Green's Theorem
Chapters 5,6
Section 7.1

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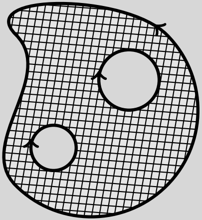
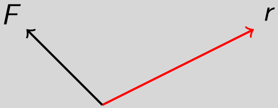


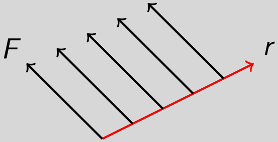
Work

Let $F = (F_1, F_2)$ be a force vector (newtons) and $r = (r_1, r_2)$ be a displacement vector (meters).

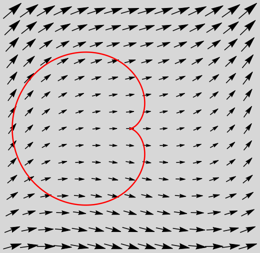


The work (newton meters = joules) done by the force F while a point particle is displaced by r is defined to be

$$F \cdot r = F_1 r_1 + F_2 r_2.$$



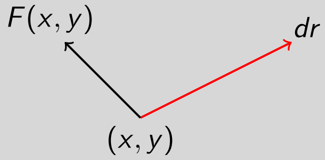
What if the particle doesn't move in a straight line and the force isn't constant?



$$F = (y^2 + 3, x^2 + y)$$

$$r = (x, y) = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \le t \le 2\pi$$

Break the path up into small pieces



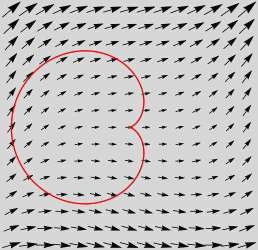
$$\begin{aligned} F(x,y) &= (F_1(x,y), F_2(x,y)) \\ dr &= (dx, dy) = (x'(t)dt, y'(t)dt) \\ F(x,y)\cdot dr &= F_1(x,y)dx + F_2(x,y)dy \end{aligned}$$

$$\text{work done} = F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$$

Sum everything up

$$\text{total work done} = \int_a^b F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$$

Example 1: How much work is done?



$$F = (y^2 + 3, x^2 + y)$$

$$r = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \le t \le 2\pi$$

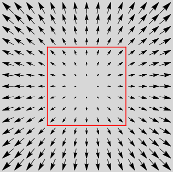
total work $= -5\pi/2 \approx -7.85398$

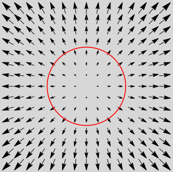
Example 2: How much work is done?

$$G = (x, y)$$

$$r1 = \begin{cases} (1, t) & -1 \le t \le 1 \\ (-t, 1) & -1 \le t \le 1 \\ (-1, -t) & -1 \le t \le 1 \\ (t, -1) & -1 \le t \le 1 \end{cases}$$

$$r2 = (\cos(t), \sin(t)) \quad 0 \le t \le 2\pi$$



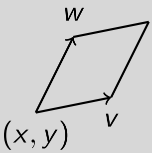


If $r(t)$ is a loop, then we have

$$\begin{aligned}\int_r G \cdot dr &= \int_a^b x(t)x'(t) + y(t)y'(t)dt \\ &= \int_a^b r(t) \cdot r'(t)dt \\ &= \frac{1}{2}\int_a^b \frac{d}{dt}|r(t)|^2 dt \\ &= |r(b)|^2 - |r(a)|^2 = 0\end{aligned}$$

Question: What makes the force field $G = (x, y)$ special compared to $F = (y^2 + 3, x^2 + y)$?

Green's Theorem: The fundamental idea

$P =$ 

$v = (v_1, v_2) \quad w = (w_1, w_2)$

$$W := \int_{\partial P} F_1(x, y)dx + F_2(x, y)dy = ??$$

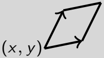
Green's Theorem: The fundamental idea

$$
\begin{aligned}
W \approx\ & F_1(x, y)v_1 + F_2(x, y)v_2 \\
& + F_1(x + v_1, y + v_2)w_1 \\
& + F_2(x + v_1, y + v_2)w_2 \\
& - F_1(x, y)w_1 - F_2(x, y)w_2 \\
& - F_1(x + w_1, y + w_2)v_1 \\
& - F_2(x + w_1, y + w_2)v_2
\end{aligned}
$$

Green's Theorem: The fundamental idea

$$
\begin{aligned}
W \approx\ & F_1 v_1 + F_2 v_2 \\
& + \left(F_1 + \frac{\partial F_1}{\partial x} v_1 + \frac{\partial F_1}{\partial y} v_2 \right) w_1 \\
& + \left(F_2 + \frac{\partial F_2}{\partial x} v_1 + \frac{\partial F_2}{\partial y} v_2 \right) w_2 \\
& - F_1 w_1 - F_2 w_2 \\
& - \left(F_1 + \frac{\partial F_1}{\partial x} w_1 + \frac{\partial F_1}{\partial y} w_2 \right) v_1 \\
& - \left(F_2 + \frac{\partial F_2}{\partial x} w_1 + \frac{\partial F_2}{\partial y} w_2 \right) v_2 \\
& = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) (v_1 w_2 - v_2 w_1)
\end{aligned}
$$

Green's Theorem: The fundamental idea

If $P =$ 

is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{Area}(P)$$

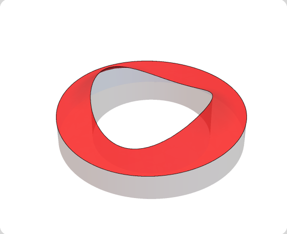
We define

$$\mathrm{curl}(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{pmatrix}$$

Double Integrals

Suppose that $A \subseteq \mathbb{R}^2$ is a *closed* region and $f : A \to \mathbb{R}$ is a function. Then

$$\int_A f(x,y)dxdy = \text{Volume under the graph of } f.$$



$$f(x,y) = 4x^2 e^{-x^2-y^2} + 1 \qquad 2 \leq x^2 + y^2 \leq 5$$

We can take *closed* to mean that $\partial A \subseteq A$ in practice, but in theory, precisely defining *closed* is a subtle issue.

Changing Coordinates

$$\int_{2\leq x^2+y^2\leq 5}(4x^2e^{-x^2-y^2}+1)dxdy$$
$$= 4\int_{2\leq x^2+y^2\leq 5} x^2e^{-x^2-y^2}dxdy + 21\pi$$

We want to change to polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dx = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \sin\theta dr + r\cos\theta d\theta$$

Parallelogram rules

$$drdr = 0 \quad \text{(parallelogram has zero area)}$$
$$d\theta dr = -drd\theta \quad \text{(parallelogram has reverse orientation)}$$

$$\begin{aligned} dxdy &= (\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta) \\ &= r\cos^2\theta drd\theta - r\sin^2\theta d\theta dr \\ &= r(\cos^2\theta + \sin^2\theta)drd\theta = rdrd\theta \end{aligned}$$

$$\int_{2\le x^2+y^2\le 5} x^2 e^{-x^2-y^2}\,dxdy = \int_0^{2\pi}\int_2^5 r^3 e^{-r^2}\cos^2\theta\,drd\theta$$

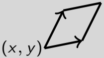
Green's Theorem

Let $A \subseteq \mathbb{R}^2$ be a closed region and F a vector field on A. Then

$$\int_{\partial A} F_1(x,y)dx + F_2(x,y)dy = \int_A \mathrm{curl}(F)dxdy$$

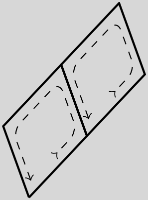
Important: You need to orient the boundary ∂A in the correct way! Boundary components for internal holes are oriented clockwise and outside boundary components are oriented counterclockwise.

Proof:

If $P =$ 

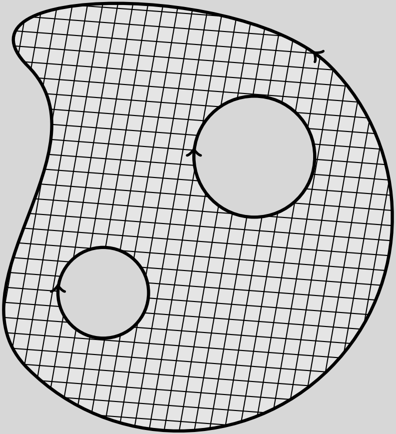
is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \mathrm{curl}(F)\mathrm{Area}(P)$$



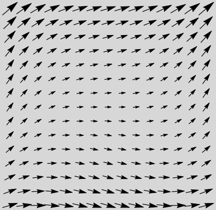
$$\int_{\gamma} F \cdot dr = -\int_{-\gamma} F \cdot dr$$

Proof:



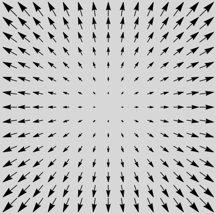
What makes G special compared to F

$F = (y^2 + 3, x^2 + y)$



$\mathrm{curl}(F) = 2x - 2y$

$G = (x, y)$



$\mathrm{curl}(G) = 0$

Example

Let $A \subseteq \mathbb{R}^2$ be a closed region. Then

$$\int_{\partial A} x dy = \int_A 1 dx dy = \text{area of } A$$

Therefore you can compute the area of A as a line integral around its boundary.

Potentials

Suppose that $A \subseteq \mathbb{R}^2$ is a closed region and $f : A \to \mathbb{R}$ is smooth function.

$\mathrm{curl}(\mathrm{grad}(f)) = 0$.

$(x, y) = \mathrm{grad}(x^2/2 + y^2/2)$.

Suppose that F is a vector field and $F = \mathrm{grad}(f)$. We call f a *potential* for F.

$$\begin{aligned}\mathrm{Work} = \int_\gamma F \cdot dr = \int_\gamma \mathrm{grad}(f) \cdot dr = \int_a^b \mathrm{grad}(f)(\gamma(t)) \cdot \gamma'(t) dt \\ = \int_b^a \frac{d}{dt} f(\gamma(t)) dt = f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

If a potential exists, work equals difference in potential.

Existence of potentials

Question: Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\mathrm{curl}(F) = 0$. When does a potential exist?

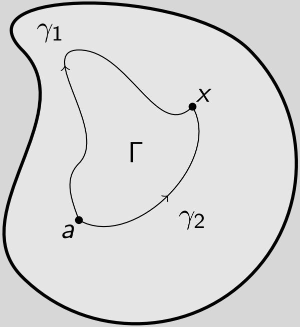
Potential formula: Fix $a \in A$. Then the potential is given by

$$f(x) = \int_\gamma F \cdot dr$$

where γ is a path in A from a to x.

Question: When is the right hand side independent of γ?

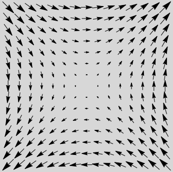
If A has no holes, then potentials always exist.



$$\int_{\gamma_2} F \cdot dr - \int_{\gamma_1} F \cdot dr = \int_{\Gamma} \mathrm{curl}(F) dx dy = 0$$

Example

Consider the vector field $F = (y, x)$.

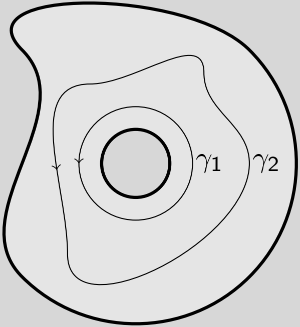


The potential is given by

$$f(a, b) = \int_{(0,0)}^{(a,b)} ydx + xdy$$

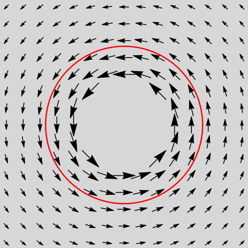
Using the path $x = ta, y = tb$ we get $f(a, b) = ab$.

If A has holes, then a potential *may not* exist.



$$\text{work around hole} = \int_{\gamma_1} F \cdot dr = \int_{\gamma_2} F \cdot dr$$

Example



$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$$\gamma(t) = (\cos(t), \sin(t))$$

$\mathrm{curl}(F) = 0$. Potential doesn't exist because the force field does 2π work around the origin.

Theorem

Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\operatorname{curl}(F) = 0$. If the work done by F around each hole is zero, then a potential exists.