Recap:

We recall the Yoneda lemma. It states that for any $F : C \to Set$ and $a \in C$ we have

$$[C, Set](C(a,-), F) \simeq F a$$

Moreover, each natural transformation

$$\alpha : C(a,-) \Rightarrow F$$

has the form

$$\alpha_x (f :: a \to x) = (F f)\ (\alpha_a\, id_a)$$

Let's see what happens if $F = C(b,-)$:

$$[C, Set](C(a,-), C(b,-)) \simeq C(b,a)$$

I.e.

$$\forall x. (a \to x) \to b \to x \simeq b \to a$$

and

$$\alpha_x (f :: a \to x) = C(b, f)\ (\alpha_a\, id_a)$$
$$= f \circ (\alpha_a\, id_a)$$

This works because $\alpha_a\, id_a :: b \to a$ so $\alpha_x f :: b \to x$ as desired

$$C(a,x) \xrightarrow{\alpha_x} C(b,x)$$

With this we can define a contravariant functor

$$F : C \to [C, Set]$$
$$F\ a = C(a,-)$$
$$F\ f = \lambda h.\ h \circ f$$

$$\begin{tikzcd}
a \arrow[r, "F"] & C(a,-) \arrow[d, Rightarrow, "Ff"] \\
b \arrow[u, "f"] \arrow[r, "F"] & C(b,-)
\end{tikzcd}$$

This is called the Yoneda embedding of C.

(1) Express the Co-Yoneda embedding in Haskell.

Solution:

The Co-Yoneda embedding states that

$$[C, Set](C(-,a), C(-,b)) \simeq C(a,b)$$

In Haskell this is

$$\forall x. (x \to a) \to x \to b \cong a \to b$$

② Let

$$\varphi :: \forall x.\left(\left((a \to x) \to (b \to x)\right)\right) \to (b \to a)$$

$$\varphi\ f = \lambda u.\ f\ id_a\ u = f\ id_a$$

$$\psi :: (b \to a) \to \left(\forall x.\ (a \to x) \to (b \to x)\right)$$

$$\psi\ f = \lambda g_x .\ g_x \circ f$$

Show that φ and ψ are inverses from one another.

Solution:

$$\begin{aligned}
\varphi \circ \psi\ f &= \varphi\ \lambda g_x .\ g_x \circ f \\
&= (\lambda g_x .\ g_x \circ f)\ id_a \\
&= id_a \circ f \\
&= f \ \checkmark
\end{aligned}$$

$$\begin{aligned}
\psi \circ \varphi\ f &= \psi\,(f\ id_a) \\
&= \lambda g_x .\ g_x \circ (f\ id_a) \\
&= \lambda g_x .\ C(b, g_x)\ (f\ id_a)
\end{aligned}$$

Yoneda embedding

$$\begin{aligned}
&= \lambda g_x .\ f\ g_x \\
&= f \ \checkmark
\end{aligned}$$

3

Work out the Yoneda embedding for a Monoid.

Solution:

Let M be a monoid category with a single object a.

The embedding says that

$$[M, \mathrm{Set}](M(a,-), M(a,-)) \simeq M(a,a)$$

i.e. the number of natural transformations from the morphisms of M to itself matches the number of morphisms in M.

This is because there are $\# M(a,a)$ choices to map the identity, and all other maps follow from the naturality condition.

(4)

Apply the covariant Yoneda embedding to a preorder category.

Solution:

Take $b \leq a$. By the embedding we get

$$\begin{tikzcd}
a \arrow[r, "F"] & C(a,-) \arrow[d, Rightarrow, "\varphi"] \\
b \arrow[u, "\leq"] \arrow[r, "F"] & C(b,-)
\end{tikzcd}$$

So $b \leq a \iff \exists\, \varphi : C(a,-) \Rightarrow C(b,-)$

$$\Updownarrow$$

$$\exists\, \varphi :: \forall x.\ (a \to x) \to (b \to x)$$

$$\Updownarrow$$

$$\forall x.\ a \leq x \Rightarrow b \leq x$$

⑤ Embed the functor category $[\mathcal{C}, \mathcal{D}]$ into Set.

Solution:

Define the contravariant functor

$$e : [\mathcal{C}, \mathcal{D}] \to [[\mathcal{C}, \mathcal{D}], \mathrm{Set}]$$

$$e\,F = [\mathcal{C}, \mathcal{D}](F, -)$$

$$e(\alpha : G \Rightarrow F) = \lambda \beta : [\mathcal{C}, \mathcal{D}](F, -) \,.\, \beta \circ \alpha$$

I.e. we get