Math 3201 lecture 5 ①

update rule forward difference method : $\underline{u^{(j+1)} = A u^{(j)} + s^{(j)}}$ (1)

Implementation Book Timothy Sauer

Froward difference for heat PDE is stable if $\dfrac{\Delta t\, D}{(\Delta x)^2} \leq 1/2$.

(prove later)

$j=1 \longrightarrow u^{(1)} = A u^{(0)} + s^{(0)}$

$j=2 \longrightarrow u^{(2)} = A\left(A u^{(0)} + s^{(0)}\right) + s^{(1)}$

$= A^2 \underbrace{u^{(0)}}_{f} + A s^{(0)} + s^{(1)}$

$\vdots$

$j=N \longrightarrow$ ~~$u^{(N)} = A^N u^{(0)}$~~

$u^{(N)} = \underline{\underline{A^N}} f + A^{N-1} s^{(0)} + A^{N-2} s^{(1)} + \cdots + s^{(N-1)}$

Convergence of Eq. (1) depends on e-values of matrix A.

Reason: Let $U^{(j)} = [u(x_i, t_j)]$ be the exact solution.

let $u^{(j)} = [u_{i,j}]$ be the approximation of heat.

Error $= u^{(j)} - U^{(j)} = \left(A u^{(j-1)} + \cancel{s^{(j-1)}}\right) - \left(A U^{(j-1)} + \cancel{s^{(j-1)}}\right)$

$e^{(j)} = e^{(j)} = A\left(u^{(j-1)} - U^{(j-1)}\right)$

$e^{(j)} = A\, e^{(j-1)}$

To ensure that errors $e^{(j)}$ are not amplified, the spetral radius of A $\rho(A) < 1$, where $\rho(A) = \max\{|\lambda_i|\}$.

Thus, for $\underset{N\to\infty}{u^{(N)}}$, matrix $\underset{N\to\infty}{A^N}$ is bounded: because $\lambda_i^N \longrightarrow$ Constant.

$\rho(A) < 1$ means that $|\lambda_i| < 1$ for $i = 1, \dots, M-1$.

If $|\lambda_i| > 1$: every time step $u^{(j)}$ will get larger and larger.

But, physically, we expect to get temperature u smaller and smaller.

we conclude $|\lambda_i| < 1$, provided that $\dfrac{\Delta t\, D}{(\Delta x)^2} \leq 1/2$.

Solve heat PDE using "Backward difference method"

$$\frac{\partial u}{\partial t} \cong \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t}$$

Recall:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

update rule: $u_{i,j} - u_{i,j-1} = \delta \left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right)$, $\delta = \dfrac{\Delta t\, D}{(\Delta x)^2}$

$$\underbrace{-\delta u_{i-1,j} + (1+2\delta) u_{i,j} - \delta u_{i+1,j}}_{\text{unknowns in time step } j} = \underbrace{u_{i,j-1}}_{\text{Knowns from time } j-1}$$

Backward method is implicit.

BC. (for simplicity): $\begin{cases} u_{0,j} = l_j = 0 \\ u_{M,j} = r_j = 0 \end{cases}$

Matrix form

$$\underbrace{\begin{bmatrix} 1+2\delta & -\delta & 0 & 0 & \\ -\delta & 1+2\delta & -\delta & 0 & \\ 0 & -\delta & 1+2\delta & -\delta & \\ & & \ddots & \ddots & \\ & & & -\delta & 1+2\delta \end{bmatrix}}_{B} \begin{bmatrix} u_{1,j} \\ \vdots \\ \vdots \\ u_{M-1,j} \end{bmatrix} = \begin{bmatrix} u_{1,j-1} \\ \vdots \\ \vdots \\ u_{M-1,j-1} \end{bmatrix}$$

$$\underline{B}\, u^{(j)} = u^{(j-1)}$$

$$j=1 \longrightarrow B\, u^{(1)} = u^{(0)} = f$$

Recall: $u^{(0)} = [u_{i,0}] = f_i = f$
$i = 0, \ldots, M$

$$u^{(1)} = B^{-1} f$$

$$\vdots$$

$$u^{(N)} = (B^{-1})^N f$$

The convergence of backward method depends on e-values of B^{-1}.
The backward method is stable if e-values of B^{-1} $|\lambda_i| < 1$.
for $i = 1, \ldots, M-1$

Stability of forward and backward methods:

Convergence of both rules depends on e-values of tridiagonal matrices.
We now derive formula for the e-values of symmetric tridiagonal matrics.

Let
$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & \\ & & \ddots & \ddots & \\ & & & b & a \end{bmatrix}_{M \times M}$$

we show that $\boxed{\lambda = a + 2b \cos \frac{\pi K}{M+1}}$ for $K = 1, \ldots, M$ are e-values of matrix A $(A\underline{x} = \lambda \underline{x})$.

$(+)$

Proof:
we wish to solve: $\underset{\sim}{A}\,\underline{x} = \lambda\,\underline{x}$ for $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}$ and λ, using finite difference method.

$$a x_1 + b x_2 = \lambda x_1 \qquad ; \; x_0 = 0$$
$$b x_1 + a x_2 + b x_3 = \lambda x_2$$
$$\vdots$$
$$b x_{M-1} + a x_M = \lambda x_M \qquad x_{M+1} = 0$$

$$b x_{j-1} + a x_j + b x_{j+1} = \lambda x_j$$

$$b x_{j-1} + (a-\lambda) x_j + b x_{j+1} = 0 \qquad (*)$$

Eq. (*) has a solution $\underline{x}$ for only special values ~~of~~ λ with formula (+).

Recall: to solve $c_1 y'' + c_2 y' + c_3 y = 0$

try: $y = e^{qx}$

$$\underbrace{\left(c_1 q^2 + c_2 q + c_3\right)}_{\text{look at roots and write general solution}} \underbrace{e^{qx}}_{\neq 0} = 0$$

We seek a trial solution $x_j = r^j$ where r is to be found.

$$= e^{j \ln r}$$

substituting trial solution in Eq. (*):

$$b r^{j-1} + (a-\lambda) r^j + b r^{j+1} = 0$$

$$\underbrace{r^{j-1}}\underbrace{\left(b + (a-\lambda) r + b r^2\right)}_{\longrightarrow\ \text{we wish to solve for } r.} = 0$$

Take $r^{j-1} \neq 0 \;\forall r$

$$b + (a-\lambda) r + b r^2 = 0$$

$$r_{1,2} = \frac{-(a-\lambda) \pm \sqrt{(a-\lambda)^2 - 4b^2}}{2b}$$ but this is cumbersome!

Instead, we write roots r_1, r_2:

$$b(r-r_1)(r-r_2) = 0$$

$$\left.\begin{aligned} b\left(r^2 - (r_1+r_2) r + r_1 r_2\right) &= 0 \\ b r^2 + (a-\lambda) r + b &= 0 \end{aligned}\right\} \longrightarrow r_1 r_2 = 1 \longrightarrow \boxed{r_2 = 1/r_1}$$

$$\boxed{-b(r_1+r_2) = a-\lambda}$$

Now, general solution for equation (*) is:

$$x_j = \alpha r_1^{\,j} + \beta r_2^{\,j}$$

Try BCs: $x_0 = 0$

$$\hookrightarrow x_0 = \alpha + \beta = 0 \longrightarrow \boxed{\beta = -\alpha}$$

$$x_j = \alpha \left(r_1^{\,j} - r_2^{\,j}\right)$$

$$x_{M+1} = 0 \longrightarrow x_{M+1} = \alpha\left(r_1^{M+1} + r_2^{M+1}\right) = 0$$

If $\alpha = 0 \longrightarrow$ General solution is $x_j = 0$ (not useful)

If $\left(r_1^{M+1} + r_2^{M+1}\right) = 0 \longrightarrow$ This will lead to find e-values.