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Gaussian Integral

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The Gaussian integral, also called the probability integral and closely related to the erf function, is the integral of the one-dimensional Gaussian function over $(-\infty, \infty)$. It can be computed using the trick of combining two one-dimensional Gaussians

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)} \quad (1)$$

$$= \sqrt{\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)} \quad (2)$$

$$= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx}. \quad (3)$$

Here, use has been made of the fact that the variable in the integral is a dummy variable that is integrates out in the end and hence can be renamed from x to y. Switching to polar coordinates then gives

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta} \quad (4)$$

$$= \sqrt{2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^{\infty}} \quad (5)$$

$$= \sqrt{\pi}. \quad (6)$$

There also exists a simple proof of this identity that does not require transformation to polar coordinates (Nicholas and Yates 1950).

The integral from 0 to a finite upper limit a can be given by the continued fraction

$$\int_0^a e^{-t^2}\,dt = \tfrac{1}{2}\sqrt{\pi}\,\operatorname{erf}(a) \quad (7)$$

$$= \tfrac{1}{2}\sqrt{\pi} - \frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\ldots}, \quad (8)$$

where $\operatorname{erf} x$ is erf (the error function), as first stated by Laplace, proved by Jacobi, and rediscovered by Ramanujan (Watson 1928; Hardy 1999, pp. 8-9).

The general class of integrals of the form

$$I_n(a) \equiv \int_0^{\infty} e^{-ax^2}\,x^n\,dx \quad (9)$$

can be solved analytically by setting

$$x \equiv a^{-1/2}\,y \quad (10)$$

$$dx = a^{-1/2}\,dy \quad (11)$$

$$y^2 = a\,x^2. \quad (12)$$

Then

$$I_n(a) = a^{-1/2}\int_0^{\infty} e^{-y^2}\left(a^{-1/2}\,y\right)^n\,dy \quad (13)$$

$$= a^{-(n+1)/2}\int_0^{\infty} e^{-y^2}\,y^n\,dy. \quad (14)$$

For $n = 0$, this is just the usual Gaussian integral, so

$$I_0(a) = \frac{\sqrt{\pi}}{2}\,a^{-1/2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}. \quad (15)$$

For $n = 1$, the integrand is integrable by quadrature,

$$I_1(a) = a^{-1}\int_0^{\infty} e^{-y^2}\,y\,dy = a^{-1}\left[-\tfrac{1}{2}e^{-y^2}\right]_0^{\infty} = \tfrac{1}{2}a^{-1}. \quad (16)$$

To compute $I_n(a)$ for $n > 1$, use the identity

$$-\frac{\partial}{\partial a}I_{n-2}(a) = -\frac{\partial}{\partial a}\int_0^{\infty} e^{-ax^2}\,x^{n-2}\,dx \quad (17)$$

$$= -\int_0^{\infty} -x^2\,e^{-ax^2}\,x^{n-2}\,dx \quad (18)$$

$$= \int_0^{\infty} e^{-ax^2}\,x^n\,dx \quad (19)$$

$$= I_n(a). \quad (20)$$

For $n = 2s$ even,

$$I_n(a) = \left(-\frac{\partial}{\partial a}\right)I_{n-2}(a) \quad (21)$$

$$(22)$$

$$= \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}$$

$$= \ldots = \left(-\frac{\partial}{\partial a}\right)^{n/2} I_0(a) \quad (23)$$

$$= \frac{\partial^{n/2}}{\partial a^{n/2}} I_0(a) \quad (24)$$

$$= \frac{\sqrt{\pi}}{2} \frac{\partial^{n/2}}{\partial a^{n/2}} a^{-1/2}, \quad (25)$$

so

$$\int_0^\infty x^{2s} e^{-a x^2} dx = \frac{\left(s-\frac{1}{2}\right)!}{2 a^{s+1/2}} \quad (26)$$

$$= \frac{(2s-1)!!}{2^{s+1} a^s} \sqrt{\frac{\pi}{a}}, \quad (27)$$

where $n!!$ is a double factorial. If $n = 2s+1$ is odd, then

$$I_n(a) = \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a) \quad (28)$$

$$= \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}(a) \quad (29)$$

$$= \ldots = \left(-\frac{\partial}{\partial a}\right)^{(n-1)/2} I_1(a) \quad (30)$$

$$= \frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} I_1(a) \quad (31)$$

$$= \frac{1}{2} \frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} a^{-1}, \quad (32)$$

so

$$\int_0^\infty x^{2s+1} e^{-a x^2} dx = \frac{s!}{2 a^{s+1}}. \quad (33)$$

The solution is therefore

$$\int_0^\infty e^{-a x^2} x^n dx = \begin{cases} \frac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\frac{\pi}{a}} & \text{for } n \text{ even} \\ \frac{\left[\frac{1}{2}(n-1)\right]!}{2 a^{(n+1)/2}} & \text{for } n \text{ odd.} \end{cases} \quad (34)$$

The first few values are therefore

$$I_0(a) = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (35)$$

$$I_1(a) = \frac{1}{2a} \quad (36)$$

$$I_2(a) = \frac{1}{4a} \sqrt{\frac{\pi}{a}} \quad (37)$$

$$I_3(a) = \frac{1}{2a^2} \quad (38)$$

$$I_4(a) = \frac{3}{8a^2} \sqrt{\frac{\pi}{a}} \quad (39)$$

$$I_5(a) = \frac{1}{a^3} \quad (40)$$

$$I_6(a) = \frac{15}{16a^3} \sqrt{\frac{\pi}{a}}. \quad (41)$$

A related, often useful integral is

$$H_n(a) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-a x^2} x^n dx, \quad (42)$$

which is simply given by

$$H_n(a) = \begin{cases} \frac{2 I_n(a)}{\sqrt{\pi}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (43)$$

The more general integral of $x^n e^{-a x^2 + b x}$ has the following closed forms,

$$\int_{-\infty}^\infty x^n e^{-a x^2 + b x} dx = i^{-n} a^{-(n+1)/2} \sqrt{\pi} e^{b^2/(4a)} U\left(-\tfrac{1}{2} n; \tfrac{1}{2}; -b^2/4a\right) \quad (44)$$

$$= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!} \frac{(2b)^{n-2k}}{(4a)^{n-k}} \quad (45)$$

$$= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! (2a)^{k-n} b^{n-2k} \quad (46)$$

for integer $n > 0$ (F. Pilolli, pers. comm.). For (45) and (46), $a, b \in \mathbb{C} - \{0\}$ (the punctured plane), $\mathfrak{R}[a] > 0$, and $(-1)!! = 1$. Here, $U(a; b; x)$ is a confluent hypergeometric function of the second kind and $\binom{n}{k}$ is a binomial coefficient.

SEE ALSO:
Erf, Gauss Integral, Gaussian Function, Leibniz Integral Rule, Normal Distribution

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