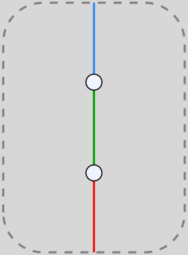
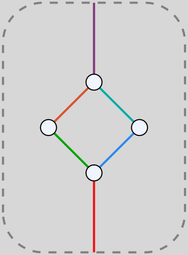
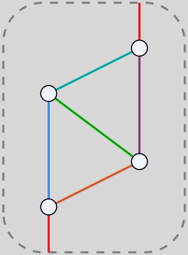
Computing Bimodule Associators in the Brauer-Picard 3-Category

Daniel Barter (ANU) Jacob Bridgeman (PI) Corey Jones (OSU)

The Brauer-Picard 3-Category

- Objects are fusion categories.
- 1-morphisms are bimodule categories.
- 2-morphisms are bimodule functors.
- 3-morphisms are natural transformations.

The Brauer-Picard 3-Category

- Objects are fusion categories.
- 1-morphisms are bimodule categories.
- 2-morphisms are bimodule functors.
- 3-morphisms are natural transformations.

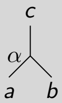
  We are going to give informal definitions of these structures. To make things more precise, we need to work with semi-simple rigid $*$-tensor categories.

Fusion Category

Objects $a, b, c \ldots$

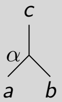
Fusion Category

Objects $a, b, c \ldots$

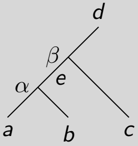
generated by trivalent verticies 

Fusion Category

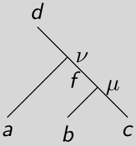
Objects $a, b, c...$

generated by trivalent verticies 

We impose the relation

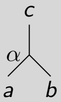


$$= \sum_{\mu,\nu} F^{\mu,\nu}_{\alpha,\beta}$$

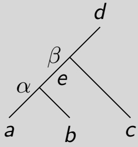


Fusion Category

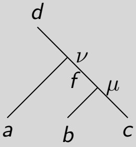
Objects a, b, c...

generated by trivalent verticies 

We impose the relation



$$= \sum_{\mu,\nu} F^{\mu,\nu}_{\alpha,\beta}$$



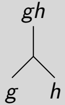
The tensor F must satisfy the pentagon equation, also known as the 3-2 Pachner equation.

Example: $\mathbf{Vec}(G)$: G-graded vector spaces

Let G be a finite group and $g, h, k \in G$.

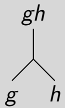
Example: **Vec**(G): G-graded vector spaces

Let G be a finite group and $g, h, k \in G$.

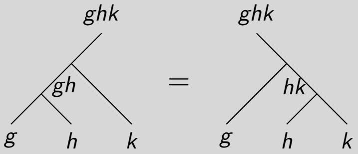
Vec(G) is generated by 

Example: **Vec**(G): G-graded vector spaces

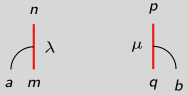
Let G be a finite group and $g, h, k \in G$.

Vec(G) is generated by 

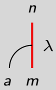
and the generators satisfy the relation



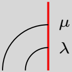
Bimodules

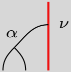
Generated by trivalent vertices 

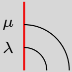
Bimodules

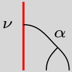
Generated by trivalent vertices 

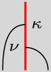
We impose the relations



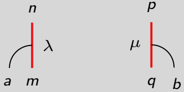
$$= \sum_{\alpha,\nu} L^{\alpha,\nu}_{\lambda,\mu}$$




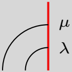
$$= \sum_{\alpha,\nu} R^{\alpha,\nu}_{\lambda,\mu}$$


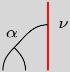
$$= \sum_{\nu,\kappa} C^{\nu,\kappa}_{\lambda,\mu}$$


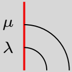
Bimodules

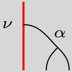
Generated by trivalent vertices 

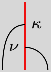
We impose the relations



$$= \sum_{\alpha,\nu} L^{\alpha,\nu}_{\lambda,\mu}$$


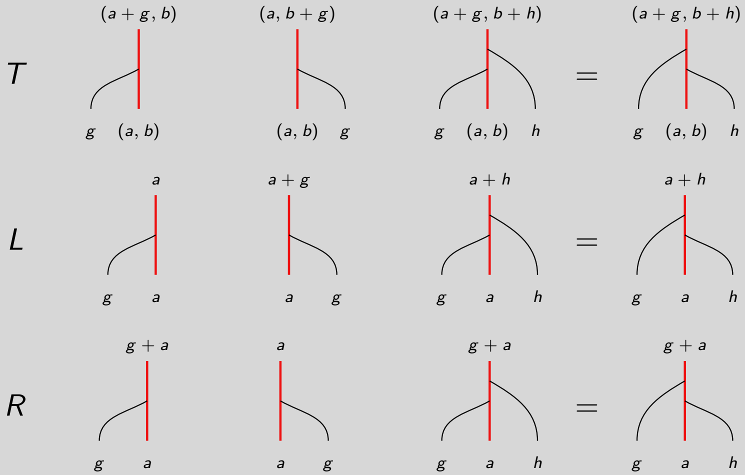


$$= \sum_{\alpha,\nu} R^{\alpha,\nu}_{\lambda,\mu}$$


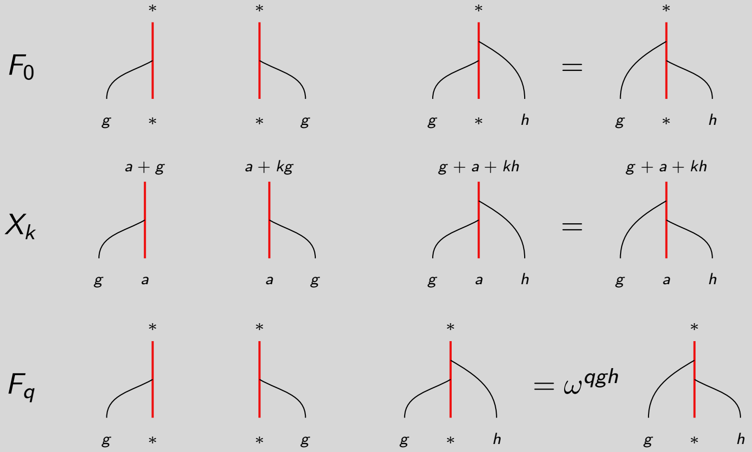
$$= \sum_{\nu,\kappa} C^{\nu,\kappa}_{\lambda,\mu}$$


The tensors L, R, C satisfy a dizzying number of coherence equations.

$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$–$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ bimodules



$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})-\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ bimodules



ω is a p-th root of unity.

Bimodule Functors $M \to N$

Let A, B be fusion categories and $A \curvearrowright M, N \curvearrowleft B$ bimodules.

Bimodule Functors $M \to N$

Let A, B be fusion categories and $A \curvearrowright M, N \curvearrowleft B$ bimodules.

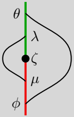
Generated by bivalent verticies 

Bimodule Functors $M \to N$

Let A, B be fusion categories and $A \curvearrowright M, N \curvearrowleft B$ bimodules.

Generated by bivalent verticies 

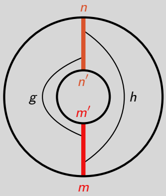
- We impose the relation



$$= \sum_{\eta} A^{\eta}_{\theta,\phi,\lambda,\mu,\zeta}$$

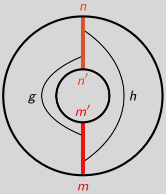
2-Bimodule Annular Category

$\mathbb{C}\Bigg\{$



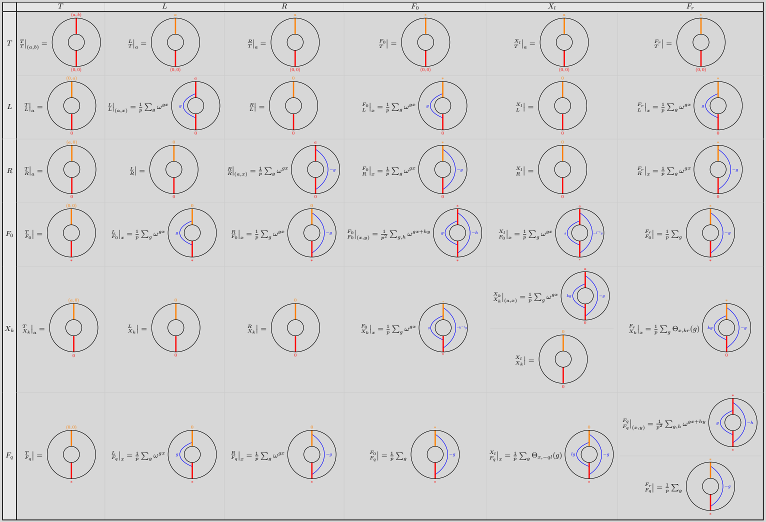
$\Bigg\}$

2-Bimodule Annular Category

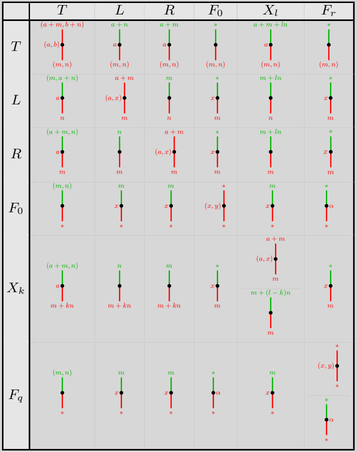
$\mathbb{C}\Bigg\{$  $\Bigg\}$

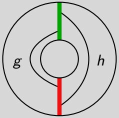
$$\{\text{bimodule functors}\} \leftrightarrow \left\{\begin{matrix}\text{annular category}\\ \text{representations}\end{matrix}\right\}$$

$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$-Bimodule Functor idempotents: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



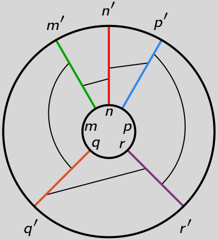
$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$-Bimodule Functor representations



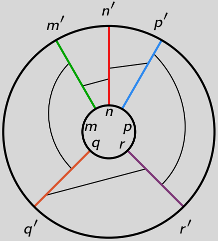
Acting with 

	T	L	R	F_0	X_l	F_r
T	$(a+m+g,b+n+h)$ (a,b) $(m+g,n+h)$	$a+n+h$ a $(m+g,n+h)$	$a+m+g$ a $(m+g,n+h)$	$*$ $(m+g,n+h)$	$a+m+g+l(n+h)$ a $(m+g,n+h)$	ω^{-rgn} $*$ $(m+g,n+h)$
L	$(m+g,a+n+h)$ a $n+h$	ω^{-gx} $a+m+h$ (a,x) $m+h$	$m+g$ $n+h$	ω^{-gx} $*$ x $m+h$	$m+g+l(n+h)$ $n+h$	$\omega^{-g(x+mr)}$ $*$ x $m+h$
R	$(a+m+g,n+h)$ a $m+g$	$n+h$ $m+g$	ω^{hx} $a+m+g$ (a,x) $m+g$	ω^{hx} $*$ x $m+g$	$m+g+l(n+h)$ $m+g$	$\omega^{h(x+r(m+g))}$ $*$ x $m+g$
F_0	$(m+g,n+h)$ $*$	ω^{-gx} $m+h$ x $*$	ω^{hx} $m+g$ x $*$	ω^{-gx+hy} $*$ (x,y) $*$	ω^{-gx} $m+g+lh$ x $*$	$\omega^{hr(\alpha+g)}$ $*$ $\alpha+g$ $*$
X_k	$(a+m+g,n+h)$ a $m+g+k(n+h)$	$n+h$ $m+g+k(n+h)$	$m+g$ $m+g+k(n+h)$	ω^{-gx} $*$ x $m+g+kh$	ω^{hx} $a+m+g+kh$ (a,x) $m+g+kh$ $m+(l-k)n+g+lh$ $m+g+kh$	$\Theta_{x,kr}(h)\omega^{hr(g+m)}$ $*$ x $m+g+kh$
F_q	ω^{qgn} $(m+g,n+h)$ $*$	$\omega^{g(mq-x)}$ $m+h$ x $*$	$\omega^{h(x-q(m+g))}$ $m+g$ x $*$	$\omega^{-qh(\alpha+g)}$ $*$ $\alpha+g$ $*$	$\Theta_{x,-ql}(h)\omega^{-qh(g+m)}$ $m+g+lh$ x $*$	ω^{-gx+hy} $*$ (x,y) $*$ $\omega^{h(r-q)(\alpha+g)}$ $*$ $\alpha+g$ $*$

5-Bimodule Annular Category

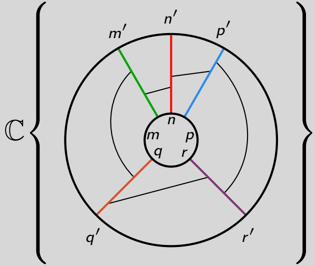
$\mathbb{C}$ {  }

5-Bimodule Annular Category

$\mathbb{C}\Bigg\{$  $\Bigg\}$

$$\left\{\begin{array}{l}\text{Bimodule functors}\\ Q \otimes_A R \to M \otimes_B N \otimes_C P\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{annular category}\\ \text{representations}\end{array}\right\}$$

5-Bimodule Annular Category



$$\begin{Bmatrix}\text{Bimodule functors} \\ Q \otimes_A R \to M \otimes_B N \otimes_C P\end{Bmatrix} \leftrightarrow \begin{Bmatrix}\text{annular category} \\ \text{representations}\end{Bmatrix}$$

These categories are called *sphere categories* by Morrison and Walker in *Blob Homology*.

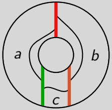
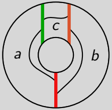
$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ Brauer-Picard Ring

$\otimes_{\mathbb{Z}/p\mathbb{Z}}$	T	L	R	F_0	X_l	F_r
T	$p \cdot T$	T	$p \cdot R$	R	T	R
L	$p \cdot L$	L	$p \cdot F_0$	F_0	L	F_0
R	T	$p \cdot T$	R	$p \cdot R$	R	T
F_0	L	$p \cdot L$	F_0	$p \cdot F_0$	F_0	L
X_k	T	L	R	F_0	X_{kl}	$F_{k^{-1}r}$
F_q	L	T	F_0	R	F_{ql}	$X_{q^{-1}r}$

$\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ Brauer-Picard Ring

$\otimes_{\mathbb{Z}/p\mathbb{Z}}$	T	L	R	F_0	X_l	F_r
T	$p \cdot T$	T	$p \cdot R$	R	T	R
L	$p \cdot L$	L	$p \cdot F_0$	F_0	L	F_0
R	T	$p \cdot T$	R	$p \cdot R$	R	T
F_0	L	$p \cdot L$	F_0	$p \cdot F_0$	F_0	L
X_k	T	L	R	F_0	X_{kl}	$F_{k^{-1}r}$
F_q	L	T	F_0	R	F_{ql}	$X_{q^{-1}r}$

The explicit isomorphisms that underlie this table are 3-bimodule annular category representations.

Acting with  and 

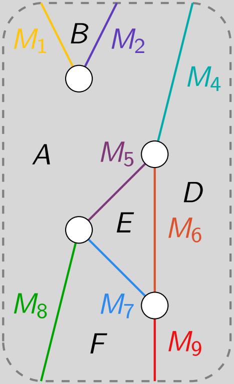
	Decomposition	Basis vectors	Action
T	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-c\mu}$
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		$\omega^{-cr(b+n)}$
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-cq(a+m)}$
L	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-c\mu}$
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		$\omega^{-cr(b+n)}$
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		ω^{-aqs}

	Decomposition	Basis vectors	Action
R	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		ω^{brs}
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-c\mu}$
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-cq(a+m)}$
F_0	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		ω^{brs}
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-c\mu}$
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		ω^{-aqs}
X_x	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, x = kl$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r, x = q^{-1}r$		$\omega^{-c(q(a+m)+br)}$
F_y	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r, y = k^{-1}r$		$\omega^{\frac{br(a+m)}{k}}$
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, y = ql$		ω^{-anq}

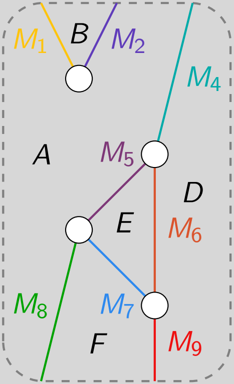
	Decomposition	Basis vectors	Action
T	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-c\mu}$
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		$\omega^{-cr(b+n)}$
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-cq(a+m)}$
L	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		$\omega^{-c\mu}$
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		$\omega^{-cr(b+n)}$
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} L$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} T$		ω^{aqs}

	Decomposition	Basis vectors	Action
R	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$T \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		ω^{-brs}
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-c\mu}$
	$R \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-cq(a+m)}$
F_0	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$L \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r$		ω^{-brs}
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		$\omega^{-c\mu}$
	$F_0 \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l$		
	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_0$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} R$		ω^{aqs}
X_x	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, x = kl$		
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r, x = q^{-1}r$		$\omega^{-c(q(a+m)+br)}$
F_y	$X_k \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} F_r, y = k^{-1}r$		$\omega^{-\frac{br(a+m)}{k}}$
	$F_q \otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})} X_l, y = ql$		ω^{anq}

Domain Wall Structures



Domain Wall Structures



Domain Wall Structure algorithm: Assign annular category representations to all the internal holes. The DWS algorithm computes the resulting compound annular category representation.

Domain Wall Structure Algorithm:

1. Construct a compound defect by filling the holes in the domain wall structure with vectors from the corresponding annular category representations, subject to the labels on the internal edges agreeing.

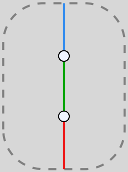
Domain Wall Structure Algorithm:

1. Construct a compound defect by filling the holes in the domain wall structure with vectors from the corresponding annular category representations, subject to the labels on the internal edges agreeing.
2. Quotient out the bubble action for each internal cavity.

Domain Wall Structure Algorithm:

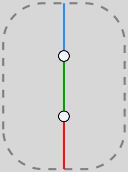
1. Construct a compound defect by filling the holes in the domain wall structure with vectors from the corresponding annular category representations, subject to the labels on the internal edges agreeing.
2. Quotient out the bubble action for each internal cavity.
3. Compute all relevant idempotent actions on the quotient representation. This lets us decompose the quotient representation into simple annular category representations.

Examples

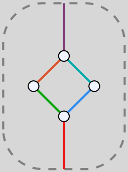


functor composition.

Examples

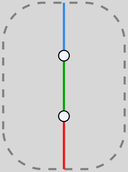


functor composition.

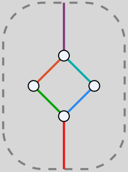


functor relative tensor product.

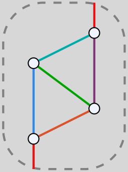
Examples



functor composition.



functor relative tensor product.



bimodule associator.

Papers:

Domain walls in topological phases and the Brauer-Picard ring for $\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$, arXiv:1806.01279

Fusing binary interface defects in topological phases: The $\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})$ *case*, arXiv:1810.09469

Computing defects associated to bounded domain wall structures: The $\mathbb{Z}/p\mathbb{Z}$ *case*, arXiv:1901.08069