Chebyshev's idea:

It moves data points towards the end points in a way that the numerator of interpolation error is minimised:

$$E = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c)$$

$$\min\{x_i\} \leqslant c \leqslant \max\{x_i\} \qquad i=1,\dots,n.$$

Chebyshev's theorem:

Given n data points on interval $[a,b]$, the interpolation error is minimised if n data points are moved to Chebyshev's points (roots):

$$x_i = \frac{b-a}{2} \cos\left(\frac{(2i-1)\pi}{2n}\right) + \frac{b+a}{2} \qquad i=1,\dots,n$$

This way (moving data points) the numerator of error is minimised with upper bound,

$$(x-x_1)(x-x_2)\cdots(x-x_n) \leqslant \frac{(b-a)^n}{2^{2n-1}}$$

(see book for proof.)

Example:

Interpolate $f(x) = \sin(x)$ at four equally spaced points on $[0, \frac{\pi}{2}]$.

Find four chebyshev's base points (roots) for the interpolation.

Find an upper bound error for Chebyshev's interpolation.

solution:

x_i	y_i			
0	0			
		0.9549		
$\frac{\pi}{6}$	0.5		-0.2443	
		0.6990		-0.1139
$2\pi/6$	$\sqrt{3}/2$		-0.4232	
		0.2559		
$3\pi/6$	1			

$$P_3(x) = 0 + 0.9549\,x - 0.2443\left(x-\frac{\pi}{6}\right)x - 0.1139\,x\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{3}\right).$$

Chebyshev's roots:

$$V_i = \frac{\frac{\pi}{2}-0}{2}\cos\left(\frac{(2i-1)\pi}{2\times 4}\right) + \frac{\pi/2+0}{2}$$

$$V_1 = \frac{\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{8} \qquad V_2 = \frac{\pi}{4} + \frac{\pi}{4}\cos\frac{3\pi}{8} \qquad V_3 = \frac{\pi}{4} + \frac{\pi}{4}\cos\frac{5\pi}{8}$$

$$V_4 = \frac{\pi}{4} + \frac{\pi}{4}\cos\frac{7\pi}{8}$$

Now, you can find Chebyshev's interpolating polynomial $P_3(V)$, using Newton's divided differences, as above.

Max error for interpolation (chebyshev's method):

$$\left|\sin(x) - P_3(V)\right| = \frac{|(x-V_1)(x-V_2)\cdots(x-V_4)|\,|f^{(4)}(c)|}{4!}$$

$$\leq \frac{\left(\frac{\pi}{2}-0\right)^4}{4!\,2^7}\,1 \simeq 0.00198$$

Spline Interpolation:

In polynomial interpolation: a single polynomial is used to interpolate all data points; (n data points)

In spline interpolation: (n−1) low-degree polynomials are use to interpolate n data points.

Example:

Given four data points (1,2), (2,1), (4,4), (5,3) find the simplest spline that interpolates the points.

simplest : linear spline.

x_i	y_i
1	2
2	1

−1

x_i	y_i
2	1
4	4

3/2

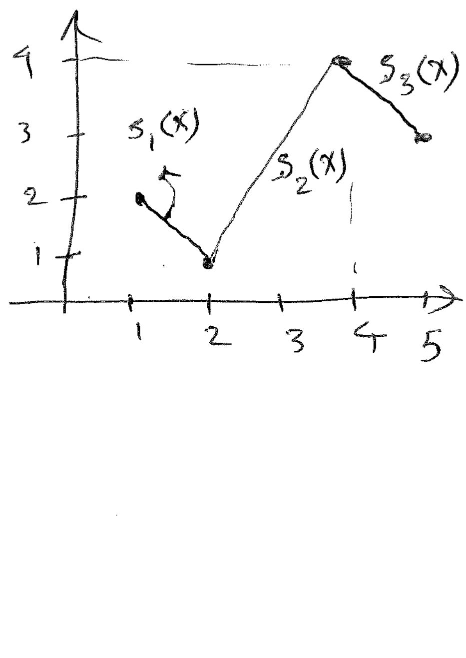
x_i	y_i
4	4
5	3

−1

$S_1(x) = 2 - (x-1)$ on $[1,2]$

$S_2(x) = 1 + 3/2\,(x-2)$ on $[2,4]$

$S_3(x) = 4 - (x-4)$ on $[4,5]$



Spline properties:

For a given n data points (x_i, y_i), where $i = 1, \dots, n$, interpolating spline $S(x)$ is a set of $(n-1)$ polynomials that satisfies:

1. Spline is Continous: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ $\quad i = 1, \dots, n-1$

2. Spline is smooth: $S'_{i+1}(x_i) = S'_i(x_i)$ $\quad i = 2, \dots, n-1$

3. Spline segments have same Curvature where they meet: $S''_{i-1}(x_i) = S''_i(x_i)$ $\quad i = 2, \dots, n-1$

Given these properties:

*) linear segments: meet only ~~...~~ 1. First and Second derivatives are Constants.

*) quadratic segments: meet 1, 2. ~~Third~~ Second derivative is Constant

*) ~~Cubic~~ Cubic segments: Satisfy 1, 2 and 3.

Note: Higher order polynomials (> 3) can satisfy all properties, but cubic segments are enough.

4. Natural spline: $S''_1(x_1) = S''_{n-1}(x_n) = 0$: at end points x_1, x_n.

Theorem:

For a set of $n \geq 2$ data points (x_i, y_i), with distinct x_i, there is a unique natural cubic spline interpolates the points:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad \text{for } i = 1, \dots, n-1$$

We want to find coefficients b_i, c_i, d_i ; (as $a_i = y_i$)

$S_1(x) = y_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3$ on $[x_1, x_2]$

$S_2(x) = y_2 + b_2(x-x_2) + c_2(x-x_2)^2 + d_2(x-x_2)^3$ on $[x_2, x_3]$

$\vdots$

$S_{n-1}(x) = y_{n-1} + b_{n-1}(x-x_{n-1}) + c_{n-1}(x-x_{n-1})^2 + d_{n-1}(x-x_{n-1})^3$ on $[x_{n-1}, x_n]$

① First property gives $(n-1)$ equations:

$$\cancel{S_{n-1}(x_n)}$$

$$\left.\begin{aligned}
S_1(x_2) &= y_2 \longrightarrow y_2 = y_1 + b_1(x_2-x_1) + c_1(x_2-x_1)^2 + d_1(x_2-x_1)^3 \\
&\vdots \\
S_{n-1}(x_n) &= y_n \longrightarrow y_n = y_{n-1} + b_{n-1}(x_n-x_{n-1}) + c_{n-1}(x_n-x_{n-1})^2 + d_{n-1}(x_n-x_{n-1})^3
\end{aligned}\right\} (1)$$

② Second property gives $(n-2)$ equations:

$$\left.\begin{aligned}
S_1'(x_2) - S_2'(x_2) &= 0 \longrightarrow b_1 + 2c_1(x_2-x_1) + 3d_1(x_2-x_1)^2 - b_2 = 0 \\
&\vdots \\
S_{n-2}'(x_{n-1}) - S_{n-1}'(x_{n-1}) &= 0 \longrightarrow \sim
\end{aligned}\right\} (2)$$

③ Third property gives $(n-2)$ equations:

$$\left.\begin{aligned}
S_1''(x_2) - S_2''(x_2) &= 0 \longrightarrow 2c_1 + 6d_1(x_2-x_1) - 2c_2 = 0 \\
&\vdots \\
S_{n-2}''(x_{n-1}) - S_{n-1}''(x_{n-1}) &= 0 \longrightarrow 2c_{n-2} + 6d_{n-2}(x_{n-1}-x_{n-2}) - 2c_{n-1} = 0
\end{aligned}\right\} (3)$$

④ Fourth property gives 2 equations:

$$\left.\begin{aligned}
S''(x_1) &= 0 \longrightarrow 2c_1 = 0 \\
S_{n-1}''(x_n) &= 0 \longrightarrow 2c_n = 0
\end{aligned}\right\} (4)$$