HW2

(1) ① zero padding for x:

$$A = [0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 3\ \ 2\ \ 1\ \ 0\ \ 0]_{1\times M}$$

② reverse y[n], and move from left to right, multiply by A:

$$[-2\ \ 0\ \ 2 \quad \rightarrow \quad -2\ \ 0\ \ 2]_{1\times(M+2N-2)}$$

$$\Rightarrow [2\ \ 4\ \ 4\ \ 4\ \ 0\ \ -4\ \ -4\ \ -4\ \ -2]_{1\times(M+N-1)} \rightarrow \text{full}$$

$$[4\ \ 4\ \ 4\ \ 0\ \ -4\ \ -4\ \ -4]_{1\times M} \rightarrow \text{same}$$

(2) ① zero padding for g:

$$g = \begin{bmatrix} 0 & 0 & \cdots & & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & & & & \cdots & 0 & 0 \\ \vdots & & 1 & 3 & 2 & 0 & 4 & & \vdots \\ & & 1 & 0 & 3 & 2 & 3 & & \\ & & 0 & 4 & 1 & 0 & 5 & & \\ & & 2 & 3 & 2 & 1 & 4 & & \\ \vdots & & 3 & 1 & 0 & 4 & 2 & & \vdots \\ 0 & 0 & \cdots & & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & & & & \cdots & 0 & 0 \end{bmatrix}_{(M+2N-2)\times(M+2N-2)}$$

② 180° rotate f and move from top left to bottom right, multiply by A.

$$\begin{bmatrix} 1 & 0 & -1 & \rightarrow & & & \\ 2 & 0 & -2 & & & & \\ 1 & 0 & -1 & & & & \\ & & & \vdots & & & \\ & & & \rightarrow & 1 & 0 & -1 \\ & & & & 2 & 0 & -2 \\ & & & & 1 & 0 & -1 \end{bmatrix}_{(M+2N-2)\times(M+2N-2)}$$

$$\begin{bmatrix} -1 & -3 & -1 & 3 & -2 & 0 & 4 \\ -3 & -6 & -4 & 4 & -4 & 2 & 11 \\ -3 & -7 & -6 & 3 & -6 & 4 & 15 \\ -3 & -11 & -4 & 8 & -10 & 3 & 17 \\ -7 & -11 & 2 & 5 & -10 & 6 & 15 \\ -8 & -5 & 6 & -4 & -6 & 9 & 8 \\ -3 & -1 & 3 & -3 & -2 & 4 & 2 \end{bmatrix}_{(M+N-1)\times(M+N-1)} \rightarrow \text{full}$$

$$\boxed{\begin{matrix} -6 & -4 & 4 & -4 & 2 \\ -7 & -6 & 3 & -6 & 4 \\ -11 & -4 & 8 & -10 & 3 \\ -11 & 2 & 5 & -10 & 6 \\ -5 & 6 & -4 & -6 & 9 \end{matrix}} \rightarrow \text{same}$$

(3) matlab 运行 $y = \text{imfilter}(x, y, \text{'conv'}, \text{'full'})$

or 'same'

可以发现结果与前面一致。

交换性: $x*y = y*x$

pad(y[n]) = [0 0 0 0 0 0 2 0 -2 0 0 0 0 0 0]

x[n] = [1 2 3 4 3 2 1] ⟶

⇒ $y*x = [2\ 4\ 4\ 4\ 0\ -4\ -4\ -4\ -2]$

结合性: $z = [1\ 2\ 3]$

$(x*y)*z = x*(y*z)$

$(x*y) = [2\ 4\ 4\ 4\ 0\ -4\ -4\ -4\ -2]$

$(x*y)*z = [2\ 8\ 18\ 24\ 20\ 8\ -12\ -24\ -22\ -16\ -6]$

$(y*z) = [2\ 4\ 4\ -4\ -6]$

$x*(y*z) = [2\ 8\ 18\ 24\ 20\ 8\ -12\ -24\ -22\ -16\ -6] = (x*y)*z$

分配性: $x*y + x*z = x*(y+z)$

$(x*z) = [1\ 4\ 10\ 16\ 20\ 20\ 14\ 8\ 3]$

$x*y + x*z = [3\ 8\ 14\ 20\ 20\ 16\ 10\ 4\ 1]$

$y+z = [3\ 2\ 1]$

$x*(y+z) = [3\ 8\ 14\ 20\ 20\ 16\ 10\ 4\ 1] = x*y + x*z$

(4) 对于1维：$\mathrm{len}(x * y) = L + l - 1$, $x: L$, $y: l$

对于2维：$\dim(x * y) = (M+m-1) \times (N+n-1)$, $x: M \times N$, $y: m \times n$

2. (1) a. Linear: H close under

① Addition:

$$H[f(t) + g(t)] = H[f(t)] + H[g(t)]$$

② Scalar multiplication:

$$H[a f(t)] = a H[f(t)]$$

b. Time-Invariant:

If $H[f(t)] = g(t)$, then $H[f(t-a)] = g(t-a)$ for any $a \in \mathbb{R}$

(2)

Let

$$A = \begin{bmatrix} -2 & 0 & 2 & & & & 0 \\ & -2 & 0 & 2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 0 & & & & -2 & 0 & 2 \end{bmatrix}_{(M+N-1) \times (M+2N-2)}$$

$$\vec{b} = [0 \; 0 \; 1 \; 2 \; 3 \; 4 \; 3 \; 2 \; 1 \; 0 \; 0]^T_{1 \times (M+2N-2)}$$

then $z[n] = x[n] * y[n] = A \cdot \vec{b}$

(3) 是的，将卷积核写为每行是其中一个信号从左到右移共M+N-1步，每行只移动一步，从而构成A，将A与zero padding后的另一信号相乘，即得卷积。此时A是一个线性变换。

(4) δ: unit impulse

$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$, h denote the response of δ, $h = H(\delta[n])$

$$y[n] = H(x[n]) = H\left(\sum_{k=-\infty}^{\infty} x[k] \delta[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k] H(\delta[n-k]) \longrightarrow \text{linearity } (x[k] \text{ as scalar})$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \longrightarrow \text{Time-invariant}$$

$$= (x * h)[n]$$

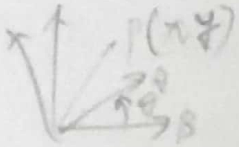
∴ 我们可以通过 x, h 得到 $y[n]$

3. Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = [f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)] - 4f(x,y)$$

$\downarrow$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$p[x,y]$ Rotation for $(-\theta)$ (B rotate θ to A)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$${}^A p = C_{AB}\,{}^B p \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$$

$$\therefore \nabla^2 f(x', y') = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \qquad \frac{\partial x}{\partial x'} = c\theta \quad \frac{\partial x}{\partial y'} = -s\theta$$

$$\frac{\partial y}{\partial x'} = s\theta \quad \frac{\partial y}{\partial y'} = c\theta$$

$$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}c\theta + \frac{\partial f}{\partial y}s\theta\right)$$

$$= c\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial x'} + c\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial x'} + s\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial x'} + s\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial x'}$$

$$= c^2\theta\frac{\partial^2 f}{\partial x^2} + s^2\theta\frac{\partial^2 f}{\partial y^2} + 2c\theta s\theta\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y'^2} = \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}\right)$$

$$= \frac{\partial}{\partial y'}\left(-s\theta\frac{\partial f}{\partial x} + c\theta\frac{\partial f}{\partial y}\right)$$

$$= -s\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial y'} - s\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial y'} + c\theta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial y'} + c\theta\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial y'}$$

$$= s^2\theta\frac{\partial^2 f}{\partial x^2} + c^2\theta\frac{\partial^2 f}{\partial y^2} - 2c\theta s\theta\frac{\partial^2 f}{\partial x \partial y}$$

$$\therefore \nabla^2 f(x', y') = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f(x, y)$$

$\therefore$ Q.E.D.

4. a. $\text{Sobel}(f) = -f(x-1, y+1) - 2f(x, y+1) - f(x+1, y+1)$
$+ f(x-1, y-1) + 2f(x, y-1) + f(x+1, y-1)$

b. $\text{Sobel}(f) = -f(x-1, y+1) - 2f(x-1, y) - f(x-1, y-1)$
$+ f(x+1, y+1) + 2f(x+1, y) + f(x+1, y-1)$

结果见 matlab

5. (1) 均值：直接把所有像素相加再除以总个数

(2) 中值：a、对于每个像素，记录其个数到数组 L，且像素总数为 N，最大 level 为 lmax

b.
```
cnt = 0, lmid = 0,
from l = [0, lmax]:
    cnt = cnt + L(l)
    if cnt ≥ N/2
        lmid = l;
        break;
```

改为
```
if cnt≥/2
  if lmax+1 is even
    lmid = (l[i]+l[i-1])/2
  else
    lmid = l[i]
```

=> lmid 即为所求中值

且该算法时间复杂度为 $O(N)$