MATH3201 Lecture 11

Recall: Given n data points (x_i, y_i) with distinct x_i, there are infinetely many degree $\geq n$ polynomials interpolate the points.

Example: How many polynomials of degree $d = 0, 1, 2, 3, 4$ interpolate four data points $(-1,-5)$, $(0,-1)$, $(2,1)$, and $(3,11)$?

Use Newton's divided differences:

x_i	y_i			
-1	-5			
		4		
0	-1		-1	
		1		1
2	1		3	
		10		
3	11			

$$P_3(x) = -5 + 4(x+1) - 1(x+1)x + 1(x+1)x(x-2)$$

~~Notes:~~

There exists only one degree 3 polynomial $P_3(x)$ interpolates the data.

There are no polynomials of degree 0, 1, and 2.

There are infintely many degree 4 polynomials in the following form

$$P_4(x) = P_3(x) + c(x+1)x(x-2)(x-3)$$

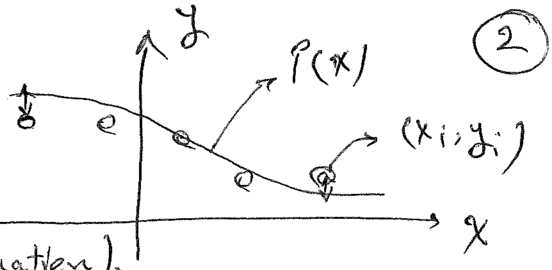
where $c \neq 0$

Notes: advantages of Newton's divided differences:

1) Nested calculations: much less work is required to obtain the interpolating polynomial, compare to Lagrange method.

2) Real-time approach: New data points can be easily added to the interpolating polynomial.

Interpolation Error:



Assume: $f(x)$ generates data points (x_i, y_i)

$P(x)$ is interpolating polynomial (fit equation).

E = Interpolation error $= |f(x) - P(x)|$

Definition:

For given n data points (x_i, y_i), where $i = 1, \dots, n$, there exists exactly one polynomial of degree at most $(n-1)$ interpolating data with error:

$$|f(x) - P(x)| = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c) \qquad (1)$$

where $\min\{x_i\} \leq c \leq \max\{x_i\}$.

Eq (1) can be used to estimate maximum error of interpolation.

Example: Given five data points produced by function $f(x) = e^x$ at $-1, -0.5, 0, 0.5, 1$, find an upper bound error at $x = 0.25$ and 0.75 for a polynomial that interpolates these points.

Interpolation error: $$E = |f(x) - P(x)| = \frac{(x+1)(x+0.5)(x-0)(x-0.5)(x-1)}{5!} f^{(5)}(c)$$

where $-1 \leq c \leq 1$.

we know $f^{(5)}(c) = e^c$. $f^{(5)}(c)$ is maximum when $c = 1$.

Now, the maximum interpolation error

$$\cancel{E_{max}}\, E = \left| e^{x} - P(x) \right| \leqslant \frac{(x+1)(x+0.5)(x)(x-0.5)(x-1)}{5!}\, e$$

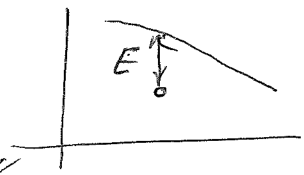
At $x = 0.25$: $\cancel{E_{max}}\, E \leqslant \dfrac{(0.25+1)(0.25+0.5)(0.25)(0.25-0.5)(0.25-1)}{5!}\, e$

$\leqslant 0.000995$

At $x = 0.75$: $\cancel{E_{max}}\ E \leqslant \dfrac{(0.75+1)(0.75+0.5)(0.75)(0.75-0.5)(0.75-1)}{5!}\, e$

$\leqslant 0.002323$

$E(0.75) > E(0.25)$:



Interpolation error seems to be smaller close to the ~~center~~ middle (i.e., 0) of interpolation interval.

Runge's phenomena:

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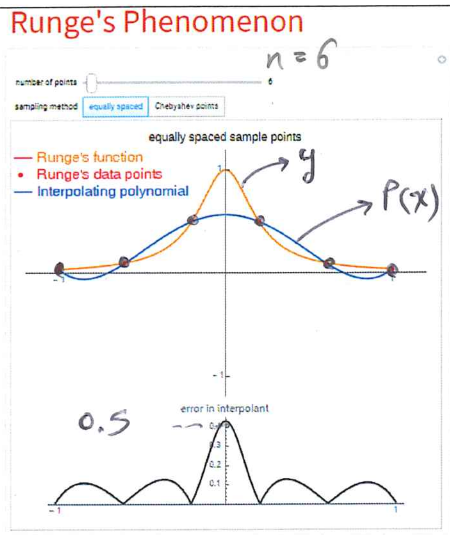
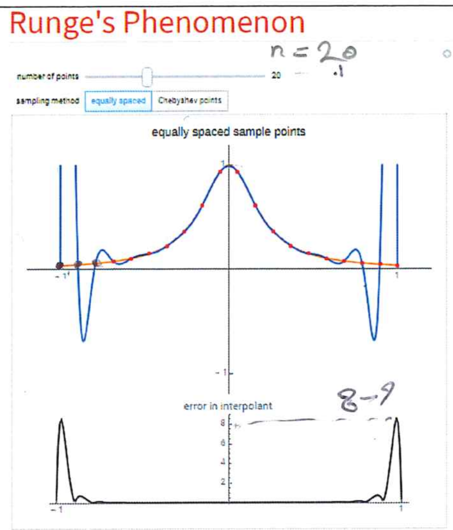
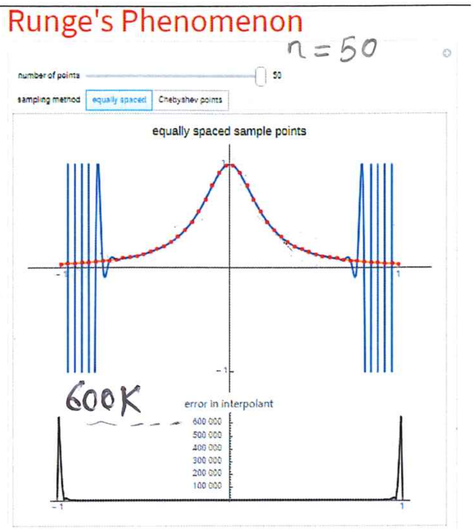
Runge's function $y = \frac{1}{1+25x^2}$

$$y = \frac{1}{1 + 25x^2}$$

Test in Mathematica: http://demonstrations.wolfram.com/RungesPhenomenon/







A. Interpolating polynomial of degree 5 passes through the 6 equally spaced data points generated by the Runge's function. This shows that 5th order polynomial is not a good approximation for the Runge's function. May be polynomial should take more data points to get closer to the original curve.

B. The 19th order polynomial interpolates 20 equally spaced data points from Runge's function. At the end points the error is large (~ 9).

C. The 49th order polynomial interpolates 50 equally spaced data points from Runge's function. At the end points the polynomial wiggle and error is very large (~ 700,000).

Runge's phenomenon:

For evenly spaced data points, interpolation polynomial p(x) (blue curve) wiggle near the ends of interpolation boundaries. (error is large).

Solution:

Move some of the data points towards the boundries so that interpolation error decreases.

~~**Note:**~~ Conclusion:

~~The~~ Runge's phenomenon suggests that higher-degree interpolation is a bad idea.

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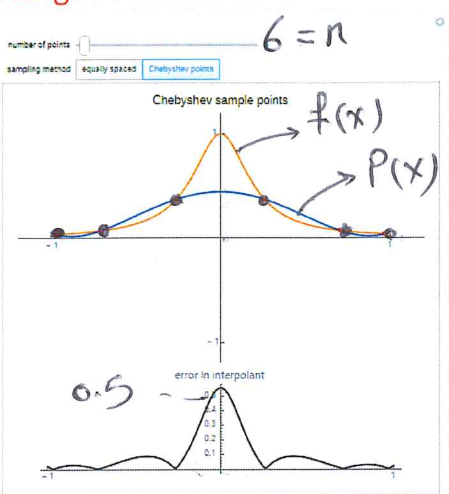
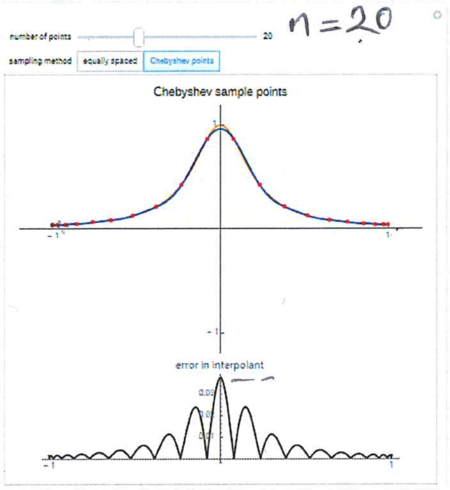
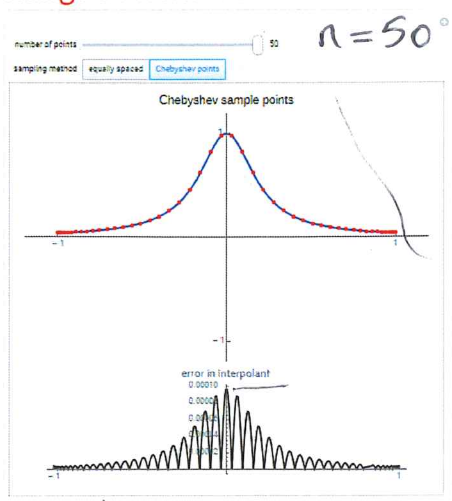
Runge's Phenomenon



Runge's Phenomenon



Runge's Phenomenon



Chebyshev's interpolating polynomial has solved the Runge's phenomenon. ✓

Table: Chebyshev's interpolating polynomial $P_3(v)$ is evaluated at six points in the following table. Table shows that the interpolation errors at all points are below the maximum error bound (~0.00198).

x_i	sin(x)	$P_3(v)$	$\lvert\sin(x)-P_3(v)\rvert$
1	0.8415	0.8408	0.0007
2	0.9093	0.9097	0.0004
3	0.1411	0.1420	0.0009
4	-0.7568	-0.7555	0.0013
14	0.9906	0.9917	0.0011
100	0.8269	0.8261	0.0008

Figure. Show that the Chebyshev error (dashed curve) is a bit smaller and is distributed more evenly throughout the interpolation interval. Note: in Panel b: there were 10 data points.

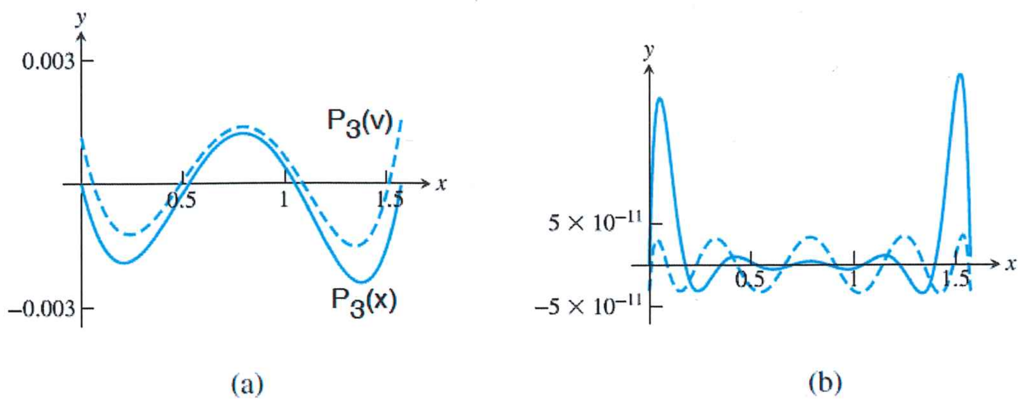


Figure 3.11 Interpolation error for approximating $f(x) = \sin x$. (a) Interpolation error for degree 3 interpolating polynomial with evenly spaced base points (solid curve) and Chebyshev base points (dashed curve). (b) Same as (a), but degree 9.