wave equation : $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

CFL Condition for the stability of numerical method:

$$\gamma = c\,\frac{\Delta t}{\Delta x} \le 1$$ is stable. ; c: wave speed

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Example: Use the finite difference method and model the motion of a string (i.e., wave) for time 0-2 units and space 0-4 units.

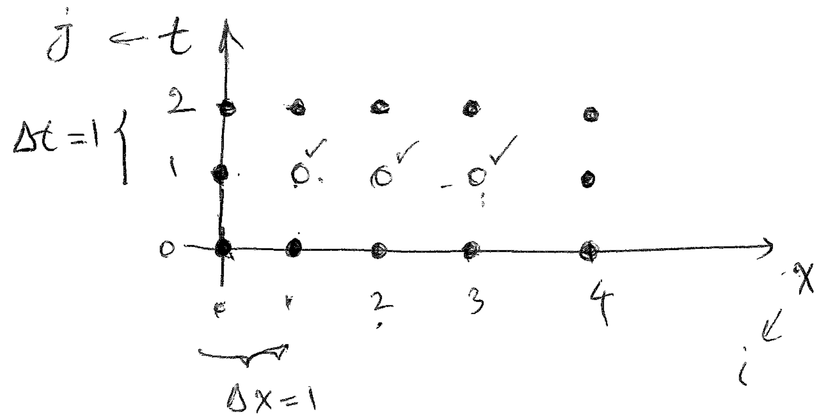
$$\frac{\partial^2 u}{\partial t^2} = 9\,\frac{\partial^2 u}{\partial x^2}$$

BCs:
$$\begin{cases} u(0,1) = -0.99 \\ u(x,0) = \cos(x) \\ u(x,2) = \cos(x+6) \\ u(4,1) = -0.84 \end{cases}$$

time step $\Delta t = 1$
space step $\Delta x = 1$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2} = 9\,\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{(\Delta x)^2}$$

update rule : $-16u_{i,j} + 9u_{i-1,j} + 9u_{i+1,j} - u_{i,j+1} - u_{i,j-1} = 0$
$(\Delta x = \Delta t = 1)$



●: Known values for solution

○: unknown value of solution to be calculated.

For $i=1$; $j=1$: $-16u_{1,1} + 9u_{0,1} + 9u_{2,1} - u_{1,2} - u_{1,0} = 0$

For $i=2$; $j=1$: $-16u_{2,1} + 9u_{1,1} + 9u_{3,1} - u_{2,2} - u_{2,0} = 0$

For $i=3$; $j=1$: $-16u_{3,1} + 9u_{2,1} + 9u_{4,1} - u_{3,2} - u_{3,0} = 0$

↓ Replace BCs.

$$-16u_{1,1} + 9u_{2,1} = 10.2$$
$$-16u_{2,1} + 9u_{1,1} + 9u_{3,1} = -0.57$$
$$-16u_{3,1} + 9u_{2,1} = 5.66$$

↓ Matrix form

$$\begin{bmatrix} -16 & 9 & 0 \\ 9 & -16 & 9 \\ 0 & 9 & -16 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 10.2 \\ -0.57 \\ 5.66 \end{bmatrix} \checkmark$$

Stability of this example:

Q: Given wave speed $(c^2 = 9)$: do you think your numerical solution will approach exact solution?

Exact solution $u(x,t) = \cos(x+3t)$

No, because the CFL condition does not hold.

Algebric solution for wave equation:

Let $u(x,t)$: be a function that models the movement of a wave propagates in 1D, in both directions, with speed c, while retaining a fixed shape. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

we show $u(x,t)$ is two waves moving at same speed in opposite directions.

Note:

$f(x,t)$: models a wave moving to right, same ~~properties~~ properties above.

An observer moving at the same speed c to the right see the unchanging picture described as: $F(z)$

$$f(x,t) = F(z) \quad \text{where} \quad z = x - ct$$

To solve wave equation we use ansatz:

$z = x - ct$; denotes distance along the wave in right direction

$\eta = x + ct$: " in left direction.

using ansatz and chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \cancelto{1}{\frac{\partial z}{\partial x}} + \frac{\partial u}{\partial \eta} \cdot \cancelto{1}{\frac{\partial \eta}{\partial x}} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z} \cdot \cancelto{-c}{\frac{\partial z}{\partial t}} + \frac{\partial u}{\partial \eta} \cdot \cancelto{c}{\frac{\partial \eta}{\partial t}} = c\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial z}\right)$$

Take second derivatives:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \eta}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial z \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial z}\right) c\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial z}\right) = c^2\left(\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial z \partial \eta} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \xrightarrow[\text{in}]{\text{results}} \frac{\partial^2 u}{\partial z \partial \eta} = 0$$

$$\frac{\partial^2 u}{\partial z\,\partial \eta} = 0 \qquad (1)$$

Integrate (1) with respect to z : $\dfrac{\partial u}{\partial \eta} = f(z)$ $\quad (*)$

Integrate $(*)$ with respect to η : $u = f(z) + g(\eta)$

$$u(x,t) = f(x-ct) + g(x+ct) \qquad (2)$$

wave u is the sum of waves travelling to right and left.

General solution solution to Eq. (2) by determining $f(z)$ and $g(\eta)$.

Let IC: $u(x,0) = h(x)$

$$\frac{\partial u}{\partial t}(x,0) = K(x) \quad *$$

From Equation (2) and ICs, we have:

$$h(x) = f(x) + g(x) \qquad (3)$$

$$K(x) = \frac{\partial u(x,0)}{\partial t} = -c\frac{\partial}{\partial x}f(x) + c\frac{\partial}{\partial x}g(x)$$

integrate

$$c\,g(x) - c\,f(x) = \int_a^x K(v)\,dv$$

From (3): $c\,g(x) + c f(x) = c\,h(x)$

Add +

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$$2c\,g(x) = c\,h(x) + \int_a^x K(v)\,dv$$

$$2c f(x) = c\,h(x) - \int_a^x K(v)\,dv$$

$$g(x) = \tfrac{1}{2} h(x) + \frac{1}{2c}\int_a^x K(v)\,dv$$

$$f(x) = \tfrac{1}{2} h(x) - \frac{1}{2c}\int_a^x K(v)\,dv$$

$$u(x,t) = \frac{1}{2}\Big(h(x+ct) + h(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} K(v)\,dv$$

D Alembert's formula.

Example: