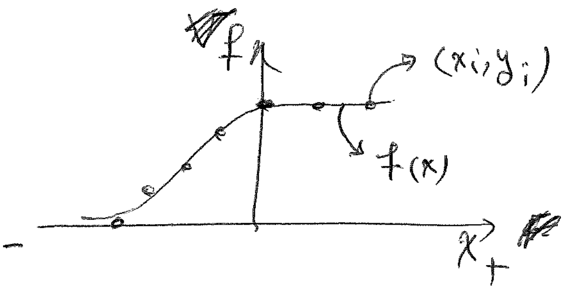
Interpolation:

We aim to find functions that pass through given data points,
i.e., we find a model (interpolating eq. or fit eq.) to represent
the data.



Definition for interpolation:

function $y = f(x)$ interpolates data points (x_i, y_i), where $1 \leq i \leq n$,
if $f(x_i) = y_i \quad \forall i$.

Note:

$f(x)$ must be a function. Every x_i corresponds to a single
y_i (x_i should be distinct).

We introduce two interpolation methods: 1 Polynomials.

2 Spline.

Polynomial Interpolation:

Reasons: 1) straight forward mathematical properies.

2) evaluate complicated functions using elementary operaters (addition, subtraction and multiplication).

Lagrange interpolating polynomial:

Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$, it defines a polynomial that interpolates the points:

$$P(x) = \sum_{i=1}^{n} y_i \, L_i(x) \quad \text{where}$$

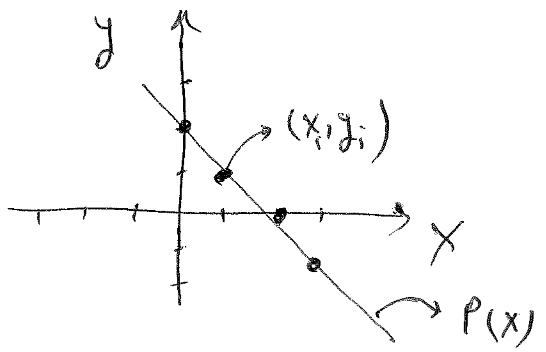
$$L_i(x) = \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Example: Given four data points $(0,2)$, $(1,1)$, $(2,0)$ and $(3,-1)$ find a polynomial interpolates the data, using Lagrange method.

$$P(x) = 2\,\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + 1\,\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} +$$

$$0\,\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} - 1\,\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} =$$

$$= -x + 2$$



Newton's divided differences:

x_i	$f(x_i)$		
x_1	$f(x_1)$		
$\vdots$			
✓ x_k	$f(x_k)$	$f[x_k\ \ x_{k+1}] = \dfrac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$	$f[x_k\ \ x_{k+1}\ \ x_{k+2}] = \dfrac{f[x_{k+1}\ x_{k+2}] - f[x_k\ x_{k+1}]}{x_{k+2} - x_k}$
x_{k+1}	$f(x_{k+1})$	$f[x_{k+1}\ \ x_{k+2}] = \dfrac{f(x_{k+2}) - f(x_{k+1})}{x_{k+2} - x_{k+1}}$	
✓ x_{k+2}	$f(x_{k+2})$		
$\vdots$			

Main Theorem:

Let (x_i, y_i) where $i = 1, \ldots, n$ be n data points with distinct x_i. There exist one and only one polynomial P of degree at most $(n-1)$ that $P(x_i) = y_i$.

Proof: see the book.

Example:

Use Newton's divided differences method to fit a polynomial to three data points $(0, 1)$, $(2, 2)$, and $(3, 4)$.

x_i	y_i		
0	1	$\dfrac{2-1}{2-0} = \dfrac{1}{2}$	$\dfrac{2-1/2}{3-0} = 1/2$
2	2	$\dfrac{4-2}{3-2} = 2$	
3	4		

x_i	y_i		
0	1	1/2	1/2
2	2	2	
3	4		

~~$P(x) = y_1 +$~~

$$P_2(x) = 1 + 1/2\,(x-0) + 1/2\,(x-0)(x-2)$$

$$= \frac{1}{2}x^2 - 1/2\,x + 1$$

Newton's divided differences:

$$P(x) = f(x_1) + f[x_1\ x_2](x - x_1) + f[x_1\ x_2\ x_3](x-x_1)(x-x_2)$$

Notes:

1) $P(x)$ is degree-2 polynomial. This implies (from main theorem) there is no degree 0 or 1 interpolating polynomial for these three data points.

2) Lagrange method will give you the same polynomial.

Example: Three data points $(0,1), (2,2), (3,4) \longrightarrow P_2(x)$.

Lets add $(1,0)$ to the data points and calculate the interpolating polynomial.

x_i	y_i			
0	1			
		1/2		
2	2		1/2	
		2		-1/2
3	4		0	
		2		
1	0			

$$P_3(x) = \underbrace{1 + \frac{1}{2}(x-0) + 1/2\,(x-0)(x-2)}_{P_2(x)} \underbrace{- \frac{1}{2}(x-0)(x-2)(x-3)}_{\text{due to } (1,0)} =$$

How many degree 3 polynomials interpolate the same three data points $(0,1)$, $(2,2)$, $(3,4)$?

One way: add one more data point; e.g., $(1,0)$

then you obtain a $P_3(x)$.

If I change the fourth point, I will get a different $P_3(x)$.

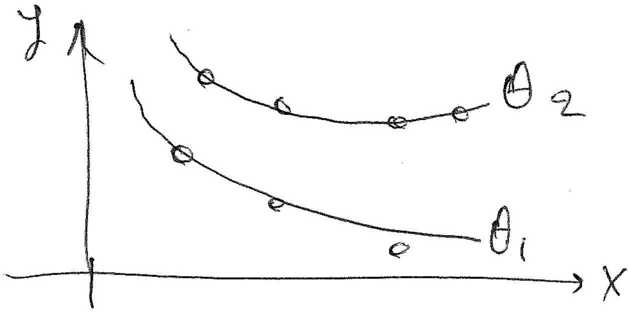
Thus, any degree 3 polynomial of following form interpolates the three data points:

$$P_3(x) = P_2(x) + C\,(x-0)(x-2)(x-3)$$

where $C \neq 0$.

Theorem: Given n data points (x_i, y_i) with distinct x_i, there are infinitely many degree $\geq n$ polynomials interpolates the points.

Example



3D fitting.