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From Sauer:

initial interval $[a, b]$

$$S_{[a,b]} = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

half interval $c = \frac{b-a}{2}$

~~$S_{[a_0,b_0]}$~~

$$S_{[a,b]} - S_{[a,c]} - S_{[c,b]} < 3\,\mathrm{TOL} \cdot \underline{\cancel{g}}\ ?$$

original interval: $b_0 - a_0 = h$

first half: $b_1 - a_1 = \frac{1}{2}h$

second half: $b_2 - a_2 = \frac{1}{4}h$

...

~~So when we~~ $\frac{b_1 - a_1}{b_0 - a_0} = \frac{\frac{1}{2}h}{h} = \frac{1}{2}$

$$\frac{b_2 - a_2}{b_1 - a_1} = \frac{\frac{1}{4}h}{\frac{1}{2}h} = \frac{1}{4}\cdot\frac{2}{1} = \frac{1}{2}$$

So when we half the interval ~~goes~~ the required tolerance goes down by $\frac{1}{2}$.

So we need to ~~mu~~ multiply our criterion by $\frac{1}{2}$.

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$S_{[a,b]} - S_{[a,c]} - S_{[c,b]} < 3 \cdot TOL \cdot \frac{1}{2}$

```
%Program 5.2 Adaptive Quadrature
% Computes approximation to definite integral
% Inputs: Matlab function f, interval [a0,b0],
%  error tolerance tol0
% Output: approximate definite integral
function int=adapquad(f,a0,b0,tol0)
int=0; n=1; a(1)=a0; b(1)=b0; tol(1)=tol0; app(1)=trap(f,a,b);
while n>0            % n is current position at end of the list
    c=(a(n)+b(n))/2; oldapp=app(n);
    app(n)=trap(f,a(n),c);app(n+1)=trap(f,c,b(n));
    if abs(oldapp-(app(n)+app(n+1)))<3*tol(n)
        int=int+app(n)+app(n+1);        % success
        n=n-1;                          % done with interval
    else                                % divide into two intervals
        b(n+1)=b(n); b(n)=c;            % set up new intervals
        a(n+1)=c;
    tol(n)=tol(n)/2; tol(n+1)=tol(n);
    n=n+1;                         % go to end of list, repeat
        end
    end

    function s=trap(f,a,b)
    s=(f(a)+f(b))*(b-a)/2;
```

half

check

adapt tolerance

from Sauer

Adaptive quadrature with Simpson's rule

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = S_{[x_i,x_{i+1}]} - \frac{h^5}{90} f^{(4)}(c_0) \quad 1$$

Apply to two halfs $[x_i, u_i]$ and $[u_i, x_{i+1}]$

we get

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = S_{[x_i,u_i]} - \frac{h^5}{32}\frac{f^{(4)}(c_1)}{90}$$

$$+ S_{[u_i,x_{i+1}]} - \frac{h^5}{32}\frac{f^{(4)}(c_2)}{90} \qquad c_1 \approx c_2$$

$$= S_{[x_i,u_i]} + S_{[u_i,x_{i+1}]} - \frac{h^5}{16}\frac{f^{(4)}(c_1)}{90} \quad 2$$

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① − ②

$$\Rightarrow S_{[x_i,x_{i+1}]} - S_{[x_i,u_i]} - S_{[u_i,x_{i+1}]}$$

$$= \frac{h^5}{90} f^{(4)}(c_0) - \frac{h^5}{16} \frac{f^{(4)}(c_1)}{90}$$

$c_1 \approx c_0$

$$\approx \frac{15}{16} \frac{f^{(4)}(c_0)}{90}$$

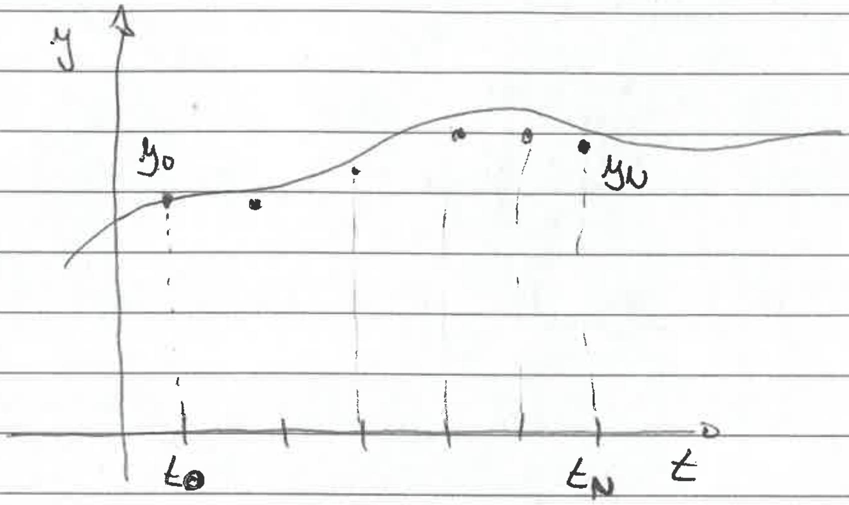
So $S_{[x_i,x_{i+1}]} - S_{[x_i,u_i]} - S_{[u_i,x_{i+1}]}$ is 15 times the error of the approximation $S_{[x_i,u_i]} + S_{[u_i,x_i]}$ so the tolerance criterion changes to

$$\left| S_{[x_i,x_{i+1}]} - S_{[x_i,u_i]} - S_{[u_i,x_{i+1}]} \right| < 15 \cdot Tol$$

It is tradition to replace 15 by 10 to make the algorithm more conservative.

ODEs

$$\frac{dy}{dt} = f(x,t) \quad , \quad y(0) = y_0$$



y_N is approximation to $y(t_N)$

$t_N = t_0 + Nh \qquad h = \frac{t_N - t_0}{N} \qquad$ h is fixed

Consider the initial value problem

$$\frac{dy}{dt} = -2t - y \qquad y(0) = -1$$

Let $f(t,y) = -2t - y$

we have $\frac{dy}{dt} = f(t,y)$

Here, $f(t,y)$ is known and $y(t_0) = y_0$

is known.

We can now expand $y(t)$ about t_0 using Taylor series.

$$y(t_0+h) = y(t_0) + h\underbrace{y'(t_0)}_{=f(\cancel{t,y}) = f(t_0, y(t_0))} + \underbrace{O(h^2)}_{\text{ignore}}$$

$$\approx y(t_0) + hf(t_0, y(t_0))$$

$$= y_0 + hf(t_0, y_0)$$

$$= y(t_1) = y_1$$

$\Rightarrow$ general formula

$$y_{n+1} = y_n + f(t_n, y_n)\cdot h$$

We can also use a finite difference for y'

$$\frac{dy}{dt} = f(t, y(t)) \quad \text{and} \quad \frac{dy}{dt} \approx \frac{y(t_0+h) - y(t_0)}{h}$$

$$\Rightarrow \frac{y(t_0+h) - y(t_0)}{h} \approx f(t_0, y(t_0)) \quad \| \cdot h \quad \| + y(t_0)$$

$$y(t_0+h) = y(t_0) + f(t_0, y(t_0))\cdot h$$

which is the same as from using Taylor series.

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$$\frac{dy}{dt} = \underbrace{-2t - y}_{f(t,y)} \qquad y(0) = -1$$

Can express initial condition as

$$(t_0, y_0) = (0, -1)$$

Let's use $h = 0.1$

$$t_1 = t_0 + 0.1 = 0.1$$

$$t_2 = t_1 + 0.1 = 0.2$$

$$\vdots$$

$$t_n = t_0 + h \cdot n$$

$$y_0 = -1$$

$$y_1 = y_0 + 0.1 \cdot f(t_0, y_0)$$

$$= -1 + 0.1\left(-2 \cdot 0 - (-1)\right)$$

$$= -0.9$$

So $(t_1, y_1) = (0.1, -0.9)$

Continuing gives $(0.2, -0.83)$, $(0.3, -0.787)$, $(0.4, -0.768)$, $(0.5, -0.77147)$, $(0.6, -0.794323)$, $(0.7, -0.8348907)$, $(0.8, -0.8914)$