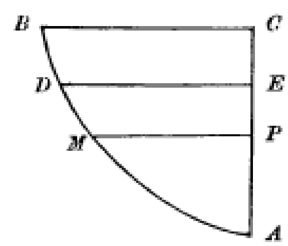
IX.

RESOLUTION OF A MECHANICS PROBLEM.

Journal für die reine und angewandte Mathematik, herausgegeben von Crelle, Bd. I, Berlin 1826.



Let $BDMA$ be an arbitrary curve. Let BC be a horizontal line and CA a vertical line. Suppose that a point mass subject to gravity moves along the curve, with a certain point D as its starting point. Let τ be the time which has elapsed when the object reaches a given point A, and let a be the height EA. The quantity τ will be a certain function of a, which depends on the shape of the curve. Conversely, the shape of the curve will depend on this function. We will examine how, using a definite integral, we can find the equation of the curve for which τ is a given continuous function of a.

Let $AM = s$, $AP = x$, and let t be the time it takes for the object to travel along the arc DM. According to the rules of mechanics, we have $-\frac{ds}{dt} = \sqrt{a-x}$, thus $dt = -\frac{ds}{\sqrt{a-x}}$. It follows that, when we integrate from $x = a$ to $x = 0$,

$$\tau = -\int_a^0 \frac{ds}{\sqrt{a-x}} = \int_0^a \frac{ds}{\sqrt{a-x}},$$

where the notation $\int_\alpha^\beta$ means that the limits of the integral are $x = \alpha$ and $x = \beta$. Let us now consider the given function

$$\tau = \varphi a.$$

We have

$$\varphi a = \int_0^a \frac{ds}{\sqrt{a-x}},$$

an equation from which we must determine s as a function of x. Instead of this equation, we will consider the more general one

$$\varphi a = \int_0^a \frac{ds}{(a-x)^n},$$

from which we will seek to derive the expression of s in terms of x.

Letting $\Gamma\alpha$ denote the function

$$\Gamma\alpha = \int_0^1 dx \left(\log\frac{1}{x}\right)^{\alpha-1},$$

we have, as is well known,

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma\alpha.\Gamma\beta}{\Gamma(a+\beta)},$$

where α and β must be greater than zero. Letting $\beta = 1-n$, we find

$$\int_0^1 \frac{y^{\alpha-1}dy}{(1-y)^n} = \frac{\Gamma\alpha.\Gamma(1-n)}{\Gamma(\alpha+1-n)},$$

from which we obtain, by taking $z = ay$,

$$\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \frac{\Gamma\alpha.\Gamma(1-n)}{\Gamma(\alpha+1-n)}a^{\alpha-n}.$$

Multiplying by $\dfrac{da}{(x-a)^{1-n}}$ and integrating from $a=0$ to $a=x$, we find

$$\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \frac{\Gamma\alpha.\Gamma(1-n)}{\Gamma(\alpha+1-n)}\int_0^x \frac{a^{\alpha-n}da}{(x-a)^{1-n}}.$$

Taking $a = xy$, we have

$$\int_0^x \frac{a^{\alpha-n}da}{(x-a)^{1-n}} = x^\alpha \int_0^1 \frac{y^{\alpha-n}dy}{(1-y)^{1-n}} = x^\alpha \frac{\Gamma(\alpha-n+1)\Gamma n}{\Gamma(\alpha+1)},$$

thus

$$\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \Gamma n.\Gamma(1-n)\frac{\Gamma\alpha}{\Gamma(\alpha+1)}x^\alpha.$$

Now according to a well-known property of the function Γ, we have

$$\Gamma(\alpha+1) = \alpha\Gamma\alpha;$$

substituting, we have

$$\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \frac{x^\alpha}{\alpha}\Gamma n.\Gamma(1-n).$$

Multiplying by $\alpha\varphi\alpha.d\alpha$ and integrating with respect to α, we find

$$\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{\left(\int \varphi\alpha.\alpha z^{\alpha-1}d\alpha\right)dz}{(a-z)^n} = \Gamma n.\Gamma(1-n)\int \varphi\alpha.x^\alpha d\alpha.$$

Letting

$$\int \varphi\alpha.x^\alpha d\alpha = fx,$$

we differentiate to obtain

$$\int \varphi\alpha.\alpha x^{\alpha-1} d\alpha = f'x,$$

thus

$$\int \varphi\alpha.\alpha z^{\alpha-1} d\alpha = f'z;$$

therefore

$$\int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{f'z.dz}{(a-z)^n} = \Gamma n.\Gamma(1-n) fx,$$

or, since $\Gamma n.\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$,

(1) $$fx = \frac{\sin n\pi}{\pi} \int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{f'z.dz}{(a-z)^n}.$$

Using this equation, it will be easy to derive the value of s from the equation

$$\varphi a = \int_0^a \frac{ds}{(a-x)^n}.$$

If we multiply this equation by $\dfrac{\sin n\pi}{\pi} \dfrac{da}{(x-a)^{1-n}}$, and take the integral from $a=0$ to $a=x$, we have

$$\frac{\sin n\pi}{\pi} \int_0^x \frac{\varphi a.da}{(x-a)^{1-n}} = \frac{\sin n\pi}{\pi} \int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{ds}{(a-x)^n},$$

thus, by virtue of equation (1),

$$s = \frac{\sin n\pi}{x} \int_0^x \frac{\varphi a.da}{(x-a)^{1-n}}.$$

Now let $n = \frac{1}{2}$. We obtain

$$\varphi a = \int_0^a \frac{ds}{\sqrt{a-x}}$$

and

$$s = \frac{1}{\pi} \int_0^x \frac{\varphi a.da}{\sqrt{x-a}}.$$

This equation gives the arc s as a function of the abscissa x, and therefore the curve is completely determined.

We will now apply the expression that has been found to a few examples.

I. Letting

$$\varphi a = \alpha_0 a^{\mu_0} + \alpha_1 a^{\mu_1} + \cdots + \alpha_m a^{\mu_m} = \Sigma \alpha a^{\mu}$$

the value of s will be

$$s = \frac{1}{\pi} \int_0^x \frac{da}{\sqrt{x-a}} \Sigma \alpha a^{\mu} = \frac{1}{\pi} \Sigma \left(\alpha \int_0^x \frac{a^{\mu} da}{\sqrt{x-a}} \right).$$

If we let $a=xy$, we will have

$$\int_0^x \frac{a^\mu da}{\sqrt{x-a}} = x^{\mu+\frac{1}{2}} \int_0^1 \frac{y^\mu dy}{\sqrt{1-y}} = x^{\mu+\frac{1}{2}} \frac{\Gamma(\mu+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{3}{2}\right)},$$

so

$$s = \frac{\Gamma\left(\frac{1}{2}\right)}{\pi} \Sigma \frac{\alpha\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{3}{2}\right)} x^{\mu+\frac{1}{2}},$$

or, since $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$,

$$s = \sqrt{\frac{x}{\pi}} \left[\alpha_0 \frac{\Gamma(\mu_0+1)}{\Gamma\left(\mu_0+\frac{3}{2}\right)} x^{\mu_0} + \alpha_1 \frac{\Gamma(\mu_1+1)}{\Gamma\left(\mu_1+\frac{3}{2}\right)} x^{\mu_1} + \cdots + \alpha_m \frac{\Gamma(\mu_m+1)}{\Gamma\left(\mu_m+\frac{3}{2}\right)} x^{\mu_m}\right].$$

If we assume that e.g. $m=0$, $\mu_0=0$, i.e. that the curve to be found is an isochrone, we find

$$s = \sqrt{\frac{x}{\pi}} \alpha_0 \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\alpha_0}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \sqrt{\frac{x}{\pi}} = \frac{2\alpha_0}{x}\sqrt{x},$$

but $s=\frac{2\alpha_0}{\pi}\sqrt{x}$ is the well-known equation of the cycloid.

II. Letting

φa from $a=0$ to $a=a_0$ be equal to $\varphi_0 a$
φa from $a=a_0$ to $a=a_1$ be equal to $\varphi_1 a$
φa from $a=a_1$ to $a=a_2$ be equal to $\varphi_2 a$
....
φa from $a=a_{m-1}$ to $a=a_m$ be equal to $\varphi_m a$,

we will have

$$\pi s = \int_0^x \frac{\varphi_0 a.da}{\sqrt{a-x}}, \quad \text{from } x=0 \text{ to } x=a_0,$$

$$\pi s = \int_0^{a_0} \frac{\varphi_0 a.da}{\sqrt{a-x}} + \int_{a_0}^x \frac{\varphi_1 a.da}{\sqrt{a-x}}, \quad \text{from } x=a_0 \text{ to } x=a_1,$$

$$\pi s = \int_0^{a_0} \frac{\varphi_0 a.da}{\sqrt{a-x}} + \int_{a_0}^{a_1} \frac{\varphi_1 a.da}{\sqrt{a-x}} + \int_{a_1}^x \frac{\varphi_2 a.da}{\sqrt{a-x}}, \quad \text{from } x=a_1 \text{ to } x=a_2,$$

....

$$\pi s = \int_0^{a_0} \frac{\varphi_0 a.da}{\sqrt{a-x}} + \int_{a_0}^{a_1} \frac{\varphi_1 a.da}{\sqrt{a-x}} + \cdots + \int_{a_{m-2}}^{a_{m-1}} \frac{\varphi_{m-1} a.da}{\sqrt{a-x}} + \int_{a_{m-1}}^x \frac{\varphi_m a.da}{\sqrt{a-x}},$$

from $x=a_{m-1}$ to $x=a_m$,

where it must be noted that the functions $\varphi_0 a$, $\varphi_1 a$, $\varphi_2 a \ldots \varphi_m a$ must satisfy

$$\varphi_0 a_0 = \varphi_1 a_0,\ \varphi_1 a_1 = \varphi_2 a_1,\ \varphi_2 a_2 = \varphi_3 a_2, \quad \text{etc.},$$

because the function φa must necessarily be continuous.