Explicit Euler:

$$y_{n+1} = y_n + h\underbrace{f(t_n, y_n)}_{-\alpha y_n}$$

$$= y_n + h(-\alpha y_n) = (1-\alpha h)y_n$$

$$y_1 = (1-h\alpha)y_0$$

$$y_2 = (1-h\alpha)y_1 = (1-h\alpha)^2 y_0$$

$$\vdots$$

$$y_n = (1-\alpha h)^n y_0$$

Study the behaviour at a point t^* and set $t^* = t_n$. $h = \frac{t^*}{n}$ and $n \to \infty$

we know $e^t = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n$

using this we have

$$y(t^*) \sim y_n = y_0\left(1 - \alpha\frac{t^*}{n}\right)^n$$

now $\lim_{n\to\infty} y_n = y_0 e^{-\alpha t^*}$

to get this convergence we need:

$$L = \lim_{n\to\infty} \left|\frac{y_{n+1}}{y_n}\right| < 1$$

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$$\Rightarrow \left| \frac{y_0\left(1-\alpha\frac{t^*}{n}\right)^{n+1}}{y_0\left(1-\alpha\frac{t^*}{n}\right)^{n}} \right| = \left|1-\alpha\frac{t^*}{n}\right| < 1$$

$$\Rightarrow |1-\alpha h| < 1$$

$<0 \quad >0$

$$-(1-\alpha h) < 1 \qquad\qquad 1-\alpha h < 1$$

$$-1+\alpha h < 1 \qquad\qquad 1 < 1+\alpha h$$

$$\alpha h < 2 \qquad\qquad 0 < \alpha h$$

$$0 < \alpha h < 2 \quad \text{or} \quad 0 < h < \frac{2}{\alpha}$$

This puts an upper limit on h for the method to be stable.

Now use implicit Euler method.

$y' = -\alpha y \quad , \quad y(0) = y_0 \quad , \quad y = y_0 e^{-\alpha t}$

$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) = y_n + h(-\alpha y_{n+1})$

~~y_{n+1}~~

$$y_{n+1} + h\alpha y_{n+1} = y_n$$

$$(1+\alpha h) y_{n+1} = y_n$$

$$y_{n+1} = \frac{y_n}{(1+\alpha h)}$$

$$y_1 = \frac{y_0}{(1+\alpha h)}, \quad y_n = \frac{y_0}{(1+\alpha h)^n}$$

again : $y(t^*) \sim y_n = \frac{y_0}{\left(1+\alpha \frac{t^*}{n}\right)^n}$ and $n \to \infty$

$$= y_0 e^{-\alpha t^*}$$

$\frac{1}{(1+\alpha h)^n}$ does not ~~grow~~ "~~explode~~" grow without bound as we have $\alpha > 0$ and h is fixed and $h > 0$

To ensure convergence we need $\frac{1}{(1+\alpha h)} < 1$ which we have for ~~any~~ as $\alpha > 0$ and any $h > 0$.

This is called absolute stability. This is desirable in numerical methods.

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Statements on stiffness:

- If a numerical method with a finite region of absolute stability, applied to a system with any initial condition is forced to use, ~~to~~ in a certain interval of integration, a steplength which is excessively small in relation to the smoothness of the exact solution in that intervall, then the system is said to be stiff in that intervall
- Stiffness occurs when stability requirements, rather than those of accuracy, constrain the stepsize.
- Stiffness occurs when some components of the solution decay much more rapidly than others

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Systems of ODEs

Most ODEs of higher order can be rewritten as a system of first order ODEs. We can therefore use known methods to solve these.

Example equation for the pendulum

$\ddot{\theta} + \sin\theta = 0$ (dimensionless form)

write as system of ODEs

let $y_1 = \theta$

$y_2 = \dot{\theta}$ $\quad \dot{y}_2 = \ddot{\theta}$

System of ODEs:

$\dot{y}_2 = -\sin y_1$ $\quad (-\sin\theta)$

$\dot{y}_1 = y_2$ $\quad (\dot{\theta})$

$\theta(0) = \theta_0 \Rightarrow y_1(0) = \theta_0$

$\dot{\theta}(0) = \dot{\theta}_0 \Rightarrow y_2(0) = \dot{\theta}_0$

given.