General Solution for Eq. (*) ; $x_j = \alpha r_1^j + \beta r_2^j$

BCs

① $x_0 = 0 \longrightarrow \alpha + \beta = 0 \longrightarrow \boxed{\beta = -\alpha}$

$$x_j = \alpha (r_1^j - r_2^j)$$

② $x_{M+1} = 0 \longrightarrow x_{M+1} = \alpha \left(r_1^{M+1} + r_2^{M+1} \right) = 0$

1) If $\alpha = 0 \rightarrow$ ~~x_{M+1}~~ $x_j = 0$ (not useful)

2) If $\left(r_1^{M+1} + r_2^{M+1} \right) = 0 \longrightarrow$ we will find e-values.

$$\left(\frac{r_1}{r_2}\right)^{M+1} - 1 = 0 \rightarrow \left(r_1^2\right)^{M+1} - 1 = 0$$

$$* \quad r_1^{2M+2} - 1 = 0$$

One obvious solution is $r_1 = 1$. Because we know $r_1 r_2 = 1 \longrightarrow r_2 = 1$ and $x_j = \alpha(1^j - 1^j) = 0$ (Not useful).

So, we look at complex roots.

we set: $1 = e^{2\pi i K}$ K: integer

$$e^{2\pi i K} = r_1^{2(M+1)} \longrightarrow r_1 = e^{\left(\frac{2\pi i K}{2(M+1)}\right)} =$$

$$\boxed{r_1 = e^{\frac{\pi i K}{M+1}}}$$

$$\boxed{r_2 = e^{-\frac{\pi i K}{M+1}}}$$

$$x_j = \alpha \left(e^{\frac{\pi i K j}{M+1}} - e^{-\frac{\pi i K j}{M+1}} \right) \times \frac{2i}{2i}$$

where $K = 1, \dots, M$ $\quad x_j = (2i\alpha) \sin\left(\frac{\pi K j}{M+1}\right) = (\alpha^*) \sin\left(\frac{\pi K j}{M+1}\right)$

Recall: $Ax = \lambda x \qquad x \equiv u \equiv \text{heat}$

Recall: $(a-\lambda) = -b(r_1+r_2)$

$$\lambda = a + b\left(e^{\frac{\pi i k}{M+1}} + e^{-\frac{\pi i k}{M+1}}\right)$$

$$\lambda = a + 2b\cos\frac{\pi K}{M+1}, \quad K = 1, \dots, M \qquad (1)$$

How do we use e_value formula (1)?

Recall: Forward difference method applied for heat equation is stable

if $\frac{D\,\Delta t}{(\Delta x)^2} \leq 1/2$

D: diffusion constant

Δt: time step

Δx: space step.

This condition is a requirement to have $|\lambda_i| < 1$.

Forward Method:

$$u^{(j+1)} = A\,u^{(j)} + s^{(j)}$$

$\vdots$

$$j = N \qquad u^{(N)} = A^N \underline{u}^{(0)} + A^{N-1} s^{(0)} + \cdots$$

Matrix A^N is bounded if $|\lambda|^N < 1$.

$$A = \begin{bmatrix} 1-2\delta & \delta & 0 & \\ \delta & 1-2\delta & \delta & 0 \\ & & \ddots & \\ 0 & \cdots & \delta & 1-2\delta \end{bmatrix} \longrightarrow a = 1-2\delta$$

$b = \delta$

$\downarrow$

$$\lambda = (1-2\delta) + 2\delta\cos\frac{\pi K}{M+1}$$

$K = 1, \dots, M$

$K \neq 0$

$K \neq M+1$

$$\lambda = 1 + 2\delta\left(\cos\frac{\pi K}{M+1} - 1\right)$$

$$\lambda = 1 + 2\delta\left(\cos\frac{\pi k}{M+1} - 1\right)$$

$$k = 1, \dots, M \longrightarrow -1 < \cos\frac{\pi k}{M+1} < 1 \longrightarrow -2 < \left(\cos\frac{\pi k}{M+1} - 1\right) < 0$$

The lowest e-value must be: $\lambda_{min} > 1 - 4\delta$

The highest e-value must be: $\lambda_{max} < 1$

For stability we want $|\lambda| < 1$ means that: $1 - 4\delta > -1 \rightarrow$

$$4\delta > 2 \rightarrow \delta < 1/2 \longrightarrow \frac{D\Delta t}{(\Delta x)^2} < 1/2$$

If $\delta = 1/2$: $\lambda = 1 + \left(\cos\frac{\pi k}{M+1} - 1\right) = \cos\frac{\pi k}{M+1} < 1 \rightarrow \lambda = \frac{1}{2}$ is ok.

$$\delta = \frac{\Delta t\, D}{(\Delta x)^2} \leq \frac{1}{2}$$

How about stability of backward method for heat equation?

Recall: $B u^{(j)} = u^{(j-1)}$

$\vdots$

$$j = N \qquad u^{(N)} = (B^{-1})^N \underline{u^{(0)}}$$

Matrix $(B^{-1})^N$ is bounded: if $|\text{e-values of } B^{-1}| < 1$.
$N \to \infty$

$$B = \begin{bmatrix} 1+2\delta & -\delta & 0 & & \\ -\delta & 1+2\delta & -\delta & 0 & \\ 0 & & \ddots & & \\ 0 & - & -\delta & 1+2\delta \end{bmatrix}$$

$a = 1 + 2\delta$

$b = -\delta$

Note: E-values of C^{-1} are the inverse of e-values of C.

$$C\underline{x} = \lambda \underline{x}$$

$$C^{-1}(C\underline{x} = \lambda\underline{x}) \longrightarrow I\underline{x} = \lambda C^{-1}\underline{x} \rightarrow C^{-1}x = \frac{1}{\lambda}x$$

E-values of B: $\lambda = a + 2b\cos\frac{\pi K}{M+1}$ $K = 1, \dots, M$

$$\lambda = 1 + 2\delta\left(1 - \cos\frac{\pi K}{M+1}\right)$$

Since $\cos\frac{\pi K}{M+1} < 1$, then e-value of matrix B: $|\lambda_B| > 1$.

Thus, e-values of B^{-1} are < 1. $\rightarrow |\lambda_{B^{-1}}| < 1$.

↳ Conclusion: Implicit backward method for heat equation is unconditionally stable.

Wave Equation:
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (2)$$

This PDE models the propagation of wave along a string.

$u(x,t)$: solution ~ describes how a wave propagates in space x and time t.

C: Wave speed.

BCs: Initial ~~sp~~ shape and speed of wave.

Standard grid: For space: M grid intervals: $x_i = x_0 + i\Delta x$

$i = 0, \dots, M$

$$\Delta x = \frac{x_M - x_0}{M}$$

For time: N grid intervals: $t_j = j\Delta t$

$$\Delta t = \frac{t_N}{N} \qquad t_0 = 0 \qquad j = 0, \dots, N$$

Exact solution   Approximate solution

$$u(x_i, t_j) \sim u_{i,j}$$

Replace continuous derivatives with discrete approximations:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2 \Delta t^2}{(\Delta x)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} + u_{i,j-1} = \delta^2 u_{i+1,j} + u_{i,j}(2-2\delta) + \delta^2 u_{i-1,j}.$$

For implementation in Matlab see book: Sauer.

The wave Equation (2) is stable if $\delta = \frac{c\Delta t}{\Delta x} \leq 1.$ *

This Condition is called CFL → Lewy

Courant   Friedrichs

$\delta = \frac{c\Delta t}{\Delta x} \leq 1$ ~~means~~ Condition means that the distance travelled by the wave $(c \cdot \Delta t)$ should not be greater than space step Δx.

Example: