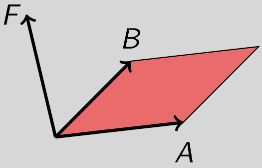
Higher Dimensional Generalizations of Green's Theorem
Chapters 7,8

Daniel Barter

Flux

Let $F = (F_1, F_2, F_3), A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ be vectors.



The flux of F through the paralellagram $A \wedge B$ is exactly

$$F \cdot (A \times B) = \det(F, A, B) = \det \begin{pmatrix} F_1 & F_2 & F_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

Recall: Green's Theorem

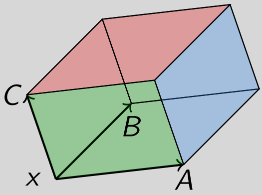
Let F be a vector field on $\mathbb{R}^2$ and $P =$  a very small paralellagram, then

$$\int_{\partial P} F_1(x, y)dx + F_2(x, y)dy = \mathrm{curl}(F)\mathrm{Area}(P).$$

Question: Is there an analog of Green's theorem for Flux?

Divergence Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and P be the parallelepiped spanned by the vectors A, B and C.



The flux of F through P (with an everywhere outward facing normal vector) is

$$\begin{aligned} \text{flux} = & \det(F(x), B, A) + \det(F(x + C), A, B) \\ & + \det(F(x), A, C) + \det(F(x + B), C, A) \\ & + \det(F(x), C, B) + \det(F(x + A), B, C) \end{aligned}$$

Divergence Theorem: local version

When P is very small, we have

$$\begin{aligned}\mathrm{flux} \approx &\det(DF(x)C, A, B)\\ &-\det(DF(x)B, A, C)\\ &+\det(DF(x)A, B, C)\\ &=\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)\mathrm{vol}(P)\end{aligned}$$

$$\mathrm{div}(F) = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

So, the flux though the small parallelepiped P is exactly $\mathrm{div}(F)\mathrm{vol}(P)$.

Flux through a surface

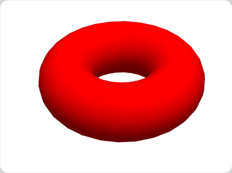
Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface. Choose a parameterization $x(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ of Σ. Then the flux of F through Σ is

$$\int_\Sigma F \cdot d\Sigma = \int_\Sigma \det\left(F(x(s,t)), \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t} \right) dsdt$$

Example

Consider a torus $T \subseteq \mathbb{R}^3$. We can parameterize it by

$$x_1(\theta, \phi) = (2 + \cos\theta)\cos\phi$$
$$x_2(\theta, \phi) = (2 + \cos\theta)\sin\phi$$
$$x_3(\theta, \phi) = \sin\theta$$



The flux of $F = (F_1, F_2, F_3)$ through T is

$$\int_{-\pi}^{\pi} \int_{0}^{2\pi} \det\left(F, \frac{\partial x}{\partial \theta}, \frac{\partial x}{\partial \phi}\right) d\theta d\phi = 0$$

Divergence Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $A \subseteq \mathbb{R}^3$ a closed 3-dimensional region with boundary the surface Σ. Then

$$\int_\Sigma F \cdot d\Sigma = \int_A \mathrm{div}(F) dxdydz$$

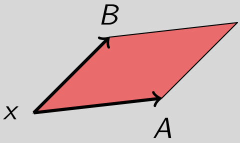
Example

Let $F = (x/3, y/3, z/3)$, $A \subseteq \mathbb{R}^3$ be a closed 3-dimensional region and $\Sigma = \partial A$. Then $\mathrm{div}(F) = 1$ so

$$\text{Volume of } A = \int_A 1 dxdydz = \int_\Sigma F \cdot d\Sigma$$

3D version of Green's Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and choose a small parallelogram $P = A \wedge B$.



The work done by F around P is

$$\begin{aligned} &F(x) \cdot A + F(x+A) \cdot B - F(x) \cdot B - F(x+B) \cdot A \\ &= (DF(x)A) \cdot B - (DF(x)B) \cdot A \\ &= \det(\mathrm{curl}(F), A, B) \end{aligned}$$

The work done by F around P is the flux of $\mathrm{curl}(F)$ through P.

3D version of Green's Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface with boundary curve γ. Then

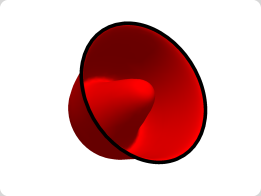
$$\int_\gamma F \cdot d\gamma = \int_\Sigma \mathrm{curl}(F) \cdot d\Sigma$$

where

$$\mathrm{curl}(F) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Example

Consider the surface $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), 2\cos^2(\theta)\sin^3(2\theta))$ bounded by $(\cos(t), \sin(t), 0)$.



Compute the work done by $F = (-y, x, 0)$ around the boundary circle.

Moral of the Story

Whenever you have a field that can be integrated over d-dimensional parallelograms, you should integrate it over the boundary of a small $d+1$-dimensional parallelogram. This way, you can discover the various different versions of Green's theorem as you need them without having to memorize lots of complicated formulas.

There is a common generalization of all these theorem's called Stoke's Theorem. Come to office hours if you want to learn about it :D