}{c^2}-\frac{a-1}{(c+1)^2}+\frac{(a-1)(a-2)}{2(c+2)^2}-\dots}.$$

From this equation, we obtain, by setting $c=1$,

$$L(1+a)=a-\frac{a(a-1)}{2^2}+\frac{a(a-1)(a-2)}{2.3^2}-\dots,$$

so by writing $-a$ for a,

$$L(1-a)=-\left(a+\frac{a(a+1)}{2^2}+\frac{a(a+1)(a+2)}{2.3^2}+\dots\right),$$

and by putting $a-1$ instead of a,

$$La=(a-1)-\frac{(a-1)(a-2)}{2^2}+\frac{(a-1)(a-2)(a-3)}{2.3^2}-\dots;$$

we obtain from this

$$\begin{aligned}&L(1-a)-La=\pi.\cot\pi a\\&=-\left(2a-1+\frac{a(a+1)-(a-1)(a-2)}{2^2}+\frac{a(a+1)(a+2)+(a-1)(a-2)(a-3)}{2.3^2}+\dots\right).\end{aligned}$$

If we substitute $a=1$ in equation (2), we will have

$$[L(c+1)-L(1)]\frac{\Gamma(1).\Gamma c}{\Gamma(c+1)}=\frac{L(1+c)}{c}=1-\frac{(c-1)}{2^2}+\frac{(c-1)(c-2)}{2.3^2}-\dots$$

as before. By setting $c=0$, it follows that

$$\frac{L(1)}{0}=\frac{0}{0}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\frac{\pi^2}{6}.$$

We have seen that

$$\int_0^1 x^{a-1}(1-x)^{c-1}l\left(\frac{1}{x}\right)dx=[L(a+c)-La]\frac{\Gamma a.\Gamma c}{\Gamma(a+c)}.$$

By logarithmically differentiating this equation, we obtain

$$\frac{\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^2dx}{\int_0^1 x^{a-1}(1-x)^{c-1}l\left(\frac{1}{x}\right)dx}=-\frac{\frac{dL(a+c)}{da}-\frac{dL(a)}{da}}{L(a+c)-La}+L(a+c)-L(a).$$

Now we have $\frac{dLa}{da} = -\Sigma \frac{1}{a^2}$; letting $\Sigma \frac{1}{a^2} = L'(a)$, we will have

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^2 .dx$$
$$= \left[(L'(a+c) - L'a) + (L(a+c) - La)^2\right] \frac{\Gamma a.\Gamma c}{\Gamma(a+c)}.$$

If we denote $\Sigma \frac{1}{a^3}$ by $L''a$, $L\frac{1}{a^4}$ by $L'''a$, and so on, we obtain by repeated differentiation

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^3 dx$$
$$= \left[2\left(L''(a+c) - L''a\right) + 3\left(L'(a+c) - L'a\right)\left(L(a+c) - La\right) + \left(L(a+c) - La\right)^3\right] \frac{\Gamma a.\Gamma c}{\Gamma(a+c)},$$
$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^4 dx$$
$$= \text{ etc.}$$

By differentiating equation (2) with respect to a, we have

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^2 dx$$
$$= 2\left(\frac{1}{a^3} - \frac{c-1}{1}.\frac{1}{(a+1)^3} + \frac{(c-1)(c-2)}{1.2}.\frac{1}{(a+2)^3} - \frac{(c-1)(c-2)(c-3)}{1.2.3}.\frac{1}{(a+3)^3} + \dots\right),$$
$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^3 dx$$
$$= 2.3\left(\frac{1}{a^4} - \frac{c-1}{1}.\frac{1}{(a+1)^4} + \frac{(c-1)(c-2)}{1.2}.\frac{1}{(a+2)^4} - \frac{(c-1)(c-2)(c-3)}{1.2.3}.\frac{1}{(a+3)^4} + \dots\right),$$

and in general

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1} dx$$
$$= \Gamma\alpha\left(\frac{1}{a^\alpha} - \frac{c-1}{1}.\frac{1}{(a+1)^\alpha} + \frac{(c-1)(c-2)}{1.2}.\frac{1}{(a+2)^\alpha} - \frac{(c-1)(c-2)(c-3)}{1.2.3}.\frac{1}{(a+3)^\alpha} + \dots\right).$$

Now the function $\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1} dx$ can be expressed by the functions Γ, L, L', L'',$\dots L^{(\alpha-1)}$, so the sum of the infinite series

$$\frac{1}{a^\alpha} - \frac{c-1}{1}.\frac{1}{(a+1)^\alpha} + \frac{(c-1)(c-2)}{1.2}.\frac{1}{(a+2)^\alpha} - \dots$$

can be expressed in terms of these same functions.

There are still other integrals that can be expressed by the same functions. Indeed, letting

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx = \varphi(a,\, c),$$

we obtain by successively differentiating with respect to c,

$$\int_0^1 x^{a-1}(1-x)^{c-1}l(1-x)\left(l\frac{1}{x}\right)^{\alpha-1}dx = \varphi' c,$$

$$\int_0^1 x^{a-1}(1-x)^{c-1}[l(1-x)]^2\left(l\frac{1}{x}\right)^{\alpha-1}dx = \varphi'' c,$$

$$\int_0^1 x^{a-1}(1-x)^{c-1}[l(1-x)]^3\left(l\frac{1}{x}\right)^{\alpha-1}dx = \varphi''' c,$$

and in general

$$\int_0^1 x^{a-1}(1-x)^{c-1}[l(1-x)]^{\beta-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx = \varphi^{(\beta-1)} c.$$

Now we have $\varphi(a,\, c) = (-1)^{\alpha-1}\dfrac{d^{\alpha-1}\frac{\Gamma a}{\Gamma(a+c)}}{da^{\alpha-1}}$, so substituting this value, we obtain the following general expression,

$$\int_0^1 x^{a-1}(1-x)^{c-1}[l(1-x)]^n(lx)^m dx = \frac{d^{m+n}\frac{\Gamma a.\Gamma c}{\Gamma(a+c)}}{da^m.dc^n},$$

and this function is, as we have just seen, expressible by the functions Γ, L, L', L'', $\ldots L^{(n-1)}\ldots L^{(m-1)}$.

We know that

(A) $$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1}dx = \Gamma\alpha.$$

By differentiating with respect to α, we have

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} ll\left(\frac{1}{x}\right)dx = \frac{d\Gamma\alpha}{d\alpha} = \frac{\frac{d\Gamma\alpha}{\Gamma\alpha}\Gamma\alpha}{d\alpha} = \Gamma\alpha.\frac{dl\Gamma\alpha}{d\alpha},$$

but $\dfrac{dl\Gamma\alpha}{d\alpha} = L\alpha - C$, so

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} ll\left(\frac{1}{x}\right)dx = \Gamma\alpha\cdot(L\alpha - C);$$

by differentiating again, we have

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} \left(ll\frac{1}{x}\right)^2 dx = \Gamma\alpha \left[(L\alpha - C)^2 - L'\alpha\right].$$

A general expression for the function

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} \left(ll\frac{1}{x}\right)^n dx$$

can easily be found as follows. By differentiating equation (A) n times successively, we will have:

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} \left(ll\frac{1}{x}\right)^n dx = \frac{d^n\Gamma\alpha}{d\alpha^n}.$$

Now $\dfrac{dl\Gamma\alpha}{d\alpha} = L\alpha - C$, so

$$l\Gamma\alpha = \int (L\alpha - C)d\alpha \text{ and } \Gamma\alpha = e^{\int [L\alpha - C]d\alpha},$$

and therefore

$$\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} \left(ll\frac{1}{x}\right)^n dx = \frac{d^n e^{\int (L\alpha - C)d\alpha}}{d\alpha^n},$$

which is expressible in terms of the functions Γ, L, L', $L'' \ldots L^{n-1}$.

If we substitute e^y for x, we have $l\dfrac{1}{x} = -y$, $ll\dfrac{1}{x} = l(-y)$, $dx = e^y dy$; therefore

$$\int_{-\infty}^0 (-y)^{\alpha-1}[l(-y)]^n e^y dy = \frac{d^n e^{\int (L\alpha - C)d\alpha}}{d\alpha^n},$$

or by changing y to $-y$

$$\int_\infty^0 y^{\alpha-1}(ly)^n e^{-y} dy = -\frac{d^n e^{\int (L\alpha - C)d\alpha}}{d\alpha^n},$$

Taking $y = z^{\frac{1}{\alpha}}$, we have $y^{\alpha-1}dy = \dfrac{1}{\alpha}d(y)^\alpha = \dfrac{1}{\alpha}dz$, $ly = \dfrac{1}{\alpha}lz$, $e^{-y} = e^{-\left(\frac{1}{z^\alpha}\right)}$, and therefore

$$\int_0^\infty (lz)^n e^{-(z^\alpha)} dz = \alpha^{n+1} \frac{d^n e^{\int (L\alpha - C)d\alpha}}{d\alpha^n}.$$

If we substitute α instead of $\dfrac{1}{\alpha}$, then by setting $n = 0$, we have

$$\int_0^\infty e^{-x^\alpha} dx = \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right);$$

by setting $n = 1$, we have

$$\int_0^\infty \ln\left(\frac{1}{x}\right) e^{-x^\alpha} dx = -\frac{1}{\alpha^2}\Gamma\left(\frac{1}{\alpha}\right)\left[L\left(\frac{1}{\alpha}\right) - C\right].$$

For example, if $\alpha = 2$, then we have

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad \text{and} \quad \int_0^\infty \ln\left(\frac{1}{x}\right) e^{-x^2}\,dx = \frac{1}{4}\sqrt{\pi}(C + 2\log 2),$$

noting that $L\left(\frac{1}{2}\right) = -2\log 2$. We must remember that the constant C is equal to $0.57721566\ldots$

If we substitute $x = y^n$ in equation (A), we find

$$\int_0^1 y^{n-1}\left(l\frac{1}{y}\right)^{\alpha-1} dy = \frac{\Gamma\alpha}{n^\alpha}, \quad \text{when } n \text{ is positive,}$$

$$\int_\infty^1 y^{n-1}\left(l\frac{1}{y}\right)^{\alpha-1} dy = \frac{\Gamma\alpha}{n^\alpha}, \quad \text{when } n \text{ is negative.}$$

Differentiating this equation with respect to α, we have, when n is positive,

$$\int_0^1 y^{n-1}\left(l\frac{1}{y}\right)^{\alpha-1} ll\left(\frac{1}{y}\right) dy = \frac{\Gamma\alpha}{n^\alpha}(L\alpha - C - \log n).$$

Letting $y = e^{-x}$, we find

$$\int_0^\infty e^{-nx}x^{\alpha-1}lx.dx = \frac{\Gamma\alpha}{n^\alpha}(L\alpha - C - \log n),$$

a result which can also be easily deduced from the equation

$$\int_0^\infty e^{-x^\alpha}l\left(\frac{1}{x}\right) dx = -\frac{1}{\alpha^2}\Gamma\left(\frac{1}{\alpha}\right)\left[L\left(\frac{1}{\alpha}\right) - C\right].$$