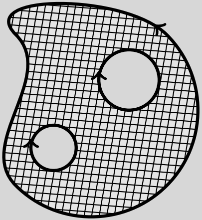
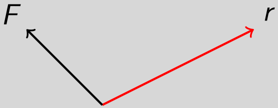
Green's Theorem
Chapters 5,6
Section 7.1

Daniel Barter



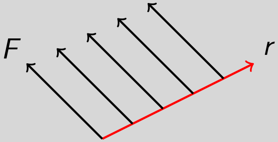
Work

Let $F = (F_1, F_2)$ be a force vector (newtons) and $r = (r_1, r_2)$ be a displacement vector (meters).

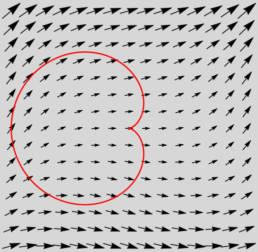


The work (newton meters = joules) done by the force F while a point particle is displaced by r is defined to be

$$F \cdot r = F_1 r_1 + F_2 r_2.$$

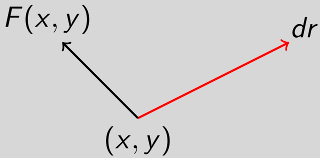


What if the particle doesn't move in a straight line and the force isn't constant?



$$F = (y^2 + 3, x^2 + y)$$
$$r = (x, y) = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \leq t \leq 2\pi$$

Break the path up into small pieces



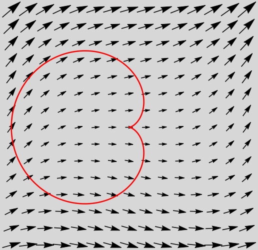
$$F(x,y) = (F_1(x,y), F_2(x,y))$$
$$dr = (dx, dy) = (x'(t)dt, y'(t)dt)$$
$$F(x,y) \cdot dr = F_1(x,y)dx + F_2(x,y)dy$$

work done $= F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$

Sum everything up

$$\text{total work done} = \int_a^b F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$$

Example 1: How much work is done?



$$F = (y^2 + 3, x^2 + y)$$
$$r = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \le t \le 2\pi$$

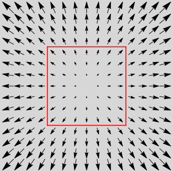
total work $= -5\pi/2 \approx -7.85398$

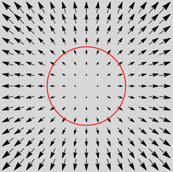
Example 2: How much work is done?

$$G = (x, y)$$

$$r1 = \begin{cases} (1, t) & -1 \leq t \leq 1 \\ (-t, 1) & -1 \leq t \leq 1 \\ (-1, -t) & -1 \leq t \leq 1 \\ (t, -1) & -1 \leq t \leq 1 \end{cases}$$

$$r2 = (\cos(t), \sin(t)) \quad 0 \leq t \leq 2\pi$$





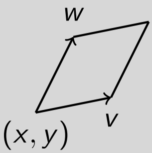
If $r(t)$ is a loop, then we have

$$\begin{aligned}
\int_r G \cdot dr &= \int_a^b x(t)x'(t) + y(t)y'(t)dt \\
&= \int_a^b r(t) \cdot r'(t)dt \\
&= \frac{1}{2}\int_a^b \frac{d}{dt}|r(t)|^2 dt \\
&= |r(b)|^2 - |r(a)|^2 = 0
\end{aligned}$$

Question: What makes the force field $G = (x, y)$ special compared to $F = (y^2 + 3, x^2 + y)$?

Green's Theorem: The fundamental idea

$P =$



$v = (v_1, v_2) \quad w = (w_1, w_2)$

$$W := \int_{\partial P} F_1(x, y)dx + F_2(x, y)dy = ??$$

Green's Theorem: The fundamental idea

$$\begin{aligned} W \approx{} & F_1(x,y)v_1 + F_2(x,y)v_2 \\ & + F_1(x+v_1, y+v_2)w_1 \\ & + F_2(x+v_1, y+v_2)w_2 \\ & - F_1(x,y)w_1 - F_2(x,y)w_2 \\ & - F_1(x+w_1, y+w_2)v_1 \\ & - F_2(x+w_1, y+w_2)v_2 \end{aligned}$$

Green's Theorem: The fundamental idea

$$
\begin{aligned}
W \approx\ & F_1 v_1 + F_2 v_2 \\
& + \left(F_1 + \frac{\partial F_1}{\partial x} v_1 + \frac{\partial F_1}{\partial y} v_2\right) w_1 \\
& + \left(F_2 + \frac{\partial F_2}{\partial x} v_1 + \frac{\partial F_2}{\partial y} v_2\right) w_2 \\
& - F_1 w_1 - F_2 w_2 \\
& - \left(F_1 + \frac{\partial F_1}{\partial x} w_1 + \frac{\partial F_1}{\partial y} w_2\right) v_1 \\
& - \left(F_2 + \frac{\partial F_2}{\partial x} w_1 + \frac{\partial F_2}{\partial y} w_2\right) v_2 \\
& = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)(v_1 w_2 - v_2 w_1)
\end{aligned}
$$

Green's Theorem: The fundamental idea

If $P =$ (x, y)  is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{Area}(P)$$

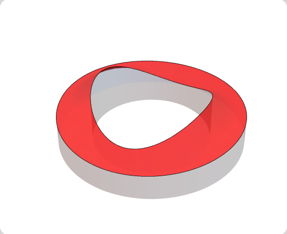
We define

$$\mathrm{curl}(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{pmatrix}$$

Double Integrals

Suppose that $A \subseteq \mathbb{R}^2$ is a *closed* region and $f : A \to \mathbb{R}$ is a function. Then

$$\int_A f(x, y)dxdy = \text{Volume under the graph of } f.$$



$$f(x, y) = 4x^2 e^{-x^2-y^2} + 1 \qquad 2 \leq x^2 + y^2 \leq 5$$

We can take *closed* to mean that $\partial A \subseteq A$ in practice, but in theory, precisely defining *closed* is a subtle issue.

Changing Coordinates

$$\int_{2\leq x^2+y^2\leq 5}(4x^2e^{-x^2-y^2}+1)dxdy$$
$$= 4\int_{2\leq x^2+y^2\leq 5}x^2e^{-x^2-y^2}dxdy + 21\pi$$

We want to change to polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dx = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \sin\theta dr + r\cos\theta d\theta$$

Parallelogram rules

$$drdr = 0 \quad \text{(parallelogram has zero area)}$$
$$d\theta dr = -drd\theta \quad \text{(parallelogram has reverse orientation)}$$

$$\begin{aligned} dxdy &= (\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta) \\ &= r\cos^2\theta drd\theta - r\sin^2\theta d\theta dr \\ &= r(\cos^2\theta + \sin^2\theta)drd\theta = rdrd\theta \end{aligned}$$

$$\int_{2\le x^2+y^2\le 5} x^2 e^{-x^2-y^2}\,dxdy = \int_0^{2\pi}\int_2^5 r^3 e^{-r^2}\cos^2\theta\,drd\theta$$

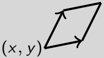
Green's Theorem

Let $A \subseteq \mathbb{R}^2$ be a closed region and F a vector field on A. Then

$$\int_{\partial A} F_1(x,y)dx + F_2(x,y)dy = \int_A \mathrm{curl}(F)dxdy$$

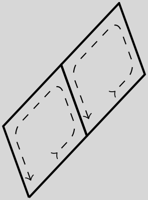
Important: You need to orient the boundary ∂A in the correct way! Boundary components for internal holes are oriented clockwise and outside boundary components are oriented counterclockwise.

Proof:

If $P =$ 

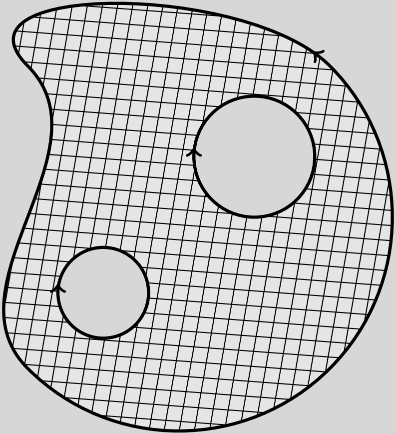
is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \mathrm{curl}(F)\mathrm{Area}(P)$$



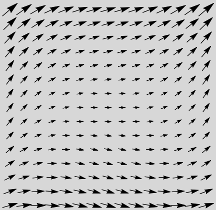
$$\int_{\gamma} F \cdot dr = -\int_{-\gamma} F \cdot dr$$

Proof:



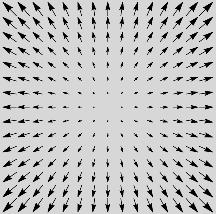
What makes G special compared to F

$F = (y^2 + 3, x^2 + y)$



$\mathrm{curl}(F) = 2x - 2y$

$G = (x, y)$



$\mathrm{curl}(G) = 0$

Example

Let $A \subseteq \mathbb{R}^2$ be a closed region. Then

$$\int_{\partial A} x dy = \int_A 1 dx dy = \text{area of } A$$

Therefore you can compute the area of A as a line integral around its boundary.

Potentials

Suppose that $A \subseteq \mathbb{R}^2$ is a closed region and $f : A \to \mathbb{R}$ is smooth function.

$\mathrm{curl}(\mathrm{grad}(f)) = 0$.

$(x, y) = \mathrm{grad}(x^2/2 + y^2/2)$.

Suppose that F is a vector field and $F = \mathrm{grad}(f)$. We call f a *potential* for F.

$$\begin{aligned}\mathrm{Work} &= \int_\gamma F \cdot dr = \int_\gamma \mathrm{grad}(f) \cdot dr = \int_a^b \mathrm{grad}(f)(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_b^a \frac{d}{dt} f(\gamma(t)) dt = f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

If a potential exists, work equals difference in potential.

Existence of potentials

Question: Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\mathrm{curl}(F) = 0$. When does a potential exist?

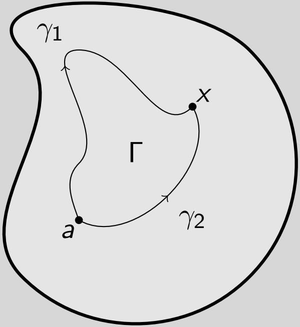
Potential formula: Fix $a \in A$. Then the potential is given by

$$f(x) = \int_\gamma F \cdot dr$$

where γ is a path in A from a to x.

Question: When is the right hand side independent of γ?

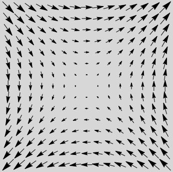
If A has no holes, then potentials always exist.



$$\int_{\gamma_2} F \cdot dr - \int_{\gamma_1} F \cdot dr = \int_{\Gamma} \mathrm{curl}(F) dxdy = 0$$

Example

Consider the vector field $F = (y, x)$.

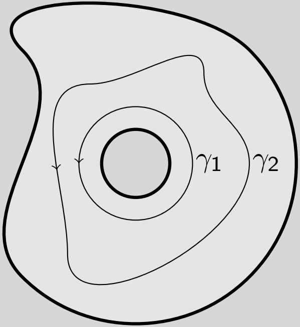


The potential is given by

$$f(a,b) = \int_{(0,0)}^{(a,b)} ydx + xdy$$

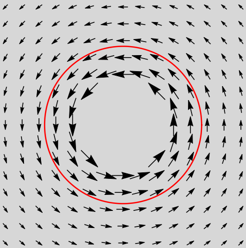
Using the path $x = ta, y = tb$ we get $f(a,b) = ab$.

If A has holes, then a potential *may not* exist.



$$\text{work around hole} = \int_{\gamma_1} F \cdot dr = \int_{\gamma_2} F \cdot dr$$

Example



$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$$\gamma(t) = (\cos(t), \sin(t))$$

$\mathrm{curl}(F) = 0$. Potential doesn't exist because the force field does 2π work around the origin.

Theorem

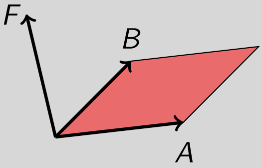
Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\mathrm{curl}(F) = 0$. If the work done by F around each hole is zero, then a potential exists.

Higher Dimensional Generalizations of Green's Theorem
Chapters 7,8

Daniel Barter

Flux

Let $F = (F_1, F_2, F_3), A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ be vectors.



The flux of F through the paralellagram $A \wedge B$ is exactly

$$F \cdot (A \times B) = \det(F, A, B) = \det \begin{pmatrix} F_1 & F_2 & F_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

Recall: Green's Theorem

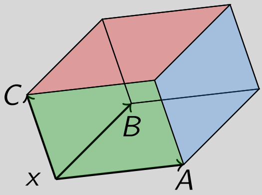
Let F be a vector field on $\mathbb{R}^2$ and $P =$  a very small paralellagram, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \mathrm{curl}(F)\mathrm{Area}(P).$$

Question: Is there an analog of Green's theorem for Flux?

Divergence Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and P be the parallelepiped spanned by the vectors A, B and C.



The flux of F through P (with an everywhere outward facing normal vector) is

$$\begin{aligned} \text{flux} = &\det(F(x), B, A) + \det(F(x + C), A, B) \\ &+ \det(F(x), A, C) + \det(F(x + B), C, A) \\ &+ \det(F(x), C, B) + \det(F(x + A), B, C) \end{aligned}$$

Divergence Theorem: local version

When P is very small, we have

$$\begin{aligned}\mathrm{flux} \approx & \det(DF(x)C, A, B) \\ & - \det(DF(x)B, A, C) \\ & + \det(DF(x)A, B, C) \\ & = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{vol}(P)\end{aligned}$$

$$\mathrm{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

So, the flux though the small parallelepiped P is exactly $\mathrm{div}(F)\mathrm{vol}(P)$.

Flux through a surface

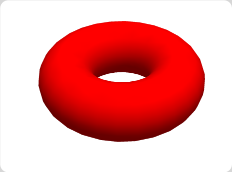
Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface. Choose a parameterization $x(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ of Σ. Then the flux of F through Σ is

$$\int_\Sigma F \cdot d\Sigma = \int_\Sigma \det\left(F(x(s,t)), \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right) dsdt$$

Example

Consider a torus $T \subseteq \mathbb{R}^3$. We can parameterize it by

$$\begin{aligned} x_1(\theta, \phi) &= (2 + \cos\theta)\cos\phi \\ x_2(\theta, \phi) &= (2 + \cos\theta)\sin\phi \\ x_3(\theta, \phi) &= \sin\theta \end{aligned}$$



The flux of $F = (F_1, F_2, F_3)$ through T is

$$\int_{-\pi}^{\pi} \int_{0}^{2\pi} \det\left(F, \frac{\partial x}{\partial \theta}, \frac{\partial x}{\partial \phi}\right) d\theta d\phi = 0$$

Divergence Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $A \subseteq \mathbb{R}^3$ a closed 3-dimensional region with boundary the surface Σ. Then

$$\int_\Sigma F \cdot d\Sigma = \int_A \mathrm{div}(F)\, dxdydz$$

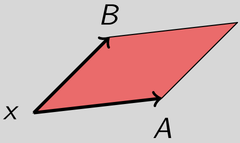
Example

Let $F = (x/3, y/3, z/3)$, $A \subseteq \mathbb{R}^3$ be a closed 3-dimensional region and $\Sigma = \partial A$. Then $\mathrm{div}(F) = 1$ so

$$\text{Volume of } A = \int_A 1 dxdydz = \int_\Sigma F \cdot d\Sigma$$

3D version of Green's Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and choose a small parallelogram $P = A \wedge B$.



The work done by F around P is

$$
\begin{aligned}
&F(x) \cdot A + F(x + A) \cdot B - F(x) \cdot B - F(x + B) \cdot A \\
&= (DF(x)A) \cdot B - (DF(x)B) \cdot A \\
&= \det(\mathrm{curl}(F), A, B)
\end{aligned}
$$

The work done by F around P is the flux of $\mathrm{curl}(F)$ through P.

3D version of Green's Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface with boundary curve γ. Then

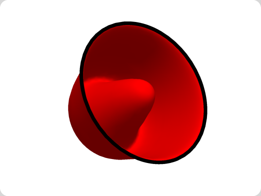
$$\int_\gamma F \cdot d\gamma = \int_\Sigma \mathrm{curl}(F) \cdot d\Sigma$$

where

$$\mathrm{curl}(F) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Example

Consider the surface $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), 2\cos^2(\theta)\sin^3(2\theta))$ bounded by $(\cos(t), \sin(t), 0)$.



Compute the work done by $F = (-y, x, 0)$ around the boundary circle.

Moral of the Story

Whenever you have a field that can be integrated over d-dimensional parallelograms, you should integrate it over the boundary of a small $d+1$-dimensional parallelogram. This way, you can discover the various different versions of Green's theorem as you need them without having to memorize lots of complicated formulas.

There is a common generalization of all these theorem's called Stoke's Theorem. Come to office hours if you want to learn about it :D