Reminder

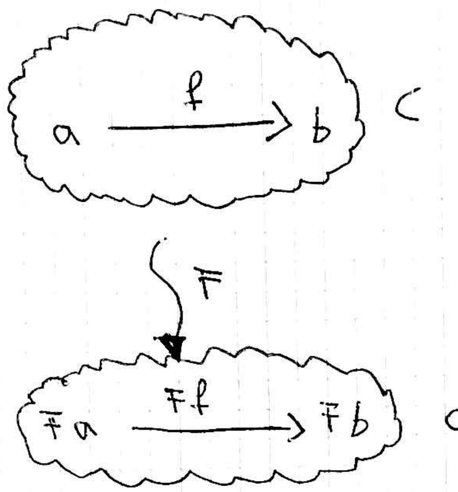
Let $C = \langle Obj, Morph \rangle$, $C' = \langle Obj', Morph' \rangle$ be categories. A functor F from C to C' is a map

$$F :: \begin{cases} Obj \to Obj' \\ Morph \to Morph' \end{cases}$$

s.t.

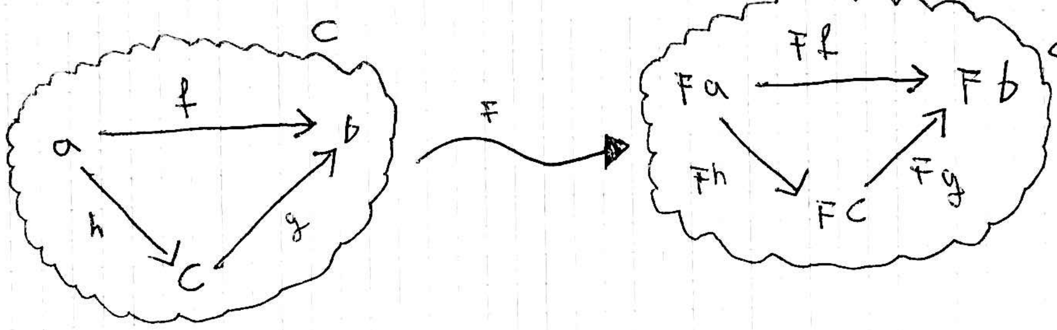
F1) $\forall a, b \in Obj.\ \forall f \in \text{Hom-Set}_C(a,b).\ Ff \in \text{Hom-Set}_{C'}(Fa, Fb)$

i.e.



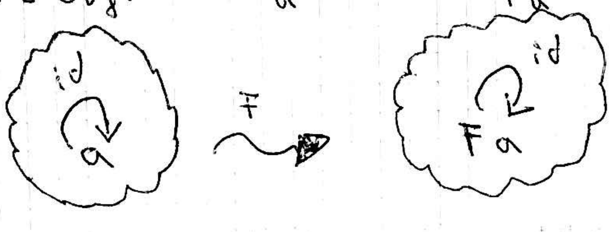
F2) $\forall f \in Morph.$ if $f = g \circ h$ then $Ff = Fg \circ Fh$

i.e.



F3) $\forall a \in Obj.\ F\, id_a = id_{Fa}$

i.e.



(1)

Let F be a map s.t.

$$F a = \text{Maybe } a$$

$$F\,(f :: a \to b) = \lambda\, x :: \text{Maybe } a \,.\, \text{Nothing}$$

Is F a functor?

Solution

F1 obviously holds ✓

F2 also holds since $Fg \circ Fh \equiv \text{Nothing} \equiv Ff$ ✓

F3 does not hold since e.g.

$$F\, id_{int}\ 5 = \text{Nothing} \neq \text{Maybe } 5 = id_{\text{Maybe int}}\ 5$$

$\Rightarrow$ F is not a functor

(2)

Fix a type r and let Reader_r be a map s.t.

$$\text{Reader}_r\ a = r \to a$$

$$\text{Reader}_r\ (f :: a \to b) = \lambda\, g :: r \to a \,.\, f \circ g$$

Show that Reader_r is a functor.

Solution

F1 is obvious ✓

To justify F2 take $f :: a \to b$, $h :: a \to c$, $g :: c \to b$ s.t. $f = g \circ h$. We show that $Ff = Fg \circ Fh$.

Take $\Sigma :: r \to a$, then

$$
\begin{aligned}
(Fg \circ Fh)\,\Sigma &= Fg\,(Fh\,\Sigma) \\
&= Fg\,(h \circ \Sigma) \\
&= g \circ (h \circ \Sigma) \\
&= (g \circ h) \circ \Sigma \\
&= F\,(g \circ h)\,\Sigma \quad \checkmark
\end{aligned}
$$

For F3, take a tye a. We see that

$$
\begin{aligned}
F\,id_a &= \lambda e :: r \to a \,.\, id_a \circ e \\
&= \lambda e :: r \to a \,.\, e \\
&= id_{Fa} \quad \checkmark
\end{aligned}
$$

$\square$

(4)

Let F be the map with

$$F\ a = \text{List}\ a$$

$$F\ f\ [\,] = [\,]$$

$$F\ f\ (\text{Cons}\ x\ xs) = \text{cons}\ fx\ (F\ f\ xs)$$

Show that F is a functor.

<u>Solution</u>

$F1$ is obvious ✓

To show $F2$ take $f :: a \to b$, $h :: a \to c$, $g :: c \to b$ with $f = g \circ h$. We show that $Ff = Fg \circ Fh$. To do so, we prove:

$$\forall xs.\ F\ f\ xs = (Fg \circ Fh)\ xs$$

by induction.

Base case $(xs = [\,])$:

$$F\ f\ [\,] = [\,] = F\ g\ [\,] = (Fg \circ Fh)\ [\,] \quad ✓$$

Inductive step:

We show that

$$\forall x. \forall xs.\ F\ f\ xs = (Fg \circ Fh)\ xs \implies F\ f\ (\text{cons}\ x\ xs) = (Fg \circ Fh)\ (\text{Cons}\ x\ xs)$$

Which is straightforward since

$$(F\,g \circ F\,h)\ (\mathrm{Cons}\ x\ xs)$$

$$= F\,g\ (F\,h\ (\mathrm{Cons}\ x\ xs))$$

$$= F\,g\ (\mathrm{Cons}\ (h\,x)\ (F\,h\ xs))$$

$$= \mathrm{Cons}\ (g\ (h\ x))\ (F\,g\ (F\,h\ xs))$$

$$= \mathrm{Cons}\ (f\ x)\ (F\,g \circ F\,h\ \ xs)$$

$$\overset{\mathrm{I.H}}{=} \mathrm{Cons}\ (f\,x)\ (F\ f\ xs)$$

$$= F\ f\ (\mathrm{Cons}\ x\ xs)\ \checkmark$$

We show F3 by induction too:

Base case ([]):

$$F\ id\ [] = [] = id\ []\ \checkmark$$

Inductive step:

We show that

$$\forall x.\ \forall xs.\ \ F\ id\ xs = xs \Rightarrow F\ id\ (\mathrm{Cons}\ x\ xs) = \mathrm{Cons}\ x\ xs$$

Which is trivial:

$$F\ id\ (\mathrm{Cons}\ x\ xs) = \mathrm{Cons}\ x\ (F\ id\ xs)$$

$$\overset{\mathrm{I.H.}}{=} \mathrm{Cons}\ x\ xs$$

$\square$