

An Implementation of Sin and Cos Using Gal's Accurate Tables

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This document describes the implementation of functions `Sin` and `Cos` in Principia. The goals of that implementation are to be portable (including to machines that do not have a fused multiply-add instruction), achieve good performance, and ensure correct rounding.

Overview

The implementation follows the ideas described by [GB91] and uses accurate tables produced by the method presented in [SZ05]. It guarantees correct rounding with a high probability. In circumstances where it cannot guarantee correct rounding, it falls back to the (slower but correct) implementation provided by the CORE-MATH project [SZG22] [ZSG+24]. More precisely, the algorithm proceeds through the following steps:

- perform argument reduction using Cody and Waite's algorithm in double precision (see [Mul+10, p. 379]);
- if argument reduction loses too many bits (i.e., the argument is close to a multiple of $\frac{\pi}{2}$), fall back to `cr_sin` or `cr_cos`;
- otherwise, uses accurate tables and a polynomial approximation to compute `Sin` or `Cos` with extra accuracy;
- if the result has a “dangerous rounding configuration” (as defined by [GB91]), fall back to `cr_sin` or `cr_cos`;
- otherwise return the rounded result of the preceding computation.

In this document we assume a base-2 floating-point number system with M significand bits¹ similar to the IEEE formats. We define a real function `m` and an integer function `e` denoting the *significand* and *exponent* of a real number, respectively:

$$x = \pm m(x) \times 2^{e(x)} \quad \text{with} \quad 2^{M-1} \leq m(x) \leq 2^M - 1$$

Note that this representation is unique. Furthermore, if x is a floating-point number, `m(x)` is an integer.

The distance between 1 and the next larger floating-point number is:

$$\epsilon_M := 2^{1-M}$$

and the distance between 1 and the next smaller floating-point number is $\frac{\epsilon_M}{2}$. The *unit of the last place* of x is defined as:

$$u(x) := 2^{e(x)}$$

In particular, $u(1) = \epsilon_M$.

We ignore the exponent bias, overflow and underflow as they play no role in this discussion.

¹In binary64, $M = 53$.

Argument Reduction

Given an argument x , the purpose of argument reduction is to compute a pair of floating-point numbers $(\hat{x}, \delta\hat{x})$ such that:

$$\begin{cases} \hat{x} + \delta\hat{x} \cong x \pmod{\frac{\pi}{2}} \\ \hat{x} \text{ is approximately in } \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\ |\delta\hat{x}| < u(\hat{x}) \end{cases}$$

Approximation of π

We approximate $\frac{\pi}{2}$ as the sum of two floating-point numbers:

$$\frac{\pi}{2} \cong C + \delta C$$

where C is obtained by truncating $\frac{\pi}{2}$ to κ_1 significand bits:

$$C := \left\lfloor 2^{-\kappa_1} m\left(\frac{\pi}{2}\right) \right\rfloor 2^{\kappa_1} u\left(\frac{\pi}{2}\right)$$

and δC is defined as $\left\lceil \frac{\pi}{2} - C \right\rceil$. Obviously we have:

$$0 < \frac{\pi}{2} - C < 2^{\kappa_1} u\left(\frac{\pi}{2}\right)$$

but if κ_1 is chosen to cut the significand of $\frac{\pi}{2}$ at a place where it has zeroes, we can actually have a stricter bound:

$$\frac{\pi}{2} - C < 2^{\kappa_2} u\left(\frac{\pi}{2}\right) \quad \text{with} \quad \kappa_2 \leq \kappa_1$$

and therefore:

$$u\left(\frac{\pi}{2} - C\right) < \frac{2^{\kappa_2} u\left(\frac{\pi}{2}\right)}{m\left(\frac{\pi}{2} - C\right)} \leq 2^{\kappa_2 - M + 1} u\left(\frac{\pi}{2}\right)$$

Since the function u is always a power of 2 this implies:

$$u\left(\frac{\pi}{2} - C\right) = 2^{\kappa_2 - M} u\left(\frac{\pi}{2}\right)$$

and:

$$\left| \frac{\pi}{2} - C - \delta C \right| \leq \frac{1}{2} u\left(\frac{\pi}{2} - C\right) = 2^{\kappa_2 - M - 1} u\left(\frac{\pi}{2}\right)$$

In other words, we have a representation with a significand that has effectively $2M - \kappa_2$ bits and is such that multiplying C by an integer less than or equal to 2^{κ_1} is exact. The representation of $\frac{\pi}{2}$ has three zeroes after the 18th bit of its significand, so by taking $\kappa_1 = 18$ we have $\kappa_2 = 14$.

Argument Reduction for Small Angles

If $|x| < \left\lceil \frac{\pi}{4} \right\rceil$ then $\hat{x} = x$ and $\delta\hat{x} = 0$.

Argument Reduction for Medium Angles

If $|x| < \left\lceil 2^{\kappa_1} \frac{\pi}{2} \right\rceil$ then we compute:

Argument Reduction for Large Angles

Accurate Tables and Their Generation

Computation of the Functions

References

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