Example: Solve BVP below using finite difference methods with five equidistant nodes.

$$\text{BVP: } \begin{cases} y'' = 4y \\ y(0) = 1, \quad y(1) = 3 \end{cases}$$

Answer:

First, replace continuous derivatives with discrete approximations:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4y_i = 0$$

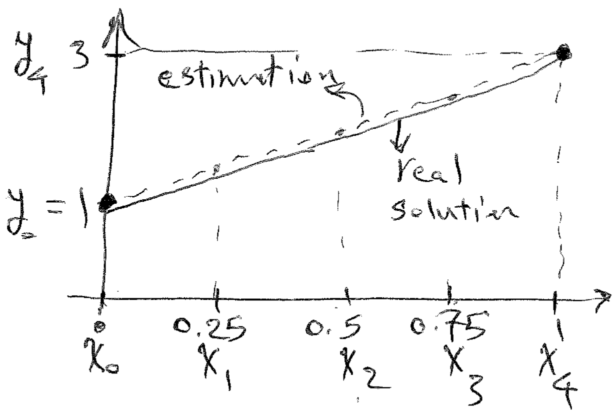
$$(*) \quad y_{i-1} + y_i(-2 - 4h^2) + y_{i+1} = 0 \quad \text{: Finite difference approximation}$$

Next, evaluate discrete form equation over standard grid at discrete intervals

BCs: $y(x_0 = 0) = y_0 = 1$

$y(x_4 = 1) = y_4 = 3$

$h = \frac{1-0}{4} = 1/4$



For $i=1$ at x_1: $\cancel{y_0}^{1} + y_1(-2 - 4h^2) + y_2 = 0$

For $i=2$ at x_2: $y_1 + y_2(-2 - 4h^2) + y_3 = 0$

at x_3: $y_2 + y_3(-2 - 4h^2) + \cancel{y_4}^{3} = 0$

$$\Rightarrow \quad \begin{aligned} -\tfrac{9}{4} y_1 + y_2 &= -1 \\ y_1 - \tfrac{9}{4} y_2 + y_3 &= 0 \\ y_2 - \tfrac{9}{4} y_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} -9/4 & 1 & 0 \\ 1 & -9/4 & 1 \\ 0 & 1 & -9/4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

solve this using Guassian elimination

$y_1 = 1.0249$

$y_2 = 1.3061$

$y_3 = 1.9138$

(2)

Exact solution: for the BVP:

$y'' - 4y = 0$ characteristic eq: $r^2 - 4 = 0 \longrightarrow r = \pm 2$

(two distinct roots)

General Solution $y(x) = c_1 e^{2x} + c_2 e^{-2x}$

B.C. $\begin{cases} y(0) = 1 \\ y(1) = 3 \end{cases} \longrightarrow \begin{cases} c_1 = 0.39 \\ c_2 = 0.61 \end{cases}$

specific solution $y(x) = 0.39\, e^{2x} + 0.61\, e^{-2x}$

$y(0.25) = Y_1 = 1.0181$

$y(0.5) = Y_2 = 1.2961$

$y(0.75) = Y_3 = 1.9049$

i	x_i	y_i	Y_i
0	0	~~1.2024~~ 1	1
1	0.25	~~1.0064~~ 1.0249	1.0181
2	0.5	1.3061	1.2961
3	0.75	1.9138	1.9049
4	1	3	3

$\lesssim 10^{-2}$ error

BVP can be non-linear.

After substituting of discrete approximations for derivatives, two possible situations: $\begin{cases} \text{linear BVP} \\ \text{non-linear BVP.} \end{cases}$

For non-linear BVP: we take one more step, ~~to solve~~ compare with solving linear BVPs.

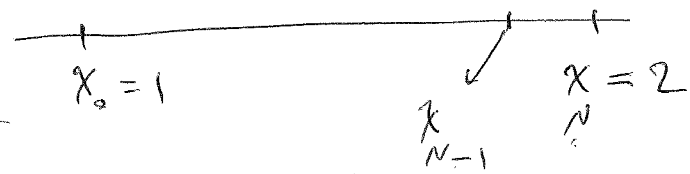
Example: BVP: $\begin{cases} y'' = 18y^2 \\ y(1) = 1/3, \quad y(2) = 1/12 \end{cases}$

First: replace continuous derivatives with discrete approximations:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 18 y_i^2$$

we use our grid: $\Delta x = h = \frac{2-1}{N} = 1/N$

BC: $\begin{cases} x_0 = 1 \longrightarrow y_0 = 1/3 \\ x_N = 2 \longrightarrow y_N = 1/12 \end{cases}$



For $i=1$: $y_2 - 2y_1 + \overset{1/3}{\cancel{y_0}} = 18h^2 y_1^2$

For $i=2$: $y_3 - 2y_2 + y_1 = 18h^2 y_2^2$

$\vdots$

For $i=N-1$: $\overset{1/12}{\cancel{y_N}} - 2y_{N-1} + y_{N-2} = 18h^2 y_{N-1}^2$

$$\begin{cases} y_2 - 2y_1 - 18h^2 y_1^2 = -1/3 \\ y_3 - 2y_2 - 18h^2 y_2^2 + y_1 = 0 \\ \vdots \\ -2y_{N-1} - 18h^2 y_{N-1}^2 + y_{N-2} = -1/12 \end{cases} \quad (*)$$

we cannot write the system above as $A\underline{x} = b$ because of non-linear terms: $y_1^2, y_2^2, \dots, y_{N-1}^2$.

we take take another step: we write the system $(*)$ as $F(\underline{y}) = 0$ which we can carry out by Newton's method.

$$F(\underline{y}) = 0 \quad \text{where } \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{pmatrix}$$

$$\left.\begin{aligned} F_1 &= +1/3 - 2y_1 - 18h^2 y_1^2 + y_2 = 0 \\ F_2 &= y_1 - 2y_2 - 18h^2 y_2^2 + y_3 = 0 \\ &\vdots \\ F_{N-1} &= 1/12 + y_{N-2} - 2y_{N-1} - 18h^2 y_{N-1}^2 = 0 \end{aligned}\right\}$$

we solve this set using Newton's method.

y_i at iteration $(j+1)$: $y^{(j+1)} = y^{(j)} - J^{-1}(y^{(j)})\, F(y^{(j)})$

we calculate J for $N = 4$:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \frac{\partial F_1}{\partial y_3} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \frac{\partial F_2}{\partial y_3} \\ \frac{\partial F_3}{\partial y_1} & \frac{\partial F_3}{\partial y_2} & \frac{\partial F_3}{\partial y_3} \end{pmatrix} = \begin{pmatrix} -2 - 36y_1h^2 & 1 & 0 \\ 1 & -2 - 36y_2h^2 & 1 \\ 0 & 1 & -2 - 36y_3h^2 \end{pmatrix}$$

start with an initial guess: $y^{(0)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

(Note: we will look at interpolation later)

$y^{(0)}$ = linear interpolation between 1/3 and 1/12 (B.C.)

$y^{(0)}$ = other values.

Book: Timothy Sauer
see for Matlab implementation example.

Estimation error is $O(h^2)$ which is coming from the derivative approximations.