An SEIR Epidemic Model with Vaccination of Newborns, Vaccination Effectiveness and Waning Immunity

Zhiwen Cao

October 2021

1 Introduction

A SEIR model divides total population into 4 subpopulations. In addition to the three groups susceptible, infectious, and recovered that are in the SIR model, there is an extra compartment E which stands for exposed. Individuals in this subpopulation are already infected but are not infectious, thus they are unable to transmit the disease.

The modified model that is discussed by this report is adapted from the paper *On an SEIR Epidemic Model with Vaccination of Newborns and Periodic Impulsive Vaccination with Eventual On-Line Adapted Vaccination Strategies to the Varying Levels of the Susceptible Subpopulation*, which investigates a SEIR epidemic model under two vaccination effort strategies. The first strategy is to introduce a new variable, fraction of vaccinated newborns, to the SEIR model. The other one is to implement periodic impulsive vaccination. This report will focus on the first strategy proposed by [1].

2 Model explanation

2.1 Original Model

[1] considers that the population under study is homogeneously mixed and that all susceptible individuals have the same probability of being infected. The mathematical representation of the SEIR model considered in this case is given by the following system of differential equations:

$$\begin{aligned} \dot{S} &= (1-q)A - (\beta I + \mu)S \\ \dot{E} &= \beta SI - (\mu + \sigma)E \\ \dot{I} &= \sigma E - (\mu + \gamma + \alpha)I \\ \dot{R} &= qA + \gamma I - \mu R \\ S(0) \geq 0, E(0) &\geq 0, I(0) \geq 0, R(0) \geq 0 \end{aligned} \tag{1}$$

The sum of four subpopulations at the initial time is assumed to be unity, so that the subpopulations of the model are fractions of the total initial population. The symbols used for each variable or parameter are as follows (with the assumption that all constants are non-negative):

S: Proportion of susceptible individuals
E: Proportion of exposed individuals
I: Proportion of infectious individuals
R: Proportion of recovered individuals
A: Proportion of newborn individuals per unit of time
β: Transmission rate
μ: Natural mortality rate
σ: Inverse of the latent period
γ: Recovery rate

α: Mortality rate caused by infection
q: Fraction of vaccinated newborns

Note when certain constants are 0 there is not much to analyze or model.
We can note the following observations by [1] (page 5) (all of the following is per one unit of time):

βSI: Proportion of individuals who were originally exposed but are no longer susceptible
μS: Proportion of individuals who die from causes not related to the infection
σE: Proportion of individuals that are infected and susceptible which are removed from βSI
μE: Proportion of individuals who are exposed but die of unrelated causes
γI: Proportion of individuals who recover from the disease
αI: Proportion of individuals who die from the infection
γI: Proportion of individuals who enter the recovered group
μR: Proportion of individuals who are withdrawn from the recovered group due to natural causes

2.2 Modification

The model proposed in [1] assumes that every vaccine is effective against infection and that newborns receive lifetime immunity upon vaccination. As such, the original model directly adds the proportion of vaccinated newborns into the recovered group, and also removes that proportion from the susceptible group. However, in reality vaccine cannot guarantee perpetual immunity and there is a risk for the recovered being infected again. Furthermore, certain individuals may not be able to reach the threshold of antibodies to develop immunity after vaccination. Therefore, In addition to the above parameters, the modified model adds two parameters ρ and ϕ. ρ indicates the proportion of vaccinated newborns who successfully developed immunity, and ϕ indicates rate at which recovered individuals lose immunity and enter the susceptible subpopulation. For simplicity, the impact of each newly added parameter is studied separately with fixed q. $\dot{N}$, obtained by summing up $\dot{S}$, $\dot{E}$, $\dot{I}$, and $\dot{R}$, is also included to investigate how the total population changes when certain parameters change.
Therefore, the following system of equations is the modified version

$$\begin{aligned}
\dot{S} &= (1-\rho q)A - (\beta I + \mu)S + \phi R \\
\dot{E} &= \beta SI - (\mu+\sigma)E \\
\dot{I} &= \sigma E - (\mu+\gamma+\alpha)I \\
\dot{R} &= \rho q A + \gamma I - \mu R - \phi R \\
\dot{N} &= A - N\mu - \alpha I \\
S(0) \geq 0, E(0) &\geq 0, I(0) \geq 0, R(0) \geq 0
\end{aligned} \tag{2}$$

3 Model Analysis

3.1 Original Model

[1] claims that for any $q \in [0,1]$, The solution of the model is non-negative for all time for any given non-negative initial conditions. In addition, If $\lim_{x\to\infty} supI(t) \leq Amin(1/\alpha, 1/\mu)$, then the solution is bounded as well. This ensures the global stability of the model with non-negative initial conditions.

According to [1], The disease free equilibrium is obtained by setting all equations in (2) to 0 and setting I to be zero.

$$\begin{aligned}
EQ^0(q) &= (S^0(q), E^0, I^0, R^0(q)) \\
&= (\frac{(1-q)A}{\mu}, 0, 0, \frac{qA}{\mu})
\end{aligned} \tag{3}$$

The endemic equilibrium is obtained by setting all equations in (2) to 0 while keeping a non-zero I.

$$\begin{aligned} EQ^*(q) &= (S^*(q), E^*(q), I^*(q), R^*(q)) \\ where \quad S^*(q) &= \frac{(\mu+\sigma)(\mu+\gamma+\alpha)}{\beta\sigma} \\ E^*(q) &= \frac{(1-q)A}{\mu+\sigma} - \frac{\mu(\mu+\gamma+\alpha)}{\sigma\beta} \\ I^*(q) &= \frac{(1-q)\sigma A}{(\mu+\sigma)(\mu+\gamma+\alpha)} - \frac{\mu}{\beta} \\ R^*(q) &= \frac{A}{\mu}(\frac{(1-q)\gamma\sigma}{(\mu+\sigma)(\mu+\gamma+\alpha)} + q) - \frac{\gamma}{\beta} \end{aligned} \tag{4}$$

The endemic equilibrium is only obtainable if the transmission rate is larger than a critical value, such that

$$\beta \geq \beta_c(q) = \frac{\mu(\mu+\sigma)(\mu+\gamma+\alpha)}{A\sigma(1-q)}$$

If the transmission rate equals the critical value then the endemic equilibrium point coincides with the disease free equilibrium point.

3.2 Modified Model

Using the similar method, the modified model has two equilibrium points as well

$$\begin{aligned} EQ^0(\rho,\phi) &= (S^0(\rho,\phi), E^0, I^0, R^0(\rho,\phi)) \\ &= (\frac{A(1-\rho q)}{\mu} + \frac{\phi\rho q A}{\mu(\mu+\phi)}, 0, 0, \frac{\rho q A}{\mu+\phi}) \end{aligned} \tag{5}$$

The endemic equilibrium is obtained by setting all equations in (2) to 0 while keeping a non-zero I.

$$\begin{aligned} EQ^*(\rho,\phi) &= (S^*(\rho,\phi), E^*(\rho,\phi), I^*(\rho,\phi), R^*(\rho,\phi)) \\ S^*(\rho,\phi) &= \frac{(1-\rho q)A + \phi R^*}{\beta I^* + \mu} \\ E^*(\rho,\phi) &= \frac{\beta S^* I^*}{\mu+\sigma} \\ I^*(\rho,\phi) &= \frac{\sigma E^*}{\mu+\gamma+\alpha} \\ R^*(\rho,\phi) &= \frac{\rho q A + \gamma I^*}{\mu+\phi} \end{aligned} \tag{6}$$

Calculation:

$$I^* = \frac{\frac{\sigma\beta S^* I^*}{\mu+\sigma}}{\mu+\gamma+\alpha}$$

$$1 = \frac{\sigma\beta S^*}{(\mu+\sigma)(\mu+\gamma+\alpha)}$$

$$S^* = \frac{B}{\sigma\beta}$$

$$\text{where} \quad B = (\mu+\sigma)(\mu+\gamma+\alpha)$$

$$\frac{B}{\sigma\beta} = \frac{(1-\rho q)A + \phi R^*}{\beta I^* + \mu}$$

$$\phi R^* = \frac{\phi\rho q A + \phi\gamma I^*}{\mu+\phi}$$

$$\frac{B}{\sigma\beta} = \frac{(1-\rho q)A + \frac{\phi\rho q A + \phi\gamma I^*}{\mu+\phi}}{\beta I^* + \mu}$$

Simplification yields :

$$I^* = \frac{BD - C\sigma\beta}{G - BF}$$

$$\text{where} \quad C = \mu A - \mu\rho q A + \phi A$$

$$D = \mu^2 + \phi\mu$$

$$F = \mu\beta + \phi\beta$$

$$G = \sigma\beta\phi\gamma$$

$$R^* = \frac{\rho q A + \frac{\gamma BD - \gamma C\sigma\beta}{G-BF}}{\mu+\phi}$$

$$= \frac{\rho q A}{\mu+\phi} + \frac{\gamma BD - \gamma C\sigma\beta}{(\mu+\phi)(G-BF)}$$

$$E^* = \frac{\beta\frac{B}{\sigma\beta}\frac{BD-C\sigma\beta}{G-BF}}{\mu+\sigma}$$

Summary:

$$
\begin{aligned}
S^*(\rho,\phi) &= \frac{B}{\sigma\beta} \\
E^*(\rho,\phi) &= \frac{B(BD - C\sigma\beta)}{\sigma(G - BF)(\mu+\sigma)} \\
I^*(\rho,\phi) &= \frac{BD - C\sigma\beta}{G - BF} \\
R^*(\rho,\phi) &= \frac{\rho q A}{\mu+\phi} + \frac{\gamma BD - \gamma C\sigma\beta}{(\mu+\phi)(G-BF)} \\
\text{where} \quad B &= (\mu+\sigma)(\mu+\gamma+\alpha) \\
C &= \mu A - \mu\rho q A + \phi A \\
D &= \mu^2 + \phi\mu \\
F &= \mu\beta + \phi\beta \\
G &= \sigma\beta\phi\gamma
\end{aligned}
\tag{7}
$$

By observation, if $\beta = \frac{BD}{C\sigma}$, then the endemic equilibrium point is the same as the disease-free one. In addition, because $G - BF < 0$, in order for the endemic equilibrium point to exist, the following inequality needs to be satisfied:

$$
\beta > \beta_c(\rho,\phi) = \frac{BD}{C\sigma} \tag{8}
$$

The Jacobian matrix is:

$$
J = \begin{bmatrix}
-\beta I^* - \mu & 0 & -\beta S^* & \phi \\
\beta I^* & -(\mu+\sigma) & \beta S^* & 0 \\
0 & \sigma & -(\mu+\gamma+\alpha) & 0 \\
0 & 0 & \gamma & -(\mu+\phi)
\end{bmatrix} \tag{9}
$$

Compute the Jacobian matrix at the disease free equilibrium point;

$$
J_{EQ_0} = \begin{bmatrix}
-\mu & 0 & -\beta(\frac{A(1-\rho q)}{\mu} + \frac{\phi\rho q A}{\mu(\mu+\phi)}) & \phi \\
0 & -(\mu+\sigma) & \beta(\frac{A(1-\rho q)}{\mu} + \frac{\phi\rho q A}{\mu(\mu+\phi)}) & 0 \\
0 & \sigma & -(\mu+\gamma+\alpha) & 0 \\
0 & 0 & \gamma & -(\mu+\phi)
\end{bmatrix} \tag{10}
$$

Then calculate the eigenvalue:

$$
\begin{vmatrix}
-\mu-\lambda & 0 & -X & \phi \\
0 & -Y-\lambda & X & 0 \\
0 & \sigma & -Z-\lambda & 0 \\
0 & 0 & \gamma & -W-\lambda
\end{vmatrix} = 0 \tag{11}
$$

$$
\begin{aligned}
where \quad X &= \beta(\frac{A(1-\rho q)}{\mu} + \frac{\phi\rho q A}{\mu(\mu+\phi)}) \\
Y &= \mu+\sigma \\
Z &= \mu+\gamma+\alpha \\
W &= \mu+\phi
\end{aligned}
$$

Which expands as

$$
\begin{aligned}
(-\mu-\lambda)[(-Y-\lambda)(-Z-\lambda)(-W-\lambda) - \sigma(X(-W-\lambda))] = 0 \\
(\mu+\lambda)[-(Y+\lambda)(Z+\lambda)(W+\lambda) + \sigma(X(W+\lambda))] = 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\lambda_1 &= -\mu < 0 \\
-(Y+\lambda)(Z+\lambda)(W+\lambda) + \sigma(X(W+\lambda)) &= 0
\end{aligned}
\tag{13}
$$

Rearrange:

$$\lambda^3 + (Y+Z+W)\lambda^2 + (YZ+YW+ZW-\sigma X)\lambda + YZW - \sigma XW = 0 \tag{14}$$

Because without actual value, it is difficult to calculate λ. The Routh–Hurwitz criterion for third order polynomials described in [2] is applied with coefficients:

$$\begin{aligned} a_1 &= Y+Z+W \\ a_2 &= YZ+YW+ZW-\sigma X \\ a_3 &= YZW-\sigma XW \end{aligned} \tag{15}$$

Based on the Routh-Hurwitz criterion for third order polynomials, the three conditions:

$$\begin{aligned} a_1 &> 0 \\ a_3 &> 0 \\ a_1 a_2 &> a_3 \end{aligned}$$

must be satisfied for the system to be locally stable at disease-free equilibrium point.
$a_1 > 0$ is satisfied because $Y > 0, Z > 0$, and $W > 0$.
$a_3 > 0$ is satisfied if $\frac{YZ}{\sigma X} > 1$. That is:

$$\beta < \frac{\mu(\mu+\sigma)(\mu+\gamma+\alpha)}{\sigma(A(1-\rho q)+\frac{\phi\rho qA}{(\mu+\phi)})} = \frac{BD}{C\sigma} \tag{16}$$

In order to satisfy the second condition $a_1 a_2 > a_3$, the following inequality has to be valid:

$$\begin{aligned} Y^2Z + Y^2W + YZ^2 + 2YZW + Z^2W + ZW^2 + W^2Y - \sigma YX - \sigma XZ > 0 \\ Y + \frac{YW}{Z} + Z + 2W + \frac{ZW}{Y} + \frac{W^2}{Y} + \frac{W^2}{Z} > \frac{\sigma X(Y+Z)}{YZ} \end{aligned}$$

Rearranging:

$$\frac{YW}{Z} + \frac{ZW}{Y} + \frac{W^2}{Y} + \frac{W^2}{Z} + Y + Z + 2W > \frac{\sigma X(Y+Z)}{YZ} \tag{17}$$

Because $\frac{YZ}{\sigma X} > 1$, the inequality is valid. Therefore, the third condition is satisfied. As a result of the Routh-Hurwitz criterion, all the eigenvalues λ have negative real part if $\frac{YZ}{\sigma X} > 1$, which leads to local stability.

4 Numerical Analysis

Both original and modified models are solved numerically in MATLAB via ode45. Time interval is from 0 to 800 days. Both models are evaluated under normalization to unity. Values of parameters and initial conditions are provided in [1] as follows:

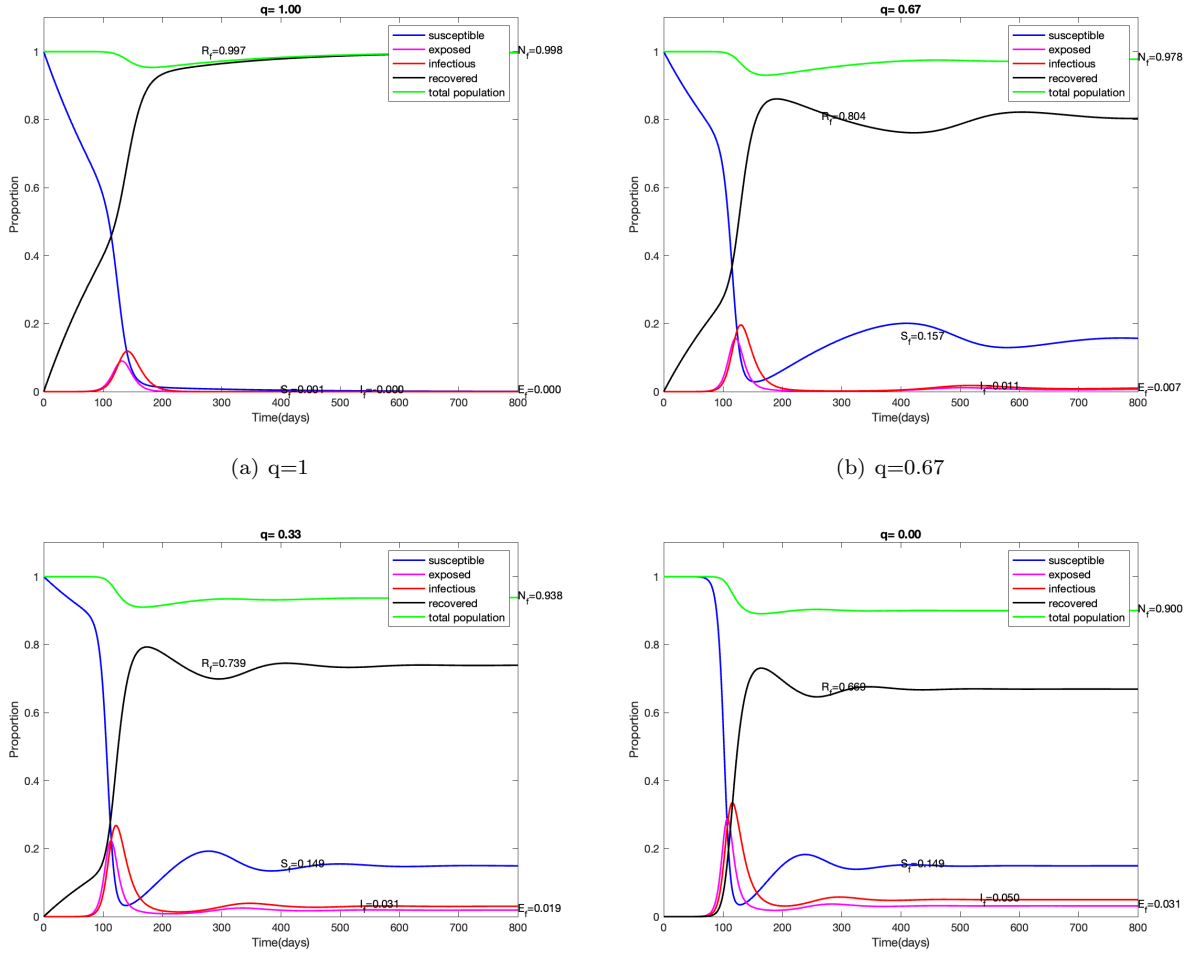
$$\begin{aligned} \beta &= 0.57 \quad day^{-1} \\ \sigma &= 0.13 \quad day^{-1} \\ \gamma &= 0.067 \quad day^{-1} \\ \alpha &= 0.01 \quad day^{-1} \\ N(0) &= 4 \times 10^7 \\ A &= 0.005 \quad (individual \times day)^{-1} \\ \mu &= 0.005 \quad day^{-1} \\ S(0) &= 1 - 1/N(0) \\ E(0) &= R(0) = 0 \\ I(0) &= 1/N(0) \end{aligned} \tag{18}$$

4.1 Original Model

When $q = 1$, as times goes to infinity, N_f, R_f approaches 1, and E_f, I_f approach 0. This is reasonable because when all newborns are protected against infection, the entire population will eventually enter the recovered subpopulation ,which means each individual has developed immunity. As q decreases, the number of vaccinated newborns also decreases. Presumably the recovered subpopulation will decrease in size.

When $q = 0$, which means no vaccination is implemented, population in the end of 800 days is 90 percent of the original population because of death caused by infection, and less than 70 percent of the population has successfully developed immunity.

Animation of how change in q affects each compartment.



(a) q=1

(b) q=0.67

(c) q=0.33

(d) q=0

Figure 1: Solution trajectories of original model with different q

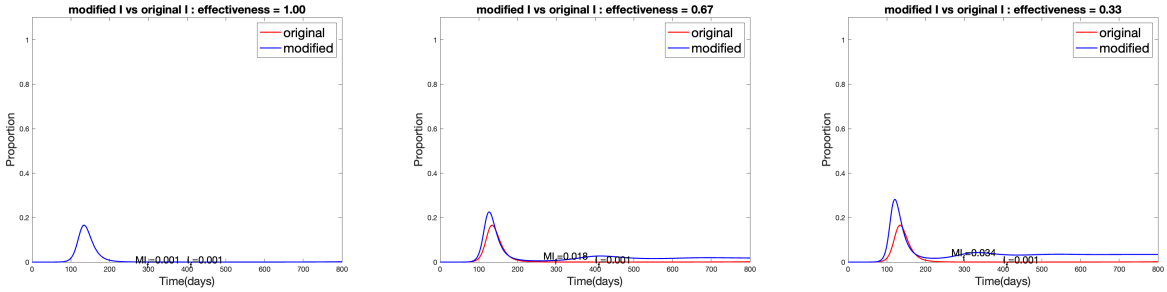
4.2 Modified Model

Fix q to be 0.8 and keep all other parameters the same.

4.2.1 Change effectiveness of vaccination

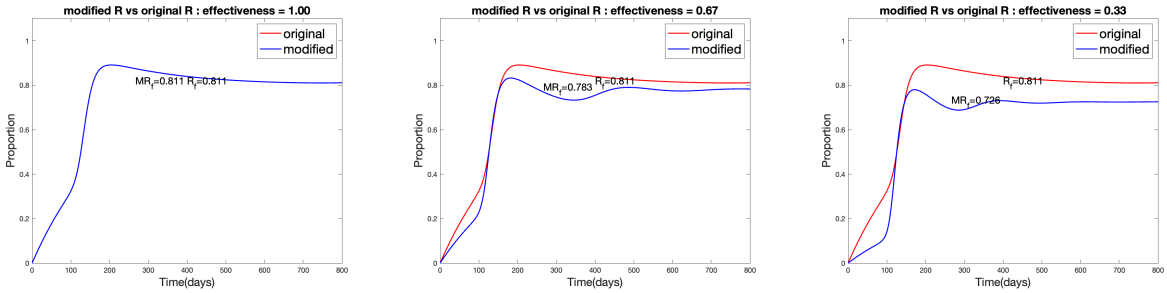
Suppose there is no loss of immunity. When $\rho = 1$, meaning each vaccine is effective, the solution to the modified model should be the same as the original one proposed in [1].. As ρ decreases, the infection

subpopulation is expected to increase, while the recovered one is expected to decrease correspondingly. The trajectories of I and R are shown in the graph below, where MI_f, I_f and MR_f, R_f represent final proportion of the infected individuals in the modified model and original model, final proportion of the recovered individuals in the modified model and original model, respectively.
Animation of how change in ρ affects S, E, I, R, N, and overall system.



(a) ρ=1 (b) ρ=0.67 (c) ρ=0.33

Figure 2: Comparison of I of original model and modified model with different effectiveness

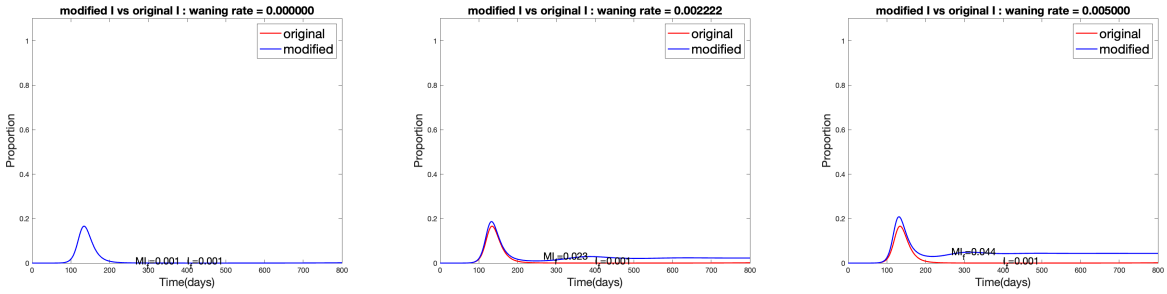


(a) ρ=1 (b) ρ=0.67 (c) ρ=0.33

Figure 3: Comparison of R of original model and modified model with different effectiveness

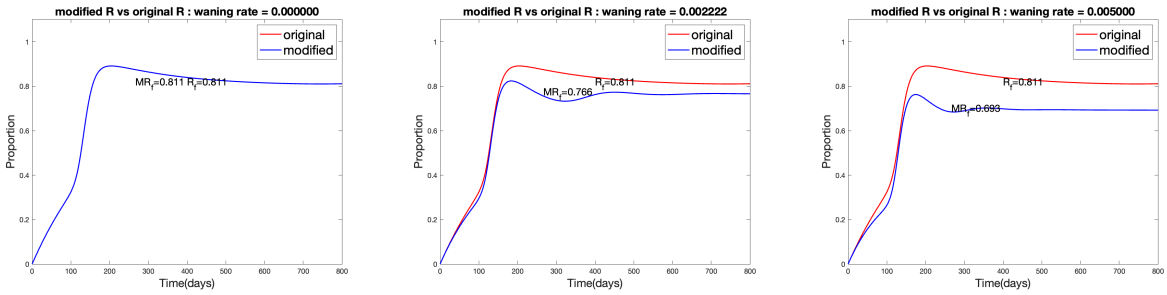
4.2.2 Change rate of losing immunity

Suppose each vaccination is effective for a certain period of time after injection. Similarly, if $\phi = 0$, the modified model is the same as the original model. Because ϕ indicates the rate of recovered individuals becoming susceptible, $\frac{1}{\phi}$ is the actual needed time. Higher value of ϕ which means losing immunity faster should cause infection subpopulation to increase and recovered subpopulation to decrease. The maximum value of ϕ is set to be $\frac{1}{200}\ day^{-1}$. The trajectories of I and R are shown in the graph below.
animation of how change of ϕ affects S, E, I, R, N, and overall system.



(a) ϕ=0 (b) ϕ=0.0022 (c) ϕ=0.005

Figure 4: Comparison of I of original model and modified model with different waning rate



(a) ϕ=0 (b) ϕ=0.0022 (c) ϕ=0.005

Figure 5: Comparison of R of original model and modified model with different waning rate

To visualize how change in ρ or ϕ contributes to the change of solution, taking I as an example, both graphs are plotted in the same figure as shown below, where M_eI_f, MI_f represent final proportion of infected individuals in the modified model under changing of ρ and final proportion of infected individuals in the modified model under changing of ϕ, respectively.
Animated version: S, E, I, R.

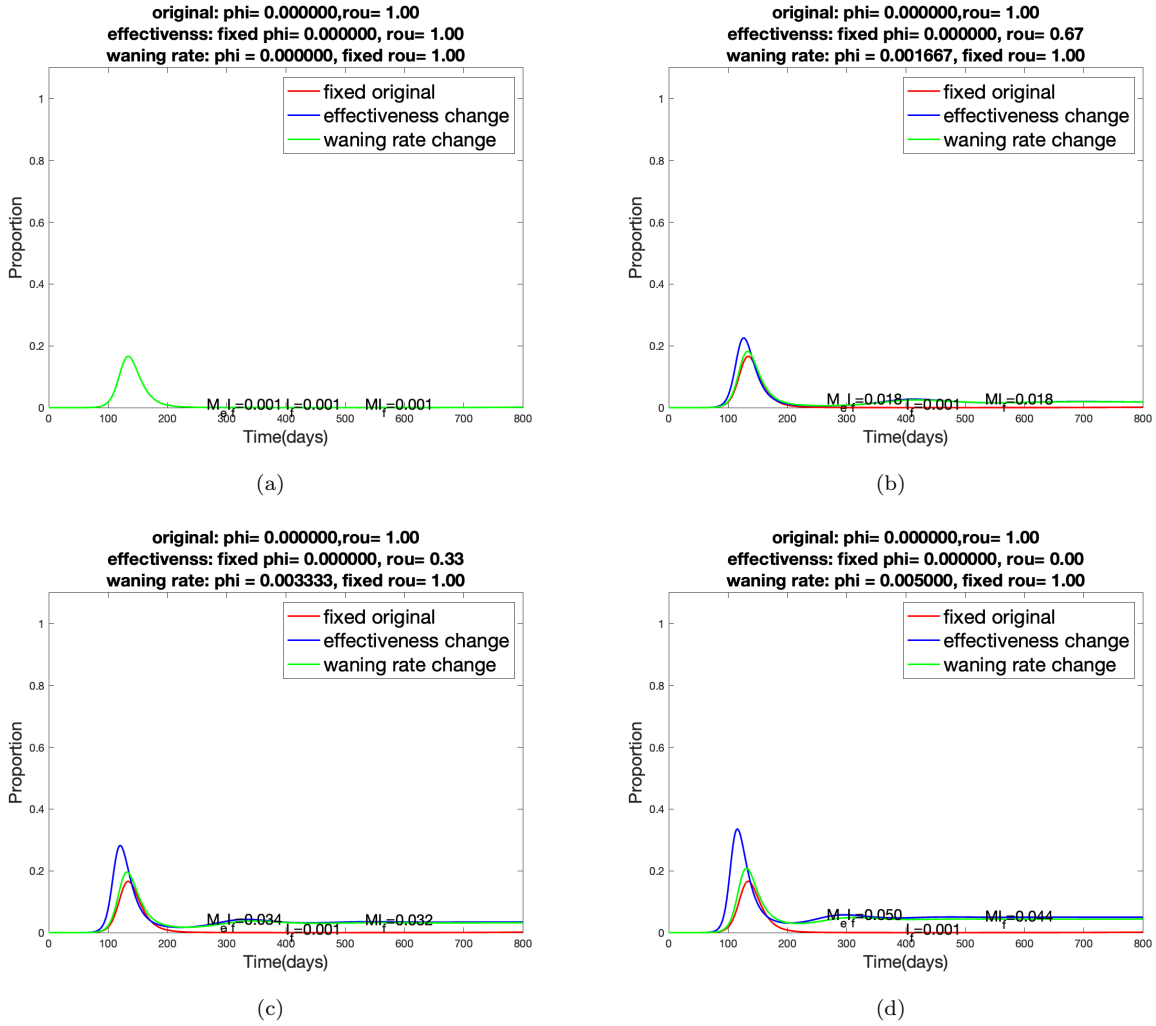
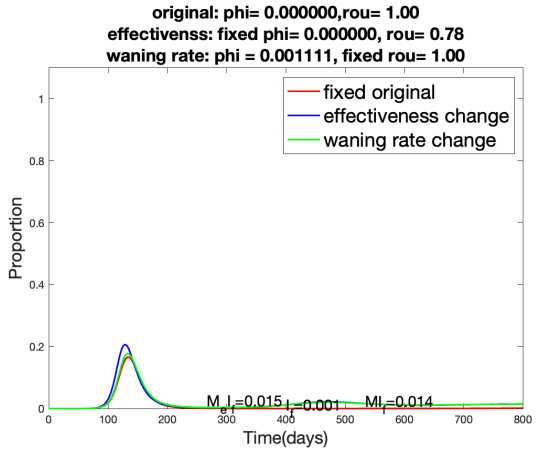
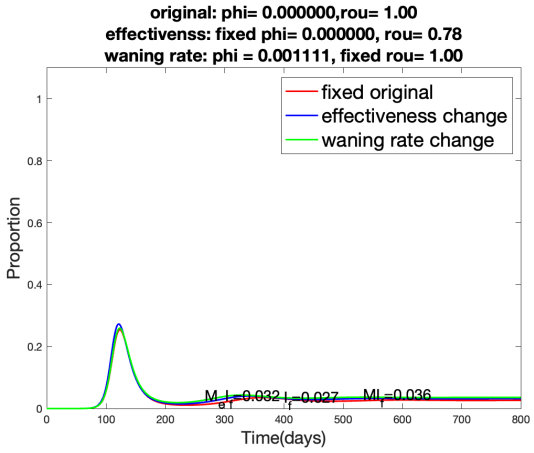


Figure 6: Change of I under different ρ or ϕ

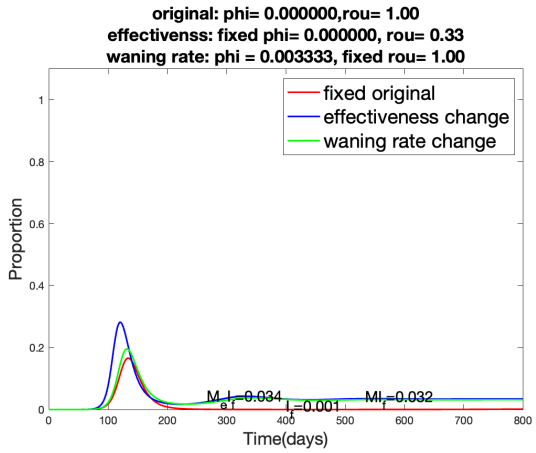
In addition, the effect of ρ and ϕ also differs under different values of q. Figure 7 shows solution trajectory of I, and figure 8 shows solution trajectory of R. When q has small value, meaning a small fraction of newborns are vaccinated, the impact of changing effectiveness of vaccination or changing waning rate reduces compared to large q. When q is large, meaning more individuals in the population are vaccinated at the end of the time interval, changing ρ or ϕ will affect this large group of vaccinated individuals, contributing significant change to the solution trajectory, whereas changing ρ or ϕ has less impact on the population in which the majority is not vaccinated.



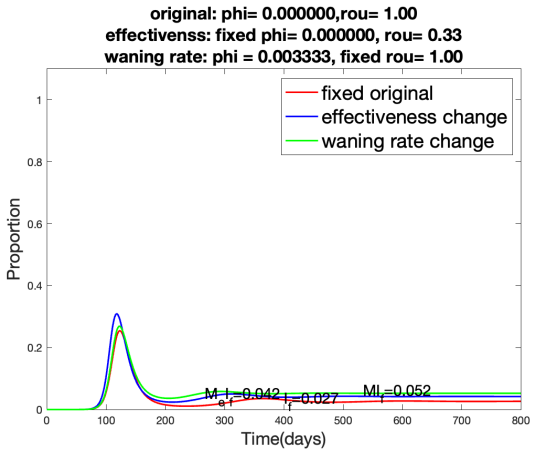
(a) q=0.8



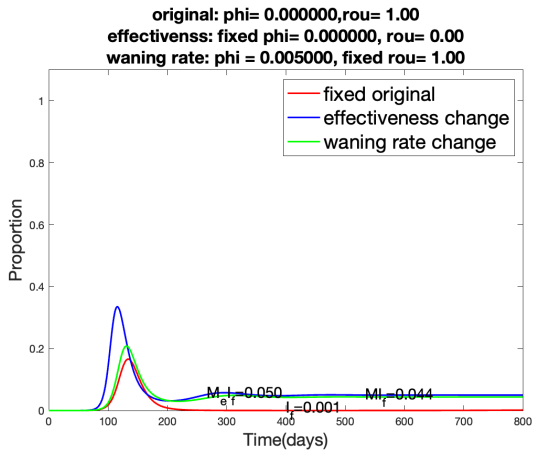
(b) q=0.4



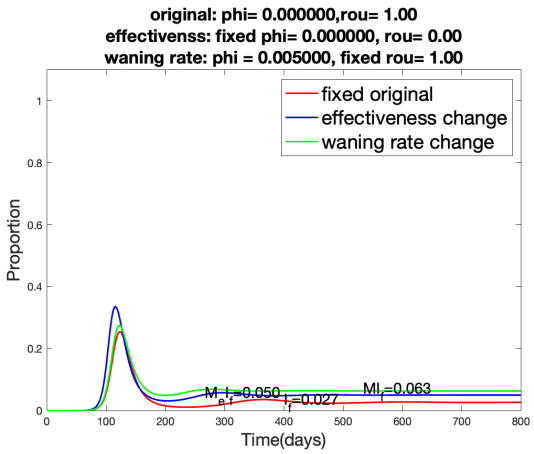
(c) q=0.8



(d) q=0.4

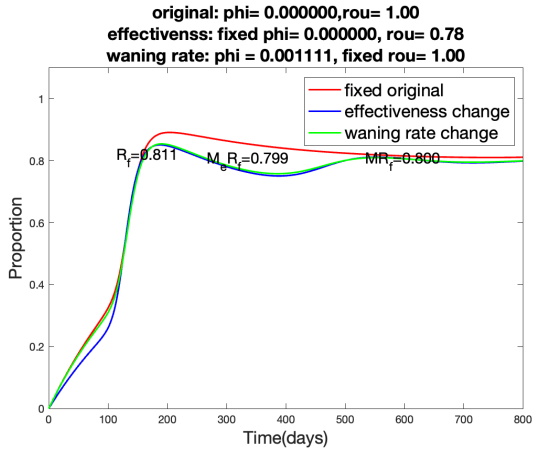


(e) q=0.8

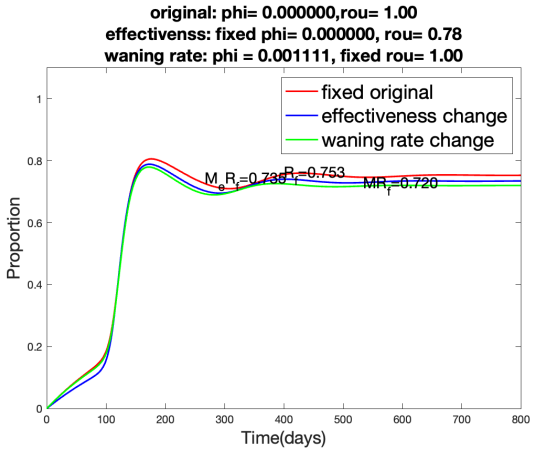


(f) q=0.4

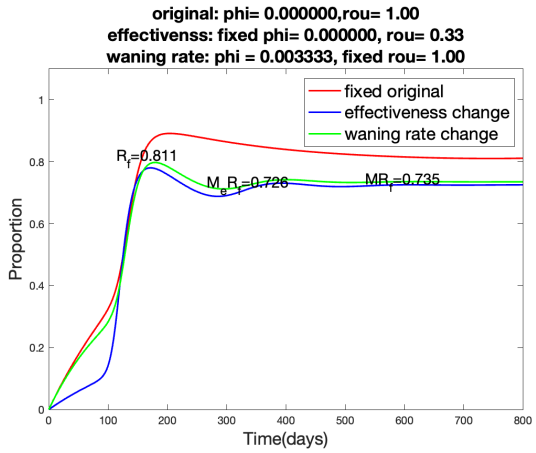
Figure 7: Change of I under different ρ or ϕ with varying q



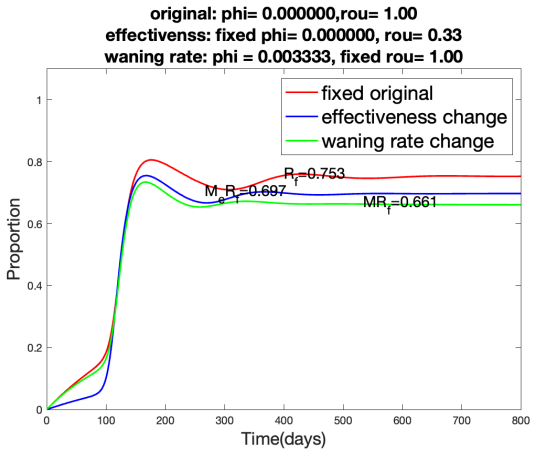
(a) q=0.8



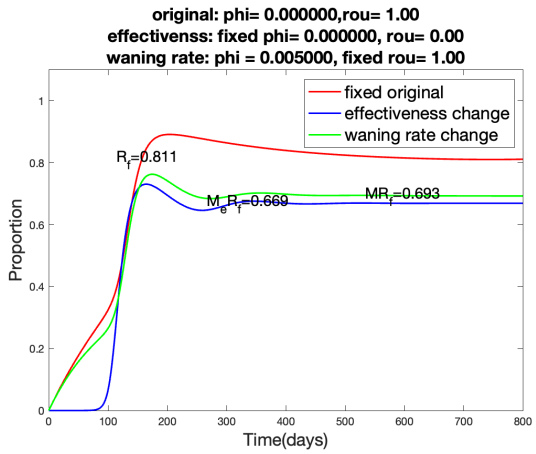
(b) q=0.4



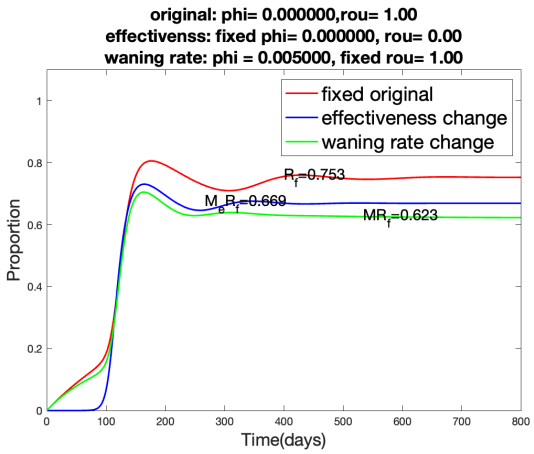
(c) q=0.8



(d) q=0.4



(e) q=0.8



(f) q=0.4

Figure 8: Change of R under different ρ or ϕ with varying q

5 Conclusion

In this report, the vaccination of newborns in SEIR proposed by *On an SEIR Epidemic Model with Vaccination of Newborns and Periodic Impulsive Vaccination with Eventual On-Line Adapted Vaccination Strategies to the Varying Levels of the Susceptible Subpopulation* is simulated. Building on this model, two new parameters, ρ, the effectiveness of vaccination, and ϕ, the rate of losing immunity after vaccination, are introduced to investigate how each of them affect the model. By calculating the critical transmission rate $\beta_c = \frac{BD}{C\sigma}$, we are able to determine the existence and stability of endemic or disease free equilibrium points. By applying the same method used to obtain the disease-free equilibrium point in [1] and the Routh-Hurwitz criterion, conditions for disease-free equilibrium point of the modified model to be locally stable is investigated. Graphs of solutions to the modified model indicate that reducing vaccination effectiveness and increasing the rate of lost immunity will have more significant impact on the population with a large fraction of vaccinated newborns, whereas impact on the population with a small fraction of vaccinated newborns is reduced.

6 Reference

[1] Etxeberria-Etxaniz, M., Alonso-Quesada, S., & De la Sen, M. (2020). On an SEIR epidemic model with vaccination of newborns and periodic impulsive vaccination with eventual on-line adapted vaccination strategies to the varying levels of the susceptible subpopulation. *Applied Sciences, 10*(22), 8296. https://doi.org/10.3390/app10228296

[2] Wikimedia Foundation. (2021, May 27). Routh–Hurwitz Stability Criterion. Wikipedia. Retrieved November 1, 2021, from https://en.wikipedia.org/wiki/Routh-Hurwitz_stability_criterion.