we have $(n-1)+(n-2)+(n-3)+2 = 3n-3$ equations

with $\underbrace{3n-3}_{3(n-1)}$ unknowns.

$b_i, c_i, d_i \longrightarrow n-1$ segments

we set $\delta_i = x_{i+1} - x_i$

$$\Delta_i = y_{i+1} - y_i$$

Equations (3) can be solved for d_i : $d_i = \dfrac{c_{i+1} - c_i}{3\delta_i}$ (5) $i=1,\dots,n-1$

Equations (1) can be solved for b_i :

$$b_i = \frac{\Delta_i}{\delta_i} - c_i\delta_i - d_i\delta_i^2$$

$$= \frac{\Delta_i}{\delta_i} - c_i\delta_i - \frac{\delta_i}{3}(c_{i+1} - c_i)$$

$$= \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}(2c_i + c_{i+1}) \qquad (6)$$

Substituting (5) and (6) into (2) results in $n-2$ equation in $c_2, \dots, c_{n-1}$;

$$\delta_1 c_1 + 2(\delta_1 + \delta_2)c_2 + \delta_2 c_3 = 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right)$$

$$\vdots$$

$$\delta_{n-2}c_{n-2} + 2(\delta_{n-2} + \delta_{n-1})c_{n-1} + \delta_{n-1}c_n = 3\left(\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right)$$

Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & & \\ \delta_1 & 2(\delta_1+\delta_2) & \delta_2 & & \\ 0 & \delta_2 & 2(\delta_2+\delta_3) & \delta_3 & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ \vdots \\ 3\left(\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right) \\ 0 \end{bmatrix}$$

Solving this matrix form (Gauss elimination or LU decomposition), you will get c_i, b_i, d_i (from (5) and (6)).

Then, you have segments $S_i(x)$ for cubic spline.

Example:

Find natural cubic spline that interpolates three data points: $(0, 3)$, $(1, -2)$, and $(2, 1)$.

$\delta_1 = \delta_2 = 1$, $\Delta_1 = -5$, $\Delta_2 = +3$

Matrix: $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3(3+5) = 24 \\ 0 \end{bmatrix}$$

Solution: $$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$ Natural spline: $c_1 = c_3 = 0$

$$d_1 = \frac{c_2 - c_1}{3\delta_1} = 2, \quad d_2 = \frac{c_3 - c_2}{3\delta_2} = -2$$

$$b_1 = \frac{\Delta_1}{\delta_1} - \frac{\delta_1}{3}(2\cancelto{0}{c_1} + c_2) = -5 - \frac{1}{3}(6) = -7$$

$$b_2 = \frac{\Delta_2}{\delta_2} - \frac{\delta_2}{3}(2c_2 + \cancelto{0}{c_3}) = 3 - \frac{1}{3}(12) = -1$$

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$ (3)

$i = 1, 2$

$$S_1(x) = 3 - 7(x) + 0 \cdot x^2 + 2x^3 \quad \text{on } [0, 1]$$

$$S_2(x) = -2 - 1(x-1) + 6(x-1)^2 - 2(x-1)^3 \quad \text{on } [1, 2]$$

Question: Assume n data points (x_i, y_i) from a function $f(x)$ are given. You are asked to estimate the function value at x_k ~~where~~ $(\min\{x_i\} < x_k < \max\{x_i\})$.

Which $S_i(x)$ will you compute?

Example:

Given the velocity of a rocket over time in following table, find the velocity at time $t = 16$ s, using cubic spline interpolation.

t (s)	v (m/s)
0	0
10	227
15	367
20	517
22.5	602
30	901

16

→ segment ~~...~~ $v(t)$

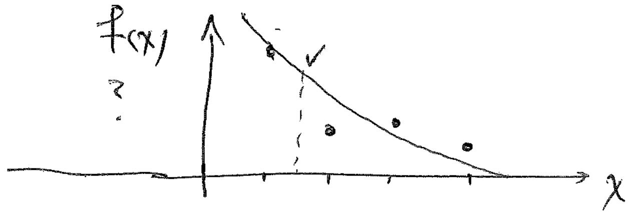
Will solve this on Friday workshop.

Extrapolation:

For a given n data points (x_i, y_i), where $i = 1, \dots, n$, produced by a function $f(x)$:

Interpolation: estimates the value of $f(x)$ within the range of x_i.

Extrapolation: estimates the " " beyond the " ".



Richardson extrapolation method:

Idea is given a sequence of approximations, it combines the approximations to obtain much better estimation.

Richardson extrapolation for differentiation:

Improves the convergence rate of a sequence of approximations for differentiation with central formula.

$$f'(x) \sim \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Error: $E_t = O(h^2) \longrightarrow E_t = c_2 h^2 + c_3 h^3 + \cdots$

$$E_t \propto h^2$$

$E_t \simeq K h^2$, where K is constant over h.

Recall: $\text{Error}_t = (\text{True value} - \text{approximate value})$

$$E_t = V_t - V_a \longrightarrow V_t = V_a + E_t \longrightarrow \underbrace{V_t}_{\text{unknown}} \simeq \underbrace{V_a(h)}_{\text{known}} + \underbrace{Kh^2}_{\text{unknown}}$$

$V_a(h)$: approximate value with step size h.

$$\left.\begin{aligned} V_t &\simeq V_a(h) + Kh^2 \\ V_t &\simeq V_a\left(\tfrac{h}{2}\right) + K\left(\tfrac{h}{2}\right)^2 \end{aligned}\right\} \text{ finding } V_t$$

$$\begin{aligned} V_t &\simeq V_a(h) + Kh^2 \\ 4V_t &\simeq 4V_a\left(\tfrac{h}{2}\right) + Kh^2 \quad - \\ \hline -3V_t &\simeq V_a(h) - 4V_a(h/2) \end{aligned}$$

$$\longrightarrow V_t \simeq V_a\left(\frac{h}{2}\right) + \frac{V_a\left(\frac{h}{2}\right) - V_a(h)}{3}$$

Richardson extrapolation for order n approximation $O(h^n)$

True value : $V_t \simeq V_a(h) + Kh^n$

$$\left.\begin{aligned} V_t &\simeq V_a(h) + Kh^n \\ V_t &\simeq V_a\left(\tfrac{h}{2}\right) + K\left(\tfrac{h}{2}\right)^n \end{aligned}\right\} \longrightarrow \begin{aligned} V_t &\simeq V_a(h) + Kh^n \\ -\quad 2^n V_t &\simeq 2^n V_a\left(\tfrac{h}{2}\right) + Kh^n \\ \hline V_t(1-2^n) &\simeq V_a(h) - 2^n V_a\left(\tfrac{h}{2}\right) \end{aligned}$$

Extrapolated approximation : $V_t \simeq V_a\left(\frac{h}{2}\right) + \dfrac{V_a(h/2) - V_a(h)}{2^n - 1}$

Use this for integration.