①

Let t and t' be distinct terminal objects. We show that there is a unique isomorphism between them. Since both objects are terminal, $\exists!$ arrow $t \to t'$ and $t' \to t$, i.e.

$$\begin{tikzcd} t \arrow[loop above] \arrow[r, "f"] & t' \arrow[loop above] \arrow[l, bend left, "g"] \end{tikzcd}$$

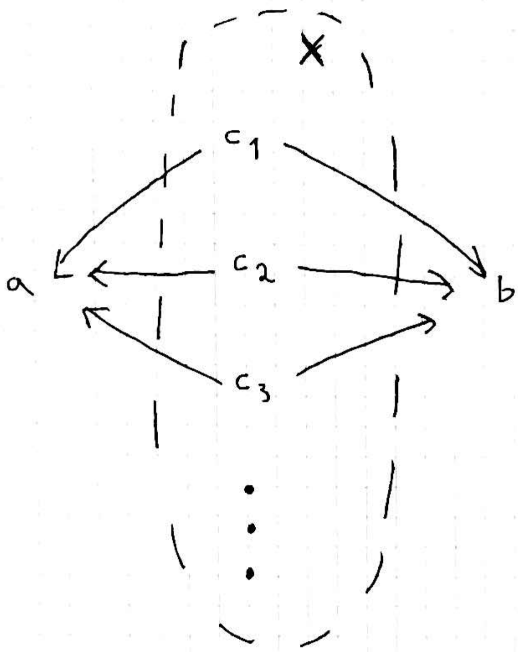
By the axioms of a category, $f \circ g$ must exist, and it is a morphism of type $t' \to t'$. There is only one such morphism, namely $id_{t'}$. The same reasoning shows $g \circ f = id_t$. $\square$

(2)

Let $\langle S, \preccurlyeq \rangle$ be a poset and $a, b \in S$. It is easy to see that if $a \preccurlyeq b$ (or $b \preccurlyeq a$), then $a \times b = a$ (or $a \times b = b$). Hence, from now on, we assume that $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$. Define the set

$$X := \{c : c \preccurlyeq a \text{ and } c \preccurlyeq b\}$$

I.e.



WLOG, $X \neq \emptyset$ since otherwise $a \times b$ is ill-defined. We now let c_{MAX} be the $c \in X$ s.t. $\forall c \in X.\ c \preccurlyeq c_{MAX}$. We claim that

(i) if c_{MAX} exists, it is unique

(ii) $a \times b$ is well-defined $\Leftrightarrow$ c_{MAX} is well-defined

(iii) $a \times b = c_{MAX}$ (provided it exists)

Proof of (i):

If c_{MAX} and c_{MAX}' are maxima of X, then in particular $c_{MAX} \preccurlyeq c_{MAX}'$ and $c_{MAX}' \preccurlyeq c_{MAX}$. By the poset axioms we get $c_{MAX} = c_{MAX}'$.

Δ

Proof of (ii $\Rightarrow$)

Assume by contradiction that $a \times b$ exists but c_{MAX} does not. Then

$$\forall c \in X \; \exists x_c \in X . \; x_c \not\ll c \qquad (*)$$

In the diagram, we get

$$\begin{tikzcd}
& X_{a\times b} \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] \arrow[d, dashed, "m"] & \\
& a \times b \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] & \\
a & & b
\end{tikzcd}$$

But note that by $(*)$, the morphism m cannot exist.

Δ

The proof of (ii $\Leftarrow$) follows from (iii) Δ.

Proof of (iii):

By (ii $\Rightarrow$) we know that c_{MAX} exists and by (i) we know it is unique. Assume by $\lightning$ that $a \times b = c_i \neq c_{MAX}$. Then $c_i \ll c_{MAX}$ but $c_{MAX} \not\ll c_i$. But then, just like in (ii $\Rightarrow$) the morphism m does not exist Δ.

From this, we conclude that

$$a \times b = \max \{ c \in S : c \ll a, c \ll b \}.$$

in particular, if $\ll$ is linear we get

$$a \times b = \min \{ a, b \}$$

③ We use the same type of reasoning as in ex.2 and get

$$a \times^{co} b = \min\{c \in S : a \preceq c,\ b \preceq c\}$$

If $\langle S, \preceq \rangle$ is linear, we can see that

$$a \times^{co} b = \max\{a, b\}$$

⑤ We get

$$\begin{tikzcd}
\text{int} \arrow[dr, "i"] \arrow[ddr, bend right, "i'"'] & & \text{bool} \arrow[dl, "j"'] \arrow[ddl, bend left, "j'"] \\
& \text{Either int bool} \arrow[d, dashed, "m"] & \\
& \text{int} &
\end{tikzcd}$$

Where i and j are the canonical projections, $i' = id$ and $j' = \mathbb{1}_{\{False\}}$

We show that the factorizing m exists and that it is unique.

Let then $m ::$ Either int bool $\longrightarrow$ int with

$$m \ (\text{int } x) = x$$

$$m \ (\text{bool } b) = \mathbb{1}_{\{False\}}(b)$$

Then clearly

$$i' = m \circ i \,, \qquad (*)$$
$$j' = m \circ j$$

Obviously, any other definition of m would not obey $(*)$, so m is unique $\square$.

(6) Informally, int with i', j' cannot be better than Either because j' loses information.

(7) Define m :: Either int bool $\to$ int with

$$m \ (\text{int } x) = \begin{cases} n & n < 0 \\ n+2 & n \geq 0 \end{cases}$$

$$m \ (\text{bool } b) = \mathbb{1}_{\{\text{False}\}}(b)$$

Just like before, $*$ holds and m is unique.

(8) Take the type (Either int bool, $\{0,1\}$) :

$$\begin{tikzcd}
\text{int} \arrow[dr, "i"] \arrow[ddr, bend right, "i'"'] & & \text{bool} \arrow[dl, "j"'] \arrow[ddl, bend left, "j'"] \\
& (\text{Either int, bool}, \{0,1\}) \arrow[d, dashed, "m"] & \\
& \text{Either int bool} &
\end{tikzcd}$$

where $i = x \mapsto (\text{int } x, 0)$, $j = b \mapsto (\text{bool } b, 0)$.

Note now that there are at least 2 valid candidates for m, namely

$$m\ (\text{int } x,\ 0) = \text{int } x$$

$$m\ (\text{int } x,\ 1) = \text{int } 42$$

$$m\ (\text{bool } b,\ 0) = \text{bool } b$$

$$m\ (\text{bool } b,\ 1) = \text{bool True}$$

and

$$m'\ (\text{int } x,\ 0) = \text{int } x$$

$$m'\ (\text{int } x,\ 1) = \text{bool True}$$

$$m'\ (\text{bool } b,\ 0) = \text{bool } b$$

$$m'\ (\text{bool } b,\ 1) = \text{int } 42$$