Review

- 导数

$$f'(x_0) \triangleq \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$f'_{\pm}(x_0) \triangleq \lim_{\Delta x \to 0^{\pm}} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- $f'(x_0)$存在 $\Leftrightarrow f'_-(x_0), f'_+(x_0)$均存在且相等.

- 导数的几何、物理意义

- 可微 $\Leftrightarrow$ 可导 $\Rightarrow$ 连续

- $y = f(x), f'(x)$也记为$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

- f 在 x_0 可微, 则 $\quad f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x.$

- f, g 在 x_0 可导, $c \in \mathbb{R}$, 则

   $(1)(f + g)'(x_0) = f'(x_0) + g'(x_0);$

   $(2)(cf)'(x_0) = cf'(x_0);$

   $(3)(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0);$

   $(4)\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$

- 多个因子连乘的函数求导时先取对数再两端求导.

- (链式法则)$\varphi(x)$在x_0可导, $f(u)$在 $u_0 = \varphi(x_0)$可导,则

$$h(x) = f(\varphi(x))在x_0可导, 且$$

$$h'(x_0) = f'(\varphi(x_0)) \cdot \varphi'(x_0), 即 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

- (一阶微分形式的不变性)$u = \varphi(x)$在x_0可微, $y = f(u)$

在$u_0 = \varphi(x_0)$可微,则$y = f(\varphi(x))$在x_0可微, 且

$$\mathrm{d}y = f'(\varphi(x_0))\varphi'(x_0)\mathrm{d}x = f'(u_0)\mathrm{d}u.$$

无论将u视为中间变量还是自变量,都有$\mathrm{d}y = f'(u)\mathrm{d}u.$

- (反函数求导)$\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}.$

$$c' = 0, \qquad (x^{\alpha})' = \alpha x^{\alpha-1},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x,$$

$$(\tan x)' = \sec^2 x, \qquad (\cot x)' = -\csc^2 x,$$

$$(\sec x)' = \sec x \tan x, \qquad (\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad \arctan x = \frac{1}{1+x^2}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}, \qquad \text{arc} \cot x = \frac{-1}{1+x^2}$$

$$(a^x)' = a^x \ln a, \qquad (e^x)' = e^x$$

$$(\log_a x)' = \frac{1}{x \ln a}, \qquad (\ln x)' = \frac{1}{x}$$

$$\left(\ln \left| x + \sqrt{x^2 \pm a^2} \right| \right)' = \frac{1}{\sqrt{x^2 \pm a^2}}$$

- 隐函数求导
- 参数函数求导

§3. 高阶导数

Def. $y = f(x).$

二阶导(函)数: $y''(x) = \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f''(x) \triangleq (f'(x))',$

三阶导(函)数: $y'''(x) = \dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = f'''(x) \triangleq (f''(x))',$

$\vdots$

$n+1$阶导(函)数: $y^{(n+1)}(x) = \dfrac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}} = f^{(n+1)}(x) \triangleq (f^{(n)}(x))'.$

Def. $f \in C^n(a,b):$ f在(a,b)上n阶可导, 且$f^{(n)} \in C(a,b).$

Question. $f \in C^n[a,b]$如何定义?

Ex. 求 $\sin^{(n)} x, \cos^{(n)} x.$

解: $\sin' x = \cos x = \sin(x + \dfrac{\pi}{2}),$

$\sin''(x) = -\sin x = \sin(x + 2 \cdot \dfrac{\pi}{2}),$

$\sin''' x = -\cos x = \sin(x + \dfrac{3\pi}{2}),$

$\sin^{(4)} x = \sin x = \sin(x + \dfrac{4\pi}{2}),$

$\vdots$

$\sin^{(n)} x = \sin(x + \dfrac{n\pi}{2}).$ 同理, $\cos^{(n)} x = \cos(x + \dfrac{n\pi}{2}).$

Ex. $y = \ln(1+x)$, 求 $y^{(n)}$.

解： $y' = \dfrac{1}{1+x} = (1+x)^{-1}$,

$y'' = -(1+x)^{-2}$,

$y''' = 2\,!(1+x)^{-3}$,

$\vdots$

$y^{(n)} = (-1)^{n-1}(n-1)\,!(1+x)^{-n}$.

Thm. 设 $f(x)$ 与 $g(x)$ 在点 x 处有 n 阶导数, $c \in \mathbb{R}$, 则

(1) $(f + g)^{(n)}(x) = f^{(n)}(x) + g^{(n)}(x)$;

(2) $(cf)^{(n)}(x) = c \cdot f^{(n)}(x)$;

(3) $(f \cdot g)^{(n)}(x) = \sum\limits_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x)$. (Leibniz公式)

Proof of (3). $n = 1$ 时, $(fg)' = f'g + fg'$, 结论成立.

设 $n = m$ 时结论成立, 即

$$(f \cdot g)^{(m)}(x) = \sum_{k=0}^{m} C_m^k f^{(k)}(x) g^{(m-k)}(x),$$

则 $n = m + 1$ 时,

$$(f \cdot g)^{(m+1)}(x) = \left(\sum_{k=0}^{m} C_m^k f^{(k)}(x) g^{(m-k)}(x) \right)'$$

$$= \sum_{k=0}^{m} C_m^k f^{(k+1)}(x) g^{(m-k)}(x) + \sum_{k=0}^{m} C_m^k f^{(k)}(x) g^{(m+1-k)}(x)$$

$$= \sum_{k=1}^{m} C_m^{k-1} f^{(k)}(x) g^{(m+1-k)}(x) + f^{(m+1)}(x) g(x)$$

$$+ \sum_{k=1}^{m} C_m^k f^{(k)}(x) g^{(m+1-k)}(x) + f(x) g^{(m+1)}(x)$$

$$= \sum_{k=1}^{m} (C_m^{k-1} + C_m^k) f^{(k)}(x) g^{(m+1-k)}(x) + f^{(m+1)}(x) g(x) + f(x) g^{(m+1)}(x)$$

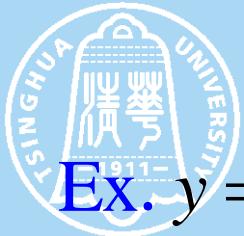$$= \sum_{k=0}^{m+1} C_{m+1}^k f^{(k)}(x) g^{(m+1-k)}(x). \ \square$$

Ex. $y = \dfrac{1}{x^2 - x - 2}$, 求 $y^{(n)}$.

解: $y = \dfrac{1}{(x+1)(x-2)} = \dfrac{1}{3}\left(\dfrac{1}{x-2} - \dfrac{1}{x+1}\right).$

$y^{(n)} = \dfrac{1}{3}\left(\dfrac{1}{x-2}\right)^{(n)} - \dfrac{1}{3}\left(\dfrac{1}{x+1}\right)^{(n)}$

$= \dfrac{1}{3}(-1)^n n!(x-2)^{-(n+1)} - \dfrac{1}{3}(-1)^n n!(x+1)^{-(n+1)}.\ \square$

Ex. $y = (\arcsin x)^2$, 求 $y^{(n)}(0)$.

解：$y' = 2(\arcsin x)\big/\sqrt{1-x^2}$, $\sqrt{1-x^2}\,y' = 2\arcsin x$,

两边对 x 求导, 得 $-xy'\big/\sqrt{1-x^2} + \sqrt{1-x^2}\,y'' = 2\big/\sqrt{1-x^2}$,

$$xy' + (x^2 - 1)y'' = -2,$$

$$y' + xy'' + 2xy'' + (x^2 - 1)y''' = 0,$$

$$xy^{(n+1)} + ny^{(n)} + (x^2 - 1)y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} = 0, n \geq 2.$$

令 $x = 0$, 得 $y^{(n+2)}(0) = n^2 y^{(n)}(0)$, $y'(0) = 0$, $y''(0) = 2$, $y'''(0) = 0$,

故 $\quad y^{(n)}(0) = \begin{cases} 0, & n = 2k-1, \\ 2^{2k-1}\big((k-1)!\big)^2, & n = 2k. \end{cases}$

Ex. $x^2 + xy + y^2 = 1$ 确定了隐函数 $y = y(x)$, 求 $y''(x)$.

解：视 $x^2 + xy + y^2 = 1$ 中 $y = y(x)$, 两边对 x 求导, 得

$$2x + y + xy' + 2yy' = 0, \quad y' = -\frac{2x+y}{x+2y}.$$

于是

$$y'' = -\frac{(2x+y)'(x+2y) - (2x+y)(x+2y)'}{(x+2y)^2}$$

$$= -\frac{(2+y')(x+2y) - (2x+y)(1+2y')}{(x+2y)^2}$$

$$= \frac{3(xy'-y)}{(x+2y)^2} = \frac{-6(x^2+xy+y^2)}{(x+2y)^3} = \frac{-6}{(x+2y)^3}. \square$$

Ex. $y = 2x + \sin x$, 求 $x''(y)$.

解法一： 视 $y = 2x + \sin x$ 中 $x = x(y)$, 两边对 y 求导, 得

$$1 = 2x'(y) + \cos x \cdot x'(y).$$

再对 y 求导, 得 $\qquad 0 = 2x'' - \sin x \cdot (x')^2 + \cos x \cdot x''.$

解得 $\qquad x'(y) = \dfrac{1}{2 + \cos x}, \quad x'' = \dfrac{\sin x \cdot (x')^2}{2 + \cos x} = \dfrac{\sin x}{(2 + \cos x)^3}.$

解法二： $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}} = \dfrac{1}{2 + \cos x},$

$\dfrac{d^2 x}{dy^2} = \dfrac{d}{dy}\left(\dfrac{dx}{dy}\right) = \dfrac{d}{dx}\left(\dfrac{dx}{dy}\right) \cdot \dfrac{dx}{dy} = \dfrac{d}{dx}\left(\dfrac{1}{2 + \cos x}\right) \cdot \dfrac{1}{2 + \cos x}$

$\qquad = -\dfrac{-\sin x}{(2 + \cos x)^2} \cdot \dfrac{1}{2 + \cos x} = \dfrac{\sin x}{(2 + \cos x)^3}.$ □

Ex. $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$, 求 $y'(x), y''(x)$.

解： $y'(x) = \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{a \sin t}{a(1 - \cos t)} = \dfrac{\sin t}{1 - \cos t}.$

$y''(x) = \dfrac{d(\dfrac{dy}{dx})}{dx} = \dfrac{\dfrac{d}{dt}\left(\dfrac{\sin t}{1 - \cos t}\right)}{\dfrac{dx}{dt}} = \dfrac{\dfrac{\cos t(1 - \cos t) - \sin^2 t}{(1 - \cos t)^2}}{a(1 - \cos t)}$

$= \dfrac{-1}{a(1 - \cos t)^2}.\ \square$

Ex. 证明 $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ 任意阶可导,并求 $f^{(n)}(x)$.

Proof. $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{e^{-1/x^2}}{x} = \lim\limits_{t \to \infty} \dfrac{t}{e^{t^2}} = 0.$

$$f'(x) = \begin{cases} 0, & x = 0, \\ 2x^{-3}e^{-x^{-2}}, & x \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} 0, & x = 0, \\ (-6x^{-4} + 4x^{-6})e^{-x^{-2}}, & x \neq 0. \end{cases}$$

归纳可证

$$f^{(n)}(x) = \begin{cases} 0, & x = 0, \\ P_{3n}(\dfrac{1}{x})e^{-x^{-2}}, & x \neq 0. \end{cases}$$

$P_{3n}(\cdot)$为$3n$次多项式. □

作业：习题**3.3**
    No. 3(单),4(3),5(3),6,7