W6 L1

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General derivation of ~~Run~~.
Runge-Kutta methods

Use fundamental theorem of calculus:

$$y(t+h) - y(t) = h\int_0^1 y'(t+\theta h)\,d\theta$$

$$\Rightarrow y(t+h) = y(t) + h\underbrace{\int_0^1 y'(t+\theta h)\,d\theta}_{\text{use quadrature to approximate the integral.}} \qquad y' = f(t,y)$$

As we know $f(t,y)$ we can use values at several points to approximate $f(t,y)$

This is using nodes and weights.

$$\int_0^1 y'(t+\theta h)\,d\theta \approx \sum_{i=1}^{m} \underset{\text{weights}}{\gamma_i}\, y'(\underbrace{t+\alpha_i h}_{\text{nodes}})$$

$$\Rightarrow y(t+h) = y(t) + h\sum_{i=1}^{m} \gamma_i\, y'(t+\alpha_i h)$$

$$= y(t) + h\sum_{i=1}^{m} \gamma_i\, f(t+\alpha_i h, \underbrace{y(t+\alpha_i h)}_{\text{unknown}})$$

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(9.4)

$$y(t+\alpha_i h) = y(t) + h\int_0^{\alpha_i} y'(t+\theta h)\,d\theta$$

$i=1,\dots,m$

use quadrature again, but use same α_i.

$$\int_0^{\alpha_i} y'(t+\theta h)\,d\theta \approx \sum_{j=1}^{m}{}' \beta_{ij}\, y'(t+\alpha_j h)$$

Here $\alpha_i = \sum_{j=1}^{m}{}' \beta_{ij} \qquad i=1,\dots,m$

Now denote $y'(t+\alpha_i h) = k_i$

$i=1,\dots,m$

we get $y(t+h) = y(t) + h\sum_{i=1}^{m}{}' \gamma_i k_i$

where $k_i = f(t+\alpha_i h,\ y(t) + h\sum_{j=1}^{m}{}' \beta_{ij} k_j)$

$i=1,\dots,m$

$$\Rightarrow \quad y_{n+1} = y_n + h\sum_{i=1}^{m}{}' \gamma_i k_i$$

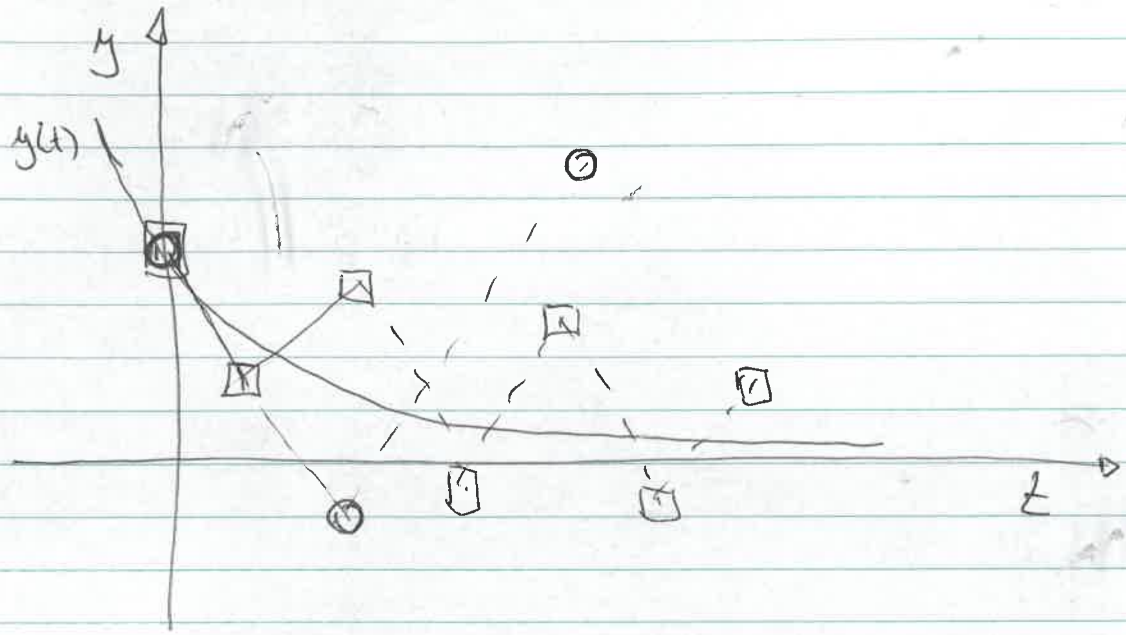
$$k_i = f(t+\alpha_i h,\ y_n + h\sum_{j=1}^{m}{}' \beta_{ij} k_j)$$

$i=1,\dots,m$

Stiff problems – implicit methods

$y'(t) = -15\,y(t) \qquad t \geq 0,\ y(0) = 1$

exact solution $y(t) = e^{-15t}$ with $\displaystyle y(t) \underset{t \to 0}{\longrightarrow} 0$



○ Euler with $h = \frac{1}{4}$ diverges.

□ Euler with $h = \frac{1}{8}$ oscillates.

So far $y_{n+1} = y_n + h f(t_n, y_n; h)$ explicit

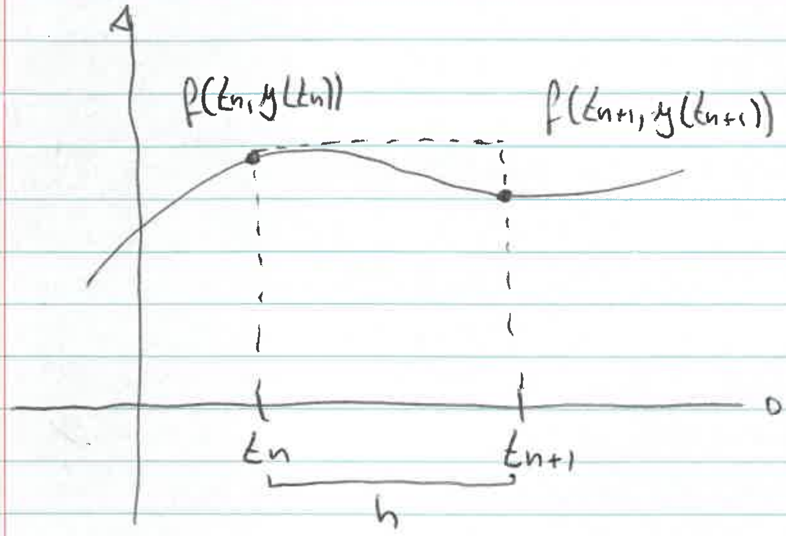
$y = x^2 + 1$ or $y = x^3 + 3x - 2$ $\quad y = f(x_1, x_2, \dots, x_n)$

$2x^3 y + y^3 = x + 3$ implicit

Use fundamental theorem of calculus:

$$\int_{t_n}^{t_{n+1}} f(t, y)\, dt = y(t_{n+1}) - y(t_n)$$

To get update rule for $y(t_{n+1})$ we can approximate the integral by a rectangle.

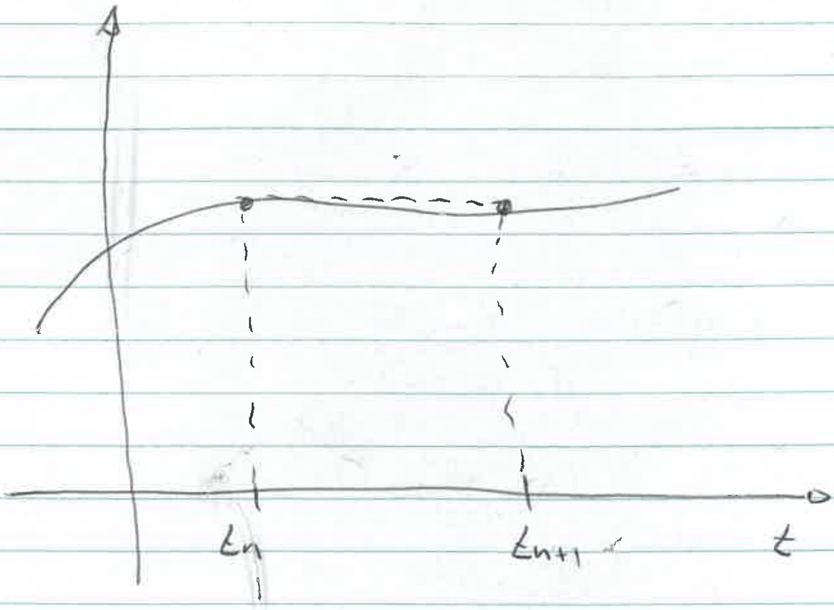


Can use height at left side: $f(t_n, y(t_n))$

$$\Rightarrow y(t_{n+1}) - y(t_n) \approx h f(t_n, y(t_n))$$

$$\Rightarrow y(t_{n+1}) \approx y(t_n) + h f(t_n, y(t_n)) \quad \text{explicit}$$

which is Euler's method. It is sometimes called forward Euler due to this.



Can also use the height at the right side, so at t_{n+1}. Then we get

$$y(t_{n+1}) - y(t_n) \approx h f(t_{n+1}, y(t_{n+1}))$$

$$\Rightarrow y(t_{n+1}) = y(t_n) + h f(t_{n+1}, y(t_{n+1})) \to O(h^2)$$

$$\Rightarrow y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \text{ implicit}$$

This is the implicit Euler method, which also called the backward Euler due to this. This leads to a system of equations to solve, so need something like Newton or similar to do this. This means implicit method require more effort but are essential for some problems.

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$$y' = \cancel{t}\, y^3$$

$$y_{n+1} = y_n + h\, \cancel{t}_{n+1} \cdot y_{n+1}^3$$

$$h t_{n+1} y_{n+1}^3 + y_n - y_{n+1} = 0$$

So we need to solve this cubic equation for y_{n+1} at each step.

Example of stiff ODE:

$y' = -\alpha y \quad \alpha > 0, \; y(0) = y_0$

$\Rightarrow \quad y = y_0 e^{-\alpha t}$