Example: Solve the wave equation using algebric methods

$$\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2} \qquad c = 3 : \text{wave speed.}$$

$$u(x,0) = x^3 + x^2 - 5x + 4 + \cos(x) + 5\sin(x) \qquad : h(x)$$

$$\frac{\partial u}{\partial t}(x,0) = -9x^2 + 6x - 27 - 15\cos(x) - 3\sin(x) \qquad : K(x)$$

General solution

$$u(x,t) = \frac{1}{2}\Big(h(x+ct) + h(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} K(v)\,dv.$$

$$u(x,t) = \frac{1}{2}\Big((x+3t)^3 + (x+3t)^2 - 5(x+3t) + 4 + \cos(x+3t) + 5\sin(x+3t)$$
$$+ (x-3t)^3 + (x-3t)^2 - 5(x-3t) + 4 + \cos(x-3t) + 5\sin(x-3t)\Big)$$
$$+ \frac{1}{6}\int_{x-3t}^{x+3t}\Big(-9v^2 + 6v - 27 - 15\cos(v) - 3\sin(v)\Big)\,dv =$$

$$= x^3 + x^2 + 27xt^2 - 5x + 9t^2 + 4 + \frac{1}{2}\Big(\cos(x-3t) + \cos(x+3t)\Big) +$$
$$\frac{5}{2}\Big(\sin(x-3t) + \sin(x+3t)\Big) +$$
$$+ \frac{1}{6}\Big[-3v^3 + 3v^2 - 27v - 15\sin(v) + 3\cos(v)\Big]_{x-3t}^{x+3t} =$$

$$u(x,t) = (x+3t)^3 - 7(x+3t) + (x-3t)^3 - 2(x-3t) + 4 + \cos(x+3t)$$
$$+ 5\sin(x+3t)$$

The solution describes how the wave propagates in space(x) and time(t).

Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Describes the steady-states ~~where~~ which are time-independent.

Example: $u(x,y)$: heat

Laplace Eq. models the distribution of heat on a surface whose boundaries are being held at a specific temperature.

Note:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{Poisson Eq.}$$

Solving Poisson Equation using Finite Difference Methods:

$$\nabla^2 u = f$$

Note:

Dirichlet BCs: specify the value of solution u on the boundaries.

Neumann BCs: specify the value of a derivative at the boundary.

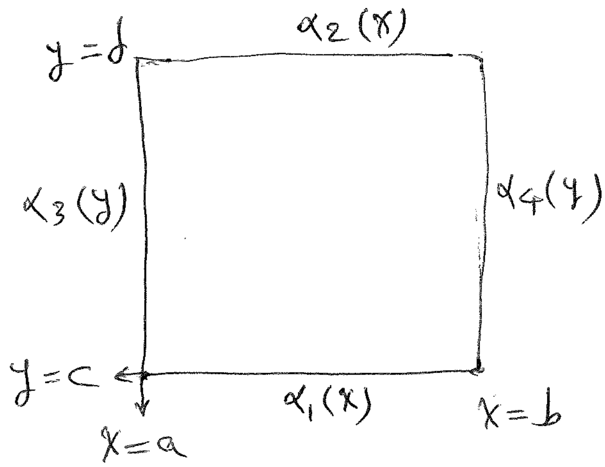
Look at solving Poisson Eq:

BCs: $u(x,c) = \alpha_1(x)$

$u(x,d) = \alpha_2(x)$

$u(a,y) = \alpha_3(y)$

$u(b,y) = \alpha_4(y)$



std grid:

M grid points in x-direction: $x_i = a + i\Delta x$, $\Delta x = \dfrac{b-a}{M}$,

$i = 0, \dots, M$

N gird points in y-direction: $y_j = c + j\Delta y$, $\Delta y = \dfrac{d-c}{N}$

$j = 0, \dots, N$

Note: we take $\Delta x = \Delta y$.

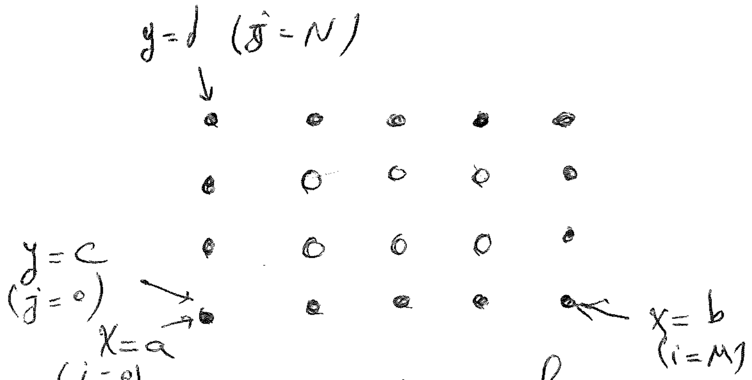
$$\underbrace{u_{i,j}}_{\text{Approximation}} \sim \underbrace{u(x_i, y_j)}_{\text{Exact solution}}$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} - f_{i,j}(\Delta x)^2 = 0$$

BCs:

$$\left.\begin{aligned} u_{i,0} &= \alpha_1(x_i) \\ u_{i,N} &= \alpha_2(x_i) \end{aligned}\right\} i = 1, \dots, M-1$$

$$\left.\begin{aligned} u_{0,j} &= \alpha_3(y_j) \\ u_{M,j} &= \alpha_4(y_j) \end{aligned}\right\} j = 1, \dots, N-1$$



●: Known values of solution

O: unknowns to be calculated

Example: Solve poisson equation

$$\nabla^2 u = f \quad \text{where } 0 \le x \le 5, \quad 0 \le y \le 4, \quad \Delta x = \Delta y,$$

$u(x,0) = 1$

$u(x,4) = 2$

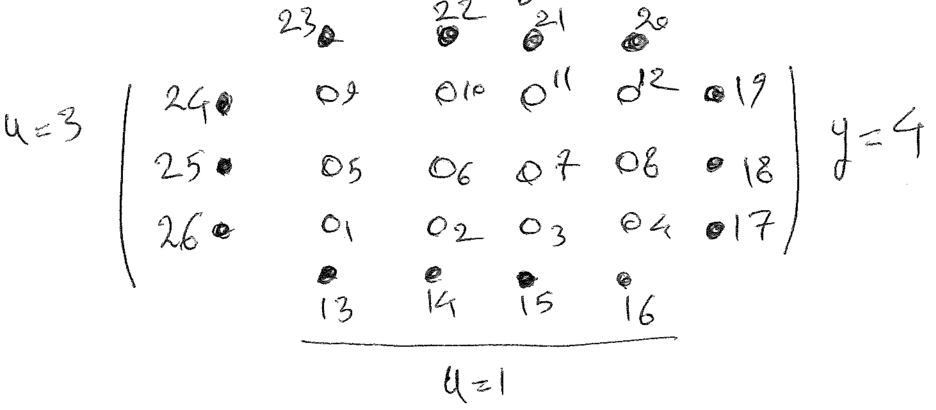
$u(0,y) = 3$

$u(5,y) = 4$

we have 4×3 (12) integral grid points.

General Solution: $u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} - h^2 f_{i,j} = 0$

$h = \Delta x = \Delta y$



Point 1: $-4u_1 + u_5 + \cancel{u_{26}}^{\,3} + \cancel{u_{13}}^{\,1} + u_2 - h^2 f_1 = 0$

$u_{1,1}$

$$-4u_1 + u_2 + u_5 = h^2 f_1 - 4$$

Point 5: $-4u_5 + u_9 + \cancel{u_{25}}^{\,3} + u_1 + u_6 = h^2 f_5$

$u_{1,2}$

$$u_1 - 4u_5 + u_6 + u_9 = h^2 f_5 - 3$$

$\vdots$

we will get 12 equations

Then, we write the 12 equations in matrix form.

(5)

If we continue, we get the following system of equations: in the matrix form:

12 rows

for 12 interval points

	1	2	3	4	5	6	7	8	9	10	11	12		RHS
1	-4	1			1								=	h^2 f-3-1 → BC
2	1	-4	1			1							=	h^2 f -1
3		1	-4	1			1						=	h^2 f -1
4			1	-4				1					=	h^2 f -1 -4
5	1				-4	1			1				=	h^2 f -3
6		1			1	-4	1			1			=	h^2 f
7			1			1	-4	1			1		=	h^2 f
8				1			1	-4				1	=	h^2 f -4
9					1				-4	1			=	h^2 f -3 -2
10						1			1	-4	1		=	h^2 f - 2
11							1			1	-4	1	=	h^2 f -2 -3
12								1			1	-4	=	h^2 f -2 -4

$\underbrace{\qquad}_{A}$ $\underbrace{\qquad}_{b}$

BC

BC

we then write this as: $A\underline{u} = b$, where b: contains all RHS and BCs.

A: has a sparse structure (many zeros).