



Review

• Thm.(微积分基本定理)

$f \in R[a,b]$, $F(x) = \int_a^x f(t)dt$ ($a \leq x \leq b$), 则

- (1) $F \in C[a,b]$;
- (2) 若 f 在 $x_0 \in [a,b]$ 连续, 则 F 在 x_0 可导, 且 $F'(x_0) = f(x_0)$;
- (3) 若 $f \in C[a,b]$, 则 F 是 f 在 $[a,b]$ 上的一个原函数. 若 G 为 f 的任一个原函数, 则

$$\int_a^b f(t)dt = G(b) - G(a) \triangleq G(x) \Big|_a^b. \quad (\text{Newton-Leibniz})$$

$$\bullet F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$



§ 4. 不定积分换元法与分部积分

1. 换元积分法

$$\begin{aligned}\frac{d}{dx} f(\varphi(x)) &= f'(\varphi(x)) \cdot \varphi'(x) \\ \Rightarrow \int f'(\varphi(x)) \varphi'(x) dx &= f(\varphi(x)) + C.\end{aligned}$$

与 $\int f'(u) du = f(u) + C$ 比较, 得

$$\int f'(\varphi(x)) \varphi'(x) dx = \int f'(\varphi(x)) d\varphi(x) = f(\varphi(x)) + C$$

$$\underline{\underline{u = \varphi(x)}} \int f'(u) du = f(u) + C = f(\varphi(x)) + C$$

——第一换元法(凑微分法)

$$\begin{aligned}\int f'(u) du &\underline{\underline{u = \varphi(x)}} \int f'(\varphi(x)) \varphi'(x) dx \quad \text{——第二换元法} \\ &= g(x) + C = g(\varphi^{-1}(u)) + C\end{aligned}$$



Ex. $\int 2xe^{x^2} dx = \int e^{x^2} dx^2 = e^{x^2} + C$

Ex. $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d \sin x = \ln |\sin x| + C.$

Ex. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d(x/a)}{\sqrt{1 - (x/a)^2}} = \arcsin \frac{x}{a} + C, (a > 0).$

Ex. $\int \frac{dx}{x(1+x^5)} = \int \frac{x^4 dx}{x^5(1+x^5)} = \frac{1}{5} \int \frac{dx^5}{x^5(1+x^5)}$

$$= \frac{1}{5} \int \left(\frac{1}{x^5} - \frac{1}{1+x^5} \right) dx^5 = \ln|x| - \frac{1}{5} \ln|1+x^5| + C.$$



Ex. $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{d \sin x}{1 - \sin^2 x}$

$$= \frac{1}{2} \int \left(\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \right) d \sin x$$
$$= \frac{1}{2} \ln(1 + \sin x) - \frac{1}{2} \ln(1 - \sin x) + C$$
$$= \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C$$
$$= \ln |\sec x + \tan x| + C$$



Ex. $\int \sqrt{a^2 - x^2} dx (a > 0)$

解: 令 $x = a \sin t, |t| \leq \frac{\pi}{2}$, 则 $dx = a \cos t dt, \sqrt{a^2 - x^2} = a \cos t$,

$$\begin{aligned}\int \sqrt{a^2 - x^2} dx &= \int a^2 \cos^2 t dt = \frac{1}{2} a^2 \int (1 + \cos 2t) dt \\ &= \frac{1}{2} a^2 \left(t + \frac{1}{2} \sin 2t \right) + C\end{aligned}$$

$$t = \arcsin \frac{x}{a}, \quad \frac{1}{2} \sin 2t = \sin t \sqrt{1 - \sin^2 t} = \frac{x}{a^2} \sqrt{a^2 - x^2}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C. \square$$



Ex. $\int \frac{dx}{1 + \sqrt[3]{1+x}}$

解: 令 $t = \sqrt[3]{1+x}$, 则 $x = t^3 - 1$, $dx = 3t^2 dt$,

$$\begin{aligned} \int \frac{dx}{1 + \sqrt[3]{1+x}} &= 3 \int \frac{t^2 dt}{1+t} = 3 \int \left(t - 1 + \frac{1}{1+t} \right) dt \\ &= 3 \left(\frac{1}{2} t^2 - t + \ln|1+t| \right) + C \\ &= 3 \left(\frac{1}{2} \sqrt[3]{(1+x)^2} - \sqrt[3]{1+x} + \ln \left| 1 + \sqrt[3]{1+x} \right| \right) + C. \end{aligned}$$

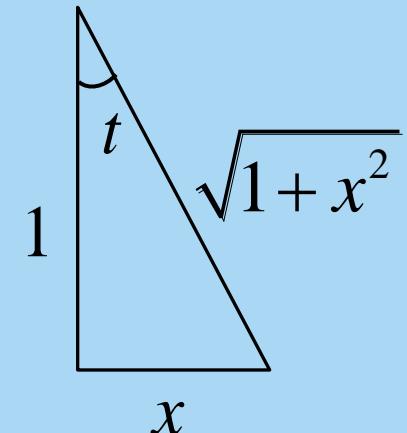


Ex. $\int \frac{dx}{x^2 \sqrt{1+x^2}}$

解: 令 $x = \tan t, |t| < \frac{\pi}{2}$,

$$\int \frac{dx}{x^2 \sqrt{1+x^2}} = \int \frac{\sec^2 t dt}{\tan^2 t \sec t}$$

$$= \int \frac{\cos t dt}{\sin^2 t} = \int \frac{d \sin t}{\sin^2 t} = -\frac{1}{\sin t} + C = -\frac{\sqrt{1+x^2}}{x} + C.$$





2. 分部积分法

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

$$\Rightarrow \int u'(x)v(x)dx + \int u(x)v'(x)dx = \int (u(x)v(x))' dx$$

$$\Rightarrow \int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

也记作 $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x).$

——分部积分法



Ex. $\int x \cos x dx = \int x d(\sin x)$

$$= x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

Ex. $\int \ln x dx = x \ln x - \int x d(\ln x)$

$$= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$$

Ex. $\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$

$$= x \arcsin x - \frac{1}{2} \int \frac{dx^2}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C.$$



Ex. $\int \sqrt{x^2 + a^2} dx = x\sqrt{x^2 + a^2} - \int \frac{x \cdot 2x}{2\sqrt{x^2 + a^2}} dx$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2} dx + \int \frac{a^2}{\sqrt{x^2 + a^2}} dx$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \int \frac{1}{\sqrt{x^2 + a^2}} dx$$

$$= \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left(x + \sqrt{x^2 + a^2}\right) + C.$$

Ex. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + C.$



Ex. $J_k = \int ((x+a)^2 + b^2)^{-k} dx$

解: $J_1 = \int \frac{1}{(x+a)^2 + b^2} dx = \frac{1}{b} \int \frac{1}{\left(\frac{x+a}{b}\right)^2 + 1} d\frac{x+a}{b}$
 $= \frac{1}{b} \arctan \frac{x+a}{b} + C.$

$$\begin{aligned} J_k &= \int ((x+a)^2 + b^2)^{-k} d(x+a) \\ &= \frac{x+a}{((x+a)^2 + b^2)^k} + 2k \int \frac{(x+a)^2 + b^2 - b^2}{((x+a)^2 + b^2)^{k+1}} d(x+a) \\ &= (x+a)((x+a)^2 + b^2)^{-k} + 2kJ_k - 2kb^2J_{k+1}. \end{aligned}$$

$$J_{k+1} = \frac{1}{2kb^2} \left((x+a)((x+a)^2 + b^2)^{-k} + (2k-1)J_k \right). \square$$



Ex. $I = \int e^{ax} \cos bx dx$.

解法一: $I = \frac{1}{b} \int e^{ax} d \sin bx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx dx$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} \int e^{ax} d \cos bx$$
$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} \left(e^{ax} \cos bx - a \int e^{ax} \cos bx dx \right)$$
$$= \frac{e^{ax}(b \sin bx + a \cos bx)}{b^2} - \frac{a^2}{b^2} I$$
$$I = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2} + C.$$



解法二：定义 $e^{(a+ib)} = e^a(\cos b + i \sin b)$, $\forall a, b \in \mathbb{R}$;

$$(x(t) + iy(t))' = x'(t) + iy'(t);$$

$$\int (x(t) + iy(t)) dt = \int x(t) dt + i \int y(t) dt.$$

可以验证 $e^{\lambda_1 + \lambda_2} = e^{\lambda_1} \cdot e^{\lambda_2}$, $\forall \lambda_1, \lambda_2 \in \mathbb{C}$; $(e^{\lambda t})' = \lambda e^{\lambda t}$, $\forall \lambda \in \mathbb{C}$;

$$\int e^{(a+ib)x} dx = \frac{1}{a+ib} e^{(a+ib)x} + C = \frac{(a-ib)}{a^2+b^2} e^{ax} (\cos bx + i \sin bx) + C$$

比较实部虚部, 得

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + C,$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + C. \quad \square$$



3. 联合积分法

Ex. $I = \int e^{ax} \cos bx dx, J = \int e^{ax} \sin bx dx.$

解: $I = \frac{1}{b} \int e^{ax} d \sin bx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx dx$

$$J = -\frac{1}{b} \int e^{ax} d \cos bx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int \cos bx e^{ax} dx$$

即 $I = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} J, \quad J = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I.$

联立得 $I = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2} + C,$

$$J = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C. \square$$



$$\text{Ex. } I = \int \frac{dx}{1+x^2+x^4}, J = \int \frac{x^2 dx}{1+x^2+x^4}.$$

解：

$$I + J = \int \frac{(1+x^2)dx}{1+x^2+x^4} = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2 + \frac{1}{x^2} + 1} = \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 3} = A(x) + C,$$

$$\text{其中, } A(x) = \begin{cases} \frac{1}{\sqrt{3}} \arctan \frac{x^2 - 1}{\sqrt{3}x} + \frac{\pi}{\sqrt{3}}, & x > 0 \\ \frac{\pi}{2\sqrt{3}}, & x = 0 \\ \frac{1}{\sqrt{3}} \arctan \frac{x^2 - 1}{\sqrt{3}x}, & x < 0 \end{cases}$$



$$\begin{aligned} J - I &= \int \frac{(x^2 - 1)dx}{1 + x^2 + x^4} = \int \frac{\left(1 - \frac{1}{x^2}\right)dx}{x^2 + \frac{1}{x^2} + 1} = \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} \\ &= \frac{1}{2} \ln \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| + C = \frac{1}{2} \ln \frac{x^2 - x + 1}{x^2 + x + 1} + C \end{aligned}$$

$$I = \frac{1}{2} A(x) - \frac{1}{4} \ln \frac{x^2 - x + 1}{x^2 + x + 1} + C, \quad J = \frac{1}{2} A(x) + \frac{1}{4} \ln \frac{x^2 - x + 1}{x^2 + x + 1} + C. \quad \square$$

Remark. 通常记为 $I = \frac{1}{\sqrt{3}} \arctan \frac{x^2 - 1}{\sqrt{3}x} - \frac{1}{4} \ln \frac{x^2 - x + 1}{x^2 + x + 1} + C.$



$$\text{Ex. } I = \int \frac{\cos^3 x dx}{\cos x + \sin x}, J = \int \frac{\sin^3 x dx}{\cos x + \sin x}.$$

$$\text{解: } I + J = \int \left(1 - \frac{1}{2} \sin 2x\right) dx = x + \frac{1}{4} \cos 2x + C$$

$$I - J = \int \frac{(\cos x - \sin x)\left(1 + \frac{1}{2} \sin 2x\right) dx}{\cos x + \sin x}$$

$$\begin{aligned} &= \int \frac{\cos 2x \cdot \left(1 + \frac{1}{2} \sin 2x\right) dx}{1 + \sin 2x} = \frac{1}{2} \int \frac{\left(1 + \frac{1}{2} \sin 2x\right) d \sin 2x}{1 + \sin 2x} \\ &= \frac{1}{4} \int \left(1 + \frac{1}{1 + \sin 2x}\right) d \sin 2x = \frac{1}{4} \sin 2x + \frac{1}{4} \ln(1 + \sin 2x) + C. \quad \square \end{aligned}$$



作业：习题5.4 No.3-7(单)