Example: Given $f(x) = 5xe^{-2x}$, extrapolate the value of $f'(0.35)$ with central difference formula with step sizes 0.25 and 0.125.

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

$$h = 0.25 \rightarrow f'(0.35) \approx \frac{f(0.35+0.25) - f(0.35-0.25)}{2\times 0.25} = \frac{f(0.6)-f(0.1)}{0.5} =$$

$$= 0.9880 \quad (1)$$

$$h = 0.125 \rightarrow f'(0.35) \approx \frac{f(0.35+0.125) - f(0.35-0.125)}{2\times 0.125} = \frac{f(0.475)-f(0.225)}{0.25}$$

$$= 0.8040 \quad (2)$$

We use two approximations above and extrapolate a better estimation:

$$V_t \approx V_a\left(\frac{h}{2}\right) + \frac{V_a(h/2) - V_a(h)}{3} = 0.8040 + \frac{0.8040 - 0.9880}{3} = 0.7427 \quad (3)$$

Exact value: $f'(0.35) = 0.7449$

Extrapolated approximation in (3) is closer to exact value.

Note: If you want even better approximation, you can repeat extrapolation with $h = \frac{0.125}{2}$.

Numerical integration:

1. Newton-Cotes formulas:

estimate integration at equally spaced points,

1) Trapezoid rule

2) Simpson's rule

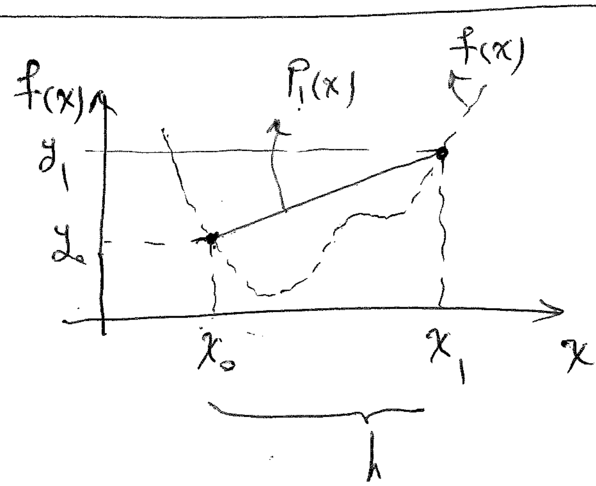
3) Romberg rule. (extrapolation).

2. Adaptive quadrature.

numerical integration is based on evaluating integrand by adapting size of intervals.

Trapezoid rule:

$$\int_{x_0}^{x_1} f(x)\,dx$$



Idea is to replace ~~the~~ f(x) with line interpolating the integration interval (P(x)) then calculate the area $\underbrace{\text{below the line.}}_{\text{trapezoid.}}$

Trapezoid rule:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12} f''(c)$$

where $x_0 \leq c \leq x_1$, $h = x_1 - x_0$.

$$f(x) = P_1(x) + E(x)$$

$$= \underbrace{y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}}_{P_1(x)} + \underbrace{\frac{(x-x_0)(x-x_1)}{2!} f''(c)}_{E(x)}$$

Integrating both sides:

$$\int_{x_0}^{x_1} f(x)\, dx = \int_{x_0}^{x_1} P_1(x) + \int_{x_0}^{x_1} E(x)$$

Assume: $\omega_1 = -x + x_1 \qquad \omega_2 = x - x_0$

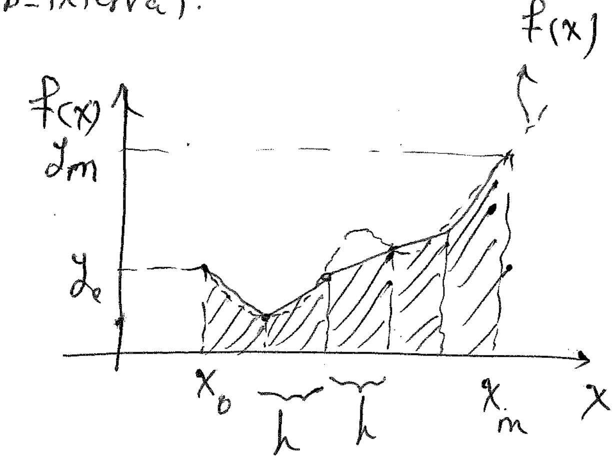
$$\int_{x_0}^{x_1} P_1(x)\, dx = y_0 \int_0^h \frac{\omega_1}{h}\, d\omega_1 + y_1 \int_0^h \frac{\omega_2}{h}\, d\omega_2 = y_0 \frac{h}{2} + y_1 \frac{h}{2}$$

$$= \frac{h}{2}(y_0 + y_1)$$

$$\int_{x_0}^{x_1} E(x)\, dx = \frac{f''(c)}{2} \int_{x_0}^{x_1} \underbrace{(x-x_0)}_{u}(x-x_1)\, dx = \frac{f''(c)}{2} \int_0^h u(u-h)\, du = \frac{-h^3}{12} f''(c)$$

Composite Trapezoid rule:

1) Divide into subintervals
2) Apply Trapezoid rule for every sub-interval.
3) Sum up all sub-areas.



Assum:

$$h = x_{i+1} - x_i$$

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b$$

For every sub-interval: $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\, dx = \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] - \frac{h^3}{12} f''(c_i) \qquad i = 0, \to m-1$

$$x_i \leq c_i \leq x_{i+1}$$

Now, for entire interval $[a, b]$:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right] - \underbrace{\sum_{i=0}^{m-1} \frac{h^3}{12} f''(c_i)}$$

Theorem:

Let $f(x)$ be Continous on interval $[a, b]$.

let $x_1, \ldots, x_m$ be points in $[a, b]$ and

Constants $a_1, \ldots, a_m > 0$.

There is a number c : $a < c < b$

$$a_1 f(x_1) + a_2 f(x_2) + \cdots + a_m f(x_m) = (a_1 + a_2 + \cdots + a_m) f(c)$$

using this theorem

$$\frac{h^3}{12} m f''(c), \quad a < c < b$$

$$mh = b - a$$

$$\frac{(b-a)h^2}{12} f''(c)$$

Composite Trapezoid rule :

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right] - \frac{(b-a)h^2}{12} f''(c)$$

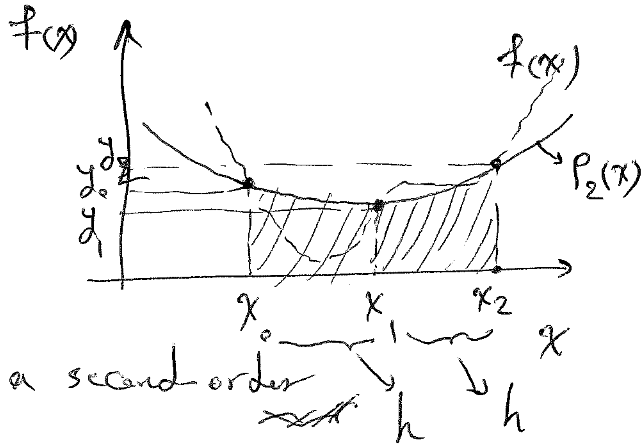
$a < c < b$.

The error is second-order in h : $O(h^2)$

Simpson's rule:

Similar to Trapezoid rule, but replaces the degree one interpolating line with a parabola.

$$\int_{x_0}^{x_2} f(x)\,dx$$



For 3 points (x_0, x_1, x_2) we can find a second-order Polynomial that interpolates the points.

$$f(x) = P_2(x) + E(x)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx + \int_{x_0}^{x_2} E(x)\,dx$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = y_0\int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx + y_1\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx +$$

$$y_2\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx = y_0\frac{h}{3} + y_1\frac{4h}{3} + y_2\frac{h}{3}$$

where $h = x_1 - x_0 = x_2 - x_1$

$$\int_{x_0}^{x_2} E(x)\,dx = -\frac{h^5}{90} f^{(4)}(c) \qquad x_0 \leq c \leq x_2$$

Simpson's rule: $\displaystyle\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] - \frac{h^5}{90} f^{(4)}(c)$