(1)

Let M, N be monoids and $h: M \to N$ be a bijective map s.t.

$$\forall a, b \in M, \; ha \cdot hb = h(a \cdot b)$$

~~Solution:~~ Show that

$$h e_M = e_N$$

Solution:

$$\begin{aligned} h e_M &= h e_M \cdot e_N \\ &= h e_M \cdot h \circ h^{-1} e_N \\ &= h(e_M \cdot h^{-1} e_N) \\ &= h \circ h^{-1} e_N \\ &= e_N \end{aligned}$$

$\square$

②

Let $M = ([\mathbb{Z}], ++)$, $N = (\mathbb{Z}, \cdot)$ be monods and

$$h : M \to N$$

be a homomorphism.

a) What is $h[\,]$?

b) Assume from now on that $h[x] = x \ \forall x$. What is $h[1,2,3,4]$?

c) Evaluate $\# h^{-1}(\{12\})$

d) Is there any other homomorphism $i : M \to N$?

Solution:

a) $h\, e_M = e_N$ so $h[\,] = 1$

b) $h[1,2,3,4] = h([1] ++ [2] ++ [3] ++ [4])$

$$= h[1] \cdot h[2] \cdot h[3] \cdot h[4]$$

$$= 1 \cdot 2 \cdot 3 \cdot 4$$

$$= 24$$

c) $\# h^{-1}(\{12\}) = \# \{ x \in [\mathbb{Z}] : \prod x = 12 \} = \infty$

d) Yes, any $f : \mathbb{Z} \to \mathbb{Z}$ can be extended to a homomorphism by setting

$$i[x] = f(x) \quad \forall x.$$

(3)

Let M be the free monoid generated by "x".

What is M isomorphic to?

Solution:

$(M, ++) \simeq (\mathbb{N}_0, +)$ ∎

Intuition:

$$\underbrace{[x, x, \dots, x]}_{n\text{-times}} \xmapsto{h} n$$

and

$$h \; \underbrace{[x, \dots, x]}_{n\text{-times}} ++ \underbrace{[x, \dots, x]}_{m\text{-times}} = n + m$$