Review

Thm.(Fermat)x_0是f的极值点,$f'(x_0)$存在,则$f'(x_0)=0$.

Thm.(Darboux)f在$[a,b]$上可导,$f'_+(a) \neq f'_-(b)$,则对介于$f'_+(a)$与$f'_-(b)$之间的任意实数λ,$\exists \xi \in (a,b), s.t. f'(\xi) = \lambda$.

Thm.(Rolle)$f \in C[a,b]$,f在(a,b)可导.若$f(a)=f(b)$,则存在$\xi \in (a,b), s.t. f'(\xi)=0$.

Thm.(Cauchy)$f,g \in C[a,b]$,f,g在(a,b)可导,且$\forall t \in (a,b)$,有$g'(t) \neq 0$. 则存在$\xi \in (a,b), s.t.$

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Thm.(Lagrange) $f \in C[a,b], f$ 在 (a,b) 可导, 则 $\exists \xi \in (a,b), s.t.$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Remark. $f \in C[a,b], f$ 在 (a,b) 可导, 则

(1) $\exists \xi \in (a,b), s.t. \quad f(b) - f(a) = f'(\xi)(b - a).$

(2) $\forall x, x_0 \in [a,b], \exists$ 介于 x 与 x_0 之间的 $\xi, s.t.$

$$f(x) - f(x_0) = f'(\xi)(x - x_0).$$

(3) $\forall x_0, x_0 + \Delta x \in [a,b], \exists \theta \in (0,1), s.t.$

$$f(x_0 + \Delta x) - f(x_0) = f'(x_0 + \theta \Delta x) \cdot \Delta x.$$

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§2. L'Hospital法则

Thm. f, g 在 $(x_0, x_0 + \rho)$ 中可导, $g'(x) \neq 0$, $\displaystyle\lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = A$, 则

$(1)(0/0型)\ \displaystyle\lim_{x \to x_0+} f(x) = \lim_{x \to x_0+} g(x) = 0 \Rightarrow \lim_{x \to x_0+} \frac{f(x)}{g(x)} = A.$

$(2)(\infty/\infty型)\ \displaystyle\lim_{x \to x_0+} g(x) = \infty \Rightarrow \lim_{x \to x_0+} \frac{f(x)}{g(x)} = A.$

Remark. (1)极限过程 $\displaystyle\lim_{x \to x_0+}$ 替换成 $\displaystyle\lim_{x \to x_0-}$ 或 $\displaystyle\lim_{x \to x_0}$, 定理仍成立.

(2)A替换成 $+\infty, -\infty$ 或 ∞, 定理仍然成立.

Question. $\lim\limits_{x \to x_0+} \dfrac{f(x)}{g(x)} = \mathrm{A}$,能否推出 $\lim\limits_{x \to x_0+} \dfrac{f'(x)}{g'(x)} = \mathrm{A}$？否！

反例：$\lim\limits_{x \to 0} \dfrac{x^2 \sin(1/x)}{x} = 0, \lim\limits_{x \to 0} \dfrac{\left(x^2 \sin(1/x)\right)'}{(x)'}$ 不存在.

Proof.(1) $\lim\limits_{x \to x_0+} f(x) = \lim\limits_{x \to x_0+} g(x) = 0,$ 不妨设 $f(x_0) = g(x_0) = 0.$

由Cauchy中值定理, $\exists \xi_x \in (x_0, x), s.t.$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi_x)}{g'(\xi_x)}, \ 且 \ x \to x_0^+ 时, \xi_x \to x_0^+.$$

于是 $\lim\limits_{x \to x_0+} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to x_0+} \dfrac{f'(\xi_x)}{g'(\xi_x)} = \lim\limits_{x \to x_0+} \dfrac{f'(x)}{g'(x)} = \mathrm{A}.$

$(2)\ \lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)}=\mathrm{A},\ 则\forall\varepsilon\in(0,1),\exists\delta>0,s.t.$

$$\left|\dfrac{f'(x)}{g'(x)}-\mathrm{A}\right|<\varepsilon,\quad\forall x\in(x_0,x_0+\delta).$$

由Cauchy中值定理, $\forall x\in(x_0,x_0+\delta),\exists\ {\color{red}\xi_x\in(x,x_0+\delta)},s.t.$

$$\dfrac{f(x)-f(x_0+\delta)}{g(x)-g(x_0+\delta)}=\dfrac{f'(\xi_x)}{g'(\xi_x)},\ \dfrac{\dfrac{{\color{red}f(x)}}{{\color{red}g(x)}}-\dfrac{f(x_0+\delta)}{g(x)}}{1-\dfrac{g(x_0+\delta)}{g(x)}}=\dfrac{f'(\xi_x)}{g'(\xi_x)},$$

$$\dfrac{f(x)}{g(x)}\ {\color{red}-\mathrm{A}}\ =\dfrac{f(x_0+\delta)}{g(x)}+\dfrac{f'(\xi_x)}{g'(\xi_x)}\ {\color{red}-\mathrm{A}}\ -\dfrac{f'(\xi_x)}{g'(\xi_x)}\dfrac{g(x_0+\delta)}{g(x)},$$

$$\left| \frac{f(x)}{g(x)} - A \right| \leq \left| \frac{f(x_0 + \delta)}{g(x)} \right| + \left| \frac{f'(\xi_x)}{g'(\xi_x)} - A \right| + \left| \frac{f'(\xi_x)}{g'(\xi_x)} \right| \cdot \left| \frac{g(x_0 + \delta)}{g(x)} \right|$$

$$\lim_{x \to x_0+} g(x) = \infty, \text{则} \exists 0 < \delta_1 < \delta, s.t.$$

$$\left| \frac{f(x_0 + \delta)}{g(x)} \right| < \varepsilon, \quad \left| \frac{g(x_0 + \delta)}{g(x)} \right| < \varepsilon, \quad \forall x \in (x_0, x_0 + \delta_1).$$

因此，
$$\left| \frac{f(x)}{g(x)} - A \right| \leq \varepsilon + \varepsilon + \left(|A| + \varepsilon \right) \varepsilon < \left(|A| + 3 \right) \varepsilon,$$
$$\forall x \in (x_0, x_0 + \delta_1).$$

故 $\lim_{x \to x_0+} \frac{f(x)}{g(x)} = A.$ □

Thm. f, g 在 $(a, +\infty)$ 中可导, $g'(x) \neq 0$, $\lim\limits_{x \to +\infty} \dfrac{f'(x)}{g'(x)} = A$, 则

(1)(0 / 0型) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = 0 \Rightarrow \lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = A.$

(2)(∞ / ∞型) $\lim\limits_{x \to +\infty} g(x) = \infty \Rightarrow \lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = A.$

Remark.(1)极限过程 $\lim\limits_{x \to +\infty}$ 替换成 $\lim\limits_{x \to -\infty}$ 或 $\lim\limits_{x \to \infty}$, 定理仍成立.

(2)A替换成 $+\infty, -\infty$ 或 ∞, 定理仍然成立.

$Proof.$ 不妨设 $a > 0.$ 令 $\varphi(t) = f(\frac{1}{t}), \psi(t) = g(\frac{1}{t}), t \in (0, \frac{1}{a}).$

$$\lim_{t \to 0+} \varphi(t) = \lim_{x \to +\infty} f(x), \quad \lim_{t \to 0+} \psi(t) = \lim_{x \to +\infty} g(x).$$

由上一定理,

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{t \to 0+} \frac{\varphi(t)}{\psi(t)} = \lim_{t \to 0+} \frac{\varphi'(t)}{\psi'(t)} = \lim_{t \to 0+} \frac{f'(1/t) \cdot \frac{-1}{t^2}}{g'(1/t) \cdot \frac{-1}{t^2}}$$

$$= \lim_{t \to 0+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = A. \square$$

Ex. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1.$

Ex. $a>1, b>0,$ 则 $\lim\limits_{x\to +\infty}\dfrac{x^b}{a^x}=\lim\limits_{x\to +\infty}\dfrac{bx^{b-1}}{a^x\ln a}=\cdots=0.$

Ex. $\lambda>0, P(x)=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x+a_0,$ 则

$$\lim\limits_{x\to \infty}\dfrac{P(x)}{e^{\lambda x}}=0.$$

Ex. $\alpha>0,$ 则 $\lim\limits_{x\to +\infty}\dfrac{\ln x}{x^\alpha}=\lim\limits_{x\to +\infty}\dfrac{1/x}{\alpha x^{\alpha-1}}=\lim\limits_{x\to +\infty}\dfrac{1}{\alpha x^\alpha}=0.$

$$\text{Ex.} \lim_{x\to\infty}(1+1/x)^{x^2}e^{-x} \boxed{= \lim_{x\to\infty}\left((1+1/x)^x\right)^x e^{-x} \overset{?}{=} \lim_{x\to\infty}e^x e^{-x}=1}$$

$$= \lim_{x\to\infty}\exp\left\{x^2\ln(1+1/x)-x\right\}$$

$$= \exp\left\{\lim_{x\to\infty}x^2\ln(1+1/x)-x\right\}$$

$$= \exp\left\{\lim_{t\to0}\frac{\ln(1+t)-t}{t^2}\right\} = \exp\left\{\lim_{t\to0}\frac{1/(1+t)-1}{2t}\right\}$$

$$= \exp\left\{\lim_{t\to0}\frac{-1/(1+t)^2}{2}\right\} = e^{-1/2}.$$

$$\text{Ex.} \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} (2 \cos x - x \sin x)} = 0.$$

法二: $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{x - \sin x}{x^2}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{2x} = \lim_{x \to 0} \frac{\sin x}{2} = 0.$$

Ex. $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$$

$$=\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=2\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{x^2}{3x^2}=\frac{1}{3}.$$

Remark. 适时分离！等价因子替换！

Remark. L'Hospital 法则并非万能！

Ex. 判断正误.

$$\lim_{x \to 0} \frac{\ln(1 + e^x)}{x} = \lim_{x \to 0} \frac{\left(\ln(1 + e^x)\right)'}{x'} = \lim_{x \to 0} \frac{\dfrac{e^x}{1 + e^x}}{1} = \frac{1}{2} \qquad \times$$

Ex. $\displaystyle\lim_{x \to 0+} x^{x^x - 1} = \lim_{x \to 0+} e^{(x^x - 1)\ln x} = \lim_{x \to 0+} e^{(e^{x \ln x} - 1)\ln x}$ 0^0 型

$$= \exp\left\{ \lim_{x \to 0+} (e^{x \ln x} - 1)\ln x \right\} = \exp\left\{ \lim_{x \to 0+} \frac{e^{x \ln x} - 1}{x \ln x} \cdot x \ln^2 x \right\}$$

$$= \exp\left\{ \lim_{x \to 0+} \frac{e^{x \ln x} - 1}{x \ln x} \cdot \lim_{x \to 0+} x \ln^2 x \right\} = e^{1 \cdot 0} = 1.$$

$$\text{Ex.} \lim_{x \to +\infty} \frac{x^{\ln x}}{(\ln x)^x} = \lim_{x \to +\infty} e^{(\ln x)^2 - x \ln \ln x} \qquad \frac{\infty}{\infty} \text{型}$$

$$= \exp\left\{ \lim_{x \to +\infty} \left((\ln x)^2 - x \ln \ln x \right) \right\}$$

$$= \exp\left\{ \lim_{x \to +\infty} x \left(\frac{(\ln x)^2}{x} - \ln \ln x \right) \right\}$$

$$= \exp\left\{ \lim_{x \to +\infty} x \cdot \lim_{x \to +\infty} \left(\frac{(\ln x)^2}{x} - \ln \ln x \right) \right\}$$

$$= \exp\left\{ +\infty \cdot \left(0 - \infty \right) \right\} = e^{-\infty} = 0.$$

$$\text{Ex.} \lim_{x \to +\infty} \left(\frac{a^x - 1}{(a-1)x} \right)^{1/x} \quad (a > 0, a \neq 1) \qquad \infty^0 \text{型}, 0^0 \text{型}$$

$$\text{解：原式} = \lim_{x \to +\infty} \exp \left\{ \frac{1}{x} \ln \frac{a^x - 1}{(a-1)x} \right\}$$

$$= \exp \left\{ \lim_{x \to +\infty} \frac{\ln |a^x - 1| - \ln |x| - \ln |a - 1|}{x} \right\}$$

$$= \exp \left\{ \lim_{x \to +\infty} \left(\frac{a^x \ln a}{a^x - 1} - \frac{1}{x} \right) \right\} = \begin{cases} \exp\{\ln a - 0\} = a, & (a > 1) \\ \exp\{0 - 0\} = 1, & (0 < a < 1) \end{cases}.$$

Ex. f 在 $(0, +\infty)$ 上可导, $a > 0$.

(1) $\lim\limits_{x \to +\infty} (af(x) + f'(x)) = l$, 则 $\lim\limits_{x \to +\infty} f(x) = l / a$.

(2) $\lim\limits_{x \to +\infty} (af(x) - f'(x)) = l$, $\left| f(x) \right| \le M$, 则 $\lim\limits_{x \to +\infty} f(x) = l / a$.

(3) $\lim\limits_{x \to +\infty} (af(x) + 2\sqrt{x} f'(x)) = l$, 则 $\lim\limits_{x \to +\infty} f(x) = l / a$.

Proof.(1)

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{ax} f(x)}{e^{ax}} = \lim_{x \to +\infty} \frac{e^{ax}(af(x) + f'(x))}{ae^{ax}}$$

$$= \lim_{x \to +\infty} \frac{af(x) + f'(x)}{a} = l / a.$$

(2) $\left| f(x) \right| \le M$,则

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{-ax} f(x)}{e^{-ax}} = \lim_{x \to +\infty} \frac{e^{-ax}(-af(x) + f'(x))}{-ae^{-ax}}$$

$$= \lim_{x \to +\infty} \frac{af(x) - f'(x)}{a} = \frac{l}{a}.$$

(3)

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{a\sqrt{x}} f(x)}{e^{a\sqrt{x}}} = \lim_{x \to +\infty} \frac{e^{a\sqrt{x}} \left(\frac{a}{2\sqrt{x}} f(x) + f'(x) \right)}{\frac{a}{2\sqrt{x}} e^{a\sqrt{x}}}$$

$$= \lim_{x \to +\infty} \frac{af(x) + 2\sqrt{x} f'(x)}{a} = \frac{l}{a}. \square$$

$$\text{Ex. } \lim_{x \to +\infty} \left(\tan \frac{\pi x}{2x+1} \right)^{1/x} \qquad \infty^0 \text{型}$$

$$= \exp\left\{ \lim_{x \to +\infty} \frac{1}{x} \ln \tan \frac{\pi x}{2x+1} \right\} = \exp\left\{ \lim_{x \to +\infty} \frac{\dfrac{\pi}{(2x+1)^2}}{\tan \dfrac{\pi x}{2x+1} \cdot \cos^2 \dfrac{\pi x}{2x+1}} \right\}$$

$$= \exp\left\{ \lim_{x \to +\infty} \frac{2\pi}{(2x+1)^2 \sin \dfrac{2\pi x}{2x+1}} \right\} = \exp\left\{ \lim_{x \to +\infty} \frac{2\pi}{(2x+1)^2 \sin \dfrac{\pi}{2x+1}} \right\}$$

$$= \exp\left\{ 2 \lim_{x \to +\infty} \frac{\dfrac{\pi}{2x+1}}{\sin \dfrac{\pi}{2x+1}} \cdot \lim_{x \to +\infty} \frac{1}{2x+1} \right\} = e^0 = 1. \square$$

$$\text{Ex.} \lim_{x \to 0} \frac{e^{(1+x)^{1/x}} - (1+x)^{e/x}}{x^2}$$

$$\frac{0}{0} \text{型}$$

$$\text{解:} \left(e^{(1+x)^{1/x}} \right)' = e^{(1+x)^{1/x}} \cdot \left((1+x)^{1/x} \right)'$$

$$= e^{(1+x)^{1/x}} \cdot \left(e^{\frac{1}{x}\ln(1+x)} \right)'$$

$$= e^{(1+x)^{1/x}} \cdot (1+x)^{1/x} \cdot \left(\frac{1}{x}\ln(1+x) \right)'$$

$$\left((1+x)^{e/x} \right)' = \left(e^{\frac{e}{x}\ln(1+x)} \right)' = e(1+x)^{e/x} \cdot \left(\frac{1}{x}\ln(1+x) \right)'$$

$$\lim_{x \to 0} \left(\frac{1}{x} \ln(1+x) \right)' = \lim_{x \to 0} \frac{\dfrac{x}{1+x} - \ln(1+x)}{x^2}$$

$$= \lim_{x \to 0} \frac{1 - \dfrac{1}{1+x} - \ln(1+x)}{x^2} = \lim_{x \to 0} \frac{\dfrac{1}{(1+x)^2} - \dfrac{1}{1+x}}{2x}$$

$$= \lim_{x \to 0} \frac{\dfrac{-x}{(1+x)^2}}{2x} = \lim_{x \to 0} \frac{\dfrac{-1}{(1+x)^2}}{2} = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{e^{(1+x)^{1/x}} - (1+x)^{e/x}}{x^2}$$

$$= \lim_{x \to 0} \frac{\left(e^{(1+x)^{1/x}}(1+x)^{1/x} - e(1+x)^{e/x}\right) \cdot \left(\dfrac{1}{x}\ln(1+x)\right)'}{2x}$$

$$= \lim_{x \to 0} \frac{\left(e^{(1+x)^{1/x}}(1+x)^{1/x} - e(1+x)^{e/x}\right)}{2x} \cdot \lim_{x \to 0} \left(\frac{1}{x}\ln(1+x)\right)'$$

$$= -\frac{1}{4} \lim_{x \to 0} \frac{\left(e^{(1+x)^{1/x}} (1+x)^{1/x} - e(1+x)^{e/x} \right)}{x}$$

$$= -\frac{1}{4} \lim_{x \to 0} \left(e^{(1+x)^{1/x}} (1+x)^{2/x} + e^{(1+x)^{1/x}} (1+x)^{1/x} - e^2 (1+x)^{e/x} \right)$$

$$\cdot \lim_{x \to 0} \left(\frac{1}{x} \ln(1+x) \right)'$$

$$= \frac{1}{8} \lim_{x \to 0} \left(e^e e^2 + e^e e - e^2 e^e \right) = \frac{1}{8} e^{e+1}. \square$$

作业：习题4.2 No. 2,3,4