Memo Date:

1. $1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$

$\because \iint_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$

$= \int_0^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_0^{\infty} d\theta = \pi$

and $\iint_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$

$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\therefore$ let $z = \frac{x-\mu}{\sqrt{2}\sigma}$ $dz = dx \cdot \frac{1}{\sqrt{2}\sigma}$

$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-z^2)\, dz \cdot \sqrt{2}\sigma$

$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$

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Memo Date:

2. u, v 相同维度列向量，$\vec{u}^T\vec{v} = tr(\vec{v}\vec{u}^T)$

Let $\vec{u} = [u_1, \cdots, u_n]^T$

$\vec{v} = [v_1, \cdots, v_n]^T$

$$\vec{u}^T\vec{v} = [u_1, \cdots, u_n] \cdot [v_1, \cdots v_n]^T$$

$$= \sum_{i=1}^{n} u_i v_i$$

$$\vec{v}\vec{u}^T = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot [u_1, \cdots, u_n]$$

$$= \begin{bmatrix} v_i u_j \end{bmatrix}_{i,j} \quad i = 1 \sim n,\ j = 1 \sim n$$

$$\therefore tr(\vec{v}\vec{u}^T) = \sum_{i=1 \sim n} u_i v_i = \vec{u}^T\vec{v}$$

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Memo

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4. $\vec{x} \sim N(\vec{\mu}, \Sigma)$, $\vec{\mu} = E[\vec{x}] = \int_{-\infty}^{\infty} \vec{x} p(\vec{x}) d\vec{x}$

$$RHS = \int_{-\infty}^{\infty} \vec{x} p(\vec{x}) d\vec{x}$$

$$= \int_{-\infty}^{\infty} \vec{x} \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right) d\vec{x}$$

let $\vec{y} = \vec{x} - \vec{\mu}$ $\vec{x} = \vec{y} + \vec{\mu}$ $d\vec{x} = d\vec{y}$

$$RHS = \underbrace{\frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \vec{y} \exp\left(-\frac{1}{2} \vec{y}^T \Sigma^{-1} \vec{y}\right) d\vec{y}}_{I_1}$$

$$+ \underbrace{\frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \vec{\mu} \exp\left(-\frac{1}{2} \vec{y}^T \Sigma^{-1} \vec{y}\right) d\vec{y}}_{I_2}$$

let $f(\vec{y}) = \vec{y} \exp\left(-\frac{1}{2} \vec{y}^T \Sigma^{-1} \vec{y}\right)$

so $I_1 = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\vec{y}) d\vec{y}$

and $f(-\vec{y}) = -\vec{y} \exp\left[-\frac{1}{2}(-\vec{y})^T \Sigma^{-1} (-\vec{y})\right]$

$= -f(\vec{y})$

$\therefore$ f is an odd function

$\therefore I_1 = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\vec{y}) d\vec{y} = 0$

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$\because$ Σ is positive definite and symmetric

$\therefore$ we can write $\Sigma = U \Lambda U^T$ for U and Λ

~~U contains~~

U is a full rank orthogonal matrix containing eigenvectors of Σ as its columns, and ~~A~~ Λ is diagonal matrix containing Σ's eigen values

and we define $\Lambda^{\frac{1}{2}} \in R^{n \times n}$ to be diagonal matrix whose entries are square roots of corresponding entries of ~~A~~ Λ, $\Lambda = \Lambda^{\frac{1}{2}} (\Lambda^{\frac{1}{2}})^T$

$$\therefore \quad \Sigma = U \Lambda U^T = U \Lambda^{\frac{1}{2}} (\Lambda^{\frac{1}{2}})^T U^T = B B^T$$

where $B = U \Lambda^{\frac{1}{2}}$, $\Sigma^{-1} = B^{-T} B^{-1}$

$$\therefore \quad I_2 = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \vec{\mu} \exp\left(-\frac{1}{2} \vec{y}^T B^{-T} B^{-1} \vec{y}\right) d\vec{y}$$

$$= \frac{\vec{\mu}}{(2\pi)^{\frac{n}{2}} |BB^T|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \vec{y}^T B^{-T} B^{-1} \vec{y}\right) d\vec{y}$$

let $\vec{z} = B^{-1} \vec{y}$

from change of variables formula:

if $\vec{y} \in R^n$ is a random variable vector with joint density function $f_{\vec{y}} : R^n \to R$, $\vec{z} = H(\vec{y})$ where H is a bijective differentiable function, then $\vec{z}$ has joint density $f_{\vec{z}}$ where

$$f_{\vec{z}}(\vec{z}) = f_{\vec{y}}(\vec{y}) \left| \det \begin{pmatrix} \frac{\partial y_1}{\partial z_1} & \cdots & \frac{\partial y_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial z_1} & \cdots & \frac{\partial y_n}{\partial z_n} \end{pmatrix} \right|$$

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$\vec{y} = B\vec{z}$ $\cancel{d\vec{y} = d\vec{z}\,|B|}$ $d\vec{y} = |B|\, d\vec{z}$ and $|B| = |\Sigma|^{\frac{1}{2}}$

$$\therefore I_2 = \frac{1}{(2\pi)^{N/2}|\Sigma|^{\frac{1}{2}}}\int_{-\infty}^{\infty} \vec{\mu}\exp\left(-\frac{1}{2}\vec{z}^T\vec{z}\right)|B|\, d\vec{z}$$

$$= \frac{\vec{\mu}}{(2\pi)^{N/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\vec{z}^T\vec{z}\right)d\vec{z}$$

$$= \frac{\vec{\mu}}{(2\pi)^{N/2}}\prod_{i=1}^{N}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}z_i^2\right)dz_i$$

(let $t_i = z_i\frac{1}{\sqrt{2}}$)

$$= \frac{\vec{\mu}}{(2\pi)^{N/2}}\prod_{i=1}^{N}\int_{-\infty}^{\infty}\exp\left(-t_i^2\right)\sqrt{2}\,dt_i$$

$$\cancel{= \frac{1}{(2\pi)^{N/2}}\prod_{i=1}^{N}\int_{-\infty}^{\infty}}$$

$$= \frac{\vec{\mu}}{(2\pi)^{N/2}}\prod_{i=1}^{N}\sqrt{2\pi}$$

$$= 1\times\vec{\mu} = \vec{\mu}$$

$$\therefore RHS = I_1 + I_2 = \vec{\mu}$$

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Memo Date:

5. $\vec{x} \sim N(\vec{\mu}, \Sigma)$, $\Sigma = E[x] = \int_{-\infty}^{\infty} (\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T p(\vec{x})\, d\vec{x}$

$$RHS = \int_{-\infty}^{\infty} (\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T \underbrace{\frac{1}{\sqrt{(2\pi)^N|\Sigma|}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right)}_{A} d\vec{x}$$

$$= \int_{-\infty}^{\infty} \left(\vec{x}\vec{x}^T - \vec{\mu}\vec{x}^T - \vec{x}\vec{\mu}^T + \vec{\mu}\vec{\mu}^T\right) A \cdot d\vec{x}$$

$$= \overbrace{\int_{-\infty}^{\infty} \vec{x}\vec{x}^T A\, d\vec{x}}^{I_1} \underbrace{-2\vec{\mu}^T \int_{-\infty}^{\infty} \vec{x} A\, d\vec{x}}_{I_2} \underbrace{+ \vec{\mu}\vec{\mu}^T \int_{-\infty}^{\infty} A\, d\vec{x}}_{I_3}$$

$\because$ from problem 4. $A = p(\vec{x})$

$$I_2 = -2\vec{\mu}^T \underbrace{\int_{-\infty}^{\infty} \vec{x} p(\vec{x})\, d\vec{x}}_{\vec{\mu}} = -2\vec{\mu}^T\vec{\mu} \qquad I_3 = \vec{\mu}\vec{\mu}^T$$

$\therefore RHS = I_1 - 2\vec{\mu}^T\vec{\mu} + \vec{\mu}\vec{\mu}^T = I_1 - \vec{\mu}\vec{\mu}^T$

and $I_1 = E[\vec{x}\vec{x}^T] \Rightarrow RHS = E[\vec{x}\vec{x}^T] - \vec{\mu}\vec{\mu}^T$

$\because \Sigma = E[(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T]$

$= E[\vec{x}\vec{x}^T - \vec{\mu}\vec{x}^T - \vec{x}\vec{\mu}^T + \vec{\mu}\vec{\mu}^T]$

$= E[\vec{x}\vec{x}^T] - E[\vec{\mu}\vec{x}^T] - E[\vec{x}\vec{\mu}^T] + E[\vec{\mu}\vec{\mu}^T]$

$= E[\vec{x}\vec{x}^T] - \vec{\mu}E[\vec{x}^T] - E[\vec{x}]\vec{\mu}^T + \vec{\mu}\vec{\mu}^T$

$= E[\vec{x}\vec{x}^T] - \vec{\mu}\vec{\mu}^T - \vec{\mu}\vec{\mu}^T + \vec{\mu}\vec{\mu}^T = E[\vec{x}\vec{x}^T] - \vec{\mu}\vec{\mu}^T$

$\therefore E[\vec{x}\vec{x}^T] = \Sigma + \vec{\mu}\vec{\mu}^T$

$\therefore RHS = \Sigma + \vec{\mu}\vec{\mu}^T - \vec{\mu}\vec{\mu}^T = \Sigma$

Memo Date: .

6. K个相互独立高斯变量 $x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$

$$\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})\right) \equiv \eta \prod_{k=1}^{K} \exp\left(-\frac{1}{2}(\vec{x}_k-\vec{\mu}_k)^T \Sigma_k^{-1}(\vec{x}_k-\vec{\mu}_k)\right)$$

where $\Sigma^{-1} = \sum_{k=1}^{K} \Sigma_k^{-1}$ $\quad \Sigma^{-1}\vec{\mu} = \sum_{k=1}^{K} \Sigma_k^{-1}\vec{\mu}_k$

$\because$ gaussian pdf can be written as,

$$p(\vec{x}) = \exp\left[\zeta + \eta^T\vec{x} - \frac{1}{2}\vec{x}^T\Lambda\vec{x}\right] \quad ①$$

where $\Lambda = \Sigma^{-1}$, $\vec{\eta} = \Sigma^{-1}\vec{\mu}$, $\zeta = -\frac{1}{2}(N\ln(2\pi) - \ln|\Lambda| + \vec{\eta}^T\Lambda^{-1}\vec{\eta})$

$$\therefore \prod_{k=1}^{K} p_k(\vec{x}_k) = \exp\left[\sum_{k=1}^{K}\zeta_k + \left(\sum_{k=1}^{K}\vec{\eta}_k\right)\vec{x}_k - \frac{1}{2}\vec{x}_k^T\left(\sum_{k=1}^{K}\Lambda_k\right)\vec{x}_k\right]$$

where $\sum_{k=1}^{K}\zeta_k = -\frac{1}{2}\left(KN\ln 2\pi - \sum_{k=1}^{K}\ln|\Lambda_k| + \sum_{k=1}^{K}\vec{\eta}_k^T\Lambda_k^{-1}\vec{\eta}_k\right)$

$$\therefore \prod_{k=1}^{K} p_k(\vec{x}_k) = \exp\left[\sum_{k=1}^{K}\zeta_k + \zeta_K - \zeta_K + \left(\sum_{k=1}^{K}\vec{\eta}_k\right)^T\vec{x}_k - \frac{1}{2}\vec{x}_k^T\left(\sum_{k=1}^{K}\Lambda_k\right)\vec{x}_k\right]$$

$$= \exp\left(\sum_{k=1}^{K}\zeta_k - \zeta_K\right)\exp\left(\zeta_K + \vec{\eta}_K^T\vec{x} - \frac{1}{2}\vec{x}^T\Lambda_K\vec{x}\right) \quad ②$$

where $\vec{\eta}_K = \sum_{k=1}^{K}\eta_k$, $\Lambda_K = \sum_{k=1}^{K}\Lambda_k$

and $\zeta_K = -\frac{1}{2}(N\ln 2\pi - \ln|\Lambda_K| + \vec{\eta}_K^T\Lambda_K^{-1}\vec{\eta}_K)$ ③

$\therefore$ comparing ① & ② with ③, we have

$\Lambda = \Lambda_K$ $\qquad \cancel{\vec{\eta}_K = \sum_{k=1}^{K}\eta_k}$

$\Rightarrow \Sigma^{-1} = \sum_{k=1}^{K}\Sigma_k^{-1}$ $\qquad \vec{\eta} = \vec{\eta}_K$

$\Rightarrow \Sigma^{-1}\vec{\mu} = \sum_{k=1}^{K}\Sigma_k^{-1}\vec{\mu}_k$

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