MATH3201 Assignment 3
Due: 5.00 pm 07-May-2019

Instructions for assignment preparation:
Assignment should include all your workings: formulations, steps, and explanations/justifications taken to obtain the formulations, tables, figures, and Matlab codes.
Notes:

- Figures: your plots should have clear label for every axis. Include a clear caption which includes: the main idea drawn from the figure, brief introduction of all symbols used in the figure, and addressing what settings the colors correspond to. Every figure should be references in the text where appropriate.
- Tables: include a clear caption which includes: a general title for the table, and brief introduction of all symbols used in the table. Every table should be references in the text where appropriate.
- Matlab code: your code should be clearly labelled containing comments showing what the different variables are and what a part of the code is doing.

Note:
Marks will be deducted if your assignment does not meet the instructions outlined above.

Question 3.1 [20 marks]

Consider the ordinary differential equation (ODE) boundary value problem (BVP)

$$y'' + 2y' = 0$$

with boundary conditions $y(1) = 1$ and $y(2) = 0$.

(a) Formulate a finite difference approximation for the solution $y(x)$ [5 marks].

(b) Derive the discrete approximate equations in matrix form for $N = 5$ intervals [5 marks].

(c) Repeat parts (a) and (b) for a new formulation and matrix form if the ODE is modified to $y'' + xy' = 0$, where the boundary conditions remains the same [10 marks].

Question 3.2 [40 marks]

Use the finite difference method to estimate solution for the non-linear ODE:

$$y" = \frac{2y^2}{1+x}$$

subject to boundary conditions $y(1) = 1/2$ and $y(5) = 1/6$.

Note, the exact solution is $y(x) = 1/(1+x)$.

(a) Formulate finite difference approximation (by discretizing the ODE). Then, expand your formulation for the approximation using Newton's method including the derivation of Jacobian [10 marks].

(b) Implement your formulation in part (a) in Matlab for N = 3, 5, 9, and 17 intervals [10 marks].

(c) Calculate the order of convergence (see the hint below). First, for every value of N (i.e., 3, 5, 9, and 17), calculate and tabulate the maximum absolute error $\varepsilon = \max(|y_i - Y_i|)$, where y_i and Y_i are the estimated and real values of solution, respectively, at grid points x_i with $i = 0, \ldots, N$. Then, tabulate ε/h^k for every N, where $k > 0$ is integer. To determine the order of convergence, find a suitable integer k for which ε/h^k remains constant for all N values [10 marks].

(d) Plot the exact and approximated (for N = 17 only) solutions [10 marks].

Hint for Question 3.2: Order of convergence

The order of convergence k is a value for which:

$$\frac{\varepsilon}{h^k} \approx \text{constant, as } h \to 0$$

where ε is the maximum absolute error, as defined in Question 3.2(c), and h is the approximation step-size. The maximum absolute error can be written as:

$$\varepsilon = m_1 h^n + m_2 h^{n+1} + m_3 h^{n+2} + \cdots$$

which is equivalent to

$$\varepsilon/h^k = m_1 h^{n-k} + m_2 h^{n+1-k} + m_3 h^{n+2-k} + \cdots$$

where n, $k > 0$ are integers, and m_j (for $j = 1, 2, \ldots$) are constant. For $n = k$ then

$$\frac{\varepsilon}{h^k} = m_1 + m_2 h^1 + m_3 h^2 + \cdots$$

And, as $h \to 0$,

$$\frac{\varepsilon}{h^k} \to m_1 (\text{constant})$$

It is then concluded that order of convergence k is an order of step-size h at which the estimation error remains constant.