In ~~gen~~ general we consider

$\dot{y}_1 = f_1(y_1, y_2, t) \qquad y_1(0) = A_1$

$\dot{y}_2 = f_2(y_1, y_2, t) \qquad y_2(0) = A_2$

$\Rightarrow \dot{\underline{Y}} = \underline{f}(\underline{Y}, t) \qquad \underline{Y}(0) = \underline{A}$

Solve using Euler (a better one in practice).

$y_1^{n+1} = y_1^n + h f_1(y_1^n, y_2^n, t^n)$

$y_2^{n+1} = y_2^n + h f_2(y_1^n, y_2^n, t^n)$

$\underline{y}^{n+1} = \underline{y}^n + h \underline{f}(\underline{y}^n, t^n) \qquad t_n = t_0 + hn$

$h = \dfrac{(t_N - t_0)}{N}$

Consider the second order IVP

$\ddot{x} + 3\dot{x} + 2x = 0 \quad , x(0) = 1, \dot{x}(0) = 2$

Analytically we have

$x(t) = 4e^{-t} - 3e^{-2t}$

Set $\dot{x} = y \Rightarrow \dot{y} + 3y + 2x = 0$

$\dot{y} = -2x - 3y$

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Now use $X = \begin{pmatrix} x \\ y \end{pmatrix}$ then $\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$,

and $\dot{X} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X = F(t, X)$

With initial condition $x(0) = 1$, $\dot{x}(0) = 2$

$$X_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Use RK2 Heun's method :

$$x_{n+1} = x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_n + hf(t_n, x_n))\right]$$

$$\dot{x}_n = f(t_n, x_n)$$

and for y

$$y_{n+1} = y_n + \frac{h}{2}\left[g(t_n, y_n) + g(t_{n+1}, y_n + hg(t_n, y_n))\right]$$

$$\dot{y}_n = g(t_n, y_n)$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{h}{2}\left[\begin{pmatrix} f(t_n, x_n) \\ g(t_n, y_n) \end{pmatrix} + \begin{pmatrix} f(t_{n+1}, x_n + hf(t_n, x_n)) \\ g(t_{n+1}, y_n + hg(t_n, y_n)) \end{pmatrix}\right]$$

So $X_{n+1} = X_n + \frac{h}{2}\left[F(t_n, X_n) + F(t_{n+1}, X_n + hF(t_n, X_n))\right]$

For higher order ~~in~~ system $F = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$

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Example $\dot{X} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X$, $X_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

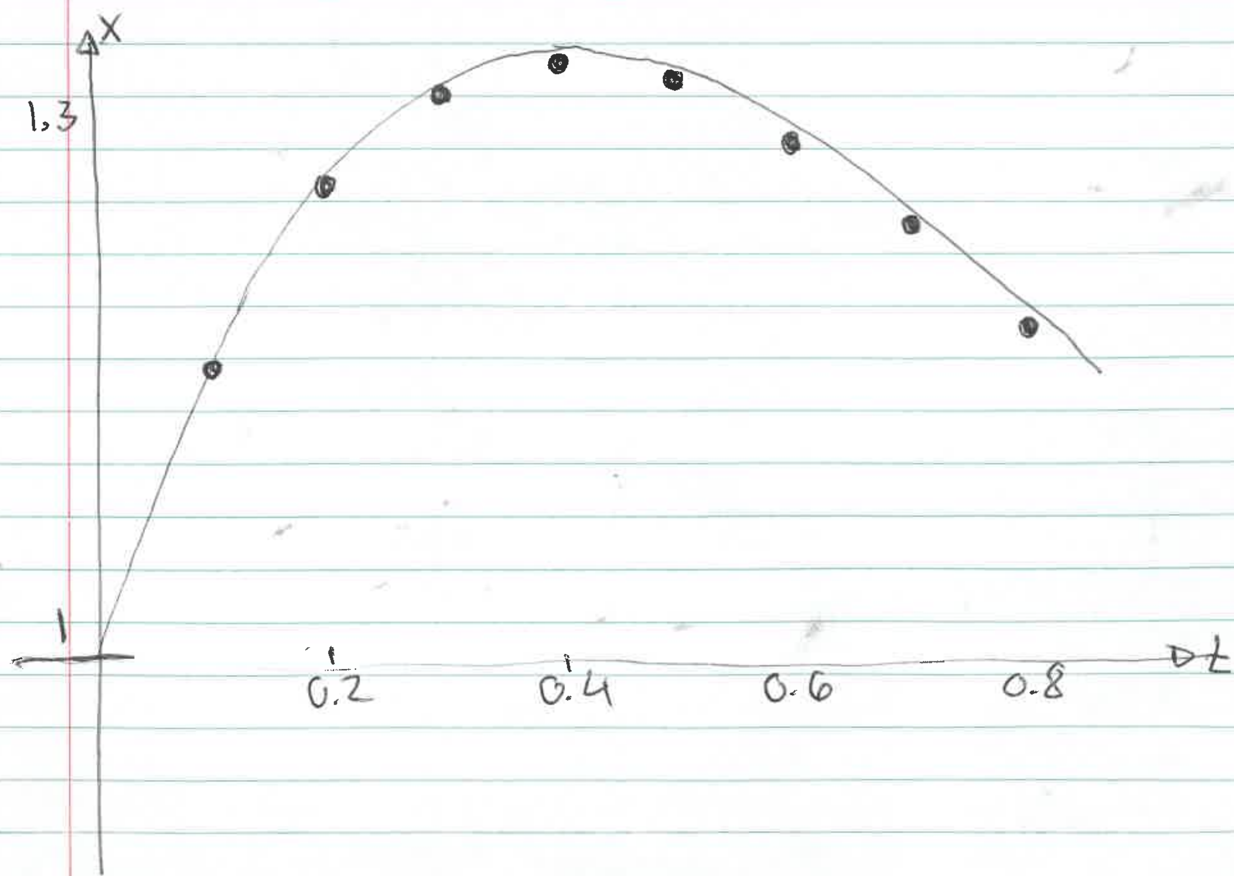
$h = 0.1$ remember $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 1$, $\dot{x}(0) = 2$

$$X_{n+1} + X_n + \frac{h}{2}\left[F(t_n, X_n) + F(t_{n+1}, X_n + hF(t_n, X_n))\right]$$

$$= X_n + 0.05\left[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X_n + \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\left(X_n + 0.1\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X_n\right)\right]$$

$X_1 = \begin{pmatrix} 1.16 \\ 1.3 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1.2589 \\ 0.7583 \end{pmatrix}$, ...

Plot analytic solution $x(t) = 4e^{-t} - 3e^{-2t}$ with the one from RK2



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Boundary value problems for ODEs — Shooting

For initial value problems we have say $y' = f(t,y)$ and initial conditions, depending on the order of the ODE:

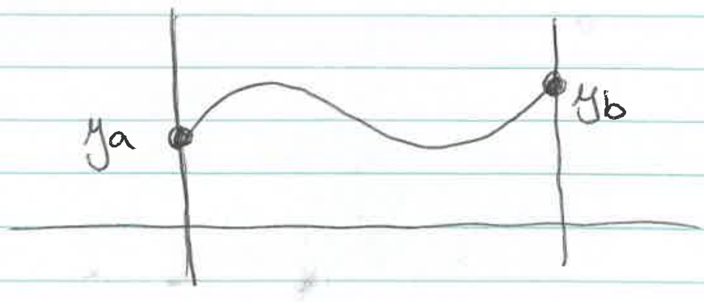
$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \ldots$$

For boundary value problems we usually have y given at several values of t, so $y(t_0) = y_0$ and $y(t_n) = y_n$, for boundaries t_0 and t_n

General problem $\frac{d^2y}{dx^2} = f\left(x, y(x), \frac{dy}{dx}(x)\right)$

$y(a) = y_a$

$y(b) = y_b$



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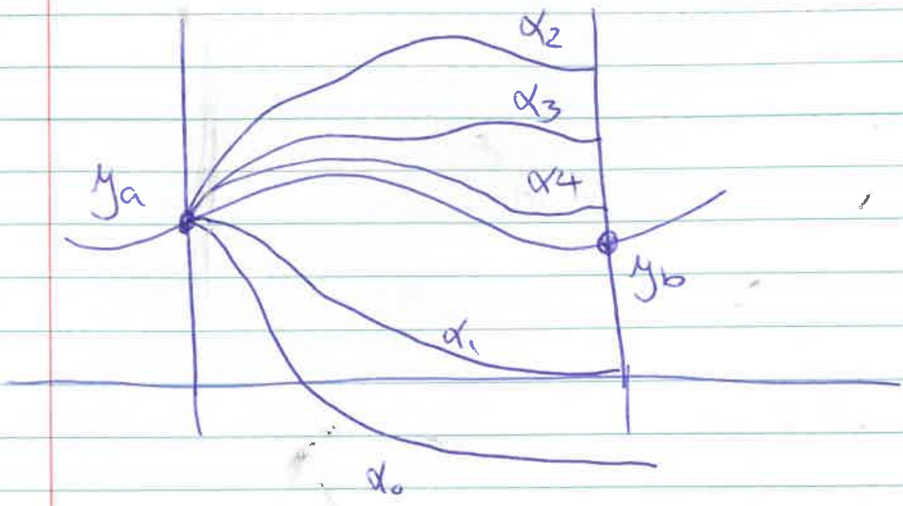

Can rewrite the second order problem as a system of first order ODEs.
However, to use the known ODE solvers, such as Runge-Kutta, we need initial values.
In shooting method we guess these initial values and by trial and error, or more sophisticated methods, we find the initial values and then use RK methods for example for the IVP.

Example:

$$u'' - \left(1 - \frac{x}{5}\right)u = x \quad , \quad u(1) = 2, \; u(3) = -1$$

For IVP we would need $u'(1)$?

let $v = u' \Rightarrow v' = u''$ so

$$v' = \left(1 - \frac{x}{5}\right)u + x$$

Now let $U = \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow$

$$U' = \begin{pmatrix} 0 & 1 \\ 1-\frac{x}{5} & 0 \end{pmatrix} U + \begin{pmatrix} 0 \\ x \end{pmatrix} = F(x, U)$$

Now guess $u'(1) = -1.5$

So for our system we have $U_0 = \begin{pmatrix} 2 \\ -1.5 \end{pmatrix}$

Using RK2 (Heun)

$$U_{n+1} = U_n + \frac{h}{2}\left[F(x_n, U_n) + F(x_n, U_n + hF(x_n, U_n))\right]$$

Using this and U_0 go to $x = 3$ we get

$u(3) \to 4.7764 \ldots 5.1018$

but we need $u(3) = -1$ our original boundary value.

So $u'(1) = -1.5$ is not correct.

Guess again $\to$ try $U_0 = \begin{pmatrix} 2 \\ -4.5 \end{pmatrix}$

. . . .

Need some way to check if our guess was good or not.

So for $y''(t) = f(t, y(t), y'(t))$ and

$y(t_0) = y_0$, $y(t_n) = y_n$

We can write

$$y''(t) = f(t, y(t), y'(t)), \; y(t_0) = y_0, \; y'(t_0) = \alpha$$

where $y'(t_0) = \alpha$ is our guess. To check ~~to~~ how good our guess is and to have a way to find the correct α we define a function $g(\alpha)$ that is the difference between $y(t_n; \alpha)$ and the ~~α~~ ~~a~~ required boundary value $y(t_n) = y_n$. $y(t_n; \alpha)$ is the solution obtained by our ODE solver for ~~initi~~ guess $y(t_n) = \alpha$.

$$g(\alpha) = y(t_n; \alpha) - y_n$$

If g has a root ~~than~~ α^*, then the solution $y(t; \alpha^*)$ of the corresponding initial value problem is also a solution of the boundary value problem. The roots of $g(\alpha)$ can be obtained by ~~Newtons~~ a root finding method ~~[illegible]~~ ~~example~~. such as bisection or secant or ~~Simp~~ other.