An Implementation of Sin and Cos Using Gal's Accurate Tables

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This document describes the implementation of functions `Sin` and `Cos` in Principia. The goals of that implementation are to be portable (including to machines that do not have a fused multiply-add instruction), achieve good performance, and ensure correct rounding.

Overview

The implementation follows the ideas described by [GB91] and uses accurate tables produced by the method presented in [SZ05]. It guarantees correct rounding with a high probability. In circumstances where it cannot guarantee correct rounding, it falls back to the (slower but correct) implementation provided by the CORE-MATH project [SZG22] [ZSG+24]. More precisely, the algorithm proceeds through the following steps:

- perform argument reduction using Cody and Waite's algorithm in double precision (see [Mul+10, p. 379]);
- if argument reduction loses too many bits (i.e., the argument is close to a multiple of $\frac{\pi}{2}$), fall back to `cr_sin` or `cr_cos`;
- otherwise, uses accurate tables and a polynomial approximation to compute `Sin` or `Cos` with extra accuracy;
- if the result has a "dangerous rounding configuration" (as defined by [GB91]), fall back to `cr_sin` or `cr_cos`;
- otherwise return the rounded result of the preceding computation.

Notation and Accuracy Model

In this document we assume a base-2 floating-point number system with M significand bits[1] similar to the IEEE formats. We define a real function $\mathfrak{m}$ and an integer function $\mathfrak{e}$ denoting the *significand* and *exponent* of a real number, respectively:

$$x = \pm\,\mathfrak{m}(x) \times 2^{\mathfrak{e}(x)} \qquad \text{with} \qquad 2^{M-1} \leq \mathfrak{m}(x) \leq 2^{M} - 1$$

Note that this representation is unique. Furthermore, if x is a floating-point number, $\mathfrak{m}(x)$ is an integer.

The distance between 1 and the next larger floating-point number is:

$$\epsilon_M \coloneqq 2^{1-M}$$

and the distance between 1 and the next smaller floating-point number is $\frac{\epsilon_M}{2}$. The *unit of the last place* of x is defined as:

$$\mathfrak{u}(x) \coloneqq 2^{\mathfrak{e}(x)}$$

In particular, $\mathfrak{u}(1) = \epsilon_M$ and:

$$\frac{x\,\epsilon_M}{2} < \frac{x}{2^M - 1} \leq \mathfrak{u}(x) \leq \frac{x}{2^{M-1}} = x\,\epsilon_M \tag{1}$$

[1] In `binary64`, $M = 53$.

We ignore the exponent bias, overflow and underflow as they play no role in this discussion.

Finally, for error analysis we use the accuracy model of [Hig02], equation (2.4): everywhere they appear, the quantities δ_i represent a roundoff factor such that $\delta_i < u = \epsilon_M/2 = 2^{-M}$ (see pages 37 and 38). We also use θ_n and γ_n with the same meaning as in [Hig02], lemma 3.1.

Approximation of $\frac{\pi}{2}$

To perform argument reduction, we need to build approximations of $\frac{\pi}{2}$ with extra accuracy and analyse the circumstances under which they may be used and the errors that they entail on the reduced argument.

We start by defining the truncation function $\mathrm{Tr}(\kappa, z)$ which clears the last κ bits of the significand of z:

$$\mathrm{Tr}(\kappa, z) := \lfloor 2^{-\kappa}\,\mathfrak{m}(z)\rfloor 2^{\kappa}\,\mathfrak{u}(z)$$

The definition of the floor function implies:

$$0 \leq z - \mathrm{Tr}(\kappa, z) < 2^{\kappa}\,\mathfrak{u}(z)$$

Furthermore if the bits that are being truncated start with exactly k zeros we have the stricter inequality:

$$2^{\kappa'-1}\,\mathfrak{u}(z) \leq z - \mathrm{Tr}(\kappa, z) < 2^{\kappa'}\,\mathfrak{u}(z) \quad \text{with} \quad \kappa' = \kappa - k \tag{2}$$

This leads to the following upper bound for the unit of the last place of the truncation error:

$$\mathfrak{u}(z - \mathrm{Tr}(\kappa, z)) < \frac{2^{\kappa'}\,\mathfrak{u}(z)}{\mathfrak{m}(z - \mathrm{Tr}(\kappa, z))} \leq 2^{\kappa'-M+1}\,\mathfrak{u}(z)$$

which can be made more precise by noting that the function $\mathfrak{u}$ is always a power of 2:

$$\mathfrak{u}(z - \mathrm{Tr}(\kappa, z)) = 2^{\kappa'-M}\,\mathfrak{u}(z) \tag{3}$$

Two-Term Approximation

In this scheme we approximate $\frac{\pi}{2}$ as the sum of two floating-point numbers:

$$\frac{\pi}{2} \simeq C_1 + \delta C_1$$

which are defined as:

$$\begin{cases} C_1 & := \mathrm{Tr}\left(\kappa_1, \dfrac{\pi}{2}\right) \\ \delta C_1 & := \left[\!\left[\dfrac{\pi}{2} - C_1 \right]\!\right] \end{cases}$$

Equation (2) applied to the definition of C_1 yields:

$$2^{\kappa_1'-1}\,\mathfrak{u}\left(\frac{\pi}{2}\right) \leq \frac{\pi}{2} - C_1 < 2^{\kappa_1'}\,\mathfrak{u}\left(\frac{\pi}{2}\right)$$

where $\kappa_1' \leq \kappa_1$ accounts for any leading zeroes in the bits of $\frac{\pi}{2}$ that are being truncated. Accordingly equation (3) yields, for the unit of the last place:

$$\mathfrak{u}\left(\frac{\pi}{2} - C_1\right) = 2^{\kappa_1'-M}\,\mathfrak{u}\left(\frac{\pi}{2}\right)$$

Noting that the absolute error on the rounding that appears in the definition of δC_1 is bounded by $\frac{1}{2}\,\mathfrak{u}\left(\frac{\pi}{2} - C_1\right)$, we obtain the absolute error on the two-term approximation:

$$\left|\frac{\pi}{2} - C_1 - \delta C_1\right| \leq \frac{1}{2}\,\mathfrak{u}\left(\frac{\pi}{2} - C_1\right) = 2^{\kappa_1'-M-1}\,\mathfrak{u}\left(\frac{\pi}{2}\right) \tag{4}$$

and the following upper bound for δC_1:

$$|\delta C_1| < 2^{\kappa_1'}(1 + 2^{-M-1})\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \tag{5}$$

This scheme gives a representation with a significand that has effectively $2M - \kappa_1'$ bits and is such that multiplying C_1 by an integer less than or equal to 2^{κ_1} is exact.

Three-Term Approximation

In this scheme we approximate $\frac{\pi}{2}$ as the sum of three floating-point numbers:

$$\frac{\pi}{2} \simeq C_2 + C_2' + \delta C_2$$

which are defined as:

$$\begin{cases} C_2 & \coloneqq \mathrm{Tr}\Big(\kappa_2, \dfrac{\pi}{2}\Big) \\ C_2' & \coloneqq \mathrm{Tr}\Big(\kappa_2, \dfrac{\pi}{2} - C_2\Big) \\ \delta C_2 & \coloneqq \Big[\!\!\Big[\dfrac{\pi}{2} - C_2 - C_2'\Big]\!\!\Big] \end{cases}$$

Equation (2) applied to the definition of C_2 yields:

$$2^{\kappa_2'-1}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \le \frac{\pi}{2} - C_2 < 2^{\kappa_2'}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \tag{6}$$

where $\kappa_2' \le \kappa_2$ accounts for any leading zeroes in the bits of $\frac{\pi}{2}$ that are being truncated. Accordingly equation (3) yields, for the unit of the last place:

$$\mathfrak{u}\Big(\frac{\pi}{2} - C_2\Big) = 2^{\kappa_2'-M}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big)$$

Similarly, equation (2) applied to the definition of C_2' yields:

$$\begin{aligned} 2^{\kappa_2''-1}\,\mathfrak{u}\Big(\frac{\pi}{2} - C_2\Big) \le \frac{\pi}{2} - C_2 - C_2' &< 2^{\kappa_2''}\,\mathfrak{u}\Big(\frac{\pi}{2} - C_2\Big) \\ 2^{\kappa_2'+\kappa_2''-M-1}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \le \qquad\qquad &< 2^{\kappa_2'+\kappa_2''-M}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \end{aligned}$$

where $\kappa_2'' \le \kappa_2$ accounts for any leading zeroes in the bits of $\frac{\pi}{2} - C_2$ that are being truncated. Note that normalization of the significand of $\frac{\pi}{2} - C_2$ effectively drops the zeroes at positions κ_2 to κ_2' and therefore the computation of C_2' applies to a significand aligned on position κ_2'.

It is straightforward to transform these inequalities using (6) to obtain bounds on C_2':

$$2^{\kappa_2'}\Big(\frac{1}{2} - 2^{\kappa_2''-M}\Big)\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) < C_2' < 2^{\kappa_2'}(1 - 2^{\kappa_2''-M-1})\,\mathfrak{u}\Big(\frac{\pi}{2}\Big)$$

Equation (3) applied to the definition of C_2' yields, for the unit of the last place:

$$\begin{aligned} \mathfrak{u}\Big(\frac{\pi}{2} - C_2 - C_2'\Big) &= 2^{\kappa_2''-M}\,\mathfrak{u}\Big(\frac{\pi}{2} - C_2\Big) \\ &= 2^{\kappa_2'+\kappa_2''-2M}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \end{aligned}$$

Noting that the absolute error on the rounding that appears in the definition of δC_2 is bounded by $\frac{1}{2}\,\mathfrak{u}\Big(\frac{\pi}{2} - C_2 - C_2'\Big)$, we obtain the absolute error on the three-term approximation:

$$\Big|\frac{\pi}{2} - C_2 - C_2' - \delta C_2\Big| \le \frac{1}{2}\,\mathfrak{u}\Big(\frac{\pi}{2} - C_2 - C_2'\Big) = 2^{\kappa_2'+\kappa_2''-2M-1}\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \tag{7}$$

and the following upper bound for δC_2:

$$|\delta C_2| < 2^{\kappa_2'+\kappa_2''-M}(1 + 2^{-M-1})\,\mathfrak{u}\Big(\frac{\pi}{2}\Big) \tag{8}$$

This scheme gives a representation with a significand that has effectively $3M - \kappa_2' - \kappa_2''$ bits and is such that multiplying C_2 and C_2' by an integer less than or equal to 2^{κ_2} is exact.

Argument Reduction

Given an argument x, the purpose of argument reduction is to compute a pair of floating-point numbers $(\hat{x}, \delta\hat{x})$ such that:

$$\begin{cases} \hat{x} + \delta\hat{x} \cong x \pmod{\dfrac{\pi}{2}} \\ \hat{x} \text{ is approximately in } \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right] \\ |\delta\hat{x}| \leq \dfrac{1}{2}\,\mathfrak{u}(\hat{x}) \end{cases}$$

Argument Reduction for Small Angles

If $|x| < \left\llbracket \frac{\pi}{4} \right\rrbracket$ then $\hat{x} = x$ and $\delta\hat{x} = 0$.

Argument Reduction Using the Two-Term Approximation

If $|x| \leq 2^{\kappa_1}\left\llbracket \frac{\pi}{2} \right\rrbracket$ we compute:

$$\begin{cases} n & = \left\lceil \left\llbracket x \left\llbracket \dfrac{2}{\pi} \right\rrbracket \right\rrbracket \right\rfloor \\ y & = x - n\ C_1 \\ \delta y & = \llbracket n\ \delta C_1 \rrbracket \\ (\hat{x}, \delta\hat{x}) & = TwoDifference\big(y, \delta y\big) \end{cases}$$

Let's first show that $|n| \leq 2^{\kappa_1}$. :

$$\begin{aligned} |x| &\leq 2^{\kappa_1}\frac{\pi}{2}(1+\delta_1) \\ |n| &\leq \left\lceil 2^{\kappa_1}\frac{\pi}{2}(1+\delta_1)\frac{2}{\pi}(1+\delta_2)(1+\delta_3) \right\rfloor \\ &\leq \left\lceil 2^{\kappa_1}(1+\gamma_3) \right\rfloor \end{aligned}$$

As long as $2^{\kappa_1}\gamma_3$ is small enough (less that 1/2), the rounding cannot cause n to exceed 2^{κ_1}. In practice we choose a relatively small value for κ_1, so this condition is met.

The product $n\ C_1$ is exact thanks to the κ_1 trailing zeroes of C_1. The subtraction $x - n\ C_1$ is exact by Sterbenz's Lemma. Finally, the last step performs an exact addition[2] using algorithm 4 of [HLB08].

To compute the overall error on argument reduction, first remember that, from equation (4) we have:

$$C_1 + \delta C_1 = \frac{\pi}{2} + \zeta \quad \text{with} \quad \left|\zeta\right| \leq 2^{\kappa_1' - M - 1}\,\mathfrak{u}\left(\frac{\pi}{2}\right)$$

[2] The more efficient $QuickTwoDifference$ is not usable here. First, note that $|y|$ is an integral multiple of $\mathfrak{u}(x)$ and therefore, when not zero, may be as small as $\mathfrak{u}(x)$. Ignoring rounding errors we have:

$$|\delta y| \geq n\ 2^{\kappa_1' - 1}\,\mathfrak{u}\left(\frac{\pi}{2}\right) \geq 2^{\kappa_1' + M - 2}\,\mathfrak{u}\left(\frac{\pi}{2}\right)\mathfrak{u}(n)$$

where we used the bound given by equation (1). Now the computation of n can result in a value that is either in the same binade or in the binade below that of x. Therefore $\mathfrak{u}(n) \geq \frac{1}{2}\,\mathfrak{u}(x)$ and the above inequality becomes:

$$|\delta y| \geq 2^{\kappa_1' + M - 3}\,\mathfrak{u}\left(\frac{\pi}{2}\right)\mathfrak{u}(x)$$

plugging $\mathfrak{u}\left(\frac{\pi}{2}\right) = 2^{1-M}$ we find:

$$|\delta y| \geq 2^{\kappa_1' - 2}\,\mathfrak{u}(x)$$

Therefore, as long as $\kappa_1' > 2$, there exist arguments x for which $|\delta y| > |y|$.

The error computation proceeds as follows:

$$\begin{aligned} y - \delta y &= x - n\ C_1 - n\ \delta C_1 (1 + \delta_4) \\ &= x - n(C_1 + \delta C_1) - n\ \delta C_1\ \delta_4 \\ &= x - n\frac{\pi}{2} - n\big(\zeta + \delta C_1\ \delta_4\big) \end{aligned}$$

from which we deduce an upper bound on the absolute error of the reduction:

$$\begin{aligned} \left| y - \delta y - \left(x - n\frac{\pi}{2}\right)\right| &\le 2^{\kappa_1} 2^{\kappa_1'} (2^{-M-1} + 2^{-M} + 2^{-2M-1})\, \mathfrak{u}\left(\frac{\pi}{2}\right) \\ &= 2^{\kappa_1 + \kappa_1' - M} \left(\frac{3}{2} + 2^{-M-1}\right) \mathfrak{u}\left(\frac{\pi}{2}\right) \\ &< 2^{\kappa_1 + \kappa_1' - M + 1}\, \mathfrak{u}\left(\frac{\pi}{2}\right) \end{aligned}$$

where we have used the upper bound for δC_1 given by equation (5).

If we want $\hat{x} + \delta\hat{x}$ to have κ_3 extra bits of accuracy, we must have:

$$2^{\kappa_1 + \kappa_1' - M + 1}\, \mathfrak{u}\left(\frac{\pi}{2}\right) \le 2^{-\kappa_3} |\mathfrak{u}(\hat{x})| \le 2^{-\kappa_3 - M + 1} |\hat{x}|$$

which leads to the following condition on the reduced angle:

$$|\hat{x}| \ge 2^{\kappa_1 + \kappa_1' + \kappa_3}\, \mathfrak{u}\left(\frac{\pi}{2}\right)$$

The rest of the implementation assumes that $\kappa_3 = 18$ to achieve correct rounding most of the time and detect cases of dangerous rounding. If we choose $\kappa_1 = 8$ we find that $\kappa_1' = 5$ (because there are three consecutive zeroes at this location in the significand of $\frac{\pi}{2}$) and the desired accuracy is obtained as long as $|\hat{x}| \ge 2^{-21} \simeq 4.8 \times 10^{-7}$.

Argument Reduction Using the Three-Term Approximation

If $|x| \le 2^{\kappa_2} \llbracket \frac{\pi}{2} \rrbracket$ we compute:

$$\begin{cases} n &= \left\lceil \left\llbracket x \left\llbracket \frac{2}{\pi} \right\rrbracket \right\rrbracket \right\rfloor \\ y &= x - n\ C_2 \\ y' &= n\ C_2' \\ \delta y &= \llbracket n\ \delta C_2 \rrbracket \end{cases}$$

It is straightforward to show, like we did in the preceding section, that:

$$|n| \le \lceil 2^{\kappa_2} (1 + \gamma_3) \rfloor$$

and therefore that $|n| \le 2^{\kappa_2}$ as long as $2^{\kappa_2} \gamma_3 < 1/2$.

The products $n\ C_2$ and $n\ C_2'$ are exact thanks to the κ_2 trailing zeroes of C_2 and C_2'. The subtraction $x - n\ C_2$ is exact by Sterbenz's Lemma.

Accurate Tables and Their Generation

Computation of the Functions

Sin

Near Zero

For $\hat{x}$ near zero we evaluate:

$$\begin{aligned}
\widehat{x^2} &= \llbracket \hat{x}^2 \rrbracket = \hat{x}^2(1+\delta_1) \\
\widehat{x^3} &= \llbracket \hat{x}\ \widehat{x^2} \rrbracket = \hat{x}^3(1+\delta_1)(1+\delta_2) \\
\hat{p} &= \llbracket a\widehat{x^2} + b \rrbracket = (a\hat{x}^2(1+\delta_1)+b)(1+\delta_3) \\
s(x) &\coloneqq \hat{x} + \left\llbracket \llbracket \widehat{x^3}\hat{p} \rrbracket + \delta\hat{x} \right\rrbracket \\
&= \hat{x} + (\hat{x}^3(1+\delta_1)(1+\delta_2)(a\hat{x}^2(1+\delta_1)+b)(1+\delta_3)(1+\delta_4) + \delta\hat{x})(1+\delta_5) \\
&= \hat{x} + a\hat{x}^3(1+\theta_5) + b\hat{x}^5(1+\theta_4) + \delta\hat{x}(1+\delta_5)
\end{aligned}$$

References

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