I.

THE TRANSCENDENTAL FUNCTIONS $\Sigma\frac{1}{a^2}, \Sigma\frac{1}{a^3}, \Sigma\frac{1}{a^4}, \ldots \Sigma\frac{1}{a^n}$ EXPRESSED AS DEFINITE INTEGRALS.

If we differentiate the function $\Sigma\frac{1}{a}$ several times, we will have

$$\frac{d\Sigma\frac{1}{a}}{da} = \frac{\Sigma d\frac{1}{a}}{da} = -\Sigma\frac{1}{a^2},$$
$$\frac{d^2\Sigma\frac{1}{a}}{da^2} = \frac{\Sigma d^2\left(\frac{1}{a}\right)}{da^2} = +2\Sigma\frac{1}{a^3},$$
$$\frac{d^3\Sigma\frac{1}{a}}{da^3} = \frac{\Sigma d^3\left(\frac{1}{a}\right)}{da^3} = -2.3\Sigma\frac{1}{a^4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{d^n\Sigma\frac{1}{a}}{da^n} = \frac{\Sigma d^n\left(\frac{1}{a}\right)}{da^n} = \pm 2.3.4\ldots n.\Sigma\frac{1}{a^{n+1}},$$

where the sign $+$ occurs when n is even, and the sign $-$ occurs when n is odd.

We then reciprocally conclude

$$\Sigma\frac{1}{a^2} = -\frac{d\Sigma\frac{1}{a}}{da}, \quad \Sigma\frac{1}{a^3} = +\frac{d^2\Sigma\frac{1}{a}}{2.da^2}, \quad \Sigma\frac{1}{a^4} = -\frac{d^3\Sigma\frac{1}{a}}{2.3.da^3} + \text{ etc.},$$
$$\Sigma\frac{1}{a^n} = \pm\frac{d^{n-1}\Sigma\frac{1}{a}}{1.2.3\ldots(n-1)da^{n-1}} = \pm\frac{d^{n-1}L(a)}{2.3\ldots(n-1)da^{n-1}}.$$

Now we have $\Sigma\frac{1}{a} = L(a) = \int_0^1 \frac{x^{a-1}-1}{x-1}dx$. Consequently, by differentiating with respect to a,

$$\frac{d\Sigma\frac{1}{a}}{da} = \int_0^1 \frac{x^{a-1}(lx)}{x-1}dx,$$
$$\frac{d^2\Sigma\frac{1}{a}}{da^2} = \int_0^1 \frac{x^{a-1}(lx)^2}{x-1}dx,$$
$$\frac{d^3\Sigma\frac{1}{a}}{da^3} = \int_0^1 \frac{x^{a-1}(lx)^3}{x-1}dx,$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{d^{n-1}\Sigma\frac{1}{a}}{da^{n-1}} = \int_0^1 \frac{x^{a-1}(lx)^{n-1}}{x-1}dx.$$

By substituting these values, we obtain

$$\Sigma\frac{1}{a^2}=-\int_0^1\frac{x^{a-1}lx}{x-1}dx,$$
$$\Sigma\frac{1}{a^3}=\frac{1}{2}\int_0^1\frac{x^{a-1}(lx)^2}{x-1}dx,$$
$$\Sigma\frac{1}{a^4}=-\frac{1}{2.3}\int_0^1\frac{x^{a-1}(lx)^3}{x-1}dx,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Sigma\frac{1}{a^{2n}}=-\frac{1}{2.3.4\ldots(2n-1)}\int_0^1\frac{x^{a-1}(lx)^{2n-1}}{x-1}dx,$$
$$\Sigma\frac{1}{a^{2n+1}}=+\frac{1}{2.3.4\ldots 2n}\int_0^1\frac{x^{a-1}(lx)^{2n}}{x-1}dx.$$

In general, for any α, we have

$$\Sigma\frac{1}{a^\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^1\frac{x^{a-1}\left(l\frac{1}{x}\right)^{\alpha-1}}{x-1}dx.$$

Denoting $\Sigma\frac{1}{a^\alpha}$ by $L(a,\,\alpha)$, we will have

$$L(a,\,\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^1\frac{x^{a-1}\left(l\frac{1}{x}\right)^{\alpha-1}}{x-1}dx+C. \tag{1}$$

By expanding $\frac{x^{a-1}}{x-1}$ into an infinite series, we obtain

$$L(a,\,\alpha)=\frac{1}{\Gamma(\alpha)}\left[\int_0^1 x^{a-2}\left(l\frac{1}{x}\right)^{\alpha-1}dx+\int_0^1 x^{\alpha-3}\left(l\frac{1}{x}\right)^{\alpha-1}dx+\int_0^1 x^{a-4}\left(l\frac{1}{x}\right)^{\alpha-1}dx+\ldots\right];$$

now $\int_0^1 x^{a-k-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx=\frac{\Gamma(\alpha)}{(a-k)^\alpha}$, therefore

$$L(a,\,\alpha)=\frac{1}{(a-1)^\alpha}+\frac{1}{(a-2)^\alpha}+\frac{1}{(a-3)^\alpha}+\cdots+C,$$

where C is a constant independent of a. To find it, we substitute $a=1$ in (1), yielding $L(1,\,\alpha)=0$ and $x^{a-1}=x^0=1$; hence

$$C=-\frac{1}{\Gamma(\alpha)}\int_0^1\frac{\left(l\frac{1}{x}\right)^{\alpha-1}}{x-1}dx.$$

We can then draw the conclusion that

$$L(a,\,\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^1\frac{x^{a-1}-1}{x-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx,$$

where α can be either positive, negative or zero. We have

$$x^{a-1}=\left(\frac{1}{x}\right)^{-a+1}=1-(a-1)\left(l\frac{1}{x}\right)+\frac{(a-1)^2}{2}\cdot\left(l\frac{1}{x}\right)^2-\frac{(a-1)^3}{2.3}\left(l\frac{1}{x}\right)^3+\text{ etc.}$$

By substituting this value, we have

$$L(a,\alpha)=\frac{1}{\Gamma(\alpha)}\left\{(a-1)\int_0^1\frac{\left(l\frac{1}{x}\right)^\alpha}{1-x}dx-\frac{(a-1)^2}{2}\int_0^1\frac{\left(l\frac{1}{x}\right)^{\alpha+1}}{1-x}dx+\frac{(a-1)^3}{2.3}\int_0^1\frac{\left(l\frac{1}{x}\right)^{\alpha+2}}{1-x}dx-\ldots\right\}.$$

Let us consider the expression $\int_0^1 \frac{\left(l\frac{1}{x}\right)^k}{1-x} dx$. By expanding $\frac{1}{1-x}$, we have

$$\int \frac{\left(l\frac{1}{x}\right)^k}{1-x} dx = \int \left(l\frac{1}{x}\right)^k dx + \int x \left(l\frac{1}{x}\right)^k dx + \int x^2 \left(l\frac{1}{x}\right)^k dx + \ldots;$$

but $\int_0^1 x^n \left(l\frac{1}{x}\right)^k dx = \frac{\Gamma(k+1)}{(n+1)^{k+1}}$, thus

$$\int_0^1 \frac{\left(l\frac{1}{x}\right)^k}{1-x} dx = \Gamma(k+1) \left(1 + \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} + \frac{1}{4^{k+1}} + \ldots\right),$$

and thus finally

$$\begin{aligned} L(a,\, \alpha) = & \frac{(a-1).\Gamma(\alpha+1)}{\Gamma(\alpha)} \left(1 + \frac{1}{2^{\alpha+1}} + \frac{1}{3^{\alpha+1}} + \frac{1}{4^{\alpha+1}} + \ldots\right) \\ & - \frac{(a-1)^2.\Gamma(\alpha+2)}{2.\Gamma(\alpha)} \left(1 + \frac{1}{2^{\alpha+2}} + \frac{1}{3^{\alpha+2}} + \frac{1}{4^{\alpha+2}} + \ldots\right) \\ & + \frac{(a-1)^3.\Gamma(\alpha+3)}{2.3.\Gamma(\alpha)} \left(1 + \frac{1}{2^{\alpha+3}} + \frac{1}{3^{\alpha+3}} + \frac{1}{4^{\alpha+3}} + \ldots\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Now we have $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma(\alpha+2) = \alpha(\alpha+1)\Gamma(\alpha)$, and in general $\Gamma(\alpha+k) = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)\Gamma(\alpha)$. Substituting these values, we obtain

$$\begin{aligned} L(a,\, \alpha) = & \frac{a-1}{1}\alpha \left(1 + \frac{1}{2^{\alpha+1}} + \frac{1}{3^{\alpha+1}} + \frac{1}{4^{\alpha+1}} + \ldots\right) \\ & - \frac{(a-1)^2}{1.2}\alpha(\alpha+1) \left(1 + \frac{1}{2^{\alpha+2}} + \frac{1}{3^{\alpha+2}} + \frac{1}{4^{\alpha+2}} + \ldots\right) \\ & + \frac{(a-1)^3}{1.2.3}\alpha(\alpha+1)(\alpha+2) \left(1 + \frac{1}{2^{\alpha+3}} + \frac{1}{3^{\alpha+3}} + \frac{1}{4^{\alpha+3}} + \ldots\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If we let a go to infinity, we have

$$L(\infty,\, \alpha) = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} + \ldots,$$

so denoting $L(\infty,\, \alpha)$ by $L'(\alpha)$, we have

$$L(a, \alpha) = \alpha.(a-1)L'(\alpha+1) - \frac{\alpha(\alpha+1)}{2}(a-1)^2 L'(\alpha+2) + \frac{\alpha(\alpha+1)(\alpha+2)}{2.3}(a-1)^3 L'(\alpha+3) - \ldots.$$

If in formula (1) we replace a with $\frac{m}{a}$, we obtain

$$L\left(\frac{m}{a},\, \alpha\right) = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{\left(x^{\frac{m}{a}-1} - 1\right)\left(l\frac{1}{x}\right)^{\alpha-1}}{x-1} dx.$$

By letting $x^{\frac{1}{a}} = y$, x becomes y^a, $dx = ay^{a-1}$, $\left(l\frac{1}{x}\right)^{\alpha-1} = a^{\alpha-1}\left(l\frac{1}{y}\right)^{\alpha-1}$, and therefore

$$L\left(\frac{m}{a},\, \alpha\right) = \frac{a^\alpha}{\Gamma(\alpha)} \int_0^1 \frac{(y^{m-a}-1)\left(l\frac{1}{y}\right)^{\alpha-1} y^{a-1}}{y^a - 1} dy = \frac{a^\alpha}{\Gamma(\alpha)} \int_0^1 \frac{y^{m-1} - y^{a-1}}{y^a - 1} \left(l\frac{1}{y}\right)^{\alpha-1} dy.$$

From this, we obtain

$$L\left(\frac{m}{a},\,\alpha\right)=-\frac{1}{\Gamma(\alpha)}\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{y-1}dy+\frac{a^\alpha}{\Gamma(\alpha)}\int_0^1\frac{y^{m-1}\left(l\frac{1}{y}\right)^{\alpha-1}}{y^a-1}dy.$$

Now if $m-1<a$, as we can assume, then the fraction $\frac{y^{m-1}}{y^a-1}$ can be expressed in terms of partial fractions of the form $\frac{A}{1-cy}$. We will therefore have

$$L\left(\frac{m}{a},\,\alpha\right)=\left\{A\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1-cy}dy+A'\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1-c'y}dy+\dots\right\}\frac{a^\alpha}{\Gamma(\alpha)}.$$

If we expand $\frac{1}{1-cy}$ as a series, we see that

$$\int\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1-cy}dy=\int\left(l\frac{1}{y}\right)^{\alpha-1}dy+c\int y\left(l\frac{1}{y}\right)^{\alpha-1}dy+c^2\int y^2\left(l\frac{1}{y}\right)^{\alpha-1}dy+\dots,$$

however $\int_0^1\left(l\frac{1}{y}\right)^{\alpha-1}y^k dy=\frac{\Gamma(\alpha)}{(k+1)^\alpha}$, so

$$\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1-cy}dy=\Gamma(\alpha)\left(1+\frac{c}{2^\alpha}+\frac{c^2}{3^\alpha}+\frac{c^3}{4^\alpha}+\dots\right),$$

and thus denoting $1+\frac{c}{2^\alpha}+\frac{c^2}{3^\alpha}+\frac{c^3}{4^\alpha}+\dots$ by $L'(\alpha,\,c)$, we will have

$$\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1-cy}.dy=\Gamma(\alpha).L'(\alpha,\,c);$$

and we will finally obtain:

$$L\left(\frac{m}{a},\,\alpha\right)=a^\alpha\left[A.L'(\alpha,\,c)+A'.L'\left(\alpha,\,c'\right)+A''.L'\left(\alpha,\,c''\right)+\text{ etc. }\right].$$

The function $L\left(\frac{m}{a},\,\alpha\right)$ can thus, when m and a are integers, be expressed in finite form using the functions $\Gamma(\alpha)$ and $L'(\alpha,\,c)$. For example, let $m=1$, $a=2$. Then we have

$$L\left(\frac{1}{2},\,\alpha\right)=\frac{2^\alpha}{\Gamma(\alpha)}\int_0^1\frac{1-y}{y^2-1}\left(l\frac{1}{y}\right)^{\alpha-1}dy=-\frac{2^\alpha}{\Gamma(\alpha)}\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{1+y}dy.$$

Therefore, we have $A=-1$ and $c=-1$, thus

$$L\left(\frac{1}{2},\,\alpha\right)=-2^\alpha.L'(\alpha,\,-1)=-2^\alpha\left(1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\frac{1}{4^\alpha}+\dots\right).$$

When α is an integer, we know that the sum of this series can be expressed in terms of the number π or by the logarithm of 2. Letting $\alpha=1$, we have $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots=\log 2$, so $L\left(\frac{1}{2},\,1\right)=L\left(\frac{1}{2}\right)=-2\log 2$.

By setting $\alpha = 2$, we have $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$, thus

$$L\left(\frac{1}{2}, 2\right) = -\frac{\pi^2}{3}.$$

In general, we can express $L\left(\frac{1}{2}, 2n\right)$ as $-M\pi^{2n}$, where M is a rational number.