Math 3201 L4

Recall: A PDE is a differential equation with more than one independent variable.

we will look at numerical methods to approximate solutions for PDEs.

Example:
$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

The heat (diffusion) equation models $\underset{\downarrow}{\underline{\text{temperature}}}$ along a finite bar
at position x at time t. $u(x,t)$.

Aim: to approximate solution $u(x,t)$.

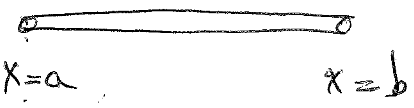
independent variables are: space x

time t

D: diffusion coefficient with represents the thermal diffusivity of material making up the bar.

Recall: heat ~~equa~~ PDE has infinitely many solutions. So, to pin down a specific solution, we need extra conditions $\begin{pmatrix} BCs \\ ICs \end{pmatrix}$

Standard boundary and initial conditions:



① ICs:

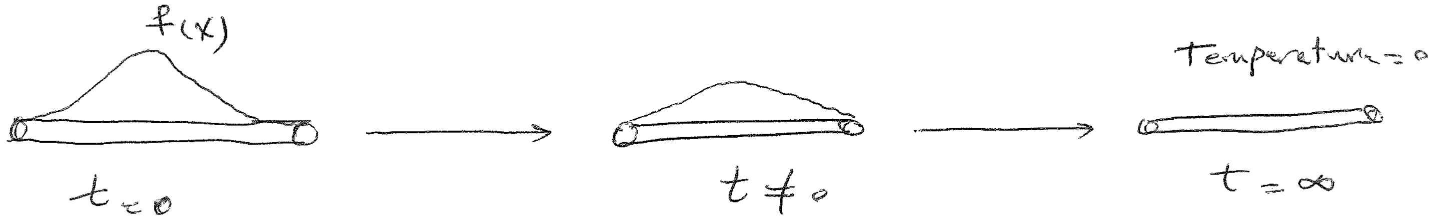
initial temperature along the bar at position x at time 0: $u(x,0) = f(x)$

all $a \leqslant x \leqslant b$

② BCs:

Temperature at either end of bar: $\left.\begin{array}{l} u(a,t) = L(t) \\ u(b,t) = r(t) \end{array}\right\}$ for all $t \geqslant 0$

Note: In many cases we take BCs: $L(t) = r(t) = 0$

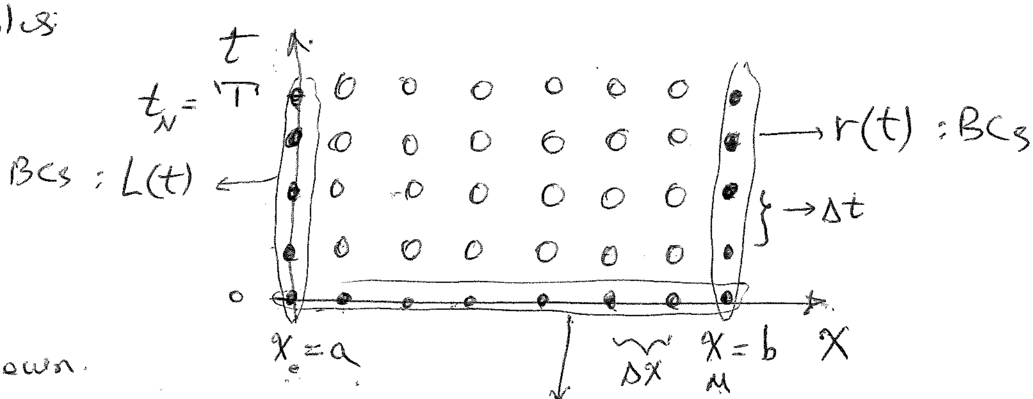


Discretise the problem:

we use finite difference methods to solve PDE for heat.

we discretise the heat equation on time interval $[0, T]$ and space interval $[a, b]$.

Grid in independent variables:



•: values of solution $u(x,t)$ known from BCs and ICs.

o: values of solution unknown.

(will be approximated by numerical methods).

ICs: $f(x)$

For discrete space points x_i: M mesh grid points.

$$x_0 = a$$
$$x_M = b$$
$$x_i = a + i\Delta x, \quad \Delta x = \frac{b-a}{M}, \quad i = 0, \dots, M$$

For discrete time points t_j: N time intervals.

$$t_0 = 0$$
$$t_N = T \quad \text{(Total time)}$$
$$t_j = j\Delta t \qquad \Delta t = \frac{T}{N}, \quad j = 0, \dots, N$$

Set notation: Exact solution $\longrightarrow$ Approximate discrete solution

$$u(x_i, t_j) \sim u_{i,j}$$

$$\text{BCs:} \begin{cases} u(x_0, t) = l(t) \longrightarrow u_{0,j} = l(t_j) = l_j \\ u(x_M, t) = r(t) \longrightarrow u_{M,j} = r(t_j) = r_j \end{cases}$$

$$\text{ICs:} \left\{ u(x,0) = f(x) \longrightarrow u_{i,0} = f(x_i) = f_i \right.$$

Discretise heat equation using "Forward Difference Method":

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

For 1^{st} derivative (t): replace forward-difference formula
For 2^{nd} " (x): " central-difference formula

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = D \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2}$$

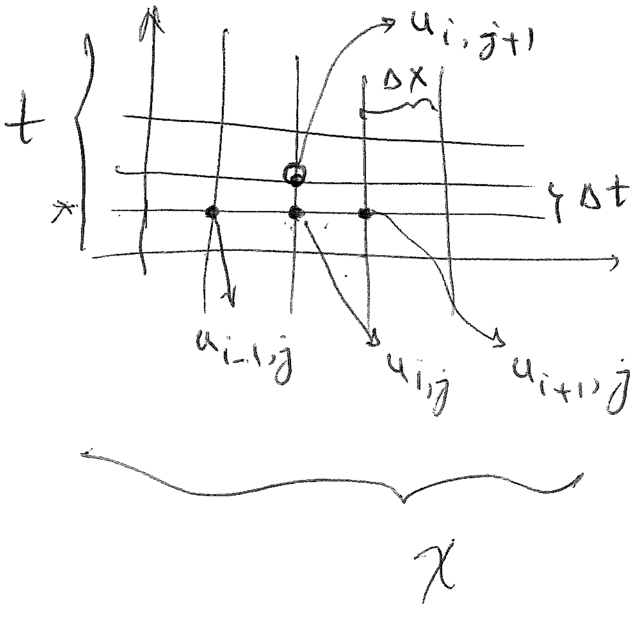
$$\Big\downarrow \; u(x_i, t_j) \sim u_{i,j}$$

$$u_{i,j+1} - u_{i,j} = \frac{D \Delta t}{(\Delta x)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right)$$

update rule:

$$\underbrace{u_{i,j+1}}_{\substack{\text{Heat at } x_i \\ \text{at time } t_{j+1}}} = \underbrace{u_{i,j}}_{\substack{\text{heat at} \\ x_i \text{ at time} \\ t_j}} + \delta \overbrace{\Big(\underbrace{u_{i+1,j}}_{\substack{\text{heat coming} \\ \text{from right} \\ x_{i+1}}} - \underbrace{2u_{i,j}}_{\substack{\text{heat} \\ \text{flows out} \\ \text{from } x_i}} + \underbrace{u_{i-1,j}}_{\substack{\text{heat comes} \\ \text{from left} \\ x_{i-1}}} \Big)}^{\text{are known from prior timestep}}, \quad \delta = \frac{\Delta t D}{(\Delta x)^2} \qquad (1)$$

The forward difference method (1) is "explicit": we can determine temperature (i.e., solution) at any given time-step $(j+1)$ directly from the previously known temperature at time step j.



$$\begin{cases} u_{i,j+1} = u_{i,j} + \delta\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \qquad \delta = \dfrac{\Delta t D}{(\Delta x)^2} \\ \text{BCs: } i=0 \ (x_0 = a) \longrightarrow u_{0,j+1} = l_{j+1} \\ \qquad\quad i = M \ (x_M = b) \longrightarrow u_{M,j+1} = r_{j+1} \\ \text{ICs: } u_{i,0} = f_i \longrightarrow f_0 = l_0 \longrightarrow (u_{0,0}) \\ \qquad\qquad\qquad\quad f_M = r_0 \longrightarrow (u_{M,0}) \end{cases}$$

write these in matrix form for positions $i = 1, \dots, M-1$ (internal grid points)

$$\underbrace{\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{M-1,j+1} \end{bmatrix}}_{\text{unknown heat}} = \underbrace{\begin{bmatrix} (1-2\delta) & \delta & 0 & 0 & 0 & & \\ \delta & (1-2\delta) & \delta & 0 & 0 & \cdots & \\ 0 & \delta & (1-2\delta) & \delta & 0 & \cdots & 0 \\ & & & & & & \\ 0 & \cdots & \cdots & & 0 & \delta & (1-2\delta) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}}_{\text{known heat}} + \underbrace{\begin{bmatrix} \delta l_j \\ 0 \\ \vdots \\ 0 \\ \delta r_j \end{bmatrix}}_{S}$$

$$\underline{u}^{(j+1)} = A\,\underline{u}^{(j)} + \underline{S}^{(j)}$$