Review

定积分的计算

- $\int_a^b F'(x)dx = F(x)\Big|_{x=a}^b$
- $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$

  $\varphi(\alpha)=a, \varphi(\beta)=b, a\le\varphi(t)\le b, \varphi\in C^1[\alpha,\beta].$
- $\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx.$

带积分余项的Taylor公式

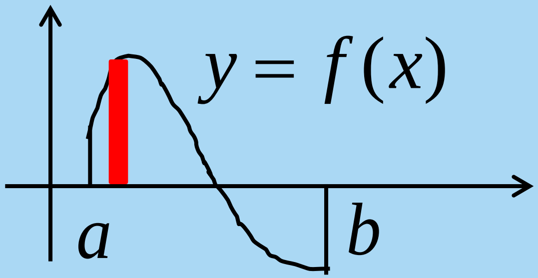
$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt.$$

§ 7.积分的应用 --微元法

- 平面区域的面积
- 曲线的弧长
- 平面曲线的曲率
- 旋转体的体积
- 旋转面的面积
- 积分在物理中的应用

- 平面图形的面积

1) $y=f(x),\ y=0,\ x\in[a,b].$



Riemann积分四部曲:

分割、取点、近似和、极限

$$S=\int_a^b |f(x)|\,dx$$

微元法: $[x,x+\Delta x]$对应窄条的面积

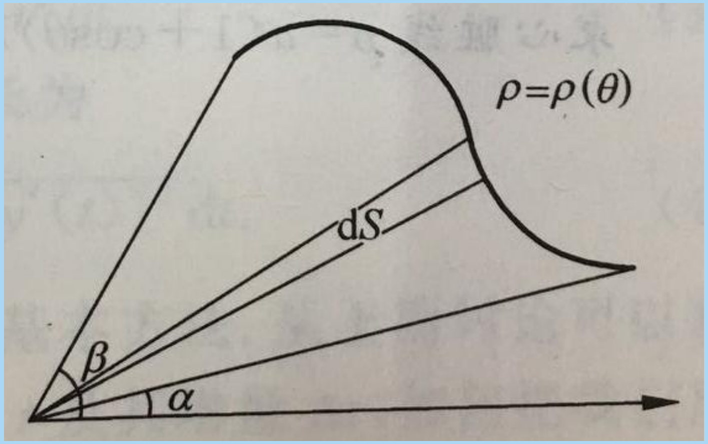
$$\Delta S\approx|f(x)|\Delta x,\quad \Delta S=|f(x)|\Delta x+o(\Delta x)$$

$$dS=|f(x)|\,dx.$$

2) $y=f_1(x), y=f_2(x), x\in[a,b].$

$$S=\int_a^b\left|f_1(x)-f_2(x)\right|dx$$

3) $\rho=\rho(\theta), \theta\in[\alpha,\beta].$



（注意ρ和θ的几何意义！）

微元法: $\Delta S\approx\dfrac{1}{2}\rho^2(\theta)\Delta\theta$

$$\Delta S=\frac{1}{2}\rho^2(\theta)\Delta\theta+o(\Delta\theta)$$

$$dS=\frac{1}{2}\rho^2(\theta)d\theta, \qquad S=\int_\alpha^\beta\frac{1}{2}\rho^2(\theta)d\theta.$$

Ex. 求 $x^{2/3}+y^{2/3}=a^{2/3}(a>0)$所围区域的面积.

解法一: 曲线关于 x 轴和 y 轴对称,因此 $S=4\int_0^a y(x)dx$.

第一象限中曲线有参数方程

$$x=a\sin^3 t, y=a\cos^3 t, t\in[0,\frac{\pi}{2}].$$

故

$$S=4\int_0^{\pi/2} a\cos^3 t\cdot 3a\sin^2 t\cos t\,\mathrm{d}t$$

$$=3a^2\int_0^{\pi/2}\sin^2 2t\cos^2 t\,\mathrm{d}t$$

$$=\frac{3}{4}a^2\int_0^{\pi/2}(1-\cos 4t)(1+\cos 2t)\,\mathrm{d}t=\frac{3}{8}\pi a^2.$$

解法二：曲线有参数方程$x=a\sin^3\theta, y=a\cos^3\theta, \theta\in[0,2\pi]$.

$$\rho^2(\theta)=x^2(\theta)+y^2(\theta)=a^2(\sin^6\theta+\cos^6\theta)$$

$$=a^2(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta)$$

$$=a^2(1-\frac{3}{4}\sin^2 2\theta)$$

$$S=\frac{1}{2}\int_0^{2\pi}\rho^2(\theta)\,\mathrm{d}\theta=2a^2\int_0^{\pi/2}(1-\frac{3}{4}\sin^2 2\theta)\,\mathrm{d}\theta$$

$$=\pi a^2-\frac{3}{4}a^2\int_0^{\pi/2}(1-\cos 4\theta)\,\mathrm{d}\theta=\frac{5}{8}\pi a^2.$$ ✗

参数方程中的θ并非辐角！

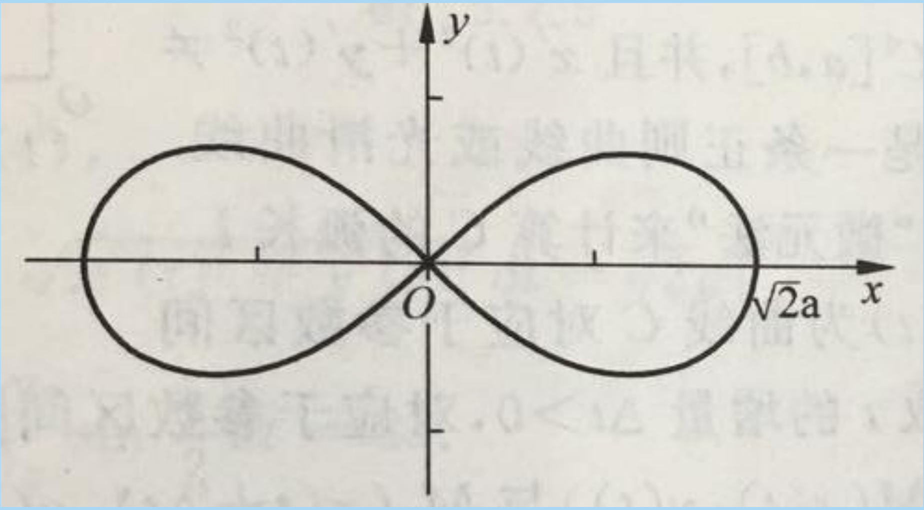
Ex. 求双纽线$\rho^2=2a^2\cos 2\theta$所围区域的面积S.

解: $\rho(-\theta)=\rho(\theta),\ \rho(\pi-\theta)=\rho(\theta)$, 故图像关于$x,y$轴对称.

$$2a^2\cos 2\theta=\rho^2\geq 0$$

$$\Rightarrow \cos 2\theta \geq 0$$

$$\Rightarrow \text{第一象限中}\ \theta\in[0,\frac{\pi}{4}]$$



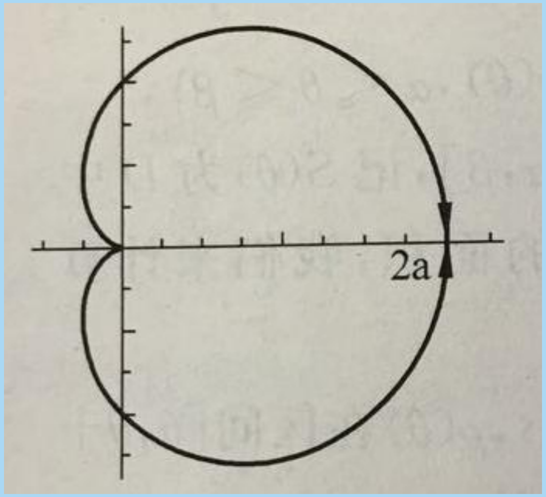
$$S=4S_1=4\int_0^{\pi/4}\frac{1}{2}\rho^2(\theta)d\theta=4\int_0^{\pi/4}a^2\cos 2\theta d\theta$$

$$=2a^2\sin 2\theta\Big|_0^{\pi/4}=2a^2.\square$$

Ex.求心脏线$\rho=a(1+\cos\theta)$所围区域的面积S.

解: $a(1+\cos\theta)=\rho\geq 0$

$$\Rightarrow \theta\in[-\pi,\pi].$$



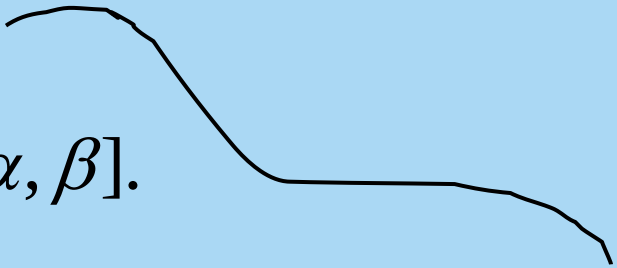
$$S=\int_{-\pi}^{\pi}\frac{1}{2}\rho^2(\theta)d\theta$$

$$=\int_0^{\pi}a^2(1+\cos\theta)^2d\theta$$

$$=\int_0^{\pi}a^2(1+2\cos\theta+\frac{1+\cos 2\theta}{2})d\theta$$

$$=\frac{3}{2}\pi a^2.\square$$

- 曲线的弧长



$$L:\ x=x(t), y=y(t), z=z(t), \quad t\in[\alpha,\beta].$$

考虑$[t,t+\Delta t]$对应的弧段

$$\Delta l \approx \sqrt{(x(t+\Delta t)-x(t))^2+(y(t+\Delta t)-y(t))^2+(z(t+\Delta t)-z(t))^2}$$

$$=\sqrt{(x'(\xi))^2+(y'(\eta))^2+(z'(\varsigma))^2}\,\Delta t_i$$

$$\approx \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,\Delta t.$$

$$l=\int_\alpha^\beta \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.$$

弧长微元$dl=\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.$

Remark. • 平面曲线$L: x=x(t), y=y(t), \alpha\le t\le\beta$,的弧长

$$l=\int_\alpha^\beta\sqrt{(x'(t))^2+(y'(t))^2}\,dt.$$

• 曲线$L: y=f(x), a\le x\le b$,的弧长 $l=\int_a^b\sqrt{1+(f'(x))^2}\,dx.$

• $\rho=\rho(\theta), \alpha\le\theta\le\beta$,的弧长（注意$\rho$和$\theta$的几何意义！）

$$x=\rho(\theta)\cos\theta, y=\rho(\theta)\sin\theta,$$

$$\sqrt{(x'(\theta))^2+(y'(\theta))^2}=\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}$$

$$l=\int_\alpha^\beta\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}\,d\theta.$$

Ex.求心脏线$\rho = a(1+\cos\theta), -\pi \le \theta \le \pi$,的弧长$L$.

解: $x = \rho(\theta)\cos\theta = a(1+\cos\theta)\cos\theta,$

$$y = \rho(\theta)\sin\theta = a(1+\cos\theta)\sin\theta, \quad \theta \in [-\pi, \pi].$$

$$L = \int_{-\pi}^{\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}\, d\theta$$

$$= 2a\int_{-\pi}^{\pi} \cos\frac{\theta}{2}\, d\theta \ = 4a\sin\frac{\theta}{2}\Big|_{-\pi}^{\pi} = 8a. \square$$

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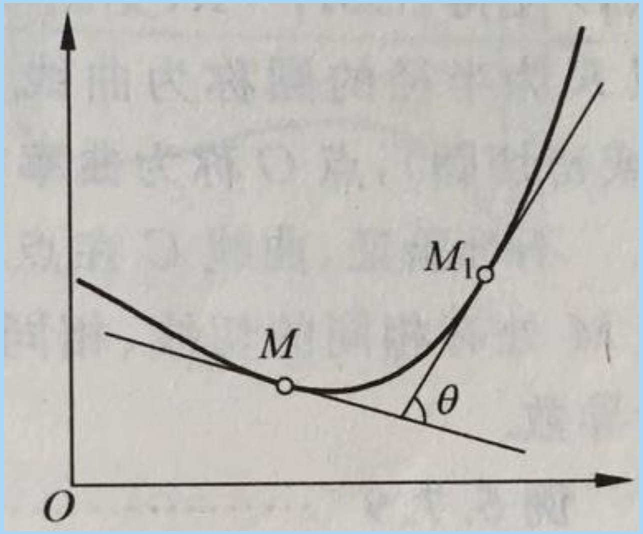
- 平面曲线的曲率 $L: \ x = x(t), y = y(t), \quad t \in [\alpha, \beta].$



$$M(x(t), y(t)), M_1(x(t+\Delta t), y(t+\Delta t)),$$

$$\sigma = MM_1 = \int_t^{t+\Delta t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2}\, d\tau$$

$$\theta = \arctan\frac{y'(t+\Delta t)}{x'(t+\Delta t)} - \arctan\frac{y'(t)}{x'(t)}$$

$$\text{曲率} k \triangleq \lim_{\Delta t \to 0} \frac{|\theta|}{\sigma} = \lim_{\Delta t \to 0} \frac{|\theta / \Delta t|}{\sigma / \Delta t} = \frac{\left| \frac{d}{dt}\left(\arctan\frac{y'(t)}{x'(t)} \right) \right|}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

$$= \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left((x'(t))^2 + (y'(t))^2 \right)^{3/2}}.$$

Remark. $y=f(x)$在点x处的曲率为$k=\dfrac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$.

Ex. 求$x=R\cos t, y=R\sin t, 0\le t\le 2\pi$的曲率.

解: $$k=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left((x'(t))^2+(y'(t))^2\right)^{3/2}}=\frac{1}{R}.\square$$

Def. 曲线C在点M处的曲率k的倒数$R=1/k$称为C在点M的曲率半径. C的凹侧与C相切的半径为R的圆称为C在点M的曲率圆(密切圆), 曲率圆的圆心称为曲率中心.

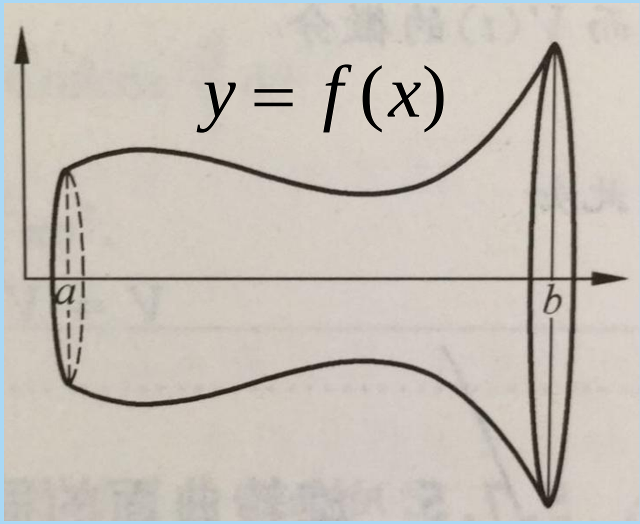
Remark. 曲线的曲率圆与曲线在切点处有相同的切线、曲率与二阶导数.

- 旋转体的体积

曲线 $y=f(x), a\le x\le b$,

绕 x 轴旋转得旋转体 Ω.

求$\mathrm{V}(\Omega)$.



微元法

$[x, x+\Delta x]$对应的薄片体积：$\Delta\mathrm{V}\approx \pi f^2(x)\Delta x$,

$$\mathrm{V}(\Omega)=\pi\int_a^b f^2(x)dx.$$

Remark. $x=f(y), c\le y\le d$, 绕y轴旋转得旋转体Ω的体积

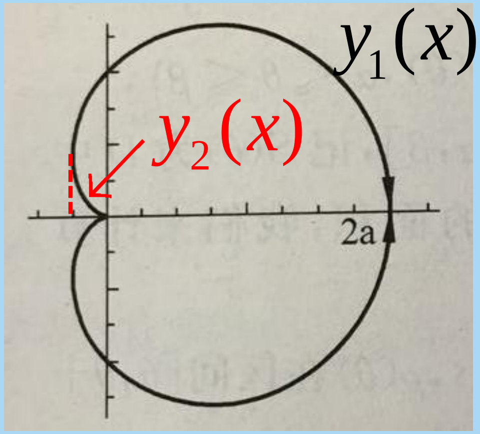
$$\mathrm{V}(\Omega)=\pi\int_c^d f^2(y)dy.$$

Ex. 求心脏线$\rho = a(1+\cos\theta)$绕 x 轴旋转所得旋转体的体积.

解: 心脏线关于 x 轴对称,因此只需考虑$0 \le \theta \le \pi$.

$$x = a(1+\cos\theta)\cos\theta$$

$$= a\left((\cos\theta + \frac{1}{2})^2 - \frac{1}{4}\right) \ge -\frac{1}{4}a.$$



$$V = \int_{-a/4}^{a} \pi y_1^2(x)\,dx - \int_{-a/4}^{0} \pi y_2^2(x)\,dx$$

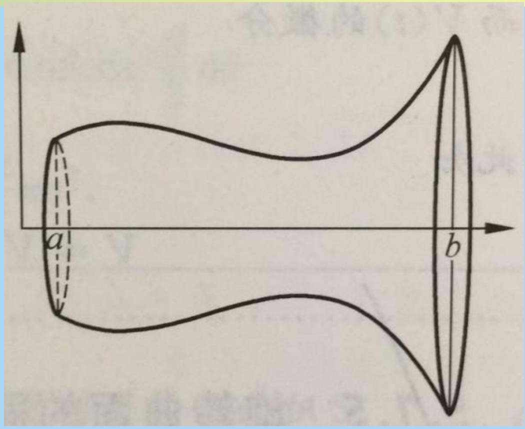
$$= \pi\int_{2\pi/3}^{0} y_1^2(\theta)x'(\theta)d\theta - \pi\int_{2\pi/3}^{\pi} y_1^2(\theta)x'(\theta)d\theta$$

$$= \pi\int_{\pi}^{0} a^2(1+\cos\theta)^2\sin^2\theta \cdot a(\cos\theta+\cos^2\theta)'d\theta = \frac{8}{3}a^3. \square$$

- 旋转面的面积

(1)求曲线$x=x(t), y=y(t), a\le t\le b$,

绕x轴旋转得旋转面Σ的面积$S(\Sigma)$.



微元法: $[t, t+\Delta t]$对应段弧σ的弧长为

$$\int_t^{t+\Delta t}\sqrt{(x'(\tau))^2+(y'(\tau))^2}\,d\tau \approx \sqrt{(x'(t))^2+(y'(t))^2}\,\Delta t$$

σ绕x轴旋转所得曲面面积 $\Delta S \approx 2\pi|y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,\Delta t$

$$S=2\pi\int_a^b |y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,dt.$$

(2)曲线$y=f(x), a\le x\le b$,绕x轴旋转得旋转面面积为

$$S=2\pi\int_a^b |y(x)|\sqrt{1+(f'(x))^2}\,dx.$$

Ex. 求心脏线$\rho = a(1+\cos\theta), a>0, 0\le\theta\le 2\pi$绕$x$轴旋转一周所得旋转面的面积$S$.

解: 心脏线关于x轴对称, 只需考虑$0\le\theta\le\pi$.

$$x=\rho\cos\theta, y=\rho\sin\theta.$$

$$S = 2\pi\int_0^{\pi} |y(\theta)|\sqrt{(x'(\theta))^2+(y'(\theta))^2}\,d\theta$$

$$= 2\pi\int_0^{\pi} \rho\sin\theta\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}\,d\theta$$

$$= 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\cdot 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta = \frac{32\pi a^2}{5}. \square$$

- 积分在物理中的应用(功, 质量, 质心, 引力)

Ex. $C: x = x(t), y = y(t), a \le t \le b$. C上点$M(x(t), y(t))$处密度为$\rho(t)$.求C的质量m与重心坐标$(\bar{x}, \bar{y})$.

分析: 平面质点系$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$的质量分别为$m_1, m_2, \cdots, m_n$.其重心坐标$(\bar{x}, \bar{y})$为

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}.$$

对C进行分割, 近似为有限个质点.

解: 分析$[t,t+\Delta t]$对应的一小段弧.

弧长微元$dl=\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

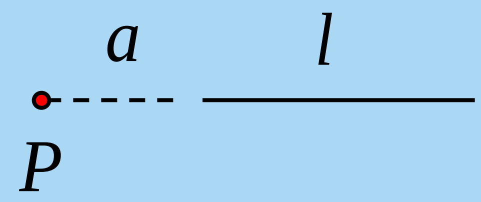
质量微元$dm=\rho(t)dl=\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt.$

因而 $$m=\int_a^b \rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt.$$

$$\bar{x}=\frac{\int_a^b x(t)\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt}{m},$$

$$\bar{y}=\frac{\int_a^b y(t)\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt}{m}.\square$$

Ex.质量为M长度为l的均匀细杆,对其延长线上距离a处质量为m的质点P的引力F.



解:取P为坐标原点,细杆所在直线为x轴.

考虑细杆上一小段$[x, x+\Delta x]$, 视之为质点, 其质量为

$\dfrac{M}{l}\Delta x$, 它对P的引力为$\Delta F \approx k\dfrac{Mm\Delta x}{lx^2}$, k为引力常数, 故

$$F = k\int_a^{a+l}\frac{Mm}{lx^2}dx = -\frac{kMm}{lx}\bigg|_a^{a+l} = \frac{kMm}{a(a+l)}.\square$$

作业：习题5.7

No.2(2,4,5), 3(1,5), 6,

7(4,5), 8(2), 9(2).