CT Model   DT Model

$$\begin{cases} \dot{p}=v \\ \dot{v}=a \\ \dot{a}=j \end{cases} \xrightarrow[\;P_i=P(idt)\;]{DT} \qquad \begin{aligned} P_{i+1} &= P_i + v_i\,dt + \tfrac{1}{2}a_i\,dt^2 + \tfrac{1}{6}j_i\,dt^3 \\ v_{i+1} &= v_i + a_i\,dt + \tfrac{1}{2}j_i\,dt^2 \\ a_{i+1} &= a_i + j_i\,dt \\ \vdots \end{aligned}$$

$$\dot{x} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} x + \overset{B}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} u$$

$$\Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$x[k] = x(kT)$$

$$\Rightarrow x[k+1] = e^{AT}x[k] + A^{-1}(e^{AT}-I)Bu[k]$$

$$A_d = e^{AT} \qquad B_d = A^{-1}(e^{AT}-I)B$$

$$\Rightarrow A_d = \begin{bmatrix} 1 & T & \tfrac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Prediction Model

$$\begin{cases} P = T_p J + B_p \\ V = T_v J + B_v \\ A = T_a J + B_a \end{cases} \qquad \begin{aligned} P &= [\,P_1,\,P_2,\,P_3\,\cdots\,]^T \\ V &= [\,v_1\;v_2\;v_3\,\cdots\,]^T \\ A &= [\,a_1\,a_2\,a_3\,\cdots\,]^T \\ J &= [\,j_0,\,j_1,\,j_2\,\cdots\,]^T \end{aligned}$$

$$B_d = A^{-1}(I + AT + \tfrac{1}{2}AT^2 - I)B$$

$$= (T + \tfrac{1}{2}T^2)B$$

$$= \begin{bmatrix} 0 \\ 0 \\ T + \tfrac{1}{2}T^2 \end{bmatrix}$$

① $\quad a_1 = a_0 + j_0 T$

$$a_2 = a_1 + j_1 T = a_0 + j_0 T + j_1 T$$

$$a_3 = a_2 + j_2 T = a_0 + j_0 T + j_1 T + j_2 T$$

$$\Rightarrow \overset{A}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}} = \overset{T_a}{\begin{bmatrix} T & & & 0 \\ T & T & & \\ T & T & T & \\ \vdots & & & \ddots \end{bmatrix}} \begin{Bmatrix} j_0 \\ j_1 \\ j_2 \\ \vdots \end{Bmatrix} + \overset{B_a}{\begin{bmatrix} a_0 \\ a_0 \\ a_0 \\ \vdots \end{bmatrix}}$$

$$\therefore\; P_{k+1} = P_k + Tv_k + \tfrac{1}{2}T^2 a_k$$

$$v_{k+1} = v_k + T a_k$$

$$a_{k+1} = a_k$$

$$\begin{bmatrix} \dot{P} \\ \dot{v} \\ \dot{a} \\ \dot{j} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} P \\ v \\ a \\ j \end{bmatrix}$$

② $\quad v_1 = v_0 + a_0 T + \tfrac{1}{2}j_0 T^2$

$$v_2 = v_1 + a_1 T + \tfrac{1}{2}j_1 T^2$$

$$= v_1 + (a_0 + j_0 T)T + \tfrac{1}{2}j_1 T^2$$

$$= v_0 + a_0 T + \tfrac{1}{2}j_0 T^2 + a_0 T + j_0 T^2 + \tfrac{1}{2}j_1 T^2$$

$$= v_0 + 2a_0 T + (\tfrac{1}{2}+1)j_0 T^2 + \tfrac{1}{2}j_1 T^2$$

$$v_3 = v_2 + a_2 T + \tfrac{1}{2}j_2 T^2$$
$$\downarrow$$
$$(a_0 + j_0 T + j_1 T)T$$

$$A_d = \exp(A) = \begin{bmatrix} 1 & T & \tfrac{1}{2}T^2 & \tfrac{1}{6}T^3 \\ 0 & 1 & T & \tfrac{1}{2}T^2 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ \\ V \end{bmatrix} = \overset{T_v}{\begin{bmatrix} \tfrac{1}{2}T^2 & & & 0 \\ (\tfrac{1}{2}+1)T^2 & \tfrac{1}{2}T^2 & & \\ (\tfrac{1}{2}+2)T^2 & (\tfrac{1}{2}+1)T^2 & \tfrac{1}{2}T^2 & \\ \vdots & & & \ddots \end{bmatrix}} \begin{Bmatrix} j_0 \\ j_1 \\ j_2 \\ \vdots \end{Bmatrix} + \overset{B_v}{\begin{bmatrix} v_0 + a_0 T \\ v_0 + 2a_0 T \\ v_0 + 3a_0 T \\ \vdots \end{bmatrix}}$$

③ $\quad P_1 = P_0 + v_0 T + \tfrac{1}{2}a_0 T^2 + \tfrac{1}{6}j_0 T^3$

$$P_2 = P_1 + v_1 T + \tfrac{1}{2}a_1 T^2 + \tfrac{1}{6}j_1 T^3$$
$$\downarrow$$
$$(v_0 + a_0 T + \tfrac{1}{2}j_0 T^2)T \qquad (a_0 + j_0 T)$$

$$P_2 = P_1 + v_1 T + \dots + 6v11$$

$$\downarrow$$

$$(v_0 + a_0 T + \tfrac{1}{2}j_0 T^2)T \qquad (a_0 + j_0 T)$$

$$P_1 = P_0 + v_0 T + \tfrac{1}{2}a_0 T^2 + \tfrac{1}{6}j_0 T^3$$

$$\downarrow$$

$$(v + a_0 T + \tfrac{1}{2}j_0 T^2)T \qquad (a_0 + j_0 T + j_1 T)$$

$$\downarrow$$

$$(a_0 + j_0 T)$$

$$\therefore \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{pmatrix} \tfrac{1}{6}T^3 & & 0 \\ (\tfrac{1}{6}+\tfrac{1}{2}+\tfrac{1}{2})T^3 & \tfrac{1}{6}T^3 & \\ (\tfrac{1}{6}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+1+\tfrac{1}{2})T^3 & (\tfrac{1}{6}+\tfrac{1}{2}+\tfrac{1}{2})T^3 & \tfrac{1}{6}T^3 \\ & \vdots & \\ & \vdots & \end{pmatrix} \begin{bmatrix} j_0 \\ j_1 \\ j_2 \\ \vdots \\ \vdots \end{bmatrix} + \begin{bmatrix} P_0 + v_0 T + \tfrac{1}{2}a_0 T^2 \\ P_0 + 2v_0 T + (\tfrac{1}{2}+1+\tfrac{1}{2})a_0 T^2 \\ P_0 + 3v_0 T + (\tfrac{1}{2}+1+\tfrac{1}{2}+2+\tfrac{1}{2})a_0 T^2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{6}T^3 & & 0 \\ (\tfrac{1}{6}+1)T^3 & \tfrac{1}{6}T^3 & \\ (\tfrac{1}{6}+1+2)T^3 & (\tfrac{1}{6}+1)T^3 & \tfrac{1}{6}T^3 \\ & \vdots & \end{bmatrix} \begin{bmatrix} j_0 \\ j_1 \\ j_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} P_0 + v_0 T + \tfrac{1}{2}a_0 T^2 \\ P_0 + 2v_0 T + \tfrac{4}{2}a_0 T^2 \\ P_0 + 3v_0 T + \tfrac{9}{2}a_0 T^2 \\ \vdots \end{bmatrix}$$

$$\underset{P}{} \qquad\qquad \underset{T_p}{} \qquad\qquad\qquad\qquad \underset{B_p}{}$$