(64)

W4 L2

$$I_{TR} - I_{EX} =$$

$$= \cancel{hf_i} + \cancel{\tfrac{1}{2} f_i' h^2} + \tfrac{1}{4} f_i'' h^3 + O(h^4)$$

$$- \left(\cancel{hf_i} + \cancel{\tfrac{1}{2} f_i' h^2} + \tfrac{1}{6} f_i'' h^3 + O(h^4) \right)$$

$$= \left(\tfrac{1}{4} - \tfrac{1}{6}\right) f_i'' h^3 + O(h^4) = \left(\tfrac{3}{12} - \tfrac{2}{12}\right) f_i'' h^3 + O(h^4)$$

$$= \tfrac{1}{12} f''(x_i)\, h^3 + O(h^4)$$

error is $O(h^3)$

$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx = \tfrac{h}{2}\left(f(x_{i+1}) + f(x_i)\right) + O(h^3)$$

This was one interval, now get error for $[a,b]$

$h = \frac{b-a}{N}$ $\quad x_i = a + ih \;,\; i = 0, \dots, N$

$$\Rightarrow \text{Total error} = N \cdot \tfrac{1}{12} f''(x_i)\, h^3$$

$$\text{error} < N \cdot \tfrac{1}{12} h^3 \max_{a<\gamma<b} |f''(\gamma)|$$

$$= N \cdot h \tfrac{1}{12} h^2 \max |\;|$$

$$= (b-a) \tfrac{1}{12} h^2 \max |\;| = \tfrac{b-a}{12} h^2 \max |\;|$$

$$\Rightarrow O(h^2)$$

Example of Trapezoidal rule

$f(x) = e^{-x^2}$. $\int_{-1}^{1} e^{-x^2} dx$, $h = 0.1$

```
function m = F(x)

   m = exp(-(x^2))
```

Now define function to add the $\frac{1}{2}(f(x_i)+f(x_{i+1}))h$ from $x=-1$ to $x=1$

```
function t = trapadd (h, first, last)

  x = first;
  s = 0;
  for j = 1:((last-first)/h)

   s = vpa(s+h*(F(x)+F(x+h))/2);

   x = x + h
  end
  t = s
```

running that gives 1.49242

The true answer is $\int_{-1}^{1} e^{-x^2} dx = 1.49365$.

We can achieve a more accurate solution by reducing the step size to 0.01.

W4 L2

$$\int_0^1 \sqrt{x^2+1}\,dx$$ use $n=5$ intervals / points

$$h = \frac{1-0}{5} = 0.2 \Rightarrow [0, 0.2],$$

$$[0.2, 0.4], [0.4, 0.6], [0.6, 0.8], [0.8, 1]$$

$$f(x_0) = f(0) = \sqrt{0^2+1} = 1$$

$$f(x_1) = f(0.2) = \sqrt{(0.2)^2+1} = 1.0198039$$

$$f(x_2) = f(0.4) = \sqrt{(0.4)^2+1} = 1.0770330$$

$$f(x_3) = f(0.6) = \sqrt{(0.6)^2+1} = 1.1661904$$

$$f(x_4) = f(0.8) = \sqrt{(0.8)^2+1} = 1.2806248$$

$$f(x_5) = f(1) = \sqrt{(1)^2+1} = 1.4142136$$

$$\int_0^1 \sqrt{x^2+1}\,dx \approx 0.2\left(\frac{1}{2}\cdot 1 + 1.0198039\right.$$

$$+ 1.0770330 + 1.1661904 + 1.2806248$$

$$\left. + \frac{1}{2}\cdot 1.4142136\right) = \underline{\underline{1.150}}$$

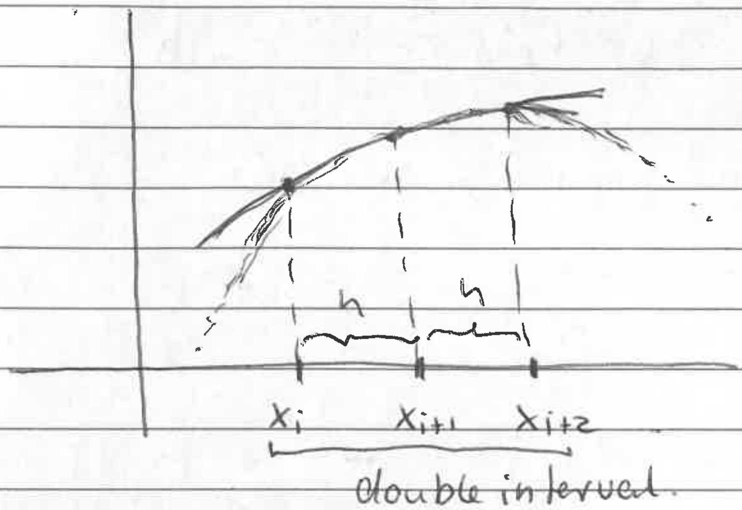
$$\int_0^1 \sqrt{x^2+1}\,dx = \underline{1.148}$$

W4 L2 (50)

Simpson's Rule:

Trapezoidal rule: 2 points, linear approximation

Simpson's rule: 3 points, quadratic approximation



$x_{i+2} - x_i = 2h$

$$\int_{x_i}^{x_{i+2}} \underbrace{f(x)}_{\text{Taylor about } x_{i+1}} dx = \int_{x_i}^{x_{i+2}} \Big\{ f(x_{i+1}) + f'(x_{i+1})(x-x_{i+1}) + \tfrac{1}{2} f''(x_{i+1})(x-x_{i+1})^2 + \tfrac{1}{6} f'''(x_{i+1})(x-x_{i+1})^3 + \tfrac{1}{24} f^{(4)}(x_{i+1})(x-x_{i+1})^4 + \underbrace{\cancel{x^5}}_{HOT} \Big\} dx$$

$$= \Big[f(x_{i+1})x + \tfrac{1}{2} f'(x_{i+1})(x-x_{i+1})^2 + \tfrac{1}{6} f''(x_{i+1})(x-x_{i+1})^3 + \tfrac{1}{24} f^{(3)}(x_{i+1})(x-x_{i+1})^4 + \tfrac{1}{120} f^{(4)}(x_{i+1})(x-x_{i+1})^5 + HOT \Big]_{x_i}^{x_{i+2}}$$

use $f_i = f(x_i)$ abbreviation

$$= f_{i+1} x_{i+2} + \tfrac{1}{2} f'_{i+1} \underbrace{(x_{i+2}-x_{i+1})}_{h}^2 + \tfrac{1}{6} f^{(2)}_{i+1} \underbrace{(x_{i+2}-x_{i+1})}_{h}^3 + \tfrac{1}{24} f^{(3)}_{i+1} \underbrace{(x_{i+2}-x_{i+1})}_{h}^4$$

$$+ \tfrac{1}{120} f^{(4)}_{i+1}(x_{i+1}) \underbrace{(x_{i+2}-x_{i+1})}_{h}^5 + HOT$$

$$- f_{i+1} x_i - \tfrac{1}{2} f'_{i+1} \underbrace{(x_i - x_{i+1})}_{-h}^2 - \tfrac{1}{6} f''_{i+1} \underbrace{(x_i - x_{i+1})}_{-h}^3 - \tfrac{1}{24} f^{(3)}_{i+1} \underbrace{(x_i - x_{i+1})}_{-h}^4$$

$$- \tfrac{1}{120} f^{(4)}_{i+1}(x_{i+1}) \underbrace{(x_i - x_{i+1})}_{-h}^5 + HOT$$

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W4 L2

$$= f_{i+1} \underbrace{(x_{i+2} - x_i)}_{2h} + \frac{1}{2} f'_{i+1} \cancel{(h^2 - h^2)} + \frac{1}{6} f''_{i+1} (h^3 + h^3)$$

$$+ \frac{1}{24} f'''_{i+1} \cancel{(h^4 - h^4)} + \frac{1}{120} f^{(4)}(x_{i+1})(h^5 + h^5)$$

$$= 2h f_{i+1} + \frac{1}{3} f''_{i+1} h^3 + \frac{1}{60} f^{(4)}(x_{i+1}) h^5$$

now use approximation for f''_{i+1} with error

$$f''_{i+1} = \frac{f_i - 2f_{i+1} + f_{i+2}}{h^2} - \frac{h^2}{12} f^{(4)}(x_{i+1})$$

$$= 2h f_{i+1} + \frac{h^3}{3} \left(\frac{f_i - 2f_{i+1} + f_{i+2}}{h^2} - \frac{h^2}{12} f^{(4)}(x_{i+1}) \right)$$

$$+ \frac{1}{60} f^{(4)}(x_{i+1}) h^5$$

$\frac{6h}{3}$

$$= 2h f_{i+1} + \frac{h}{3} f_i - \frac{2h}{3} f_{i+1} + \frac{h}{3} f_{i+2} - \frac{h^5}{36} f^{(4)}(x_{i+1})$$

$$+ \frac{h^5}{60} f^{(4)}(x_{i+1})$$

$$= \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2}) - \frac{5h^5}{180} f^{(4)}(x_{i+1}) + \frac{3h^5}{180} f^{(4)}(x_{i+1})$$

$$= \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2}) - \frac{h^5}{90} f^{(4)}(x_{i+1})$$

$$= \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2}) + O(h^5)$$ double subinterval

error over total interval $\frac{b-a}{N} = h$ $\quad N = \frac{b-a}{h}$

$$\frac{N}{2} \frac{h^5}{90} \max_{a \le \gamma \le b} f^{(4)}(\gamma) = \frac{b-a}{N} \frac{h^4}{90} \max_{a \le \gamma \le b} f^{(4)}(\gamma)$$

$$\Rightarrow O(h^4)$$