MATH3201 Lecture 2

Recall:

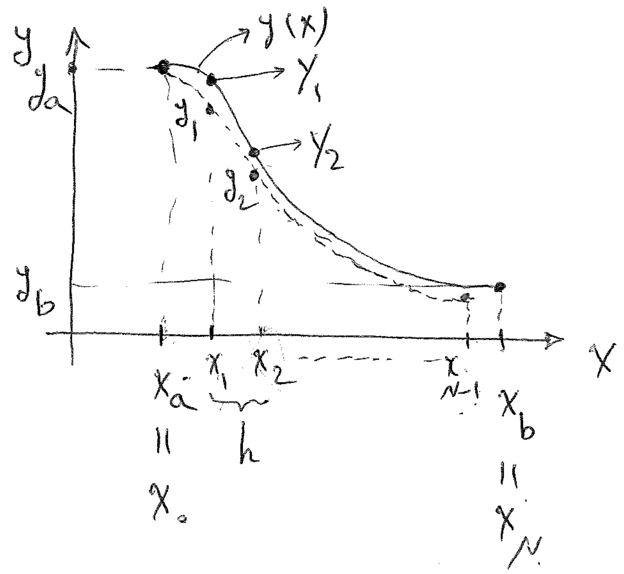
BVP is a differential equation presented along with its behaviours at boundaries. (BC.)

- shooting method ✓
- Finite difference methods.

Convert a differential equation into a closed set of equation by replacing continuous derivatives with discrete approximation.

General form:
$$\begin{cases} y'' = f(x, y, y') \\ y(x_a) = y_a, \quad y(x_b) = y_b \end{cases}$$

To discretize the equation, we set up a "standard grid".



solution: $y(x)$

standard grid:

*) $N+1$ grid points x_i where $i = 0, \dots, N$

*) N grid intervals

*) $x_i = x_a + ih$ $\qquad h = \Delta x = \frac{x_b - x_a}{N}$

Note: Grid can be defined differently depending on indexing.

standard grid will allow us to discretise the equation to a right number of simultaneous equation to be solved.

Example: BVP: $\begin{cases} y'' - y' = 0 \\ y(0) = \alpha, \quad y(1) = \beta \end{cases}$

To solve it using finite difference methods, we replace the first and second derivatives with discrete approximations.

$$y' \simeq \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} - \text{Error } o(\Delta x)^2$$

$$y'' \simeq \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2} + \text{Error } o(\Delta x)^2$$

$$\underbrace{y_i}_{\text{approximation}} \simeq \underbrace{y(x_i)}_{\text{real value of solution}}$$

BVP: $$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{y_{i+1} - y_{i-1}}{2h} = 0$$

Now, we evalute BVP over standard grid:

B.C. $\begin{cases} y(x_a) = \alpha \longrightarrow y(x_0) = y_0 = \alpha \\ y(x_b) = \beta \longrightarrow y(x_N) = y_N = \beta \end{cases}$ B.C

(*)

For $i=1$ (at x_1): $$\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{y_2 - y_0}{2h} = 0$$

For $i=2$ (at x_2): $$\frac{y_3 - 2y_2 + y_1}{h^2} - \frac{y_3 - y_1}{2h} = 0$$

$\vdots$

For $i=N-1$: $$\frac{y_N - 2y_{N-1} + y_{N-2}}{h^2} - \frac{y_N - y_{N-2}}{2h} = 0$$

interior grid points

set of simultaneous equations which are closed.

Substituting B.C into the interior discrete equations, you will have $N-1$ equations with $N-1$ unknowns: which are: $y_1, y_2 \cdots y_{N-1}$

~~Net~~ Next, is to write equations $(*)$ in form $A\underline{x} = b$

A: Coefficient matrix

X: Unknowns (variables)

b: Knowns.

For $i=1$: $y_2 - 2y_1 + y_0 - \frac{h}{2}(y_2 - y_0) = 0$
(at x_1)

Recall B.C $y_0 = \alpha$

$$-2y_1 + y_2\left(1-\frac{h}{2}\right) = -y_0 - \frac{h}{2}y_0$$

$$= -\alpha\left(1+\frac{h}{2}\right) \checkmark$$

For $i=2$: $y_3 - 2y_2 + y_1 - \frac{h}{2}(y_3 - y_1) = 0$
(at x_2)

$$\left(1+\frac{h}{2}\right)y_1 - 2y_2 + y_3\left(1-\frac{h}{2}\right) = 0 \checkmark$$

For $i=N-1$: $\underline{y_N} - 2y_{N-1} + y_{N-2} - \frac{h}{2}(\underline{y_N} - y_{N-2}) = 0$
(at x_{N-1})

B.C: $y_N = \beta$

$$y_{N-2}\left(1+\frac{h}{2}\right) - 2y_{N-1} = -\beta\left(1-\frac{h}{2}\right) \checkmark$$

Now, write this set of equations in matrix form. $A\underline{x} = b$.

$$\underbrace{\begin{bmatrix} -2 & (1-\frac{h}{2}) & 0 & \cdots & \cdots & 0 \\ (1+\frac{h}{2}) & -2 & (1-\frac{h}{2}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & (1+\frac{h}{2}) & -2 & (1-\frac{h}{2}) & \\ 0 & \cdots & \cdots & 0 & (1+\frac{h}{2}) & -2 \end{bmatrix}}_{\text{matrix } A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -\alpha(1+\frac{h}{2}) \\ 0 \\ \vdots \\ 0 \\ -\beta(1-\frac{h}{2}) \end{bmatrix}}_{b}$$

Now, you can solve $A\underline{X} = b$ using Gaussian elimination methods:

- Thomas algorithm
- Gauss-Seidel iterative method.

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Few points about the matrix form $Ax = b$:

① Matrix A has a repeating pattern: easy to code.

② Matrix A is tridiagonal: very efficient (memory / speed) for large problems.

2010's: $N = 10^6$ large

1960's: $N = 20$ "

Accuracy of approximation: is limited by the error terms $O(\Delta x^2)$

where: $\Delta x = \dfrac{x_b - x_a}{N}$.

Error estimation:

Estimations by the finite difference methods converge to solution in the second order of Δx.

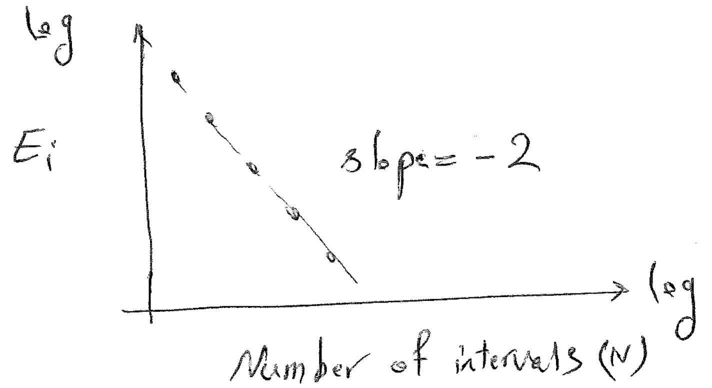
If we increase the number of grid intervals, the error will decrease.

At a grid a grid point x_i :

Y_i : real value of solution at x_i

y_i : estimation of solution at x_i

$E_i = |Y_i - y_i|$ estimation error at x_i



$E_i \approx a + b \log N$, $b \approx -2$

$E_i \simeq K N^{-2}$

Error is $o(N^{-2}) = o(h^2)$