Review

- 隐函数求导、参数函数求导

- **Thm.** 设$f(x)$与$g(x)$在点x处有n阶导数, $c\in\mathbb{R}$, 则

$$(1)\ (f+g)^{(n)}(x)=f^{(n)}(x)+g^{(n)}(x);$$

$$(2)\ (cf)^{(n)}(x)=c\cdot f^{(n)}(x);$$

$$(3)\ (f\cdot g)^{(n)}(x)=\sum_{k=0}^{n}C_n^k f^{(k)}(x)g^{(n-k)}(x).$$ (Leibniz公式)

§ 1. 微分中值定理

- (Fermat Thm) x_0是f的极值点, $f'(x_0)$存在, 则$f'(x_0)=0$.
- $f,g\in C[a,b]$, f,g在(a,b)可导,

(Rolle Thm)若$f(a)=f(b)$, 则$\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=0$.

(Lagrange Thm)$\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$.

(Cauchy Thm)若$g'(t)\neq 0$, 则$\exists\xi\in(a,b)$, $s.t.\ \dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$.

- (Darboux Thm) f在$[a,b]$上可导, $f'_+(a)\neq f'_-(b)$, 则对介于$f'_+(a)$与$f'_-(b)$之间的任意实数λ, $\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=\lambda$.

Def. f在x_0的邻域中有定义,若$\exists\rho>0, s.t.$

$$f(x)\geq(\leq)f(x_0),\quad \forall x\in U(x_0,\rho),$$

则称f在x_0取得极小(大)值,并称x_0是f的极小(大)值点.

若$\exists\rho>0, s.t.$

$$f(x)>(<)f(x_0),\quad \forall x\in U(x_0,\rho),$$

则称f在x_0取得严格极小(大)值,并称x_0是f的严格极小(大)值点.

Question. 极值与最值的区别与联系?

Thm.(Fermat) x_0是f的极值点, $f'(x_0)$存在, 则$f'(x_0)=0$.

Proof. 不妨设x_0是f的极小值点, 则$\exists \rho>0, s.t.$

$$f(x)\ge f(x_0), \quad \forall x\in U(x_0,\rho).$$

而$f'(x_0)$存在, 由极限的保序性, 有

$$f'_-(x_0)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\le 0,$$

$$f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\ge 0.$$

故 $f'(x_0)=f'_-(x_0)=f'_+(x_0)=0.\square$

Def. 若$f'(x_0)=0$, 则称x_0为f的驻点.

Thm.(Rolle) $f\in C[a,b]$, f在(a,b)可导.若$f(a)=f(b)$,则存在$\xi\in(a,b)$, s.t. $f'(\xi)=0$.

Proof. $f\in C[a,b]$,则f在$[a,b]$上有最大值M和最小值m.

若$f(a)=f(b)=M=m$,则f在$[a,b]$上恒为常数,因而$\forall x\in(a,b)$,有$f'(x)=0$.

若M与m至少有一个不等于$f(a)$,不妨设$f(a)<M$.则$\exists\xi\in(a,b)$, s.t. $f(\xi)=M$. 由Fermat定理, $f'(\xi)=0$.□

Question. Rolle定理的几何意义?

Thm.(Lagrange) $f\in C[a,b]$, f在(a,b)可导, 则$\exists\xi\in(a,b)$, s.t.

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

Proof. 令$h(x)=f(x)-\dfrac{f(b)-f(a)}{b-a}(x-a)$,

则$h\in C[a,b]$, h在(a,b)可导, $h(a)=h(b)=f(a)$. 由Rolle定理,

$\exists\xi\in(a,b)$, s.t. $h'(\xi)=f'(\xi)-\dfrac{f(b)-f(a)}{b-a}=0.$ □

Question. Lagrange中值定理的几何意义?

Remark. $f \in C[a,b]$, f在(a,b)可导, 则

(1)$\exists \xi \in (a,b), s.t.\quad f(b)-f(a)=f'(\xi)(b-a).$

(2)$\forall x, x_0 \in [a,b]$, $\exists$介于x与x_0之间的ξ, $s.t.$

$$f(x)-f(x_0)=f'(\xi)(x-x_0).$$

(3)$\forall x_0, x_0+\Delta x \in [a,b], \exists \theta \in (0,1), s.t.$

$$f(x_0+\Delta x)-f(x_0)=f'(x_0+\theta\Delta x)\cdot\Delta x.$$

Thm.(Cauchy) $f, g \in C[a,b]$, f, g在(a,b)可导, 且$\forall t \in (a,b)$, 有$g'(t)\neq 0$. 则存在$\xi \in (a,b), s.t.$ $\dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. 由Lagrange中值定理, $\exists \eta \in (a,b), s.t.$

$$g(b)-g(a)=g'(\eta)(b-a)\neq 0.$$

令$h(x)=f(x)-\dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$, 则 $h\in C[a,b]$,

h在(a,b)可导, $h(a)=h(b)=f(a)$. 由Rolle定理,

$$\exists \xi \in (a,b), s.t.\ h'(\xi)=f'(\xi)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(\xi)=0. \square$$

Question. Cauchy中值定理的几何意义?

Remark. Lagrange中值定理是Cauchy中值定理的特殊情形.

Thm. f在$[a,b]$上可导.

(1)若$f'_{+}(a)f'_{-}(b)<0$,则$\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=0$.

(2)若$f'_{+}(a)\neq f'_{-}(b)$,则对介于$f'_{+}(a)$与$f'_{-}(b)$之间的任意实数λ, $\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=\lambda$. (Darboux)

Proof.(1)不妨设

$$f'_{+}(a)=\lim_{x\to a+}\frac{f(x)-f(a)}{x-a}>0,\ f'_{-}(b)=\lim_{x\to b-}\frac{f(x)-f(b)}{x-b}<0.$$

因此$\exists a<x_1<x_2<b$, $s.t.\ f(x_1)>f(a),\ f(x_2)>f(b)$. 于是$f$在$[a,b]$上的最大值在某点$\xi\in(a,b)$处取得.由Fermat定理,

$$f'(\xi)=0.$$

(2) $f'_+(a) \neq f'_-(b)$, 则对介于$f'_+(a)$与$f'_-(b)$之间的任意λ, 令

$$g(x) = f(x) - \lambda x.$$

则 $g'_+(a)g'_-(b) = (f'_+(a) - \lambda)(f'_-(b) - \lambda) < 0.$

由(1)中结论, $\exists \xi \in (a,b), s.t. g'(\xi) = 0$, 即$f'(\xi) = \lambda$.□

Ex. 若$f'(x)$在(a,b)上恒为0, 则$f(x)$在(a,b)上为常数.

Proof. $\forall x_1, x_2 \in (a,b)$, 由Lagrange中值定理, $\exists$介于x_1, x_2之间的ξ, $s.t.$

$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) = 0.$$

故$f(x)$在(a,b)上为常数.□

Ex. $\dfrac{x}{1+x} < \ln(1+x) < x \quad (x > -1, x \neq 0).$

Proof. 由Lagrange中值定理, $\forall x > -1, \exists \theta \in (0,1), s.t.$

$$\ln(1+x) = \ln(1+x) - \ln 1 = \frac{x}{1+\theta x}.$$

当$x > -1, x \neq 0$时, $\dfrac{x}{1+x} < \dfrac{x}{1+\theta x} < x.$ □

Remark. $\dfrac{1}{n+1} < \ln(1+\dfrac{1}{n}) < \dfrac{1}{n} \quad (n \in \mathbb{N}).$

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Ex. $x^4+2x^3+6x^2-4x-5=0$恰有两个不同的实根.

Proof. 令$f(x)=x^4+2x^3+6x^2-4x-5$,则

$$\lim_{x\to\pm\infty} f(x)=+\infty.$$

于是$\exists a<0<b, s.t. f(a)>0, f(b)>0$. 而$f(0)=-5<0$,

由介值定理, $f(x)=0$至少有两个相异实根.

假设$f(x)=0$至少有3个相异实根. 由Rolle定理, $f'(x)$至少有2个相异实根, $f''(x)$至少有1个实根. 但

$$f''(x)=12x^2+12x+12>0,$$

矛盾.故$f(x)=0$恰有两个相异实根.□

Ex. $f,g\in C[a,b]$, f,g在(a,b)内可导, $f(a)=f(b)=0$.证明:

$\exists\xi\in(a,b), s.t.\ f'(\xi)+g'(\xi)f(\xi)=0.$

Proof.令$h(x)=f(x)e^{g(x)}$,则h在(a,b)内可导,$h(a)=f(b)=0.$

由Rolle定理,$\exists\xi\in(a,b), s.t.$

$$h'(\xi)=(f'(\xi)+g'(\xi)f(\xi))e^{g(\xi)}=0.$$

$$f'(\xi)+g'(\xi)f(\xi)=0.\square$$

Remark. 辅助函数.

Question. 什么辅助函数求导可出现

$(1)\ f'(x)+g'(x)f(x)?$ $(2)\ f''(x)+2f'(x)+f(x)?$

$(3)\ f''(x)-2f'(x)+f(x)?$ $(4)\ f''(x)-f(x)?$

$(5)\ f''(x)-f'(x)?$ $\cdots$

$$(1)\left(f(x)e^{g(x)}\right)'=(f'(x)+f(x)g'(x))e^{g(x)},$$

$$(2)(3)\ (f(x)e^{\pm x})''=\left((f'(x)\pm f(x))e^{\pm x}\right)'$$

$$=(f''(x)\pm 2f'(x)+f(x))e^{\pm x},$$

$$(4)\ \left(e^{\pm x}(f'(x)\mp f(x))\right)'=e^{\pm x}(f''(x)-f(x))$$

$$(5)\ (f'(x)-f(x))'=f''(x)-f'(x).$$

Ex. $f\in C[0,1]$, 在$(0,1)$上可导, $f(1)=0$, 则$\exists\xi\in(0,1)$, s.t.

$$f'(\xi)=(1-\xi^{-1})f(\xi).$$

分析: $f'(\xi)=(1-\xi^{-1})f(\xi)\Leftrightarrow \xi f'(\xi)=(\xi-1)f(\xi)$

$$\Leftrightarrow \xi f'(\xi)+f(\xi)-\xi f(\xi)=0\Leftrightarrow \left(xf(x)\right)'\Big|_{x=\xi}-\xi f(\xi)=0$$

Proof: 令$h(x)=xf(x)e^{-x}$, 由$f(1)=0$, 有$h(0)=h(1)=0$.

由Rolle定理, $\exists\xi\in(0,1)$, s.t.

$$h'(\xi)=e^{-\xi}\left(f(\xi)+\xi f'(\xi)-\xi f(\xi)\right)=0.$$

$$f(\xi)+\xi f'(\xi)-\xi f(\xi)=0.$$

$$f'(\xi)=(1-\xi^{-1})f(\xi).\square$$

Ex. f 在 $[0,1]$ 上二阶可导, $f(0)=f(1)=0$, 则 $\exists\xi\in(0,1)$, s.t.

$$f''(\xi)=2f'(\xi)/(1-\xi).$$

分析: $\left(a(x)e^{b(x)}\right)'=\left(a'(x)+a(x)b'(x)\right)e^{b(x)}.$

取 $a(x)=f'(x)$, 则 $b'(x)=-2/(1-x)$, $b(x)=2\ln(1-x)$.

Proof. $f(0)=f(1)=0$, 则 $\exists\eta\in(0,1)$, s.t. $f'(\eta)=0$.

令 $h(x)=(1-x)^2f'(x)$, 则 h 在 $[0,1]$ 上可导, $h(\eta)=h(1)=0$,

由Rolle定理, $\exists\xi\in(\eta,1)\subset(0,1)$, s.t.

$$h'(\xi)=(1-\xi)^2f''(\xi)-2(1-\xi)f'(\xi)=0.$$

$$f''(\xi)=2f'(\xi)/(1-\xi).\square$$

Ex. f,g在$[a,b]$上二阶可导, $f(a)=f(b)=g(a)=g(b)=0$,

$g''(x)\neq 0(x\in(a,b))$. 则$\exists\xi\in(a,b)$, s.t. $\dfrac{f(\xi)}{g(\xi)}=\dfrac{f''(\xi)}{g''(\xi)}$.

Proof. 令$h(x)=f'(x)g(x)-f(x)g'(x)$.由$f(a)=f(b)=g(a)$

$=g(b)=0$,得$h(a)=h(b)=0$. 由Rolle定理,$\exists\xi\in(a,b)$, s.t.

$$h'(\xi)=f''(\xi)g(\xi)-f(\xi)g''(\xi)=0.$$

若$g(\xi)=0$,则$\exists\xi_1\in(a,\xi),\xi_2\in(\xi,b)$, s.t. $g'(\xi_1)=g'(\xi_2)=0$.

从而$\exists\xi_3\in(\xi_1,\xi_2)$, s.t. $g''(\xi_3)=0$, 与$g''(x)\neq 0(x\in(a,b))$矛盾.

故$g(\xi)\neq 0$, $\dfrac{f(\xi)}{g(\xi)}=\dfrac{f''(\xi)}{g''(\xi)}$. □

Ex. f, g在$[a,b]$上可导, $g'(x) \neq 0 (x \in (a,b))$. 则$\exists \xi \in (a,b)$, s.t.

$$\frac{f(a)-f(\xi)}{g(\xi)-g(b)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof. 令$h(x) = f(a)g(x) + g(b)f(x) - f(x)g(x)$, 则

$$h(a) = h(b) = f(a)g(b).$$

由Rolle定理, $\exists \xi \in (a,b)$, s.t. $h'(\xi) = 0$, 即

$$f(a)g'(\xi) + f'(\xi)g(b) - f(\xi)g'(\xi) - f'(\xi)g(\xi) = 0$$

$$\Leftrightarrow g'(\xi)(f(a)-f(\xi)) = f'(\xi)(g(\xi)-g(b))$$

$$\Leftrightarrow \frac{f(a)-f(\xi)}{g(\xi)-g(b)} = \frac{f'(\xi)}{g'(\xi)}. \square$$

Ex. f 在$[0,1]$上可导, $f(0)=0$, $f(x)>0(0<x<1)$, 则

$$\exists \xi \in (0,1), s.t.\ \frac{2f'(\xi)}{f(\xi)} = \frac{f'(1-\xi)}{f(1-\xi)}.$$

Proof. 令$h(x)=f^2(x)f(1-x)$. 由$f(0)=0$得$h(0)=h(1)=0$.

由Rolle定理, $\exists \xi \in (0,1), s.t.$

$$h'(\xi) = 2f(\xi)f'(\xi)f(1-\xi) - f^2(\xi)f'(1-\xi) = 0.$$

$$2f'(\xi)f(1-\xi) - f(\xi)f'(1-\xi) = 0.$$

又 $f(x)>0(0<x<1)$, 故 $\dfrac{2f'(\xi)}{f(\xi)} = \dfrac{f'(1-\xi)}{f(1-\xi)}$. □

$$\left(\alpha(x)e^{\beta(x)}\right)' = (\alpha'(x)+\alpha(x)\beta'(x))e^{\beta(x)}$$

Ex. f在[0,1]上二阶可导, $f'(0)=0$.则$\exists\xi\in(0,1)$, s.t.

$$f'(\xi)-(\xi-1)^2 f''(\xi)=0.$$

Proof. 令$g(x)=\begin{cases} e^{1/(x-1)} f'(x), & x\neq 1 \\ 0, & x=1 \end{cases}$, 则 $g\in C[0,1]$, g在(0,1)

上可导, $g(0)=e^{-1}f'(0)=0=g(1)$,

$$g'(x)=\left(f''(x)-\frac{1}{(x-1)^2}f'(x)\right)e^{1/(x-1)}.$$

由Rolle定理,$\exists\xi\in(0,1)$, s.t. $g'(\xi)=0$. 从而有

$$f''(\xi)-\frac{1}{(\xi-1)^2}f'(\xi)=0. \square$$

Ex. $f\in C[a,b]$, 在 (a,b) 内可导, $a>0$, 则

(1) $\exists \xi\in(a,b),\ s.t.\ \ f(b)-f(a)=\xi f'(\xi)\ln\dfrac{b}{a}$.

(2) $\lim\limits_{n\to+\infty} n\left(\sqrt[n]{x}-1\right)=\ln x,\ \forall x>0.$

Proof. (1)由Cauchy中值定理, $\exists \xi\in(a,b),\ s.t.$

$$\frac{f(b)-f(a)}{\ln b-\ln a}=\frac{f'(\xi)}{1/\xi}.$$

(2) 由(1), 任意给定 $x>0$, $\exists \xi_n$ 介于1与 x 之间, $s.t.$

$$n\left(\sqrt[n]{x}-1\right)=n\left(\sqrt[n]{x}-\sqrt[n]{1}\right)=n\cdot\xi_n\cdot\frac{1}{n}\xi_n^{1/n-1}\ln x=\xi_n^{1/n}\ln x.$$

$\lim\limits_{n\to\infty}x^{1/n}=1$, 由夹挤原理 $\lim\limits_{n\to\infty}\xi_n^{1/n}=1$. 故 $\lim\limits_{n\to+\infty} n\left(\sqrt[n]{x}-1\right)=\ln x$. □

Ex. $f\in C[0,1]$, 在$[0,1]$上二阶可导, $f(x)>0(x\in[0,1])$, $f'(0)=f'(1)=0$, 则$\exists\xi\in(0,1)$, $s.t.\ f(\xi)f''(\xi)-2\left(f'(\xi)\right)^2=0.$

Proof. 令$h(x)=f'(x)/f^2(x)$. 由$f'(0)=f'(1)=0$, 得

$$h(0)=h(1)=0.$$

由Rolle定理, $\exists\xi\in(0,1)$, $s.t.$

$$h'(\xi)=\frac{f''(\xi)f^2(\xi)-2f(\xi)\left(f'(\xi)\right)^2}{f^4(\xi)}=0.$$

$$f(\xi)f''(\xi)-2\left(f'(\xi)\right)^2=0.\square$$

Ex. f在$[0,+\infty)$上可导, $f'(x)$单调递减, $f(0)=0$, 则

$$f(x_1+x_2)\le f(x_1)+f(x_2)\quad (x_1,x_2>0).$$

Proof. 不妨设$0<x_1\le x_2$, 由$f(0)=0$, 得

$$f(x_1+x_2)-f(x_1)-f(x_2)$$

$$=f(x_1+x_2)-f(x_2)-(f(x_1)-f(0))$$

$$=f'(x_2+\theta_1x_1)x_1-f'(\theta_2x_1)x_1\quad (0<\theta_1,\theta_2<1)$$

$$\le 0\quad (f'\text{单调递减}).\square$$

Ex. $f\in C[a,b]$, f在(a,b)上可导, $f(x)+f'(x)<\varepsilon\ (a<x<b)$

$f(a)<\varepsilon$. 证明: $f(x)<\varepsilon\ (a<x<b)$.

证法一: 用Lagrange中值定理. 令$h(x)=e^x(f(x)-\varepsilon)$, 则

$\forall x\in(a,b),\exists\xi\in(a,x),s.t.$

$$h(x)=h(a)+e^{\xi}(f(\xi)+f'(\xi)-\varepsilon)(x-a)<h(a)<0,$$

故$f(x)<\varepsilon$.□

证法二: 用Cauchy中值定理. $\forall x\in(a,b),\exists\xi\in(a,x),s.t.$

$$\frac{e^x f(x)-e^a f(a)}{e^x-e^a}=\frac{e^{\xi}(f(\xi)+f'(\xi))}{e^{\xi}}<\varepsilon,$$

故$e^x f(x)<e^a f(a)+\varepsilon(e^x-e^a)<\varepsilon e^x$, 从而$f(x)<\varepsilon$.□

作业：习题4.1

No. 2,5,11,13,14