Review

- 连续点与间断点

  初等函数的连续性

- 当$x \to 0$时, $\arcsin x \sim x$, $\arctan x \sim x$.

- 闭区间上连续函数的性质

  零点定理　　介值定理　　有界性定理

  最大最小值定理　一致连续性定理

- f在I上非一致连续$\Leftrightarrow$

$$\exists \varepsilon_0 > 0, \exists x_n, y_n \in I, \lim_{n\to\infty}(x_n - y_n) = 0, s.t. \left| f(x_n) - f(y_n) \right| \geq \varepsilon_0.$$

§ 1. 导数

Def.（导数, 左、右导数）

$$(1)\ f'(x_0) \triangleq \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0};$$

$$(2)\ f'_-(x_0) \triangleq \lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0};$$

$$(3)\ f'_+(x_0) \triangleq \lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}.$$

Question. 导数的几何意义? 切线的斜率.

Question. 可导的几何意义? 光滑性

Remark. 导函数 $f'(x)$.

Ex.(1) $c' = 0$, (2) $(\sin x)' = \cos x$, (3) $(\cos x)' = -\sin x$,

(4) $(a^x)' = a^x \ln a$, (5) $(\log_a x)' = \dfrac{1}{x \ln a}$, (6) $(x^\alpha)' = \alpha x^{\alpha-1}$.

Proof.(1) $f(x) \equiv c$,则

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = 0.$$

$$(2)(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{2\sin\frac{h}{2}\cos(x+\frac{h}{2})}{h}$$

$$= \lim_{h \to 0} \frac{2\sin\frac{h}{2}}{h} \cdot \lim_{h \to 0} \cos(x+\frac{h}{2}) = 1 \cdot \cos x = \cos x.$$

$$(3)(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\frac{h}{2}\sin(x+\frac{h}{2})}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\frac{h}{2}}{h}\cdot\lim_{h\to 0}\sin(x+\frac{h}{2}) = -\sin x.$$

$$(4)(a^x)' = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x\lim_{h\to 0}\frac{a^h-1}{h} = a^x\ln a.$$

特别地, $(e^x)' = e^x$.

$$(5)(\log_a x)' = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \frac{1}{\ln a}\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$= \frac{1}{x\ln a}\lim_{h\to 0}\frac{\ln(1+h/x)}{h/x} = \frac{1}{x\ln a}.$$

特别地, $(\ln x)' = \dfrac{1}{x}.$

$$(6)(x^\alpha)'=\lim_{h\to 0}\frac{(x+h)^\alpha-x^\alpha}{h}$$

$$x\neq 0\text{时},(x^\alpha)'=x^\alpha\lim_{h\to 0}\frac{(1+h/x)^\alpha-1}{h}=x^\alpha\lim_{h\to 0}\frac{\alpha h/x}{h}=\alpha x^{\alpha-1}.$$

$$x=0\text{时},\ f'(0)=\lim_{h\to 0}\frac{h^\alpha-0^\alpha}{h}=\lim_{h\to 0}h^{\alpha-1}$$

$$=\begin{cases}1 & \alpha=1\\ \text{不存在} & \alpha<1\\ 0 & \alpha>1\text{且}x^{\alpha-1}\text{在}(-\delta,\delta)\text{有定义}\\ \text{不存在} & \alpha>1\text{且}x^{\alpha-1}\text{在}(-\delta,0)\text{无定义}\end{cases}$$

综上， $(x^\alpha)'=\alpha x^{\alpha-1}$ （$x^{\alpha-1}$有意义时成立）.

Thm. $f'(x_0)$存在 $\Leftrightarrow$ $f'_-(x_0)$, $f'_+(x_0)$均存在且相等.

f在x_0可导时, $f'(x_0)=f'_-(x_0)=f'_+(x_0)$.

Ex. $f(x)=\begin{cases} x+1, & x\le 0, \\ e^x, & x>0. \end{cases}$ 求$f'(0)$.

$$f'_-(0)=\lim_{x\to 0-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0-}\frac{x+1-1}{x}=1.$$

$$f'_+(0)=\lim_{x\to 0+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0+}\frac{e^x-1}{x}=1=f'_-(0).$$

故$f'(0)=1$.□

Thm. f在x_0可导 $\Rightarrow$ f在x_0连续

Proof. f在x_0可导, 记$\rho(x)=\dfrac{f(x)-f(x_0)}{x-x_0}-f'(x_0)$, 则

$$\lim_{x\to x_0}\rho(x)=0,$$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\rho(x)(x-x_0).$$

于是$\lim\limits_{x\to x_0}f(x)=f(x_0)$, 即$f$在$x_0$连续.□

Ex. $f(x)=x^2D(x)$的可导性质？$D(x)$为Dirichlet函数.

解：$f'(0)=\lim\limits_{x\to 0}\dfrac{x^2D(x)-0}{x}=0.$

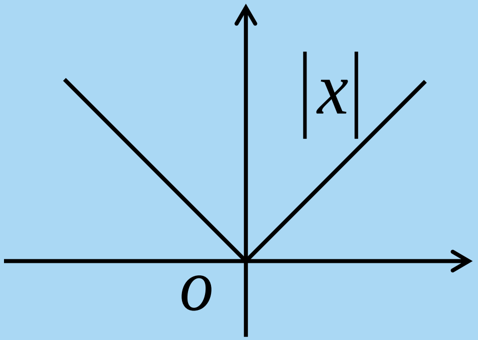
$f(x)$在任一$x_0\neq 0$处不连续, 因而不可导.□

Ex. $f(x)=|x|$在$x_0=0$是否可导？

连续$\not\Rightarrow$可导

$$\lim_{x\to 0^{\pm}}\frac{|x|-0}{x}=\pm 1,\quad f(x)=|x|\text{在}x_0=0\text{不可导.}$$

$$(|x|)'=\begin{cases}1 & x>0,\\ \text{不存在} & x=0,\\ -1 & x<0.\end{cases}$$



Remark. 利用级数可以构造处处连续处处不可导的例子.

Question. 导数的物理意义?

t	$f(t)$	$f'(t)$
时间	位移	速度
时间	速度	加速度

Def. 记$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$,若存在常数$\alpha$, s.t.

$$\Delta f(x_0)=\alpha\Delta x+o(\Delta x)\quad(\Delta x\to 0),$$

则称f在x_0可微,并称$\mathrm{d}f(x_0)=\alpha\Delta x\triangleq\alpha\,\mathrm{d}x$为$f$在点$x_0$的微分.

Thm. f在x_0可微$\Leftrightarrow$ f在点x_0可导.

Proof. 设f在x_0可微, 则$\exists\alpha\in\mathbb{R}$, $s.t.$

$$\Delta f(x_0)=\alpha\Delta x+o(\Delta x)\quad(\Delta x\to 0).$$

因此 $$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta f(x_0)}{\Delta x}=\alpha+\lim_{\Delta x\to 0}\frac{o(\Delta x)}{\Delta x}=\alpha.$$

设f在点x_0可导. 记$\rho(x)=\dfrac{f(x)-f(x_0)}{x-x_0}-f'(x_0)$,则

$$\lim_{x\to x_0} \rho(x) = 0,$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \rho(x)(x - x_0)$$

$$= f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \quad (x \to x_0).$$

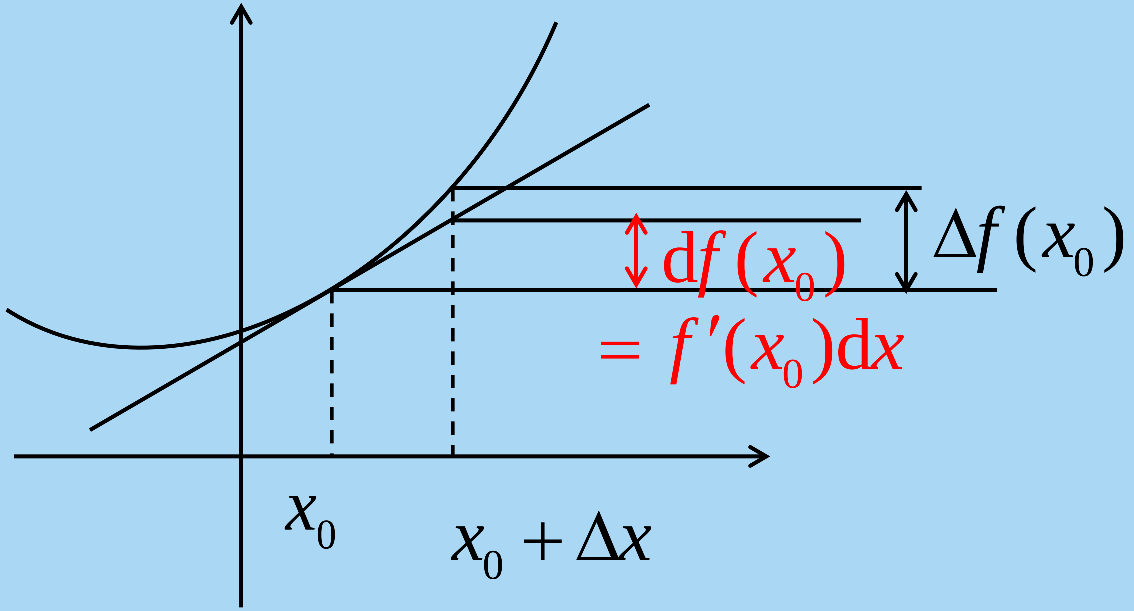
故f在x_0可微.□

Remark. $y = f(x)$在x_0可微,

$$\Delta f(x_0) = \mathrm{d}f(x_0) + o(\Delta x) \quad (\Delta x \to 0),$$

$$\mathrm{d}y = \mathrm{d}f(x_0) = \alpha \Delta x = \alpha \mathrm{d}x$$

则$f'(x_0) = \alpha = \dfrac{\mathrm{d}f(x_0)}{\Delta x} = \dfrac{\mathrm{d}y}{\mathrm{d}x}(x_0).$



Remark. f在x_0可微, 则

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x.$$

Question. 可导与可微等价, 为什么需要给两个定义?

可微的概念是“以直代曲”, 便于推广到多元函数.

§ 2. 求导法则

$$\mathrm{d}(f+g)=\mathrm{d}f+\mathrm{d}g$$
$$\mathrm{d}(cf)=c\,\mathrm{d}f$$
$$\mathrm{d}(fg)=g\,\mathrm{d}f+f\,\mathrm{d}g$$
$$\mathrm{d}\left(\frac{f}{g}\right)=\frac{g\,\mathrm{d}f-f\,\mathrm{d}g}{g^2}$$

Thm. f,g在x_0可导, $c\in\mathbb{R}$,则

(1) $(f+g)'(x_0)=f'(x_0)+g'(x_0)$;

(2) $(cf)'(x_0)=cf'(x_0)$;

(3) $(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0)$;

(4) $\left(\dfrac{f}{g}\right)'(x_0)=\dfrac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$.

特别地, $\left(\dfrac{1}{g}\right)'(x_0)=\dfrac{-g'(x_0)}{g^2(x_0)}$.

$$\text{Proof.}(3)(fg)'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}$$

$$=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0+h)g(x_0)}{h}$$

$$+\lim_{h\to 0}\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0)}{h}$$

$$=\lim_{h\to 0}f(x_0+h)\cdot\lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}$$

$$+g(x_0)\cdot\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

$$=f(x_0)g'(x_0)+f'(x_0)g(x_0).$$ (可导$\Rightarrow$连续)

$$(4)\left(\frac{f}{g}\right)'(x_0)=\lim_{h\to 0}\left(\frac{f(x_0+h)}{g(x_0+h)}-\frac{f(x_0)}{g(x_0)}\right)\Big/ h$$

$$=\lim_{h\to 0}\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{h}\cdot\frac{1}{g(x_0)g(x_0+h)}$$

$$=\frac{1}{g^2(x_0)}\lim_{h\to 0}\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{h}$$

$$=\frac{1}{g^2(x_0)}\left\{\lim_{h\to 0}\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0)}{h}+\lim_{h\to 0}\frac{f(x_0)g(x_0)-f(x_0)g(x_0+h)}{h}\right\}$$

$$=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}.\square$$

Ex. $(\tan x)' = \sec^2 x,\qquad (\cot x)' = -\csc^2 x,$

$(\sec x)' = \sec x \tan x,\qquad (\csc x)' = -\csc x \cot x.$

Proof. $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x};$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

Thm.(复合函数求导的链式法则)$\varphi(x)$在x_0可导, $f(u)$在$u_0=\varphi(x_0)$可导,则$h(x)=f(\varphi(x))$在x_0可导, 且

$$h'(x_0)=f'(\varphi(x_0))\cdot\varphi'(x_0).$$

即$\mathrm{d}f(\varphi(x))=f'(\varphi(x))\mathrm{d}\varphi(x)=f'(\varphi(x))\cdot\varphi'(x)\mathrm{d}x.$

Proof.

令$g(u)=\begin{cases}\dfrac{f(u)-f(u_0)}{u-u_0}, & u\neq u_0,\\ f'(u_0), & u=u_0.\end{cases}$ 则$\lim\limits_{u\to u_0}g(u)=f'(u_0)$,

$$h'(x_0)=\lim_{x\to x_0}\frac{f(\varphi(x))-f(\varphi(x_0))}{x-x_0}=\lim_{x\to x_0}g(\varphi(x))\cdot\frac{\varphi(x)-\varphi(x_0)}{x-x_0}$$

$$=f'(\varphi(x_0))\cdot\varphi'(x_0).\square$$

Remark. $u=\varphi(x)$在x可导, $y=f(u)$在$u=\varphi(x)$可导, 则 $y=f(\varphi(x))$在x可导, 且

$$y'(x)=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Remark.(一阶微分形式的不变性) $u=\varphi(x)$在x_0可微, $y=f(u)$在$u_0=\varphi(x_0)$可微, 则$y=f(\varphi(x))$在x_0可微, 且

$$\mathrm{d}y=f'(\varphi(x_0))\varphi'(x_0)\mathrm{d}x=f'(u_0)\mathrm{d}u.$$

无论将u视为中间变量还是自变量, 都有$\mathrm{d}y=f'(u)\mathrm{d}u$.

Ex. $f(x)=\ln|x|$, 求 $f'(x)$.

解. $x>0$ 时, $f(x)=\ln x$, $f'(x)=\dfrac{1}{x}$.

$x<0$ 时, $f(x)=\ln(-x)$, $f(x)$ 是 $\ln u$ 与 $u=-x$ 的复合函数.

由链式法则,

$$f'(x)=\frac{1}{-x}(-x)'=\frac{1}{x}.$$

综上, $(\ln|x|)'=\dfrac{1}{x}$. □

Ex. $f(x)=\left(\dfrac{x+1}{x-1}\right)^{3/2}$，求$f'(x)$.　　$x\in(-\infty,-1]\cup(1,+\infty)$

解. 令$g(u)=u^{3/2}$, $h(x)=\dfrac{x+1}{x-1}$，则$f(x)=g(h(x))$,

$$g'(u)=\frac{3}{2}u^{1/2},$$

$$h'(x)=\frac{(x+1)'(x-1)-(x+1)(x-1)'}{(x-1)^2}=\frac{-2}{(x-1)^2}$$

$$f'(x)=g'(h(x))h'(x)$$

$$=\frac{3}{2}\left(\frac{x+1}{x-1}\right)^{1/2}\cdot\frac{-2}{(x-1)^2}=\frac{-3}{(x-1)^2}\left(\frac{x+1}{x-1}\right)^{1/2}.\square$$

Ex. $f(x)=\ln\left|x+\sqrt{x^2\pm a^2}\right|$, 求 $f'(x)$.

解. $$f'(x)=\frac{\left(x+\sqrt{x^2\pm a^2}\right)'}{x+\sqrt{x^2\pm a^2}}=\frac{1+\dfrac{2x}{2\sqrt{x^2\pm a^2}}}{x+\sqrt{x^2\pm a^2}}=\frac{1}{\sqrt{x^2\pm a^2}}.\square$$

Ex. $f(x)=u(x)^{v(x)}, u(x)>0, u(x), v(x)$ 可导, 求 $f'(x)$.

解. $$f'(x)=\left(e^{v(x)\ln u(x)}\right)'=e^{v(x)\ln u(x)}\cdot\left(v(x)\ln u(x)\right)'$$

$$=u(x)^{v(x)}\cdot\left(v'(x)\ln u(x)+v(x)\frac{u'(x)}{u(x)}\right)$$

$$=u(x)^{v(x)}\ln u(x)\cdot v'(x)+v(x)u(x)^{v(x)-1}u'(x).\square$$

对数求导法

Ex. $f(x)=f_1(x)f_2(x)\cdots f_n(x)$, 求$f'(x)$.

解: $\ln|f(x)|=\ln|f_1(x)|+\ln|f_2(x)|+\cdots+\ln|f_n(x)|$,

两边对x求导, 得 $$\frac{f'(x)}{f(x)}=\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+\cdots+\frac{f_n'(x)}{f_n(x)}.$$

$$f'(x)=f(x)\left(\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+\cdots+\frac{f_n'(x)}{f_n(x)}\right)$$

$$=\sum_{k=1}^{n}f_1(x)\cdots f_{k-1}(x)f_k'(x)f_{k+1}(x)\cdots f_n(x).\square$$

Remark. 多个因子连乘的函数求导时先取对数再两端求导可简化计算. ($f(x_0)=0$时结论仍成立. 如何处理?)

Ex. $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$，求$f'(x)$.

解: $$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}x\sin\frac{1}{x}=0.$$

$$f'(x)=\begin{cases} 2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}. \ \square$$

Question. (1) f在$[a,b]$可导, f'在$[a,b]$上一定连续吗?

(2) $f\in C[a,b]$, f在(a,b)可导, $f'_+(a)$与$f'_-(b)$是否存在?

Thm.(反函数求导)设f在(a,b)严格单调且连续, $x_0\in(a,b)$, $f'(x_0)\neq 0$, 则$x=f^{-1}(y)$在$y_0=f(x_0)$处可导, 且

$$(f^{-1})'(y_0)=1/f'(x_0).$$

Proof. f在(a,b)严格单调且连续, 则其反函数$x=f^{-1}(y)$也严格单调且连续. 当$y\to y_0$, 且$y\neq y_0$时, 有$x\neq x_0$, 且$x\to x_0$.

$$(f^{-1})'(y_0)=\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\lim_{x\to x_0}\frac{x-x_0}{f(x)-f(x_0)}$$

$$=\frac{1}{\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}}=\frac{1}{f'(x_0)}.\square$$

Remark. $\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}$.

Ex. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $\arctan x = \dfrac{1}{1+x^2}$,

$(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$, $\operatorname{arc}\cot x = \dfrac{-1}{1+x^2}$.

解: (1) $y = \arcsin x$ 与 $x = \sin y$ 互为反函数, 因此

$$(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(2) $y = \arctan x$ 与 $x = \tan y$ 互为反函数, 因此

$$(\arctan x)' = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Def.(隐函数) $F(x,y)=0$确定的函数$y=y(x)$称为隐函数.

Ex. $xy-e^x+e^y=0$确定隐函数$y=y(x)$, 求$y'(x)$.

解：视方程$xy-e^x+e^y=0$中$y=y(x)$, 两边对x求导, 得

$$y+xy'(x)-e^x+e^y y'(x)=0.$$

解得 $y'(x)=\dfrac{e^x-y}{x+e^y}$.□

Ex. 求曲线$x^2+y\cos x-2e^{xy}=0$在点$M(0,2)$处的切线方程.

解: 视方程$x^2+y\cos x-2e^{xy}=0$中$y=y(x)$, 两边对x求导, 得

$$2x+y'\cos x-y\sin x-2(y+xy')e^{xy}=0,$$

$$y'=\frac{2ye^{xy}+y\sin x-2x}{\cos x-2xe^{xy}}.$$

将$x=0, y(0)=2$代入, 得

$$y'(0)=4.$$

故曲线在点$M(0,2)$处的切线方程为

$$y=4x+2.\square$$

Ex. $x=\varphi(t)$严格单调且连续，其反函数$t=\varphi^{-1}(x)$也严格单调且连续. 于是，参数方程$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$确定了函数

$$y=\psi(t)=\psi(\varphi^{-1}(x)).$$

求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解：由复合函数求导的链式法则，

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\psi'(t)\cdot(\varphi^{-1})'(x)=\frac{\psi'(t)}{\varphi'(t)}.\square$$

Ex. $y=y(x)$由参数方程$\begin{cases} x=t+e^t \\ y=t^2+e^{2t} \end{cases}$确定,求$\dfrac{dy}{dx}$.

解:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\frac{2(t+e^{2t})}{1+e^t}.\square$$

作业：习题3.1 No. 5,9,15

习题3.2 No. 4-10（单）