①

Define a natural transformation from the Maybe functor to the List functor.

Solution

I need to find an $\eta_a :: \text{Maybe } a \to \text{List } a$ s.t. the following diagramm commutes $\forall f$:

$$
\begin{tikzcd}
\text{Maybe } a \arrow[r, "\eta_a"] \arrow[d, "\text{fmap } f"] & \text{List } a \arrow[d, "\text{fmap } f"] \\
\text{Maybe } b \arrow[r, "\eta_b"] & \text{List } b
\end{tikzcd}
$$

Let

$$\eta_a \text{ Nothing} = []$$

$$\eta_a (\text{Just } x) = [x]$$

Then

$$\text{fmap } f \circ \eta_a \text{ Nothing} = [] = \eta_b \circ \text{fmap } f \text{ Nothing} \checkmark$$

$$\begin{aligned}
\text{fmap } f \circ \eta_a (\text{Just } x) &= \text{fmap } f\ [x] = [f\,x] \\
&= \eta_b \left(\text{Just } (f\,x)\right) \\
&= \eta_b \circ \text{fmap } f \left(\text{Just } x\right) \checkmark
\end{aligned}$$

② Define 2 natural transformations between Reader () and List.

Solution

We wont $\varphi_a, \beta_a :: (() \to a) \to \text{List } a$ s.t. the following diagramm commutes $\forall f$:

$$
\begin{tikzcd}
a \arrow[d, "f"] \\
b
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
() \to a \arrow[r, "\varphi_a"] \arrow[d, "\text{fmap } f"'] & \text{List } a \arrow[d, "\text{fmap } f"] \\
() \to b \arrow[r, "\varphi_b"] & \text{List } b
\end{tikzcd}
$$

Let

$$\varphi_a \;\; x = [x\;()],$$

$$\beta_a \;\; \_ = [\,]$$

then

$$\text{fmap } f \circ \varphi_a \;\; x = [\, f\;(x\,())\,] = \varphi_b \circ \text{fmap } f \;\; x \quad \checkmark$$

$$\text{fmap } f \circ \beta_a \;\; x = [\,] = \beta_b \circ \text{fmap } f \;\; x \quad \checkmark$$

There are ∞ many lists of (), namely [ ], [()], [(),()], ...

③ Define 2 natural transformations between Reader Bool and Maybe.

Solution:

$$
\begin{tikzcd}
\text{Bool} \to a \arrow[r, "\alpha_a / \beta_a"] \arrow[d, "\text{fmap } f"'] & \text{Maybe } a \arrow[d, "\text{fmap } f"] \\
\text{Bool} \to b \arrow[r, "\alpha_b / \beta_b"] & \text{Maybe } b
\end{tikzcd}
$$

Let $\alpha_a \; g = \text{Just } (g \text{ True}) \quad \forall \, g,$

$\beta_a \; g = \text{Just } (g \text{ False})$

Then

$$\text{fmap } f \circ \alpha_a \; g = \text{fmap } f \; (\text{Just } (g \text{ True}))$$
$$= \text{Just } (f \circ g \text{ True})$$

and

$$\alpha_b \circ \text{fmap } f \; g = \alpha_b \; f \circ g$$
$$= \text{Just } (f \circ g \text{ True})$$

Same goes for β. $\square$

④ Show that horizontal composition of natural transformations satisfies the naturality condition.

Solution:

Let C, D, E be categories and $F, F' :: C \to D$, $G, G' :: D \to E$ be functors. Also, let $\alpha :: F \to F'$, $\beta :: G \to G'$ be natural transformations. I.e.

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

We will show that $\exists$ a natural transformation of the form

$$\begin{tikzcd} C \arrow[r, bend left, "G \circ F"] \arrow[r, bend right, "G' \circ F'"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & E \end{tikzcd}$$

by diagram chasing. We will do this by defining $\beta * \alpha$ and then showing that the following diagram commutes:

$$\begin{tikzcd} G \circ F\, a \arrow[r, "\beta * \alpha_{G \circ F a}"] \arrow[d, "G \circ F\, f"'] & G' \circ F'\, a \arrow[d, "G' \circ F'\, f"] \\ G \circ F\, b \arrow[r, "\beta * \alpha_{G' \circ F' b}"'] & G' \circ F'\, b \end{tikzcd}$$

$\forall a :: C,\ b :: C,$

$f :: a \to b$

By the naturality of α, the following commutes:

$$\begin{tikzcd}
Fa \arrow[r, "\alpha_a"] \arrow[d, "Ff"'] & F'a \arrow[d, "F'f"] \\
Fb \arrow[r, "\alpha_b"'] & F'b
\end{tikzcd}$$

which in turn gives rise to 2 other commuting diagrams, by applying G and G':

$$\begin{tikzcd}
G \circ Fa \arrow[r, "G \circ \alpha_a"] \arrow[d, "G \circ Ff"'] & G \circ F'a \arrow[d, "G \circ F'f"] \\
G \circ Fb \arrow[r, "G \circ \alpha_b"'] & G \circ F'b
\end{tikzcd}
\qquad
\begin{tikzcd}
G' \circ Fa \arrow[r, "G' \circ \alpha_a"] \arrow[d, "G' \circ Ff"'] & G' \circ F'a \arrow[d, "G' \circ F'f"] \\
G' \circ Fb \arrow[r, "G' \circ \alpha_b"'] & G' \circ F'b
\end{tikzcd}$$

But these can be linked with β:

$$\begin{tikzcd}
& G' \circ Fa \arrow[rr, "G' \circ \alpha_a"] \arrow[dd, "G' \circ Ff"' near start] & & G' \circ F'a \arrow[dd, "G' \circ F'f"] \\
G \circ Fa \arrow[ur, "\beta_{Fa}"] \arrow[rr, "G \circ \alpha_a" near end, crossing over] \arrow[dd, "G \circ Ff"'] & & G \circ F'a \arrow[ur, "\beta_{F'a}"] & \\
& G' \circ Fb \arrow[rr, "G' \circ \alpha_b" near start] & & G' \circ F'b \\
G \circ Fb \arrow[ur, "\beta_{Fb}"'] \arrow[rr, "G \circ \alpha_b"'] & & G \circ F'b \arrow[ur, "\beta_{F'b}"'] \arrow[from=uu, "G \circ F'f" near start, crossing over] &
\end{tikzcd}$$

By chasing the diagram, we find 2 equal definitions for $\beta * \alpha$:

$$\beta * \alpha_{G \circ F a} = \beta_{F'a} \circ G \circ \alpha_a$$

$$= G' \circ \alpha_a \circ \beta_{Fa} \quad \square$$