Composite Simpson's rule:

1. Divide the interval into sub-intervals.
2. Apply Simpson's rule for sub-intervals.
3. Sum up all sub-areas. This sum is the approximate integral.

Recall: in Simpson's rule we used midpoints of interval that used in Trapezoid rule.

So, we have for entire interval

$$a = x_0 < x_1 < x_2 \cdots < x_{2m} = b$$

For every interval:

$$\int_{x_{2i}}^{x_{2i+2}} f(x)\,dx \simeq \frac{h}{3}\left[f(x_{2i}) + 4 f(x_{2i+1}) + f(x_{2i+2})\right] - \frac{h^5}{90} f^{(4)}(c_i)$$

where $x_{2i} \leq c_i \leq x_{2i+2}$ $\quad h = x_{i+1} - x_i$

Composite Simpson's rule:

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left[y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}\right] - \underbrace{\frac{h^5}{90} \sum_{i=0}^{m-1} f^{(4)}(c_i)}_{\frac{h^5}{90} m f^{(4)}(c)}$$

$a \leq c \leq b$

Since $m2h = (b-a) \longrightarrow -\dfrac{(b-a)h^4}{180} f^{(4)}(c)$

Example: Calculate four-panel (four-interval) approximation for the following integral using ~~T/S~~ Composite Trapezoid and simpson's rule.

(2)

$$\int_1^2 \ln(x)\,dx$$

(1) Trapezoid rule:

Four intervals $\longrightarrow h = 1/4$

i	x_i
0	1
1	5/4
2	6/4
3	7/4
4	2

$$\int_1^2 \ln(x)\,dx \simeq \frac{1/4}{2}\left[f(x_0) + f(x_4) + 2\sum_{i=1}^{3} f(x_i)\right]$$

$$\simeq \frac{1}{8}\left(\ln 1 + \ln 2 + 2\left[\ln(5/4) + \ln(6/4) + \ln(7/4)\right]\right) \simeq 0.3837$$

$$\text{Error} = \frac{(b-a)h^2}{12}\left|f''(c)\right| = \frac{1/16}{12}\left(\frac{1}{c^2}\right) \leqslant \frac{1}{(16)(12)(1^2)} = 0.0052$$

$1 \leqslant c \leqslant 2$

For $c=1$ error will be max

(2) Simpson's rule:

$h = 1/8$

i	x_i
0	1
1	9/8
2	5/4
3	11/8
4	6/4
5	13/8
6	7/4
7	15/8
8	2

$$\int_1^2 \ln(x)\,dx \simeq \frac{1/8}{3}\left[f(x_0) + f(x_8) + 4\sum_{i=1}^{4} y_{2i-1} + 2\sum_{i=1}^{3} y_{2i}\right] =$$

$$\simeq \frac{1}{24}\left[\ln(1) + \ln(2) + 4\left(\ln\frac{9}{8} + \ln\frac{11}{8} + \ln\frac{13}{8} + \ln\frac{15}{8}\right) + 2\left(\ln\frac{5}{4} + \ln\frac{6}{4} + \ln\frac{7}{4}\right)\right] \simeq 0.386292$$

$$\text{Error} = \frac{(b-a)h^4}{180}\left|f^{(4)}(c)\right| = \frac{(1/8)^4}{180}\,\frac{6}{c^4} \leqslant 0.000008$$

$1 \leqslant c \leqslant 2$

for $c=1$

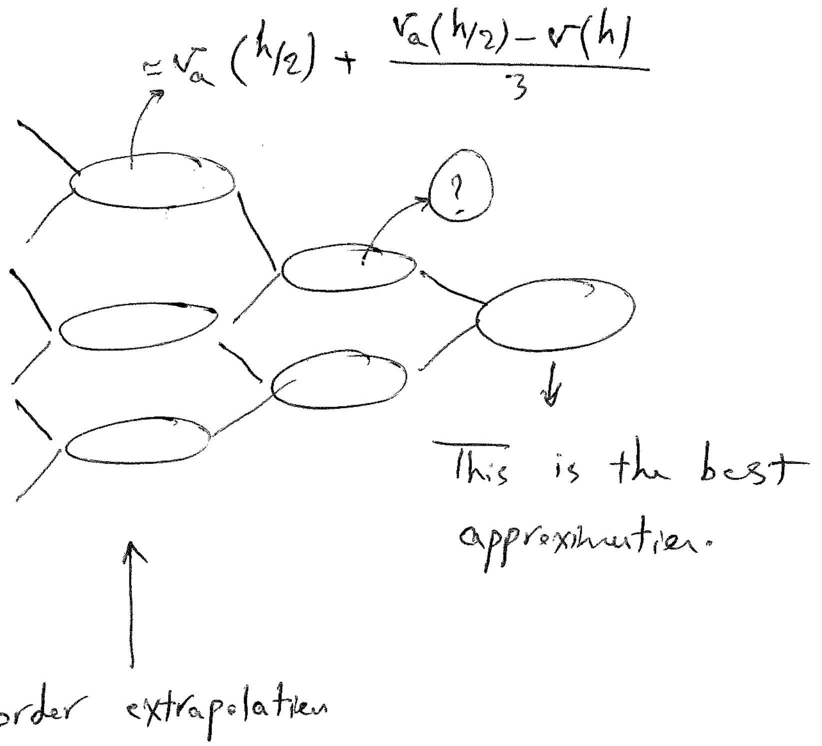
Error is max

Exact value: $\int_1^2 \ln(x)\,dx = x\ln(x)\Big|_1^2 - \int_1^2 dx = 2\ln 2 - \ln 1 - 1 = 0.386294$

Romberg Integration:

Idea: approximate the integration over step sizes h, $\frac{h}{2}$, $\frac{h}{4}$, ..., then extrapolate to obtain better approximations again and again.

	step size	
n intervals	; h	$\longrightarrow V_a(h)$
2n	; $h/2$	$\longrightarrow V_a(h/2)$
4n	; $h/4$	$\longrightarrow V_a(h/4)$
8n	; $h/8$	$\longrightarrow V_a(h/8)$



~~Calculating 4th order extrapolation:~~

For integral $\int_a^b f(x)\,dx$.

$h_1 = b - a$

$h_2 = (b-a)/2$

$\vdots$

$h_j = (b-a)/2^{j-1}$

$$\begin{array}{cccc} R_{11} & & & \\ & R_{22} & & \\ R_{21} & & R_{33} & \\ & R_{32} & & R_{44} \\ R_{31} & & R_{43} & \\ & R_{42} & & \\ R_{41} & & & \\ \vdots & & & \end{array}$$

Trapezoid rule is used:

since $h_2 = h_1/2$

$$R_{11} \simeq \frac{h_1}{2}\left[f(a) + f(b)\right]$$

$$R_{21} \simeq \frac{h_2}{2}\left[f(a) + f(b) + 2 f\left(\frac{a+b}{2}\right)\right] = \frac{1}{2}R_{11} + h_2 f\left(\frac{a+b}{2}\right)$$

$\vdots$

$$R_{j1} \simeq \frac{1}{2}R_{j-1,1} + h_j \sum_{i=1}^{2^{(j-2)}} f\left(a + (2i-1)h_j\right) \qquad (*)$$

Equation (*) is a model to Calculate Composite Trapezoid rule incrementally by decreasing step size h. 4

Now, we aim to calculate ~~approxim~~ extrapolations:

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right) + c_2 h^2 + c_4 h^4 + \dots$$

Error order: $O(h^2)$

Composite Trapezoid rules: (first column) have error order $O(h^2)$

So, first order extrapolation (second column) R_{j2} are 4^{th} order in h. ($\propto c_4 h^4$) and so on.

$$R_{22} \simeq R_{21} + \frac{R_{21} - R_{11}}{3} \qquad R_{32} \simeq R_{31} + \frac{R_{31} - R_{21}}{3} \qquad R_{42} \simeq R_{41} + \frac{R_{41} - R_{31}}{3}$$

The error in second column is 4^{th} order in h, we can extrapolate third column:

$$R_{33} \simeq R_{32} + \frac{R_{32} - R_{22}}{4^2 - 1} \quad , \quad R_{43} = \dots$$

Recall: Extrapolation with error $O(h^n)$:

$$\begin{cases} V_t \simeq V_a(h) + K h^n \\ V_t \simeq V_a(h/4) + K\left(\frac{h}{4}\right)^n \end{cases} \longrightarrow \quad \begin{aligned} V_t &\simeq V_a(h) + K h^n \\ 4^n V_t &\simeq 4^n V_a\left(\tfrac{h}{4}\right) + K h^n \quad \ominus \end{aligned}$$

$$V_t(1-4^n) \simeq V_a(h) - 4^n V_a(h/4)$$

$$V_t \simeq V_a\left(\frac{h}{4}\right) + \frac{V(h/4) - V_a(h)}{4^n - 1}$$