高数自测 level 1

1. 左极限 $f(0^-) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (3x+2) = 2$

右极限 $f(0^+) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (x^2+1) = 1$

$\because f(0^-) \neq f(0^+)$，$\therefore \lim\limits_{x\to 0} f(x)$ 不存在

左极限 $f(1^-) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2+1) = 2$

右极限 $f(1^+) = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \left(\frac{2}{x}\right) = 2$

$\therefore \lim\limits_{x\to 2} f(x) = 2$

2. 由定义，左导数 $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{x}{x} = 1$

右导数 $f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{\ln(1+x)}{x} = 1$

$\therefore f'(0) = 1$

3. (1) $\lim\limits_{n\to\infty} \ln \dfrac{2n^2+3n+1}{n^2-5n-3} = \lim\limits_{n\to\infty} \ln \dfrac{2+\frac{3}{n^2}+1}{1-\frac{5}{n}-\frac{3}{n^2}} = \ln 2$

(2) $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^3+2x^2-x-2} = \lim\limits_{x\to 1} \dfrac{(x-1)(x-2)}{(x-1)(x^2+3x+2)} = \lim\limits_{x\to 1} \dfrac{x-2}{x^2+3x+2} = -\dfrac{1}{6}$

(3) $\lim\limits_{x\to 0} \left(\dfrac{2-x}{2}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0} \left[\left(1-\frac{x}{2}\right)^{-\frac{2}{x}}\right]^{-\frac{1}{2}} = e^{-\frac{1}{2}}$

(4) $\lim\limits_{x\to 0} \dfrac{\arctan x^2}{1-\cos 2x} = \lim\limits_{x\to 0} \dfrac{x^2}{\frac{1}{2}\cdot(2x)^2} = \dfrac{1}{2}$

$\left(\begin{array}{l} \arctan u \sim u \\ 1-\cos u \sim \frac{1}{2}u^2 \end{array}\right)$

(5) $\lim\limits_{x\to+\infty} x\left(\sqrt{1+x^2}-x\right) = \lim\limits_{x\to+\infty} \dfrac{x}{\sqrt{1+x^2}+x} = \lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}+1} = \dfrac{1}{2}$

(6) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x-x-1}{x(e^x-1)}$ (通分)

$= \lim\limits_{x\to 0} \dfrac{e^x-x-1}{x^2}$ ($e^x-1 \sim x$)

$= \lim\limits_{x\to 0} \dfrac{e^x-1}{2x}$ (洛必达法则)

$= \dfrac{1}{2}$ ($e^x-1 \sim x$)

(7) $\lim\limits_{x\to 0^+} x^{\sin x}$

$= e^{\lim\limits_{x\to 0^+} \sin x \cdot \ln x}$

$= e^{\lim\limits_{x\to 0^+} x \cdot \ln x}$ ($\sin x \sim x$)

$= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}}$

$= e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}}$ (洛必达法则)

$= e^{\lim\limits_{x\to 0^+} (-x)}$

$= 1$

4. (1) $f(x) = \dfrac{(x+1)(x-1)}{(x-2)(x-1)}$

$\therefore$ $f(x)$ 的连续区间 $(-\infty, 1) \cup (1, 2) \cup (2, +\infty)$

且 $f(x)$ 在 $x=1$ 处 可去间断；

$x=2$ 处 无穷间断.

(2) $f(x) = \cos\dfrac{1}{x}$

$f(x)$ 的连续区间 $(-\infty, 0) \cup (0, +\infty)$

且 $f(x)$ 在 $x=0$ 振荡间断.

5. (1) $y' = \dfrac{1}{1+(e^{-x})^2} \cdot (e^{-x})' = -\dfrac{e^{-x}}{1+e^{-2x}}$

(2) $y' = \dfrac{1}{\sin x}(\sin x)' = \cot x$

6. (1) $dy = e^{-x^2} d(-x^2) = -2x e^{-x^2} dx$

(2) $dy = \dfrac{1}{x+\sqrt{1+x^2}} d(x+\sqrt{1+x^2}) = \dfrac{1}{\sqrt{1+x^2}} dx$

7. (1) $y = 1 - xe^y$

$\therefore$ $y' = -e^y - xe^y \cdot y'$

$\therefore$ $y' = -\dfrac{e^y}{1+xe^y}$

(2) $\sin x + \sin y = \sin(xy)$

$\therefore$ $\cos x + \cos y \cdot y' = \cos(xy)\cdot(xy)' = \cos(xy)\cdot(y+xy')$

$\therefore$ $y' = \dfrac{y\cos(xy) - \cos x}{\cos y - x\cos(xy)}$

8. $\dfrac{dy}{dt} = \dfrac{1}{1+t^2}$

$\dfrac{dx}{dt} = \dfrac{t}{1+t^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1}{t}$

9. (1) $y = x^4 - 2x^3$

$\therefore y' = 4x^3 - 6x^2,\quad y'' = 12x^2 - 12x$

$y'' = 0 \Rightarrow x = 0, 1$

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,+\infty)$
y''	$+$	0	$-$	0	$+$
y	凹	拐点	凸	拐点	凹

$\therefore$ 凹区间 $(-\infty,0]$, $[1,+\infty)$ ，拐点 $(0,0)$, $(1,-1)$

凸区间 $[0,1]$

(2) $y = \ln(1+x^2) + 1$

$\therefore y' = \dfrac{2x}{1+x^2}$

$y'' = \dfrac{2(1-x^2)}{(1+x^2)^2}$ ，$y'' = 0 \Rightarrow x = \pm 1$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
y''	$-$	0	$+$	0	$-$
y	凸	拐点	凹	拐点	凸

$\therefore$ 凹区间 $[-1,1]$

凸区间 $(-\infty,-1]$, $[1,+\infty)$

拐点 $(\pm 1,\ 1+\ln 2)$

10. $y=(x-1)\sqrt[3]{x^2}$

$y'=x^{\frac{2}{3}}+\frac{2}{3}(x-1)x^{-\frac{1}{3}}$

$=\frac{2}{3}x^{-\frac{1}{3}}(\frac{5}{2}x-1)$

x	$(-\infty,0)$	0	$(0,\frac{2}{5})$	$\frac{2}{5}$	$(\frac{2}{5},+\infty)$
y'	$+$	不存在	$-$	0	$+$
y	$\nearrow$	极大值	$\downarrow$	极小值	$\nearrow$

∴ 单调递增区间 $(-\infty,0]$，$[\frac{2}{5},+\infty)$

单调递减区间 $[0,\frac{2}{5}]$

极大值点 $(0,0)$，极小值点 $(\frac{2}{5},-\frac{3}{5}(\frac{2}{5})^{\frac{2}{3}})$

11. $y'=3x^2-6x-9=3(x+1)(x-3)$

∴ $y'=0\Rightarrow$ 驻点 $x=-1,3$

∵ 端点函数值 $y(-4)=-75$

$y(4)=-19$

可能极值点函数值 $y(-1)=6$

$y(3)=-26$

∴ 最大值 $y(-1)=6$

最小值 $y(-4)=-75$

12.(1) 令 $f(x)=(1+x)\ln(1+x)-\arctan x$，$x>0$

$\therefore f(0)=0$，

$f'(x)=\ln(1+x)+1-\dfrac{1}{1+x^2}>0$

$\therefore f(x)$ 在 $[0,+\infty)$ 上单调递增.

$\therefore f(x)>f(0)=0$

$\therefore \ln(1+x)>\dfrac{\arctan x}{1+x}$，$x>0$

(2) 令 $f(x)=\arctan x+\operatorname{arccot} x$

$\therefore f'(x)=\dfrac{1}{1+x^2}-\dfrac{1}{1+x^2}=0$

$\therefore f(x)\equiv C$，

又 $\because f(0)=0+\dfrac{\pi}{2}=\dfrac{\pi}{2}=C$

$\therefore f(x)\equiv\dfrac{\pi}{2}$

13. (1) $\displaystyle\int\frac{x}{x^2+1}dx=\frac{1}{2}\int\frac{1}{x^2+1}d(x^2+1)=\frac{1}{2}\ln(x^2+1)+C$

(2) $\displaystyle\int\frac{1}{x^2-x-2}dx$

$\displaystyle=\int\frac{1}{(x+1)(x-2)}dx$

$\displaystyle=\frac{1}{3}\int\left(\frac{1}{x-2}-\frac{1}{x+1}\right)dx$ （裂项）

$\displaystyle=\frac{1}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C$

(3) $\int \frac{x^3}{x^2+1} dx$

$= \int \left(x - \frac{x}{x^2+1}\right) dx$

$= \frac{1}{2}x^2 - \int \frac{x}{x^2+1} dx$

$= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$

(4) $\int \sin x \cos^2 x \, dx$

$= -\int \cos^2 x \, d(\cos x)$

$= -\frac{1}{3}\cos^3 x + C$

(5) $\int \sin^2 x \, dx$

$= \frac{1}{2}\int (1-\cos 2x) dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(6) $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{1}{\cos x} d(\cos x)$

$= -\ln|\cos x| + C$

(7) $\int \tan^2 x \, dx$

$= \int (\sec^2 x - 1) dx$

$= \tan x - x + C$

(8) $\int \frac{1}{\sqrt{e^x-1}}\,dx$

$= \int \frac{1}{u}\,\frac{2u}{1+u^2}\,du$ （令 $u=\sqrt{e^x-1}$，$dx=\frac{2u}{1+u^2}du$）

$= 2\int \frac{1}{1+u^2}\,du$

$= 2\arctan u + C$

$= 2\arctan\sqrt{e^x-1} + C$

(9) $\int x e^x\,dx$

$= \int x\,de^x$

$= xe^x - \int e^x\,dx$ （分部积分）

$= (x-1)e^x + C$

(10) $\int x\ln x\,dx$

$= \int \ln x\,d(\frac{1}{2}x^2)$

$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\,\frac{1}{x}\,dx$ （分部积分）

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

(11) $\int e^{\sqrt{x}}\,dx$

$= \int e^u\,2u\,du$ （令 $u=\sqrt{x}$，$dx=2u\,du$）

$= 2(u-1)e^u + C$ （利用 13.(9) 结果

$= 2(\sqrt{x}-1)e^{\sqrt{x}} + C$

14. (1) $\int_0^1 (2x+1)^2 dx$

$= \int_1^3 u^2 \frac{1}{2} du$ (令 $u = 2x+1$, $dx = \frac{1}{2} du$)

$= \frac{1}{6} u^3 \Big|_1^3$

$= \frac{13}{3}$

(2) $\int_0^1 \arctan x \, dx$

$= x \arctan x \Big|_0^1 - \int_0^1 x \cdot d(\arctan x)$ (分部积分)

$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{1}{1+x^2} d(1+x^2)$

$= \frac{\pi}{4} - \frac{1}{2} \ln(1+x^2) \Big|_0^1$

$= \frac{\pi}{4} - \frac{1}{2} \ln 2$

(3) $\int_0^1 e^{\arcsin x} dx$

$= \int_0^{\frac{\pi}{2}} e^u \cos u \, du$ （令 $u = \arcsin x$, $x = \sin u$, $dx = \cos u \, du$）

令 $I = \int_0^{\frac{\pi}{2}} e^u \cos u \, du$

$= \int_0^{\frac{\pi}{2}} e^u \, d \sin u$

$= e^u \sin u \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^u \sin u \, du$

$= e^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^u \, d(\cos u)$

$= e^{\frac{\pi}{2}} + e^u \cos u \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^u \cos u \, du$

$= e^{\frac{\pi}{2}} - 1 - I$

$\therefore I = \frac{1}{2}(e^{\frac{\pi}{2}} - 1)$

$\therefore \int_0^1 e^{\arcsin x} dx = I = \frac{1}{2}(e^{\frac{\pi}{2}} - 1)$

15. (1) $\int_{-1}^{1} (x^2 + x^3 \sin^2 x) \, dx$

$= \int_{-1}^{1} \underbrace{x^2}_{\text{偶函数}} dx + \int_{-1}^{1} \underbrace{x^3 \sin^2 x}_{\text{奇函数}} dx$

$= 2\int_0^1 x^2 dx + 0$

$= \frac{2}{3} x^3 \Big|_0^1$

$= \frac{2}{3}$

(2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\arctan^3 x + \cos^2 x)\, dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\arctan^3 x}_{\text{奇函数}}\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^2 x}_{\text{偶函数}}\, dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$= \int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx$

$= \left(x + \frac{1}{2}\sin 2x\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

16. (1) $\int \frac{1}{x^2+2x+5}\, dx = \int \frac{1}{(x+1)^2+4}\, dx = \frac{1}{2}\arctan\frac{x+1}{2} + C$

$\therefore \int_{-\infty}^{+\infty} \frac{1}{x^2+2x+5}\, dx = \frac{1}{2}\arctan\frac{x+1}{2}\Big|_{-\infty}^{+\infty} = \frac{\pi}{2}$

(2) $\int_0^{+\infty} e^{-ax}\, dx = -\frac{1}{a} e^{-ax}\Big|_0^{+\infty} = \frac{1}{a}$

(3) $x=0$ 是瑕点，$\therefore \int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_0^1 \frac{1}{x^2}\, dx$

$\because \int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_0^1 = +\infty$

$\therefore \int_{-1}^{1} \frac{1}{x^2}\, dx$ 发散

(4) $x=0$ 是瑕点，$\int_{-1}^{1} x^{-\frac{2}{3}}\, dx = \int_{-1}^{0} x^{-\frac{2}{3}}\, dx + \int_0^1 x^{-\frac{2}{3}}\, dx$

$\int_{-1}^{0} x^{-\frac{2}{3}}\, dx = 3x^{\frac{1}{3}}\Big|_{-1}^{0} = 3$，$\int_0^1 x^{-\frac{1}{3}} = 3x^{\frac{1}{3}}\Big|_0^1 = 3$

$\therefore \int_{-1}^{1} x^{-\frac{2}{3}}\, dx = 6$