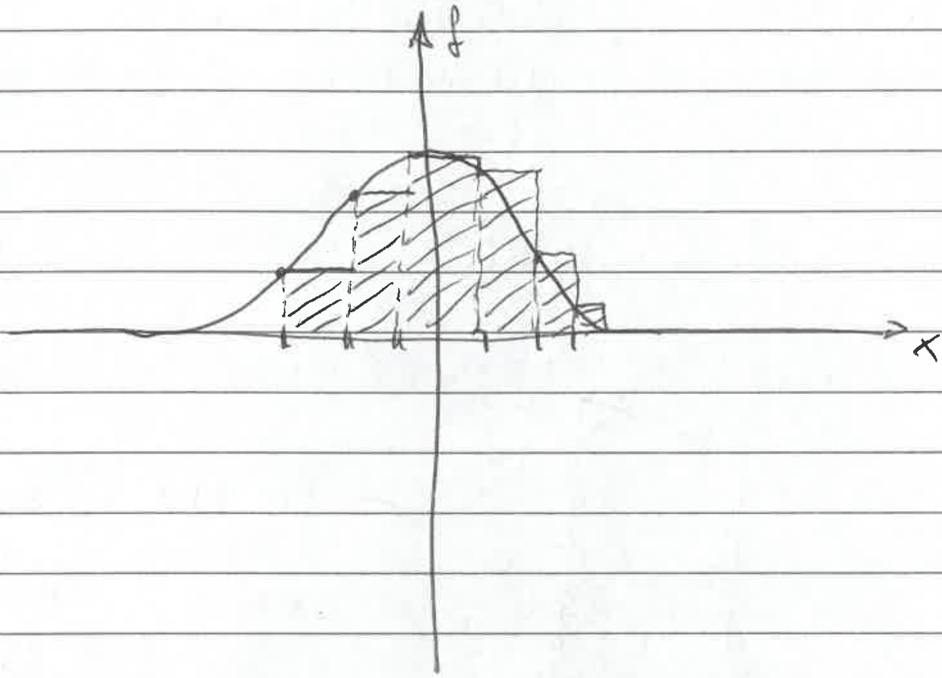
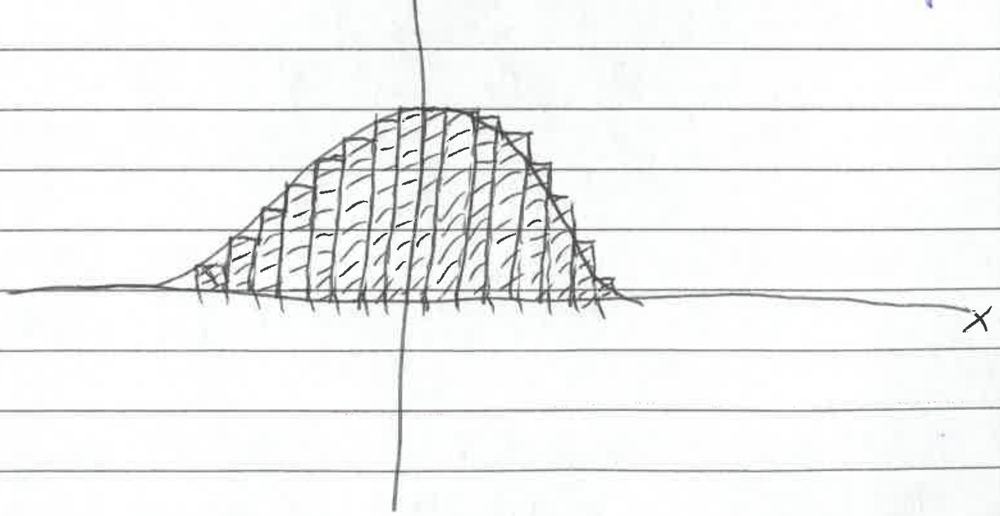
Integration:

Want to integrate functions such as e^{-x^2}

$$\int_{-1}^{1} e^{-x^2}\,dx$$



increase number of intervals

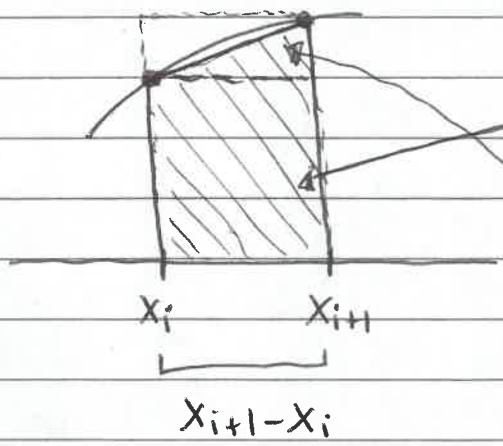


So to compute $\int_a^b f(x)\,dx$ divide interval $[a,b]$ into n equal subintervals

$$[a,b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i]$$

Add the area of rectangle with height at point x_i, $f(x_i)$. The smaller the subintervals the better is the approximation.

Trapezoidal Rule:



$$f(x_i)(x_{i+1}-x_i) + \frac{(f(x_{i+1})-f(x_i))(x_{i+1}-x_i)}{2}$$

$$\Rightarrow \quad \text{area} = f(x_i)(x_{i+1}-x_i) + \frac{1}{2} f(x_{i+1})(x_{i+1}-x_i) - \frac{1}{2} f(x_i)(x_{i+1}-x_i)$$

$$= \frac{1}{2}\left(f(x_{i+1}) + f(x_i)\right)(x_{i+1}-x_i)$$

So instead of heigt $f(x_{i+1})$ or $f(x_i)$ we use the average height. ~~between those~~

We can also use Taylor to derive the Trapezoidal rule. This will also give us the error.

$x \in [x_i, x_{i+1}]$

$f(x) = f(x_i) + f'(x_i)(x - x_i)$ use difference to approximate $f'(x_i)$

$\approx f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$ now ~~use~~ denote $f_i = f(x_i)$

$= f_i + (f_{i+1} - f_i)\theta$ (*) where $\theta = \frac{x - x_i}{\underbrace{x_{i+1} - x_i}_{h}} = \frac{x - x_i}{h}$

Here we assume that the distance between all the points is the same, so h constant.

Now integrate over one intervall $[x_i, x_{i+1}]$

exact area $= \int_{x_i}^{x_{i+1}} f(x)\,dx$

$\frac{x - x_i}{h} = \theta$

$x = x_i + \theta h \quad 0 \leq \theta \leq 1$

$\frac{dx}{d\theta} = h \quad dx = h\,d\theta$

integral bounds

$\frac{x_i - x_i}{h} = 0$

$\frac{x_{i+1} - x_i}{h} = 1$

do substitution

$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_0^1 f(x_i + \theta h)\,h\,d\theta$ (*)

$= \int_0^1 [f_i + (f_{i+1} - f_i)\theta]\,h\,d\theta$ abbreviate $f(x_i)$ with f_i, etc

$= h\left[f_i\theta + \frac{1}{2}(f_{i+1} - f_i)\theta^2\right]_0^1 = h\left[f_i + \frac{1}{2}(f_{i+1} - f_i)\right]$

W4 L1

$$= h\left[\frac{1}{2}(f_i + f_{i+1})\right]$$

$$= \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right)\underbrace{(x_{i+1} - x_i)}_{=h}$$

which is Trapezoidal rule.

This is for one intervall. To get the whole integral sum over all intervalls.

$[a,b]$, $h = \frac{b-a}{N}$ so N intervalls.

$$\int_a^b f(x)\,dx \sim h\left[\frac{1}{2}(f_0 + f_1) + \frac{1}{2}(f_1 + f_2) + \cdots + \frac{1}{2}(f_{N-1} + f_N)\right]$$

$$= h\left[\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{N-1} + \frac{1}{2}f_N\right]$$

This is called the composite Trapezoidal rule.

Error analysis of Trapezoidal rule.

Use single interval again first.

I_{TR} = Integral using Trapezoidal rule

$$= \frac{1}{2}\left(f_i + f_{i+1}\right)h = \frac{h}{2}\left(f(x_i) + \underbrace{f(x_i+h)}_{\text{Taylor expansion}}\right)$$

$$= \frac{h}{2}\left[f(x_i) + \underbrace{f(x_i) + f'(x_i)h + \frac{1}{2}f''(x_i)h^2 + O(h^3)}\right]$$

$$= \frac{h}{2}\left[2f(x_i) + f'(x_i)h + \frac{1}{2}f''(x_i)h^2 + O(h^3)\right]$$

$$\text{Exact integral} = I_{EX} = \int_{x_i}^{x_{i+1}} \underbrace{f(x)}_{\text{Taylor}}\,dx$$

$$= \int_{x_i}^{x_{i+1}} \left(f(x_i) + f'(x_i)(x-x_i) + \frac{1}{2}f''(x_i)(x-x_i)^2 + O((x-x_i)^3)\right)dx$$

$$= \left[f(x_i)x + \frac{1}{2}f'(x_i)(x-x_i)^2 + \frac{1}{6}f''(x_i)(x-x_i)^3 + O((x-x_i)^4)\right]_{x_i}^{x_{i+1}}$$

use $f_i = f(x_i)$ again

$$= \left(f_i x_{i+1} + \frac{1}{2}f_i'\underbrace{(x_{i+1}-x_i)}_{h}^2 + \frac{1}{6}f_i''\underbrace{(x_{i+1}-x_i)}_{h}^3 + O\left(\underbrace{(x_{i+1}-x_i)}_{h}^4\right)\right)$$

$$- \left(f_i x_i + \frac{1}{2}f_i'\overset{=0}{(x_i-x_i)} + \frac{1}{6}f_i''\overset{=0}{(x_i-x_i)}^3 + O\left(\overset{=0}{(x_i-x_i)}^4\right)\right)$$

$$= f_i\underbrace{(x_{i+1}-x_i)}_{h} + \frac{1}{2}f_i'h^2 + \frac{1}{6}f_i''h^3 + O(h^4)$$

$$= f_i h + \frac{1}{2}f_i'h^2 + \frac{1}{6}f_i''h^3 + O(h^4)$$