状态估计 hw2

Monday, April 27, 2020 20:18

1. (1) Consider DT system $\begin{cases} x_k = x_{k-1} + v_k + w_k & w_k \sim N(0, Q) & x_k \in \mathbb{R} \\ y_k = x_k + n_k & n_k \sim N(0, R) & v_k \in \mathbb{R} \end{cases}$

which could represent a cart moving back and forth along the x-axis. The initial state $\check{x}_0$, is unknown

Set up the system of eqns for the batch LSE:

$$(H^T W^{-1} H)\hat{x} = H^T W^{-1} z, \qquad LL^T \hat{x} = Ld$$

max time step $k=5$, all noises uncorrelated

Will unique sol. exist?

Let lifted data vector $\qquad A=1,\ C=1$

$$z = \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ \hline y_0 \\ \vdots \\ y_k \end{bmatrix} \quad x = \begin{bmatrix} x_0 \\ \vdots \\ x_k \end{bmatrix} \quad H = \begin{bmatrix} -A_0 & 1 & & \\ & \ddots & \ddots & \\ & & -A_{k-1} & 1 \\ \hline C_0 & & & \\ & \ddots & & \\ & & & C_k \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ \hline 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}_{11\times 6}$$

$k=5$

$$W = \left[\begin{array}{ccc|ccc} Q_1 & & & & & \\ & \ddots & & & & \\ & & Q_k & & & \\ \hline & & & R_0 & & \\ & & & & \ddots & \\ & & & & & R_k \end{array}\right] = \left[\begin{array}{ccc|ccc} Q & & & & & \\ & \ddots & & & & \\ & & Q & & & \\ \hline & & & R & & \\ & & & & \ddots & \\ & & & & & R \end{array}\right]_{11\times 11}$$

$\because$ the system is LTI

$\therefore$ We need to calculate rank of M

$$M = [C_0^T \ \ A_0^T C_1^T \ \ A_0^T A_1^T C_2^T \cdots A_0^T \cdots A_{k-1}^T C_k^T] \qquad \begin{matrix} A_0 = \cdots = A_{k-1} = A \\ C_0 = \cdots = C_k = C \end{matrix}$$

$$= [C \ \ AC \ \ (A^T)^2 C \ \ (A^T)^3 C \ \ (A^T)^4 C \ \ (A^T)^5 C]$$

$$= [1 \ \cdots \ 1]_{1\times 6}$$

$\because$ $\mathrm{rank}(M) = 1 \neq N(k+1) = 6$

$\therefore$ $H^T W^{-1} H$ is not invertible and $\hat{x}$ doesn't exist

2. (2) Using the same system as P1, set $Q=R=1$ and show that

$\therefore$ $H^T W^{-1} H$ is not invertible and ...

2.(2) Using the same system as P1, set $Q=R=1$ and show that

$$H^T W^{-1} H = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

What will be the sparsity pattern of the Cholesky factor L, such that $LL^T = H^T W^{-1} H$?

$$W = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{11\times 11} \qquad W^{-1} = W = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$\therefore$ $LHS = H^T W H = H^T H$

$$= \begin{array}{c} \\ 6\left\{\begin{bmatrix} \overbrace{\begin{matrix} -1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & -1 \\ & & & 1 \end{matrix}}^{A} & \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{B} \end{bmatrix}\right. \\ \quad\ \underbrace{\qquad\qquad}_{5}\ \ \underbrace{\qquad\qquad}_{6} \end{array} \cdot \begin{bmatrix} \overbrace{\begin{matrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{matrix}}^{6} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{bmatrix} \begin{matrix} \}5 \ C \\ \\ \}6 \ D \end{matrix}$$

$$= \begin{bmatrix} A & B \end{bmatrix} \cdot \begin{bmatrix} C \\ D \end{bmatrix}$$

$$= AC + BD$$

$$AC = \begin{bmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

$BD = I(6) \cdot I(6) = I(6)$

$$\therefore\ LHS = AC + BD = AC + \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{6\times 6} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & -1 & 3 & -1 & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{bmatrix} = RHS$$

$$L = \begin{bmatrix} L_0 & & \\ L_{10} & L_1 & \\ & L_{21} & L_2 \end{bmatrix}$$

$$L = \begin{bmatrix} L_0 & & & & & \\ L_{10} & L_1 & & & & \\ & L_{21} & L_2 & & & \\ & & \ddots & \ddots & & \\ & & & L_{4,3} & L_4 & \\ & & & & L_{5,4} & L_5 \end{bmatrix}$$

$$L_0 L_0^T = C_0^T R_0^{-1} C_0 + A_0^T Q_1^{-1} A_0$$

$$= 1 + 1 = 2 \quad \Rightarrow L_0 = \sqrt{2} = 1.4142$$

$$L_{10} L_0^T = -Q_1^{-1} A_0 = -1 \Rightarrow L_{10} = -\frac{\sqrt{2}}{2} = -0.7071$$

$$L_1 L_1^T = -L_{10} L_{10}^T + Q_1^{-1} + C_1^T R_1^{-1} C_1 + A_1^T Q_2^{-1} A_1$$

$$= -\frac{1}{2} + 1 + 1 + 1 = \frac{5}{2} \Rightarrow L_1 = 1.5811$$

$$L_{21} L_1^T = -Q_2^{-1} A_1 = -1 \Rightarrow L_{21} = -0.6325$$

$$L_2 L_2^T = -L_{21} L_{21}^T + Q_2^{-1} + C_2^T R_2^{-1} C_2 + A_2^T Q_3^{-1} A_2$$

$$= -0.4 + 1 + 1 + 1 = 2.6 \Rightarrow L_2 = 1.6125$$

$$L_{3,2} L_2^T = -Q_3^{-1} A_2 = -1 \Rightarrow L_{32} = -0.6202$$

$$L_3 L_3^T = -L_{32} L_{32}^T + Q_3^{-1} + C_3^T R_3^{-1} C_3 + A_3^T Q_4^{-1} A_3$$

$$= -0.3846 + 1 + 1 + 1 = 2.6164 \Rightarrow L_3 = 1.6172$$

$$L_{4,3} L_3^T = -Q_4^{-1} A_3 = -1 \Rightarrow L_{4,3} = -0.6183$$

$$L_4 L_4^T = -L_{43} L_{43}^T + Q_4^{-1} + C_4^T R_4^{-1} C_4 + A_4^T Q_5^{-1} A_4$$

$$= -0.3823 + 1 + 1 + 1 = 2.6177 \Rightarrow L_4 = 1.6179$$

$$L_{5,4} L_4^T = -Q_5^{-1} A_5 = -1 \Rightarrow L_{5,4} = -0.6181$$

$$L_5 L_5^T = -L_{54} L_{54}^T + Q_5^{-1} + C_5^T R_5^{-1} C_5$$

$$= -0.3820 + 1 + 1 = 1.6180 \Rightarrow L_5 = 1.2720$$

$$\therefore L = \begin{bmatrix} 1.4142 & & & & & \\ -0.7071 & 1.5811 & & & & \\ & 0.6325 & 1.6125 & & & \\ & & -0.6202 & 1.6172 & & \\ & & & -0.6183 & 1.6179 & \\ & & & & -0.6181 & 1.2720 \end{bmatrix}$$

3.(b) show that

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$$\begin{bmatrix} 1 & & & & \\ A & 1 & & & \\ A^2 & A & 1 & & \\ & & & \ddots & \\ A^k & A^{k-1} & A^{k-2} & \cdots A & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ -A & 1 & & & \\ & -A & 1 & & \\ & & -A & \ddots & 1 \\ & & & -A & 1 \end{bmatrix}$$

let LHS $= A^{-1}$ RHS $= B$

$\therefore$ to prove $A^{-1} = B$

is to prove $I = AB$

$$\text{RHS} = A \cdot B = \begin{bmatrix} 1 & & & & & \\ A-A & 1 & & & & \\ A^2-A^2 & A-A & 1 & & & \\ A^3-A^3 & A^2-A^2 & A-A & 1 & & \\ \vdots & & & & \ddots & \\ A^k-A^k & A^{k-1}-A^{k-1} & A^{k-2}-A^{k-2} & A^{k-3}-A^{k-3} \cdots & A-A & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I = \text{LHS}$$

$\therefore$ proved.