(88) WS L3

More general way of deriving methods of second order.

Take $y_{n+1} = y_n + hF(t_n, y_n; h)$

Use $F(t,y;h) = \gamma_1 f(t,y) + \gamma_2 f(t+\alpha h, y+\beta h f(t,y))$

$\gamma_1 = 1$ and $\gamma_2 = 0 \Rightarrow$ Euler

$\gamma_1 = 0$, $\gamma_2 = 1$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$

$\Rightarrow$ midpoint rule.

To get other methods expand $F(t,y;h)$ in a Taylor series about $h=0$

$$F(t,y;h) = F(t,y;0) + \frac{dF}{dh}(t,y;0)h + O(h^2)$$

$$= \gamma_1 f(t,y) + \gamma_2 f(t,y) + \gamma_2 h\left(\frac{\partial f}{\partial t}\frac{d}{dh}(t+\alpha h) + \frac{\partial f}{\partial y}\frac{d}{dh}(y+\beta h f(t,y))\right) + O(h^2)$$

$$= \gamma_1 f(t,y) + \gamma_2 f(t,y) + \gamma_2 h\left(\frac{\partial f}{\partial t}\alpha + \frac{\partial f}{\partial y}\beta f(t,y)\right) + O(h^2)$$

$$\underbrace{= \gamma_1 f(t,y) + \gamma_2 f(t,y) + \gamma_2 \alpha h \frac{\partial f}{\partial t} + \gamma_2 \beta h \frac{\partial f}{\partial y} f(t,y) + O(h^2)}_{*}$$

$$y_{n+1} = y_n + h\,(*)$$

$$= y_n + h\left[(\gamma_1+\gamma_2) f(t,y)\right] + h^2\left[\alpha\gamma_2 \frac{\partial f}{\partial t} + \beta\gamma_2 \frac{\partial f}{\partial y} f(t,y)\right] + O(h^3)$$

Now compare to Taylor to find $\gamma_1, \gamma_2, \alpha$ and β.

$$y_{n+1} = y_n + hf(t,y) + \frac{h^2}{2}\left(\frac{\partial f(t,y)}{\partial t} + \frac{\partial f(t,y)}{\partial y} f(t,y)\right) + O(h^3)$$

$$\Rightarrow \left.\begin{aligned} \gamma_1 + \gamma_2 &= 1 \\ \gamma_2\alpha &= \tfrac{1}{2} \\ \gamma_2\beta &= \tfrac{1}{2} \end{aligned}\right\}$$

4 unknowns

3 equations

$\Rightarrow$ infinite number of solutions

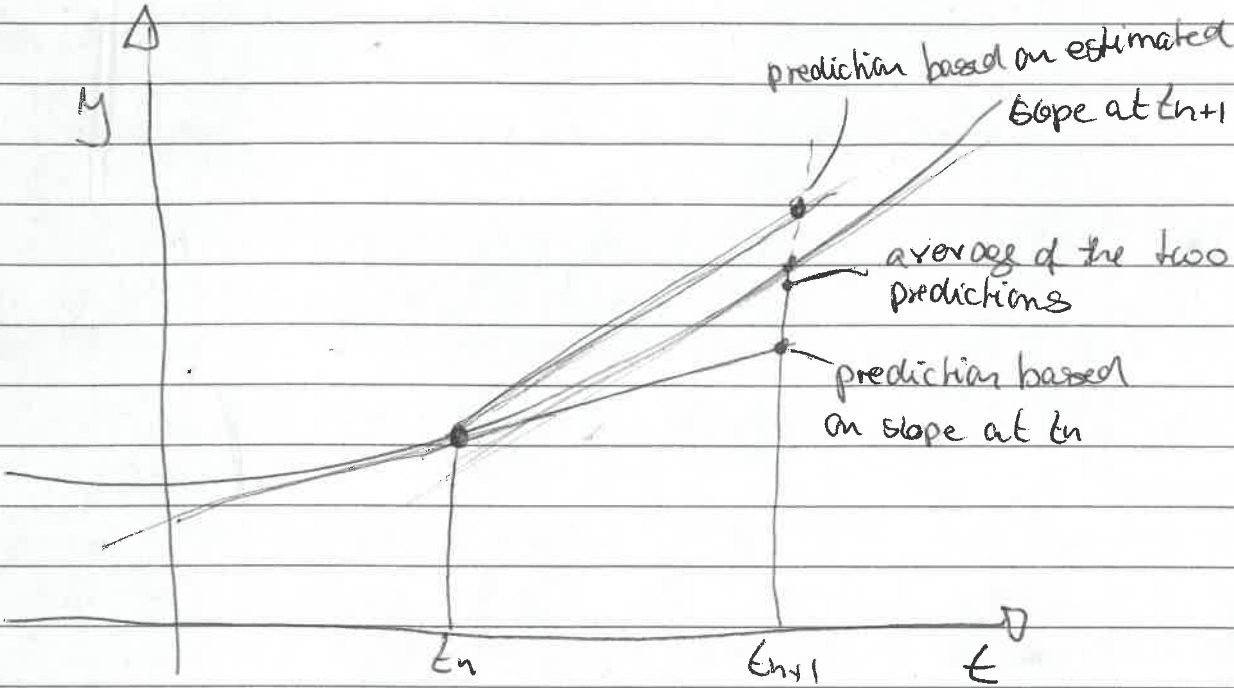
choose an obvious one.

$$\gamma_1 = \frac{1}{2},\ \gamma_2 = \frac{1}{2},\ \alpha = 1,\ \beta = 1$$

RK2 of global error $O(h^2)$ (local $O(h^3)$)

$$y_{n+1} = y_n + h\left(\frac{1}{2} f(t_n, y_n) + \frac{1}{2} f(t_n + h, y_n + hf(t_n, y_n))\right) + O(h^3).$$

RK2 (Heun)



$$y_{n+1} = y_n + \frac{h}{2}\Big(\underbrace{f(t_n, y_n)}_{\text{slope at } t_n} + \underbrace{f(\overbrace{t_n+h}^{=t_{n+1}}, y_n + f(t_n, y_n))}_{\text{Slope at } t_{n+1}}\Big)$$

Slope at ~~estima~~ t_{n+1}

$$y_{n+1} = y_n + \frac{1}{2}h\left(\text{Slope-left} + \text{Slope-right}\right)$$

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W5 L3

RK4 is 4th order Runge-Kutta method and widely used. It is simple to implement but much more accurate than RK2 or midpoint.

$$y_{n+1} = y_n + h \cdot \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$ ← same as midpoint

$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$

$k_4 = f(t_n + h, y_n + hk_3)$

Local error of $O(h^5)$, global error $O(h^4)$

k_1 = slope at t_n

k_2 = slope at estimated midpoint from k_1

k_3 = slope at estimated midpoint from k_2

k_4 = slope at t_{n+1} from k_3

$\Rightarrow$ System of none linear equations of m dimension to solve.

For $\alpha_1 = 0$ and $\beta_{ij} = 0$ for $j \geq i \Rightarrow$ explicit method

Coefficients can be arranged in a so called Butcher tableaux

$$\begin{array}{c|cccc} \alpha_1 & \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \alpha_2 & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \alpha_m & \beta_{m1} & \cdots & \cdots & \beta_{mm} \\ \hline & \gamma_1 & \cdots & \cdots & \gamma_m \end{array}$$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ \alpha_2 & \beta_{21} & \ddots & 0 & \\ \vdots & \vdots & \ddots & \ddots & \\ \alpha_m & \beta_{m1} & \cdots & \beta_{mm-1} & 0 \\ \hline & \gamma_1 & \cdots & \cdots & \gamma_m \end{array}$$

Q6

Euler

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad y_{n+1} = y_n + hf(t_n, y_n)$$

Midpoint

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \qquad y_{n+1} = y_n + hf\left(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf(t_n, y_n)\right)$$

RK2 (Heun's method)

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

RK4

$$\begin{array}{c|cccc} 0 & 0 & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Backward Euler (implicit)

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$