Review

- inf A, sup A, min A, max A

- $\lim\limits_{n\to\infty} a_n = \text{A}$ 的 ε-N 语言描述

$\Leftrightarrow \forall \varepsilon > 0, \exists \text{N} \in \mathbb{N}, \text{s.t. } \text{当} n > \text{N} \text{时}, \text{有} a_n \in (\text{A}-\varepsilon, \text{A}+\varepsilon)$

$\Leftrightarrow \forall \varepsilon > 0, \exists \text{N} \in \mathbb{N}, \text{s.t. } \text{当} n \geq \text{N} \text{时}, \text{有} \text{A}-\varepsilon \leq a_n \leq \text{A}+\varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \exists \text{N} \in \mathbb{N}, \text{s.t. } \text{当} n > \text{N} \text{时}, \text{有} |a_n - \text{A}| < \varepsilon/2$

$\Leftrightarrow \forall \varepsilon \in (0,1), \exists \text{N} \in \mathbb{N}, \text{s.t. } \text{当} n \geq \text{N} \text{时}, \text{有} |a_n - \text{A}| \leq 2\varepsilon.$

$\Leftrightarrow \forall k \in \mathbb{N}, \exists \text{N} = \text{N}(k) \in \mathbb{N}, \text{s.t. } \text{当} n \geq \text{N} \text{时}, \text{有} |a_n - \text{A}| \leq \dfrac{1}{2^k}.$

- ε-N语言中N的选取.

  (1) 放缩法求解不等式 $|a_n - \text{A}| < \varepsilon$

  $$|a_n - \text{A}| < \cdots < \boxed{n\text{的简单表达式}} < \varepsilon$$

  (2) 分段法确定N

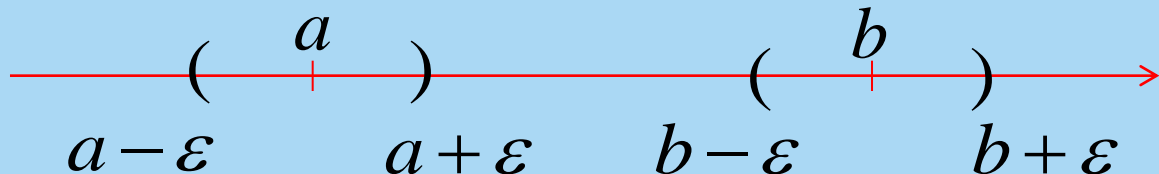 $$\text{N} = \max\{N_1, N_2, \cdots, N_k\}$$

- 记住一些基本结论.

  $$\lim_{n \to \infty} \sqrt[n]{n} = 1, \qquad \lim_{n \to \infty} a_n \text{存在} \Rightarrow \lim_{n \to \infty} e^{a_n} = e^{\lim_{n \to \infty} a_n},$$

  $$\lim_{n \to \infty} \frac{\ln n}{n} = 0, \qquad \lim_{n \to \infty} a_n \text{存在且非零} \Rightarrow \lim_{n \to \infty} \ln a_n = \ln\left(\lim_{n \to \infty} a_n\right)$$

§3. 收敛列的性质

Prop1. 收敛列的极限唯一.

Proof. 假设 a, b 均为 $\{a_n\}$ 的极限, 且 $a \neq b$. $\forall 0 < \varepsilon < \dfrac{|a-b|}{2}$,

由 $\lim\limits_{n \to \infty} a_n = a, \exists N_1,$ 当 $n > N_1$ 时, $|a_n - a| < \varepsilon,$

由 $\lim\limits_{n \to \infty} a_n = b, \exists N_2,$ 当 $n > N_2$ 时, $|a_n - b| < \varepsilon.$

令 $N = \max\{N_1, N_2\} + 1,$ 则

$$|a - b| = \left|(a - a_N) - (b - a_N)\right| \leq \left|a - a_N\right| + \left|b - a_N\right|$$

$$< 2\varepsilon < |a - b|, \text{矛盾.} \square$$

$$\lim_{n\to\infty} a_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{s.t. } \forall n > N, \text{有} |a_n - A| < \varepsilon.$$

$$\lim_{n\to\infty} a_n \neq A \Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \text{s.t.} |a_n - A| > \varepsilon.$$

Prop2. 在数列中添加、删除有限项, 或者改变有限项的值, 不改变数列的敛散性与极限值.

Def. $0 < n_1 < n_2 < \cdots < n_k < \cdots$ 为一列自然数, 称 $\{a_{n_k}\}$ 为 $\{a_n\}$ 的一个子列.

Prop3. (收敛列的任意子列具有相同的极限)

$$\lim_{n\to\infty} a_n = a \Rightarrow \lim_{k\to\infty} a_{n_k} = a. \qquad (n_k \geq k)$$

Crollary. $\lim_{k\to\infty} a_{n_k} = a \neq b = \lim_{k\to\infty} a_{m_k} \Rightarrow \{a_n\}$ 发散.

Ex. $\left\{(-1)^n\right\}$ 发散.

Question. $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1} = A \overset{?}{\Longrightarrow} \lim\limits_{n\to\infty} a_n = A.$ *Yes*!

Prop4. 收敛列一定有界.

   Proof. 设 $\lim\limits_{n\to\infty} a_n = a.$ 对 $\varepsilon = 1, \exists N,$ 当 $n > N$ 时, $|a_n - a| < 1.$

因此, $|a_n| = |(a_n - a) + a| \le |a_n - a| + |a| < |a| + 1, \forall n > N.$

令 $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\},$ 则 $|a_n| \le M, \forall n \in \mathbb{N}.$ □

Question. 有界列是否必为收敛列? *No*!

Def. 若 $\lim\limits_{n\to\infty} a_n = 0,$ 则称 $\left\{a_n\right\}$ 为无穷小数列.

Prop5. $\{a_n\}$ 为无穷小列, $\{b_n\}$ 为有界列, 则 $\{a_n b_n\}$ 为无穷小列.

Prop6. (极限的保序性) $\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b.$

(1) 若 $a < b,$ 则 $\exists N,$ 当 $n > N$ 时有 $a_n < b_n.$

(2) 若 $\exists N,$ 当 $n > N$ 时有 $a_n \le b_n,$ 则 $a \le b.$

Proof. (1)

$$a-\varepsilon \quad (\quad \overset{a}{|} \quad) \quad\quad b-\varepsilon \quad (\quad \overset{b}{|} \quad) \quad b+\varepsilon$$

(2) 反设 $a > b.$ 由(1)中结论, $\exists N_1,$ 当 $n > N_1$ 时有 $a_n > b_n.$ 矛盾. □

Question.
$$\left.\begin{array}{l} \lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b \\ a_n < b_n, \quad \forall n \end{array}\right\} \overset{?}{\Rightarrow} a < b \quad (\times)$$

Prop7. (极限的四则运算) 若$\{a_n\}$与$\{b_n\}$都收敛,则

(1) $\forall c \in \mathbb{R}, \lim\limits_{n \to \infty}\left(ca_n\right) = c\lim\limits_{n \to \infty}a_n$;

(2) $\lim\limits_{n \to \infty}\left(a_n \pm b_n\right) = \lim\limits_{n \to \infty}a_n \pm \lim\limits_{n \to \infty}b_n$;

(3) $\lim\limits_{n \to \infty}\left(a_n \cdot b_n\right) = \lim\limits_{n \to \infty}a_n \cdot \lim\limits_{n \to \infty}b_n$;

(4) $\lim\limits_{n \to \infty}b_n \neq 0$时, $\lim\limits_{n \to \infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n \to \infty}a_n}{\lim\limits_{n \to \infty}b_n}$.

Question. 能否推广到$\lim\limits_{n \to \infty}a_n, \lim\limits_{n \to \infty}b_n$为$\infty$或$\pm\infty$的情形?

Remark. (3)$a > 0, a \cdot \pm\infty = \pm\infty$; $0 \cdot \infty, 0 \cdot \pm\infty$没有意义.

(4)$a \neq \infty, \dfrac{a}{\infty} = 0$; $a \neq 0, \dfrac{a}{0} = \infty$; $\dfrac{0}{0}, \dfrac{(\pm)\infty}{(\pm)\infty}$没有意义.

$\text{Proof}.$ (3) $\{b_n\}$收敛,则有界, $\exists M > 0, s.t.$

$$|b_n| < M, \quad \forall n.$$

因 $\lim\limits_{n \to \infty} a_n = a, \lim\limits_{n \to \infty} b_n = b, \ \forall \varepsilon > 0, \exists N_1, N_2, s.t.$

$$|a_n - a| < \varepsilon, \quad \forall n > N_1,$$

$$|b_n - b| < \varepsilon, \quad \forall n > N_2.$$

当 $n > N = \max\{N_1, N_2\}$时,

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$$

$$\leq |a_n - a||b_n| + |a||b_n - b|$$

$$\leq (M + |a|)\varepsilon.$$

(4) $\lim\limits_{n\to\infty} b_n = b \neq 0,$ 对 $\varepsilon_0 = |b|/2, \exists N_1,$ 当 $n > N_1$ 时, $|b_n - b| < \varepsilon_0.$

因此, $\quad |b_n| = |b_n - b + b| \geq |b| - |b_n - b|$

$$> |b| - \varepsilon_0 = |b|/2, \quad \forall n > N_1.$$

因 $\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b, \forall \varepsilon > 0, \exists N_2, N_3, s.t.$

$$|a_n - a| < \varepsilon, \ \forall n > N_2; \quad |b_n - b| < \varepsilon, \ \forall n > N_3.$$

当 $n > N = \max\{N_1, N_2, N_3\}$ 时,

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \frac{|ba_n - ab_n|}{|bb_n|} \leq \frac{2}{b^2} |ba_n - ab + ab - ab_n|$$

$$\leq \frac{2}{b^2} \left(|b| |a_n - a| + |a| |b - b_n| \right) \leq \frac{2}{b^2} \left(|b| + |a| \right) \varepsilon. \square$$

Prop8. (夹挤原理) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a$, 且 $\exists n_0, s.t.$

$$a_n \leq x_n \leq b_n, \quad \forall n > n_0.$$

则 $\lim\limits_{n\to\infty} x_n = a.$

Proof. 因 $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a, \forall \varepsilon > 0, \exists N_1, N_2, s.t.$

$$-\varepsilon < a_n - a < \varepsilon, \quad \forall n > N_1,$$

$$-\varepsilon < b_n - a < \varepsilon, \quad \forall n > N_2.$$

又 $n > n_0$ 时, $a_n \leq x_n \leq b_n$, 令 $N = \max\{n_0, N_1, N_2\}$, 则

$$-\varepsilon < a_n - a \leq x_n - a \leq b_n - a < \varepsilon, \quad \forall n > N. \square$$

Ex. 设 $b > 0, a > 1$. 证明: $\lim\limits_{n \to \infty} \dfrac{n^b}{a^n} = 0$.

Proof. 法一: $\ln \dfrac{n^b}{a^n} = \ln \dfrac{\left(\sqrt[n]{n}\right)^{nb}}{\left(\sqrt[k]{a}\right)^{nb}} = nb\left(\ln \sqrt[n]{n} - \ln \sqrt[b]{a}\right)$

$$\lim_{n \to \infty} \ln \frac{n^b}{a^n} = \lim_{n \to \infty} nb \cdot \lim_{n \to \infty}\left(\ln \sqrt[n]{n} - \ln \sqrt[b]{a}\right)$$

$$= +\infty \cdot (\ln 1 - \ln \sqrt[b]{a})$$

$$= -\infty$$

$$\lim_{n \to \infty} \frac{n^b}{a^n} = \lim_{n \to \infty} \exp\left(\ln \frac{n^b}{a^n}\right) = \exp\left(\lim_{n \to \infty} \ln \frac{n^b}{a^n}\right) = \exp(-\infty) = 0.$$

法二：$(1)b=k$ 为整数时. 令 $d=a-1$, 则 $d>0$, 当 $n\geq 2k$ 时

$$\left|\frac{n^k}{a^n}-0\right|=\frac{n^k}{(1+d)^n}<\frac{n^k}{C_n^{k+1}d^{k+1}}=\frac{n^k(k+1)!}{n(n-1)\cdots(n-k)d^{k+1}}$$

$$=\frac{(k+1)!}{n(1-\frac{1}{n})\cdots(1-\frac{k}{n})d^{k+1}}\leq\frac{2^k(k+1)!}{nd^{k+1}}$$

$$\forall\varepsilon>0,\exists N=2k+\left[\frac{2^k(k+1)!}{\varepsilon d^{k+1}}\right],当 n>N 时,\left|\frac{n^k}{a^n}-0\right|<\varepsilon.$$

故 $\lim\limits_{n\to\infty}\dfrac{n^k}{a^n}=0.$

$(2)b > 0$不为整数时. $0 < \dfrac{n^b}{a^n} < \dfrac{n^{[b]+1}}{a^n}$.

由(1)中结论, $\lim\limits_{n\to\infty} \dfrac{n^{[b]+1}}{a^n} = 0$. 由夹挤原理, $\lim\limits_{n\to\infty} \dfrac{n^b}{a^n} = 0$. □

Ex. $a > 0$, 证明: $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.

Proof. 当$a \geq 1$时, $\forall n > [a] + 1$, 有 $1 < \sqrt[n]{a} < \sqrt[n]{n}$, 而 $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$,

由夹挤原理, $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.

当$0 < a < 1$时, $\lim\limits_{n\to\infty} \sqrt[n]{a} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{1/a}} = \dfrac{1}{\lim\limits_{n\to\infty} \sqrt[n]{1/a}} = 1$. □

Question. 错在哪里？

$$1 = \lim_{n \to \infty} (\underbrace{\frac{1}{n} + \frac{1}{n} + \cdots \frac{1}{n}}_{n\text{个}})$$

$$= \underbrace{\lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} \frac{1}{n} + \cdots + \lim_{n \to \infty} \frac{1}{n}}_{n\text{个}} = 0 + 0 + \cdots 0 = 0$$

无穷个0相加

Question. 谁对谁错？错在哪里？

$$（1）\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=\lim_{n\to\infty}\left(\underbrace{\sqrt[n]{1}\cdot\sqrt[n]{\frac{1}{2}}\cdot\sqrt[n]{\frac{1}{3}}\cdots\sqrt[n]{\frac{1}{n}}}_{n个}\right)$$

$$=\underbrace{\lim_{n\to\infty}\sqrt[n]{1}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{1}{2}}\cdots\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}}}_{n个}=1\cdot1\cdots1={\color{red}1}$$

$$（2）\quad n!>\left(\frac{n}{2}\right)^{n/2},0<\frac{1}{\sqrt[n]{n!}}<\frac{1}{\sqrt{n/2}},\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}={\color{red}0}.$$

作业：习题**1.3 No. 4,6,8**

No.8 $\dfrac{1}{2\sqrt{n}} < \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\cdots\cdot\dfrac{2n-1}{2n} < \dfrac{1}{\sqrt{2n+1}}$