Recap:

Let $D : I \to C$ be a diagram.

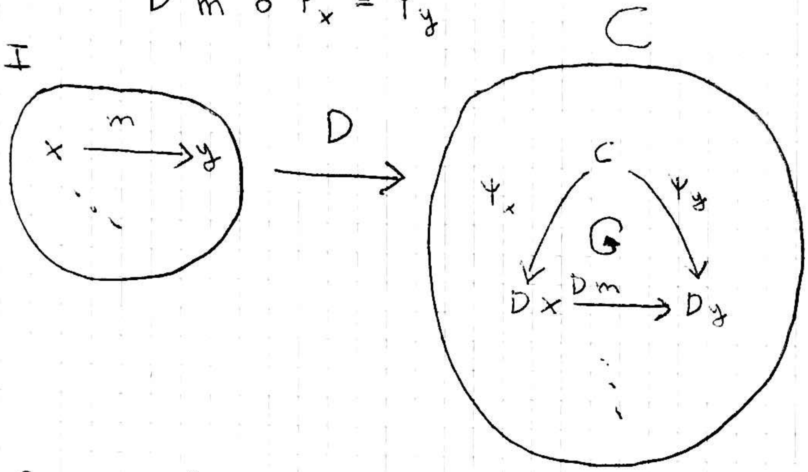
A cone to D is a pair $\langle c, \{\psi_i\}_{i \in I} \rangle$ where

- $c \in C$ (the apex of the cone)
- $\forall i \in I$, ψ_i has signature

$$\psi_i : c \to Di$$

and it satisfies: $\forall$ morphism $m : x \to y$ in I, we have

$$Dm \circ \psi_x = \psi_y$$



The limit of D is a cone $\langle \mathrm{Lim}\, D, \{\phi_i\} \rangle$ s.t.

$\forall$ cone $\langle c, \{\psi_i\} \rangle$ $\exists!\, u : c \to \mathrm{Lim}\, D$ s.t. $\phi_x \circ u = \psi_x$ $\forall x \in I$

$$\begin{tikzcd}
 & c \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_x"'] \arrow[ddr, bend left, "\psi_y"] & \\
 & \mathrm{Lim}\, D \arrow[dl, "\phi_x"'] \arrow[dr, "\phi_y"] & \\
Dx \arrow[rr, "Dm"] & & Dy
\end{tikzcd}$$

(2)

Let C be a category with initial object init_C, and let $I = C$, $D = \mathrm{id}_C$.
Show that

$$\mathrm{Lim}\, D = \mathrm{init}_C$$

Solution:

We structure the proof as follows:

a) $\mathrm{Lim}\, D$ exists $\Rightarrow \Psi_{\mathrm{Lim}\, D} = \mathrm{id}_{\mathrm{Lim}\, D}$

b) $\mathrm{Lim}\, D$ exists $\Rightarrow \mathrm{Lim}\, D = \mathrm{init}_C$

c) $\mathrm{Lim}\, D$ exists

Proof of a):

Since $D = \mathrm{id}$ we have that $\forall x \xrightarrow{f} y$:

$$\begin{tikzcd}
 & \mathrm{Lim}\, D \arrow[dl, bend right, "\Psi_x"'] \arrow[dr, bend left, "\Psi_y"] & \\
x \arrow[rr, "f"] & & y
\end{tikzcd}$$

So taking $x = \mathrm{Lim}\, D$, $f = \Psi_y$ we get

$$\begin{tikzcd}
 & \mathrm{Lim}\, D \arrow[dl, bend right, "\Psi_{\mathrm{Lim}\, D}"'] \arrow[dr, bend left, "\Psi_y"] & \\
\mathrm{Lim}\, D \arrow[rr, "\Psi_y"] & & y
\end{tikzcd}$$

meaning that

$$\cancel{\Psi_{\mathrm{Lim}\, D} = \Psi_y}$$

$$\Psi_y = \Psi_y \circ \Psi_{\mathrm{Lim}\, D}$$

$\Rightarrow \Psi_{\mathrm{Lim}\, D} = \mathrm{id}_{\mathrm{Lim}\, D}$

Δ

Proof of b):

Recall that the intal object is s.t. $\exists!$ morphism init $\dashrightarrow x \ \forall x$. ~~Note that taking~~ Since Lim D is a cone we know that there is at least 1 morphism to every x. We now show that there is at most 1 such morphism. Let $f : \text{Lim } D \to y$, taking $x = \text{Lim } D$, we get

$$\begin{tikzcd}
& \text{Lim } D \arrow[dl, bend right, "\psi_{\text{Lim } D}"'] \arrow[dr, bend left, "\psi_y"] \arrow[d, phantom, "\circlearrowright"] & \\
\text{Lim } D \arrow[rr, "f"] & & y
\end{tikzcd}$$

So $f \circ \psi_{\text{Lim } D} = \psi_y$

But we know that $\psi_{\text{Lim } D} = id_{\text{Lim } D}$ so $f = \psi_y$ △

Proof of c):

It suffices to show that ~~Lim D~~ init_C is a cone. This is straightforward since $D = id$. △

□

③

Let S be a non-empty set and C be the category whose objects are $\mathcal{P}(S)$ and $x \to y \iff x \subset b$.

- What is a pullback in C?
- What is a pushout in C?
- What are the initial and terminal objects?

Solution

The pullback of $a \to b \leftarrow c$ is $a \cap c$.

Proof.

Assume by contradiction that $x \neq a \cap c$ is the pullback of $a \to b \leftarrow c$:

$$\begin{tikzcd}
 & a \cap c \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & x \arrow[dl] \arrow[dr] & \\
a \arrow[dr] & \circlearrowright & c \arrow[dl] \\
 & b &
\end{tikzcd}$$

Then $\exists e \in x \setminus a \cap c$. But $e \in a$ and $e \in c$ so $e \in a \cap c$ and $e \notin a \cap c$, a contradiction $\lightning$ Δ

~~The pushout of a~~

- The pushout of $a \leftarrow b \rightarrow c$ is $a \cup c$.

Proof

Assume by $\lightning$ that $x \neq a \cup c$ is the pushout of $a \leftarrow b \rightarrow c$:

$$
\begin{tikzcd}
 & b \arrow[dl, bend right] \arrow[dr, bend left] & \\
a \arrow[dr] \arrow[ddr, bend right] & & c \arrow[dl] \arrow[ddl, bend left] \\
 & x \arrow[d] & \\
 & a \cup c &
\end{tikzcd}
$$

Take an $e \in a \cup c \setminus x$. Then $e \in a \Rightarrow e \in x$. Thus $e \in x$ and $e \notin x$ $\lightning$

- The initial object of C is $\emptyset$ and the terminal one is S.

(4)

What is a coequalizer?

A co-equalizer is an equalizer in the opposite category.

I.e. it is the "best" object coeq st.

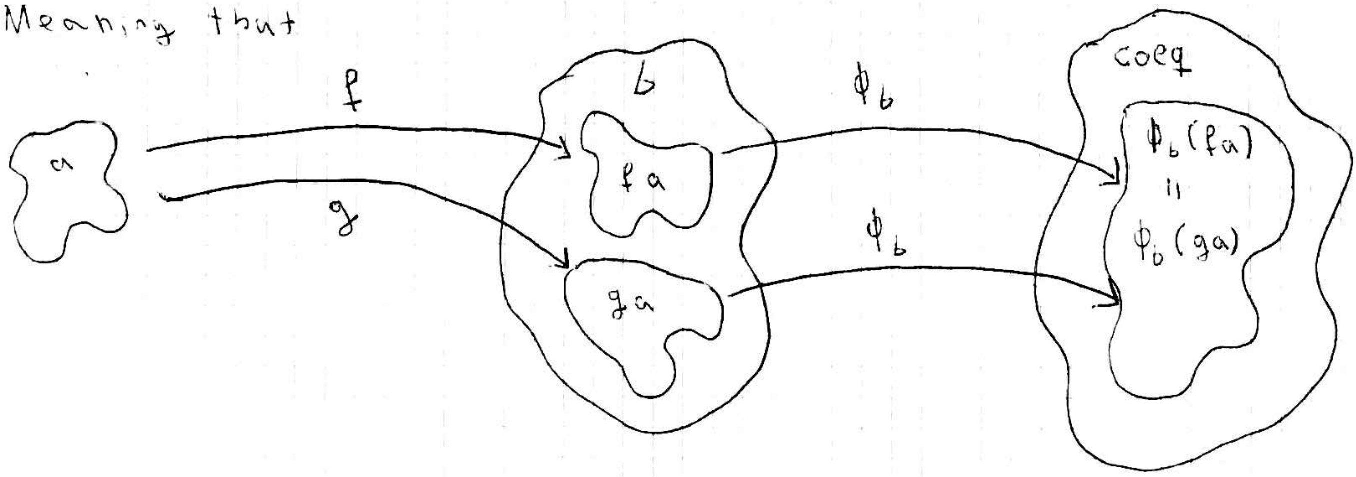
$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, bend right, "\phi_a"'] & b \arrow[d, "\phi_b"] \\ & \text{coeq} \end{tikzcd}$$

$\Rightarrow \phi_b \circ f = \phi_a, \quad \phi_b \circ g = \phi_a$

$\Rightarrow \phi_b \circ f = \phi_b \circ g$

$\Rightarrow \phi_b \Big|_{fa} = \phi_b \Big|_{ga}$

Meaning that



(5)

Show that in a category with a terminal object, a pullback towards the terminal object is a product.

Solution:

Consider the pullback $a \xrightarrow{f} t \xleftarrow{g} b$ where t is terminal.

$$\begin{tikzcd}
 & a \times b \arrow[dl, bend right, "\psi_a"'] \arrow[dr, bend left, "\psi_b"] & \\
a \arrow[dr, dashed, "f"] & & b \arrow[dl, dashed, "g"] \\
 & t &
\end{tikzcd}$$

The only thing we must show is that

$$f \circ \psi_a = g \circ \psi_b$$

Everything else follows from the def of $a \times b$.

Note that since t is terminal $\exists!$ morphism $m: a \times b \dashrightarrow t$.

Then by the axioms of a category:

$$\left.\begin{array}{l} m = f \circ \psi_a \\ m = g \circ \psi_b \end{array}\right\} \Rightarrow f \circ \psi_a = g \circ \psi_b$$

as desired. $\square$

⑥

Show that the pushout from an initial object is the coproduct.

Solution:

Let i be the initial object and take $a, b \in C$:

$$\begin{tikzcd}
 & i \arrow[dl, dashed, "\psi_a"'] \arrow[dr, dashed, "\psi_b"] & \\
a \arrow[dr, "\phi_a"'] & & b \arrow[dl, "\phi_b"] \\
 & a \oplus b &
\end{tikzcd}$$

~~Really~~ It is sufficient to show that $\phi_a \circ \psi_a = \phi_b \circ \psi_b$.

Since i is initial $\exists ! \, m : i \dashrightarrow a \oplus b$ so by the axioms of a category:

$$\phi_a \circ \psi_a = m,$$

$$\phi_b \circ \psi_b = m$$

yielding $\phi_a \circ \psi_a = \phi_b \circ \psi_b$ $\square$.