



Review

Thm.(确界原理) 非空有上界的集合必有上确界.

Thm.(单调收敛原理) 单调有界列必收敛.

Thm.(闭区间套定理) 若闭区间列 $[a_n, b_n]$ 满足条件:

$$(1) [a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \dots),$$

$$(2) \lim_{n \rightarrow \infty} (b_n - a_n) = 0,$$

则 $\exists! \xi \in \mathbb{R}, s.t. \xi \in \bigcap_{n \geq 1} [a_n, b_n]; \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \xi.$

Thm.(Bolzano-Weirstrass定理) 有界列必有收敛子列.

Thm.(Cauchy收敛原理) 收敛列 \Leftrightarrow Cauchy列.



§ 1. 函数的极限

$$N(x_0, \delta) := (x_0 - \delta, x_0 + \delta),$$

$$U(x_0, \delta) := N(x_0, \delta) \setminus \{x_0\}$$

Def. (函数在一点的极限) 设 f 在 $U(x_0, \rho)$ 中有定义, $A \in \mathbb{R}$.
若 $\forall \varepsilon > 0, \exists \delta \in (0, \rho), s.t.$

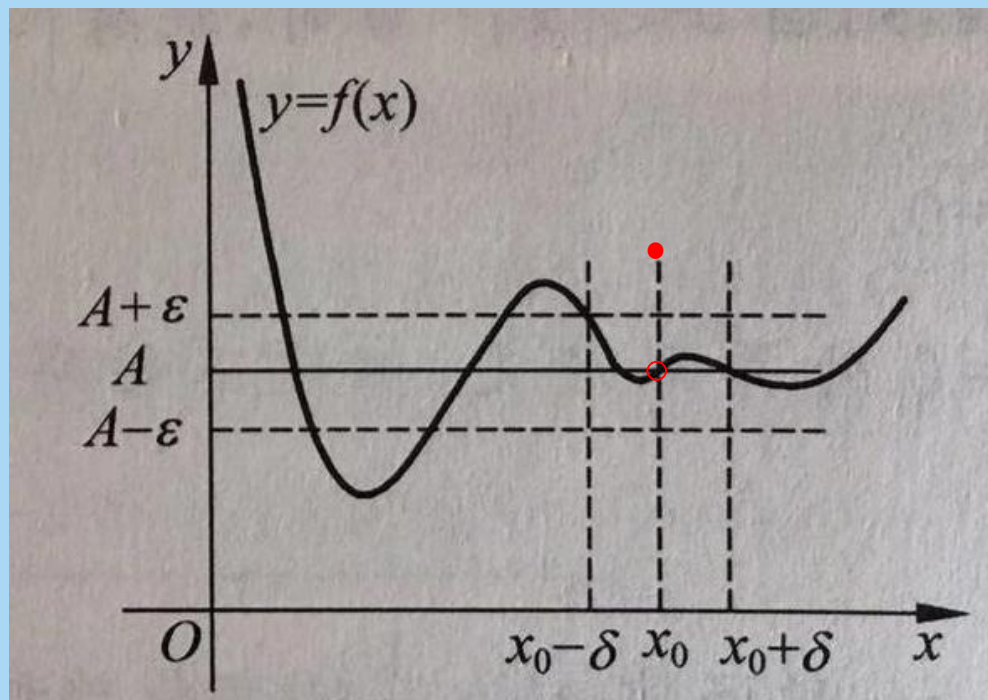
$$|f(x) - A| < \varepsilon, \quad \forall x \in U(x_0, \delta),$$

则称 $f(x)$ 在点 x_0 处有极限 A , 或者当 x 趋于 x_0 时, $f(x)$ 趋于 A .
记作 $\lim_{x \rightarrow x_0} f(x) = A$, 或 $f(x) \rightarrow A (x \rightarrow x_0)$.



Remark. $\lim_{x \rightarrow x_0} f(x)$ 与 f 在 x_0 的定义无关.

Question. $\lim_{x \rightarrow x_0} f(x) = A$ 的几何意义?

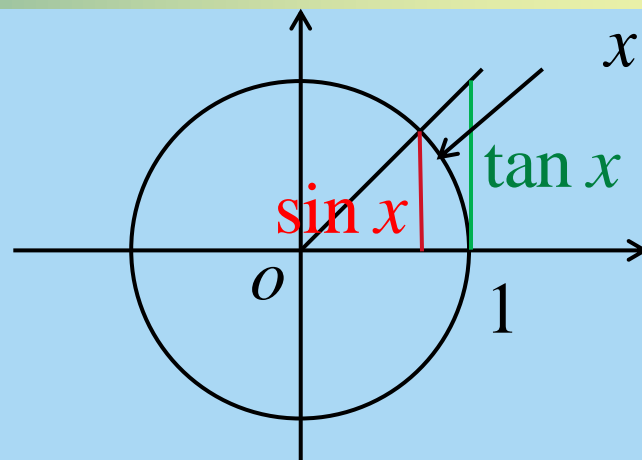


Question. 如何用 $\epsilon - \delta$ 语言描述 $\lim_{x \rightarrow x_0} f(x) \neq A$?



$$|\sin x| \leq |x|, \forall x \in \mathbb{R}.$$

$$|x| \leq |\tan x|, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Ex. $\lim_{x \rightarrow x_0} \cos x = \cos x_0.$

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$, 当 $0 < |x - x_0| < \delta$ 时, 有

$$\begin{aligned} |\cos x - \cos x_0| &= \left| 2 \sin \frac{x + x_0}{2} \sin \frac{x - x_0}{2} \right| \\ &\leq 2 \left| \sin \frac{x - x_0}{2} \right| \leq 2 \cdot \frac{|x - x_0|}{2} = |x - x_0| < \delta = \varepsilon. \square \end{aligned}$$

Ex. $\lim_{x \rightarrow x_0} \sin x = \sin x_0.$



Ex. $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = \underline{0}$.

$$\left| x \sin \frac{1}{x} \right| \leq |x|.$$

Ex. $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - x} = \underline{-1}$.

分析: $\frac{x^2 - 3x + 2}{x^2 - x} = \frac{(x-1)(x-2)}{x(x-1)} = \frac{x-2}{x}, \forall x \neq 1.$

Proof. 当 $|x-1| < \frac{1}{2}$ 时, $|x| > \frac{1}{2}$. $\forall \varepsilon > 0, \exists \delta = \min\{\frac{1}{2}, \frac{\varepsilon}{4}\}, s.t.$

$$\left| \frac{x^2 - 3x + 2}{x^2 - x} - (-1) \right| = 2 \left| \frac{x-1}{x} \right| < 4|x-1| < \varepsilon,$$

$$\forall 0 < |x-1| < \delta. \square$$



Def.(右极限) 设 f 在 $(x_0, x_0 + \rho)$ 中有定义, $A \in \mathbb{R}$. 若 $\forall \varepsilon > 0$, $\exists \delta \in (0, \rho)$, s.t.

$$|f(x) - A| < \varepsilon, \quad \forall x_0 < x < x_0 + \delta,$$

则称 $f(x)$ 在点 x_0 处有右极限 A , 或者当 x 趋于 x_0^+ 时, $f(x)$ 趋于 A . 记作 $\lim_{x \rightarrow x_0^+} f(x) = A$, 或 $f(x) \rightarrow A (x \rightarrow x_0^+)$.



Def.(左极限) 设 f 在 $(x_0 - \rho, x_0)$ 中有定义, $A \in \mathbb{R}$. 若 $\forall \varepsilon > 0$, $\exists \delta \in (0, \rho)$, s.t.

$$|f(x) - A| < \varepsilon, \quad \forall x_0 - \delta < x < x_0,$$

则称 $f(x)$ 在点 x_0 处有左极限 A , 或者当 x 趋于 x_0^- 时, $f(x)$ 趋于 A . 记作 $\lim_{x \rightarrow x_0^-} f(x) = A$, 或 $f(x) \rightarrow A (x \rightarrow x_0^-)$.



Thm. $\lim_{x \rightarrow x_0} f(x) = A \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) = A.$

Proof. 略.

Ex. $\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$

$\lim_{x \rightarrow 0^+} \text{sgn}(x) \underline{= 1}, \lim_{x \rightarrow 0^-} \text{sgn}(x) \underline{= -1}, \lim_{x \rightarrow 0} \text{sgn}(x) \underline{\text{不存在}}.$

Ex. $\lim_{x \rightarrow 0^-} e^{\frac{1}{x}} \underline{= 0}, \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} \underline{= +\infty}, \lim_{x \rightarrow 0} e^{\frac{1}{x}} \underline{\text{不存在}}.$



Def.(函数在无穷远点的极限)

(1) 设 $|x| > a$ 时 f 有定义, $A \in \mathbb{R}$. 若 $\forall \varepsilon > 0, \exists M > 0, s.t.$

$$|f(x) - A| < \varepsilon, \quad \forall |x| > M,$$

则称当 x 趋于 ∞ 时, $f(x)$ 有极限 A . 记作

$$\lim_{x \rightarrow \infty} f(x) = A, \text{ 或 } f(x) \rightarrow A (x \rightarrow \infty).$$

(2) 设 $x > a$ 时 f 有定义, $A \in \mathbb{R}$. 若 $\forall \varepsilon > 0, \exists M > 0, s.t.$

$$|f(x) - A| < \varepsilon, \quad \forall x > M,$$

则称当 x 趋于 $+\infty$ 时, $f(x)$ 有极限 A . 记作

$$\lim_{x \rightarrow +\infty} f(x) = A, \text{ 或 } f(x) \rightarrow A (x \rightarrow +\infty).$$



(3) 设 $x < a$ 时 f 有定义, $A \in \mathbb{R}$. 若 $\forall \varepsilon > 0, \exists M > 0, s.t.$

$$|f(x) - A| < \varepsilon, \quad \forall x < -M,$$

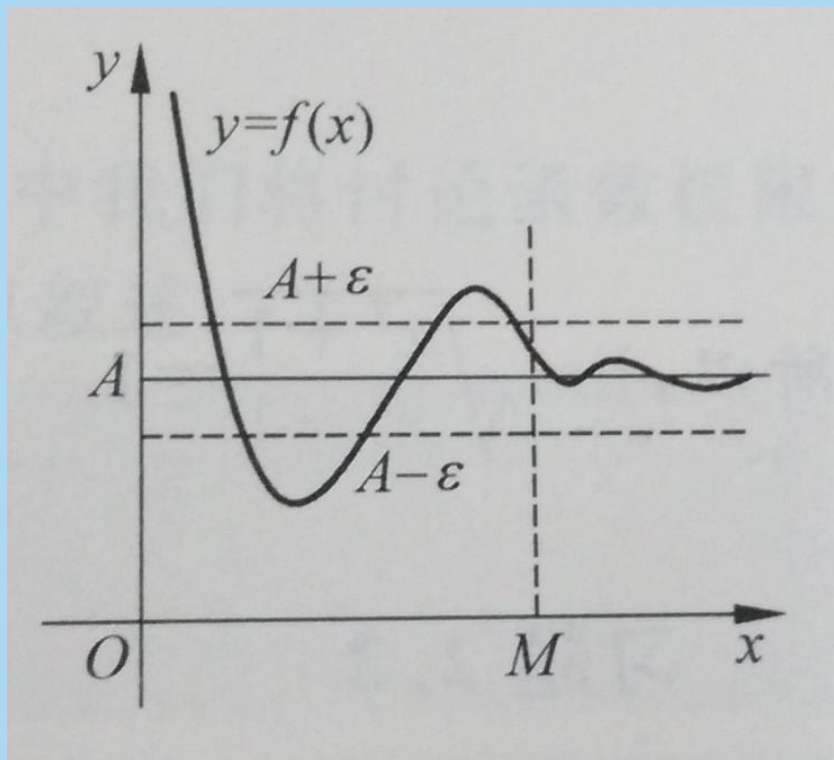
则称当 x 趋于 $-\infty$ 时, $f(x)$ 有极限 A . 记作

$$\lim_{x \rightarrow -\infty} f(x) = A, \text{ 或 } f(x) \rightarrow A (x \rightarrow -\infty).$$

Question. 如何用 $\varepsilon - \delta$ 语言描述 $\lim_{x \rightarrow +\infty} f(x) \neq A$?



Question. $\lim_{x \rightarrow +\infty} f(x) = A$ 的几何意义?





Ex. $\lim_{x \rightarrow +\infty} \sqrt{\frac{x^2+1}{x^2-1}} = \underline{\quad 1 \quad}$

Proof. $\forall \varepsilon > 0, \exists M = \max\{\sqrt{2}, \frac{2}{\varepsilon}\}, s.t.$

$$\begin{aligned} \left| \sqrt{\frac{x^2+1}{x^2-1}} - 1 \right| &= \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^2-1}} \\ &= \frac{2}{\sqrt{x^2-1}(\sqrt{x^2+1} + \sqrt{x^2-1})} < \frac{2}{|x|} < \varepsilon, \forall x > M. \end{aligned}$$

故 $\lim_{x \rightarrow +\infty} \sqrt{\frac{x^2+1}{x^2-1}} = 1. \square$



§ 2. 函数极限的性质

$$\lim_{x \rightarrow x_0} f(x), \quad \lim_{x \rightarrow x_0+} f(x), \quad \lim_{x \rightarrow x_0-} f(x),$$

$$\lim_{x \rightarrow \infty} f(x), \quad \lim_{x \rightarrow +\infty} f(x), \quad \lim_{x \rightarrow -\infty} f(x).$$

以 $\lim_{x \rightarrow x_0} f(x)$ 为例叙述函数极限的性质, 其他情形类似.

Prop1. 若 $\lim_{x \rightarrow x_0} f(x)$ 存在, 则极限值唯一.

Prop2. 若 $\lim_{x \rightarrow x_0} f(x)$ 存在, 则 $\exists \delta > 0, M > 0, s.t.$

$$|f(x)| < M, \quad \forall x \in U(x_0, \delta). \quad (\text{局部有界})$$



Prop3.(保序性) $\lim_{x \rightarrow x_0} f(x) = A, \lim_{x \rightarrow x_0} g(x) = B.$

(1)若 $A > B$, 则 $\exists \delta > 0, s.t.$

$$f(x) > g(x), \quad \forall x \in U(x_0, \delta).$$

(2)若 $\exists \delta > 0, s.t.$

$$f(x) \geq g(x), \quad \forall x \in U(x_0, \delta),$$

则 $A \geq B.$



Prop4.(四则运算) $\lim_{x \rightarrow x_0} f(x) = A, \lim_{x \rightarrow x_0} g(x) = B.$

$$(1) \forall c \in \mathbb{R}, \lim_{x \rightarrow x_0} cf(x) = cA;$$

$$(2) \lim_{x \rightarrow x_0} (f(x) \pm g(x)) = A \pm B;$$

$$(3) \lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = AB;$$

$$(4) B \neq 0 \text{ 时}, \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Remark. A, B可取
 $+\infty, -\infty$ 或 ∞ ,只要
右端运算有意义.

$$\text{Ex. } \lim_{x \rightarrow x_0} \tan x = \tan x_0, \quad \lim_{x \rightarrow x_0} \cot x = \cot x_0,$$

$$\lim_{x \rightarrow x_0} \sec x = \sec x_0, \quad \lim_{x \rightarrow x_0} \csc x = \csc x_0.$$

条件?



Prop5.(夹挤原理)若

$$\left. \begin{array}{l} f(x) \leq g(x) \leq h(x), \quad \forall x \in U(x_0, \rho) \\ \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = A \end{array} \right\} \Rightarrow \lim_{x \rightarrow x_0} g(x) = A.$$

Ex. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$ $\frac{0}{0}$ 型极限

Proof. $\forall 0 < |x| < \frac{\pi}{2}$, 有 $|\sin x| \leq |x| \leq |\tan x|$,

$$\cos x \leq \frac{|\sin x|}{|x|} \leq 1, \quad \cos x \leq \frac{\sin x}{x} = \frac{|\sin x|}{|x|} \leq 1.$$

而 $\lim_{x \rightarrow 0} \cos x = 1$, 由夹挤原理, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1. \square$



Prop6.(单调收敛原理)

(1) f 在 (a, b) 上的单增有上界, 则 $\lim_{x \rightarrow b^-} f(x) = \sup_{a < x < b} f(x)$;

(2) f 在 (a, b) 上的单减有下界, 则 $\lim_{x \rightarrow b^-} f(x) = \inf_{a < x < b} f(x)$;

(3) f 在 (a, b) 上的单增有下界, 则 $\lim_{x \rightarrow a^+} f(x) = \inf_{a < x < b} f(x)$;

(4) f 在 (a, b) 上的单减有上界, 则 $\lim_{x \rightarrow a^+} f(x) = \sup_{a < x < b} f(x)$.

Proof. 只证(1), 其它情形同理可证. $\{f(x) : x \in (a, b)\}$

非空有上界, 从而有上确界

$$A = \sup \{f(x) : x \in (a, b)\}.$$



由上确界的定义,

$$\forall \varepsilon > 0, \exists x_1 \in (a, b), \text{ s.t. } f(x_1) > A - \varepsilon,$$

且 $f(x) \leq A, \quad \forall x \in (a, b).$

$f \uparrow$, 则 $\forall x \in (x_1, b)$, 有

$$A - \varepsilon < f(x_1) \leq f(x) \leq A.$$

故 $\lim_{x \rightarrow b^-} f(x) = A. \square$



Prop7.
$$\left. \begin{array}{l} \lim_{x \rightarrow x_0} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(u) = A \\ g(x) \neq u_0, \forall x \neq x_0 \end{array} \right\} \Rightarrow \lim_{x \rightarrow x_0} f(g(x)) = A = \lim_{u \rightarrow u_0} f(u).$$

(复合函数的极限)

Proof. $\lim_{u \rightarrow u_0} f(u) = A$, 则 $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(u) - A| < \varepsilon, \quad \forall 0 < |u - u_0| < \delta.$$

对此 $\delta > 0$, 因 $g(x) \neq u_0, \forall x \neq x_0, \lim_{x \rightarrow x_0} g(x) = u_0, \exists \eta > 0, s.t.$

$$0 < |g(x) - u_0| < \delta, \quad \forall 0 < |x - x_0| < \eta,$$

从而 $|f(g(x)) - A| < \varepsilon, \quad \forall 0 < |x - x_0| < \eta.$

由函数极限定义, 有 $\lim_{x \rightarrow x_0} f(g(x)) = A. \square$



Remark. 复合函数的极限运算可以理解为函数极限运算的变量替换法.

Question. 条件 “ $g(x) \neq u_0, \forall x \neq x_0$ ” 是否可去? 反例?

Question. 条件 “ $g(x) \neq u_0, \forall x \neq x_0$ ” 何时可去?

$$\left. \begin{array}{l} \text{Remark. } \lim_{x \rightarrow x_0} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(u) = f(u_0) \end{array} \right\} \Rightarrow \lim_{x \rightarrow x_0} f(g(x)) = f(u_0) = f\left(\lim_{x \rightarrow x_0} g(x)\right).$$

Question. $x \rightarrow +\infty$ 时, $x^b, a^x, \ln x, x^x$ 的增长速度? ($a > 1, b > 0$)



Thm. 证明: (a, b) 上的单调函数在每一点处左右极限都存在.

Proof. 不妨设 f 在 (a, b) 上单增, $x_0 \in (a, b)$, 往证 $\lim_{x \rightarrow x_0^-} f(x)$ 存在 (同理可证 $\lim_{x \rightarrow x_0^+} f(x)$ 存在).

$f \uparrow, \{f(x) : x \in (a, x_0)\}$ 非空有上界 $f(x_0)$, 从而有上确界

$$A = \sup \{f(x) : x \in (a, x_0)\}.$$

由上确界的定义,

$$\forall \varepsilon > 0, \exists x_1 \in (a, x_0), \text{ s.t. } f(x_1) > A - \varepsilon,$$

且
$$f(x) \leq A, \quad \forall x \in (a, x_0).$$

$f \uparrow, \forall x \in (x_1, x_0)$, 有 $A - \varepsilon < f(x_1) \leq f(x) \leq A$. 故 $\lim_{x \rightarrow x_0^-} f(x) = A. \square$



Ex. $\lim_{x \rightarrow +\infty} \frac{\log_a x}{x^b} = 0 \quad (a > 1, b > 0).$

Proof. $0 < \frac{\ln x}{x} \leq \frac{\ln([x]+1)}{[x]} \leq \frac{\ln 2}{[x]} + \frac{\ln[x]}{[x]}, \quad \forall x > 1.$

$$\lim_{x \rightarrow +\infty} \left(\frac{\ln 2}{[x]} + \frac{\ln[x]}{[x]} \right) = \lim_{x \rightarrow +\infty} \frac{\ln 2}{[x]} + \lim_{x \rightarrow +\infty} \frac{\ln[x]}{[x]} = 0.$$

由夹挤原理, $\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0$. 而 $\lim_{x \rightarrow +\infty} \frac{\log_a x}{x} = \frac{1}{\ln a} \lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0$.

$$\lim_{x \rightarrow +\infty} \frac{\log_a x}{x^b} = \lim_{y \rightarrow +\infty} \frac{\log_a y^{1/b}}{y} = \frac{1}{b} \lim_{y \rightarrow +\infty} \frac{\log_a y}{y} = 0. \square$$



Remark. $\lim_{x \rightarrow 0^+} x^b \log_a x = 0 \ (a > 1, b > 0)$.

Ex. $\lim_{x \rightarrow +\infty} \frac{x^b}{a^x} = 0 \ (a > 1, b > 0)$.

Proof. $0 < \frac{x^b}{a^x} \leq \frac{([x]+1)^b}{a^{[x]}} \leq \frac{(2[x])^b}{a^{[x]}} \leq \frac{2^b [x]^b}{a^{[x]}}, \forall x > 1$.

$\lim_{x \rightarrow +\infty} \frac{2^b [x]^b}{a^{[x]}} = 2^b \lim_{x \rightarrow +\infty} \frac{[x]^b}{a^{[x]}} = 0$. 由夹挤原理, $\lim_{x \rightarrow +\infty} \frac{x^b}{a^x} = 0$. \square

Ex. $\lim_{x \rightarrow +\infty} \frac{a^x}{x^x} = 0 \ (a > 0, a \neq 1)$.

Proof. $\lim_{x \rightarrow +\infty} \frac{a^x}{x^x} = \lim_{y \rightarrow +\infty} e^{x(\ln a - \ln x)} = e^{(+\infty) \cdot (-\infty)} = e^{-\infty} = 0$. \square



Ex. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$

1^∞ 型极限

Proof. $\left(1 + \frac{1}{[x]+1}\right)^{[x]} \leq \left(1 + \frac{1}{x}\right)^x \leq \left(1 + \frac{1}{[x]}\right)^{[x]+1}, \forall x > 1.$

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]+1} = \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]}\right) = e,$$

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]} = \frac{\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]+1}}{\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{[x]+1}\right)} = e,$$

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由夹挤原理, $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$.

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow -\infty} \left(\frac{x}{1+x}\right)^{-x} = \lim_{x \rightarrow -\infty} \left(\frac{x}{1+x}\right)^{-(x+1)} \cdot \lim_{x \rightarrow -\infty} \frac{x}{1+x}$$

$$= \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{-(1+x)}\right)^{-(x+1)} \cdot 1 = \lim_{y \rightarrow +\infty} \left(1 + \frac{1}{y}\right)^y = e.$$

综上, $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$. \square

Remark. $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$. 常用以处理 1^∞ 型极限.



Thm. f 在 $U(x_0, \rho)$ 中有定义, 则以下命题等价:

(1) $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in U(x_0, \delta)$, 有 $|f(x) - f(y)| < \varepsilon$;

(2) $\exists A \in \mathbb{R}$, 对 $U(x_0, \rho)$ 中任意收敛到 x_0 的点列 $\{x_n\}$, 有

$$\lim_{n \rightarrow \infty} f(x_n) = A;$$

(3) $\lim_{x \rightarrow x_0} f(x) = A$.

Remark. (1) \Leftrightarrow (3) (函数极限的Cauchy收敛原理)

Remark. (2) \Leftrightarrow (3) (用数列的极限来研究函数的极限)



Proof. (1) \Rightarrow (2): 设 $x_n \in U(x_0, \rho)$, $\lim_{n \rightarrow \infty} x_n = x_0$. $\forall \varepsilon > 0$, 由(1),

$\exists \delta > 0$, $\forall x, y \in U(x_0, \delta)$, 有 $|f(x) - f(y)| < \varepsilon$.

对此 δ , 因 $\lim_{n \rightarrow \infty} x_n = x_0$, $\exists N$, s.t. $x_n \in U(x_0, \delta)$, $\forall n > N$.

于是 $|f(x_n) - f(x_m)| < \varepsilon$, $\forall n, m > N$.

故 $\{f(x_n)\}$ 为Cauchy列, 收敛, $\exists A \in \mathbb{R}$, s.t. $\lim_{n \rightarrow \infty} f(x_n) = A$.

设 $\lim_{n \rightarrow \infty} y_n = x_0$, 同理 $\lim_{n \rightarrow \infty} f(y_n) = B$. 只要证 $A = B$ 即可.

构造点列 $\{z_n\}$: $z_{2n-1} = x_n$, $z_{2n} = y_n$, 则 $\lim_{n \rightarrow \infty} z_n = x_0$, $\{f(z_n)\}$

收敛, 且 $A = \lim_{n \rightarrow \infty} f(z_{2n-1}) = \lim_{n \rightarrow \infty} f(z_n) = \lim_{n \rightarrow \infty} f(z_{2n}) = B$.



(2) \Rightarrow (3):

设 $\lim_{x \rightarrow x_0} f(x) \neq A$. 则 $\exists \varepsilon_0 > 0, \forall n \in \mathbb{N}, \exists x_n \in U(x_0, \frac{1}{n}), s.t.$

$$|f(x_n) - A| > \varepsilon_0.$$

此时, $\lim_{n \rightarrow \infty} x_n = x_0$, 但 $\lim_{n \rightarrow \infty} f(x_n) \neq A$, 与(2)矛盾.

(3) \Rightarrow (1): 略. \square



Remark. $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = x_0$, 则

- $\lim_{n \rightarrow \infty} f(x_n) = A \neq B = \lim_{n \rightarrow \infty} f(y_n) \Rightarrow \lim_{x \rightarrow x_0} f(x)$ 不存在;
- $\lim_{n \rightarrow \infty} f(x_n)$ 不存在 $\Rightarrow \lim_{x \rightarrow x_0} f(x)$ 不存在.

Ex. Dirichlet函数 $D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$

$\lim_{x \rightarrow x_0^-} D(x)$ 不存在, $\lim_{x \rightarrow x_0^+} D(x)$ 不存在, $\lim_{x \rightarrow x_0} D(x)$ 不存在,

$\forall x_0 \in \mathbb{R}.$



Ex. $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ 不存在.

Proof. $x_n = \frac{1}{2n\pi}, y_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi},$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} y_n = 0,$$

而 $\lim_{n \rightarrow +\infty} \sin \frac{1}{x_n} = 0, \lim_{n \rightarrow +\infty} \sin \frac{1}{y_n} = 1,$ 故 $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ 不存在. \square



Ex. (1) $\lim_{x \rightarrow x_0} e^x = e^{x_0}$, (2) $\lim_{x \rightarrow x_0} \ln x = \ln x_0$.

Proof. $\forall \{x_n\}, x_n \rightarrow x_0$, 有 $\lim_{n \rightarrow \infty} e^{x_n} = e^{x_0}, \lim_{n \rightarrow \infty} \ln x_n = \ln x_0$. \square

Thm. $\lim_{x \rightarrow x_0} u(x) = a, \lim_{x \rightarrow x_0} v(x) = b, a^b$ 有意义, 则 $\lim_{x \rightarrow x_0} u(x)^{v(x)} = a^b$.

Proof. $\lim_{x \rightarrow x_0} u(x)^{v(x)} = \lim_{x \rightarrow x_0} e^{v(x) \ln u(x)}$
 $= e^{\lim_{x \rightarrow x_0} (v(x) \ln u(x))} = e^{\lim_{x \rightarrow x_0} v(x) \cdot \lim_{x \rightarrow x_0} \ln u(x)} = e^{b \ln a} = a^b$. \square

Remark. $\lim_{x \rightarrow x_0} \sqrt{u(x)} = \sqrt{\lim_{x \rightarrow x_0} u(x)}, \lim_{x \rightarrow x_0} a^{u(x)} = a^{\lim_{x \rightarrow x_0} u(x)}, \dots$



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Ex. $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$

Proof.
$$\begin{aligned} \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}} &= \lim_{x \rightarrow 0} \left(1 - 2 \sin^2 \frac{x}{2} \right)^{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0} \left(1 - 2 \sin^2 \frac{x}{2} \right)^{\frac{1}{-2 \sin^2 \frac{x}{2}} \cdot \frac{-2 \sin^2 \frac{x}{2}}{x^2}} \\ &= \left\{ \lim_{x \rightarrow 0} \left(1 - 2 \sin^2 \frac{x}{2} \right)^{\frac{1}{-2 \sin^2 \frac{x}{2}}} \right\}^{-\frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2} = e^{-\frac{1}{2}}. \square \end{aligned}$$



作业：

习题2.2 No. 3(单),7,8

习题2.3 No. 6(单), 7(单),8,9.