Variational Ridging in Sea Ice Models

Andrew Roberts

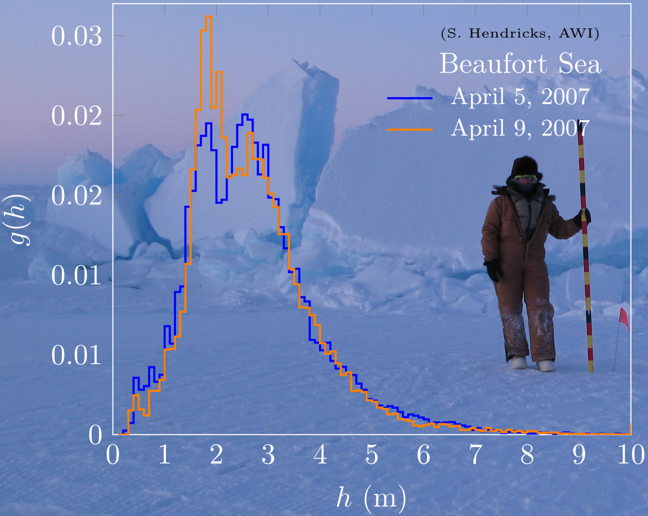
Naval Postgraduate School

Elizabeth Hunke, Samy Kamal, William Lipscomb, Wieslaw Maslowski

C23E: Integrating Observations and Models to Better Understand a Changing Arctic Sea Ice Cover

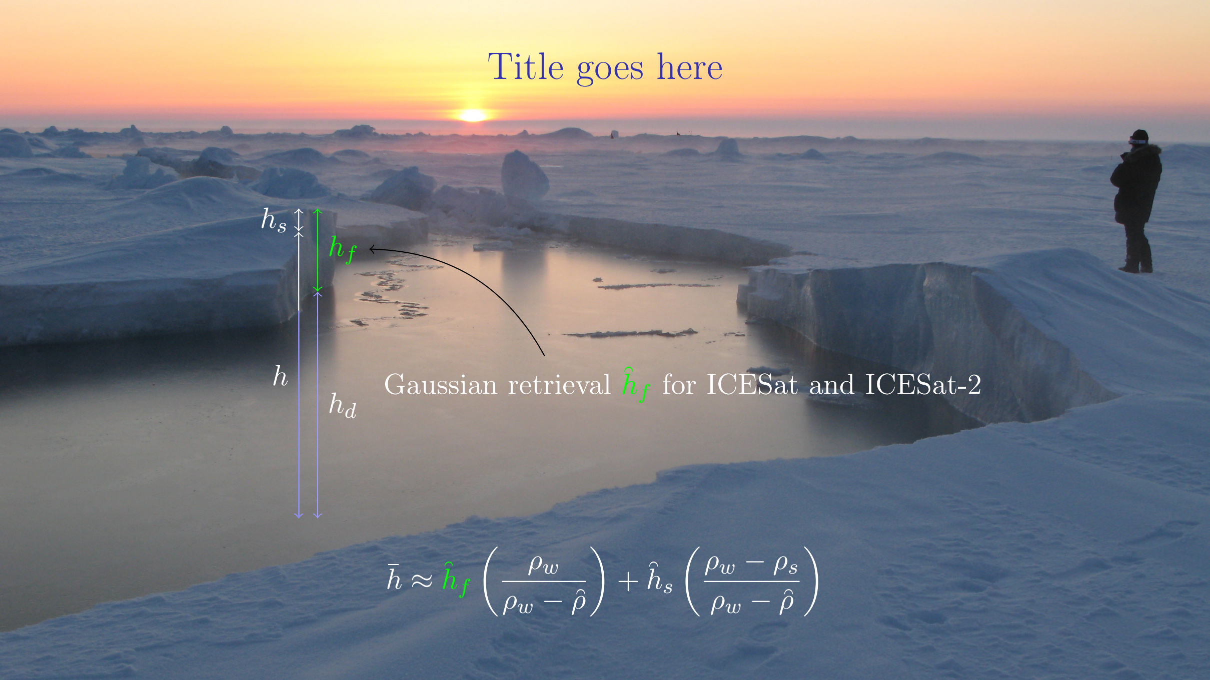
Fall AGU, New Orleans, 12 December, 2017

Sea Ice Thickness Distribution

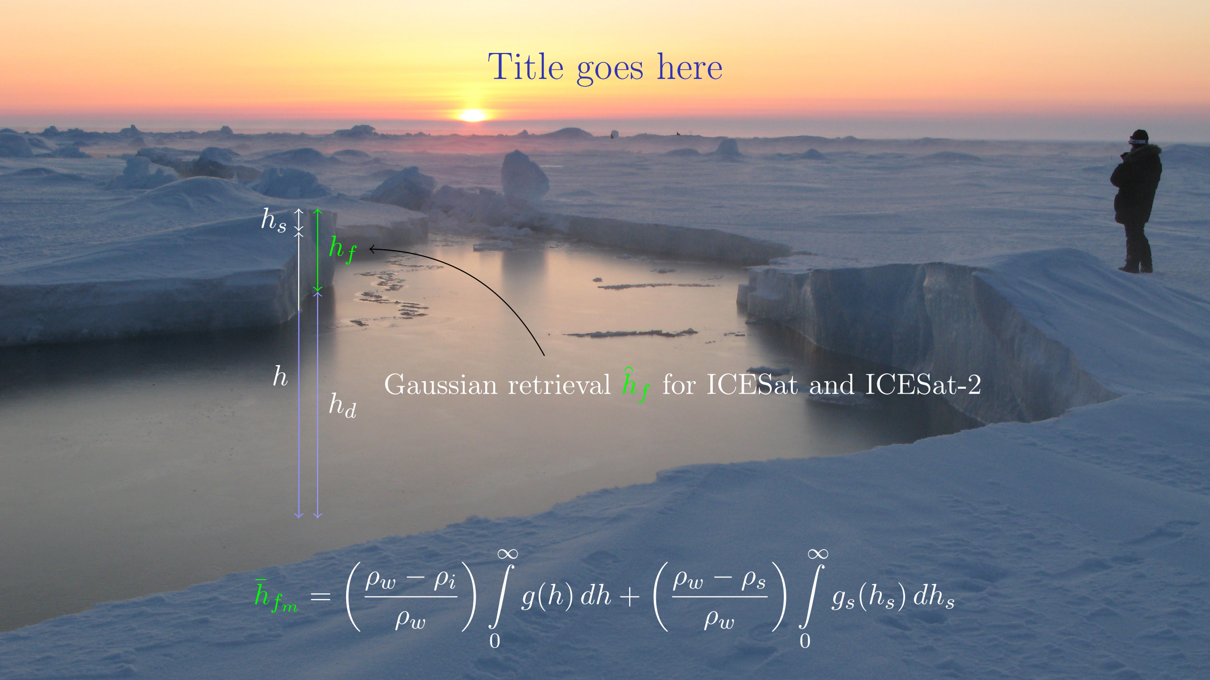


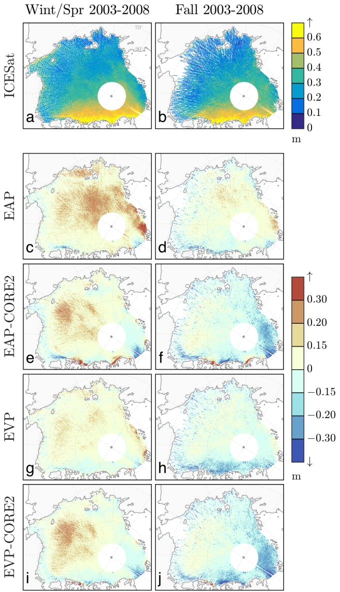
$$m = \rho \int_0^\infty g(h)\, h\, dh$$

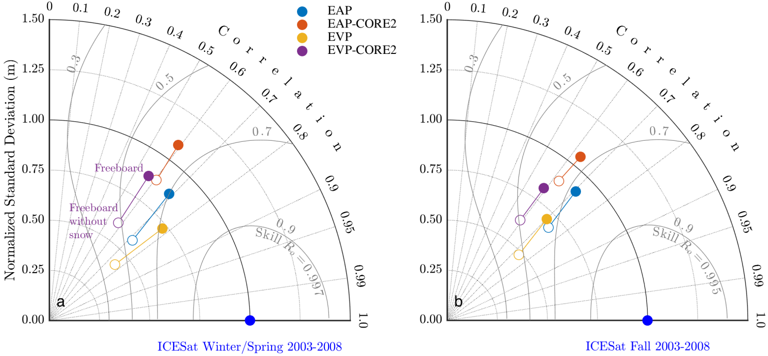
Title goes here



$$\bar{h} \approx \hat{h}_f \left(\frac{\rho_w}{\rho_w - \hat{\rho}} \right) + \hat{h}_s \left(\frac{\rho_w - \rho_s}{\rho_w - \hat{\rho}} \right)$$


Title goes here
h_s
h_f
h
h_d
Gaussian retrieval $\hat{h}_f$ for ICESat and ICESat-2
$$\bar{h}_{f_m} = \left(\frac{\rho_w - \rho_i}{\rho_w}\right)\int_0^{\infty} g(h)\,dh + \left(\frac{\rho_w - \rho_s}{\rho_w}\right)\int_0^{\infty} g_s(h_s)\,dh_s$$


Wint/Spr 2003-2008
Fall 2003-2008
ICESat
EAP
EAP-CORE2
EVP
EVP-CORE2
a
b
c
d
e
f
g
h
i
j
0.6
0.5
0.4
0.3
0.2
0.1
0
m
0.30
0.20
0.15
0
−0.15
−0.20
−0.30
m


EAP
EAP-CORE2
EVP
EVP-CORE2
Normalized Standard Deviation (m)
Correlation
Freeboard
Freeboard without snow
Skill $R_o = 0.997$
Skill $R_o = 0.995$
a
b
ICESat Winter/Spring 2003-2008
ICESat Fall 2003-2008

Sea ice momentum and continuity

Sea Ice Momentum Equation

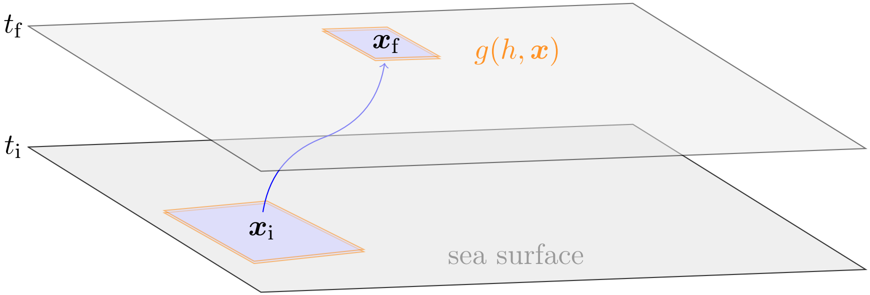
$$m\,\frac{d\dot{\boldsymbol{x}}}{dt} = m\mathrm{f}\,\mathbf{k}\times\dot{\boldsymbol{x}} + \boldsymbol{F}_{\mathrm{b}} + \nabla\cdot\bar{\boldsymbol{\sigma}}$$

for the body force $\boldsymbol{F}_{\mathrm{b}} = \boldsymbol{\tau}_a + \boldsymbol{\tau}_w - m\hat{\mathrm{g}}\nabla\eta$, Cauchy Stress $\boldsymbol{\sigma}$ where $\bar{\boldsymbol{\sigma}} = J\boldsymbol{\sigma}\boldsymbol{J}^{-T}$ and compressive strength $\boldsymbol{\sigma} = P\boldsymbol{f}$ for yield criteria $\boldsymbol{f}$

Sea Ice Continuity Equation

$$\frac{dg}{dt} = \Theta_h + \Psi_h - g(\nabla\cdot\dot{\boldsymbol{x}})$$

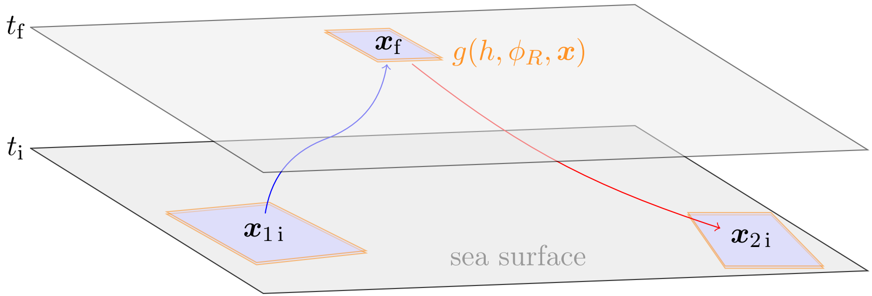
Introduction to Variational Ridging



$$\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \int_A \left(\nabla \cdot \bar{\boldsymbol{\sigma}} - m \, \frac{d\dot{\boldsymbol{x}}}{dt} \right) \cdot \delta \boldsymbol{x} \; dA \; dt = 0$$

Non-Conservative Stationary Ridging Action

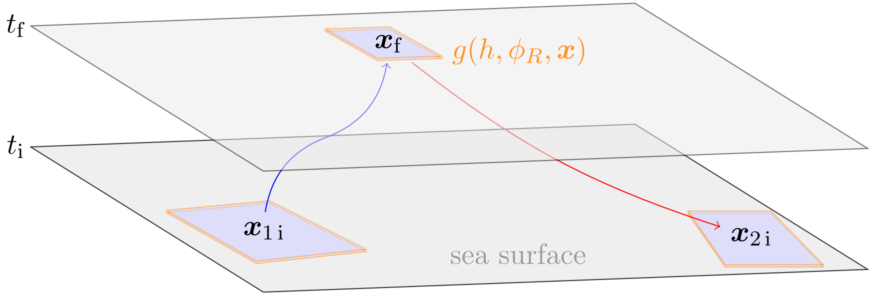
$$\mathfrak{f}\colon\, \mathfrak{g}(h, \phi, s, \varepsilon, \boldsymbol{x}) \mapsto g(h, \phi_R, \boldsymbol{x})$$



$$\mathcal{S}[\boldsymbol{x}_1, \boldsymbol{x}_2] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \int_A \Omega[\boldsymbol{x}_1, \boldsymbol{x}_2]\, dA\, dt$$

Galley, C. R., D. Tsang, and L. C. Stein (2014), The principle of stationary nonconservative action for classical mechanics and field theories, arXiv:1412.3082

Non-Conservative Stationary Ridging Action



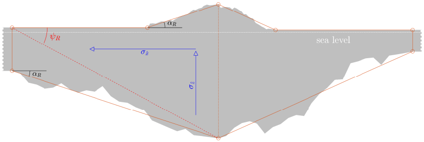
$$\mathcal{S}[\boldsymbol{x}_1, \boldsymbol{x}_2] = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \int_A \Omega[\boldsymbol{x}_1, \boldsymbol{x}_2] \, dA \, dt$$

For the non-conservative Lagrangian:

$$\Omega[\boldsymbol{x}_1, \boldsymbol{x}_2] = \mathcal{L}(\boldsymbol{x}_1, \partial_\chi \boldsymbol{x}_1, \boldsymbol{X}, t) - \mathcal{L}(\boldsymbol{x}_2, \partial_\chi \boldsymbol{x}_2, \boldsymbol{X}, t) + \mathcal{K}(\boldsymbol{x}_1, \boldsymbol{x}_2, \partial_\chi \boldsymbol{x}_1, \partial_\chi \boldsymbol{x}_2, \boldsymbol{X}, t)$$

Conservative Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, Non-Conservative Potential $\mathcal{K}$

Coarse-Grained Ridge Morphology



System of 14 equations:

- Archimedes' Principle
- Conservation of Volume
- Conservation of Mass
- Geometric Constraints
- Highly Permeable Material

Boundary conditions:

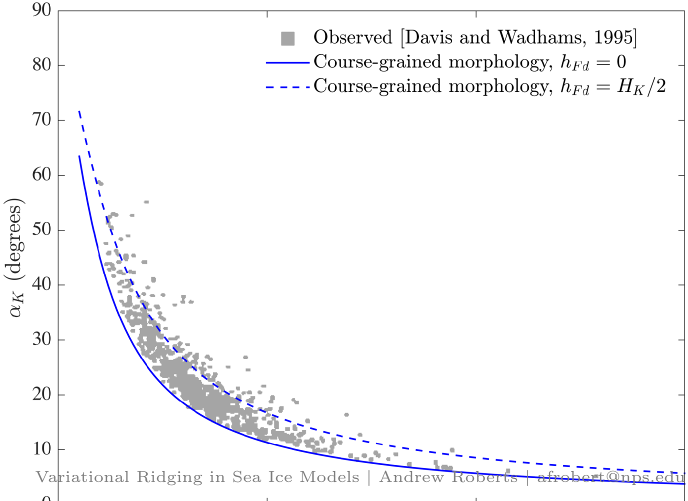
- h_{Fi}, h_{Fs}, θ_R

State variables:

- ϵ_{R_I}, α_R, ϕ_R

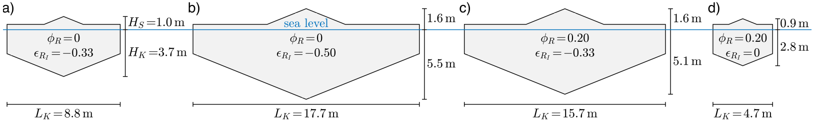
Coarse-Grained Ridge Morphology

Comparison with submarine data



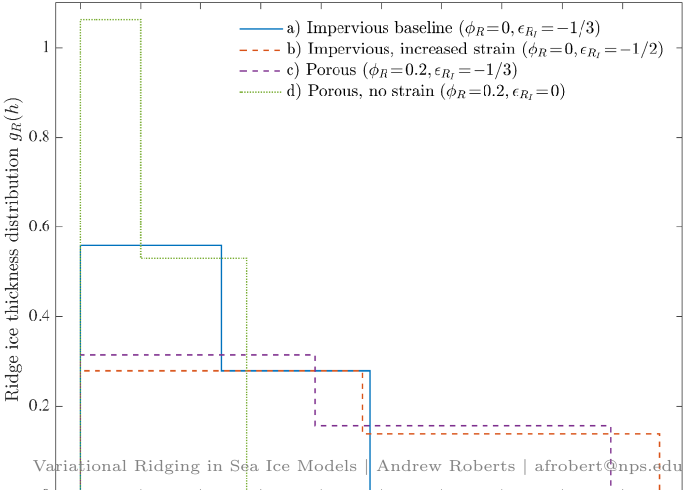
Coarse-Grained Ridge Morphology

Effect of strain ϵ_{R_I} and porosity ϕ_R



Coarse-Grained Ridge Morphology

Effect of strain ϵ_{R_I} and porosity ϕ_R



Coarse-Grained Ridge Mechanics

Applying the principle of non-conservative stationary action gives:

$$0 = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \int_A \left(K_{p_1}\, \delta\mathcal{V}_1 - K_{p_2}\, \delta\mathcal{V}_2 \right) dA\, dt$$

Virtual potential energy along each path $\delta\mathcal{V}_{1,2}$

Coefficients of Passive Stress $K_{p_{1,2}}$ along each path

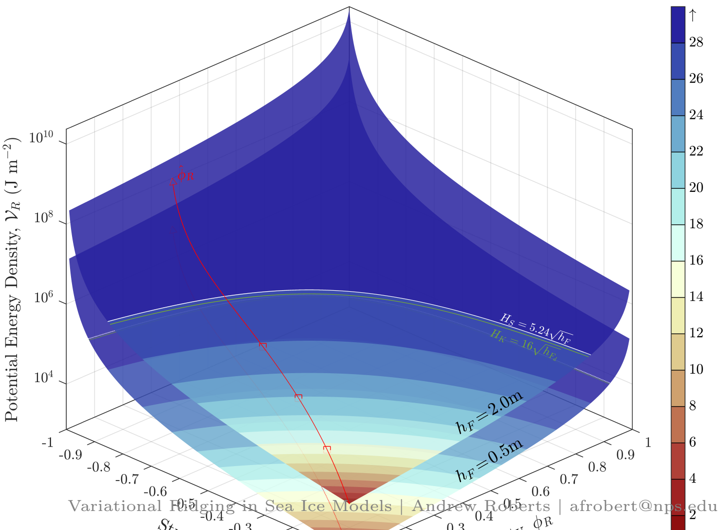
In the physical limit of $\delta\mathcal{V}_1 \to \delta\mathcal{V}_2$, $K_{p_1} \to K_{p_2}$ is constant, which occurs when the following condition is met:

$$\frac{\partial \sigma_{\hat{x}}}{\partial \alpha_R} = 0$$

In this circumstance, $K_p = 3$. Compare this with Hopkins et al. (1991), who obtained the range $2.4 \leq K_p \leq 4.6$ with finite element modeling.

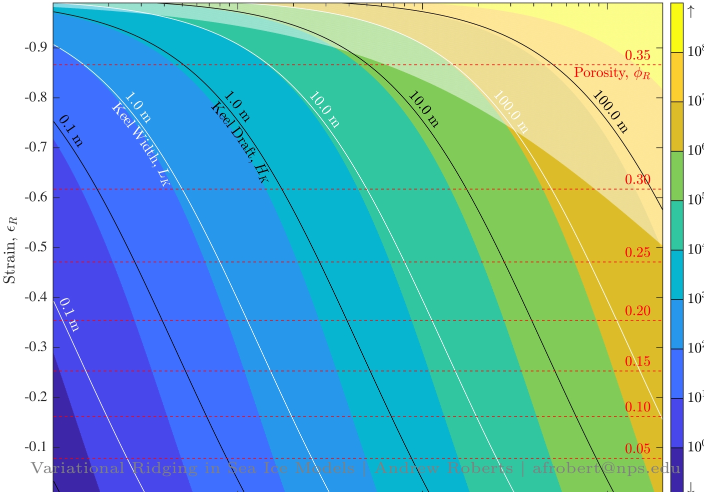
Coarse-Grained Ridge Mechanics

State space of ridges



Coarse-Grained Ridge Mechanics

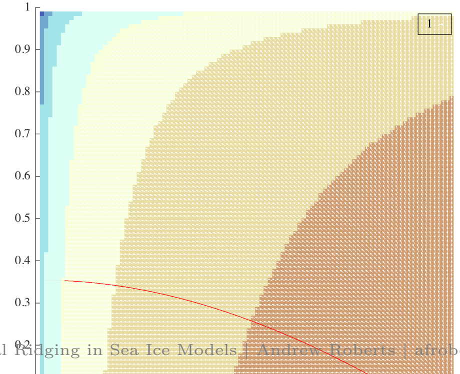
Ridge states with identical energetics (from previous slide)



Coarse-Grained Ridge Mechanics

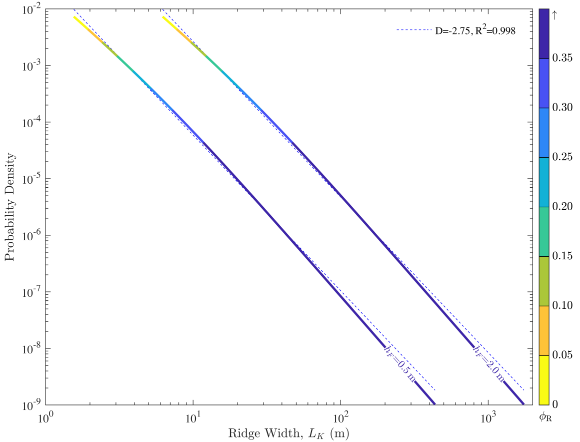
Finding a path through the ϵ_{R_I} and ϕ_R state space.

$$0 = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \int_A \left(\frac{\partial \sigma_{\hat{x}}}{\partial \epsilon_{R_I}} + \frac{\partial \sigma_{\hat{x}}}{\partial \phi_R} \frac{\partial \phi_R}{\partial \epsilon_{R_I}} + \cancelto{0}{\frac{\partial \sigma_{\hat{x}}}{\partial \alpha_R}} \frac{\partial \alpha_R}{\partial \epsilon_{R_I}} \right) \delta \epsilon_{R_I} \, dA \, dt$$



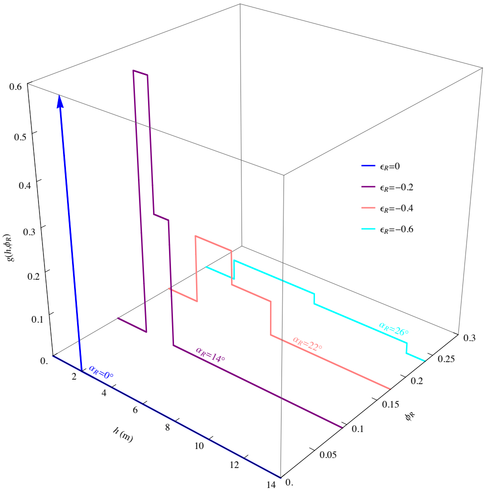
Coarse-Grained Ridge Mechanics

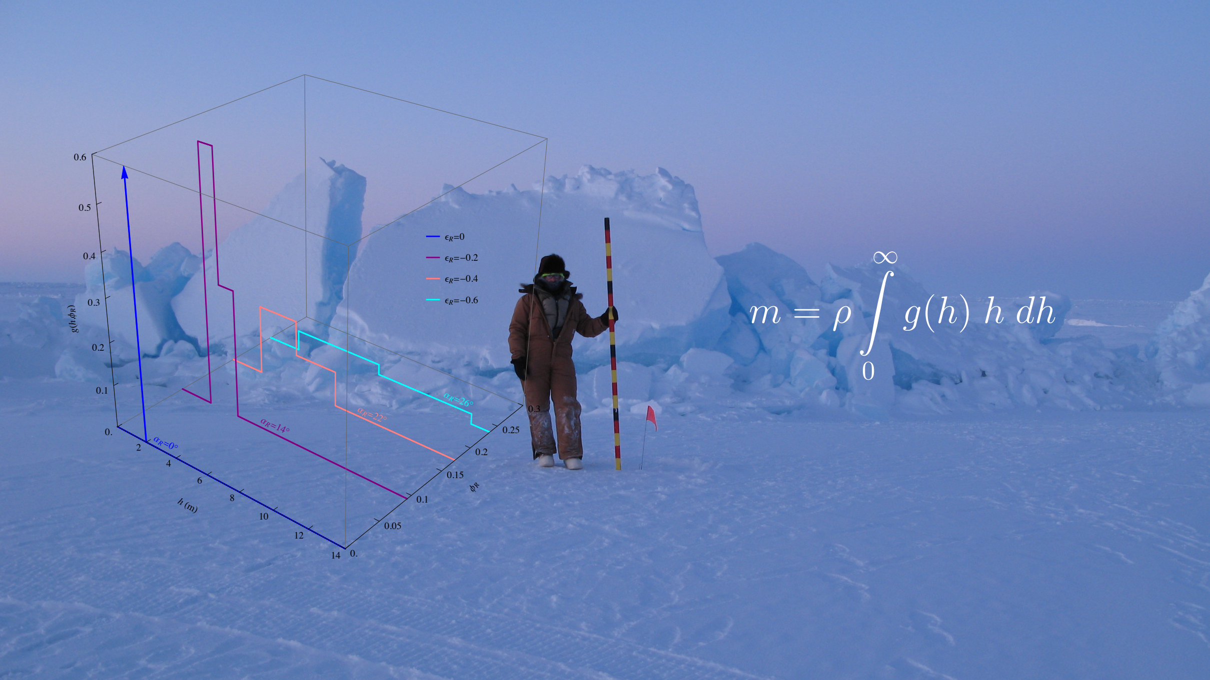
Derived distributions from the Coarse-Grain Model



Coarse-Grained Ridge Mechanics

Derived distributions from the Coarse-Grain Model

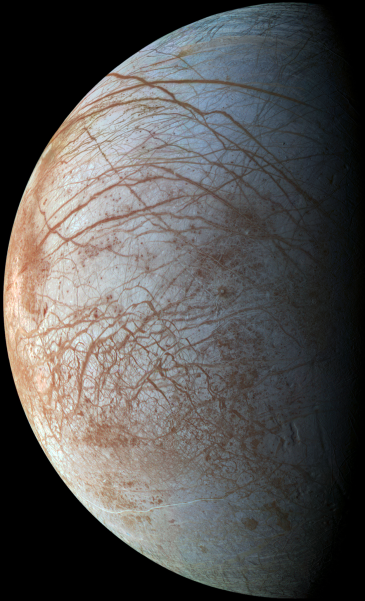




0.6
0.5
0.4
0.3
0.2
0.1
0.
$g(h,\phi_R)$
2
4
6
8
10
12
14
h (m)
0.
0.05
0.1
0.15
0.2
0.25
0.3
ϕ_R
ϵ_R=0
ϵ_R=−0.2
ϵ_R=−0.4
ϵ_R=−0.6
α_R=0°
α_R=14°
α_R=22°
α_R=26°
$m = \rho \int_0^\infty g(h)\ h\ dh$

Conclusions

- By considering sea ice ridging at the level of the action, rather than in Newton's second law, new insights become possible.
- A simple coarse-grained model is able to reproduce many desired features of ridges due to the underlying stationary assumption.
- Using this method, we are able to replace $g(h, \boldsymbol{x})$ with $g(h, \phi_R, \boldsymbol{x})$ to represent macro-porosity of sea ice of $A(\boldsymbol{x})$.
- We do not presume that all sea ice is isostatic.
- The method provides new equations for redistribution Ψ and compressive strength P based on first principles.
- The next stage of this work is to apply it to CICE.



Office of Naval Research

Department of Energy

National Science Foundation

With thanks to Christopher Horvat, Adrian Turner, Jennifer Hutchings, Jackie Richter-Menge and Peter Wadhams.

afrobert@nps.edu