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Gaussian Integral

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The Gaussian integral, also called the [probability integral](#) and closely related to the [erf](#) function, is the integral of the one-dimensional [Gaussian function](#) over $(-\infty, \infty)$. It can be computed using the trick of combining two one-dimensional Gaussians

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)} \tag{1}$$

$$= \sqrt{\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)} \tag{2}$$

$$= \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx} . \tag{3}$$

Here, use has been made of the fact that the variable in the integral is a [dummy variable](#) that is integrates out in the end and hence can be renamed from x to y . Switching to [polar coordinates](#) then gives

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta} \tag{4}$$

$$= \sqrt{2\pi \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty}} \tag{5}$$

$$= \sqrt{\pi} . \tag{6}$$

There also exists a simple proof of this identity that does not require transformation to [polar coordinates](#) (Nicholas and Yates 1950).

The integral from 0 to a finite upper limit a can be given by the [continued fraction](#)

$$\int_0^a e^{-t^2} dt = \frac{1}{2} \sqrt{\pi} \operatorname{erf}(a) \tag{7}$$

$$= \frac{1}{2} \sqrt{\pi} - \frac{e^{-a^2}}{2a} - \frac{1}{a} - \frac{2}{2a} - \frac{3}{a} - \frac{4}{2a} + \dots , \tag{8}$$

where $\operatorname{erf} x$ is [erf](#) (the error function), as first stated by Laplace, proved by Jacobi, and rediscovered by Ramanujan (Watson 1928; Hardy 1999, pp. 8-9).

The general class of integrals of the form

$$I_n(a) \equiv \int_0^{\infty} e^{-ax^2} x^n dx \tag{9}$$

can be solved analytically by setting

$$x \equiv a^{-1/2} y \tag{10}$$

$$dx = a^{-1/2} dy \tag{11}$$

$$y^2 = ax^2 . \tag{12}$$

Then

$$I_n(a) = a^{-1/2} \int_0^{\infty} e^{-y^2} (a^{-1/2} y)^n dy \tag{13}$$

$$= a^{-(n+1)/2} \int_0^{\infty} e^{-y^2} y^n dy . \tag{14}$$

For $n = 0$, this is just the usual Gaussian integral, so

$$I_0(a) = \frac{\sqrt{\pi}}{2} a^{-1/2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} . \tag{15}$$

For $n = 1$, the integrand is integrable by quadrature,

$$I_1(a) = a^{-1} \int_0^{\infty} e^{-y^2} y dy = a^{-1} \left[-\frac{1}{2} e^{-y^2}\right]_0^{\infty} = \frac{1}{2} a^{-1} . \tag{16}$$

To compute $I_n(a)$ for $n > 1$, use the identity

$$-\frac{\partial}{\partial a} I_{n-2}(a) = -\frac{\partial}{\partial a} \int_0^{\infty} e^{-ax^2} x^{n-2} dx \tag{17}$$

$$= -\int_0^{\infty} -x^2 e^{-ax^2} x^{n-2} dx \tag{18}$$

$$= \int_0^{\infty} e^{-ax^2} x^n dx \tag{19}$$

$$= I_n(a) . \tag{20}$$

For $n = 2s$ even,

$$I_n(a) = \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a) \tag{21}$$

$$\tag{22}$$

erf

THINGS TO TRY:

= erf

= erf of 1/((3+x)sqrtx)

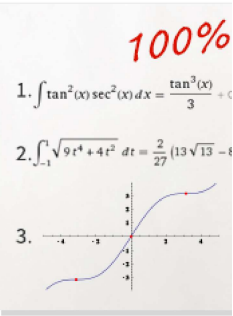
= integral erf(e^(-4x)) from 0 to 2

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Some Gaussian
Integrals

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$$= \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}$$

$$= \dots = \left(-\frac{\partial}{\partial a}\right)^{n/2} I_0(a) \quad (23)$$

$$= \frac{\partial^{n/2}}{\partial a^{n/2}} I_0(a) \quad (24)$$

$$= \frac{\sqrt{\pi}}{2} \frac{\partial^{n/2}}{\partial a^{n/2}} a^{-1/2}, \quad (25)$$

so

$$\int_0^\infty x^{2s} e^{-ax^2} dx = \frac{(s - \frac{1}{2})!}{2 a^{s+1/2}} \quad (26)$$

$$= \frac{(2s-1)!!}{2^{s+1} a^s} \sqrt{\frac{\pi}{a}}, \quad (27)$$

where $n!!$ is a [double factorial](#). If $n = 2s + 1$ is odd, then

$$I_n(a) = \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a) \quad (28)$$

$$= \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}(a) \quad (29)$$

$$= \dots = \left(-\frac{\partial}{\partial a}\right)^{(n-1)/2} I_1(a) \quad (30)$$

$$= \frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} I_1(a) \quad (31)$$

$$= \frac{1}{2} \frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} a^{-1}, \quad (32)$$

so

$$\int_0^\infty x^{2s+1} e^{-ax^2} dx = \frac{s!}{2 a^{s+1}}. \quad (33)$$

The solution is therefore

$$\int_0^\infty e^{-ax^2} x^n dx = \begin{cases} \frac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\frac{\pi}{a}} & \text{for } n \text{ even} \\ \frac{[\frac{1}{2}(n-1)]!}{2 a^{(n+1)/2}} & \text{for } n \text{ odd.} \end{cases} \quad (34)$$

The first few values are therefore

$$I_0(a) = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (35)$$

$$I_1(a) = \frac{1}{2a} \quad (36)$$

$$I_2(a) = \frac{1}{4a} \sqrt{\frac{\pi}{a}} \quad (37)$$

$$I_3(a) = \frac{1}{2a^2} \quad (38)$$

$$I_4(a) = \frac{3}{8a^2} \sqrt{\frac{\pi}{a}} \quad (39)$$

$$I_5(a) = \frac{1}{a^3} \quad (40)$$

$$I_6(a) = \frac{15}{16a^3} \sqrt{\frac{\pi}{a}}. \quad (41)$$

A related, often useful integral is

$$H_n(a) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-ax^2} x^n dx, \quad (42)$$

which is simply given by

$$H_n(a) = \begin{cases} \frac{2 I_n(a)}{\sqrt{\pi}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (43)$$

The more general integral of $x^n e^{-ax^2+bx}$ has the following closed forms,

$$\int_{-\infty}^\infty x^n e^{-ax^2+bx} dx = i^{-n} a^{-(n+1)/2} \sqrt{\pi} e^{b^2/(4a)} U\left(-\frac{1}{2}n; \frac{1}{2}; -b^2/4a\right) \quad (44)$$

$$= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k! (n-2k)!} \frac{(2b)^{n-2k}}{(4a)^{n-k}} \quad (45)$$

$$= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! (2a)^{k-n} b^{n-2k} \quad (46)$$

for integer $n > 0$ (F. Pilolli, pers. comm.). For (45) and (46), $a, b \in \mathbb{C} - \{0\}$ (the [punctured plane](#)), $\Re[a] > 0$, and $(-1)!! = 1$. Here, $U(a; b; x)$ is a [confluent hypergeometric function of the second kind](#) and $\binom{n}{k}$ is a [binomial](#)

coefficient.

SEE ALSO:

[Erf](#), [Gauss Integral](#), [Gaussian Function](#), [Leibniz Integral Rule](#), [Normal Distribution](#)

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