lookat 3 topics:

(1) Boundary value problems (BVPs).

To find solution for a differential equation when you know the solution values at the boundaries.

- for ODEs, partial differential equations (PDEs)
- ~~~~ use: { Shooting method.
  Finite difference methods.

(2) Interpolation:

To find a function that passes through given data points (fits to data).

- Use polynomials and splines.

(3) Integration:

- Newton's method, Adaptive quadrature using polynomials.

Boundary Value Problems (B.V.P.s):

In the first half of the course (IVPs):

Looked at methods for calculating the solution to an initial value problem (IVP). Recall: IVPs are a differential equation together with initial values for the solution function, specified at the left end of the solution interval (e.g. at time zero or initial position of a moving object). The approximate solution began at the left end (start of boundary) and progressed forward in the independent variable time t.

In the second half of the course (BVPs):

We look at an equally important set of problems which arises when a differential equation is presented along with boundary data, specified at both start and end of the solution interval. To approximate solutions for a boundary value problem (BVP), we look at two methods:

1) Shooting method: a combination of the IVP solvers and equation solvers from Chapter 1.
2) Finite difference methods: convert the differential equation and boundary conditions into a system of linear or nonlinear equations to be solved.
3) Finite Element Method (will not be covered in the course).

Description of BVPs:

Recall IVPs:

IVPs are differential equations where we supply initial conditions (initial value and initial slope):

$$\begin{cases} \frac{dx}{dt} = f(x,t) \\ x(t_0) = x_0 \end{cases} \qquad (*)$$

Example:

The following is an IVP coupled to initial conditions:

$$\begin{cases} \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = \sin(t) \\ y(0) = 1, y'(0) = 2 \end{cases}$$

We have seen how to express this second-order ODE in terms of an expression in like (*). This is what we are familiar with from the first half of the course, and we know a number of methods to estimate the solution.

Note (use of initial conditions):

Some ODEs (e.g. logistic $y' = cy(1-y)$) can have infinitely many solutions. By specifying **an** initial condition, we can identify which of the infinite family we are interested in (i.e. we pin down a desired solution).

Boundary value problems (BVPs):

PVPs are differential equations where we supply start and end boundary conditions:

$$\begin{cases} \frac{d^2y}{dx^2} = f(x, y(x), \frac{dy(x)}{dx}) \\ y(a) = y_a, y(b) = y_b \end{cases} \quad (1)$$

Boundary values are:

$$y(a) = y_a$$
$$y(b) = y_b$$

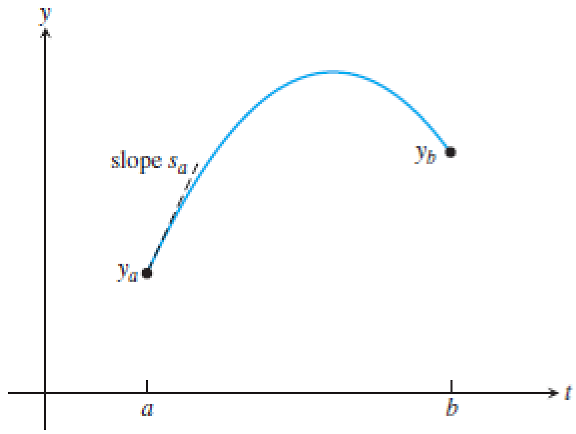

Figure 7.1 Comparison of IVP and BVP. In an initial value problem, the initial value $y_a = y(a)$ and initial slope $s_a = y'(a)$ are specified as part of the problem. In a boundary value problem, boundary values y_a and y_b are specified instead; s_a is unknown.

- Instead of initial values, we supply boundary values [e.g. y(0) = 1 and y(2) = 4].

- Equation (1) is second order, and thus two extra constraints are needed to pin down a desired solution. They are given as boundary conditions for solution $y(x)$ at a and b.

- Basically, we say that we have domain boundaries a and b. I look for a solution $y(x)$ that satisfies the equation between the points and is equal to y_a and y_b at the boundaries a and b, respectively; see Figure 7.1.

Example 1:

Find the trajectory function for an object that is thrown from the top of a 30-meter tall building and reaches the ground 4 seconds later.

Let $y(t)$ be the height at time t. We know that the gravity force is $F = -mg$ and $g = 9.81\ \text{m/s}^2$. The trajectory (i.e. solution) can be expressed as:

the solution of the IVP:

$$\begin{cases} y'' = -g \\ y(0) = 30\ \text{m} \\ y'(0) = v_0 \end{cases}$$

or the solution for BVP:

$$\begin{cases} y'' = -g \\ y(0) = 30\ \text{m} \\ y(4) = 0 \end{cases}$$

Since we don't know the initial velocity v_0, we must solve the BVP.

Integrating twice gives us the solution trajectory:

$$\text{General solution:} \qquad y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

Now, using the boundary conditions:

$$y(0) = 30 \rightarrow y_0 = 30$$

$$y(4) = 0 \rightarrow \ v_0 \sim 12.12\ \text{m/s}$$

Thus, the solution trajectory is:

$$\text{Specific solution:} \qquad y(t) = -\frac{1}{2}gt^2 + 12.12t + 30$$

Properties of BVPs:

BVPs can have many solutions or no solutions. We will examine them in the following examples.

Example 1: We would solve the following:

$$\begin{cases} y'' - y = 0 \\ y(0) = 0, y(1) = 1 \end{cases} \qquad (**)$$

From MATH1052, we try solution $y = e^{mx}$ (basically this is a trial solution). We get:

$m^2 e^{mx} - e^{mx} = 0 \rightarrow e^{mx}(m^2 - 1) = 0 \rightarrow$ where $e^{mx} \neq 0, m^2 - 1 = 0 \rightarrow m = \pm 1$.

Thus, the general solution is:

$$y(x) = Ae^{x} + Be^{-x}$$

Or, we can write its characteristic equation which has two distinct roots: $r^2 - r^0 = 0 \rightarrow r^2 = 1 \rightarrow r = \pm 1$.

Thus, the general solution is: $y(x) = Ae^{r_1 x} + Be^{r_2 x} = Ae^{x} + Be^{-x}$

Now, we choose constants A and B to satisfy the boundary conditions (BCs):
y(0) = 0 → $A + B = 0$ → B = -A
y(1) = 1 → $Ae + B/e = 1 \rightarrow Ae - A/e = 1 \rightarrow A = \frac{e}{e^2 - 1}$ (UGLY!)

Solution will be much nicer if I use hyperbolic trig functions, $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. This is because both $\sinh(x)$ and $\sinh(x)$ satisfy the differential equation (**):

$$\frac{d^2}{dx^2}\sinh(x) = \sinh(x) \text{ and } \frac{d^2}{dx^2}\cosh(x) = \cosh(x).$$

Now, given that $e^x = \cosh(x) + \sinh(x)$ and $e^{-x} = \cosh(x) - \sinh(x)$, the general solution can be written as:

$$y(x) = C\sinh(x) + D\cosh(x)$$

Applying boundary conditions $y(0) = 0$ and $y(1) = 1$, we get:

$$\mathrm{Y}(0) = 0 \rightarrow D = 0$$

$$Y(1) = 1 \rightarrow C\frac{e^1 - e^{-1}}{2} = 1 \rightarrow C = \frac{2}{e^1 - e^{-1}}$$

Replacing C and D in the general solution, we end-up with:

$$\text{Specific solution:} \quad y(x) = \frac{e^x - e^{-x}}{e^1 - e^{-1}} = \frac{\text{Sinh}(x)}{\text{Sinh}(1)}$$

Meets the constraints: if you put zero for x, then y is equal to zero. If you assign 1 to x, the y is equal to 1.
Conclusion: this example shows that BVPs look like the IVP when you supply two end conditions.

Example 2:
B.V.Ps can have infinite number of solutions.
This is unlike the IVPs where we specify the initial conditions and get one solution to the problem.
Now consider the following example:

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

One solution is: y(x) = 0.
Other solutions are: y(x) = k sin(πx), for any k (including k = 0).
There is no uniqueness of solutions for this example.
This is a Sturm–Liouville problem (studied at MTAH3403 in details). You have to take care numerically.

Example 3:
B.V.Ps can have no solution.
Now consider the following example:

$$\begin{cases} y'' = -y \\ y(0) = 0 \\ y(\pi) = 1 \end{cases}$$

The differential equation has a two-dimensional family of solutions, generated by the linearly independent solutions cos t and sin t. All solutions of the equation must have the form y(t) = a cos t + b sin t . Substituting the first boundary condition, y(0) = 0 implies that a = 0 and y(t) = b sin t . The second boundary condition 1 = y(π) = b sinπ = 0 gives a contradiction. There is no solution, and existence fails.

Shooting method to approximate a solution for a BVP

The Shooting Method solves the BVP in Equation (1) by finding the IVP that has the same solution. A sequence of IVPs is produced, converging to the correct IVP (i.e., with the correct initial slope s*):

$$\text{Correct IVP: } \begin{cases} y'' = f(x, y, y') \\ y(a) = a \\ y'(a) = s^* \end{cases}$$

The sequence begins with an initial guess for the slope s_a at initial boundary, provided to go along with the initial value y_a. The IVP that results from this initial slope is solved and compared with the boundary value y_b. By trial and error, the initial slope s is improved until the boundary value is matched to y_b (i.e. estimation error is within a desired threshold).

Formally, the solution $y(t)$ for BVP in Equation (1) is reduced to solve:

$$F(s) = [y(b) - y_b] = 0 \qquad (***)$$

where $y'(a) = s$ and $y(a)$= y_a.

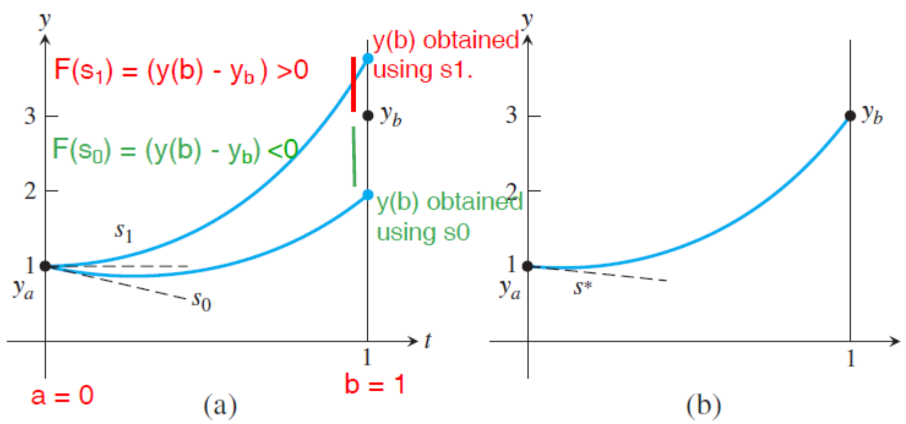

Figure 7.3 The Shooting Method. (a) To solve the BVP, the IVP with initial conditions $y(a) = y_a, y'(a) = s_0$ is solved with initial guess s_0. The value of $F(s_0)$ is $y(b) - y_b$. Then a new s_1 is chosen, and the process is repeated with the goal of solving $F(s) = 0$ for s. (b) The MATLAB command ode45 is used with root s^* to plot the solution of the BVP (7.7).

Shooting method algorithm:

The Bisection Method is then used: two values of s (s_0 and s_1) should be found for which $F(s_0)F(s_1) < 0$. This means that s_0 and s_1 bracket a root for Equation (***). A root s^* can be located within the required tolerance by the chosen equation solver (Tolerance: while (b $-$ a)/2 > TOL continues). Finally, the solution to the BVP in Equation (1) can be traced (by an IVP solver, e.g. Euler method) as the solution to the I.V.P:

Summary of shooting mehtod:
you turn the BVP problem into an IVP: you start with an initial guess for the slope (i.e. you guess $y'(0) = \alpha$, and we choose α such that y(b) = b. You shoot, over shoot, undershoot and converge on the solution where y(b) = y_b.

Shooting method implementation in Matlab:
You will use Matlab's ode45 IVP solver.
You need to write the differential equation as a first-order system in order to use Matlab's ode45 IVP solver.