



Review

- 第一换元法(凑微分法)

$$\int f'(\varphi(x))\varphi'(x)dx = \int f'(\varphi(x))d\varphi(x) = f(\varphi(x)) + C$$

- 第二换元法

$$\begin{aligned}\int f'(u)du &\stackrel{u=\varphi(x)}{=} \int f'(\varphi(x))\varphi'(x)dx \\ &= g(x) + C = g(\varphi^{-1}(u)) + C\end{aligned}$$

- 分部积分法 $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$

- 联合积分法



§ 5. 有理函数与三角有理函数的不定积分

1. 有理函数 $\frac{p(x)}{q(x)}$ (p, q 为多项式)

Thm. 有理真分式 $\frac{p(x)}{q(x)}$ 可分解成最简分式之代数和:

(1) $q(x)$ 的一次 k 重因式 $(x-a)^k$ 对应 k 项

$$\frac{A_1}{x-a}, \frac{A_2}{(x-a)^2}, \dots, \frac{A_k}{(x-a)^k};$$

(2) $q(x)$ 的二次 k 重因式 $((x+a)^2+b^2)^k$ 对应 k 项

$$\frac{B_1x+C_1}{(x+a)^2+b^2}, \frac{B_2x+C_2}{((x+a)^2+b^2)^2}, \dots, \frac{B_kx+C_k}{((x+a)^2+b^2)^k}.$$



Remark. $\frac{Bx+C}{((x+a)^2+b^2)^k}$ 可以表示成形如 $\frac{x+a}{((x+a)^2+b^2)^k}$,
 $\frac{1}{((x+a)^2+b^2)^k}$ 的简单分式的代数和.

$$\bullet \int \frac{dx}{x-a} = \ln|x-a| + C,$$

$$\bullet \int \frac{dx}{(x-a)^k} = \frac{-1}{(k-1)(x-a)^{k-1}} + C, \quad (k > 1)$$



$$\bullet \int \frac{x+a}{(x+a)^2+b^2} dx = \frac{1}{2} \ln((x+a)^2 + b^2) + C,$$

$$\bullet \int \frac{x+a}{((x+a)^2+b^2)^k} dx = \frac{-1}{2(k-1)((x+a)^2+b^2)^{k-1}}, \quad (k > 1)$$

$$\bullet J_k = \int \frac{1}{((x+a)^2+b^2)^k} dx$$

$$J_1 = \frac{1}{b} \arctan \frac{x+a}{b} + C,$$

$$J_{k+1} = \frac{1}{2kb^2} \left((x+a)((x+a)^2+b^2)^{-k} + (2k-1)J_k \right).$$



$$\text{Ex. } I = \int \frac{dx}{1+x^2+x^4}.$$

解: $1+x^2+x^4 = (x^2+1)^2 - x^2 = (x^2+x+1)(x^2-x+1)$.

$$\begin{aligned}\frac{1}{1+x^2+x^4} &= \frac{ax+b}{x^2+x+1} + \frac{cx+d}{x^2-x+1} \\ \Rightarrow a &= b = d = \frac{1}{2}, c = -\frac{1}{2}.\end{aligned}$$

$$\frac{1}{1+x^2+x^4} = \frac{1}{2} \frac{x+1}{x^2+x+1} - \frac{1}{2} \frac{x-1}{x^2-x+1}.$$



$$\int \frac{dx}{1+x^2+x^4} = \frac{1}{2} \int \frac{x+1}{x^2+x+1} dx - \frac{1}{2} \int \frac{x-1}{x^2-x+1} dx$$

$$= \frac{1}{2} \int \frac{x+1/2}{x^2+x+1} dx + \frac{1}{4} \int \frac{dx}{x^2+x+1}$$

$$- \frac{1}{2} \int \frac{x-1/2}{x^2-x+1} dx + \frac{1}{4} \int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{4} \ln(x^2+x+1) + \frac{1}{2\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}}$$

$$- \frac{1}{4} \ln(x^2-x+1) + \frac{1}{2\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + C.$$



2. 三角有理式 $R(\sin x, \cos x)$: $\sin x, \cos x$ 有限次四则运算

万能变换 $t = \tan \frac{x}{2}$, $x = 2 \arctan t$, $dx = \frac{2}{1+t^2} dt$,

$$\sin x = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}.$$

$$\int R(\sin x, \cos x) dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}.$$



$$\text{Ex. } I = \int \frac{1 + \sin x}{1 + \cos x} dx.$$

解法一：令 $t = \tan \frac{x}{2}$, 则 $x = 2 \arctan t$, $dx = \frac{2}{1+t^2} dt$,

$$\begin{aligned} \int \frac{1 + \sin x}{1 + \cos x} dx &= \int \frac{1 + \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{(1+t)^2}{2} \cdot \frac{2}{1+t^2} dt \\ &= \int \left(1 + \frac{2t}{1+t^2}\right) dt = t + \ln(1+t^2) + C = \tan \frac{x}{2} - 2 \ln \left| \cos \frac{x}{2} \right| + C. \end{aligned}$$

解法二： $I = \int \frac{1}{2 \cos^2 \frac{x}{2}} dx + \int \frac{\sin x}{1 + \cos x} dx$
 $= \tan \frac{x}{2} - \ln(1 + \cos x) + C.$



Remark. 万能变换不一定简单.

$$\begin{aligned} \text{Ex. } & \int \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &= \int \frac{\tan x}{a^2 + b^2 \tan^2 x} \cdot \frac{dx}{\cos^2 x} = \int \frac{\tan x}{a^2 + b^2 \tan^2 x} d \tan x \\ &= \frac{1}{2} \int \frac{d \tan^2 x}{a^2 + b^2 \tan^2 x} = \frac{1}{2b^2} \ln(a^2 + b^2 \tan^2 x) + C. \end{aligned}$$



3. 可化为有理式的简单无理式

$$1) \int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx \quad (ad-bc \neq 0)$$

令 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$, 则 $x = \frac{b - dt^n}{ct^n - a}$, $dx = \frac{ad - bc}{(ct^n - a)^2} nt^{n-1} dt$,

$$\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx = \int R\left(\frac{b - dt^n}{ct^n - a}, t\right) \frac{ad - bc}{(ct^n - a)^2} nt^{n-1} dt.$$



$$\text{Ex. I} = \int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

解：令 $t = \sqrt[3]{\frac{x+1}{x-1}}$, 则 $x = \frac{t^3 + 1}{t^3 - 1}$, $dx = \frac{-6t^2}{(t^3 - 1)^2} dt$.

$$I = \int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1} = \int \frac{-3}{t^3 - 1} dt = \int \left(\frac{-1}{t-1} + \frac{t+2}{t^2 + t + 1} \right) dt$$

$$= -\ln|t-1| + \frac{1}{2} \ln(t^2 + t + 1) + \sqrt{3} \arctan \frac{2t+1}{\sqrt{3}} + C$$

$$= \frac{1}{2} \ln|x-1| - \frac{3}{2} \ln \left| \sqrt[3]{\frac{x+1}{x-1}} - 1 \right| + \sqrt{3} \arctan \frac{2}{\sqrt{3}} \left(\sqrt[3]{\frac{x+1}{x-1}} - \frac{1}{2} \right) + C.$$



$$2) \int R(x, \sqrt{ax^2 + bx + c}) dx, (a \neq 0)$$

- $\int R(x, \sqrt{(x+p)^2 + q^2}) dx$ 令 $x+p = q \tan t$
- $\int R(x, \sqrt{(x+p)^2 - q^2}) dx$ 令 $x+p = q \sec t$
- $\int R(x, \sqrt{q^2 - (x+p)^2}) dx$ 令 $x+p = q \sin t$

Euler第一替换: 若 $a > 0$, 令 $\sqrt{ax^2 + bx + c} = \pm\sqrt{ax + t}$.

$$\text{此时, } x = \frac{t^2 - c}{b \mp 2\sqrt{at}}, \quad \sqrt{ax^2 + bx + c} = \frac{\sqrt{a}(t^2 - c)}{b \mp 2\sqrt{at}} + t.$$



Euler第二替换: 若 $c > 0$, 令 $\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$.

$$\text{此时, } x = \frac{-b \pm 2\sqrt{ct}}{a - t^2}, \sqrt{ax^2 + bx + c} = \frac{-b \pm 2\sqrt{ct}}{a - t^2} t \pm \sqrt{c}.$$

Euler第三替换: 若 $ax^2 + bx + c$ 有相异实根 α, β , 令

$$\sqrt{ax^2 + bx + c} = (x - \alpha)t,$$

$$\text{此时, } a(x - \alpha)(x - \beta) = (x - \alpha)^2 t^2, \quad x = \frac{a\beta - \alpha t^2}{a - t^2}.$$



$$\text{Ex. } I = \int \frac{dx}{x^2 \sqrt{4x^2 - 3x - 1}}$$

解法一： $4x^2 - 3x - 1 = (4x + 1)(x - 1)$, 令

$$\sqrt{(4x + 1)(x - 1)} = (4x + 1)t,$$

则 $x = \frac{t^2 + 1}{1 - 4t^2}$, $dx = \frac{10t dt}{(1 - 4t^2)^2}$, $\sqrt{4x^2 - 3x - 1} = \frac{5t}{1 - 4t^2}$,

$$I = \int \frac{2(1 - 4t^2)dt}{(t^2 + 1)^2} = -8 \int \frac{dt}{t^2 + 1} + 10 \int \frac{dt}{(t^2 + 1)^2} = -8I_1 + 10I_2,$$



$$\arctan t = I_1 = \frac{t}{t^2 + 1} + \int \frac{2t^2 dt}{(t^2 + 1)^2} = \frac{t}{t^2 + 1} + 2 \arctan t - 2I_2,$$

$$I_2 = \frac{1}{2} \left(\frac{t}{t^2 + 1} + \arctan t \right),$$

$$\begin{aligned} I &= -3 \arctan t + \frac{5t}{t^2 + 1} + C \\ &= -3 \arctan \frac{\sqrt{4x^2 - 3x - 1}}{4x + 1} + \frac{5\sqrt{4x^2 - 3x - 1}}{x} + C. \end{aligned}$$



解法二：令 $\sqrt{4x^2 - 3x - 1} = -2x + t$, 则 $x = \frac{1+t^2}{4t-3}$,

$$dx = \frac{2(2t^2 - 3t - 2)}{(4t-3)^2} dt, \sqrt{4x^2 - 3x - 1} = \frac{2t^2 - 3t - 2}{4t-3},$$

$$I = 2 \int \frac{(4t-3)dt}{(1+t^2)^2} = 4 \int \frac{d(t^2+1)}{(1+t^2)^2} - 6 \int \frac{dt}{(1+t^2)^2}$$

$$= -\frac{4}{1+t^2} - \frac{3t}{t^2+1} - 3 \arctan t + C$$

$$= \frac{\sqrt{4x^2 - 3x - 1} - 2x}{x} - 3 \arctan(2x + \sqrt{4x^2 - 3x - 1}) + C$$



Remark. 初等函数的原函数不一定是初等函数, 如

$$e^{x^2}, \sin x^2, \cos x^2, \frac{\sin x}{x}, \frac{\cos x}{x},$$

$$\frac{1}{\ln x}, \sqrt{1 - k^2 \sin^2 x} \quad (0 < k < 1).$$



作业：习题5.5 No.1- 4(单)