W5 L2

82 Solving analytically

$$y(t) = -2t + 2 - 3e^{-t}$$

check: $y' = -2 - 3e^{-t}(-1)$

$$= -2 + 3e^{-t}$$

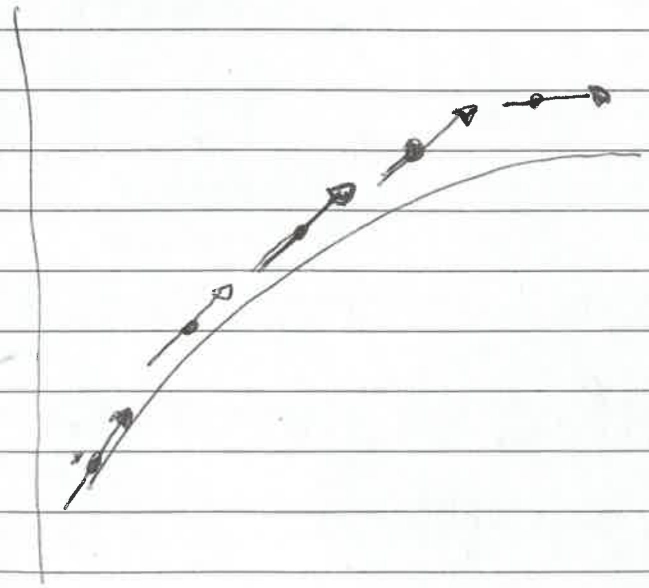
$$\cancel{= -2t - (-2t + 2 - 3e^{-t})}$$

$$= -2t + 2t - (2 - 3e^{-t})$$

$$= -2t - (-2t + 2 - 3e^{-t})$$

$$= -2t - y$$

So can compare the approximation given by Euler's method to the actual solution in $[0, 0.8]$



errors add up cumulatively.

WS 22



error : use Taylor

exact : $y(t_0+h) = y(t_0) + hy'(t_0) + \frac{1}{2}h^2 y''(t_0) + O(h^3)$

Euler : $\tilde{y}(t_0+h) = y(t_0) + hy'(t_0)$

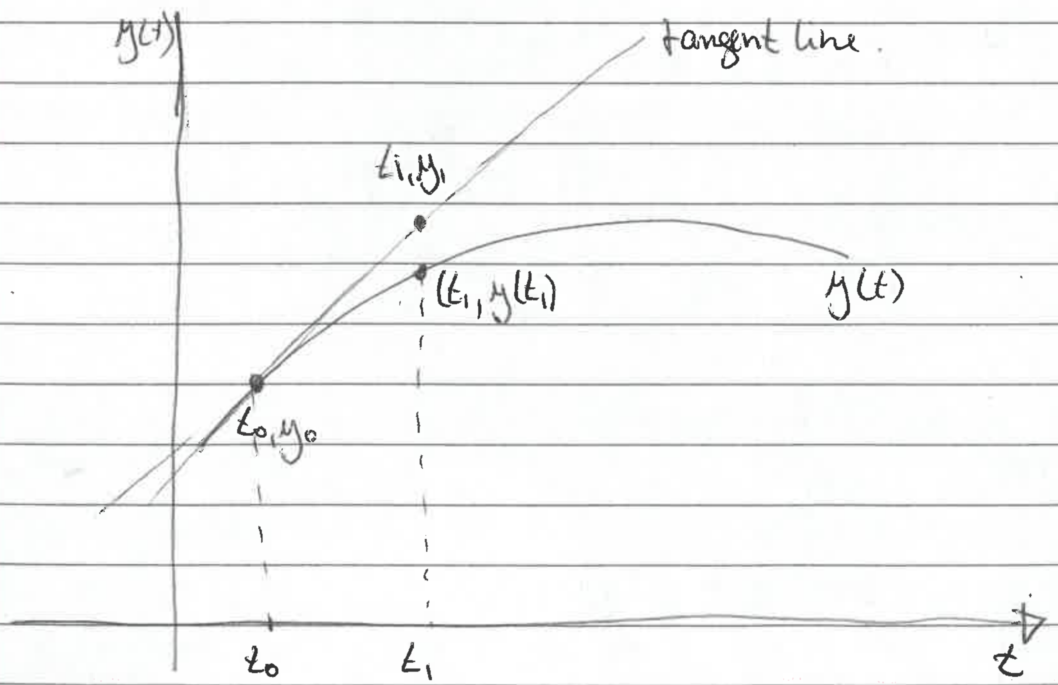
difference : $\frac{1}{2}h^2 y''(t_0) + O(h^3)$ for each step.

$\Rightarrow$ error $= \frac{1}{2}h^2 y''(\varepsilon)$ for $t_0 \leq \varepsilon \leq t_0+h$

Global error over $N$ steps :

$$N \cdot \frac{1}{2}h^2 y''(\varepsilon) \qquad N = \frac{t_N - t_0}{h}$$

$$= (t_N - t_0)\frac{h}{2} y''(\varepsilon) \Rightarrow O(h)$$



W5L2

84 Development of higher order methods via Taylor.

Euler : $y_{n+1} = y_n + h y'(t_n) + O(h^2)$

$= y_n + h f(t_n, y_n) + O(h^2)$

Euler is Taylor method of order 1.

We can find update rules for Taylor's method for $O(h^3)$, $O(h^4)$, etc

Taylor's method is not used in practice.

But it is the key to deriving the Runge-Kutta family of ODE solvers.

Finding $\frac{\partial f}{\partial t}$ $\frac{\partial f}{\partial y}$ and all their counterparts at 4th order is tedious (both numerically and analytically).

Midpoint rule : Runge-Kutta of order 2.

$$y(t_0+h) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + O(h^3)$$

Now use that we know $y'(t_0) = f(t_0, y(t_0))$

But also need $y''(t_0,$ [illegible]$)$

$$y''(t) = \frac{d^2 y}{dt^2} = \frac{df(t, y(t))}{dt} = \text{[illegible]}$$

W5 L2

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$$y''(t) = \frac{d^2y}{dt^2} = \frac{df(t,y(t))}{dt}$$

$$= \frac{\partial f(t,y(t))}{\partial t} \cdot \frac{dt}{dt} + \frac{\partial f()}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \underbrace{\frac{dy}{dt}}^{= y' = f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot f$$

$$y''(t) = f_t(t,y(t)) + f_y(t,y(t)) \cdot f(t,y(t))$$

$$\Rightarrow y(t_0+h) = y(t_0) + h f(t_0, y(t_0)) + \frac{h^2}{2}\Big(f_t(t_0,y(t_0)) + f_y(t_0,y(t_0)) \cdot f(t_0,y(t_0))\Big)$$

How do we get $f_t$ and $f_y$

First change notation: $y(t_0+h) = y_{n+1}$, $y(t_0) = y_n$
$t_0 \to t_n$

$$\Rightarrow y_{n+1} = y_n + h f(t_n,y_n) + \frac{h^2}{2}\big(f_t(t_n,y_n) + f_y(t_n,y_n) \cdot f(t_n,y_n)\big)$$

$$= y_n + h\Big[\underline{f(t_n,y_n) + \frac{h}{2} f_t(t_n,y_n) + \frac{h}{2} f_y(t_n,y_n) \cdot f(t_n,y_n)}\Big]$$

can express $f_t$ and $f_y$ differently by using Taylor in 2D.

$$f(t+a, y+b) \approx f(t,y) + a f_t(t,y) + b f_y(t,y)$$

so use expression $cf(t+a, y+b)$ to express right hand side.

$$cf(t+a, y+b) \approx cf(t,y) + ca f_t(t,y) + cb f_y(t,y)$$
$$= f(t,y) + \frac{h}{2} f_t(t,y) + \frac{h}{2} f_y(t,y) \cdot f(t,y)$$

$\Rightarrow c = 1 \quad ca = \frac{h}{2} \quad cb = \frac{h}{2} f(t,y)$

$\Rightarrow a = \frac{h}{2}, \quad b = \frac{h}{2} f(t,y)$

$\Rightarrow cf(t+a, y+b) = f(t + \frac{h}{2}, y + \frac{h}{2} f(t,y))$

$$y_{n+1} = y_n + h f(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)) + O(h^3)$$

This is a Runge-Kutta method of second order called the midpoint rule.

To get $y_{n+1}$ we need to do

$k_1 = h f(t_n, y_n)$ local error $O(h^3)$

$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{1}{2} k_1)$ global $O(h^2)$ hence second order.

$y_{n+1} = y_n + hk_2$

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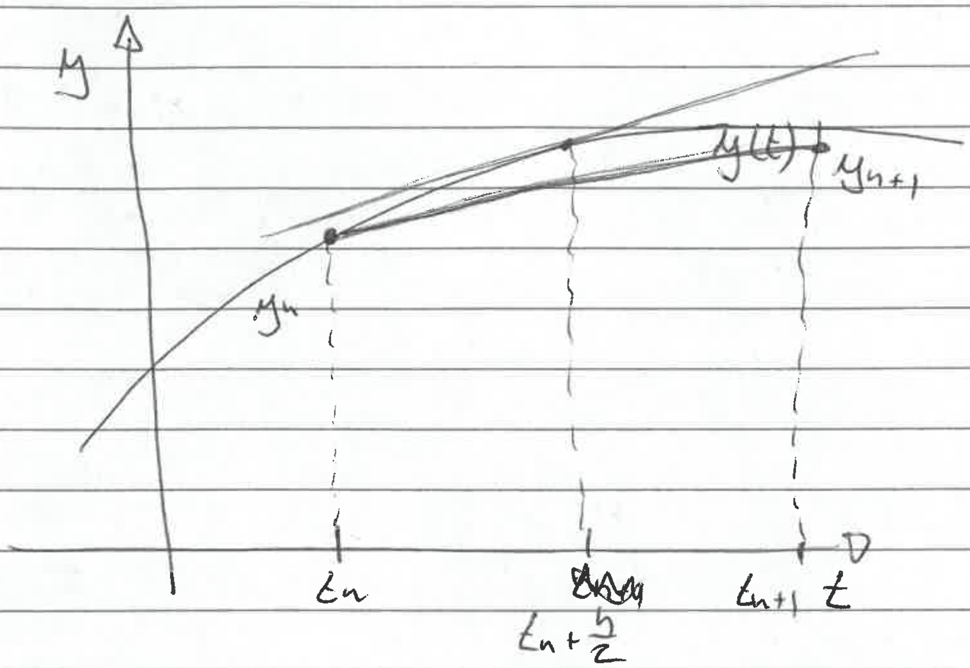
Euler uses the slope at $t_n$ to approximate $y(t_{n+1})$.




Midpoint rule uses Euler to estimate $y(t)$ at midpoint and then uses that to get the tangent at midpoint which is then used to get $y_{n+1}$.

$$y_{n+1} = y_n + h \underbrace{f\left(t_n + \frac{h}{2}, \underbrace{y_n + \frac{h}{2} f(t_n, y_n)}_{\text{Euler estimate for } y \text{ at midpoint}}\right)}_{\text{slope at midpoint}}$$