Example: $\int_2^3 \frac{1}{x+1}\,dx$

$n=4 \qquad h = \frac{b-a}{n} = \frac{3-2}{4} = 0.25$

$[2, 2.25], [2.25, 2.5], [2.5, 2.75], [2.75, 3]$

$f(x_0) = f(2) = \frac{1}{2+1} = 0.\bar{3}$

$f(x_1) = f(2.25) = \frac{1}{2.25+1} = 0.3076923$

$f(x_2) = f(2.5) = \frac{1}{2.5+1} = 0.2857142$

$f(x_3) = f(2.75) = \frac{1}{2.75+1} = 0.2\bar{6}$

$f(x_4) = f(3) = \frac{1}{3+1} = 0.25$

$$\int_2^3 \frac{1}{x+1}\,dx \approx \frac{0.25}{3}\big(0.\bar{3} + 4(0.3076923)$$

$$+ 2(0.2857142) + 4(0.2\bar{6}) + 0.25\big)$$

$$= 0.2876831$$

$$\int_2^3 \frac{1}{x+1}\,dx = 0.287682$$

70 W4 L3

$$\int_0^2 e^{x^2}\,dx = 16.45262776$$

$h = \frac{1}{2} \Rightarrow$ intervals are

$[0, \frac{1}{2}], [\frac{1}{2}, 1], [1, 1\frac{1}{2}], [1\frac{1}{2}, 2]$

Trapezoidal rule:

$$\frac{\frac{1}{2}}{2}\left(e^{(0)^2} + 2e^{(0.5)^2} + 2e^{(1)^2} + 2e^{(1.5)^2} + e^{(2)^2}\right)$$

$= 20.64455905$ error ≈ 4.19193129

Simpson's rule:

$$\frac{\frac{1}{2}}{3}\left(e^{(0)^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2}\right)$$

$= 17.35362645$ error ~ 0.90099869

Adaptive quadrature.

So far ~~Trap~~ integration methods have used equal step sizes. Smaller step sizes improve accuracy, in general. If the function we would like to integrate varies wildly we would need to use a small step size to ~~est~~ ensure accuracy in these sections. However, this would require more computing time.

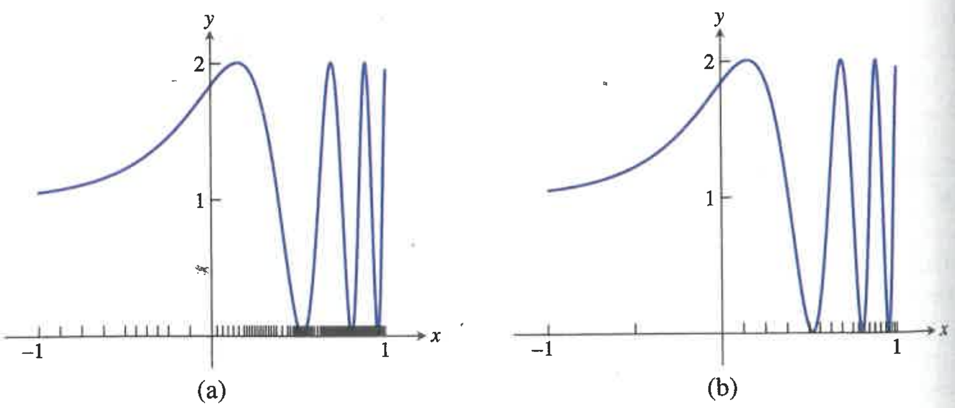


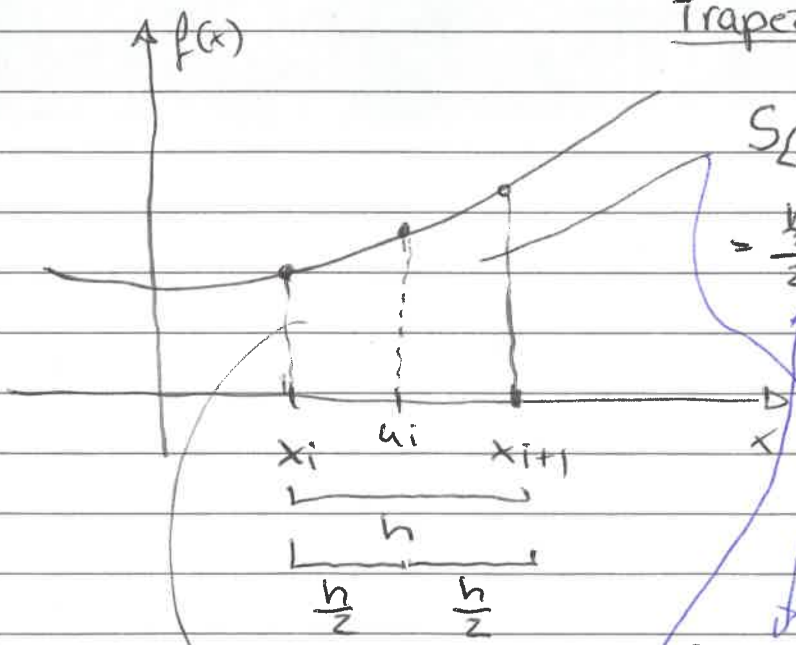
Figure 5.5 Adaptive Quadrature applied to $f(x) = 1 + \sin e^{3x}$. Tolerance is set to TOL = 0.005. (a) Adaptive Trapezoid Rule requires 140 subintervals. (b) Adaptive Simpson's Rule requires 20 subintervals.

from Sauer.

So idea is to adapt the step size, depending on the function. This way a large step size is used if the function does not vary rapidly and a small step size is used where the function changes very quickly.

Look at one single intervall first to see how this might look like.

Trapezoidal rule



$$S_{[x_i,u_i]} = \frac{\frac{h}{2}}{2}\left(f(x_i)+f(u_i)\right) = \frac{h}{4}\left(f(x_i)+f(u_i)\right)$$

$$S_{[u_i,x_{i+1}]} = \frac{h}{4}\left(f(u_i)+f(x_{i+1})\right)$$

W4 L3

73

Trapezoidal rule using 1 interval

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = S_{[x_i, x_{i+1}]} - \frac{h^3}{12} f''(c_0) \qquad (1)$$

Approximation using 2 half intervals

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = S_{[x_i, u_i]} - \frac{\left(\frac{h}{2}\right)^3}{12} f''(c_1) + S_{[u_i, x_{i+1}]} - \frac{\left(\frac{h}{2}\right)^3}{12} f''(c_2)$$

$$= S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]} - \left(\frac{h^3}{8 \cdot 12} f''(c_1) + \frac{h^3}{8 \cdot 12} f''(c_2)\right)$$

assume $c_1 \approx c_2$

$$= S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]} - \left(\frac{\cancel{2}h^3}{\cancel{8}_{4} \cdot 12} f''(c_1)\right)$$

$$= S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]} - \frac{h^3}{4} \frac{f''(c_1)}{12} \qquad (2)$$

W4 L3

74

① − ②

$$0 = S_{[x_i, x_{i+1}]} - h^3 \frac{f''(c_0)}{12} - \left(S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]} - \frac{h^3}{4} \frac{f''(c_1)}{12} \right)$$

$$= S_{[x_i, x_{i+1}]} - S_{[x_i, u_i]} - S_{[u_i, x_{i+1}]} - h^3 \frac{f''(c_0)}{12} + \frac{h^3}{4} \frac{f''(c_1)}{12}$$

$$\Rightarrow S_{[x_i, x_{i+1}]} - S_{[x_i, u_i]} - S_{[u_i, x_{i+1}]} = h^3 \frac{f''(c_0)}{12} - \frac{h^3}{4} \frac{f''(c_1)}{12}$$

assuming $c_0 \approx c_1$

$$= \frac{3}{4} h^3 \frac{f''(c_0)}{12}$$

$$\Rightarrow \text{error}\left(S_{[x_i, x_{i+1}]} - S_{[x_i, u_i]} - S_{[u_i, x_{i+1}]} \right)$$

$$\approx 3 \cdot \text{error}\left(S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]} \right)$$

$$\Rightarrow \left| S_{[x_i, x_{i+1}]} - S_{[x_i, u_i]} - S_{[u_i, x_{i+1}]} \right| < 3 \cdot \text{Tol}$$

accept $S_{[x_i, u_i]} + S_{[u_i, x_{i+1}]}$ as approximation for $\int_{x_i}^{x_{i+1}} f(x)\,dx$.

$\underbrace{S_{[x_i,x_{i+1}]} - S_{[x_i,u_i]} - S_{[u_i,x_{i+1}]}}_{*}$ is approximately three times the size of the integration error of $\underbrace{S_{[x_i,u_i]} + S_{[u_i,x_{i+1}]}}_{**}$ on $[x_i, x_{i+1}]$. Therefore, we can check if ~~the~~ $*$ is less than $3 * TOL$ for some error tolerance as an approximate way of checking if ~~the~~ $**$ approximates the integral within TOL.

If the tolerance is not met, we can subdivide the interval again.

For each half, the required error tolerance goes down by a factor of 2, while the error (for the Trapezoidal rule) should drop by a factor of $2^3 = 8$.

So this adaptive procedure should allow the original tolerance ~~for~~ to be met.