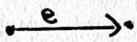
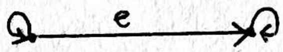
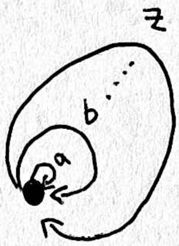
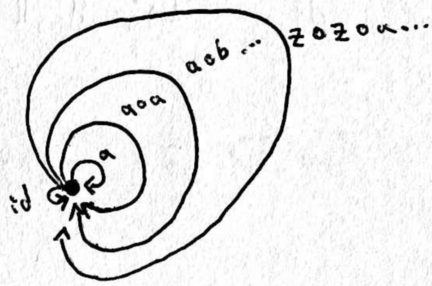
1

<table>
<tr><th></th><th>Graph G</th><th>Free category of G</th></tr>
<tr><td>a)</td><td></td><td></td></tr>
<tr><td>b)</td><td></td><td></td></tr>
<tr><td>c)</td><td></td><td></td></tr>
<tr><td>d)</td><td></td><td></td></tr>
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②

Reminder

Let S be a non-empty set and $\preccurlyeq_S$ be a binary relation in S.

Then

- $\langle S, \preccurlyeq_S \rangle$ is a preorder if

  $\preccurlyeq_S$ is reflexive, i.e. $a \preccurlyeq_S a \quad \forall a \in S$

  $\preccurlyeq_S$ is transitive, i.e. $a \preccurlyeq_S b \wedge b \preccurlyeq_S c \Rightarrow a \preccurlyeq_S c \quad \forall a,b,c \in S$

- $\langle S, \preccurlyeq_S \rangle$ is a partial order if $\langle S, \preccurlyeq_S \rangle$ is a preorder and if

  $a \preccurlyeq_S b \wedge b \preccurlyeq_S a \Rightarrow a = b \quad \forall a,b \in S$

- $\langle S, \preccurlyeq_S \rangle$ is a total order / linear order if $\langle S, \preccurlyeq_S \rangle$ is a partial order and if

  $a \preccurlyeq_S b \vee b \preccurlyeq_S a \quad \forall a,b \in S$

a) What kind of order is $\langle \mathcal{U}, \subset \rangle$? where $\mathcal{U}$ is a set of sets and $\subset$ is the inclusion relation?

Solution:

If we take $\mathcal{U}$ to be the powerset of some set S, then the solution depends upon the cardinality of S.

- If $\# S = 0$ then $\mathcal{U} = \{\emptyset\}$, so $\langle \mathcal{U}, \subset \rangle$ is clearly a total order
- Likewise, if $\# S = 1$, then $\mathcal{U} = \{\emptyset, S\}$ and again $\langle \mathcal{U}, \subset \rangle$ is a total order.
- If $\# S \geq 2$ then $\langle \mathcal{U}, \subset \rangle$ is clearly a partial order but not a total one. Indeed, we can take $x \neq y \in S$ and we see that

$$\{x\} \not\subset \{y\} \text{ and } \{y\} \not\subset \{x\}$$

b) What kind of order is this?

$\langle T, \preccurlyeq_T \rangle$ where

- T is the set of all types
- $a \preccurlyeq_T b :\Leftrightarrow$ a can be safely casted into b.

Solution:

We have that

- $\preccurlyeq_T$ is reflexive, since e.g. $int8 \preccurlyeq_T int8$ and so on
- $\preccurlyeq_T$ is transitive, since e.g. $int8 \preccurlyeq_T int16$, $int16 \preccurlyeq_T int32$ and $int8 \preccurlyeq_T int32$
- $\preccurlyeq_T$ fulfills the partial order property, since there aren't 2 distinct types that can be sofely casted into one another
- $\preccurlyeq_T$ is NOT a total order since e.g.

$$int8 \not\preccurlyeq_T String \quad \text{and} \quad string \not\preccurlyeq_T int8$$

$\Rightarrow \langle T, \preccurlyeq_T \rangle$ is a Partial order

③

Reminder

Let $G \neq \emptyset$, and $*: G^2 \to G$. We say that $\langle G, *, e \rangle$ is a (set-theore-
$e \in G$

-tical) monoid iff

M1) $*$ is associative

M2) $\forall a \in G: e * a = a * e = a$.

Now Let $B = \{\text{True}, \text{False}\}$, $\&\& : B^2 \to B$, $|| : B^2 \to B$ with

$$\text{True} \;\&\&\; \text{True} = \text{True},$$
$$\text{True} \;\&\&\; \text{False} = \text{False}$$
$$\text{False} \;\&\&\; \text{True} = \text{False}$$
$$\text{False} \;\&\&\; \text{False} = \text{False}$$

$$\text{True} \;||\; \text{True} = \text{True}$$
$$\text{True} \;||\; \text{False} = \text{True}$$
$$\text{False} \;||\; \text{True} = \text{True}$$
$$\text{False} \;||\; \text{False} = \text{False}$$

Show that

a) $\langle B, \&\&, \text{True} \rangle$ is a (set-theoretical) monoid

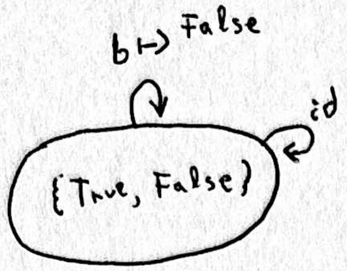
b) $\langle B, ||, \text{False} \rangle$ — " — ——

solution:

Follows by brute-force inspection :)

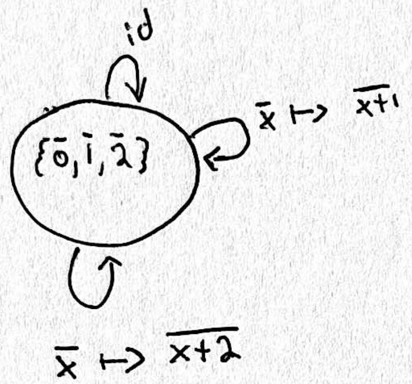
④

We represent the monoid graphically:



There are only 2 morphisms, id and $b \equiv False$. Their composition is $b \equiv False$. The morphism $b \equiv False$ composed with itself equals itself.

⑤



Where $\bar{n} := \{ m \in \mathbb{Z} \mid n \bmod 3 = m \bmod 3 \}$ is the equiv. class mod 3.