Basic Linear Algebra and Matrix Operations

Notations

Scalars (numbers): normal font lower case letters

e.g. a, b, x, y ...

$\mathbb{R}$ —— the set of all real numbers

$\mathbb{C}$ —— the set of all complex numbers

$a \in \mathbb{R}$ —— a is a real number

$b \in \mathbb{C}$ —— b is a complex number

Unless specified, a scalar is assumed real by default.

Vectors: bold lower case letters in printing.

(standard lower case letters in hand-writing)

e.g.,

$$a = [a_i]_{n \times 1} = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$$

$n \times 1$ or dropped for simplicity

a_i —— the i-th entry of a.

, where a_i are scalars.

Unless specified, we refer vectors as column vectors.

The transpose of vector a is denoted by a^T and given by

$$a^T = [a_1 \cdots a_i \cdots a_n]$$

Obviously, a^T is a length-n row vector.

Vector spaces: $\mathbb{R}^n$, $\mathbb{C}^n$

$a \in \mathbb{R}^n$ —— a is a length-n column real vector.

$a \in \mathbb{C}^n$ —— a is a length-n column complex vector.

"Real" spaces will be assumed if not specified

Special vectors:

length-n basis vector: $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← the i-th component.

length-n zero vector: $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

length-n vector of all ones.

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} e_i$$

Matrices: bold upper case letters in printing

standard upper case letters in hand-writing

An $n \times m$ matrix $A \in \mathbb{R}^{n \times m}$ (or $A \in \mathbb{C}^{n \times m}$)

called element or entry

$$A = [a_{ij}]_{n \times m} \text{ or } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

or dropped for simplicity

where $a_{ij} \in \mathbb{R}$ (or $\mathbb{C}$) for $i = 1, \cdots, n$, $j = 1, \cdots, m$.

Special matrices:

Identity matrix I

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n\times n}$$

Matrix spaces: $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$ —— A is an $m\times n$ real matrix.

$A \in \mathbb{R}^{m\times n}$ —— A is an $m\times n$ complex matrix.

Since a vector $a \in \mathbb{R}^n$ can also be viewed as an $n\times 1$ matrix, i.e., $a \in \mathbb{R}^{n\times 1}$,

$$\mathbb{R}^{n\times 1} = \mathbb{R}^n$$

and

$$\mathbb{C}^{n\times 1} = \mathbb{C}^n$$

Similarly, the space of all length-n row vectors is

$$\mathbb{R}^{1\times n} \quad \text{or} \quad \mathbb{C}^{1\times n}.$$

Any matrix $A \in \mathbb{R}^{n\times m}$ (or $\mathbb{C}^{n\times m}$) can be written in a column vector form as

$$A = [a_{(1)}, a_{(2)}, \cdots a_{(m)}],$$

where $a_{(j)} \in \mathbb{R}^n$ (or $\mathbb{C}^n$), $j = 1, \cdots, m$, are columns of A.

Similarly, we can also write $A \in \mathbb{R}^{n\times m}$ (or $\mathbb{C}^{n\times m}$) in a row column vector form as

$$A = \begin{bmatrix} (a^{(1)})^T \\ (a^{(2)})^T \\ \vdots \\ (a^{(n)})^T \end{bmatrix}$$

where $a^{(i)} \in \mathbb{R}^m$ (or $\mathbb{C}^m$), $i = 1, \cdots, n$, are rows of A.

Matrix transpose:

Given $A = [a_{ij}]_{n\times m}$, its transpose A^T is defined by

$$A^T = [a_{ji}]_{m\times n}$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longleftrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$

Obviously $(A^T)^T = A$

Hemitian (Also called Adjoint and conjugate transpose)

$$A = [a_{ij}]_{n\times m} \Longleftrightarrow A^* = [\overline{a_{ji}}]_{m\times n}$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longleftrightarrow A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \end{bmatrix}$

If $A \in \mathbb{R}^{n\times m}$, then $A^* = A^T$

If $A \in \mathbb{C}^{n\times m}$, then $A^* = \overline{A^T}$

Symmetry and Hermitian

$A \in \mathbb{R}^{n\times n}$ is symmetric if $A = A^T$

$A \in \mathbb{C}^{n\times n}$ is Hermitian if $A = A^*$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is symmetric

$A = \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is Hermitian.

Some Basic Operations of Matrices and Vectors

☆ Addition, Subtraction

Given two matrices $A=[a_{ij}]_{n\times m}$ and $B=[b_{ij}]_{n\times m}$, their summation/difference, denoted by C, is in $\mathbb{R}^{n\times m}$ and given by

$$C = A \pm B = \begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} & \cdots & a_{1m}\pm b_{1m} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & \cdots & a_{2m}\pm b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\pm b_{n1} & a_{n2}\pm b_{n2} & \cdots & a_{nm}\pm b_{nm} \end{bmatrix}$$

☆ Scalar product

Let $c\in\mathbb{R}$ be a scalar and $A\in\mathbb{R}^{n\times m}$ be a matrix.

The scalar product of c and A, denoted by B, is in $\mathbb{R}^{n\times m}$ and given by

$$B = cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & \cdots & ca_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nm} \end{bmatrix}$$

pseudo code:

```
for i=1:n
    for j=1:m
        bij = c aij
    end
end
```

ij-loop

B is obtained row-by-row

```
for j=1:m
    for i=1:n
        bij = c aij
    end
end
```

ji-loop

B is obtained column-by-column

The two implementations may have different performances, depending on how the matrices are stored. If the matrices is stored row-by-row, then ij-loop may be better, because it can save communication cost of Cache and CPU.

☆ Inner product:

- Given two vectors $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ in R^n

  their inner product is
$$\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$$

- pseudo code:

```
inner_prod = 0;
for i = 1:n
    inner_prod = inner_prod + a_i b_i ;
end
```

- If $\langle a,b \rangle = 0$, then we say a, b are perpendicular or othogonal.

☆ Matrix-Vector product:

- Let $A = [a_{ij}]_{m \times n} \in R^{m \times n}$ and $x = [x_i]_{n \times 1} \in R^n$.

  Then the matrix-vector product of A and x, denoted b, is a vector in R^m, given by

$$b = Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

- It is easy to see that
$$\langle a, b \rangle = [a_1 \ a_2 \cdots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [b_1 \ b_2 \cdots b_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$
$$= a^T b \qquad = b^T a$$

  i.e., inner product is a special case of matrix-vector product.

- If we write A in row vector form

$$(a^{(1)})^T$$
$$= \quad (a^{(2)})^T \qquad (i)^T \quad n \quad \ldots \quad ar \quad ow$$

$(a^{(m)})^T$

then

$$b = Ax = \begin{bmatrix} (a^{(1)})^T \\ (a^{(2)})^T \\ \vdots \\ (a^{(m)})^T \end{bmatrix} x = \begin{bmatrix} (a^{(1)})^T x \\ (a^{(2)})^T x \\ \vdots \\ (a^{(m)})^T x \end{bmatrix} = \begin{bmatrix} \langle a^{(1)}, x\rangle \\ \langle a^{(2)}, x\rangle \\ \vdots \\ \langle a^{(m)}, x\rangle \end{bmatrix}$$

i.e., entries of b are inner products of row vectors $a^{(i)}$ and x.

This gives an implementation of $b = Ax$.

pseudo code 1:

ij-loop

```
for i = 1:m
    b_i = 0;
    for j = 1:n
        b_i = b_i + a_ij x_j;
    end
end
```

inner product $b_i = \langle a^{(i)}, x\rangle$

- If we write A in column vector form

$A = [a_{(1)}, a_{(2)}, \cdots, a_{(n)}]$, where $a_{(j)} \in \mathbb{R}^m$, $j = 1, \cdots, n$ are columns of A,

then

$$b = Ax = [a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_{(1)} + x_2 a_{(2)} + \cdots\ x_n a_{(n)}$$

This reveals that b is a linear combination of columns $a_{(j)}$ of A.

This gives another implementation of $b = Ax$:

pseudo code 2:

ji-loop

```
set vector b = 0
for j = 1:n
    for i = 1:m
        b_i = b_i + x_j a_ij
    end
end
```

add b by $x_j a_{(j)}$.

- Comparison of two implementations.

ij - loop	ji - loop
Outer loop on rows	Outer loop on columns
Inner loop on columns	Inner loop on rows

Their efficiency depends on how the matrix A stored.

If A is stored column-wise, ji-loop is better

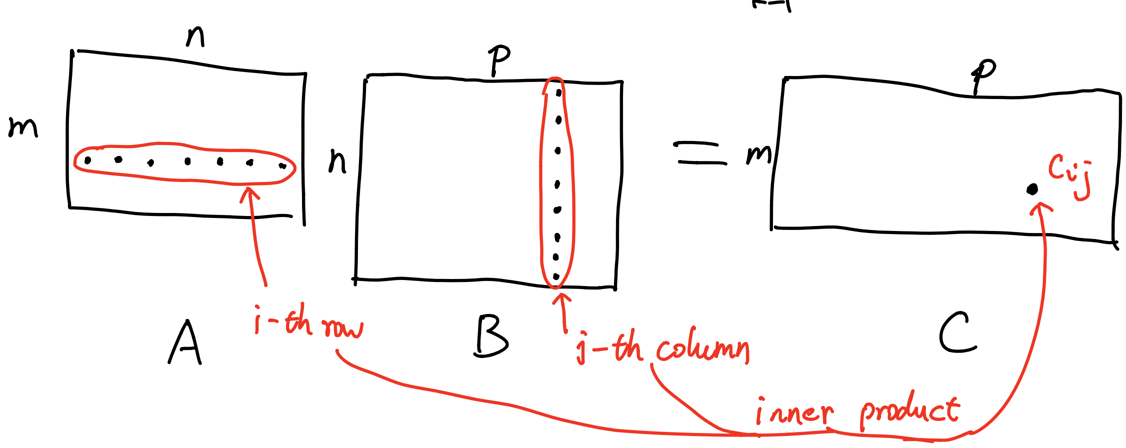
row-wise, ij-loop

★ Matrix - Matrix product

- Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{n \times p}$.

  Then the product of A and B, denoted by C, is an $m \times q$ matrix, given by

$$C = [C_{ij}] \in \mathbb{R}^{m \times p}, \quad \text{where} \quad C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



- Obviously, inner product and matrix-vector product are special cases of matrix-matrix product.
- We have different implementations.

  ① By the definition above.

```
for i = 1:m
    for j = 1:p
        Cij = 0
```

```
for j = 1:p
    for i = 1:m
        Cij = 0
```

```
        for k=1:n
            c_ij = c_ij + a_ik b_kj
        end
    end
end
```

ijk-loop

The product is computed row-by-row

```
        for k=1:n
            c_ij = c_ij + a_ik b_kj
        end
    end
end
```

jik-loop

The product is computed column-by-column

② Write A in column form and B in row form, i.e.,

$$A = [a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}], \qquad B = \begin{bmatrix} (b^{(1)})^T \\ (b^{(2)})^T \\ \vdots \\ (b^{(n)})^T \end{bmatrix}$$

Then,

$$C = AB = [a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}] \begin{bmatrix} (b^{(1)})^T \\ (b^{(2)})^T \\ \vdots \\ (b^{(n)})^T \end{bmatrix}$$

$$= \sum_{k=1}^{n} a_{(k)} (b^{(k)})^T.$$

We get another implementation

```
Initialize matrix C=0
for k=1:n
    for i=1:m
        for j=1:p
            c_ij = c_ij + a_ik b_kj
        end
    end
end
```

Add C by $a_{(k)}(b^{(k)})^T$

kij-loop.

Any permutation of the sequence (i,j,k) is a possible implementation. So we have $3! = 6$ different implementations of $C = AB$.

☆ Computational complexity analysis

- How many scalar operations are involved in the computation of inner product of $a, b \in \mathbb{R}^n$?

$$\sum_{i=1}^{n} (1+1) = 2n$$

for an i, one scalar muliplication and one scalar addition

We say the computational complexity is $O(n)$, meaning a constant multiple of n.

- Matrix-vector product of $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (1+1) = 2nm \equiv O(nm)$$

for an i, j, one multiplication one addition

- Matrix-Matrix product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$

$$\sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} (1+1) = 2nmp \equiv O(nmp)$$

☆ All these basic operations are implemented efficiently in BLAS (Basic Linear Algebra Subroutine) library, which comes with CPU. or cuBLAS, which comes with NVIDIA GPU.

The real performance depends on many factors, e.g., computational complexity, CPU Cache, ...

Matrix Partitions and block forms.

- We will often write a matrix $A \in \mathbb{R}^{m\times n}$ into a block form as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix},$$

All blocks in the same row have the same # of rows

All blocks in the same column have the same # of columns

where $A_{ij} \in \mathbb{R}^{m_i \times n_j}$ are sub-matrices of A, and $n = n_1 + n_2 + \cdots + n_p$. $m = m_1 + m_2 + \cdots + m_q$.

We call it a partition of A.

- Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$

$$A = \left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right]$$

A_{11} A_{12} A_{21} A_{22}

a valid matrix partition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{11} A_{12} A_{21} A_{22}

NOT a valid matrix partition

- Matrix additions, multiplications, transpose in matrix partition forms. are formally the same as those in standard forms
  - Transpose: $A = [A_{ij}]_{i,j} \iff A^T = [A_{ji}^T]_{j,i}$
  - Addition: $A = [A_{ij}]_{i,j}$, $B = [A_{ij}]_{i,j}$, then $A + B = [A_{ij} + B_{ij}]_{i,j}$ if $A_{ij} + B_{ij}$ makes sense.
  - Multiplication: $A = [A_{ij}]_{i=1,\, j=1}^{p,\quad q}$ $\quad B = [B_{ij}]_{i=1,\, j=1}^{q,\quad r}$

    Then $AB = \left[\sum_{k=1}^{q} A_{ik} B_{kj}\right]_{i=1,\, j=1}^{p,\quad r}$, if all block multiplications are valid.

Examples 1: We usually write $A \in \mathbb{R}^{m\times n}$ into column vector form.

$$A = [a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}], \text{ where } a_{(i)} \in \mathbb{R}^m \text{ is the } i\text{-th column.}$$

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ be a vector.

Then

$$A x = [a_{(1)} \cdots a_{(n)}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} a_{(j)} x_j \underset{\uparrow}{=} \sum_{j=1}^{n} x_j a_{(j)}$$

because x_j is a scalar

So, we see $Ax \in \mathbb{R}^m$ is a linear combination of columns of A.

Example 2: (a) We write $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n \times l}$ in block form as

$$A = [a_{(1)}\ a_{(2)} \cdots a_{(n)}] \qquad B = \begin{bmatrix} (b^{(1)})^T \\ \vdots \\ (b^{(n)})^T \end{bmatrix}$$

where $a_{(j)}$ are columns of A, $(b^{(j)})^T$ are rows of B.

Then $$AB = [a_{(1)} \cdots a_{(n)}] \begin{bmatrix} (b^{(1)})^T \\ \vdots \\ (b^{(n)})^T \end{bmatrix} = \sum_{k=1}^{n} \underbrace{a_{(k)} (b^{(k)})^T}_{\text{a rank-1 } m\times l \text{ matrix}}$$

(b). If we write into another form

$$A = \begin{bmatrix} (a^{(1)})^T \\ \vdots \\ (a^{(m)})^T \end{bmatrix} \qquad B = [b_{(1)} \cdots b_{(l)}]$$

where $(a^{(i)})^T$ are rows of A, $b_{(j)}$ are columns of B.

Then $$AB = \begin{bmatrix} (a^{(1)})^T \\ \vdots \\ (a^{(m)})^T \end{bmatrix} [b_{(1)} \cdots b_{(l)}] = \begin{bmatrix} (a^{(1)})^T b_{(1)} & \cdots & (a^{(1)})^T b_{(l)} \\ \vdots & & \vdots \\ \underbrace{(a^{(m)})^T b_{(1)}}_{\text{a number.}} & \cdots & (a^{(m)})^T b_{(l)} \end{bmatrix}$$

More concepts of matrices

Given a matrix $A = [a_{(1)}\ a_{(2)} \cdots a_{(n)}] \in \mathbb{R}^{m\times n}$ with $a_{(j)} \in \mathbb{R}^m$ columns of A,

- the range of A is

$$\operatorname{Ran}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$
$$= \{x_1 a_{(1)} + x_2 a_{(2)} + \cdots + x_n a_{(n)} \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

$\operatorname{Ran}(A)$ is also called column space of A, because all vectors in $\operatorname{Ran}(A)$ is a linear combination of column vectors of A.

- The null space of A is
$$\operatorname{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$
- The rank of A is
$$\operatorname{Rank}(A) = \dim(\operatorname{Ran}(A))$$
We must have $\operatorname{Rank}(A) \le \min\{m, n\}$.

Vector and Matrix Norms

To measure how accurate our algorithm is, we need to define metric, i.e., distance between two scalars, two vectors, or two matrices.

☆ Metric for scalars

For two scalars $x, y \in \mathbb{R}$, their distance is $|x-y|$.

☆ Metric for vectors

For two vectors $a, b \in \mathbb{R}^n$, we need to generalize the absolute function for scalars to vectors.

- For a scalar, the absolution function satisfies:

① $|x| \ge 0, \ \forall x \in \mathbb{R}$ and $|x| = 0 \Leftrightarrow x = 0$.

② $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

③ $|x+y| \le |x| + |y| \quad \forall x, y \in \mathbb{R}$

- Norm —— generalization of absolute value to vectors.

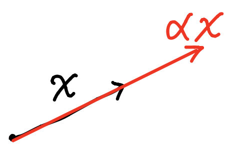
Definition: A function $\mathbb{R}^n \to \mathbb{R}$, denoted by $\|x\|$ for any $x \in \mathbb{R}^n$, is called a norm, if it satisfies:

① $\|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$ and $\|x\|=0 \Leftrightarrow x=0.$

② $\|\alpha x\| = |\alpha|\, \|x\| \quad \forall\, \alpha \in \mathbb{R},\ x \in \mathbb{R}^n$

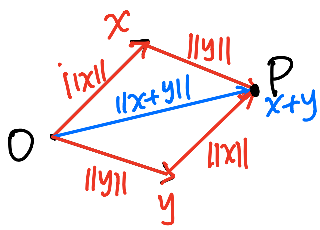
③ $\|x+y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in \mathbb{R}^n$

Remarks:
- Condition ① is to make a norm a magnitude.
- Condition ② Says that the norm of a scaling of vectors is the scaling of the norm.



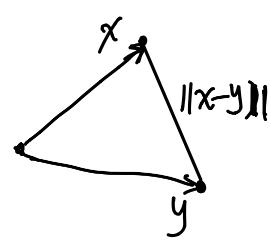
$\|\alpha x\| = |\alpha|\, \|x\|$

- Condition ③ is also known as the triangle inequality.



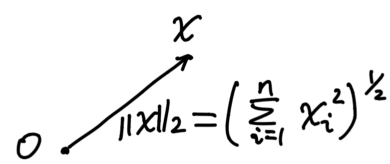
The length of indirect path (the red) from O to P is longer than that of the direct path (the blue)

- Given two vectors $x, y \in \mathbb{R}^n$, their distance is $\|x-y\|$



Example 1: Euclidean norm (2-norm)

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \forall\, x \in \mathbb{R}^n$$



$\|\cdot\|_2$ is indeed a norm, because

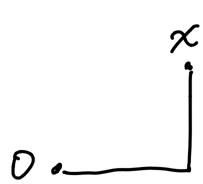
① $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \geq 0$ and $\|x\|_2 = 0 \Leftrightarrow \sum_{i=1}^n x_i^2 = 0 \Leftrightarrow x_i = 0\ \forall i \Leftrightarrow x=0.$

② $\|\alpha x\|_2 = \left(\sum_{i=1}^n (\alpha x_i)^2\right)^{1/2} = \left(\sum_{i=1}^n \alpha^2 x_i^2\right)^{1/2} = (\alpha^2)^{1/2}\left(\sum_{i=1}^n x_i^2\right)^{1/2} = |\alpha|\, \|x\|_2$

③ Need to use Cauchy-Schwartz to prove $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.
(We omit it)

Example 2: Manhattan norm (1-norm)

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \forall\, x \in \mathbb{R}^n$$



$\|x\|_1 = \sum_{i=1}^n |x_i|$

$\|\cdot\|_1$ is indeed a norm, because

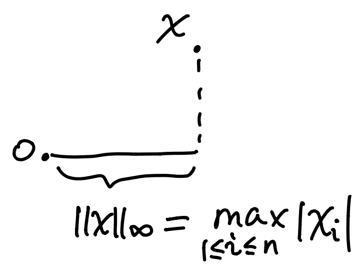
① $\|x\|_1 = \sum_{i=1}^n |x_i| \geq 0$ and $\|x\|_1 = 0 \Leftrightarrow \sum_{i=1}^n |x_i| = 0 \Leftrightarrow |x_i| = 0 \ \forall i \Leftrightarrow x = 0.$

② $\|\alpha x\|_1 = \sum_{i=1}^n |\alpha x_i| = \sum_{i=1}^n |\alpha||x_i| = |\alpha| \sum_{i=1}^n |x_i| = |\alpha| \, \|x\|_1 \qquad \forall \alpha \in \mathbb{R}, x \in \mathbb{R}^n$

③ $\|x+y\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n (|x_i| + |y_i|) = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|x\|_1 + \|y\|_1$
$\forall x, y \in \mathbb{R}^n$

Example 3: Max norm (∞-norm)

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad \forall x \in \mathbb{R}^n$$



$\boxed{\text{Question: Check } \|\cdot\|_\infty \text{ is a norm.}}$

Example 4: p-norm ($p \geq 1$)

Let p be a positive integer. Define

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

Then it can check $\|x\|_p$ is a norm for any $p \geq 1$.

p-norm indeed defines a metric for vectors.

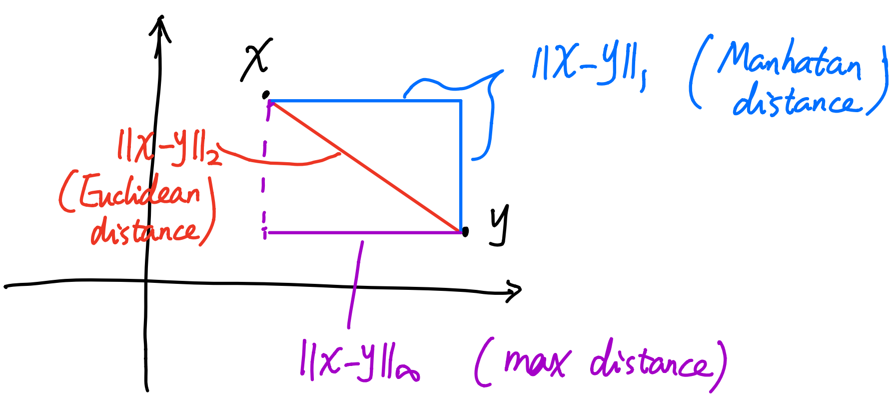
$p=1$, $\quad \|x\|_1 = \sum_{i=1}^n |x_i|$ (Manhatan norm)

$p=2$, $\quad \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ (Euclidean norm)

$p \to \infty$, $\quad \lim_{p\to\infty} \|x\|_p = \max_{1 \leq i \leq n} |x_i|$ (Max norm)

$\boxed{\text{Can you prove it?}}$

(For this reason, we denoted $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$)



✩ Metric for Matrices

Similar to vector norms, we can define matrix norm.

Definition: A function $\mathbb{R}^{m\times n} \to \mathbb{R}$, denoted by $\|A\|$ for any $A \in \mathbb{R}^{m\times n}$, is called a norm, if it satisfies:

① $\|A\| \geq 0 \quad \forall A \in \mathbb{R}^{m\times n}$ and $\|A\| = 0 \Leftrightarrow A = 0$.

② $\|\alpha A\| = |\alpha| \, \|A\| \quad \forall \alpha \in \mathbb{R}, \ A \in \mathbb{R}^{m\times n}$

③ $\|A+B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathbb{R}^{m\times n}$

- Since $m\times n$ matrices can be viewed as long vectors of length mn, i.e.,

$$\mathbb{R}^{m\times n} \longleftrightarrow \mathbb{R}^{mn}$$

we can apply norms for vectors to obtain norms for matrices.

Example 1: Frobenius norm (vector 2-norm)

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2} \qquad \forall A \in \mathbb{R}^{m\times n}$$

Example 2: (vector p-norm)

$$\|A\|_{vec,p} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p\right)^{1/p} \qquad \forall A \in \mathbb{R}^{m\times n}$$

Example 3: Consider the identity matrix

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Then $\|I\|_F = \sqrt{n}$ and $\|I\|_p = n^{\frac{1}{p}}$

Example 4: Consider a unitary matrix $U \in \mathbb{R}^{n\times n}$ (i.e., $UU^T = U^TU = I$)

Then $\|U\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij}^2\right)^{1/2} = \left(\sum_{i=1}^{n} 1\right)^{1/2} = \sqrt{n}$.

each row of U are unit vectors in 2-norm

From Example 3 and 4 we see that: The Frobenius norm for identity and unitary matrices grows with respect to n. However, they are "unit" linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, and their magnitude should not grow w.r.t. n. Therefore, these matrix norms generalized from vector norms are not suitable for linear transformations.

- We view matrices as linear transformations.

$$A: \mathbb{R}^{m\times n} \quad : \quad \underset{\substack{\uparrow \\ p\text{-norm}}}{\mathbb{R}^n} \to \underset{\substack{\uparrow \\ p\text{-norm}}}{\mathbb{R}^m}$$

So, we can define matrix operator p-norm (or simply matrix p-norm) as

$$\|A\|_p = \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{\|Ax\|_p}{\|x\|_p}$$

In other words, matrix p-norm of A is the largest magnifying factor of A acting from $\mathbb{R}^n$ with p-norm to $\mathbb{R}^m$ with p-norm.

Theorem: $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$

proof. By definition,

$$\|A\|_p = \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \left\|A\left(\frac{x}{\|x\|_p}\right)\right\|_p = \sup_{\|x\|_p\neq 0} \left\|A\frac{x}{\|x\|_p}\right\|_p$$

$$= \sup_{\left\|\frac{x}{\|x\|_p}\right\|_p=1} \left\|A\frac{x}{\|x\|_p}\right\|_p \underset{\substack{\uparrow \\ \text{Let} \\ \tilde{x}=\frac{x}{\|x\|_p}}}{=} \sup_{\|\tilde{x}\|_p=1} \|A\tilde{x}\|_p \underset{\substack{\uparrow \\ \text{use } x \\ \text{to replace } \tilde{x}}}{=} \sup_{\|x\|_p=1} \|Ax\|_p$$

It remains to prove $\sup_{\|x\|_p=1} \|Ax\|_p = \max_{\|x\|_p=1} \|Ax\|_p$. To this end, we define $S=\{x\in\mathbb{R}^n \mid \|x\|_p=1\}$.

- The p-norm $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, viewed as a function of x, is continuous, because all $|\cdot|^p$, $+$, $(\cdot)^{1/p}$ functions are continuous. Let $\{x^{(k)}\}_{k\in\mathbb{N}} \subset S$ and $\lim_{k\to\infty} x^{(k)} = x \in \mathbb{R}^n$. Then,

$$1 = \lim_{k\to+\infty} \|x^{(k)}\|_p = \|\lim_{k\to+\infty} x^{(k)}\|_p = \|x\|_p \implies x\in S.$$

  So, S is closed.
- For any $x\in S$, we have $|x_i|\le 1 \ \forall i$. Thus, S is bounded.
- Moreover, $\|Ax\|_p = \left(\sum_{i=1}^m \left|\sum_{j=1}^n a_{ij}x_j\right|^p\right)^{1/p}$ is continuous in x, because it is a composition of functions such as multiplication, addition, p-th power, p-th root, absolute value, which are all continuous.

Altogether, by using Weierstrass extreme value theorem (a continuous function on a bouned and closed set obtains its exetreme values),

$\|Ax\|_p$ can attain its supremum on S, i.e.,

$$\sup_{x\in S}\|Ax\|_p = \max_{x\in S}\|Ax\|_p \qquad \boxtimes$$

Example 1: 2-norm

because $\|y\|_2^2 = y^T y \ \forall y.$

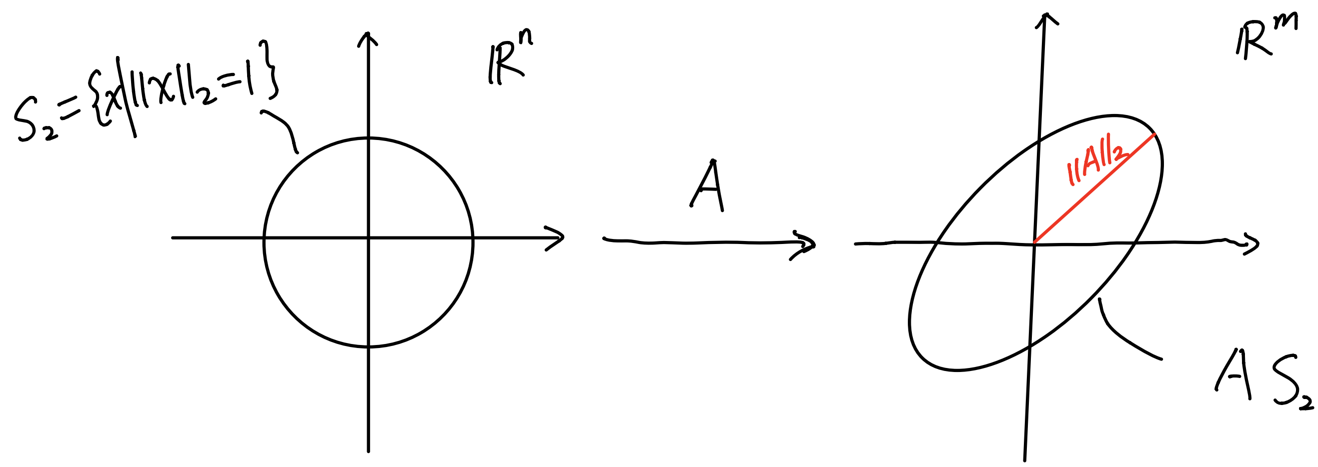
$$\|A\|_2 = \max_{\|x\|_2=1}\|Ax\|_2 \iff \|A\|_2^2 = \max_{\|x\|_2=1}\|Ax\|_2^2 = \max_{\|x\|_2=1} x^T A^T A x$$

$$= \text{maximum eigenvalue of } A^T A$$

will be proved later in eigenvalue decomposition

Therefore, $\|A\|_2 = (\text{max eigenvalue of } A^T A)^{1/2}$

Geometry:

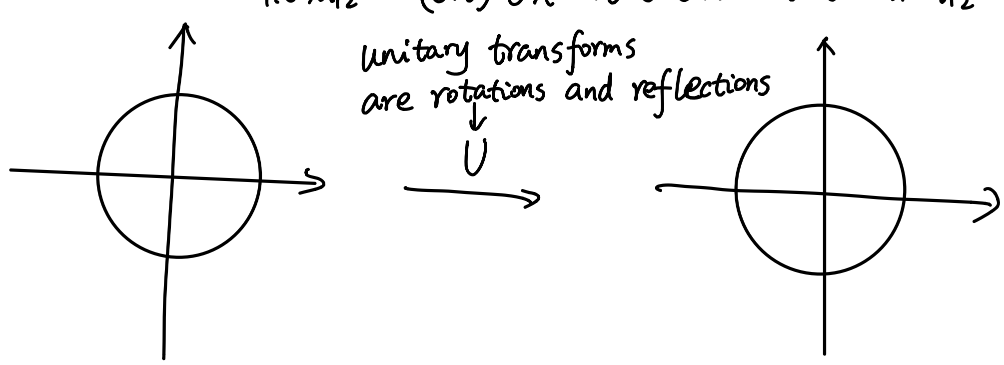


So,
$$\|I\|_2 = \max_{\|x\|_2=1}\|Ix\|_2 = \max_{\|x\|_2=1}\|x\|_2 = \max_{\|x\|_2=1} 1 = 1$$

Also, for unitary $U\in\mathbb{R}^{n\times n}$ (i.e, $UU^T = U^T U = I$)

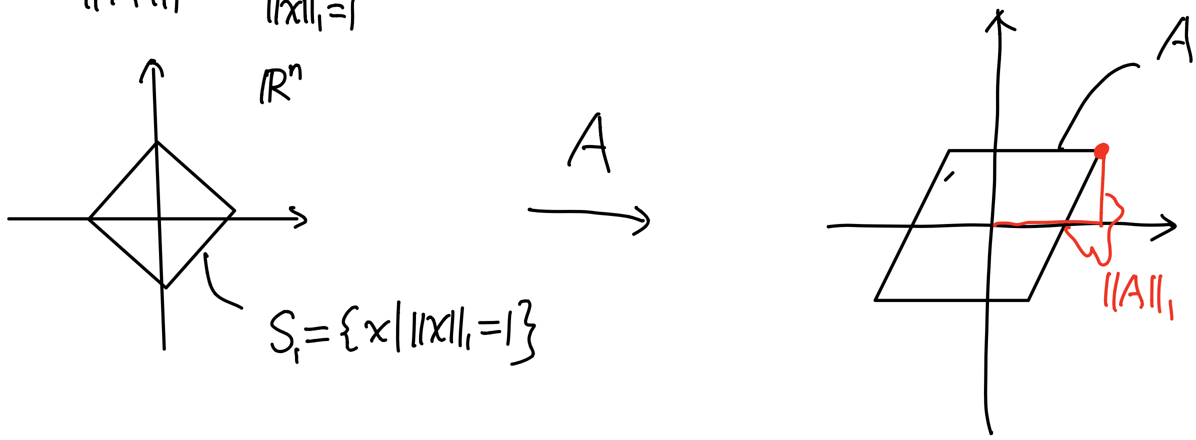
$$\|U\|_2 = \max_{\|x\|_2=1}\|Ux\|_2 = \max_{\|x\|_2=1}\|x\|_2 = 1$$

$$\|Ux\|_2^2 = (Ux)^T Ux = x^T U^T U x = x^T x = \|x\|_2^2$$



Example 2: 1-norm

$$\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1$$



Theorem: $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_{1\le j\le n} \|a_{(j)}\|_1$,

where $A=[a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}] \in \mathbb{R}^{m\times n}$

$\boxed{\|A\|_1 \text{ is the max 1-norm of columns.}}$

proof. We prove the theorem by showing

$$\max_{\|x\|_1=1} \|Ax\|_1 \le \max_{1\le j\le n} \|a_{(j)}\|_1 \quad \text{and} \quad \max_{\|x\|_1=1} \|Ax\|_1 \ge \max_{1\le j\le n} \|a_{(j)}\|_1$$

① For "$\le$":

For an arbitrary x satisfying $\|x\|_1=1$,

$$\|Ax\|_1 = \left\|[a_{(1)}\ a_{(2)}\ \cdots\ a_{(n)}]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right\|_1 = \left\|\sum_{\ell=1}^n x_\ell a_{(\ell)}\right\|_1$$

triangle inequality; because $\|a_{(\ell)}\|_1 \le \max_{1\le j\le n} \|a_{(j)}\|_1$

$$\le \sum_{\ell=1}^n \|x_\ell a_{(\ell)}\|_1 = \sum_{\ell=1}^n |x_\ell|\,\|a_{(\ell)}\|_1 \le \left(\sum_{\ell=1}^n |x_\ell|\right)\cdot\left(\max_{1\le j\le n}\|a_{(j)}\|_1\right)$$

$$= \|x\|_1 \cdot \max_{1\le j\le n} \|a_{(j)}\|_1 = \max_{1\le j\le n} \|a_{(j)}\|_1$$

So, $\|Ax\|_1 \le \max_{1\le j\le n} \|a_{(j)}\|_1 \quad \forall x: \|x\|_1=1$

Taking max over all x satisfying $\|x\|_1=1$

$$\max_{\|x\|_1=1} \|Ax\|_1 \le \max_{1\le j\le n} \|a_{(j)}\|_1$$

② For "$\ge$":

Let $j_0 = \arg\max_{1\le j\le n} \|a_{(j)}\|_1$, i.e., $\|a_{(j_0)}\|_1 = \max_{1\le j\le n} \|a_{(j_0)}\|_1$

Then $\|Ae_{j_0}\|_1 = \|a_{(j_0)}\|_1 = \max_{1\le j\le n} \|a_{(j)}\|_1$

Because e_{j_0} satisfies $\|e_{j_0}\|_1 = 1$,

$$\max_{\|x\|_1=1} \|Ax\|_1 \ge \|Ae_{j_0}\|_1$$

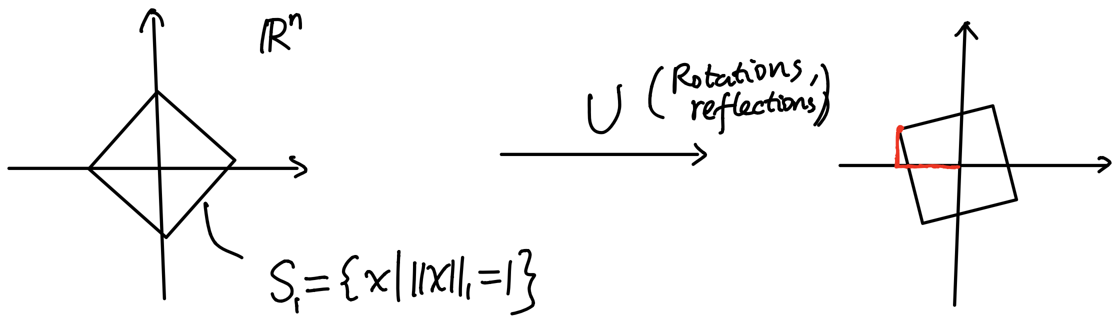
$$\Rightarrow \max_{\|x\|_1=1} \|Ax\|_1 \ge \max_{1\le j\le n} \|a_{(j)}\|_1$$

∎

For the identity matrix

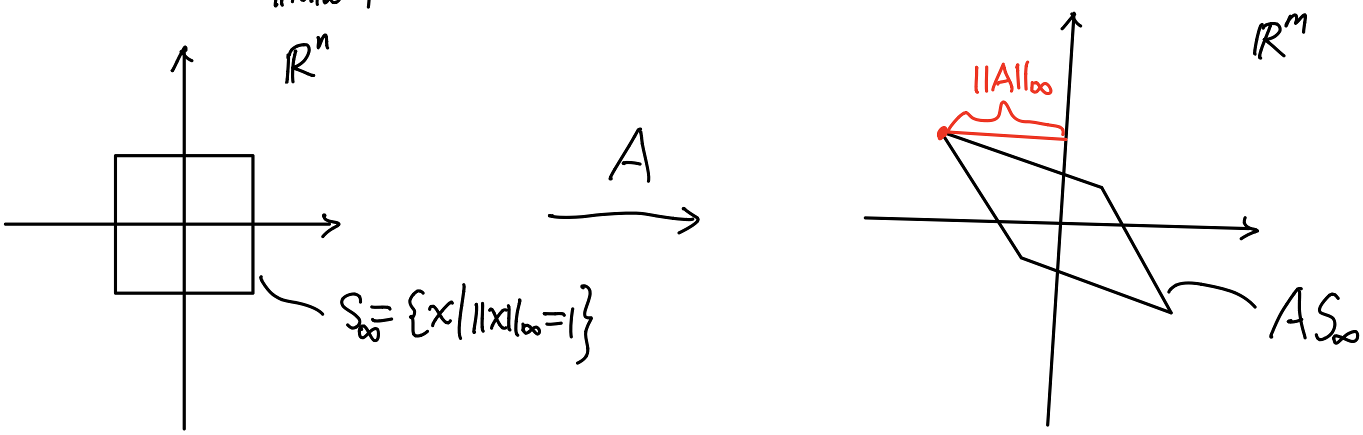
$$\|I\|_1 = \max_{\|x\|_1=1} \|Ix\|_1 = \max_{\|x\|_1=1} \|x\|_1 = 1$$

For a unitary matrix U, $\|U\|_1 \neq 1$ in general. (see the picture below)



Example 3: ∞-norm

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty$$



Theorem: $\|A\|_\infty = \max_{1 \le i \le m} \|a^{(i)}\|_1$,

$$\text{where } A = \begin{bmatrix} (a^{(1)})^T \\ (a^{(2)})^T \\ \vdots \\ (a^{(m)})^T \end{bmatrix} \in \mathbb{R}^{m \times n}$$

In other words, $\|A\|_\infty$ is the maximum 1-norm of row vectors.

$\boxed{\text{Question: Prove the theorem ?}}$

For matrix p-norms, besides ①②③ in the definition of matrix norm, the matrix p-norm also satisfies:

④ $\|Ax\|_p \le \|A\|_p \|x\|_p \quad \forall A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$

⑤ $\|AB\|_p \le \|A\|_p \|B\|_p \quad \forall A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$