A SIMPLE DEFINITE INTEGRAL EXPRESSING THE FINITE INTEGRAL $\Sigma^n \phi x$

According to the well-known theorem of *Parseval*, one may express the finite integral $\Sigma^n \phi x$ as a definite double integral, but if I am not mistaken, this finite integral has never been expressed as a simple definite integral. To do so will be the object of this memoire.

Writing ϕx for an arbitrary function of x, it is easy to see that one may always set

$$\phi x = \int e^{vx} fv.dv, \tag{1}$$

where the integral is taken between two limits of v which are independent of x, and fv denotes a function of v whose form depends on that of ϕx. Setting $\Delta x = 1$, and taking the finite integral of both sides of equation (1), one gets

$$\Sigma \phi x = \int e^{vx} \frac{fv}{e^v - 1} dv, \tag{2}$$

to which must be added an arbitrary constant. Taking a second finite integral, one obtains

$$\Sigma^2 \phi x = \int e^{vx} \frac{fv}{(e^v - 1)^2} dv. \tag{3}$$

In general, one finds that

$$\Sigma^n \phi x = \int e^{vx} \frac{fv}{(e^v - 1)^n} dv. \tag{4}$$

To complete the integral one must add to the right hand side a function of the form

$$C + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1},$$

where C, C_1, C_2 etc. are arbitrary constants.

The task is now to find the value of the definite integral $\int e^{vx} \frac{fv}{(e^v-1)^n} dv$. For this I will use a theorem of *Lagrange* (Exerc. de calc. int. T. II p. 189):

$$\frac{1}{4} \frac{e^v + 1}{e^v - 1} - \frac{1}{2v} = \int_0^{\frac{1}{0}} \frac{dt. \sin(vt)}{e^{2\pi t} - 1},$$

from which follows

$$\frac{1}{e^v - 1} = \frac{1}{v} - \frac{1}{2} + 2 \int_0^{\frac{1}{0}} \frac{dt. \sin(vt)}{e^{2\pi t} - 1}.$$

Substituting this value of $\frac{1}{e^v - 1}$ in equation (2), one gets

$$\Sigma \phi x = \int e^{vx} \frac{fv}{v} dv - \frac{1}{2} \int e^{vx} fv.dv + 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t} - 1} \int e^{vx} fv. \sin vt.dv.$$

The integral $\int e^{vx} fv. \sin vt.dv$ can be found in the following manner. Substituting $x \pm t\sqrt{-1}$ for x in equation (1), one gets:

$$\phi(x + t\sqrt{-1}) = \int e^{vx} e^{vt\sqrt{-1}} f(v).dv,$$

$$\phi(x - t\sqrt{-1}) = \int e^{vx} e^{vt\sqrt{-1}} f(v).dv.$$

Taking the difference and dividing by $2\sqrt{-1}$, one gets

$$\int e^{vx} \sin vt.fv.dv = \frac{\phi(x + t\sqrt{-1}) - \phi(x - t\sqrt{-1})}{2\sqrt{-1}}.$$

Therefore,

$$\Sigma \phi x = \int \phi x.dx - \frac{1}{2} \phi x + 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t} - 1} \frac{\phi(x + t\sqrt{-1}) - \phi(x - t\sqrt{-1})}{2\sqrt{-1}}.$$

Now to find the value of the general integral

$$\Sigma^n \phi x = \int e^{vx}.fv.\frac{dv}{(e^v-1)^n},$$

one may set

$$\frac{1}{(e^v-1)^n} = (-1)^n\left(A_{0,n}p + A_{1,n}\frac{dp}{dv} + A_{2,n}\frac{d^2p}{dv^2} + \cdots + A_{n-1,n}\frac{d^{n-1}p}{dv^{n-1}}\right),$$

where p is equal to $\frac{1}{e^v-1}$, and $A_{0,n}, A_{1,n}, \ldots$ are numerical coefficients to be determined. Differentiating the equation above, one gets

$$\frac{ne^v}{(e^v-1)^{n+1}} = (-1)^n\left(A_{0,n}\frac{dp}{dv} + A_{1,n}\frac{d^2p}{dv^2} + \cdots + A_{n-1},n\frac{d^np}{dv^n}\right).$$

Now,

$$\frac{ne^v}{(e^v-1)^{n+1}} = \frac{n}{(e^v-1)^n} + \frac{n}{(e^v-1)^{n+1}},$$

and therefore,

$$\begin{aligned}\frac{ne^v}{(e^v-1)^{n+1}} &= n(-1)^{n-1}\left(A_{0,n}p + A_{1,n}\frac{dp}{dv} + A_{2,n}\frac{d^2p}{dv^2} + \cdots + A_{n-1,n}\frac{d^{n-1}p}{dv^{n-1}}\right)\\ &\quad + n(-1)^n\left(A_{0,n+1}p + A_{1,n+1}\frac{dp}{dv} + A_{2,n}\frac{d^2p}{dv^2} + \cdots + A_{n,n+1}\frac{d^np}{dv^n}\right).\end{aligned}$$

Comparing these two expressions for $\frac{ne^v}{(e^v-1)^{n+1}}$, one deduces the following equations:

$$\begin{aligned}
A_{0,n+1} - A_{0,n} &= 0 & \Delta A_{0,n} &= 0,\\
A_{1,n+1} - A_{1,n} &= \tfrac{1}{n}A_{0,n} & \Delta A_{1,n} &= \tfrac{1}{n}A_{0,n},\\
A_{2,n+1} - A_{2,n} &= \tfrac{1}{n}A_{1,n} & \Delta A_{0,n} &= \tfrac{1}{n}A_{1,n},\\
\cdots\ \cdots\ \cdots & & \cdots &\\
A_{n-1,n+1} - A_{n-1,n} &= \tfrac{1}{n}A_{n-2,n} & \Delta A_{n-1,n} &= \tfrac{1}{n}A_{n-2,n},\\
A_{n,n+1} &= \tfrac{1}{n}A_{n-1,n},
\end{aligned}$$

from which one gets

$$A_{0,n} = 1,\ A_{1,n} = \Sigma\tfrac{1}{n},\ A_{2,n} + \Sigma\tfrac{1}{n}\Sigma\tfrac{1}{n},\ A_{3,n} = \Sigma\tfrac{1}{n}\Sigma\tfrac{1}{n}\Sigma\tfrac{1}{n},\ \text{etc.}$$

$$A_{n,n+1} = \tfrac{1}{n}\cdot\tfrac{1}{n-1}\cdot\tfrac{1}{n-2}\cdots\tfrac{1}{2}\cdot\tfrac{1}{1}\cdot A_{0,1} = \tfrac{1}{\Gamma(n+1)}.$$

The last equation serves to determine the constants in the expressions for $A_{0,n}$, $A_{1,n}$, $A_{2,n}$ etc.

Having thus determined the coefficients $A_{0,n}$, $A_{1,n}$, $A_{2,n}$ etc., substituting the value of $\frac{1}{(e^v-1)^n}$ in equation (3) gives

$$\Sigma^n\phi x = (-1)^{n-1}\int e^{vx}fv.dv\left(A_{0,n}p + A_{1,n}\frac{dp}{dv} + \cdots + A_{n-1,n}\frac{d^{n-1}p}{dv^{n-1}}\right).$$

Now, one has

$$p = \frac{1}{v} - \frac{1}{2} + 2\int_0^{\frac{1}{0}}\frac{dt.\sin vt}{e^{2\pi t}-1},$$

and differentiating gives

$$\begin{aligned}
\frac{dp}{dv} &= -\frac{1}{v^2} + 2\int_0^{\frac{1}{0}}\frac{tdt.\cos vt}{e^{2\pi t}-1},\\
\frac{d^2p}{dv^2} &= \frac{2}{v^2} - 2\int_0^{\frac{1}{0}}\frac{tdt.\sin vt}{e^{2\pi t}-1},\\
\frac{d^3p}{dv^3} &= \frac{2\cdot 3}{v^2} - 2\int_0^{\frac{1}{0}}\frac{tdt.\cos vt}{e^{2\pi t}-1}\ \text{etc.;}
\end{aligned}$$

therefore, by substitution

$$\Sigma^n \phi x = \int \left(A_{n-1,n} \frac{\Gamma n}{v^n} - A_{n-2,n} \frac{\Gamma(n-1)}{v^{n-1}} + \cdots + (-1)^{n-1} A_{0,n} \frac{1}{v} + (-1)^n . \frac{1}{2} \right) e^{vx} fv.dv$$
$$+ 2(-1)^{n-1} \iint \frac{P \sin vt.dt}{e^{2\pi t}-1} e^{vx} fv.dv + 2(-1)^{n-1} \iint_0^{\frac{1}{0}} \frac{Q \cos vt.dt}{e^{2\pi t}-1} e^{vx} fv.dv.$$

Integrating the equation $\phi x = \int e^{vx} fv.dv$, one gets:

$$\int \phi x.dx = \int e^{vx} fv \frac{dv}{v},$$
$$\int^2 \phi x dx^2 = \int e^{vx} fv \frac{dv}{v^2},$$
$$\int^3 \phi x.dx^3 = \int e^{vx} fv \frac{dv}{v^3} \quad \text{etc.}$$

Furthermore,

$$\int e^{vx} \sin vt.fv.dv = \frac{\phi(x+t\sqrt{-1}) - \phi(x-t\sqrt{-1})}{2\sqrt{-1}}.$$
$$\int e^{vx} \cos vt.fv.dv = \frac{\phi(x+t\sqrt{-1}) + \phi(x-t\sqrt{-1})}{2},$$

so by substitution one has

$$\Sigma^n \phi x = A_{n-1,n} \Gamma n \int^n \phi x.dx^n - A_{n-2,n} \Gamma(n-1) \int^{n-1} \phi x.dx^{n-1} + \cdots + (-1)^{n-1} \int \phi x.dx$$
$$+ (-1)^n \cdot \frac{\phi x}{2} + 2(-1)^{n-1} \int_0^{\frac{1}{0}} \frac{Pdt}{e^{2\pi t}-1} \frac{\phi(x+t\sqrt{-1}) - \phi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$+ 2(-1)^{n-1} \int_0^{\frac{1}{0}} \frac{Qdt}{e^{2\pi t}-1} \frac{\phi(x+t\sqrt{-1}) + \phi(x-t\sqrt{-1})}{2}$$

where

$$P = A_{0,n} - A_{2,n} t^2 + A_{4,n} t^4 - \cdots,$$
$$Q = A_{1,n} t - A_{3,n} t^3 + A_{5,n} t^5 - \cdots.$$

For example, taking $n = 2$, one gets

$$\Sigma^2 \phi x = \iint \phi x.dx^2 - \int \phi x.dx + \frac{1}{2} \phi x$$
$$- 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1} \frac{\phi(x+t\sqrt{-1}) - \phi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$- 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1} \frac{\phi(x+t\sqrt{-1}) + \phi(x-t\sqrt{-1})}{2}$$

and taking $\phi x = e^{ax}$,

$$\phi(x \pm t\sqrt{-1}) = e^{ax} e^{\pm at\sqrt{-1}}, \quad \int e^{ax} dx = \frac{1}{a} e^{ax}, \quad \iint e^{ax} dx^2 = \frac{1}{a^2} e^{ax}.$$

Substituting and dividing by e^{ax},

$$\frac{1}{(e^a-1)^2} = \frac{1}{2} - \frac{1}{a} + \frac{1}{a^2} - 2 \int_0^{\frac{1}{0}} \frac{dt.\sin at}{e^{2\pi t}-1} - 2 \int_0^{\frac{1}{0}} \frac{tdt.\cos at}{e^{2\pi t}-1}.$$

The most remarkable case is when $n = 1$. Then one has, from what we have seen:

$$\Sigma \phi x = C + \int \phi x.dx - \frac{1}{2} \phi x + 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1} \frac{\phi(x+t\sqrt{-1}) - \phi(x-t\sqrt{-1})}{2\sqrt{-1}}.$$

Assuming that the integrals $\Sigma \phi x$ and $\int \phi x dx$ vanish when $x = a$, one clearly has

$$C = \frac{1}{2} \phi a - 2 \int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1} \frac{\phi(a+t\sqrt{-1}) - \phi(a-t\sqrt{-1})}{2\sqrt{-1}},$$

and therefore

$$\Sigma\phi x = \int \phi x.dx + \frac{1}{2}\left(\phi a - \phi x\right) + 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\phi(x+t\sqrt{-1})-\phi(x-t\sqrt{-1})}{2\sqrt{-1}}$$
$$- 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

Taking $x = \infty$, and assuming that ϕx and $\int \phi x.dx$ vanish for this value of x, one has:

$$\phi a + \phi(a+1) + \phi(a+2) + \phi(a+3) + \dots$$
$$= \int_0^{\frac{1}{0}} \phi x.dx + \frac{1}{2}\phi a - 2\int_0^{\frac{1}{0}} \frac{dt}{e^{2\pi t}-1}\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

For example, if $\phi x = \frac{1}{x^2}$,

$$\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}} = \frac{-2at}{\left(a^2+t^2\right)^2},$$

and therefore

$$\frac{1}{a^2} + \frac{1}{\left(a+1\right)^2} + \frac{1}{\left(a+2\right)^2} + \cdots = \frac{1}{2a^2} + \frac{1}{a} + 4a\int_0^{\frac{1}{0}} \frac{tdt}{(e^{2\pi t}-1)(a^2+t^2)^2}.$$

Setting $a = 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6} = \frac{3}{2} + 4\int_0^{\frac{1}{0}} \frac{tdt}{(e^{2\pi t}-1)(1+t^2)^2}.$$