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Generating Perfect Numbers

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Optimized Approach

- before we get into the approach we need to understand a bit about perfect number theory.

↳ Link b/w Perfect Numbers & Mersenne Primes

- The smallest perfect number is 6.
  - ↳ $6 = 1 + 2 + 3$
  - The reason we call it a perfect number is that it equals the sum of it's proper factors (i.e. factors excluding the number itself)
  - Another way to look at this is that the sum of all factors of a perfect number is equal to twice that number
    - ↳ $6 \Rightarrow 1 + 2 + 3 + 6 = 12 = 2(6)$ ✓
- The next perfect number is 28
  - ↳ $28 = \underbrace{1 + 2 + 4 + 7 + 14}_{\text{sum of proper factors}}$
- The next perfect number is 496
  - ↳ $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$
- The next perfect number is 8128, we will refrain from listing all its proper factors to keep things concise.
- The reason we are able to find perfect number is thanks to Mersenne Primes.
- A Mersenne Prime is defined as:

$$2^n - 1 = \text{prime}$$

↳ So if $2^n - 1$ equals a prime number then it is a mersenne prime.

- Let's go through some examples to see this

n	$2^n - 1$	2^n-1 is Prime?
1	1	False
2	3	True
3	7	True
4	15	False
5	31	True
6	63	False
7	127	True
8	255	False
9	511	False
10	1023	False
11	2047	False

* generally speaking 1 is not considered a prime #.

- A pattern that may stick out to you is that $2^n - 1$ equals a prime # when n is a prime #
- However we see that 11 is a prime # but it corresponding $2^n - 1$ value, 2047, is not a prime.
- So lets consider the reverse of this pattern, if $2^n - 1$ is a prime, then n must be a prime.

↳ This statement turns out to be true.

· If we revisit our perfect numbers you will see that there is a link to Mersenne Primes

↳ $6 \Rightarrow 1 + 2 + 3 + 6$

↳ 3 is a Mersenne Prime

↳ $28 \Rightarrow 1 + 2 + 4 + 7 + 14 + 28$

↳ 7 is a Mersenne Prime

↳ $496 \Rightarrow 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$

↳ 31 is a Mersenne Prime

↳ $8128 \Rightarrow 1+2+4+8+16+32+64+127+254+508+1016+2032+4064$

↳ 127 is a Mersenne Prime

· So we can see that the Mersenne Primes match up with the factors of perfect numbers.

· Let's see if we can find a deeper pattern by visualizing this relationship.

n	2^n-1	2^n-1 Matching Perfect Number
2	3	6
3	7	28
5	31	496
7	127	8128
13	8191	?

*Note: there is only ever 1 mersenne prime in an even perfect number and every even perfect number has a mersenne prime as one of it's factors.

*Note: we have not found an odd perfect number so far. Yet no one has proved that odd perfect numbers don't exist.

$n=2$
· for $2^n-1=3$ we see that

↳ $3 \times 2 = 6$

$n=3$
· for $2^n-1=7$:

↳ $7 \times 4 = 28$

$n=5$
· for $2^n-1=31$:

↳ $31 \times 6 = 496$

$n=7$
· for $2^n-1=127$

↳ $127 \times 64 = 8128$

....

↳ what is the pattern here?.....

· We notice that

$n=2$ ↳ $3 \times 2^{n-1} = 3 \times 2^{2-1} = 3 \times 2^1 = 3 \times 2 = 6$

$n=3$ ↳ $7 \times 2^{n-1} = 7 \times 2^{3-1} = 7 \times 2^2 = 7 \times 4 = 28$

$n=5$ ↳ $31 \times 2^{n-1} = 31 \times 2^{5-1} = 31 \times 2^4 = 31 \times 16 = 496$

$n=7$ ↳ $127 \times 2^{n-1} = 127 \times 2^{7-1} = 127 \times 2^6 = 127 \times 64 = 8128$

....

· So the pattern here is that:

↳ if 2^n-1 is a Mersenne Prime then: $(2^n-1)(2^{n-1}) =$ perfect number

→ proof on next page.....

Let's Prove this.....

lets consider the perfect number 496.

Recall that we said a perfect number is a number whose sum of factors equals twice the number.

↳So: $496 \Rightarrow \underbrace{1+2+4+8+16+31+62+124+248+496}_{\text{we can notice a pattern here.}} = 992 = 2(496)$

• we can identify a pattern here by isolating the Mersenne Prime, 31, out of the sequence of sum of factors, i.e.:

$1+2+4+8+16+31+62+124+248+496 = (1+2+4+8+16) + 31(1+2+4+8+16)$

$31\times1=31$
$31\times2=62$
$31\times4=124$
$31\times8=248$
$31\times16=496$

• recall that for 496, the corresponding Mersenne Prime is $2^n-1=31$ for which $n=5$.

• So we can rewrite our sequence as

$496 \Rightarrow (1+2+4+8+16)+31(1+2+4+8+16)$
$= (1+31)\cdot(1+2+4+8+16)$
$= (32)\cdot(31) \quad \} \text{———} \quad (2^n)\cdot(2^n-1) = 2^5\cdot 2^5-1 = 32\cdot 31$
$= 992$
$= 2(496)$

• Notice how we can break this sequence of additions into $(2^n)\cdot(2^n-1)$ where n is the corresponding value such that 2^n-1 is the matching Mersenne Prime.

• we see from this example that $(2^n)\cdot(2^n-1) = 2(\text{perfect number})$.

• This implies that by factoring out 2 we get:

$$(2^{n-1})\cdot(2^n-1) = \text{perfect number}.$$

• We can even see this from our example case for which the perfect number is 496 and $n=5$; $(2^{5-1})\cdot(2^5-1) = 2^4\cdot(2^5-1) = 16\cdot 31 = 496$

• Now let's prove this formally for the general case.

⇒Proof:

① Calculate sum of factors of $2^{n-1}(2^n-1)$

$$2^{n-1}(2^n-1) \xRightarrow{\text{sum of factors}} = (1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1})(2^n-1)$$

$$+(2^n-1)+2^1(2^n-1)+2^2(2^n-1)+2^3(2^n-1)+\dots+2^{n-2}(2^n-1)+2^{n-1}(2^n-1)$$

$$= (1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1})+$$

$$(2^n-1)(1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1})$$

$$= (1+2^n-1)(1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1})$$

$$= 2^n(1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1})$$

$$\left\{\begin{array}{l} \text{let } T = 1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1} \\ \text{then } 2T = 2^1+2^2+2^3+2^4+\dots+2^{n-1}+2^n \\ \text{then } 2T-T = 2^n-1 \\ \text{so } T = 2^n-1 \end{array}\right\} \text{geometric series}$$

$$\therefore 1+2^1+2^2+2^3+\dots+2^{n-2}+2^{n-1} = 2^n-1$$

$$= 2^n(2^n-1)$$

↳ recall that the sum of factors should equal twice the original perfect number, so to get the perfect number we must factor out 2.

$$\therefore 2(2^{n-1})(2^n-1) = 2\times \text{perfect number}$$

$$\boxed{\therefore (2^{n-1})(2^n-1) = \text{perfect number as long as } 2^n-1 \text{ is a Mersenne Prime,}}$$

because the sum of factors of the expression $(2^{n-1})(2^n-1)$ equals twice the expression, which is the definition of a perfect number

⇒ Optimized Approach

- Now that we understand that if 2^n-1 is a Mersenne Prime then $(2^n-1)(2^{n-1})$ Is a perfect number
- We also saw that if 2^n-1 is a Mersenne Prime then n must be a prime.
- So we can generate our prime numbers within a our range of values a and b., let p be an entry in the list of prime numbers
- Then for each prime number, p, apply the formula: 2^n-1 and then check if the resulting value is a prime.
  if it is then we have just found a Mersenne Prime, and it's n value
- After Iterating through our list of primes, we should have a corresponding filtered list of Mersenne Primes, and their corresponding n-value.
- Then we can iterate through this filtered list and apply the formula: $(2^n-1)(2^{n-1})$ to get our perfect numbers.
  - during this step we will add bounds checking to make sure that for a perfect number x, it's $a \leq x \leq b$.

* Before we do this we must understand ~~how to gener~~

① How to generate primes.

② How to check if a number is a prime.

① Generating Primes with Sieve of Eratosthenes.

Algorithm: find primes up to N.

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For all numbers a : from 2 to sqrt(n)
    IF a is unmarked THEN
        a is prime
        for all multiples of a: (a<n)
            mark multiples as composite
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~~②~~

② Check if a number is a prime

- Start with 2
  - Check if number is divisble by 2.
  - if it is, it's not a prime

- Check for prime divisors: *optionall this can be done w/ just odd number divisors
  - if the number isn't divisible by 2, more onto the next prime number and see if its divisible
- Stop at the square root: *The smallest factor of a number greater than one cannot be greater than the square root of that number*
  - Continue checking for prime divisors until you reach a number greater than the square root of the original number.
- If you don't find any prime divisors then the number is prime.

⇒ Now we can summarize our approach: to Generate Perfect Numbers within range A & B

① Generate Prime numbers within B.
  ↳ use Sieve of Eratosthenes

② Generate Mersenne Primes from Prime numbers
  ↳ use formula $2^p - 1$, and verify if result is a prime number

③ From Mersenne Primes use $(2^n - 1) \cdot (2^{n-1})$ to Generate Perfect Numbers
  ↳ use boundary check to verify that the generated perfect number falls within A & B.