Review

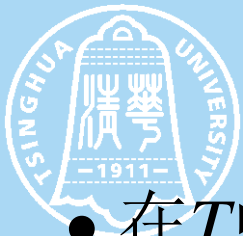- $\int_a^b f(x)dx = \mathrm{I}: \forall \varepsilon > 0, \exists \delta > 0, \text{当} |T| < \delta \text{时}, \text{无论} \xi_i \in [x_{i-1}, x_i]$ 如何取, 都有 $\left| \sum_{i=1}^n f(\xi_i)\Delta x_i - \mathrm{I} \right| < \varepsilon.$

- Darboux上和 $U(f,T) = \sum_{i=1}^n M_i \Delta x_i, \qquad M_i \triangleq \sup_{x \in [x_{i-1}, x_i]} f(x),$

  Darboux下和 $L(f,T) = \sum_{i=1}^n m_i \Delta x_i, \qquad m_i \triangleq \inf_{x \in [x_{i-1}, x_i]} f(x).$

  Riemann和 $\sigma(f,T) = \sum_{i=1}^n f(\xi_i)\Delta x_i.$

  $m(b-a) \le L(f,T) \le \sigma(f,T) \le U(f,T) \le M(b-a).$

- 在T中加入k个新分点得到T_k，则

$$0 \le U(f,T) - U(f,T_k) \le k|T|(M-m);$$

$$0 \le L(f,T_k) - L(f,T) \le k|T|(M-m).$$

- $L(f,T_1) \le U(f,T_2)$.

- Darboux上积分：$\displaystyle\overline{\int}_a^b f(x)dx = \inf\left\{U(f,T):T\text{为}[a,b]\text{的分割}\right\}$,

  Darboux下积分：$\displaystyle\underline{\int}_a^b f(x)dx = \sup\left\{L(f,T):T\text{为}[a,b]\text{的分割}\right\}$.

- $L(f,T) \le \displaystyle\underline{\int}_a^b f(x)dx \le \overline{\int}_a^b f(x)dx \le U(f,T)$.

- f 在 $[a,b]$ 有界, 则

$$f \in R[a,b];$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists [a,b] \text{的分割} T, s.t.\ U(f,T) - L(f,T) < \varepsilon;$$

$$\Leftrightarrow \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

- $[a,b]$ 上的可积函数类.

§2.Riemann积分的性质

Prop1. (线性性质)

$$f, g \in R[a,b], \alpha, \beta \in \mathbb{R} \Rightarrow \alpha f + \beta g \in R[a,b], \text{且}$$

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Proof. $\lim\limits_{|T|\to 0} \sigma(\alpha f + \beta g, T)$

$$= \lim\limits_{|T|\to 0} \alpha\sigma(f,T) + \lim\limits_{|T|\to 0} \beta\sigma(g,T). \square$$

Prop2. (积分区间的可加性) $a < b < c$, 则

$$f \in R[a,c] \Leftrightarrow f \in R[a,b] \,\&\, f \in R[b,c].$$

此时 $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$

Proof. $\Leftarrow$: 设 $f \in R[a,b], f \in R[b,c]. \forall \varepsilon > 0, \exists [a,b]$ 的分割 T_1, $[b,c]$ 的分割 $T_2, s.t.$

$$U(f,T_1) - L(f,T_1) < \varepsilon, \quad U(f,T_2) - L(f,T_2) < \varepsilon.$$

合并 T_1, T_2 的分点得到 $[a,c]$ 的分割 T, 则

$$U(f,T) - L(f,T)$$
$$= U(f,T_1) - L(f,T_1) + U(f,T_2) - L(f,T_2) < 2\varepsilon.$$

$\Rightarrow$: 设$f \in R[a,c]$. 则$\forall \varepsilon > 0, \exists [a,c]$的分割$T_0, s.t.$

$$U(f,T_0) - L(f,T_0) < \varepsilon.$$

在T_0中添加分点b得到$[a,c]$的分割T, 则

$$U(f,T) - L(f,T) \leq U(f,T_0) - L(f,T_0) < \varepsilon.$$

T在$[a,b],[b,c]$上的限制分别记为T_1, T_2, 则

$$U(f,T_i) - L(f,T_i) \leq U(f,T) - L(f,T) < \varepsilon, \quad i = 1,2.$$

故$f \in R[a,b], f \in R[b,c]$.

至此, 我们证明了 $f \in R[a,c] \Leftrightarrow f \in R[a,b] \,\&\, f \in R[b,c].$

下证 $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$

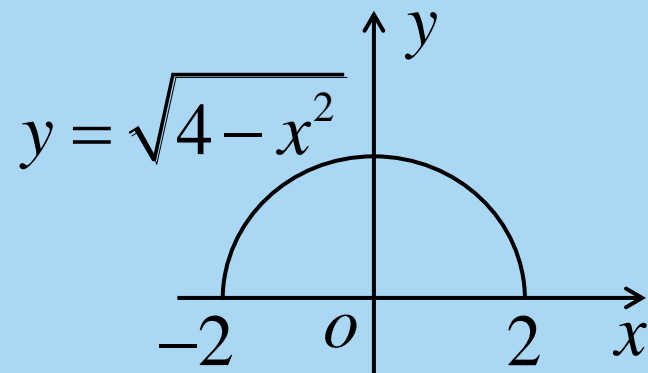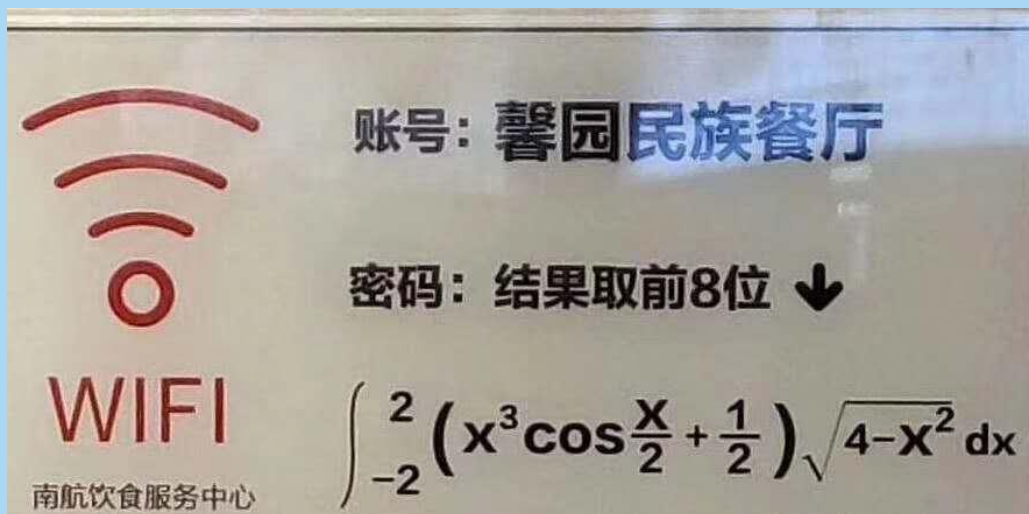设 T_1, T_2 分别为 $[a,b],[b,c]$ 上的分割, 合并 T_1, T_2 的分点得到 $[a,c]$ 上的分割 $T.$ 则

$$|T| \to 0 \Leftrightarrow |T_1| \to 0, \mathrm{i} = 1,2,$$

且

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{|T| \to 0} \sigma(f,T_1) + \lim_{|T| \to 0} \sigma(f,T_2)$$

$$= \lim_{|T| \to 0} \sigma(f,T) = \int_a^c f(x)dx. \ \square$$

Prop3. f 为 $[-a, a]$ 上的奇函数，则 $\int_{-a}^{a} f(x)\mathrm{d}x = 0$;

f 为 $[-a, a]$ 上的偶函数，则 $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\mathrm{d}x$.

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密码: 结果取前8位 ↓

WIFI

$\int_{-2}^{2} \left(x^3 \cos\frac{x}{2} + \frac{1}{2}\right) \sqrt{4-x^2}\, dx$

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$y = \sqrt{4-x^2}$

$$= 0 + \frac{1}{2} \int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x \quad = \pi = 3.1415926\cdots$$

Prop4. (单调性)$f, g \in R[a, b]$,且$f(x) \leq g(x)$,则

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

特别地,若$m \leq f(x) \leq M(a \leq x \leq b)$,则

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof. $g(x) - f(x) \geq 0 (a \leq x \leq b)$,则

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$$

$$= \lim_{|T| \to 0} \sigma(g - f, T) \geq 0. \square$$

Prop5.(积分估值) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$,且

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Proof. $U(|f|,T) - L(|f|,T) \leq U(f,T) - L(f,T)$. $\square$

Question. $|f| \in R[a,b] \overset{?}{\Rightarrow} f \in R[a,b]$

Prop6. $f, g \in R[a,b] \Rightarrow fg \in R[a,b]$.

Proof. 1)设 $f \geq 0, g \geq 0$. 记 $M = \sup_{a \leq x \leq b} f(x), N = \sup_{a \leq x \leq b} g(x)$.

任给 $T : a = x_0 < x_1 < \cdots < x_n = b$, 记

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x),$$

$$N_i = \sup_{x_{i-1} \le x \le x_i} g(x), \quad n_i = \inf_{x_{i-1} \le x \le x_i} g(x).$$

$$U(fg,T) - L(fg,T) \le \sum_{i=1}^{n} (M_i N_i - m_i n_i) \Delta x_i$$

$$= \sum_{i=1}^{n} M_i (N_i - n_i) \Delta x_i + \sum_{i=1}^{n} n_i (M_i - m_i) \Delta x_i$$

$$\le M\big[U(g,T) - L(g,T)\big] + N\big[U(f,T) - L(f,T)\big]$$

2)对任意f, 记$f^{\pm}(x) = \dfrac{1}{2}\big[|f(x)| \pm f(x)\big]$, 则

$$f^{\pm} \ge 0, \quad f^{\pm} \in R[a,b], \quad f = f^+ - f^-,$$

于是 $\quad fg = (f^+ - f^-)(g^+ - g^-)$

$$= f^+ g^+ - f^+ g^- - f^- g^+ + f^- g^- \in R[a,b]. \square$$

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Thm. 设 $f \in C[a,b], f(x) \geq 0, \int_a^b f(x)dx = 0.$ 求证 $: f(x) \equiv 0.$
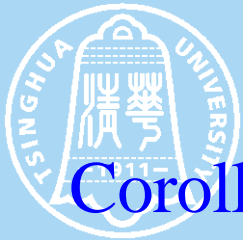
Proof. 反证. 设 $f(x)$ 在不恒为 0, 则 $\exists x_0 \in [a,b], s.t. f(x_0) > 0.$

不妨设 $x_0 \in (a,b). f \in C[a,b]$, 则 $\exists \delta > 0, s.t.$

$$f(x) > f(x_0)/2 > 0, \quad \forall x \in [x_0 - \delta, x_0 + \delta].$$

而 $f(x) \geq 0$, 于是

$$0 = \int_a^b f(x)dx = \int_a^{x_0-\delta} f(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + \int_{x_0+\delta}^b f(x)dx$$

$$\geq 0 + \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}dx + 0 \geq f(x_0)\delta > 0, 矛盾. \square$$

Corollary. 设 $f, g \in C[a,b], f(x) \geq g(x), f \neq g,$ 则

$$\int_a^b f(x)dx > \int_a^b g(x)dx.$$

Thm.(Cauchy不等式) $f, g \in R[a,b],$ 则

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

Proof. 令 $A = \int_a^b f^2(x)dx, B = \int_a^b f(x)g(x)dx, C = \int_a^b g^2(x)dx.$

则 $0 \leq \int_a^b [tf(x)+g(x)]^2 dx = At^2 + 2Bt + C, \quad \forall t \in \mathbb{R}.$

故 $(2B)^2 - 4AC \leq 0.$ □

证法二: $\left(\sigma(fg,T)\right)^2 \leq \sigma(f^2,T) \cdot \sigma(g^2,T).$

Thm.(积分第一中值定理) $f \in C[a,b], g \in R[a,b],$ g不变号,

则$\exists \xi \in [a,b], s.t.$ $\displaystyle\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx.$ (*)

特别地,$g(x) \equiv 1$时, $\displaystyle\int_a^b f(x)dx = f(\xi)(b-a).$

Proof.不妨设$g \geq 0.$记f在$[a,b]$上的最大值与最小值为$M,$

$m,$则 $\displaystyle m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$

若$\displaystyle\int_a^b g(x)dx = 0,$则$\displaystyle\int_a^b f(x)g(x)dx = 0, \forall \xi \in [a,b],$(*)成立.

若$\displaystyle\int_a^b g(x)dx > 0,$ 由连续函数的介值定理,$\exists \xi \in [a,b], s.t.$

$$f(\xi) = \frac{\displaystyle\int_a^b f(x)g(x)dx}{\displaystyle\int_a^b g(x)dx} \in [m, M]. \square$$

Question. g变号,反例?

提示:构造 $\int_a^b g(x)dx = 0, \int_a^b f(x)g(x)dx > 0.$

Ex. $f \in R[a,b], f \geq 0, \int_a^b xf(x)dx = 0,$则

$$\int_a^b x^2 f(x)dx \leq -ab\int_a^b f(x)dx.$$

Proof. $(x-a)(b-x)f(x) \geq 0, \forall x \in [a,b].$

$$0 \leq \int_a^b (x-a)(b-x)f(x)dx$$

$$= -\int_a^b x^2 f(x)dx + (a+b)\int_a^b xf(x)dx - ab\int_a^b f(x)dx$$

$$= -\int_a^b x^2 f(x)dx - ab\int_a^b f(x)dx. \square$$

Ex. $x \to 0$ 时, $f(x) = \int_{\sin x}^{x} \ln(1 + e^t)dt$ 与 x^p 是同阶无穷小, 则

$$p = \underline{\qquad}, \text{此时} \lim_{x \to 0} f(x) / x^p = \underline{\qquad}.$$

解: 由积分中值定理, 存在介于 x 与 $\sin x$ 之间的 ξ, $s.t.$

$$f(x) = (x - \sin x)\ln(1 + e^\xi) = \left(\frac{x^3}{3!} + o(x^3)\right)\ln(1 + e^\xi), x \to 0.$$

$f(x)$ 与 x^p 是同阶无穷小, 则

$$\lim_{x \to 0} \frac{f(x)}{x^p} = \ln 2 \cdot \lim_{x \to 0} \frac{\dfrac{x^3}{3!} + o(x^3)}{x^p} \text{ 存在且非0},$$

因而 $p = 3$, $\lim_{x \to 0} \dfrac{f(x)}{x^p} = \dfrac{\ln 2}{6}$. $\square$

Ex. $f \geq 0, f'' \leq 0 \Rightarrow \max\limits_{a \leq x \leq b} f(x) \leq \dfrac{2}{b-a} \displaystyle\int_a^b f(x)dx.$

几何意义?

Proof. $f'' \leq 0,$ 则 f 上凸,

$$f(x) \geq f(a) + \frac{f(b)-f(a)}{b-a}(x-a), \ \forall x \in [a,b].$$

两边在 $[a,b]$ 上积分, 得

$$\int_a^b f(x)dx \geq f(a)(b-a) + \frac{f(b)-f(a)}{b-a} \cdot \frac{1}{2}(b-a)^2$$

$$= \frac{1}{2}(b-a)(f(a)+f(b)).$$

1) 若 $\max\limits_{a \leq x \leq b} f(x) = \max\{f(a), f(b)\},$ 由 $f \geq 0,$ 有

$$f(a) + f(b) \geq \max\limits_{a \leq x \leq b} f(x), \ \int_a^b f(x)dx \geq \frac{1}{2}(b-a)\max\limits_{a \leq x \leq b} f(x).$$

2) 若 $f(c) = \max\limits_{a \le x \le b} f(x), c \in (a, b)$, 则由(1)中结论,

$$\int_a^c f(x)dx \ge \frac{c-a}{2} f(c), \qquad \int_c^b f(x)dx \ge \frac{b-c}{2} f(c),$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\ge \frac{1}{2}(b-a)f(c) = \frac{1}{2}(b-a)\max\limits_{a \le x \le b} f(x). \square$$

Ex. $\lim\limits_{n \to +\infty} \int_n^{n+\pi} \dfrac{\sin x}{x} dx = 0.$

Proof. 由积分第一中值定理, $\exists \xi_n \in [n, n+\pi], s.t.$

$$\left| \int_n^{n+\pi} \frac{\sin x}{x} dx \right| = \left| \frac{\sin \xi_n}{\xi_n} \cdot \pi \right| \le \frac{\pi}{n} \to 0 \quad (n \to +\infty). \square$$

Ex. f 在 $[0,1]$ 可导, $f(1) = 4\int_0^{1/4} e^{1-x^3} f(x)dx.$ 则 $\exists \xi \in (0,1), s.t.$

$$f'(\xi) = 3\xi^2 f(\xi).$$

Proof. 由积分第一中值定理, $\exists \eta \in (0,1), s.t.$

$$f(1) = 4\int_0^{1/4} e^{1-x^3} f(x)dx = e^{1-\eta^3} f(\eta).$$

令 $g(x) = e^{1-x^3} f(x),$ 则

$$g'(x) = e^{1-x^3}(f'(x) - 3x^2 f(x)), \ g(\eta) = g(1).$$

由 Rolle 定理, $\exists \xi \in (\eta, 1) \subset (0,1), s.t. g'(\xi) = 0,$ 即

$$f'(\xi) = 3\xi^2 f(\xi).\square$$

Ex. $f \in R[a,b]$, 则

$$\left(\int_a^b f(x)\cos xdx\right)^2 + \left(\int_a^b f(x)\sin xdx\right)^2 \le \left(\int_a^b |f(x)|dx\right)^2.$$

Proof. 由积分估值及Cauchy不等式,

$$\left(\int_a^b f(x)\cos xdx\right)^2 \le \left(\int_a^b \sqrt{|f(x)|}\cdot\sqrt{|f(x)|}\,|\cos x|dx\right)^2$$

$$\le \int_a^b |f(x)|dx \cdot \int_a^b |f(x)|\cos^2 xdx.$$

$$\left(\int_a^b f(x)\sin xdx\right)^2 \le \int_a^b |f(x)|dx \cdot \int_a^b |f(x)|\sin^2 xdx.$$

两式相加即证所需结论.□

Ex. $f \in C[a-1, b+1]$，则 $\lim\limits_{t \to 0} \int_a^b |f(x+t) - f(x)| dx = 0.$

Proof. $f \in C[a-1, b+1]$，则 f 在 $[a-1, b+1]$ 上一致连续.

$\forall \varepsilon > 0, \exists \delta \in (0,1), s.t. \forall x, y \in [a-1, b+1], |x-y| < \delta,$ 有

$$|f(y) - f(x)| < \varepsilon.$$

因此, $\forall x, \in [a, b],$ 当 $|t| < \delta$ 时,

$$|f(x+t) - f(x)| < \varepsilon,$$

$$\int_a^b |f(x+t) - f(x)| dx < \int_a^b \varepsilon dx = (b-a)\varepsilon. \square$$

Ex. $\lim\limits_{n\to+\infty}\displaystyle\int_0^1 x^n dx = 0.$

Proof. 令$a_n = \displaystyle\int_0^1 x^n dx$,则$a_n$单调下降有下界$0,\lim\limits_{n\to+\infty} a_n$存在.

$\forall \varepsilon > 0,$

$$0 \le \int_0^1 x^n dx = \int_0^{1-\varepsilon} x^n dx + \int_{1-\varepsilon}^1 x^n dx \le (1-\varepsilon)^{n+1} + \varepsilon.$$

因此, $\qquad 0 \le \lim\limits_{n\to+\infty}\displaystyle\int_0^1 x^n dx \le \varepsilon.$

由ε的任意性,$\lim\limits_{n\to+\infty}\displaystyle\int_0^1 x^n dx = 0.\square$

作业：习题**5.2 No.3,6~10**