Review

- $\lim\limits_{x\to x_0} f(x) = \mathrm{A} \in [-\infty, +\infty]$的定义与几何意义

$$\lim_{x\to x_0^{\pm}} f(x),\ \lim_{x\to\infty} f(x),\ \lim_{x\to\pm\infty} f(x)$$

- 极限的性质

唯一性, 局部有界性, 四则运算,

单调收敛原理, 复合函数的极限, 夹挤原理

- 重要不等式

$$|\sin x| \le |x|, \forall x \in \mathbb{R}. \qquad |x| \le |\tan x|, \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

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- 重要极限

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e,\qquad \lim_{x\to 0}(1+x)^{1/x}=e,$$

$$\lim_{x\to+\infty}\frac{\log_a x}{x^b}=0,\ \lim_{x\to 0^+}x^b\log_a x=0\ (a>1,b>0),$$

$$\lim_{x\to+\infty}\frac{x^b}{a^x}=0\ (a>1,b\in\mathbb{R}),\qquad \lim_{x\to+\infty}\frac{a^x}{x^x}=0\ (a>0,a\neq 1),$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\qquad \lim_{x\to x_0}e^x=e^{x_0},\qquad \lim_{x\to x_0}\ln x=\ln x_0,$$

$$\lim_{x\to x_0}u(x)^{v(x)}=\left(\lim_{x\to x_0}u(x)\right)^{\lim\limits_{x\to x_0}v(x)}\ (\text{成立的条件?})$$

- **Thm.** f在$U(x_0,\rho)$中有定义,则以下命题等价:

(1)$\forall\varepsilon>0,\exists\delta>0,\forall x,y\in U(x_0,\delta)$,有$|f(x)-f(y)|<\varepsilon$;

(2)$\exists \mathrm{A}\in\mathbb{R}$,对$U(x_0,\rho)$中任意收敛到$x_0$的点列$\{x_n\}$,有

$$\lim_{n\to\infty} f(x_n)=\mathrm{A};$$

(3)$\lim\limits_{x\to x_0} f(x)=\mathrm{A}$.

Remark. (1) $\Leftrightarrow$ (3) (函数极限的Cauchy收敛原理)

Remark. (2) $\Leftrightarrow$ (3) (用数列的极限来研究函数的极限)

§ 4. 无穷小量与无穷大量

Def.（无穷小量与无穷大量）

(1)若 $\lim\limits_{x\to x_0} f(x)=0$, 则称 $x\to x_0$ 时, $f(x)$ 是无穷小量, 记作

$$f(x)\to 0(x\to x_0);$$

(2)若 $\lim\limits_{x\to x_0} f(x)=\infty$, 则称 $x\to x_0$ 时, $f(x)$ 是无穷大量, 记作

$$f(x)\to \infty(x\to x_0);$$

(3)若 $\lim\limits_{x\to x_0} f(x)=\pm\infty$, 则称 $x\to x_0$ 时, $f(x)$ 是正（负）无穷大量, 记作 $f(x)\to \pm\infty(x\to x_0)$.

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Def. 设$x \to x_0$时, $f(x)$与$g(x)$都是无穷小量.

(1)若$\lim\limits_{x \to x_0} f(x)/g(x) = 0$, 则称$x \to x_0$时, $f(x)$是$g(x)$的高阶无穷小量, 记作$f(x) = o(g(x))\ (x \to x_0)$;

(2)若$\lim\limits_{x \to x_0} f(x)/g(x) = c \neq 0$, 则称$x \to x_0$时, $f(x)$与$g(x)$是同阶无穷小量; 特别地, 当$c = 1$时, 称$x \to x_0$时, $f(x)$与$g(x)$是等价无穷小量, 记作 $f(x) \sim g(x)\ \ (x \to x_0)$;

(3)若$\exists M > 0, \delta > 0$, 当$0 < |x - x_0| < \delta$时, 有$|f(x)/g(x)| < M$, 则记为$f(x) = O(g(x))\ (x \to x_0)$.

(4)若$\lim\limits_{x \to x_0} \dfrac{f(x)}{(x - x_0)^k} = c \neq 0$, 称$x \to x_0$时, $f(x)$是k阶无穷小量.

Def. 设$x \to x_0$时，$f(x)$与$g(x)$都是无穷大量.

(1)若$\lim\limits_{x \to x_0} f(x)/g(x) = 0$，则称$x \to x_0$时，$f(x)$是$g(x)$的低阶无穷大量，记作$f(x) = o(g(x))\ (x \to x_0)$；

(2)若$\lim\limits_{x \to x_0} f(x)/g(x) = c \neq 0$，则称$x \to x_0$时，$f(x)$与$g(x)$是同阶无穷大量；特别地，当$c = 1$时，称$x \to x_0$时，$f(x)$与$g(x)$是等价无穷大量，记作$f(x) \sim g(x)\ \ (x \to x_0)$；

(3)若$\exists M > 0, \delta > 0$，当$0 < |x - x_0| < \delta$时，有$|f(x)/g(x)| < M$，则记为$f(x) = O(g(x))\ (x \to x_0)$.

Prop. $x \to x_0$时, $f(x)=o(1),\ g(x)=O(1)$, 则$f(x)g(x)=o(1)$.

Thm. 当$x \to 0$时:

(1) $\sin x \sim \tan x \sim x$; (2) $1-\cos x \sim \frac{1}{2}x^2$; (3) $\ln(1+x) \sim x$;

(4) $e^x - 1 \sim x,\ a^x - 1 \sim x\ln a\,(a>0)$; (5) $(1+x)^\alpha - 1 \sim \alpha x$.

Proof. (1) $$\lim_{x\to 0}\frac{\sin x}{x}=1,\ \lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1.$$

(2) $$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=1.$$

$$(3)\ \lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln e=1.$$

(4) 令$u=e^x-1$,则$x=\ln(1+u)$, $x\to 0$等价于$u\to 0$,

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{u\to 0}\frac{u}{\ln(1+u)}=1.$$

$$a^x-1=e^{x\ln a}-1\sim x\ln a\quad (x\to 0).$$

$$(5)\ \frac{(1+x)^\alpha-1}{\alpha x}=\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\to 1\ (x\to 0).$$

Remark. 设$x \to x_0$时, $f(x)$与$g(x)$是等价无穷小量, 则

$$f(x) = g(x) + o(g(x)), \quad x \to x_0.$$

Proof. 记$h(x) = f(x) - g(x)$. $\lim\limits_{x \to x_0} f(x)/g(x) = 1$, 则

$$\lim_{x \to x_0} \frac{h(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - g(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)} - 1 = 0.$$

即

$$h(x) = o(g(x)), x \to x_0.$$

也即

$$f(x) = g(x) + o(g(x)), \quad x \to x_0. \square$$

Remark.

$$\sin x \sim x(x \to 0) \quad \Rightarrow \quad \sin x = x + o(x)(x \to 0)$$

$$\tan x \sim x(x \to 0) \quad \Rightarrow \quad \tan x = x + o(x)(x \to 0)$$

$$1 - \cos x \sim \frac{1}{2}x^2(x \to 0) \Rightarrow 1 - \cos x = \frac{1}{2}x^2 + o(x^2)(x \to 0)$$

$$\ln(1+x) \sim x \quad \Rightarrow \quad \ln(1+x) = x + o(x)(x \to 0)$$

$$e^x - 1 \sim x(x \to 0) \quad \Rightarrow \quad e^x - 1 = x + o(x)(x \to 0)$$

$$a^x - 1 \sim x \ln a (x \to 0) \quad \Rightarrow \quad a^x - 1 = x \ln a + o(x)(x \to 0)$$

$$(1+x)^\alpha - 1 \sim \alpha x(x \to 0) \quad \Rightarrow \quad (1+x)^\alpha - 1 = \alpha x + o(x)(x \to 0)$$

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Question. $f(x)\to 0(x\to x_0)$, 是否一定存在$k>0, s.t.$ $x\to x_0$时, $f(x)$为k阶无穷小量? 否

试考虑 $f(x)=x\sin\dfrac{1}{x}$.

Ex. $\lim\limits_{x\to 0^+}(\sin x)^{1/\ln x}$ $\left(0^0\text{型}\right)$

$$=\lim_{x\to 0^+} e^{(\ln\sin x)/\ln x}=\lim_{x\to 0^+} e^{\left(\ln\frac{\sin x}{x}+\ln x\right)/\ln x}=e. \square$$

Remark. 1)指数-对数变换. 2)利用极限典式.

Ex. $\lim\limits_{x\to 0^+} x^{\sin x}=\lim\limits_{x\to 0^+} e^{\sin x\ln x}=\lim\limits_{x\to 0^+} e^{\frac{\sin x}{x}x\ln x}=e^0=1. \square$

$\left(0^0\text{型}\right)$

Ex. $\lim_{x\to 0^+}(e^x+2x)^{1/x}$ $\left(1^\infty 型\right)$

$$=\lim_{x\to 0^+}(1+e^x+2x-1)^{\frac{1}{e^x+2x-1}\cdot\frac{e^x+2x-1}{x}}$$

$$=e^{\lim\limits_{x\to 0^+}\frac{e^x+2x-1}{x}}=e^{2+\lim\limits_{x\to 0^+}\frac{e^x-1}{x}}=e^3.\square$$

Remark. 极限运算中$o(\cdot)$的运用有时能简化计算.

Ex. $\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)$, m,n为正整数.

解: 令$x=1+t$, 则$x\to 1$等价于$t\to 0$.

$$\frac{m}{1-x^m}-\frac{n}{1-x^n}=\frac{n}{(1+t)^n-1}-\frac{m}{(1+t)^m-1}$$

$$=\frac{n}{nt+\frac{n(n-1)}{2}t^2+o(t^2)}-\frac{m}{mt+\frac{m(m-1)}{2}t^2+o(t^2)}$$

$$=\frac{n\left[mt+\frac{m(m-1)}{2}t^2+o(t^2)\right]-m\left[nt+\frac{n(n-1)}{2}t^2+o(t^2)\right]}{\left[nt+\frac{n(n-1)}{2}t^2+o(t^2)\right]\cdot\left[mt+\frac{m(m-1)}{2}t^2+o(t^2)\right]}$$

$$=\frac{\frac{mn(m-n)}{2}t^2+o(t^2)}{mnt^2+o(t^2)}\to\frac{m-n}{2}(t\to 0)\square$$

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Ex. $\lim\limits_{x\to1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$

解：令$x=1+t$，则$x\to1$等价于$t\to0$.

$$\lim_{x\to1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}=\lim_{t\to0}\frac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}$$

$$=\lim_{t\to0}\frac{(1+t)^{1/m}-1}{t/m}\cdot\lim_{t\to0}\frac{t/n}{(1+t)^{1/n}-1}\cdot\lim_{t\to0}\frac{t/m}{t/n}$$

$$=\lim_{t\to0}\frac{t/m}{t/n}=\frac{n}{m}.$$

Remark. 等价因子替换法. $t\to0$时，$(1+t)^{1/m}-1\sim\dfrac{t}{m}$,

$(1+t)^{1/n}-1\sim\dfrac{t}{n}$，则 $\lim\limits_{t\to0}\dfrac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}=\lim\limits_{t\to0}\dfrac{t/m}{t/n}$.

Ex. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^2\ln(1+x)}$

解法一: $x\to 0$时, $\tan x\sim x$, $\sin x\sim x$, $\ln(1+x)\sim x$.

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}=\lim_{x\to 0}\frac{x-x}{x^2\cdot x}=0.$$ 是否正确? ×

解法二: $$\frac{\tan x-\sin x}{x^2\ln(1+x)}=\frac{\sin x(1-\cos x)}{x^2\cos x\ln(1+x)}$$

$$=\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2/2}\cdot\frac{x}{\ln(1+x)}\cdot\frac{1}{2\cos x}\to\frac{1}{2}\ (x\to 0).$$

解法三:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2\cos x\ln(1+x)}=\lim_{x\to 0}\frac{x\cdot\frac{1}{2}x^2}{x^2\cos x\cdot x}=\frac{1}{2}.\square$$

Ex. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{\cos\sqrt{x}-1+x}$ $\left(\dfrac{0}{0}\text{型}\right)$

解: $1-\sqrt{\cos x}=1-e^{\frac{1}{2}\ln\cos x}\sim-\dfrac{1}{2}\ln\cos x=-\dfrac{1}{2}\ln(1-2\sin^2\dfrac{x}{2})$

$$\sim\sin^2\frac{x}{2}\sim\frac{x^2}{4}\quad(x\to 0).$$

$$\lim_{x\to 0}\frac{1-\sqrt{\cos x}}{\cos\sqrt{x}-1+x}=\frac{1}{4}\lim_{x\to 0}\frac{x^2}{\cos\sqrt{x}-1+x}$$

$$=\frac{1}{4}\lim_{x\to 0}\frac{x}{\dfrac{\cos\sqrt{x}-1}{x}+1}=\frac{0}{-\dfrac{1}{2}+1}=0.\square$$

$$\text{Ex. } \lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}=\lim_{x\to+\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{1+\frac{1}{x}}}=1.$$

$$\text{Ex. } \lim_{x\to+\infty}\frac{\ln(2+\sqrt{x})}{\ln(6+\sqrt[6]{x})}=\lim_{x\to+\infty}\frac{\ln\sqrt{x}+\ln(1+2/\sqrt{x})}{\ln\sqrt[6]{x}+\ln(1+6/\sqrt[6]{x})}\overset{?}{=}\lim_{x\to+\infty}\frac{\ln\sqrt{x}}{\ln\sqrt[6]{x}}=3.\square$$

$$\ln\sqrt{x}+\ln(1+2/\sqrt{x})\sim\ln\sqrt{x}\ \ (x\to+\infty)$$

$$\ln\sqrt[6]{x}+\ln(1+6/\sqrt[6]{x})\sim\ln\sqrt[6]{x}\ \ (x\to+\infty)$$

Remark.等价无穷大因子替换!

Ex. $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+2x}-\sqrt[3]{x^3-x^2}\right)$

解法一: $\lim\limits_{x\to+\infty}(\sqrt{x^2+2x}-x)$

$$=\lim_{x\to+\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim_{x\to+\infty}\frac{2}{\sqrt{1+2/x}+1}=1,$$

$\lim\limits_{x\to+\infty}(\sqrt[3]{x^3-x^2}-x)$

$$=\lim_{x\to+\infty}\frac{-x^2}{\left(\sqrt[3]{x^3-x^2}\right)^2+x\cdot\sqrt[3]{x^3-x^2}+x^2}=-\frac{1}{3}.$$

$$原式=\lim_{x\to+\infty}(\sqrt{x^2+2x}-x)-\lim_{x\to+\infty}(\sqrt[3]{x^3-x^2}-x)=\frac{4}{3}.$$

解法二：令$y=1/x$. 则$x\to+\infty$时, $y\to 0$, 且

$$\lim_{x\to+\infty}\left(\sqrt{x^2+2x}-\sqrt[3]{x^3-x^2}\right)=\lim_{y\to 0}\frac{(1+2y)^{1/2}-(1-y)^{1/3}}{y}$$

$$=\lim_{y\to 0}\left(\frac{(1+2y)^{1/2}-1}{y}-\frac{(1-y)^{1/3}-1}{y}\right)$$

$$=\lim_{y\to 0}\frac{(1+2y)^{1/2}-1}{y}-\lim_{y\to 0}\frac{(1-y)^{1/3}-1}{y}$$

$$=2\cdot\frac{1}{2}-(-1)\cdot\frac{1}{3}=\frac{4}{3}.\square$$

Ex. $\lim\limits_{x\to+\infty}\left(\sqrt[n]{(x^2+1)(x^2+2)\cdots(x^2+n)}-x^2\right)$

$$=\lim_{x\to+\infty}\left(e^{\frac{1}{n}\sum_{k=1}^{n}\ln(x^2+k)}-x^2\right)=\lim_{x\to+\infty}\left(e^{\frac{1}{n}\sum_{k=1}^{n}\left(\ln x^2+\ln(1+k/x^2)\right)}-x^2\right)$$

$$=\lim_{x\to+\infty}\left(e^{\ln x^2+\frac{1}{n}\sum_{k=1}^{n}\ln(1+k/x^2)}-x^2\right)=\lim_{x\to+\infty}x^2\left(e^{\frac{1}{n}\sum_{k=1}^{n}\ln(1+k/x^2)}-1\right)$$

$$=\lim_{x\to+\infty}\frac{x^2}{n}\sum_{k=1}^{n}\ln(1+k/x^2)\qquad\left(\lim_{x\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln(1+k/x^2)=0\right)$$

$$=\lim_{x\to+\infty}\frac{1}{n}\sum_{k=1}^{n}k\ln(1+k/x^2)^{x^2/k}=\frac{1}{n}\sum_{k=1}^{n}k=\frac{n+1}{2}.\square$$

Ex. $\lim_{x\to 0^+}\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right)^x$

解: $\lim_{x\to 0^+}\frac{\sin\sqrt{x}}{\sqrt{x}}=1, \exists\delta>0, s.t.\ \frac{3}{4}<\frac{\sin\sqrt{x}}{\sqrt{x}}\le 1, \forall x\in U(0,\delta).$

于是 $$\frac{1}{2}<\frac{2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}}{\sqrt{x}}\le 3,\ \forall x\in U(0,\delta).$$

$$原式=\lim_{x\to 0^+}e^{x\ln\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right)}$$

$$=\lim_{x\to 0^+}e^{x\ln\frac{2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}}{\sqrt{x}}+x\ln\sqrt{x}}=e^0=1.\square$$

Ex. f在$(0,+\infty)$上单调, $\lim\limits_{x\to+\infty}\dfrac{f(2x)}{f(x)}=1, a>0$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=1.$

Proof. $\lim\limits_{x\to+\infty}\dfrac{f(2^n x)}{f(x)}=\lim\limits_{x\to+\infty}\dfrac{f(2^n x)}{f(2^{n-1}x)}\cdot\dfrac{f(2^{n-1}x)}{f(2^{n-2}x)}\cdot\cdots\cdot\dfrac{f(2x)}{f(x)}=1.$

若$a\geq 1$, 则$\exists n\geq 0, s.t.\ 2^n\leq a<2^{n+1}$, 从而对充分大的$x$, 有

$$\frac{f(2^n x)}{f(x)}\leq(\geq)\frac{f(ax)}{f(x)}\leq(\geq)\frac{f(2^{n+1}x)}{f(x)}.$$

由夹挤原理, $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=1.$

若$0<a<1$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=\lim\limits_{t\to+\infty}\dfrac{f(t)}{f(t/a)}=1.$ □

Ex. $\lim_{x\to 0} f(x)=0, \lim_{x\to 0}\frac{f(x)-f(x/2)}{x}=0$, 则 $\lim_{x\to 0}\frac{f(x)}{x}=0.$

Proof. $\forall\varepsilon>0$, 由 $\lim_{x\to 0}\frac{f(x)-f(x/2)}{x}=0, \exists\delta>0, s.t.$

$$\left|f(x)-f(x/2)\right|<\varepsilon|x|,\ \ \forall 0<|x|<\delta.$$

$$|f(x)|\le\sum_{k=1}^{n}\left|f(\frac{x}{2^{k-1}})-f(\frac{x}{2^{k}})\right|+\left|f(\frac{x}{2^{n}})\right|$$

$$\le\sum_{k=1}^{n}\frac{\varepsilon|x|}{2^{k-1}}+\left|f(\frac{x}{2^{n}})\right|<2\varepsilon|x|+\left|f(\frac{x}{2^{n}})\right|,\ \ \forall 0<|x|<\delta.$$

令 $n\to+\infty$, 由 $\lim_{x\to 0} f(x)=0$ 得 $|f(x)|\le 2\varepsilon|x|$. 故 $\lim_{x\to 0}\frac{f(x)}{x}=0.\square$

Ex. $\lim\limits_{x\to\infty}(\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1})$

解: $\left|\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1}\right|$

$$=2\left|\cos\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{2}\sin\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{2}\right|$$

$$\le\left|\sqrt{x^2+1}-\sqrt{x^2-1}\right|=\left|\frac{1}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right|\to 0\,(x\to\infty).$$

故 $\lim\limits_{x\to\infty}(\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1})=0.\square$

$$\text{Ex.}\lim_{x\to 0}\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^{1/x}$$

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$$=\lim_{x\to 0}\left(1+\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1\right)\right)^{\frac{1}{\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1}\cdot\frac{\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1}{x}}$$

$$=\exp\left\{\lim_{x\to 0}\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1\right)\Big/x\right\}$$

$$=\exp\left\{\lim_{x\to 0}\frac{(e^x-1)/x+(e^{2x}-1)/x+\cdots+(e^{nx}-1)/x}{n}\right\}$$

$$=e^{(1+2+\cdots+n)/n}=e^{(n+1)/2}.\square$$

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解法二. $\lim\limits_{x\to 0}\left(\dfrac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^{1/x}$

$$=\lim_{x\to 0}\left(1+\frac{(e^x-1)+(e^{2x}-1)+\cdots+(e^{nx}-1)}{n}\right)^{1/x}$$

$$=\lim_{x\to 0}\left(1+\frac{x+o(x)+2x+o(x)+\cdots+nx+o(x)}{n}\right)^{1/x}$$

$$=\lim_{x\to 0}\left(1+\frac{(n+1)x}{2}+o(x)\right)^{1/x}=e^{\lim\limits_{x\to 0}\left(\frac{(n+1)x}{2}+o(x)\right)\Big/x}=e^{(n+1)/2}.\square$$

Ex. $\lim\limits_{n\to+\infty} n^2\left(\sqrt[n]{x}-\sqrt[n+1]{x}\right) \quad (x>0)$

$$= \lim_{n\to+\infty} n^2 x^{1/(n+1)}\left(x^{1/n(n+1)}-1\right)$$

$$= \lim_{n\to+\infty}\left(x^{1/(n+1)}\cdot\frac{x^{1/n(n+1)}-1}{1/n(n+1)}\cdot\frac{n^2}{n(n+1)}\right)$$

$$= \lim_{n\to+\infty} x^{1/(n+1)}\cdot\lim_{n\to+\infty}\frac{x^{1/n(n+1)}-1}{1/n(n+1)}\cdot\lim_{n\to+\infty}\frac{n^2}{n(n+1)} = \ln x. \square$$

Remark. $a^x-1\sim x\ln a\,(x\to 0)\Rightarrow \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a.$

Ex. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}=1$

Proof. $\dfrac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}=\dfrac{\sqrt{1+\tan x}-1}{e^x-1}-\dfrac{\sqrt{1-\tan x}-1}{e^x-1}$

$$\frac{\sqrt{1+\tan x}-1}{e^x-1}=\frac{1}{2}\cdot\frac{(1+\tan x)^{1/2}-1}{\frac{1}{2}\tan x}\cdot\frac{\tan x}{x}\cdot\frac{x}{e^x-1}\to\frac{1}{2}\quad(x\to 0)$$

同理, $\dfrac{\sqrt{1-\tan x}-1}{e^x-1}\to-\dfrac{1}{2}\quad(x\to 0)$.□

Remark. 上例可用等价因子替换法:

$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}$$

$$=\lim_{x\to 0}\frac{\sqrt{1+\tan x}-1}{e^x-1}-\lim_{x\to 0}\frac{\sqrt{1-\tan x}-1}{e^x-1}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}\tan x}{x}-\lim_{x\to 0}\frac{-\frac{1}{2}\tan x}{x}$$

$$=\frac{1}{2}-(-\frac{1}{2})=1. \square$$

Ex.(1) $\lim\limits_{x\to 0+}\dfrac{(a^x-b^x)^2}{a^{x^2}-b^{x^2}}\ (a,b>0,a\neq b)$, (2) $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}\ (a>0)$

解.(1) $$\lim_{x\to 0+}\frac{(a^x-b^x)^2}{a^{x^2}-b^{x^2}}=\lim_{x\to 0+}\frac{b^{2x}((a/b)^x-1)^2}{b^{x^2}((a/b)^{x^2}-1)}$$

$$=\lim_{x\to 0+}\frac{((a/b)^x-1)^2}{(a/b)^{x^2}-1}=\lim_{x\to 0+}\frac{(x\ln(a/b))^2}{x^2\ln(a/b)}=\ln\frac{a}{b}.$$

(2) $$\lim_{x\to a}\frac{a^{a^x}-a^{x^a}}{a^x-x^a}=\lim_{x\to a}\frac{a^{x^a}(a^{a^x-x^a}-1)}{a^x-x^a}$$

$$=\lim_{x\to a}a^{x^a}\lim_{x\to a}\frac{(a^{a^x-x^a}-1)}{a^x-x^a}=a^{a^a}\ln a.\square$$

Ex.(1) $\lim\limits_{x\to 0+} x\ln x$, (2) $\lim\limits_{x\to 0+}(x^x-1)\ln x$, (3) $\lim\limits_{x\to 0+} x^{x^x-1}$

解:(1) $$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{-\ln(1/x)}{1/x} = \lim_{y\to +\infty}\frac{-\ln y}{y} = 0.$$

(2) $$\lim_{x\to 0+}(x^x-1)\ln x = \lim_{x\to 0+}(e^{x\ln x}-1)\ln x$$

$$= \lim_{x\to 0+}\frac{e^{x\ln x}-1}{x\ln x}\cdot x(\ln x)^2 = \lim_{x\to 0+} x(\ln x)^2 = 0.$$

(3) $$\lim_{x\to 0+} x^{x^x-1} = \lim_{x\to 0+} e^{(x^x-1)\ln x} = e^{\lim\limits_{x\to 0+}(x^x-1)\ln x} = e^0 = 1.\square$$

作业：习题2.4 No. 8,9(单),12