Gaussian Quadrature:

Idea ~~is~~ evaluates the weighted sum of function values at specified points (Legendre roots) within integral domain.

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{n} c_i f(x_i)$$

c_i = weights

x_i : Legendre roots

n : number of evaluation points.

This is achieved by replacing the integrand with a polynomial that interpolates the ~~Leg~~ legendre roots.

One-point Gaussian Quadrature formula.

$$\int_a^b f(x)\,dx \simeq c_1 f(x_1) \qquad x_1 \text{ in } [a,b]$$

To calculate c_1 and x_1: I have 2 choices to change (c_1, x_1)

Thus, I can interpolate a first-order polynomial.

$$f(x) = a_0 + a_1 x$$

$$\int_a^b f(x)\,dx \simeq \int_a^b (a_0 + a_1 x)\,dx = \left[a_0 x + a_1 \frac{x^2}{2}\right]_a^b = \left[a_0(b-a) + a_1 \frac{b^2-a^2}{2}\right] \quad (1)$$

$$\int_a^b f(x)\,dx \simeq c_1 f(x_1) = c_1\left[a_0 + a_1 x_1\right] = a_0 c_1 + c_1 a_1 x_1 \quad (2)$$

$$(1),(2) \longrightarrow \begin{cases} c_1 = b-a \\ c_1 x_1 = \dfrac{b^2-a^2}{2} \end{cases} \longrightarrow \cancel{(b-a)}\, x_1 = \frac{\cancel{(b-a)}(b+a)}{2} \longrightarrow x_1 = \frac{a+b}{2}$$

$$\int_a^b f(x)\,dx \simeq c_1 f(x_1) = \underbrace{(b-a)}_{\text{wieght}}\, f\underbrace{\left(\frac{a+b}{2}\right)}_{\text{root}}$$

Two-point Gauss Quadrature rule:

$$\int_a^b f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2).$$

We use a 3^{rd}-order interpolating polynomial to approximate the integral.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\int_a^b f(x)\,dx \simeq \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx = a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right) \quad (1)$$

$$\int_a^b f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2) = a_0(c_1+c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3) \quad (2)$$

$(1),(2) \rightarrow$

$$\begin{cases} c_1 + c_2 = b-a \\ c_1x_1 + c_2x_2 = \dfrac{b^2-a^2}{2} \\ c_1x_1^2 + c_2x_2^2 = \dfrac{b^3-a^3}{3} \\ c_1x_1^3 + c_2x_2^3 = \dfrac{b^4-a^4}{4} \end{cases} \longrightarrow \text{Solutions:}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

n-point Gauss Quadrature rule:

General form is defined ~~for~~ from -1 to 1:

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{n} c_i f(x_i)$$

~~one point~~:

$$n=2 \longrightarrow \begin{cases} c_1 = c_2 = 1 \\ x_1 = (-1/\sqrt{3}) + 0 = -1/\sqrt{3} \\ x_2 = 1/\sqrt{3} \end{cases}$$

Converting integration limits:

To approximate an integral on a general interval [a,b] using Gauss Quadrature rule, first we need to transfer [a,b] back to [-1,1].

$$\int_a^b f(x)\,dx \longrightarrow \int_{-1}^{1} g(t)\,dt \simeq \sum_{i=1}^{n} c_i g(t_i)$$

Linear tranformation:

$$x = mt + c$$

expand/contract interval

transfers interval to left/right

lower limit of integral: $a = m(-1) + c$

upper " " ": $b = m(1) + c$

$$\left.\begin{aligned} a &= m(-1)+c \\ b &= m(1)+c \end{aligned}\right\} \longrightarrow m = \frac{b-a}{2}, \quad c = \frac{b+a}{2}$$

$$\left.\begin{aligned} x &= \left(\frac{b-a}{2}\right)t + \frac{b+a}{2} \\ dx &= \left(\frac{b-a}{2}\right)dt \end{aligned}\right\} \longrightarrow \int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\left(\frac{b-a}{2}\right)t + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right)dt$$

$$= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)dt$$

Example:

Consider;
$$\int_{0.1}^{1.3} 5x\, e^{-2x}\, dx$$

A) Convert integral limits from $[0.1, 1.3]$ to $[-1, 1]$.

B) Approximate the integral using 2-point Gauss Quadrature rule.
Legendre roots are: $x_1 = -0.577$ and $x_2 = 0.577$
Weights are $c_1 = c_2 = 1$.

Solution: A)

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) dt \qquad (*)$$

$$\int_{0.1}^{1.3} 5x e^{-2x}\, dx = \frac{1.3-0.1}{2}\int_{-1}^{1} f\left(\frac{1.3-0.1}{2}t + \frac{1.3+0.1}{2}\right) dt$$

$$= 0.6\int_{-1}^{1} f(0.6t + 0.7)\, dt$$

B)

$$\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2)$$

$$\int_{0.1}^{1.3} 5xe^{-2x}\,dx = 0.6\int_{-1}^{1} f(0.6t+0.7)\,dt =$$

$$= 0.6\left[c_1 f\left(0.6(-0.577)+0.7\right) + c_2 f\left(0.6(0.577)+0.7\right)\right]$$

$$= 0.6\left[\cancel{c_1}^{\,1} f(0.354) + \cancel{c_2}^{\,1} f(1.046)\right]$$

$$= 0.6\left[5(0.354)e^{-2\times 0.354} + 5(1.046)e^{-2\times 1.046}\right] \simeq 0.9101$$

Overview:

Finite difference methods:

ODES.

PDES. → { Diffusion/heat eq. | wave eq. | Laplace eq. }

Interpolation: - Lagrange polynomial
- Newton's divided difference polynomial
- Runge's phenomenon.
- chebyshev's idea.
- Spline interpolation

Extrapolation: Richardson method.

Integration: {
Newton's–Cote methods: { Trapezoid rule | Simpson's rule | Romberg rule. (used extrapolation) }

Quadratures: { Adaptive: dividing intervals. | Gaussian: weights and roots. }
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