状态估计 hw6

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1.(7.1) prove $(Cu)^\wedge \equiv C u^\wedge C^T$

$\because Ca \times Cb = C(a \times b)$

$\therefore (Cv) \times u = Cv \times CC^T u$

$= C(v \times C^T u)$

$\therefore (Cv)^\wedge u = (C v^\wedge C^T) u$

$\Rightarrow (Cv)^\wedge = C v^\wedge C^T$

$\therefore$ Q.E.D.

2.(7.2) prove $(Cu)^\wedge = (2\cos\phi + 1)u^\wedge - u^\wedge C - C^T u^\wedge$

from P1

LHS $= C u^\wedge C^T$

and from Rodrigues Formula:

$C = \cos\phi I + (1-\cos\phi) a a^T - \sin\phi a^\wedge$

$a a^T = I + a^\wedge a^\wedge$

$\therefore C = \cos\phi I + (1-\cos\phi)(I + a^\wedge a^\wedge) - \sin\phi a^\wedge$

$= \cancel{c\phi I} + I + a^\wedge a^\wedge - \cancel{c\phi I} - c\phi a^\wedge a^\wedge - s\phi a^\wedge$

$= I + (1-c\phi) a^\wedge a^\wedge - s\phi a^\wedge$

LHS $= (I + (1-c\phi) a^\wedge a^\wedge - s\phi a^\wedge) u^\wedge (I + (1-c\phi) a^\wedge a^\wedge - s\phi a^\wedge)^T$

$(a^\wedge a^\wedge)^T = (a^\wedge)^T (a^\wedge)^T = (-a^\wedge)(-a^\wedge) = a^\wedge a^\wedge \quad (a^\wedge)^T = -a^\wedge$

$= (u^\wedge + (1-c\phi) a^{\wedge 2} u^\wedge - s\phi a^\wedge u^\wedge)(I + (1-c\phi) a^\wedge a^\wedge + s\phi a^\wedge)$

$= u^\wedge + (1-c\phi) u^\wedge a^{\wedge 2} + s\phi u^\wedge a^\wedge$

$+ (1-c\phi) a^{\wedge 2} u^\wedge + (1-c\phi)^2 a^{\wedge 2} u^\wedge a^{\wedge 2} + (1-c\phi) s\phi a^{\wedge 2} u^\wedge a^\wedge$

$- s\phi a^\wedge u^\wedge - (1-c\phi) s\phi a^\wedge u^\wedge a^{\wedge 2} - s^2\phi a^\wedge u^\wedge a^\wedge$

$= \underbrace{u^\wedge + (1-c\phi)^2 a^{\wedge 2} u^\wedge a^{\wedge 2} + (1-c\phi) s\phi (a^{\wedge 2} u^\wedge a^\wedge - a^\wedge u^\wedge a^{\wedge 2}) - s^2\phi a^\wedge u^\wedge a^\wedge}$

$+ s\phi (u^\wedge a^\wedge - a^\wedge u^\wedge) + (1-c\phi)(u^\wedge a^{\wedge 2} + a^{\wedge 2} u^\wedge) \rightarrow A$

RHS $= (2c\phi + 1) u^\wedge - u^\wedge C - C^T u^\wedge$

$= (2c\phi + 1) u^\wedge - u^\wedge (I + (1-c\phi) a^{\wedge 2} - s\phi a^\wedge) - (I + (1-c\phi) a^{\wedge 2} + s\phi a^\wedge) u^\wedge$

$= 2c\phi u^\wedge + \cancel{u^\wedge} - \cancel{u^\wedge} - u^\wedge a^{\wedge 2} + c\phi u^\wedge a^{\wedge 2} + s\phi u^\wedge a^\wedge$

$- u^\wedge - a^{\wedge 2} u^\wedge + c\phi a^{\wedge 2} u^\wedge - s\phi a^\wedge u^\wedge$

$= \underbrace{(2c\phi - 1) u^\wedge}_{B} + s\phi (u^\wedge a^\wedge - a^\wedge u^\wedge) - (1-c\phi)(u^\wedge a^{\wedge 2} + a^{\wedge 2} u^\wedge)$

$\therefore$ We need to prove $A \equiv B$

or to prove

$2(1-c\phi) u^\wedge + (1-c\phi)^2 a^{\wedge 2} u^\wedge a^{\wedge 2} + (1-c\phi) s\phi (a^{\wedge 2} u^\wedge a^\wedge - a^\wedge u^\wedge a^{\wedge 2}) - s^2\phi a^\wedge u^\wedge a^\wedge + 2(1-c\phi)(u^\wedge a^{\wedge 2} + a^{\wedge 2} u^\wedge)$

$$\left(\hat{a}^2 = -1 + aa^T \quad \hat{a}^3 = -\hat{a} \quad \hat{a}^4 = -\hat{a}^2\right)$$

$$\Rightarrow 2(1-c\phi)\hat{a}^2\hat{u}\hat{a}^2 + (1-c\phi)^2\hat{a}^4\hat{u}\hat{a}^4 + (1-c\phi)s\phi(\hat{a}^4\hat{u}\hat{a}^3 - \hat{a}^3\hat{u}\hat{a}^4) - s^2\phi\,\hat{a}^3\hat{u}\hat{a}^3 + 2(1-c\phi)(\hat{a}^2\hat{u}\hat{a}^4 + \hat{a}^4\hat{u}\hat{a}^2) = 0$$

$$2(1-c\phi)\hat{a}^2\hat{u}\hat{a}^2 + (1-c\phi)^2\hat{a}^2\hat{u}\hat{a}^2 + (1-c\phi)s\phi(\hat{a}^2\hat{u}\hat{a} - \hat{a}\hat{u}\hat{a}^2) - s^2\phi\,\hat{a}\hat{u}\hat{a} - 4(1-c\phi)\hat{a}^2\hat{u}\hat{a}^2 = 0$$

$$-(1-c\phi)(1+c\phi)\hat{a}\hat{u}\hat{a}^2 + (1-c\phi)s\phi(\hat{a}^2\hat{u}\hat{a} - \hat{a}\hat{u}\hat{a}^2) - \underbrace{s^2\phi}_{(1+c\phi)(1-c\phi)}\hat{a}\hat{u}\hat{a} = 0$$

$$\Rightarrow -(1+c\phi)\hat{a}^2\hat{u}\hat{a}^2 + s\phi(\hat{a}^2\hat{u}\hat{a} - \hat{a}\hat{u}\hat{a}^2) - (1+c\phi)\hat{a}\hat{u}\hat{a} = 0$$

$$\left(\begin{array}{l} \hat{u}\hat{a} = -(u^Ta)I + au^T \\ \hat{u}\hat{a}^2 = -(u^Ta)\hat{a} + au^T\hat{a}, \quad \hat{a}a = 0 \end{array}\right)$$

$$-(1+c\phi)\hat{a}^2(-(u^Ta)\hat{a} + au^T\hat{a}) + s\phi\hat{a}^2(-(u^Ta)I + au^T)$$
$$-s\phi\hat{a}(-(u^Ta)\hat{a} + au^T\hat{a}) - (1+c\phi)\hat{a}(-(u^Ta)I + au^T) = 0$$

$$(1+c\phi)(u^Ta)\hat{a}^3 - \cancel{s\phi(u^Ta)\hat{a}^2} + \cancel{s\phi(u^Ta)\hat{a}^2} + (1+c\phi)(u^Ta)\hat{a} = 0$$

$$-(1+c\phi)\hat{a} + (1+c\phi)\hat{a} = 0$$

$$0 = 0$$

$\therefore$ Q.E.D.

3. (7.3) prove $\exp((Cu)^\wedge) = C\exp(\hat{u})C^T$

$\because \exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$

$\therefore LHS = I + (Cu)^\wedge + \frac{1}{2!}(Cu)^\wedge(Cu)^\wedge + \frac{1}{3!}\underbrace{((Cu)^\wedge)^3}_{-(Cu)^\wedge} + \frac{1}{4!}\underbrace{((Cu)^\wedge)^4}_{-((Cu)^\wedge)^2} + \cdots$

$= I + (1 - \frac{1}{3!} + \frac{1}{5!} - \cdots)(Cu)^\wedge + (\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots)((Cu)^\wedge)^2$

from P1 $(Cu)^\wedge = C\hat{u}C^T \qquad ((Cu)^\wedge)^2 = -(u^TC^TCu)I + Cuu^TC^T = -(u^Tu)I + Cuu^TC^T$

$= I + (1 - \frac{1}{3!} + \frac{1}{5!} - \cdots)(Cu)^\wedge + (\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots)(-(u^Tu)I + Cuu^TC^T)$

$RHS = C(I + \hat{u} + \frac{1}{2!}(\hat{u})^2 + \frac{1}{3!}(\hat{u})^3 + \frac{1}{4!}(\hat{u})^4 + \cdots)C^T$

$= C(I + \hat{u} + \frac{1}{2!}(\hat{u})^2 - \frac{1}{3!}\hat{u} - \frac{1}{4!}(\hat{u})^2 + \cdots)C^T$

$= C(I + (1 - \frac{1}{3!} + \frac{1}{5!} - \cdots)\hat{u} + (\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots)(\hat{u})^2)C^T$

$(\hat{u})^2 = -(u^Tu)I + uu^T$

$= I + (1 - \frac{1}{3!} + \frac{1}{5!} - \cdots)C\hat{u}C^T + (\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots)C(\hat{u})^2C^T$

$C(\hat{u})^2C^T = -(u^Tu)CC^T + C(uu^T)C^T = -(u^Tu)I + C(uu^T)C^T$

$= I + (1 - \frac{1}{3!} + \frac{1}{5!} - \cdots)(Cu)^\wedge + (\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots)(-(u^Tu)I + C(uu^T)C^T)$

$\therefore$ LHS = RHS

Q.E.D.