Recursion formula for $U_q(\mathfrak{e}_6)$ knot invariants

Thesis Defense

Knot invariants

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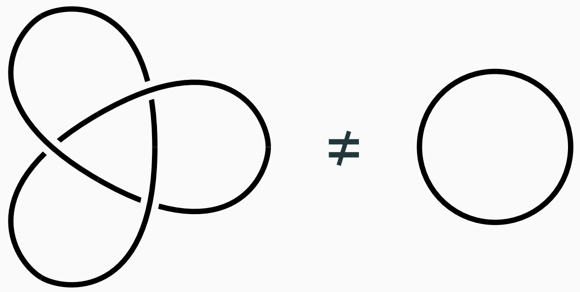
- Knot diagrams represent knots in 2D space, up to planar isotopy



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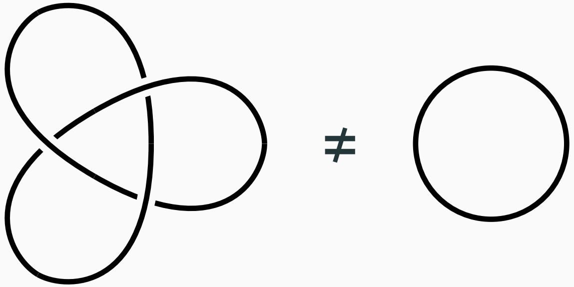

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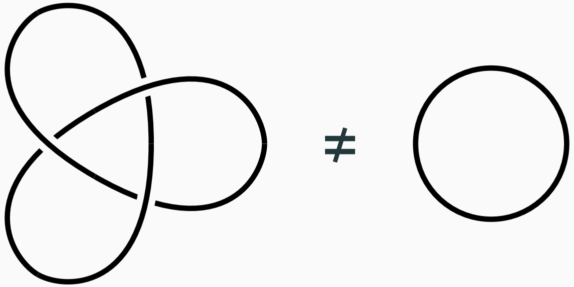

- Reidemeister moves completely characterizes diagram equivalence

Definition 1. *A **knot invariant** assigns a value to every knot diagram such that it is invariant under Reidemeister moves.*

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Definition 3. *A (ribbon) **knot invariant** is a ribbon category.*

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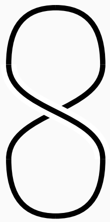

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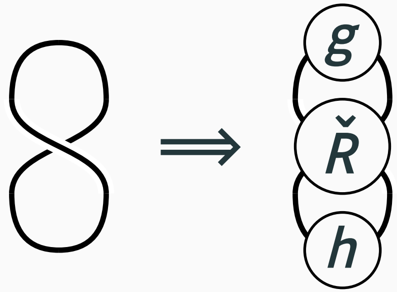

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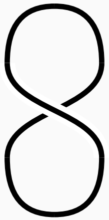

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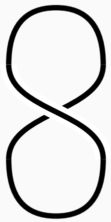

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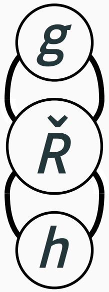

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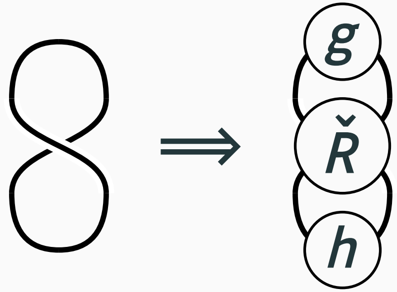

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- Plan: find a vector space equipped with tensors satisfying some equations
- Yang–Baxter equation for $R^{ji}_{k\ell} = \check{R}^{ij}_{k\ell}$: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.
  - Related to exact solutions in statistical mechanics

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  - More on this later.

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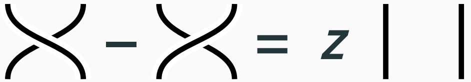

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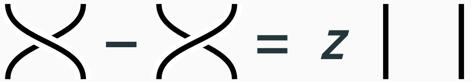

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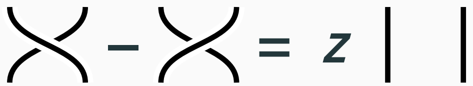

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- How do we pick a convenient splitting?

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- Group algebras $\mathbb{C}[G]$ and universal enveloping algebras $U(\mathfrak{g})$ have trivial ribbon structures, they are *cocommutative*.
- Deform $U(\mathfrak{g})$ with an infinitesimal parameter h, or $q = e^h$, to get $U_q(\mathfrak{g})$.
    - *Caveat Lector:* problems with infinite sums

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Kauffman polynomials ($\neq$ Kauffman bracket) given by B_n, C_n, D_n families.

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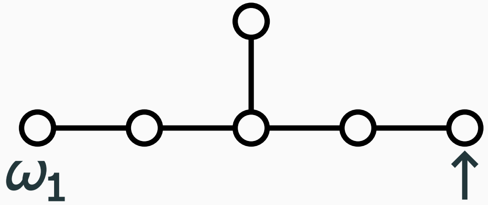
"Convenient splitting": W_i are just representations of a smaller quantum group, hence they have equally nice properties.

We consider the case with no multiplicity, i.e. splitting is canonical up to constants.

The case of $\mathfrak{d}_5 \subseteq \mathfrak{e}_6$



Fundamental representation $\dim V_{\omega_1} = 27$.

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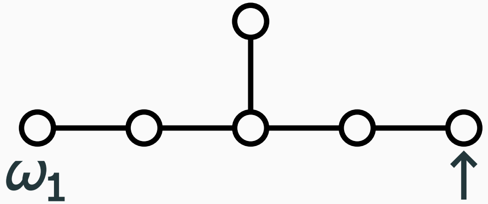

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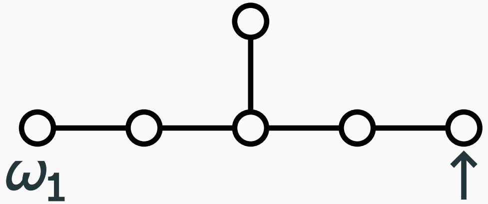

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Invariant of the trefoil:

$$q^{52}(q-1+q^{-1})^2(q+1+q^{-1})^2(q^2-1+q^{-2})$$
$$(q^3-1+q^{-3})(q^3+1+q^{-3})(q^4-1+q^{-4})$$
$$(q^{26}-q^{16}-q^{10}-q^6+q^{-10}+q^{-16}+q^{-20})$$

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Since $\mathbf{27} \cong \mathbf{10} \oplus \mathbf{16} \oplus \mathbb{1}$, the tensor $\check{R} : \mathbf{27} \otimes \mathbf{27} \to \mathbf{27} \otimes \mathbf{27}$ splits into a 9×9 matrix:

$A = q^{-1/3}$ | $B = q^{1/6}$

$C = q^{2/3}$ | $D = q^{1/6}$

$E = q^{-1/3} - q^{2/3}$ | $F = q^{-1/12}$

$G = \tau$ | $H = q^{-1/3}$

$I = q^{2/3}$ | $J = \tau^{\dagger}$

$K = (q^{-4/3} - q^{2/3} - q^{20/3} + q^{26/3})$ | $L = q^{-1/3}$

$M = (q^{-4/3} - q^{2/3})$ | $N = q^{-4/3}$

$$\tau\tau^{\dagger} = q^{-2/3}(q^4-1)^2(q^4-q^2+1)(q^4+1)$$

$\check{R}$		$\mathbf{10}\otimes$			$\mathbf{16}\otimes$			$\mathbf{1}\otimes$		
		10	**16**	**1**	**10**	**16**	**1**	**10**	**16**	**1**
$\mathbf{10}\otimes$	**10**	A								
	16				B					
	1							C		
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- Yang–Baxter equation determines the final few coefficients.

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Interpreting the trivalent node in $U_q(\mathfrak{d}_5)$

The main result involved a map $\mathbf{16} \otimes \mathbf{10} \to \overline{\mathbf{16}}$. Dually we have $\overline{\mathbf{16}} \otimes \mathbf{10} \to \mathbf{16}$.

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These are classically the **gamma matrices** acting on Weyl spinors.

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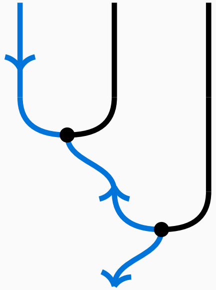
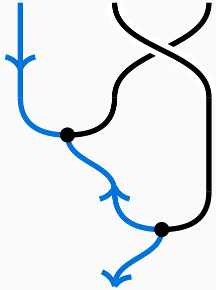
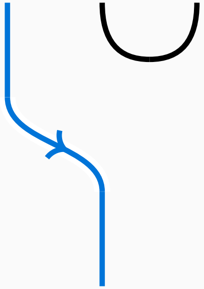
q  $+$  $= \frac{q-q^{-1}}{1-q^{-10}}$ 

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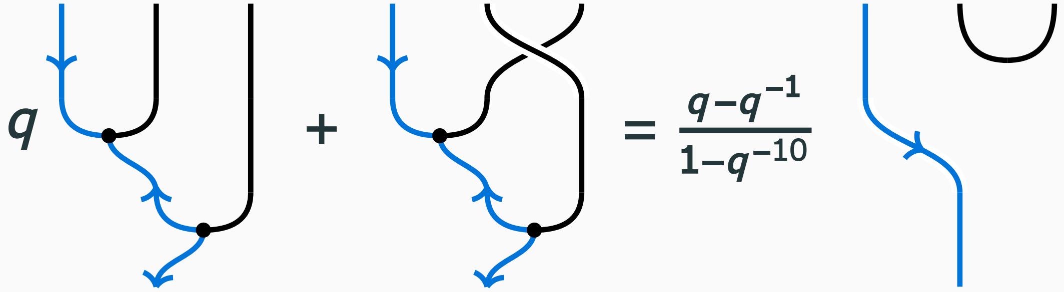

Agrees with quantized Clifford algebra of Faddeev, Reshetikhin, Takhtajan.

Further work

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As mentioned, there are three methods:

- Brute force definition
- Diagrammatic calculus
- Recursion formula to $\mathfrak{d}_n$ or $\mathfrak{a}_n$

Questions?