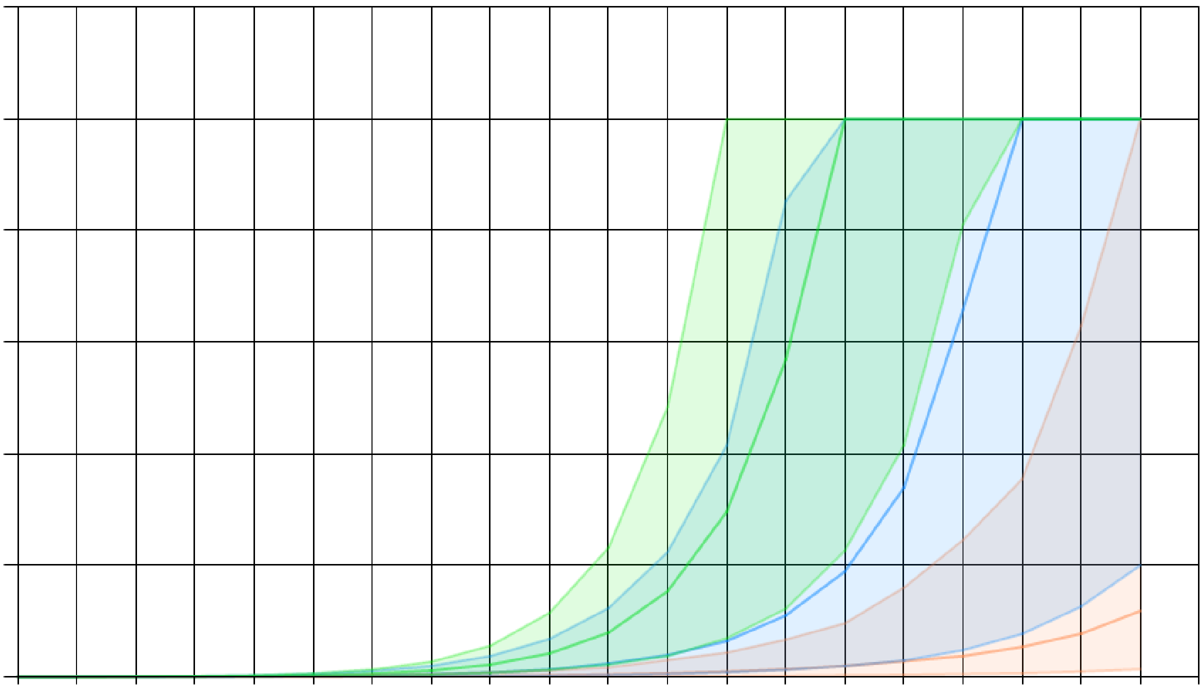
Kelly Criterion generalized

by Lorenz Auer



JKU Linz, 28 April 2022

Outline

Outline

Outline

Outline

- Introduction
- Kelly Criterion
- Generalization
- Algorithm
- Examples / Demo
- Conclusion

Outline

- Introduction
- Kelly Criterion
- Generalization
- Algorithm
- Examples / Demo
- Conclusion

Outline

- Introduction
- Kelly Criterion
- Generalization
- Algorithm
- Examples / Demo
- Conclusion

Example:

$$\text{seed capital} = 100€, \quad \text{next capital} = \text{prev capital} \begin{cases} +50\%, & \text{heads} \\ -40\%, & \text{tails} \end{cases}$$

Would you agree to play this bet 1000 times?

Example:

$$\text{seed capital} = 100€, \quad \text{next capital} = \text{prev capital} \begin{cases} +50\%, & \text{heads} \\ -40\%, & \text{tails} \end{cases}$$

Would you agree to play this bet 1000 times?

Inital Approach:

$$\begin{aligned} \text{average return} &= +5\% \\ \text{expected capital} &= 100€ \cdot 1.05^{1000} \approx 1.5 \cdot 10^{23}€ \end{aligned}$$

Initial Approach:

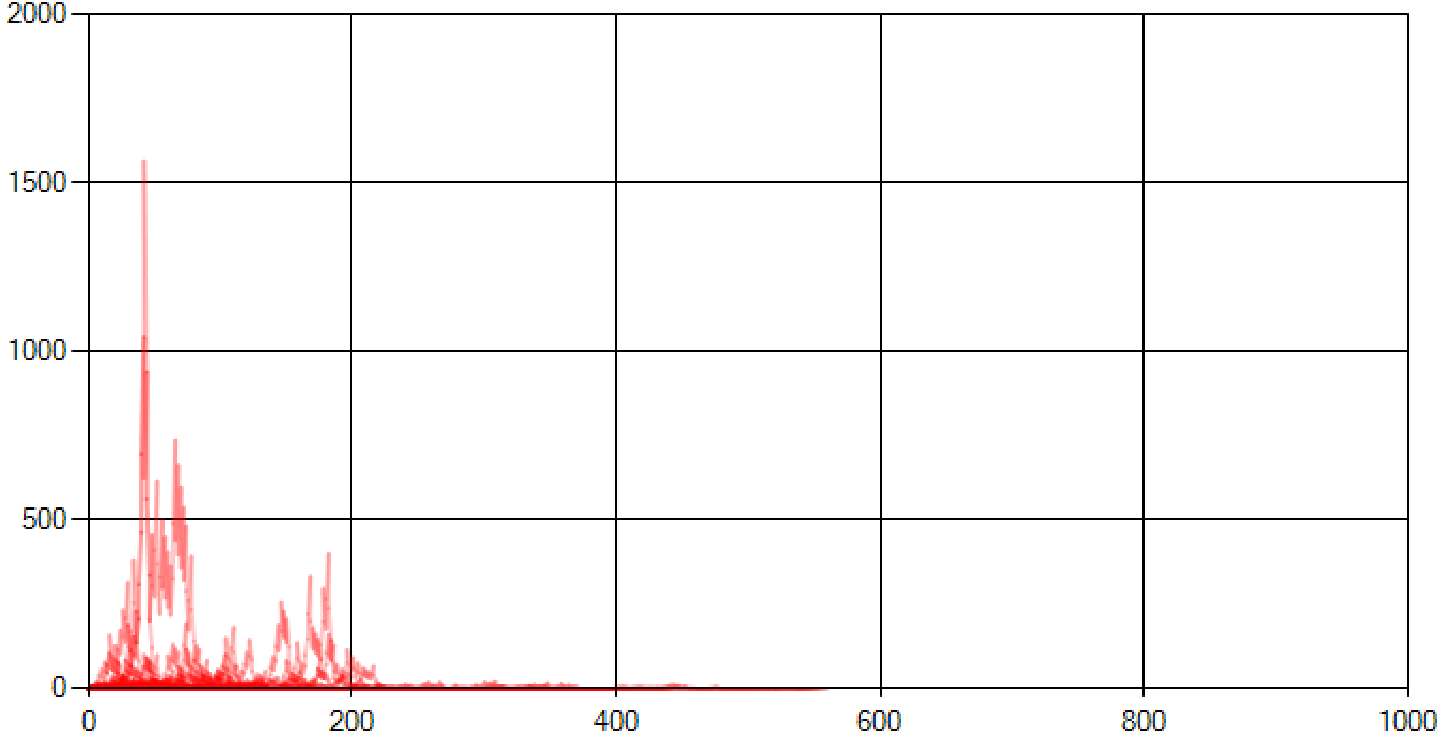


Figure 1. naive gambler, 100 runs

J.L. Kelly's Approach:

Note: wins and losses are commutative.

$$0.6 \cdot (1.5 \cdot c) = 1.5 \cdot (0.6 \cdot c)$$

J.L. Kelly's Approach:

Note: wins and losses are commutative.

$$0.6 \cdot (1.5 \cdot c) = 1.5 \cdot (0.6 \cdot c)$$

After $n \gg$ bets:

$$\underbrace{0.6}_{\text{loss}}{}^{\overbrace{n/2}^{\#\text{losses}}} \cdot \underbrace{1.5}_{\text{win}}{}^{\overbrace{n/2}^{\#\text{wins}}}$$

$$\begin{aligned} \text{expected growth} &= 1.5^{0.5} \cdot 0.6^{0.5} \approx 0.95 \\ \text{expected capital} &\approx 1.3 \cdot 10^{-21} € \end{aligned}$$

Definition. (Binary bet) *A binary bet has the following defining parameters:*

- *random variable* X *corresponding to a random event*
- *probability of winning the bet* $p \in [0, 1]$
- *odds (= payout)* $\phi > 0$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Definition. (Binary bet) *A binary bet has the following defining parameters:*

- *random variable* X *corresponding to a random event*
- *probability of winning the bet* $p \in [0, 1]$
- *odds (= payout)* $\phi > 0$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Binary Bet

Definition. (Binary bet) *A binary bet has the following defining parameters:*

- *random variable* X *corresponding to a random event*
- *probability of winning the bet* $p \in [0, 1]$
- *odds (= payout)* $\phi > 0$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Definition. (Binary bet) *A binary bet has the following defining parameters:*

- *random variable* X *corresponding to a random event*
- *probability of winning the bet* $p \in [0,1]$
- *odds (= payout)* $\phi > 0$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Definition. (Binary bet) *A binary bet has the following defining parameters:*

- *random variable* X *corresponding to a random event*
- *probability of winning the bet* $p \in [0,1]$
- *odds (= payout)* $\phi > 0$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

By placing r *on such a bet, the gambler has to pay* r *upfront. Then the random event* X *occures. If its outcome matches the desired outcome, which has the given probability* p *of happening, the gambler is payed* $\phi \cdot r$. *If the outcome does not match the desired one, the gambler gets nothing.*

Geometric mean:

$$\underbrace{(1-r)}_{\text{loss}}{}^{1-p} \cdot \underbrace{(1-r+\phi\, r)}_{\text{win}}{}^{p}$$

Kelly Criterion

Geometric mean:

$$\underbrace{(1-r)}_{\text{loss}}{}^{1-p} \cdot \underbrace{(1-r+\phi r)}_{\text{win}}{}^{p}$$

Maximize over r.

Geometric mean:

$$\underbrace{(1-r)}_{\text{loss}}{}^{1-p} \cdot \underbrace{(1-r+\phi r)}_{\text{win}}{}^{p}$$

Maximize over r.

"Kelly Criterion":

$$r = \frac{\phi p - 1}{\phi - 1}$$

1956, J. L. Kelly, *A new interpretation of information rate*

Geometric mean:

$$\underbrace{(1-r)}_{\text{loss}}{}^{1-p} \cdot \underbrace{(1-r+\phi r)}_{\text{win}}{}^{p}$$

Maximize over r.

"Kelly Criterion":

$$r = \frac{\phi p - 1}{\phi - 1}$$

1956, J. L. Kelly, *A new interpretation of information rate*

Example:

$p = 0.5$, $\phi = 2.25$

$$r = \frac{2.25 \cdot 0.5 - 1}{2.25 - 1} = \frac{0.125}{1.25} = 0.1$$

J.L. Kelly's Approach:

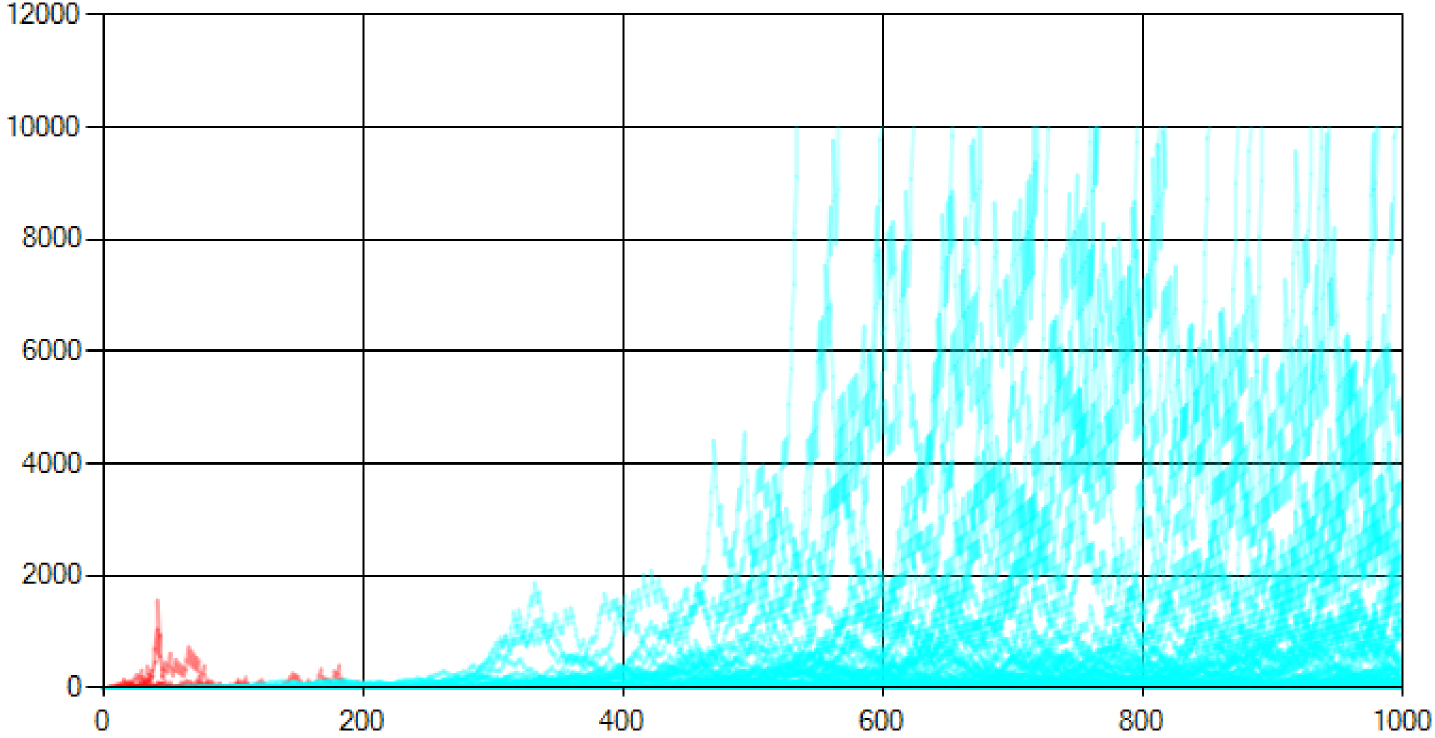


Figure 2. Simulation capped at 10^4 to prevent overflow errors

J.L. Kelly's Approach:

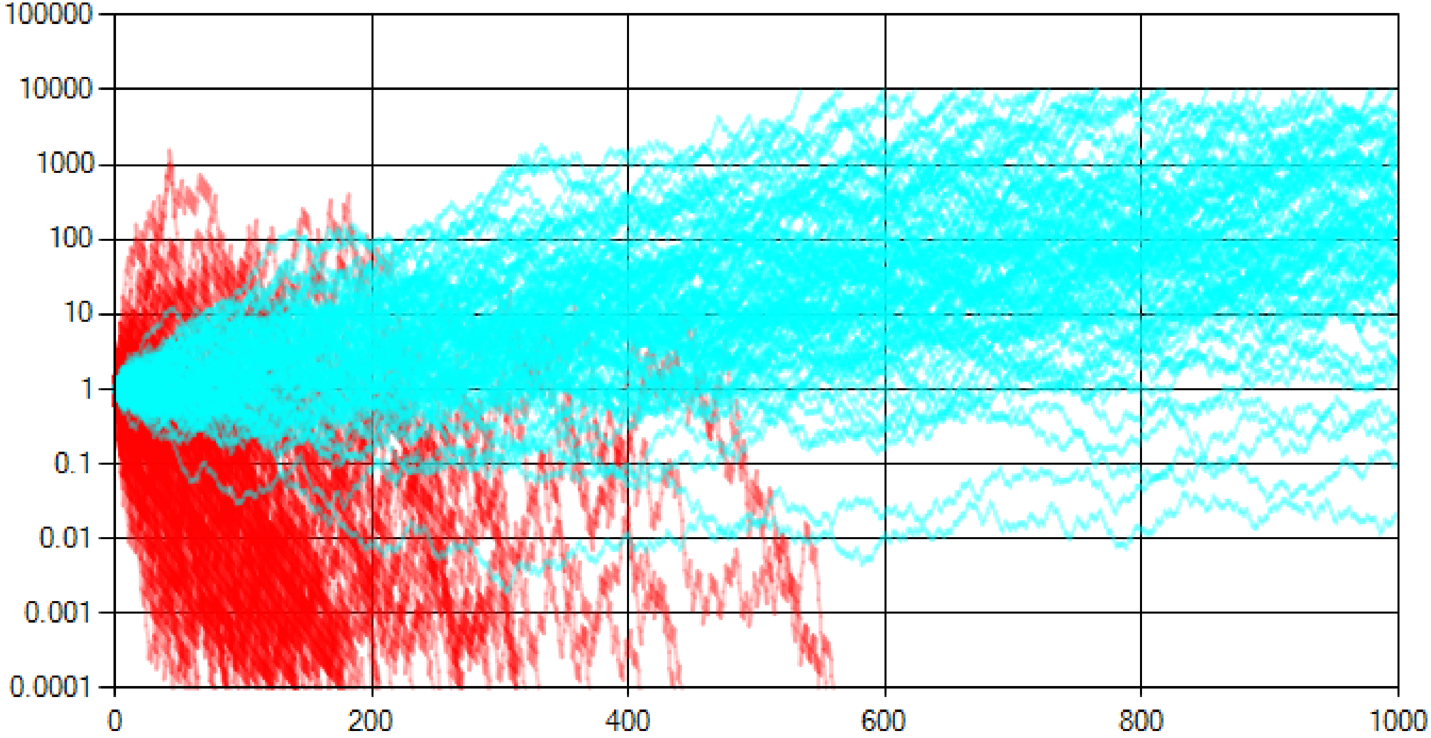


Figure 3. Logarithmic scale for better visualisation

Outline

Finite Bet

Definition. (Finite bet) *A* finite bet *has the following defining parameters:*

- *random variable* X *corresponding to a random event with* $n < \infty$ *different outcomes*
- *probabilities for each possible outcome* $p \in \mathbb{R}^n_{>0}, \|p\|_1 = 1$
- *odds (= payout) for each possible outcome* $\phi \in \mathbb{R}^n_{\geqslant 0}$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Definition. (Finite bet) *A* finite bet *has the following defining parameters:*

- *random variable* X *corresponding to a random event with* $n < \infty$ *different outcomes*
- *probabilities for each possible outcome* $p \in \mathbb{R}^n_{>0}, \|p\|_1 = 1$
- *odds (= payout) for each possible outcome* $\phi \in \mathbb{R}^n_{\geqslant 0}$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Definition. (Finite bet) *A* finite bet *has the following defining parameters:*

- *random variable* X *corresponding to a random event with* $n < \infty$ *different outcomes*
- *probabilities for each possible outcome* $p \in \mathbb{R}^n_{>0}, \|p\|_1 = 1$
- *odds (= payout) for each possible outcome* $\phi \in \mathbb{R}^n_{\geqslant 0}$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Finite Bet

Definition. (Finite bet) *A* finite bet *has the following defining parameters:*

- *random variable* X *corresponding to a random event with* $n < \infty$ *different outcomes*
- *probabilities for each possible outcome* $p \in \mathbb{R}^n_{>0}, \|p\|_1 = 1$
- *odds (= payout) for each possible outcome* $\phi \in \mathbb{R}^n_{\geqslant 0}$
- *stake* $0 \leqslant r \leqslant 1$ *chosen by the gambler in percent of their assets*

Finite Bet

Definition. (Finite bet) *A* finite bet *has the following defining parameters:*

- *random variable X corresponding to a random event with $n < \infty$ different outcomes*
- *probabilities for each possible outcome $p \in \mathbb{R}^n_{>0}, \|p\|_1 = 1$*
- *odds (= payout) for each possible outcome $\phi \in \mathbb{R}^n_{\geqslant 0}$*
- *stake $0 \leqslant r \leqslant 1$ chosen by the gambler in percent of their assets*

By placing r currency on such a bet, the gambler has to pay r upfront. Then the random event X occures and the index $1 \leqslant i \leqslant n$ is chosen according to p. The gambler then recieves $\phi_i \cdot r$ currency.

Assuming independence:

$$G(r) := \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} \left(\phi_{j,i_j} - 1 \right) r_j \right)^{\prod_{j=1}^{m} p_{j,i_j}} \qquad \text{with } \|r\|_1 \leqslant 1$$

i ... vector describing the outcome of all bets.

Assuming independence:

$$G(r) := \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} \left(\phi_{j,i_j} - 1\right) r_j \right)^{\prod_{j=1}^{m} p_{j,i_j}} \qquad \text{with } \|r\|_1 \leqslant 1$$

i ... vector describing the outcome of all bets.

$$R := \{ r \in \mathbb{R}^m_{\geqslant 0} \mid \|r\|_1 < 1 \}$$

$\overline{R}$ ... all possible bets the gambler could take.

Assuming independence:

$$G(r) := \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} \left(\phi_{j,i_j} - 1\right) r_j\right)^{\prod_{j=1}^{m} p_{j,i_j}} \qquad \text{with } \|r\|_1 \leqslant 1$$

i ... vector describing the outcome of all bets.

$$R := \{r \in \mathbb{R}^m_{\geqslant 0} \mid \|r\|_1 < 1\}$$

$\overline{R}$ ... all possible bets the gambler could take.

$$\max_{r \in \overline{R}} G(r)$$

Optimization Problem: Multiple Finite Bets

Assuming independence:

$$G(r) := \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} \left(\phi_{j,i_j} - 1\right) r_j\right)^{\prod_{j=1}^{m} p_{j,i_j}} \qquad \text{with } \|r\|_1 \leqslant 1$$

i ... vector describing the outcome of all bets.

$$R := \{r \in \mathbb{R}^m_{\geqslant 0} \mid \|r\|_1 < 1\}$$

$\overline{R}$ ... all possible bets the gambler could take.

$$\max_{r \in \overline{R}} G(r)$$

Note: $\overline{R}$ is convex.

Goal: Proof $G(r)$ is concave.

Lemma 1

Lemma 1. *Let* $a, b \in \mathbb{R}_{>0}$ *and* $p \in [0, 1]$ *then*

$$(1 + a)^p (1 + b)^{1 - p} \geqslant 1 + a^p\, b^{1 - p}$$

holds.

Lemma 2. *Let* $a,b\in\mathbb{R}_{>0}$ *and* $p\in[0,1]$ *then*

$$(1+a)^p(1+b)^{1-p}\geqslant 1+a^p\,b^{1-p}$$

holds.

Proof. Let $a,b\in\mathbb{R}_{>0}$. (w.l.o.g.: $a\geqslant b$)

$$\begin{aligned} f(p) &:= (1+a)^p(1+b)^{1-p}-1-a^p\,b^{1-p} \\ &= (1+b)\left(\frac{1+a}{1+b}\right)^p-1-b\left(\frac{a}{b}\right)^p \end{aligned}$$

by showing $f(p)\geqslant 0$ the statement will be proven. Note: $f(p)\in C^\infty(\mathbb{R})$.

For $a=b$:

$$f(p)=(1+b)-1-b=0$$

For $a \neq b$:

$$
\begin{aligned}
f(0) &= (1+b)\left(\frac{1+a}{1+b}\right)^0 - 1 - b\left(\frac{a}{b}\right)^0 = (1+b) - 1 - b = 0 \\
f(1) &= (1+b)\left(\frac{1+a}{1+b}\right)^1 - 1 - b\left(\frac{a}{b}\right)^1 = (1+a) - 1 - a = 0
\end{aligned}
$$

It is also known that:

$$
\begin{array}{rcl}
a > & b & > 0 \\
\dfrac{a}{b} & > & 1 \\
\dfrac{a}{b} > & \dfrac{1+a}{1+b} & > 1 \\
\ln\left(\dfrac{a}{b}\right) > & \ln\left(\dfrac{1+a}{1+b}\right) & > 0
\end{array}
$$

Lemma 1

Given a and b using the first derivative

$$\frac{\partial}{\partial p} f(p) = \underbrace{(1+b)\ln\left(\frac{1+a}{1+b}\right)\left(\frac{1+a}{1+b}\right)^p}_{>0} - \underbrace{b\ln\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)^p}_{>0}$$

one can explicitly calculate p' with:

$$\frac{\partial}{\partial p} f(p') = 0$$

It follows that:

$$\begin{aligned}
\frac{\partial^2}{\partial p^2} f(p') &= \underbrace{(1+b)\ln\left(\frac{1+a}{1+b}\right)^2\left(\frac{1+a}{1+b}\right)^{p'}}_{>0} - b\ln\left(\frac{a}{b}\right)^2\left(\frac{a}{b}\right)^{p'} \\
&< (1+b)\ln\left(\frac{1+a}{1+b}\right)\ln\left(\frac{a}{b}\right)\left(\frac{1+a}{1+b}\right)^{p'} - b\ln\left(\frac{a}{b}\right)^2\left(\frac{a}{b}\right)^{p'} \\
&= \ln\left(\frac{a}{b}\right)\left((1+b)\ln\left(\frac{1+a}{1+b}\right)\left(\frac{1+a}{1+b}\right)^{p'} - b\ln\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)^{p'}\right) \\
&= \ln\left(\frac{a}{b}\right) * \frac{\partial}{\partial p} f(p') = 0
\end{aligned}$$

Lemma 1

$$\frac{\partial^2}{\partial p^2} f(p') < 0$$

$f(p)$ has exactly one extremum which is a maximum at p'. Due to the mean value theorem it follows with $f(0) = 0$ and $f(1) = 0$ that $f(p) > 0$ for $p \in (0, 1)$. $\square$

Lemma 3. *Let* $n \in \mathbb{N}$. $a_1, \ldots, a_n \in \mathbb{R}_{>0}$. $b_1, \ldots, b_n \in \mathbb{R}_{>0}$. $p_1, \ldots, p_n \in \mathbb{R}_{\geqslant 0}$ *with* $\sum_{i=1}^{n} p_i = 1$. *Then*

$$\prod_{i=1}^{n} (a_i + b_i)^{p_i} \geqslant \prod_{i=1}^{n} a_i^{p_i} + \prod_{i=1}^{n} b_i^{p_i}$$

Lemma 4. *Let* $n \in \mathbb{N}$. $a_1, \ldots, a_n \in \mathbb{R}_{>0}$. $b_1, \ldots, b_n \in \mathbb{R}_{>0}$. $p_1, \ldots, p_n \in \mathbb{R}_{\geqslant 0}$ *with* $\sum_{i=1}^n p_i = 1$. *Then*

$$\prod_{i=1}^n (a_i + b_i)^{p_i} \geqslant \prod_{i=1}^n a_i^{p_i} + \prod_{i=1}^n b_i^{p_i}$$

Proof. With $c_i = \frac{b_i}{a_i} > 0$ we can simplify:

$$\begin{aligned}
\prod_{i=1}^n (a_i + b_i)^{p_i} &\geqslant \prod_{i=1}^n a_i^{p_i} + \prod_{i=1}^n b_i^{p_i} \\
\frac{\prod_{i=1}^n (a_i + b_i)^{p_i}}{\underbrace{\prod_{i=1}^n a_i^{p_i}}_{>0}} &\geqslant \frac{\prod_{i=1}^n a_i^{p_i}}{\prod_{i=1}^n a_i^{p_i}} + \frac{\prod_{i=1}^n b_i^{p_i}}{\prod_{i=1}^n a_i^{p_i}} \\
\prod_{i=1}^n \frac{(a_i + b_i)^{p_i}}{a_i^{p_i}} &\geqslant 1 + \prod_{i=1}^n \frac{b_i^{p_i}}{a_i^{p_i}} \\
\prod_{i=1}^n \left(\frac{a_i + b_i}{a_i}\right)^{p_i} &\geqslant 1 + \prod_{i=1}^n \left(\frac{b_i}{a_i}\right)^{p_i}
\end{aligned}$$

Lemma 2

$$\prod_{i=1}^{n}\left(\frac{a_i+b_i}{a_i}\right)^{p_i} \geqslant 1+\prod_{i=1}^{n}\left(\frac{b_i}{a_i}\right)^{p_i}$$

$$\prod_{i=1}^{n}\left(1+\frac{b_i}{a_i}\right)^{p_i} \geqslant 1+\prod_{i=1}^{n}\left(\frac{b_i}{a_i}\right)^{p_i}$$

$$\prod_{i=1}^{n}(1+c_i)^{p_i} \geqslant 1+\prod_{i=1}^{n} c_i^{p_i}$$

If $\exists i'\colon p_{i'}=1$:

$$\prod_{i=1}^{n}(1+c_i)^{p_i} \geqslant 1+\prod_{i=1}^{n} c_i^{p_i}$$

$$(1+c_{i'})^1 \geqslant 1+c_{i'}^1$$

$$1+c_{i'} = 1+c_{i'}$$

Induction start $n=1$: Since $p_1=1$ the inequaltiy holds due to the statement above.

Induction step $n \to n+1$:

Assume $\forall i\colon p_i \neq 1$.

$$\begin{aligned}
\prod_{i=1}^{n+1} (1+c_i)^{p_i} &= (1+c_{n+1})^{p_{n+1}} \prod_{i=1}^{n} (1+c_i)^{p_i} \\
&= (1+c_{n+1})^{p_{n+1}} \left(\prod_{i=1}^{n} (1+c_i)^{\overbrace{\frac{p_i}{1-p_{n+1}}}^{\sum_i = 1}} \right)^{\underbrace{1-p_{n+1}}_{\neq 0}} \\
\text{apply induction:} &\geqslant (1+c_{n+1})^{p_{n+1}} \left(1 + \prod_{i=1}^{n} c_i^{\frac{p_i}{1-p_{n+1}}} \right)^{1-p_{n+1}} \\
\text{Lemma 2:} &\geqslant 1 + c_{n+1}^{p_{n+1}} \cdot \prod_{i=1}^{n} c_i^{\frac{p_i \cdot (1-p_{n+1})}{1-p_{n+1}}} \\
&= 1 + c_{n+1}^{p_{n+1}} \cdot \prod_{i=1}^{n} c_i^{p_i} = 1 + \prod_{i=1}^{n+1} c_i^{p_i} \qquad \square
\end{aligned}$$

Theorem 5. *G(r) is concave on* $\overline{R}$.

Theorem 6. *G(r) is concave on* $\overline{R}$.

Proof. Let $r, r' \in R$ and $t \in (0,1)$. $r'' = r\,t + r'(1-t)$.

$$
\begin{aligned}
G(r'') &= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(\underbrace{1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1)\, r''_j}_{>0} \right)^{\prod_{j=1}^{m} p_{j,i_j}} \\
&= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1)(r_j\, t + r'_j\,(1-t)) \right)^{\prod_{j=1}^{m} p_{j,i_j}} \\
&= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + t \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r_j + (1-t) \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r'_j \right)^{\prod_{j=1}^{m} p_{j,i_j}} \\
&= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(\underbrace{t \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r_j \right)}_{>0} + \underbrace{(1-t) \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r'_j \right)}_{>0} \right)^{\underbrace{\prod_{j=1}^{m} p_{j,i_j}}_{>0, \sum_i = 1}}
\end{aligned}
$$

$$
\begin{aligned}
G(r'') &= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} \left(\underbrace{t\left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r_j\right)}_{>0} + \underbrace{(1-t)\left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r'_j\right)}_{>0} \right)^{\underbrace{\prod_{j=1}^{m} p_{j,i_j}}_{>0, \sum_i = 1}} \\
\text{Lem. } 4 \quad &\geqslant \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} \left(t\left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r_j\right)\right)^{\prod_{j=1}^{m} p_{j,i_j}} + \\
&\quad + \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} \left((1-t)\left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r'_j\right)\right)^{\prod_{j=1}^{m} p_{j,i_j}} \\
&= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} t^{\prod_{j=1}^{m} p_{j,i_j}} \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r_j\right)^{\prod_{j=1}^{m} p_{j,i_j}} + \\
&\quad + \prod_{\substack{i \in \mathbb{N}^m \\ \forall j : 1 \leqslant i_j \leqslant n_j}} (1-t)^{\prod_{j=1}^{m} p_{j,i_j}} \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1) r'_j\right)^{\prod_{j=1}^{m} p_{j,i_j}}
\end{aligned}
$$

Theorem 3

$$
\begin{aligned}
G(r'') \;\geqslant\; & \prod_{\substack{i\in\mathbb{N}^m\\ \forall j:1\leqslant i_j\leqslant n_j}} t^{\prod_{j=1}^m p_{j,i_j}} \left(1+\sum_{j=1}^m (\phi_{j,i_j}-1)r_j\right)^{\prod_{j=1}^m p_{j,i_j}} + \\
& + \prod_{\substack{i\in\mathbb{N}^m\\ \forall j:1\leqslant i_j\leqslant n_j}} (1-t)^{\prod_{j=1}^m p_{j,i_j}} \left(1+\sum_{j=1}^m (\phi_{j,i_j}-1)r'_j\right)^{\prod_{j=1}^m p_{j,i_j}} \\
= \;& t \prod_{\substack{i\in\mathbb{N}^m\\ \forall j:1\leqslant i_j\leqslant n_j}} \left(1+\sum_{j=1}^m (\phi_{j,i_j}-1)r_j\right)^{\prod_{j=1}^m p_{j,i_j}} + \\
& +(1-t) \prod_{\substack{i\in\mathbb{N}^m\\ \forall j:1\leqslant i_j\leqslant n_j}} \left(1+\sum_{j=1}^m (\phi_{j,i_j}-1)r'_j\right)^{\prod_{j=1}^m p_{j,i_j}} \\
= \;& t\,G(r)+(1-t)\,G(r')
\end{aligned}
$$

$G(r)$ is concave on R. $G(r)$ consists of a finite amount of additions and multiplications, so $G(r)$ is continuous. Further $\lim_{\|r\|_1\to 1} G(r) < \infty$, therefore the result over R can be continued to $\overline{R}$. $G(r)$ is concave on $\overline{R}$. $\square$

Corollary 7. *Over* $\overline{R}$ *any local maximum of* $G(r)$ *is a global maximum.*

Corollary 8. *Over $\overline{R}$ any local maximum of $G(r)$ is a global maximum.*

Proof. Implied directly, since $\overline{R}$ is convex and $G(r)$ is concave (Theorem 6). □

Optimization algorithms converge to gobal maximum. :)

Outline

Choice: Gradient descent algorithm

$$r_{i+1} = r_i + t\nabla G(r_i) \quad t > 0$$

Implementation

Choice: Gradient descent algorithm

$$r_{i+1} = r_i + t\nabla G(r_i) \quad t > 0$$

Pitfall: What happens if $r_{i+1} \notin \overline{R}$?

Implementation

Choice: Gradient descent algorithm

$$r_{i+1} = r_i + t\nabla G(r_i) \quad t > 0$$

Pitfall: What happens if $r_{i+1} \notin \overline{R}$?

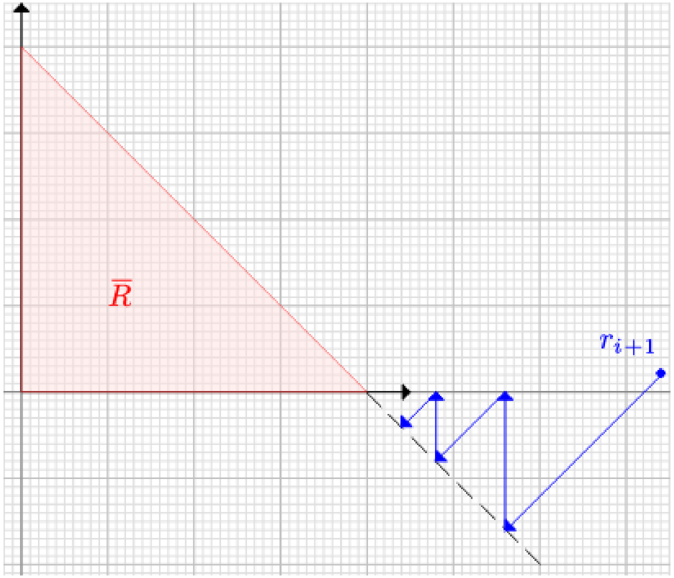
- Lower t until ok?
- Project onto surface $\partial\overline{R}$?

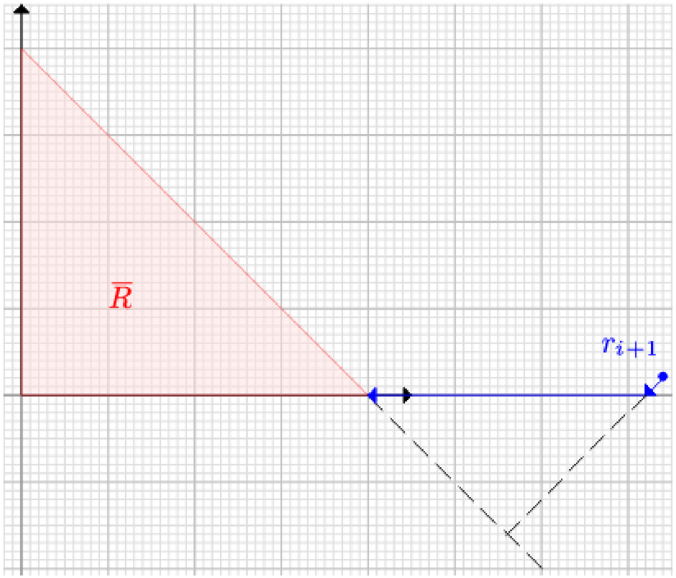
Choice: Gradient descent algorithm

$$r_{i+1} = r_i + t\nabla G(r_i) \quad t > 0$$

Pitfall: What happens if $r_{i+1} \notin \overline{R}$?

- Lower t until ok?
- Project onto surface $\partial\overline{R}$?





Calculate gradient:

$$\frac{\partial G}{\partial r_k}(r) \;=\; G(r)\cdot \sum_{\substack{i\in\mathbb{N}^m,\\ \forall j:1\leqslant i_j\leqslant n_j}} \left((\phi_{k,i_k}-1)\cdot\left(1+\sum_{j=1}^{m}(\phi_{j,i_j}-1)\,r_j\right)^{-1}\cdot\prod_{j=1}^{m}p_{j,i_j}\right)$$

Calculate gradient:

$$\frac{\partial G}{\partial r_k}(r) = G(r) \cdot \sum_{\substack{i \in \mathbb{N}^m, \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left((\phi_{k,i_k} - 1) \cdot \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1)\, r_j \right)^{-1} \cdot \prod_{j=1}^{m} p_{j,i_j} \right)$$

Note $\frac{\partial G}{\partial r_k}(r)$ is only continuous on R. There could exist $r' \in \overline{R}$ such that

$$\exists i' \in \mathbb{N}^m \colon 1 + \sum_{j=1}^{m} (\phi_{j,i'_j} - 1)\, r'_j = 0.$$

For such r' we get

$$\begin{aligned} G(r') &= \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j}} \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1)\, r'_j \right)^{\prod_{j=1}^{m} p_{j,i_j}} \\ &= \left(\underbrace{1 + \sum_{j=1}^{m} (\phi_{j,i'_j} - 1)\, r'_j}_{=0} \right)^{\prod_{j=1}^{m} p_{j,i'_j}} \prod_{\substack{i \in \mathbb{N}^m \\ \forall j: 1 \leqslant i_j \leqslant n_j \\ i \neq i'}} \left(1 + \sum_{j=1}^{m} (\phi_{j,i_j} - 1)\, r'_j \right)^{\prod_{j=1}^{m} p_{j,i_j}} = 0 \end{aligned}$$

If $r_0 = 0, G(r_0) = 1$ and $G(r_{i+1}) > G(r_i)$, $\frac{\partial G}{\partial r_k}(r)$ can always be evaluated.

Demostration:

- Optimization
- Simulation

Examlpes / Demo

Dana and Bill's fields have $\phi = (4, 3, 2, 1, 0)$ and $p = (0.2, 0.3, 0.25, 0.15, 0.1)$.

Ann and Carl's fields have $\phi = (3, 2, 1, 0)$ and $p = (0.3, 0.4, 0.2, 0.1)$.

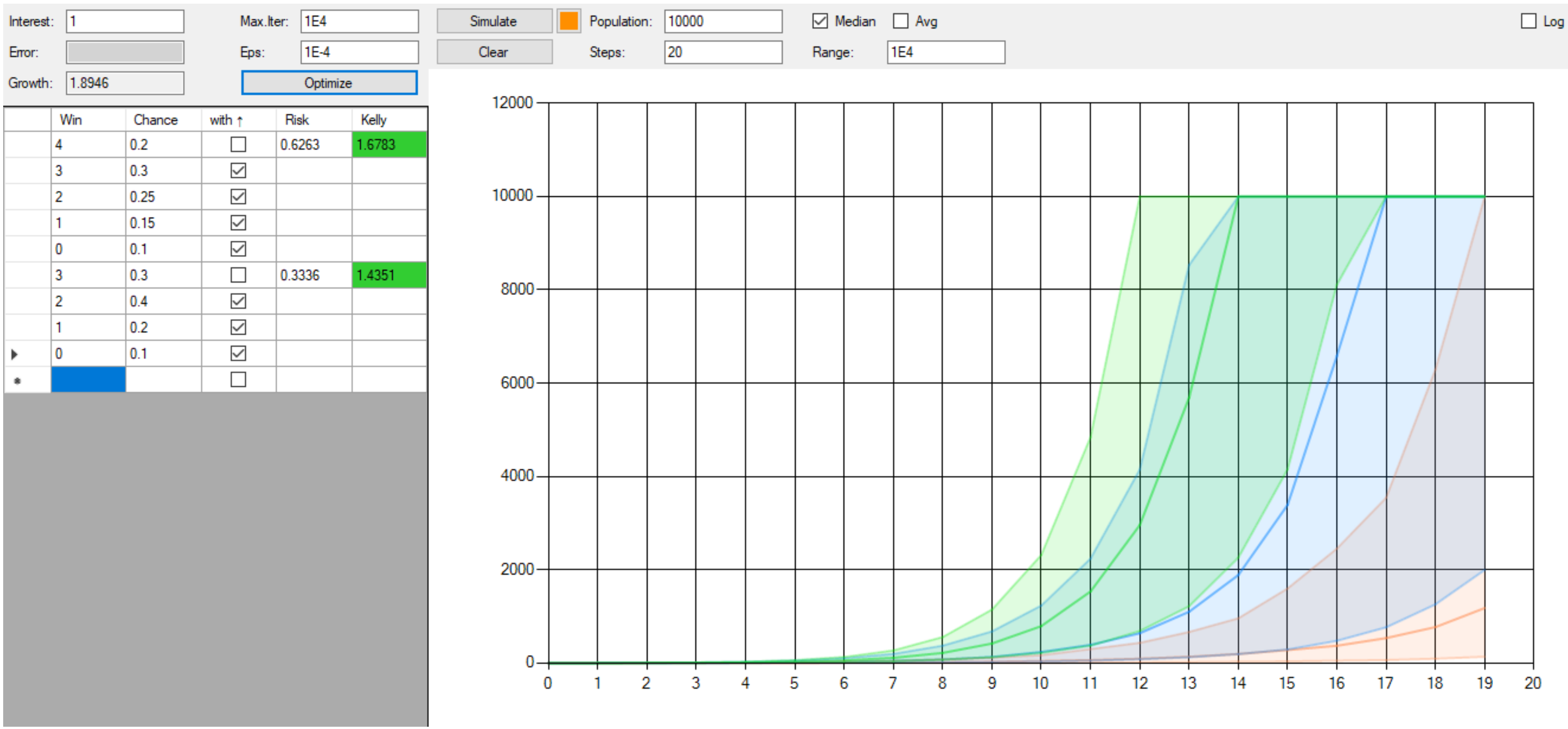


Figure 4. Ann (orange), Bill (blue), Dana and Carl (green)

Conclusion

I have developed a **proof** and an **algorithm** for applying the Kelly Criterion to multiple bets with finite outcomes.

Optimal strategies often are unintuitive.

Comparsions between those strategies yield surprising results.

Paper, software and slides are avaliabe at traxar.github.io.