Planning hw4 2

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Partially free OBVP

Modeling

$$g(s,u) = \frac{1}{T} j(t)^2$$

$$J = \frac{1}{T}\int_0^T j(t)^2 \, dt \quad \text{and } h(s(T)) = 0 \text{ for } v, a$$

state $s = [p \ v \ a]^T$ input: j

system: $\dot{s} = f_s(s,u) = [v \ a \ j]^T$

$$\dot{s} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} s + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} j$$

$$H(s,u,\lambda) = g(s,u) + \lambda^T f(s,u)$$

$$= \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j$$

$$\dot{\lambda} = -\nabla_s H(s^*, u^*, \lambda)$$

$$= -\left[\frac{\partial H}{\partial p}, \frac{\partial H}{\partial v}, \frac{\partial H}{\partial a}\right]^T$$

$$= [0, -\lambda_1, -\lambda_2]^T$$

$$\therefore \dot{\lambda} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \lambda$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau$$

and $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

$$\therefore A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{t^2}{2} & -t & 1 \end{bmatrix}$$

$$\Rightarrow \lambda(t) = e^{At} \lambda_0$$

let $\lambda_0 = \frac{1}{T}[-2\alpha, 2\beta, -2\gamma]^T$

$$\lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha t + 2\beta \\ -\alpha t^2 - 2\beta t - 2\gamma \end{bmatrix}$$

And $\lambda(T) = -\nabla h(s^*(T))$

$$= -\left[\frac{\partial h}{\partial p}, \frac{\partial h}{\partial v}, \frac{\partial h}{\partial a}\right]^T$$

$\because$ v, a partially free

$\because$ v, a patially free

$\therefore$ $\frac{\partial h}{\partial v} = 0 \quad \frac{\partial h}{\partial a} = 0$

$$\therefore \begin{cases} 2\alpha T + 2\beta = 0 \\ -\alpha T^2 - 2\beta T - 2r = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha T \\ r = \frac{1}{2}\alpha T^2 \end{cases}$$

$$\therefore \lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha(t-T) \\ -\alpha(t-T)^2 \end{bmatrix}$$

and $u^*(t) = j^*(t) = \arg\min_j H(s^*, j, \lambda)$

$$\frac{\partial H}{\partial j} = \frac{2}{T} j + \lambda_3 = 0$$

$$\Rightarrow j^* = -\frac{\lambda_3 T}{2} = \frac{\alpha(t-T)^2}{2}$$

$$\therefore s^* = \begin{bmatrix} \int_0^t v^* dt + p_0 \\ \int_0^t a^* dt + v_0 \\ \int_0^t j^* dt + a_0 \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{120}(t-T)^5 + (\frac{\alpha}{12}T^3 + \frac{a_0}{2})t^2 + (-\frac{\alpha}{24}T^4 + v_0)t + \frac{\alpha}{120}T^5 + p_0 \\ \frac{\alpha}{24}(t-T)^4 + (\frac{\alpha}{6}T^3 + a_0)t - \frac{\alpha}{24}T^4 + v_0 \\ \frac{\alpha}{6}(t-T)^3 + \frac{\alpha}{6}T^3 + a_0 \end{bmatrix}$$

and Final state $s^*(T) = [p_f, ?, ?]^T$

$$\therefore \frac{\alpha}{120}(T-T)^5 + (\frac{\alpha}{12}T^3 + \frac{a_0}{2})T^2 + (-\frac{\alpha}{24}T^4 + v_0)T + \frac{\alpha}{120}T^5 + p_0 = p_f$$

$$\frac{\alpha}{12}T^5 + \frac{a_0}{2}T^2 - \frac{\alpha}{24}T^5 + v_0 T + \frac{\alpha}{120}T^5 = p_f - p_0$$

$$\frac{\alpha}{20}T^5 + \frac{a_0}{2}T^2 + v_0 T = p_f - p_0$$

$$\Rightarrow \alpha = \frac{20}{T^5}\Delta p, \quad \Delta p = p_f - p_0 - v_0 T - \frac{a_0}{2}T^2$$

$$\therefore j^* = \frac{\alpha}{2}(t-T)^2$$

$$J = \frac{1}{T}\int_0^T j^{*2} dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{\alpha}{2}(t-T)^2\right)^2 dt$$

$$= \frac{1}{T}\int_0^T \frac{\alpha^2}{4}(t-T)^4 dt$$

$$= \frac{1}{T}\int_0^T \frac{a}{4}(t-T)^4\,dt$$

$$= \frac{1}{T}\frac{a^2}{20}(t-T)^5\Big|_0^T$$

$$= \frac{1}{T}\frac{a^2}{20}T^5$$

$$= \frac{a^2T^4}{20}$$

$$= \frac{T^4}{20}\frac{20^2\,\Delta p^2}{T^{10}}$$

$$= \frac{20}{T^6}\Delta p^2, \quad \Delta p = p_f - p_0 - v_0T - \frac{a_0}{2}T^2$$

solve for $\frac{dJ}{dT} = 0$

$$\Rightarrow \quad c_0 + c_1T + c_2T^2 + c_3T^3 + c_4T^4 = 0$$

$$c_4 = -10a_0^2$$

$$c_3 = -60a_0v_0$$

$$c_2 = -80v_0^2 - 80a_0p_0 + 80a_0p_f$$

$$c_1 = 200p_fv_0 - 200p_0v_0$$

$$c_0 = -120p_0^2 + 240p_0p_f - 120p_f^2$$

$\Rightarrow$ The 4 roots are (solved by matlab):

$$r_{0,1} = -\left(v_0 - (v_0^2 - 2a_0p_0 + 2a_0p_f)^{\frac{1}{2}}\right)/a_0$$

$$r_{2,3} = -\left(2v_0 + \sqrt{2}\,(2v_0^2 - 3a_0p_0 + 3a_0p_f)^{\frac{1}{2}}\right)/a_0$$