Recap:

Let C be a category and $a \in C$. Define

covariant functor $\vec{C}_a : C \to Set$

contravariant functor $\overleftarrow{C}_a : C \to Set$

with

$$\vec{C}_a\, x = C(a,x), \quad \overleftarrow{C}_a\, x = C(x,a)$$

$$\vec{C}_a\, f = \lambda m.\, f \circ m, \quad \overleftarrow{C}_a\, f = \lambda m.\, m \circ f$$

A functor $F : C \to D$ is representable $:\Longleftrightarrow$ $\exists a \in C : \exists \alpha, \beta$ natural tr:

$$\begin{tikzcd}
x \arrow[d, "f"] & C(a,x) \arrow[r, "\alpha_x"] \arrow[d, "\vec{C}_a f"] \arrow[dr, phantom, "\circlearrowright"] & Fx \arrow[r, "\beta_x"] \arrow[d, "Ff"] \arrow[dr, phantom, "\circlearrowright"] & C(a,x) \arrow[d, "\vec{C}_a f"] \\
y \arrow[r, phantom, "\rightsquigarrow"] & C(a,y) \arrow[r, "\alpha_y"] & Fy \arrow[r, "\beta_y"] & C(a,y)
\end{tikzcd}$$

i.e. $\alpha_x \circ \beta_x = id_{Fx}$, $\beta_x \circ \alpha_x = id_{C(a,x)}$

(i)

5. Let C be a Category and $a, x \in C$. Show that

$$\vec{C}_a \, id_x = id_{C(a,x)} \,,$$

$$\overleftarrow{C}_a \, id_x = id_{C(x,a)}$$

Solution:

$$\vec{C}_a \, id_x = \lambda m : a \to x .\ id_x \circ m = \lambda m : a \to x .\ m = id_{C(a,x)}$$

$$\overleftarrow{C}_a \, id_x = \lambda m : x \to a .\ m \circ id_x = \lambda m : x \to a .\ m = id_{C(x,a)} \quad \square$$

(2) Show that the Maybe functor is not representable.

Solution.

For Maybe to be representable, a map

$$\beta_x : \text{Maybe } x \to C(a, x)$$

needs to exist. ~~In particular~~ This is impossible since one cannot construct an $f : a \to x$ given Nothing. $\square$

~~1045~~

③ Show that the Reader functor is representable.

Solution.

The Reader functor is Haskell's hom-functor, which is trivially naturally isomorphic to itself. □

(5)

Define

$$\text{Stream } x = \text{Cons } x \ (\text{Stream } x)$$

$$\text{tabulate} :: (\mathbb{N} \to x) \to \text{Stream } x$$

$$\text{tabulate } f = \text{Cons } (f\ 0)\ (\text{tabulate } f \circ \lambda x.\ x+1)$$

$$\text{index} :: \text{Stream } x \to \mathbb{N} \to x$$

$$\text{index } (\text{Cons } b\ bs)\ n = \begin{cases} b & \text{if } n = 0 \\ \text{index } bs\ (n-1) & \text{else} \end{cases}$$

Show that

a) $\text{index } (\text{tabulate } f) = f$

b) $\text{tabulate } (\text{index } (\text{Cons } b\ bs)) = \text{Cons } b\ bs$

Solution:

Proof of a): We show by induction that

$$\text{index } (\text{tabulate } f)\ n = f\,n \qquad \forall n\ \forall f$$

B.C.

$$\text{index } (\text{tabulate } f)\ 0 = \text{index } (\text{Cons } (f\ 0)\ (\text{tabulate } f \circ (+1)))\ 0$$
$$= f\ 0 \ \checkmark$$

I.S.

$$\text{index } (\text{tabulate } f)\ (n+1) = \text{index } \text{Cons } (f\ 0)\ (\text{tabulate } f \circ (+1))\ (n+1)$$
$$= \text{index } (\text{tabulate } f \circ (+1))\ n$$
$$\overset{\text{I.H.}}{=} f \circ (+1)\ n$$
$$= f\ (n+1)$$

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Proof of b):

It is enough to show that

$$\text{tabulate}\,(\text{index}\,(\text{Cons}\;b\;bs)) = \text{Cons}\;b\;(\text{tabulate}(\text{index}\;bs))$$

$$\text{tabulate}\,(\text{index}\;\text{Cons}\;b\;bs)$$

$$\shortparallel$$

$$\text{tabulate}\;\lambda n.\begin{cases} b &, n=0 \\ \text{index}\;bs\;(n-1) &, \text{else}\end{cases}$$

$$\shortparallel$$

$$\text{Cons}\;b\;\;\text{tabulate}\;\lambda n.\begin{cases} b &, n=0 \\ \text{index}\;bs\;(n-1) &, \text{else}\end{cases} \circ (+1)$$

$$\shortparallel$$

$$\text{Cons}\;b\;\;\text{tabulate}\;\lambda n.\begin{cases} b &, n+1=0 \\ \text{index}\;bs\;n &, \text{else}\end{cases}$$

$$\shortparallel$$

$$\text{Cons}\;b\;\text{tabulate}\;\lambda n.\,\text{index}\;bs\;n$$

$$\shortparallel$$

$$\text{Cons}\;b\;\text{tabulate}\,(\text{index}\;bs)\;\;\triangle$$

(6)

The functor

$$F\,x = \text{Pair}\ x\ x$$

$$F\,f = \lambda x.\ \text{Pair}\ (f\,x)\ (f\,x)$$

is representable. What type represents it?

Implement tabulate and index.

Solution:

The type is bool.

$$\text{tabulate} : (\text{bool} \to x) \to \text{Pair}\ x\ x$$

$$\text{tabulate}\ m = \text{Pair}\ (m\ \text{True})\ (m\ \text{False})$$

$$\text{index} : \text{Pair}\ x\ x \to \text{bool} \to x$$

$$\text{index}\ (\text{Pair}\ a\ b)\ p = \begin{cases} a & p = \text{True} \\ b & p = \text{False} \end{cases}$$