



DATA SCIENCE RETREAT®  
SINCE 2014



# Introduction to Probability & Statistics

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April 2025



# Content

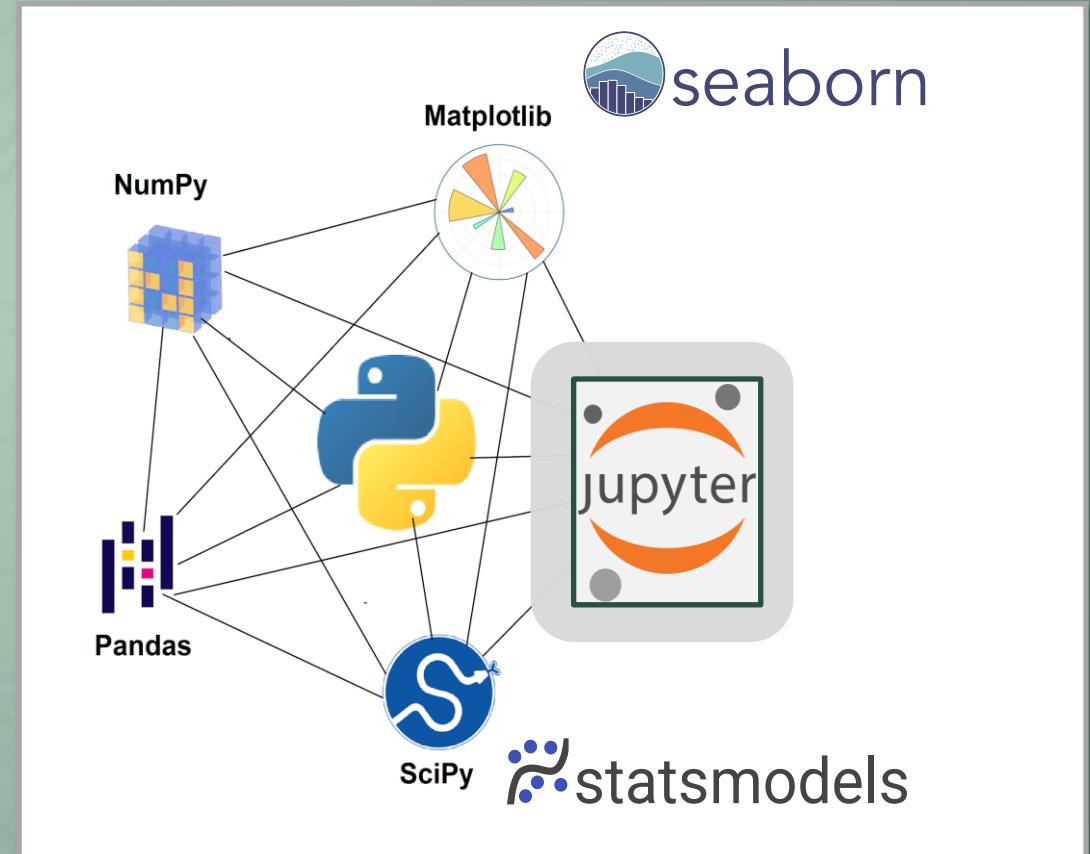
- **Technical Prerequisites**
- **Probability, Statistics and Information Theory**
- **Descriptive Statistics**
- **Combinatorics**
- **Joint and Conditional Probability**
- **Probability Distributions**
- **Structured Probabilistic (Graphical) Models**
- **Sampling, Monte Carlo**
- **Hypothesis Testing**
- **Model Estimation:**
  - Linear Regression
  - Binary Logistic Regression



# Technical Prerequisites

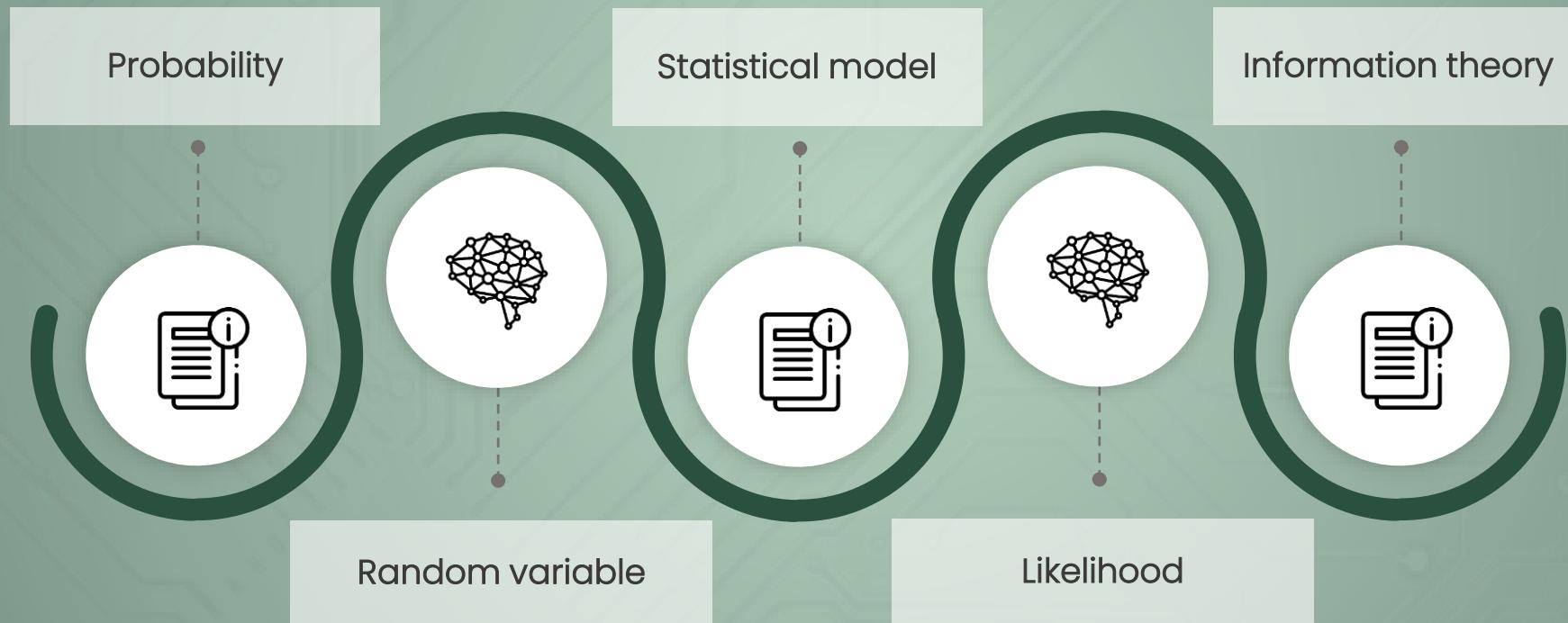
... means installation of Python environment with Jupyter

- GitHub Link:  
[https://github.com/MathforDataScience/DSR\\_Statistics1/](https://github.com/MathforDataScience/DSR_Statistics1/)
- List of packages:  
requirements.txt
- How to install:  
readme.txt
- Start Jupyter Notebook



# Probability, Statistical Models and Information Theory

Probability provides foundation for statistical models, which use probability to analyze data, make predictions, while information theory quantifies uncertainty and information content within those models.



# What is Probability?

Probability is measure quantifying the chance that random events will occur. It is expressed as number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty.

## Example

- Tossing a fair (unbiased) coin:
  - Two possible outcomes: "heads" and "tails"
  - Probability of "heads" = Probability of "tails" = 0.5 or 50%
  - Random event

## Application Areas

- Mathematics, Statistics, Finance, Gambling
- Science (including Physics)
- Artificial Intelligence/Machine Learning
- Computer Science, Game Theory, Philosophy

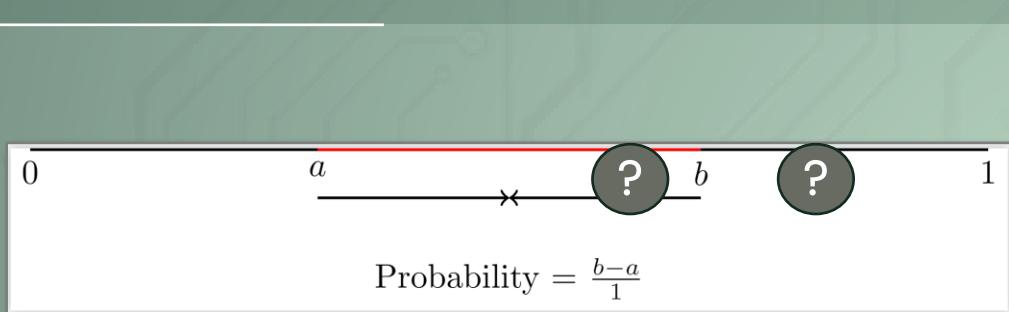
## Purpose of Probability

- Used to infer the expected frequency of events.
- Describes the underlying mechanics and regularities of complex systems.

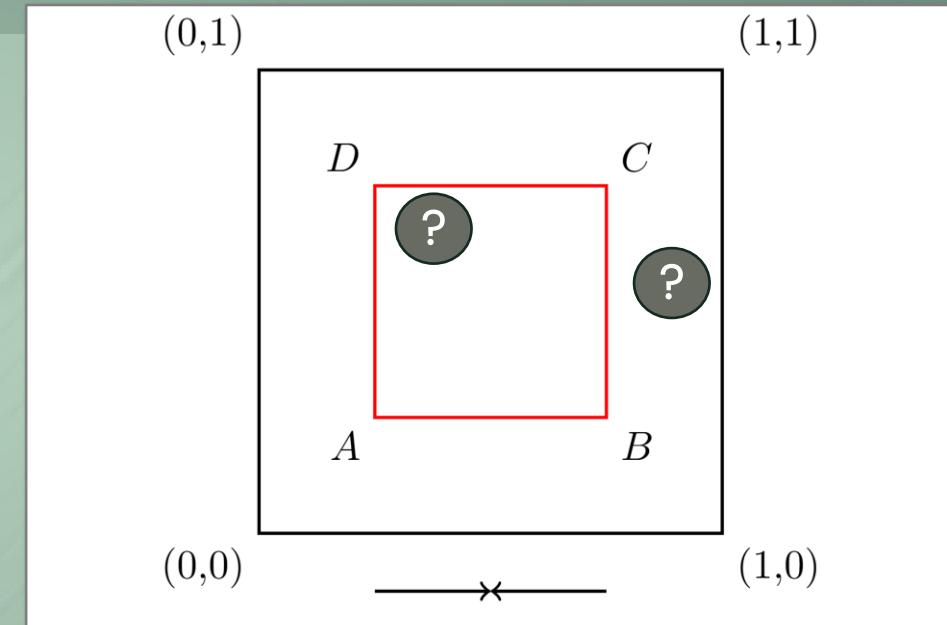


# Geometric Interpretation of Probability

... visualizes probabilities as areas (or volumes) within geometric space. Probability is proportional to size of favorable region relative to total possible region.



- Consider a point chosen at random on a line segment of length 1
- The probability that the point falls within the sub-segment is  $\frac{b-a}{1}$



$$\text{Probability} = \frac{\text{Area of } ABCD}{\text{Area of Unit Square}} = \frac{(b-a)^2}{1} = (b-a)^2$$

- Suppose a point is randomly selected within a unit square.
- The probability that the point lies within specific region (e.g., smaller square or circle) is equal to area of that region divided by area of the unit square (1)

# Random Variable

... is variable that takes on numerical values determined by the outcome of random experiment. It maps outcomes of random process to numbers.

| Aspect                   | Discrete Random Variable   | Continuous Random Variable  |
|--------------------------|--|---|
| Definition               | Takes on a countable number of distinct values.                    | Takes on an infinite number of possible values within a given range.              |
| Examples                 | Number of heads in 10 coin flips, number of students in a class.   | Height of students, time to run a marathon.                                       |
| Possible Values          | Finite or countably infinite set (e.g., $\{0, 1, 2, \dots\}$ ).    | Any value within a continuous range (e.g., $[0, \infty)$ , $(-\infty, \infty)$ ). |
| Probability Distribution | Described by a probability mass function (PMF).                    | Described by a probability density function (PDF).                                |
| Cumulative Distribution  | Step function with jumps at each possible value.                   | Smooth curve, the integral of the PDF.  |
| Sum of Probabilities     | Sum of probabilities for all possible values equals 1.             | The area under the PDF curve equals 1.  |
| Graphical Representation | Histogram or bar chart   | Continuous curve (e.g., bell curve for normal distribution).                      |
| Real-World Analogy       | Counting the number of occurrences (e.g., number of cars passing). | Measuring continuous quantities (e.g., measuring the exact weight).               |



# Examples of Random Variables

## Discrete vs. continuous

| Type of Random Variable    | Example                                     | Description  |
|----------------------------|---|--|
| Discrete Random Variable   | Number of Students in a Classroom           | Can only take specific integer values (e.g., 20, 21, 22).                        |
|                            | Number of Cars Passing Through a Toll Booth | Count of cars, taking on values like 0, 1, 2, etc.                               |
|                            | Number of Emails Received in a Day          | Exact count of emails received (e.g., 5 emails).                                 |
|                            | Number of Defective Items in a Batch        | Number of defective items, counted in whole numbers (e.g., 0, 1, 2)              |
| Continuous Random Variable | Height of Students in a Class               | Can take any value within a range, measured in units like centimeters            |
|                            | Time Taken to Run a Marathon                | Measured in hours, minutes, and seconds with infinite possible values.           |
|                            | Temperature Throughout the Day              | Can be any value within a temperature range, measured in degrees (e.g., 25.3°C). |
|                            | Amount of Milk in a Bottle                  | Volume can be measured to any degree of precision (e.g., 1.5 liters).            |

# Statistical Model

... is mathematical framework that represents relationships between variables in data and allows for analysis, inference, and prediction

## Variables:

- Dependent Variable: The outcome or response being predicted or explained.
- Independent Variables: The predictors or inputs that influence the dependent variable.

## Parameters

- Constants estimated from the data that define the model's specific form.
- Through estimated parameters the statistical model becomes generalization rule for the data sample referring to population

## Probability Distributions

Assumptions about how data is distributed (e.g., normal distribution).

## Example: Linear Regression $Y = \beta_0 + \beta_1 X + \varepsilon$

- $Y$ : Outcome, dependent variable
- $X$ : Predictor, independent variable
- $\beta_0, \beta_1$ : Parameters that are estimated
- $\varepsilon$ : Random error, unpredictable part of model



# Likelihood vs Probability

Likelihood measures how well a set of parameters explains observed data.

Probability measures the chance of a specific event occurring.

| Aspect            | Probability  | Likelihood   |
|-------------------|--|--|
| Definition        | The measure of the chance that a specific event will occur.                  | The measure of how likely a particular set of parameters is, given the observed data.  |
| Context           | Used to describe the probability of future events based on known conditions. | Used in statistical models to estimate parameters based on observed data.              |
| Focus             | Focuses on predicting the occurrence of outcomes.                            | Focuses on finding the best parameters that explain the observed data.                 |
| Example           | Probability of rolling a 6 on a fair die is 1/6.                             | Likelihood of a die being biased towards 6 given repeated observations of rolling a 6. |
| Mathematical Form | Expressed as $P(A)$ , where A is the event.                                  | Expressed as $L(\theta   X)$ , where $\theta$ are parameters and X is data.            |
| Use in Inference  | Probability helps to infer about future events or outcomes.                  | Likelihood is used to estimate the parameters of a statistical model.                  |

→ Statistical model is good predictor, i.e. generalization of data, if parameters are approximated in a way that likelihood is maximized.



# Information Theory

... is branch of applied mathematics focused on quantifying information in signals.

Main idea: Learning about unlikely event is more informative than learning about likely event.

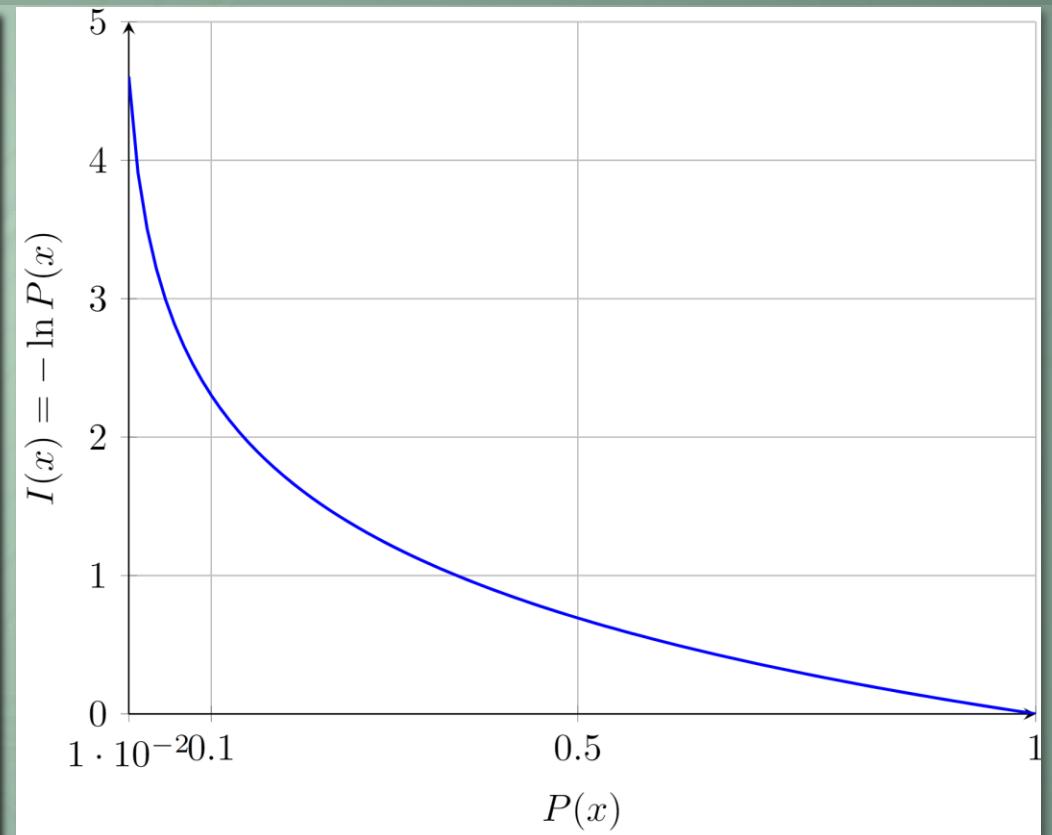
## Key Properties of Information:

- Low frequency events: Have high information content.
- High frequency events: Have low information content.
- Independent events: Information should be additive.

## Self-Information Formula

- mathematical way to quantify how much information is gained from observing outcome of random event.
- Rare events are more informative, surprising than common ones

$$I(x) = -\ln P(x)$$



# Shannon Entropy

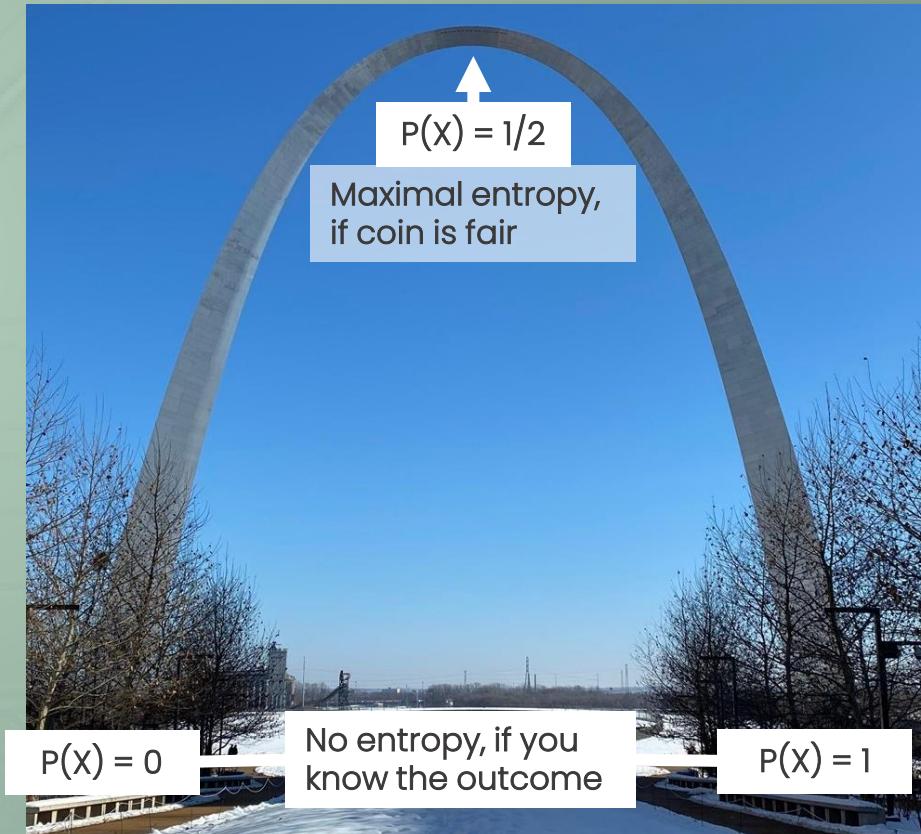
... is measure of randomness in set of possible outcomes. It quantifies expected amount of information/surprise, that event drawn from a probability distribution will provide.

- Weighted average of self-information, i.e.  $-\ln P(x)$  of all possible outcomes.
- Represents level of uncertainty/surprise inherent in outcome, examples:
  - **High Entropy:**  
Fair six-sided die has higher entropy than biased die where one number appears with very high probability.
  - **Low Entropy:**  
If die is heavily biased towards one particular number, entropy is low.

## Formula

$$H(X) = - \sum_{i=1}^n P(x_i) \ln P(x_i)$$

→ If any of the terms  $P(x_i), \ln P(x_i)$  become 0, then  $H(X)$  becomes 0



Gateway Arch, St. Louis, USA

# Shannon Entropy and Outliers

Outliers can increase the entropy if they represent rare but possible outcome, adding to the distribution's unpredictability.



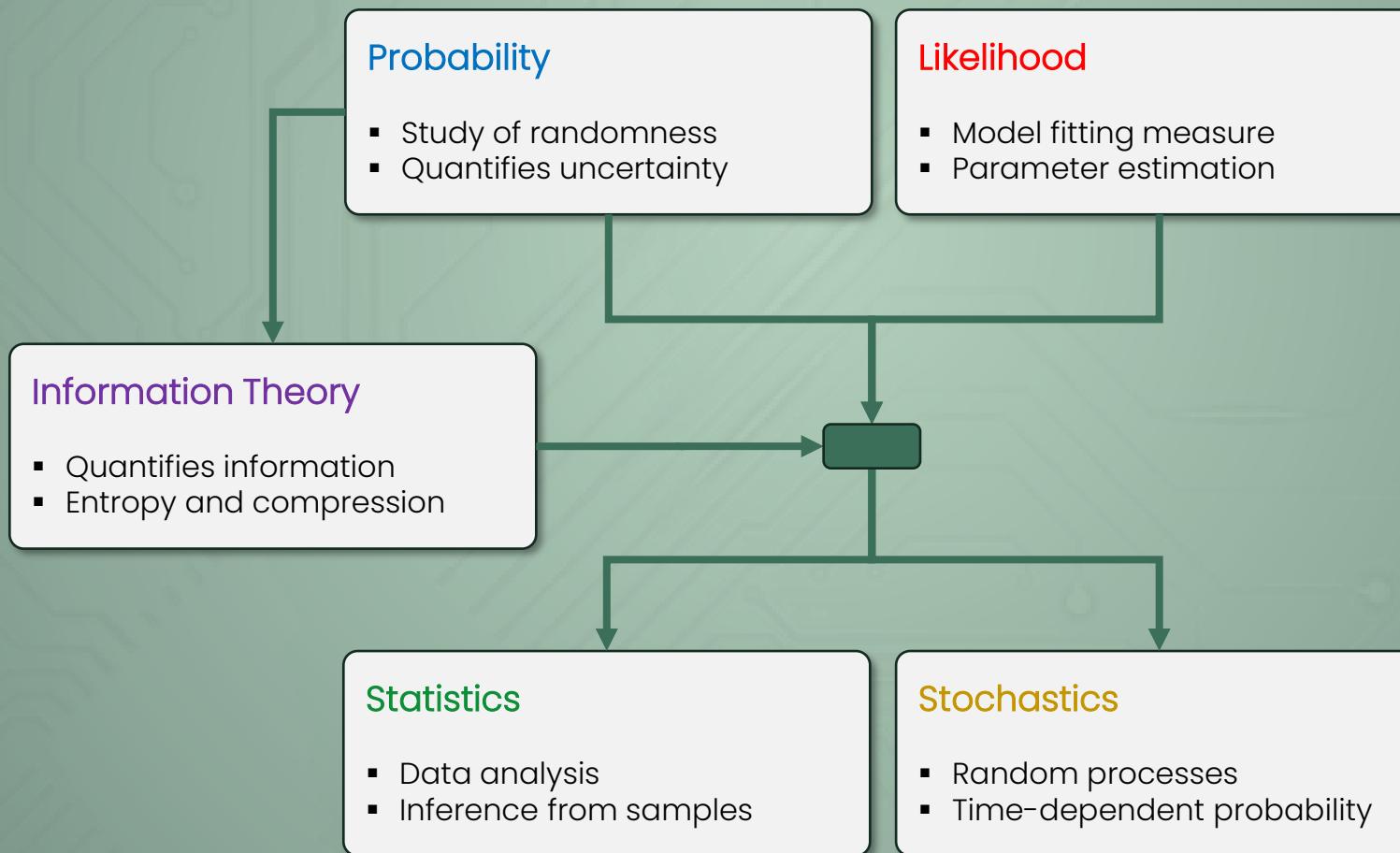
Black Swan: Symbol for rare event  
Book: The Black Swan, Taleb, 2007



"I have seen horses vomiting in front of pharmacy"  
Translation of German phrase

# Probability, Likelihood, Information, Statistics, Stochastics

Probability quantifies uncertainty, Statistics analyzes data, Stochastics applies both to time-dependent processes, Likelihood assesses model fit, Information theory measures data content



# Descriptive statistics

... involves the organization, summarization, and visualization of data. It provides simple summaries about the sample and the measures

Types of Data  
Sample vs. Population



Measures:  
Skewness and kurtosis



Graphical representation of data



Measures: central tendency, dispersion

Quantiles

Exercises

# Types of Data

## Definition and examples of Qualitative and Quantitative data

| Type of Data  | Subtype of data  | Example   |
|---|--|---|
| <b>Quantitative / Numerical:</b> <ul style="list-style-type: none"><li>• can be measured</li><li>• or counted</li></ul>   | <b>Discrete:</b> <ul style="list-style-type: none"><li>• can only take certain values</li></ul>              | <ul style="list-style-type: none"><li>• number of students in a class</li></ul>   |
|   | <b>Continuous:</b> <ul style="list-style-type: none"><li>• can take any value within certain range</li></ul> | <ul style="list-style-type: none"><li>• Age, Height, Weight, Temperature, Income</li></ul>                                  |
|   | <b>Interval</b>  | <ul style="list-style-type: none"><li>• Temperature</li><li>• IQ scores</li><li>• Years</li></ul>                           |
|   | <b>Ratio</b>   | <ul style="list-style-type: none"><li>• Probability</li></ul>   |
| <b>Qualitative / Categorical:</b> <ul style="list-style-type: none"><li>• non-numerical</li><li>• often relates to subjective qualities, characteristics, or descriptions</li></ul> | <b>Nominal:</b> <ul style="list-style-type: none"><li>• No order, hierarchy or sequence</li></ul>            | <ul style="list-style-type: none"><li>• Colors</li><li>• Types of cuisine</li><li>• Genders</li><li>• Blood types</li></ul> |
|   | <b>Ordinal:</b> <ul style="list-style-type: none"><li>• can be arranged in a particular order</li></ul>      | <ul style="list-style-type: none"><li>• Survey responses</li><li>• Educational level</li><li>• Military rank</li></ul>      |

... other types of data: binary, time-series, geodata, textual, multimedia, etc.

# Sample vs Population

A well-chosen sample should be representative of the population, allowing statisticians to draw accurate conclusions about the population based on sample data

**Population**

- Entire group of individuals or objects of interest
- Often too large to study completely
- Represented by  $N$  (size)

**Sample**

- Subset of the population
- Used to make inferences about the population
- Represented by  $n$  (size)

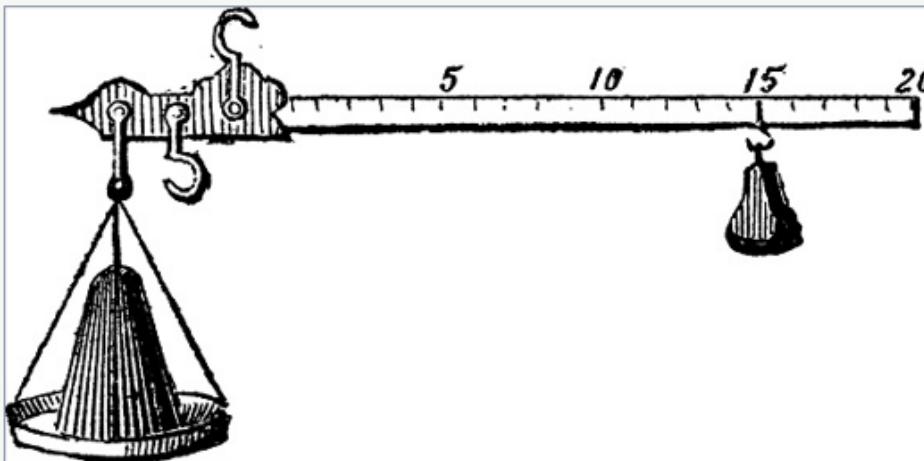


# Measures of Central Tendency

... calculate typical central points that describe or represent dataset

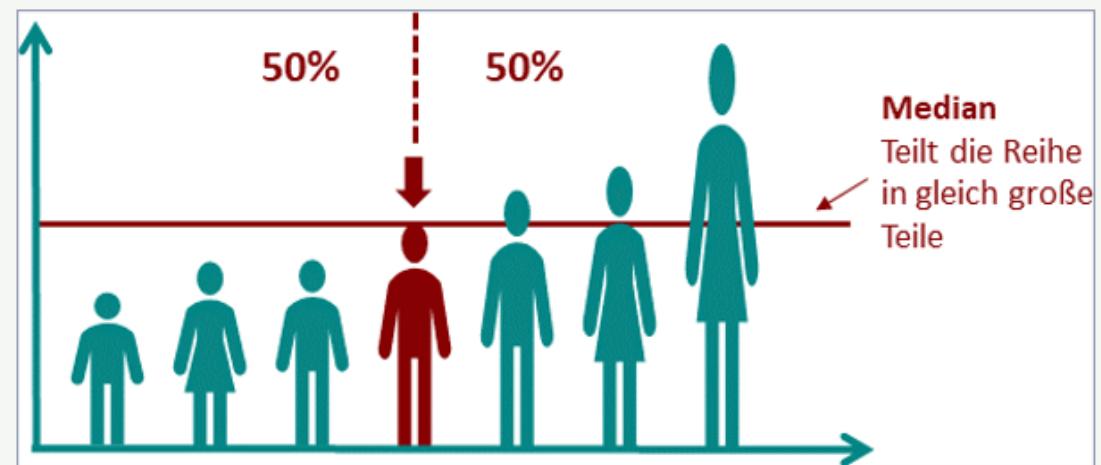
## Arithmeric Mean = Average

- Sum of all data points divided by number of data points
- E.g.:  $(1+2+3)/3 = 2$
- $\bar{X}$  = Sample Mean;  $\mu$  = Population Mean



## Median

- middle value in an ordered dataset
- E.g.: For {1, 2, 3}, the median is 2
- **Interesting because of outlier resistance!**



## ... Other measures of centrality:

- Geometric mean: e.g. population growth, investment return
- Harmonic mean: e.g. F-score (Machine Learning, binary classification, evaluation of results)

# Expectation or Expected Value

Random variable's expected value represents arithmetic mean / average of large number of independent realizations of the random variable

- X is a random variable
- with finite number of outcomes ( $x_1, x_2, \dots, x_k$ )
- occurring with probabilities ( $p_1, p_2, \dots, p_k$ ).
- **For discrete variables:**
  - $E[X] = \sum_{i=1}^k x_i p_i = x_1 p_1 + x_2 p_2 + \dots + x_n p_n$
- **For continuous variables:**
  - we compute it with an integral.

## Example: Rolling a Fair Six-Sided Die

Let X be the random variable representing the outcome of rolling a fair six-sided die.

Outcomes ( $x_i$ ): 1, 2, 3, 4, 5, 6

Probabilities ( $p_i$ ): 1/6 for each outcome (since it's a fair die)

To calculate the expected value  $E[X]$ , we'll use the formula:

$$E[X] = \sum_{i=1}^k x_i p_i = x_1 p_1 + x_2 p_2 + \dots + x_k p_k$$

$$\begin{aligned} E[X] &= 1 * (1/6) + 2 * (1/6) + 3 * (1/6) + 4 * (1/6) + 5 * (1/6) + 6 * (1/6) \\ &= (1 + 2 + 3 + 4 + 5 + 6) / 6 \\ &= 21 / 6 \\ &= 3.5 \end{aligned}$$

Therefore, the expected value of rolling a fair six-sided die is 3.5.

# Measures of Dispersion

... provide information about how data values are distributed around central tendency

| Measure of Dispersion | Definition  | Example   |
|-----------------------|---|---|
| Range                 | The difference between the highest and lowest values in a dataset | For $\{1, 2, 3, 4\}$ , the range is $4 - 1 = 3$       |
| Sum of Squares        | $SS = \sum (x - \mu)^2$   | For $\{1, 2, 3\}$ , SS: $(1+0+1) = 2$                 |
| Variance (population) | $\sigma^2 = \frac{\sum(x-\mu)^2}{n}$                              | For $\{1, 2, 3\}$ , pop. variance: $(1+0+1)/3 = 0.67$ |
| Variance (sample)     | $s^2 = \frac{\sum(x-\mu)^2}{n-1}$                                 | For $\{1, 2, 3\}$ , sample variance: $(1+0+1)/2 = 1$  |
| Standard Deviation    | The square root of the variance                                   | The square root of 0.67 is approximately 0.82         |

# Variance and Standard Deviation

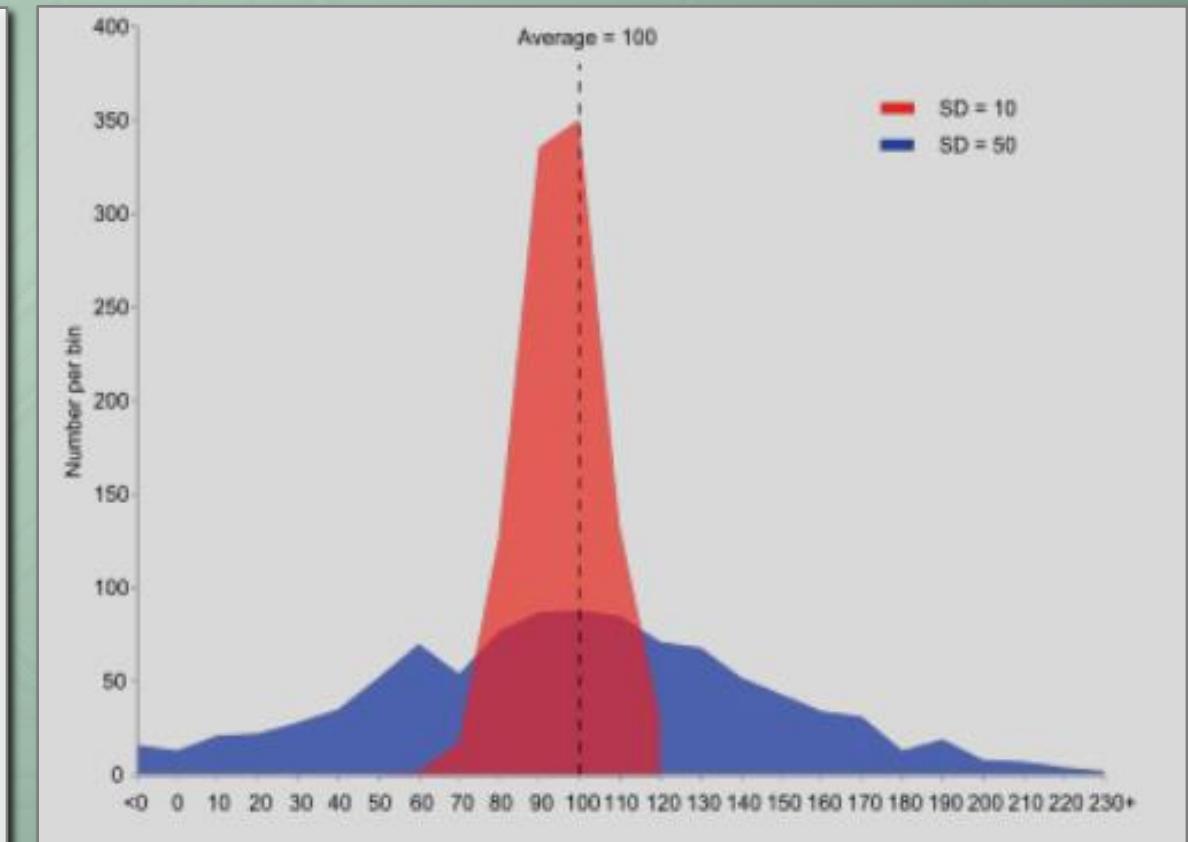
Variance  $VAR(X) = E[(X - \mu)^2]$  of standard X (function) is expected value of squared deviation from mean X;  $\mu = E[X]$

## Example with Standard Deviation = 10

- Heights of certain species of plants
- normally distributed with mean height of 50 cm
- standard deviation of 10 cm.
- Heights are spread around mean
- Most plants' heights ranging 40 - 60 cm.

## Example with Standard Deviation = 50:

- Annual income of group of people in certain profession
- normally distributed with mean income of \$200,000
- standard deviation of \$50,000.
- Incomes vary widely
- most individuals earn \$150,000 - \$250,000.



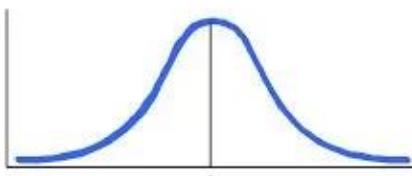
# Skewness and Kurtosis

... describe asymmetry of distributions deviation from normal distribution

## Skew

- Measure of asymmetry of the probability distribution of a real-valued random variable about its mean
- Positive (negative): Long tail on the right (left)

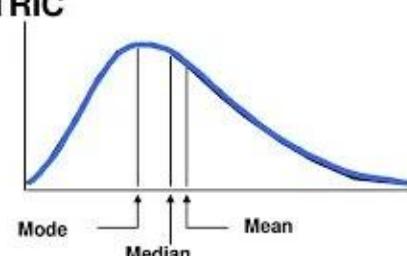
Describe the shape, center, and spread of a distribution... for shape, see below...



SYMMETRIC

Mean  
Median  
Mode

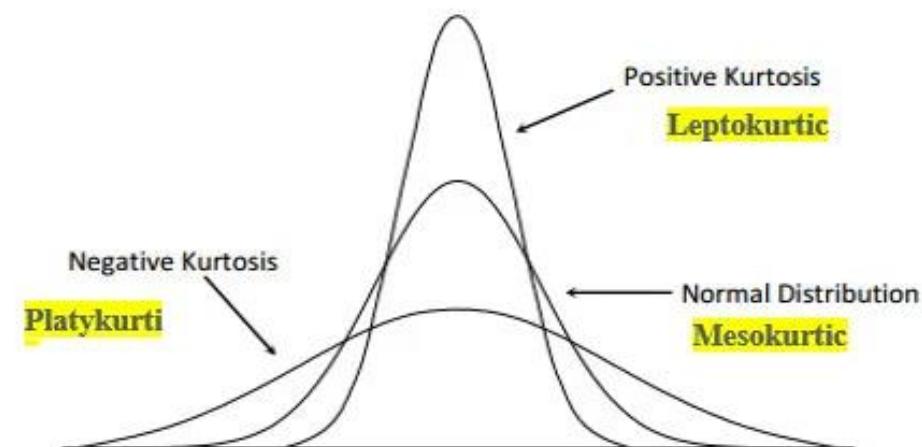
SKEWED LEFT  
(negatively)



SKEWED RIGHT  
(positively)

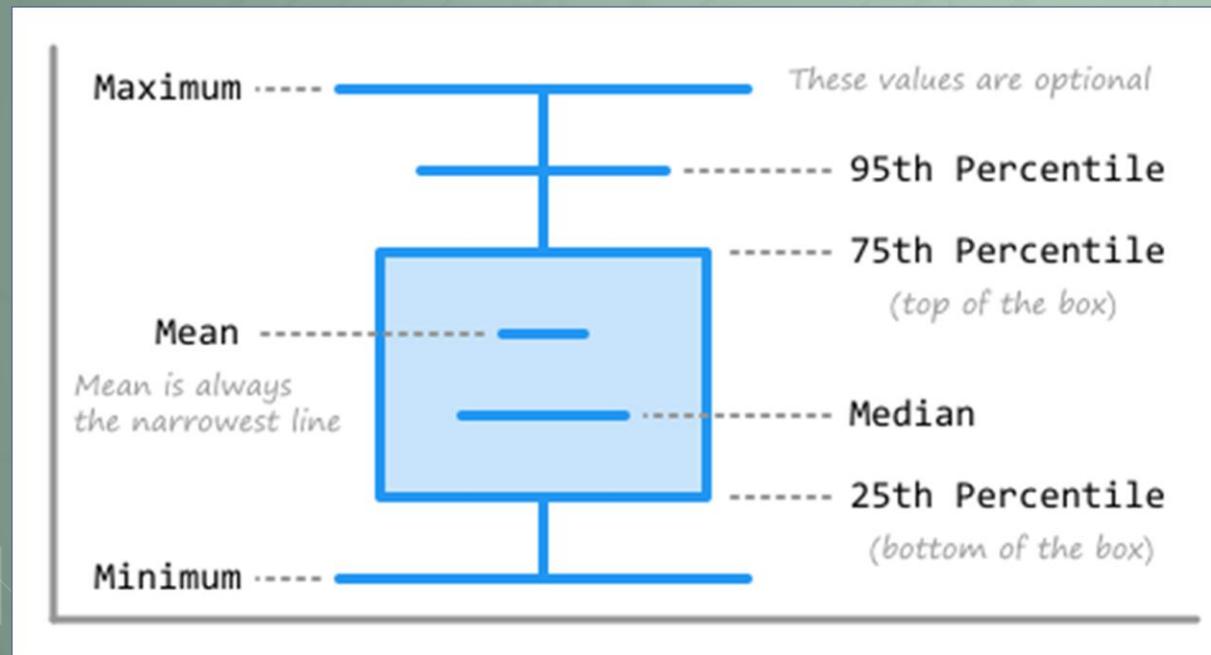
## Kurtosis

- defines how heavily the tails of a distribution differ from the tails of a normal distribution
- Outliers may facilitate positive kurtosis



# Quantiles

... statistical measures that divide a dataset into intervals of equal probability



## Quantiles ...

- are used to understand data distribution and identify outliers
- partition data into several subsets containing equal number of observations.

## Several types of quantiles:

- Quartiles: divide the data into four equal parts.
- Deciles: These divide the data into ten equal parts.
- Percentiles: These divide the data into 100 equal parts.
- Median: separates data into two halves, with 50% of the data falling below and 50% above the median.

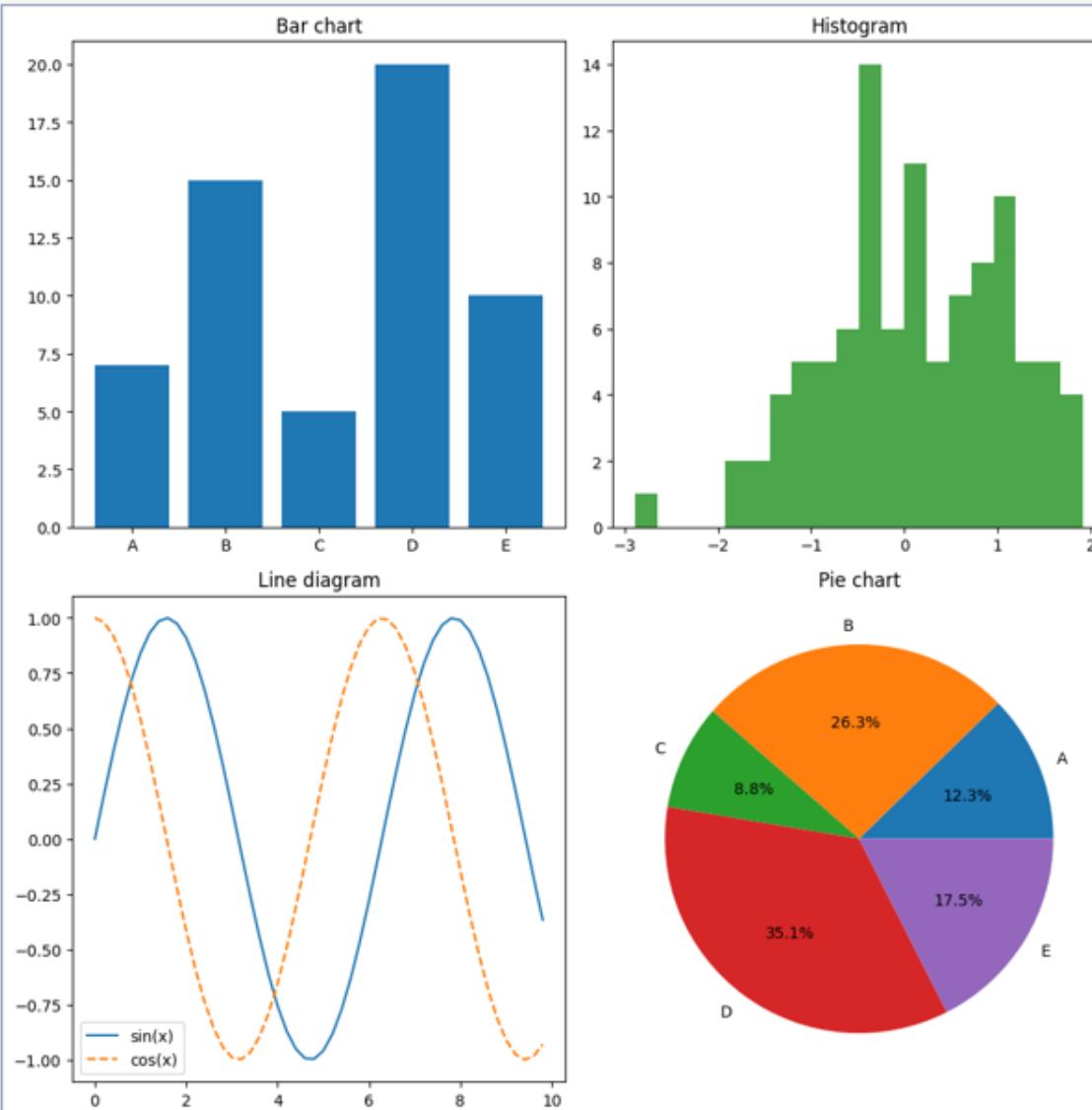
# Graphical Representation of Data

## Input:

- Categorical
- Nominal
- unordered

## Output:

- Numerical
- Continuous
- Absolute



## Input:

- Numerical
- continuous

## Output:

- Numerical
- continuous

## Input:

- Categorical
- Nominal
- ordered

## Output:

- Numerical
- Continuous
- Absolute

## Input:

- Categorical
- Nominal
- Unordered or ordered

## Output:

- Numerical
- Continuous
- relative

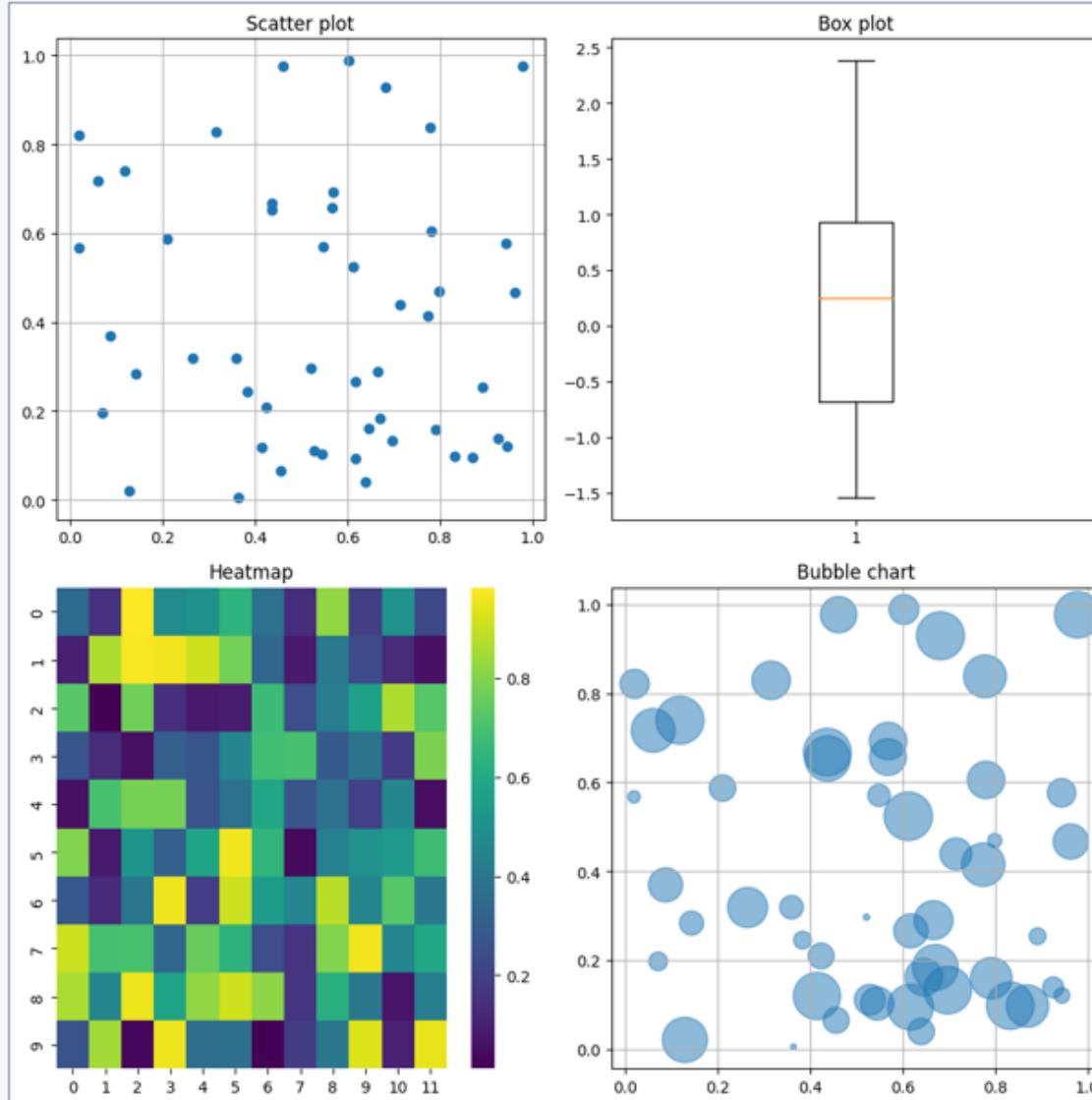
# Graphical Representation of Data

## Input:

- 2 Features
- Both numerical

## Output:

- Boolean (1;0)



## Input:

- 2 Features
- Both categorical

## Output:

- Numerical

## Input:

- 1 feature
- numerical

## Output:

- At least ordered
- Or numerical

## Input (like scatter plot):

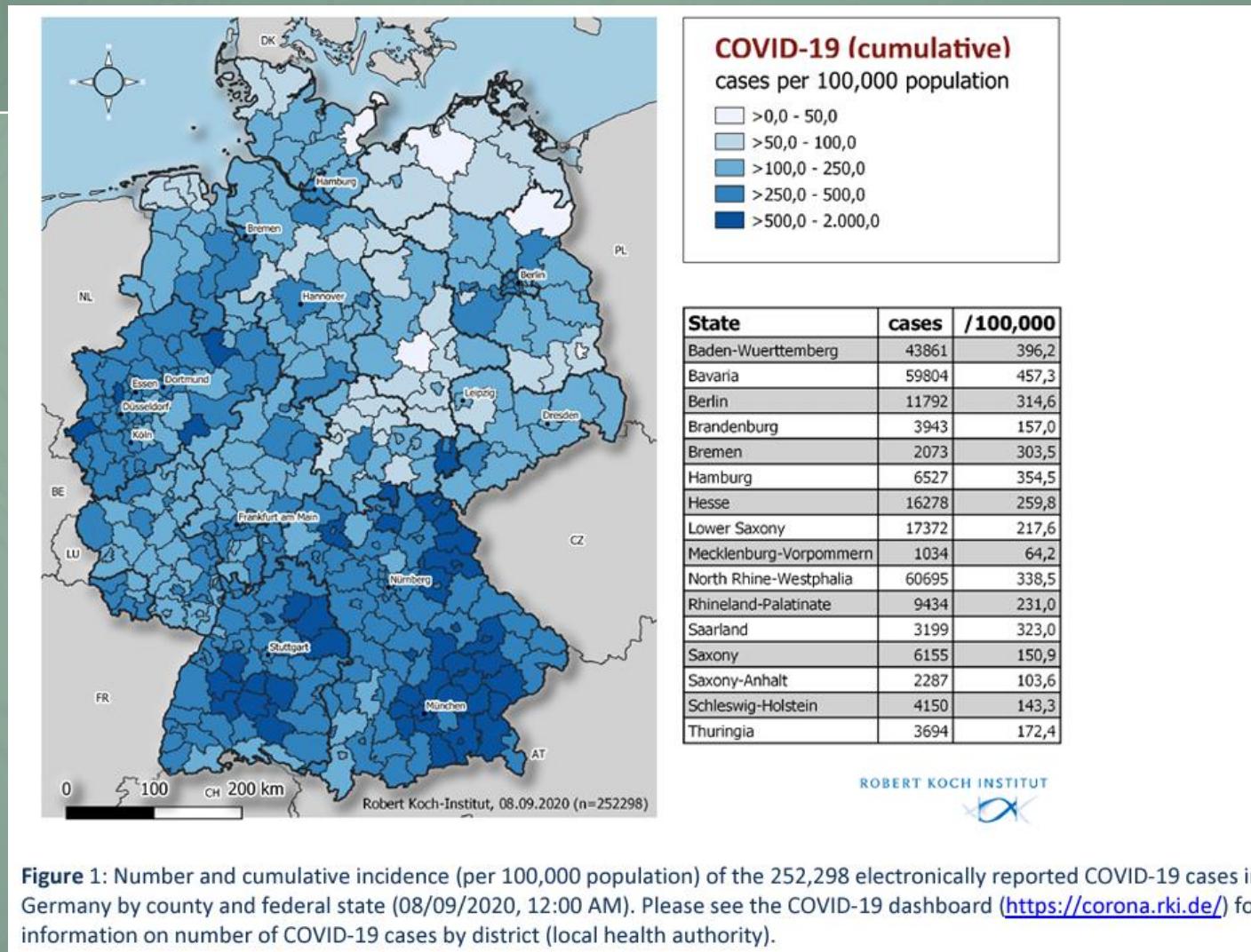
- 2 Features
- Both numerical

## Output:

- Numerical

# Graphical Representation of Data

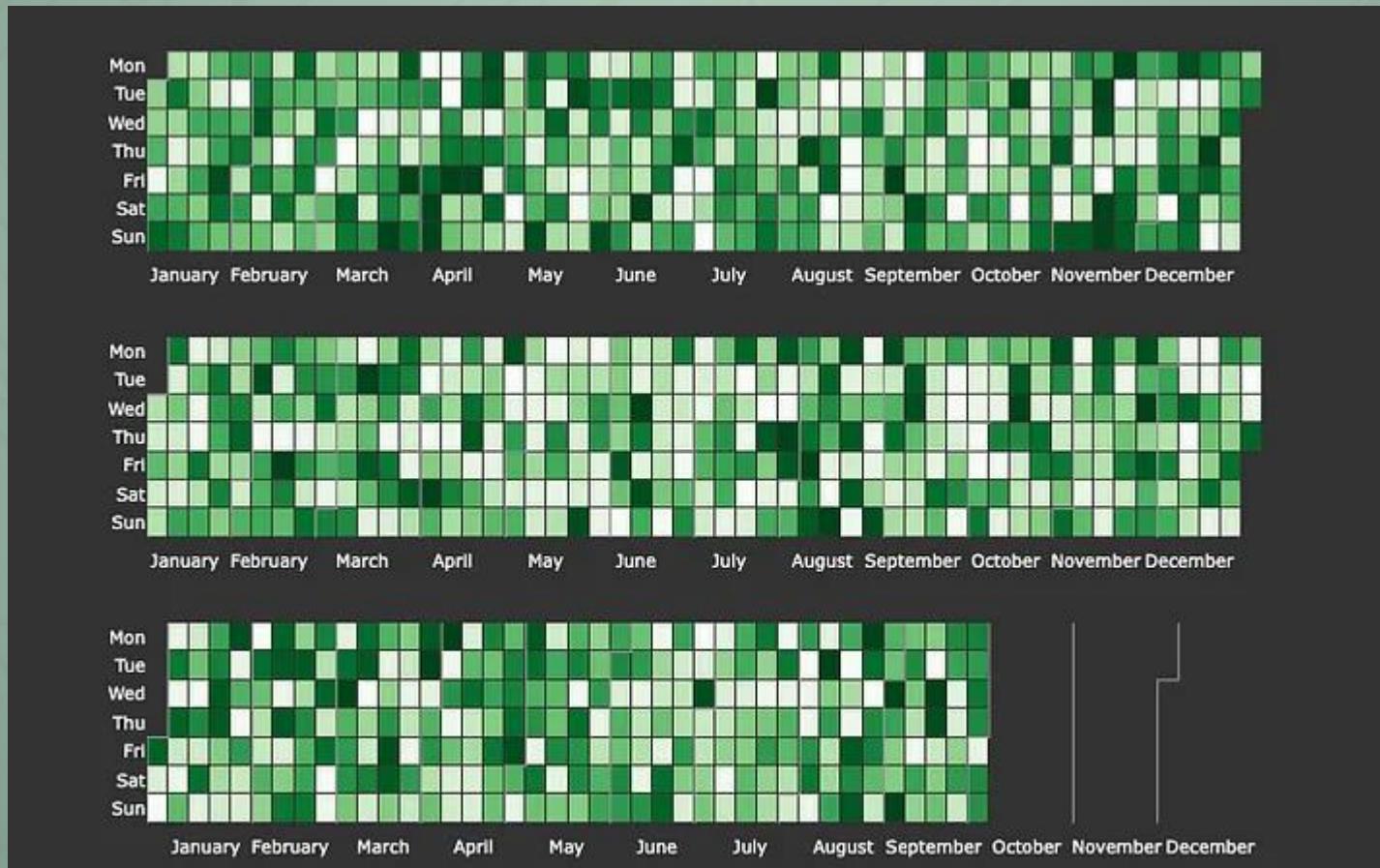
... Geovisualization with Heat Map. Example: COVID incidents Germany



# Graphical Representation of Data

... Calendar with Heat Map

Example: Visualization of visitors' prediction over the year



# Exercises

**Exercise 1:**

Question: Given the array `numbers = np.array([1, 2, 3, 4, 5, 6, 7, 8, 9, 10])`, compute the mean of the numbers.

**Exercise 2:**

Question: Given the array `numbers` from Exercise 1, compute the median of the numbers.

**Exercise 3:**

Question: Compute the variance of the numbers.

a) Use population variance b) Use sample variance

`ages = [21, 23, 25, 27, 29, 31, 33]`

b) Let's say we take a sample of 5 ages from the original group: `sample_ages = [21, 23, 27, 31, 33]`

**Exercise 4:**

Question: Given the array `numbers` from Exercise 1, compute the standard deviation of the numbers.

**Exercise 5:**

Question: Given the array `numbers` from Exercise 1, compute the first quartile (25th percentile), the second quartile (50th percentile, or median), and the third quartile (75th percentile) of the numbers.

**Exercise 6:**

show with python plots in a 2x2 subfigure matrix for ...

bar chart, histogram, line diagram, pie diagram

Scatter plot, Box plot (Box-and-whisker plot), Heatmap, bubble chart

# Combinatorics

... is concerned with the study of countable, discrete structures and includes the arrangement, combination, and permutation of sets.

Permutation



Combination



... with replacement



k-permutation

Pascal's triangle

Exercises

# Permutation

... is an arrangement of objects in a specific order.

**Table 5-1**

**Possible Rearrangements for  
Four People Sitting in a Straight Line**

| Rearrangement # | Listing               | Rearrangement # | Listing               |
|-----------------|-----------------------|-----------------|-----------------------|
| 1               | Tim, Syd, Elena, Mark | 13              | Elena, Tim, Syd, Mark |
| 2               | Tim, Syd, Mark, Elena | 14              | Elena, Tim, Mark, Syd |
| 3               | Tim, Elena, Syd, Mark | 15              | Elena, Syd, Tim, Mark |
| 4               | Tim, Elena, Mark, Syd | 16              | Elena, Syd, Mark, Tim |
| 5               | Tim, Mark, Syd, Elena | 17              | Elena, Mark, Tim, Syd |
| 6               | Tim, Mark, Elena, Syd | 18              | Elena, Mark, Syd, Tim |
| 7               | Syd, Tim, Elena, Mark | 19              | Mark, Tim, Syd, Elena |
| 8               | Syd, Tim, Mark, Elena | 20              | Mark, Tim, Elena, Syd |
| 9               | Syd, Elena, Tim, Mark | 21              | Mark, Syd, Elena, Tim |
| 10              | Syd, Elena, Mark, Tim | 22              | Mark, Syd, Tim, Elena |
| 11              | Syd, Mark, Tim, Elena | 23              | Mark, Elena, Tim, Syd |
| 12              | Syd, Mark, Elena, Tim | 24              | Mark, Elena, Syd, Tim |

## The number of permutations

- without repetition
- of a set of  $n$  objects is given by:

$$P(n) = n!$$

For  $n = 4$  there are

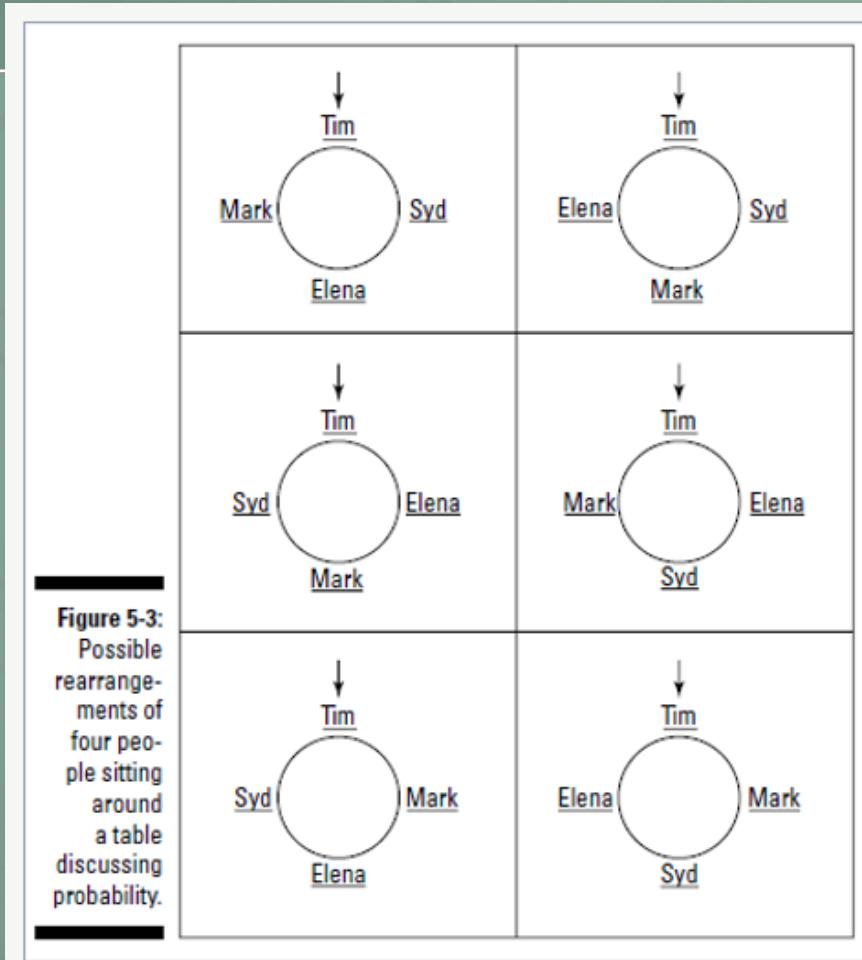
$$P(4) = 4! = 4 * 3 * 2 * 1 = 24$$

possibilities.

Here: No replacement

# Permutation

... is an arrangement of objects in a specific order.



## Example

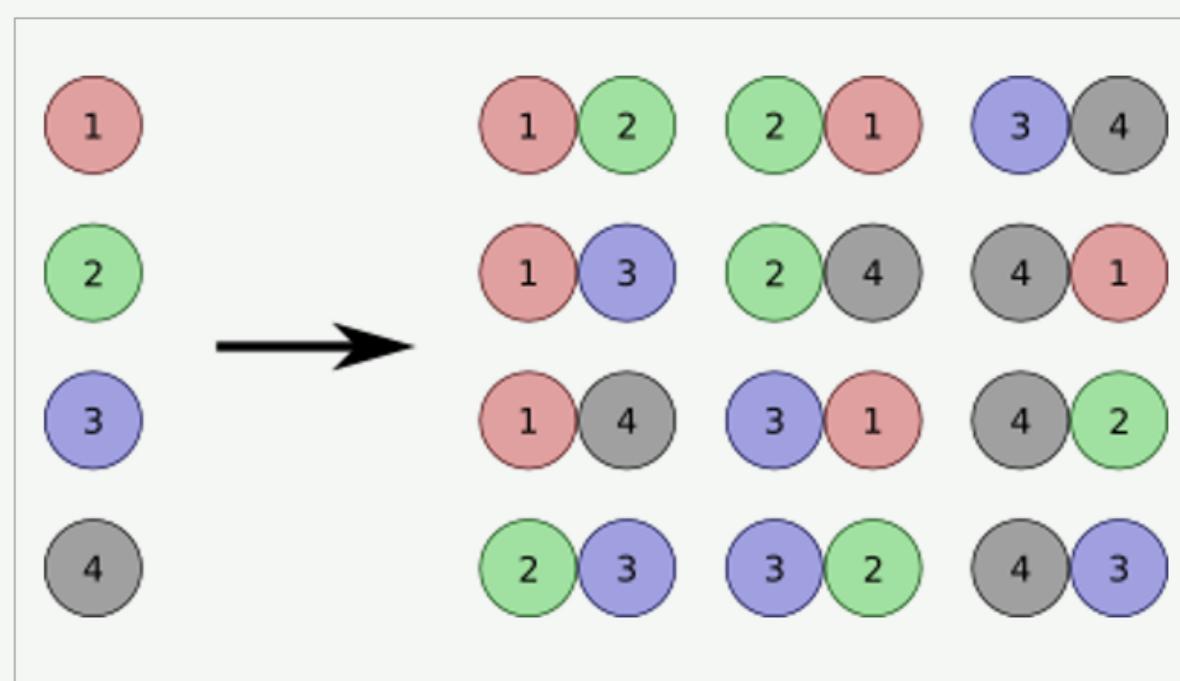
- 4 people sit at a table
- Restriction: The position of one of them is fixed.
- How many possible arrangements are there?

$$P(3) = 3! = 3 * 2 * 1 = 6$$

Here: No replacement

# K-permutation

... is an ordered arrangement of k elements selected from that set. In German it is called "Variation"



[File:Kpermutation.png - Wikimedia Commons](#)

Here: No replacement

## The number of k-permutations

- without repetition
- of a set of  $n$  objects and  $k$  selections is given by

$$P(n, k) = \frac{n!}{(n-k)!}$$

For  $n = 4$  and  $k = 2$  there are

$$P(4, 2) = \frac{4 \cdot 3 \cdot 2}{2} = \frac{24}{3 \cdot 2} = 12$$

possibilities.

Other k-permutations:

$$P(4, 1) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{24}{3 \cdot 2} = 4$$

$$P(4, 3) = \frac{4 \cdot 3 \cdot 2}{1} = \frac{24}{1} = 24$$

$$P(4, 4) = \frac{24}{0!} = 24$$

# Combination

... is a selection of items from a larger set where the order of selection does not matter.



Here: No replacement

## The number of combinations

- without repetition
- of a set of  $n$  objects and  $k$  selections is given by

$$C(n, k) = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

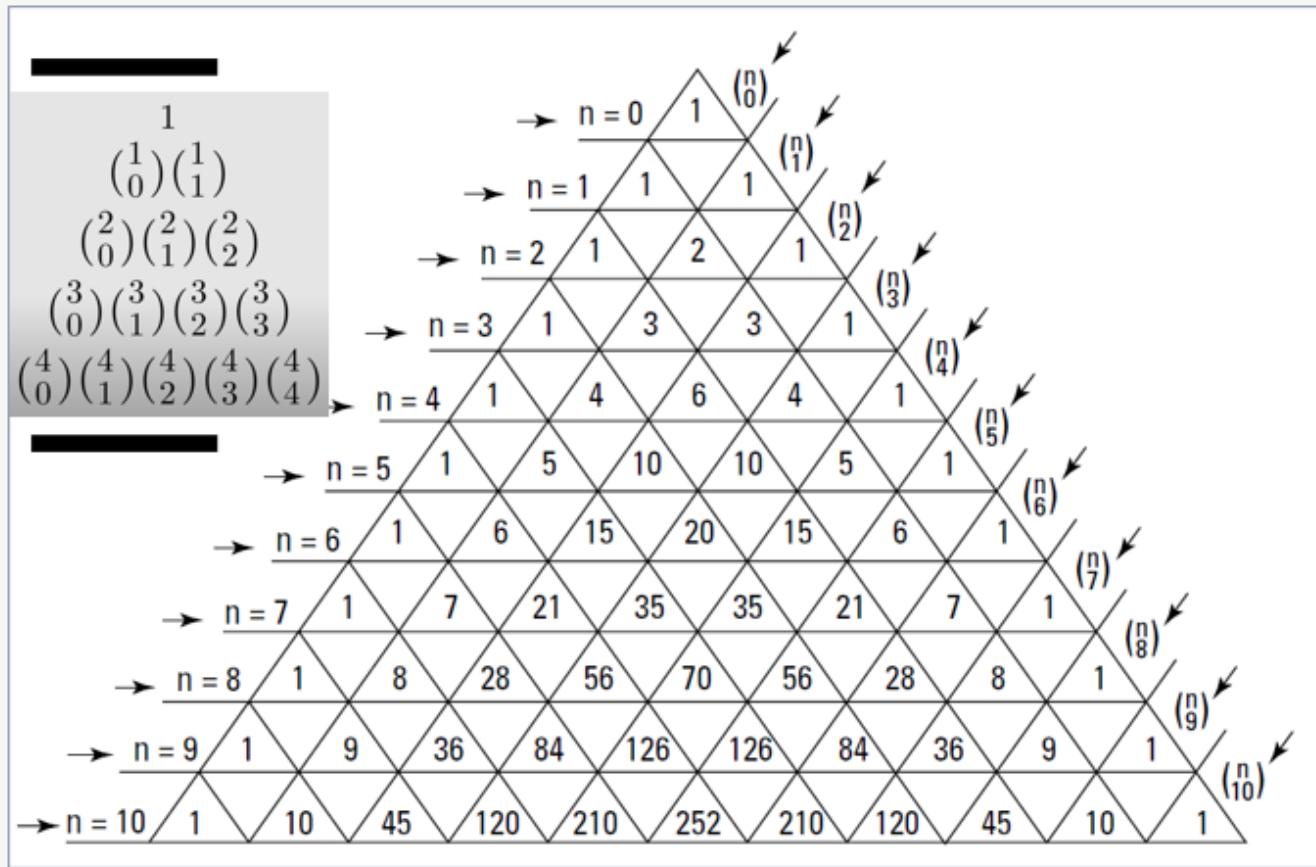
In the example of German lottery there are

- 49 balls
  - 6 selected balls
- i.e. there are

$$C(49, 6) = \binom{49}{6} = 13.983.816 \text{ possibilities.}$$

# Pascal's triangle

... is a triangular array of numbers in which each number is the sum of the two directly above it. It is used for binomial expansions, combinatorics, probability calculations, etc.



## Binomial theorem

- Coefficients from binomial expansion:

$$(a + b)^0 = 1$$

$$(a + b)^1 = 1a + 1b$$

$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

- Sum of rows  $\rightarrow 2^n$

## Coming in next chapter:

- $\binom{n}{k}$ : crucial for explaining distributions like binomial distribution

# Combinatorics with Replacement

The number of ways to choose k items from n distinct items, where each item can be chosen more than once and the order doesn't matter.

| Concept ...with Replacement | Definition   | Formula                          | Example  |
|-----------------------------|--|----------------------------------|--|
| Permutation                 | <ul style="list-style-type: none"><li>Arrangement of items where order matters</li><li>each item can be selected more than once.</li></ul>       | $\frac{n!}{k_1! * \dots * k_s!}$ | <ul style="list-style-type: none"><li>Choosing 3-digit PIN from 10 digits (0-9).</li><li>Possible permutations: <math>10^3 = 1000</math> combinations.</li></ul>           |
| k-Permutation               | <ul style="list-style-type: none"><li>Subset of k items chosen</li><li>arranged from n items</li><li>allowing repetition.</li></ul>              | $n^k$                            | <ul style="list-style-type: none"><li>Selecting, arranging 2 letters from the alphabet with repetition.</li><li>Example: <math>26^2 = 676</math> possibilities.</li></ul>  |
| Combination                 | <ul style="list-style-type: none"><li>Selection of items where order does not matter</li><li>each item can be selected more than once.</li></ul> | $\binom{n + k - 1}{k}$           | <ul style="list-style-type: none"><li>Choosing 3 fruits from basket of apples, oranges, and bananas</li><li>where repetition is allowed</li><li>10 combinations.</li></ul> |

# Exercises

## Exercise 1: Count the permutations

Write a function that returns the number of all possible permutations of a list.

```
def count_permutations(lst): # Your code here  
    print(count_permutations(['A', 'B', 'C']))
```

Additionally: Try also lists with other lengths

## Exercise 2: Generate all permutations of a list: `l1 = ['A', 'B', 'C']`

## Exercise 3: Count the k-permutations

Write a function that returns the number of all possible k-permutations of a list.

## Exercise 4: Generate k-permutations of a list

In this exercise, your task is to write a function that generates all possible k-permutations of a given list using `itertools`.

## Exercise 5: Generate all combinations of size k

Write a function that generates all possible combinations of size k from a given list of size n=8.

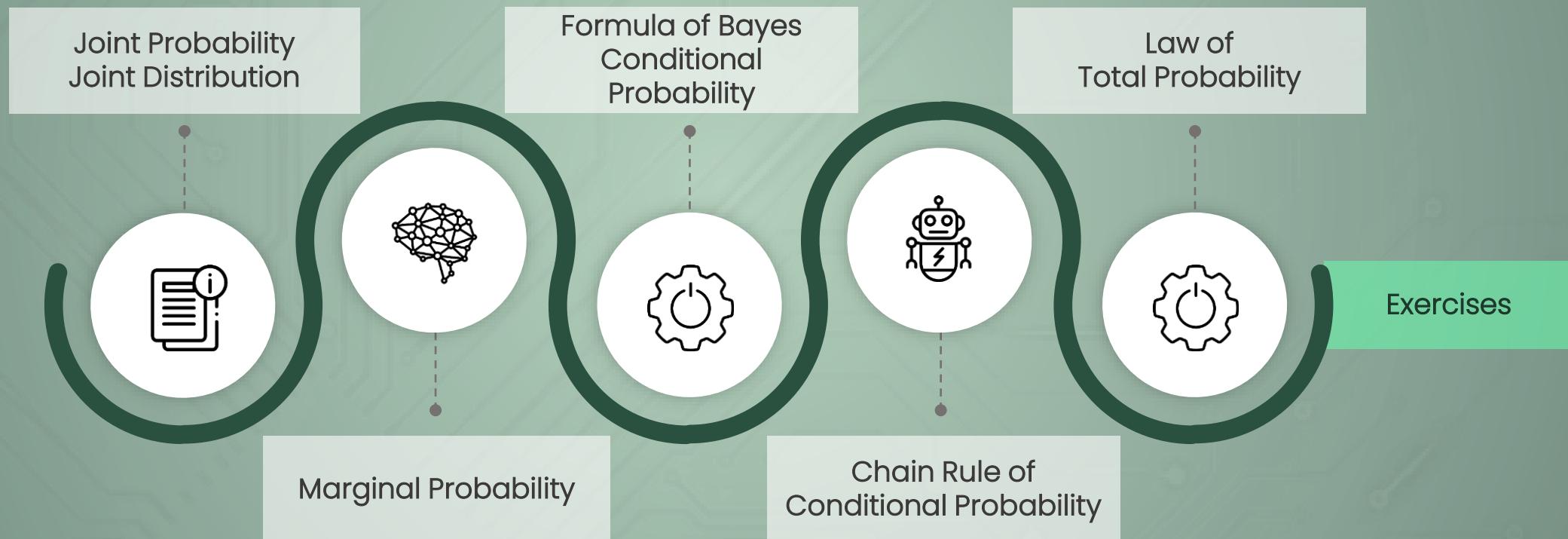
```
['apple', 'banana', 'pear', 'grapes', 'orange', 'mango', 'blueberry', 'strawberry']
```

## Exercise 6: Calculate the number of combinations

$n = 8$   $k = 2$

# Joint and Conditional probability

Bridging Events: How Joint Occurrences and Informed Probabilities interrelate



# Joint Probability

is probability of two or more events occurring simultaneously.  
It is denoted as  $P(A \text{ and } B)$  or  $P(A \cap B)$  for events A and B.

## Formula

$$P(A \text{ and } B) = P(A) * P(B|A)$$

Where:

- $P(A)$  is the probability of event A
- $P(B|A)$  is the conditional probability of B given A

## (In-) dependence

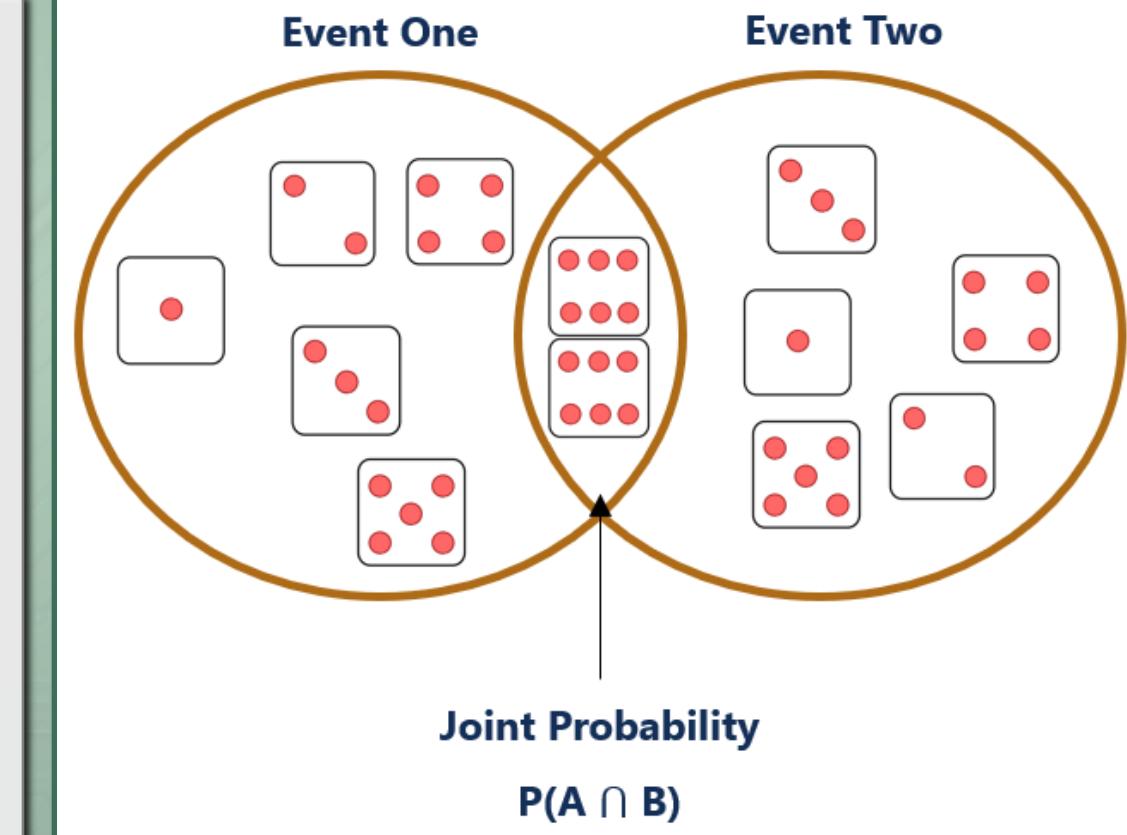
- If A and B are independent,  $P(A \text{ and } B) = P(A) * P(B)$
- Else:  $P(A \text{ and } B) \neq P(A) * P(B)$
- Mutually Exclusive: If A and B cannot occur together,  $P(A \text{ and } B) = 0$

## Joint Probability Distribution

- Describes probabilities of all possible combinations

Example: Two rolling dice  
(independent events)

$$\begin{aligned} P(\text{First die} = 6 \text{ and Second die} = 6) \\ = P(\text{First die} = 6) * P(\text{Second die} = 6) \\ = (1/6) * (1/6) \\ = 1/36 \end{aligned}$$

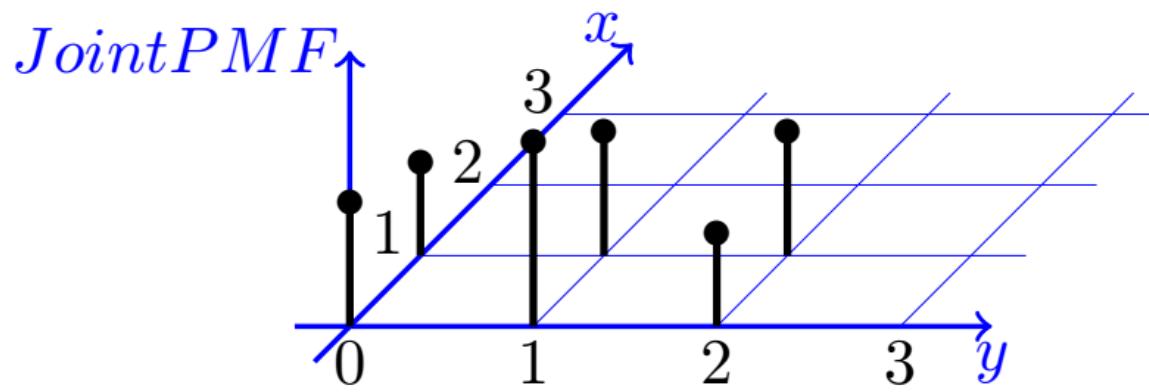


# Joint Probability Distribution

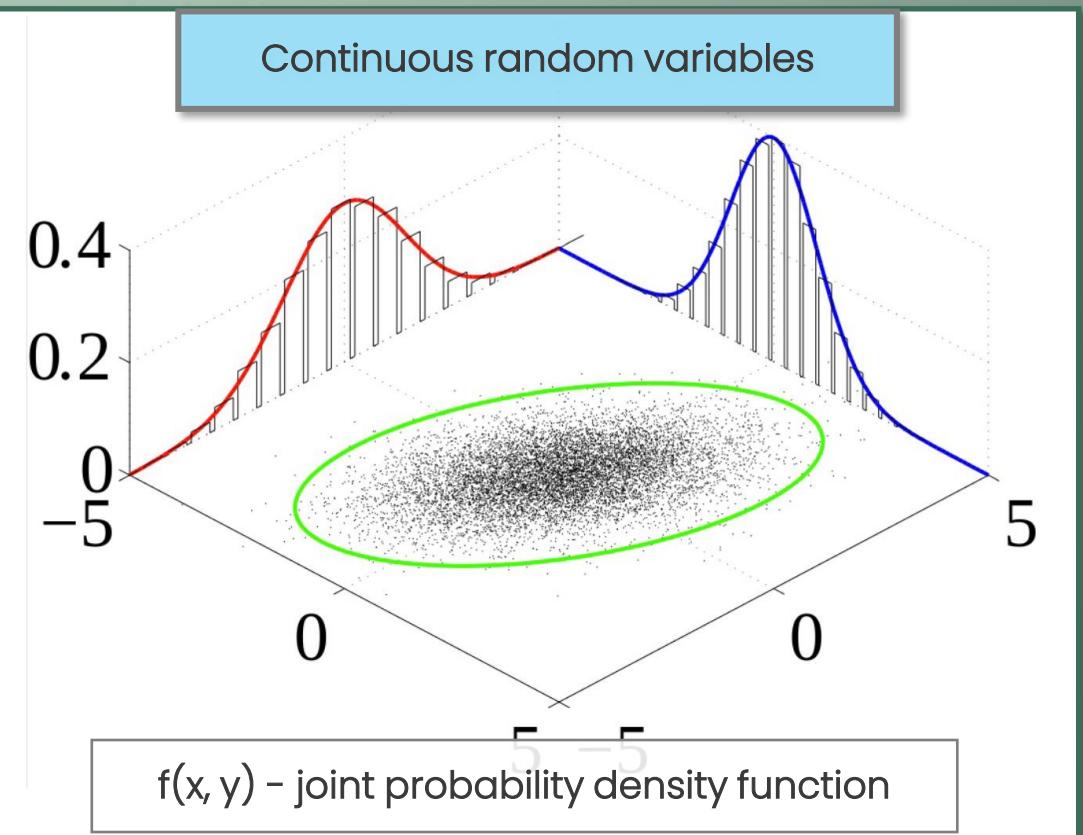
... is probability distribution that gives probability of two or more random variables occurring together. Results of random events are not added or averaged.

Discrete random variables

|       | $Y=0$ | $Y=1$ | $Y=2$ |
|-------|-------|-------|-------|
| $X=0$ | $1/6$ | $1/4$ | $1/8$ |
| $X=1$ | $1/8$ | $1/6$ | $1/6$ |



Continuous random variables



$f(x, y)$  – joint probability density function

# Marginal Probability

... is probability distribution of subset of variables from larger set of variables, ignoring the other variables.

## Key Points

- Derived from joint distribution
- Represents distribution of (summarized) single variable, regardless of others
- Useful for analyzing individual variables in multivariate scenarios
- Can be used to check for independence between variables

## Calculation

- For discrete variables X and Y:
  - Sum over all possible values of Y
  - $P(X = x) = \sum P(X = x, Y = y)$
- For continuous variables:
  - Integrate over all possible values of Y
  - $f_X(x) = \int f(x, y) dy$

## Joint Distribution

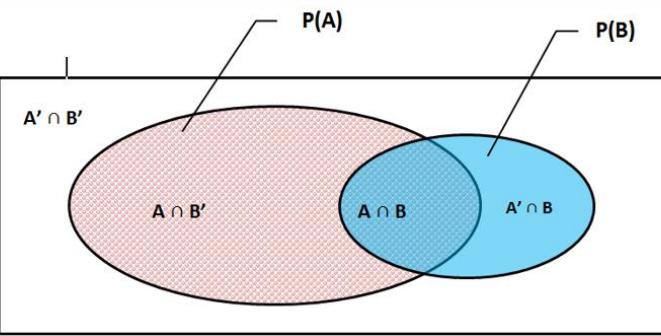


## Marginal of X

Marginal of Y

# Bayes' Theorem

... describes how to update probabilities of hypotheses when given evidence. It's used to calculate the conditional probability, i.e., the probability of event based on prior knowledge.



From conditional probabilities to Bayesian theorem:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$$

$$P(A \text{ and } B) = P(B \text{ and } A)$$

Thus:  $P(A|B)P(B) = P(B|A)P(A)$

Or:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

**LIKELIHOOD**  
the probability of "B" being TRUE given that "A" is TRUE

**PRIOR**  
the probability of "A" being TRUE

**POSTERIOR**  
the probability of "A" being TRUE given that "B" is TRUE

**P( $A|B$ )** =  $\frac{P(B|A)P(A)}{P(B)}$

The probability of "B" being TRUE

@luminousmen.com

# Table with conditional probabilities

... Use Case: How many visitors make a purchase depending on whether they visit website A or B

|                   | <u>Purchase</u>  | <u>No Purchase</u>   | <u>Total (MP)</u>  |
|-------------------|--|--|--|
| <u>Website A</u>  | 80 (A1)<br>JP: 4 %<br><b>CP: 8 %</b>                         | 920 (A2)<br>JP: 46 %<br>CP: 92 %   | <i>Total_A</i><br><b>1000</b><br>$P(A) = 50\%$   |
| <u>Website B</u>  | 90 (B1)<br>JP: 4.5 %<br>CP: 9 %                              | 910 (B2)<br>JP: 45.5 %<br>CP: 91 %   | <i>Total_B</i><br><b>1000</b><br>$P(B) = 50\%$   |
| <u>Total (MP)</u> | <u>Purchase:</u><br>170 (T1)<br>$P(\text{Purchase}) = 8.5\%$ | <u>Non-purchase:</u><br>1830 (T2)<br>$P(\text{Non-Purchase}) = \rightarrow 91.5\%$ | $\text{Total} = \text{Total}_A + \text{Total}_B$<br><b>2000</b><br>$P(\text{Total}) = 100\%$ |

$$P(\text{Purchase given Website A}) = \frac{P(\text{Website A given Purchase}) * P(\text{Purchase})}{P(\text{Website A})}$$

$$\mathbf{0.08} = \frac{80 / 170 * 0.085}{0.5}$$

## Joint probabilities

- $P(\text{A and Purchase}) = A1;$
- $P(\text{B and Purchase}) = B1;$

(JP):

$$T = 80 / 2000 = 0.04$$

$$T = 90 / 2000 = 0.045$$

## Conditional probabilities

(CP):

- $P(\text{Purchase} | A) = A1;$  TA = 80 / 1000 = 0.08  
(probability of a purchase given the user saw website A)
- $P(\text{Purchase} | B) = B1;$  TB = 90 / 1000 = 0.09  
(probability of a purchase given the user saw website B)

## Marginal probabilities

(MP):

- $P(A) = TA;$  T = 1000 / 2000 = 0.5  
(probability a visitor sees website A)
- $P(B) = TB;$  T = 1000 / 2000 = 0.5  
(probability a visitor sees website B)
- $P(\text{Purchase}) = T1;$  T = 170 / 2000 = 0.085  
(probability a visitor makes a purchase)

# Conditional vs. Joint Probability

Bridging Events: Conditional Probability is based on Joint Probability, however considers effect of new information like “Event B has taken place”

## Conditional Probability

$P(A|B)$

Probability of A given B has occurred

- Represents updated probability based on new information
- Calculates likelihood of one event considering another event's occurrence
- Formula:  $P(A|B) = P(A \cap B) / P(B)$

## Joint Probability

$P(A \cap B)$

Probability of both A and B occurring together

- Represents simultaneous occurrence of multiple events
- Calculates likelihood of all specified events happening
- Formula:  $P(A \cap B) = P(A|B) * P(B) = P(B|A) * P(A)$

## Key Points

- Conditional probability can be derived from joint probability:
- For independent events:
- Bayes' Theorem connects these concepts:

$$P(A|B) = P(A \cap B) / P(B)$$

$$P(A \cap B) = P(A) * P(B), \text{ and } P(A|B) = P(A)$$

$$P(A|B) = P(B|A) * P(A) / P(B)$$

## Reference to Combinatorics

- Sampling without Replacement:
- Sampling with Replacement:

Connected to Conditional Probability

Connected to Joint Probability

# Chain Rule of Conditional Probability

... allows to express joint probability of multiple events as product of conditional probabilities.

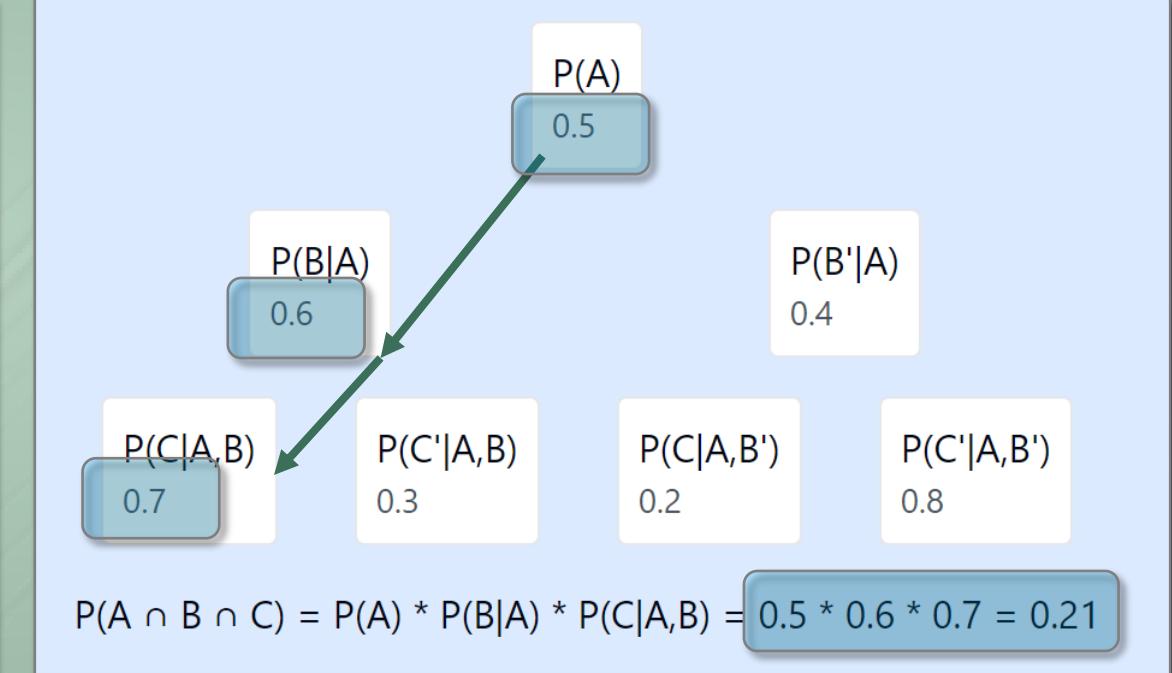
## Key Points

- Breaks down complex joint probabilities
- Useful when direct calculation of joint probability is difficult
- Order of events can affect calculation complexity

## Calculation

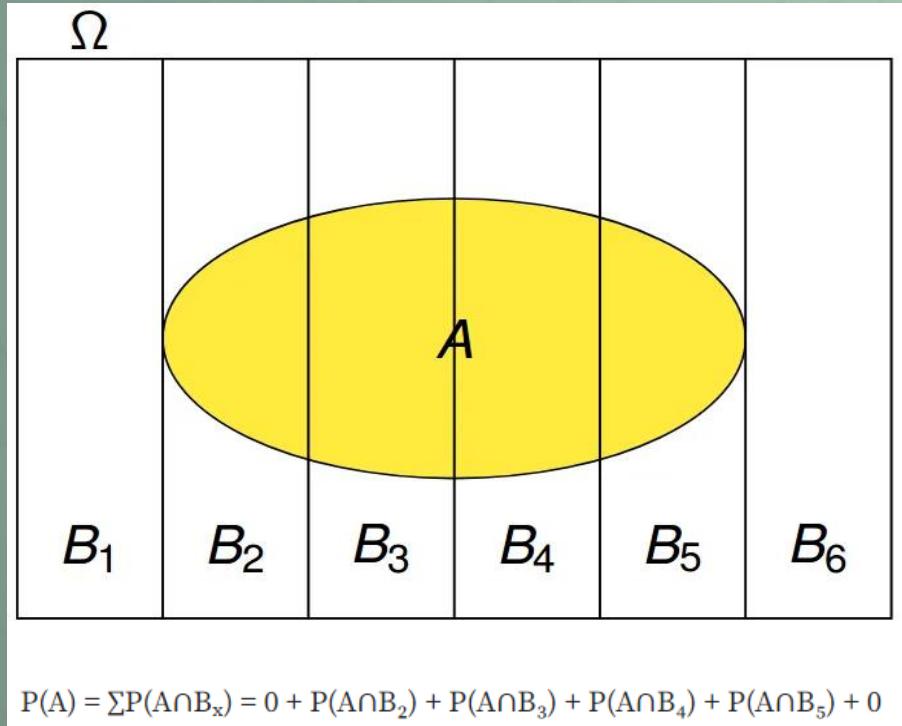
- For two events: A & B
  - $P(A \cap B) = P(A|B) * P(B)$
  - Find an example?
- More than two events  $A_1, \dots, A_n$ :
  - $P(A_n \cap \dots \cap A_1) = P(A_n | A_{n-1} \cap \dots \cap A_1) * P(A_{n-1} \cap \dots \cap A_1)$
  - Also:

$$P(A_n \cap \dots \cap A_1) = \prod_{k=1}^n P(A_k | \cap_{j=1}^{k-1} A_j)$$



# Law of Total Probability

... calculates the probability of an event by summing its conditional probabilities across all possible scenarios



$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This can be derived using the fact that  $P(A \cap B) = P(A|B)P(B)$

[Law of Total Probability] Let  $\{B_1, B_2, \dots, B_n\}$  be a partition of the sample space  $S$ , and let  $A$  be any event. Then:

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

where  $P(A|B_i)$  is the conditional probability of  $A$  given  $B_i$ , and  $P(B_i)$  is the probability of  $B_i$ .



# Exercises

## Exercise 1:

Animal Joint Probability Problem

A wildlife researcher is studying a forest habitat. They are particularly interested in the presence of two animal species:

Red foxes (F)

Barn owls (O)

Based on previous studies:

The probability of observing a red fox in a given area is 0.4

The probability of observing a barn owl in a given area is 0.3

The probability of observing both a red fox and a barn owl in the same area is 0.15

Calculate the following probabilities:

- a)  $P(F \cap O)$ : Probability of observing both a red fox and a barn owl in the same area
- b)  $P(F \cup O)$ : Probability of observing either a red fox or a barn owl (or both) in an area
- c)  $P(F | O)$ : Probability of observing a red fox, given that a barn owl has been observed
- d)  $P(O | F)$ : Probability of observing a barn owl, given that a red fox has been observed
- e)  $P(F' \cap O')$ : Probability of observing neither a red fox nor a barn owl in an area

# Exercises

## Exercise 2:

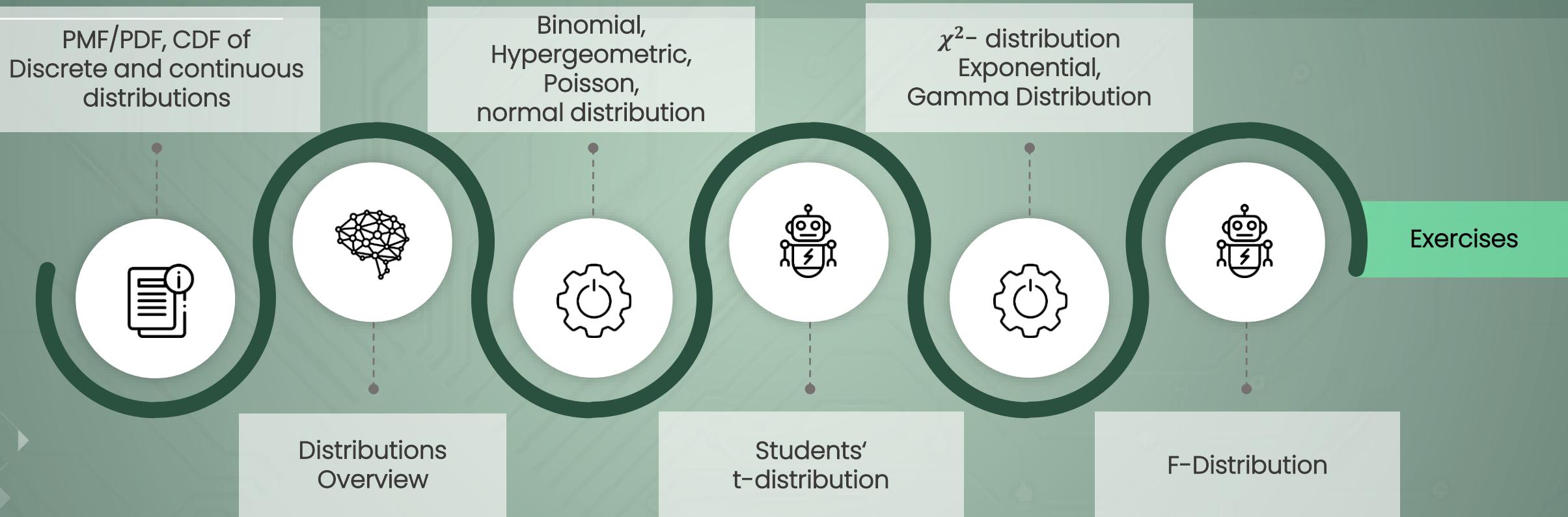
Consider the following frequency table for students' preferences for Programming Language and whether they prefer Dark Theme or not

| Programming Language | Dark Theme | Count |
|----------------------|------------|-------|
| Python               | Yes        | 90    |
| Python               | No         | 30    |
| Java                 | Yes        | 50    |
| Java                 | No         | 40    |
| JavaScript           | Yes        | 60    |
| JavaScript           | No         | 30    |

Represent this table in Python using a list of dictionaries and calculate the conditional probability of a student preferring a Dark Theme given that they prefer Python, Java, JavaScript.

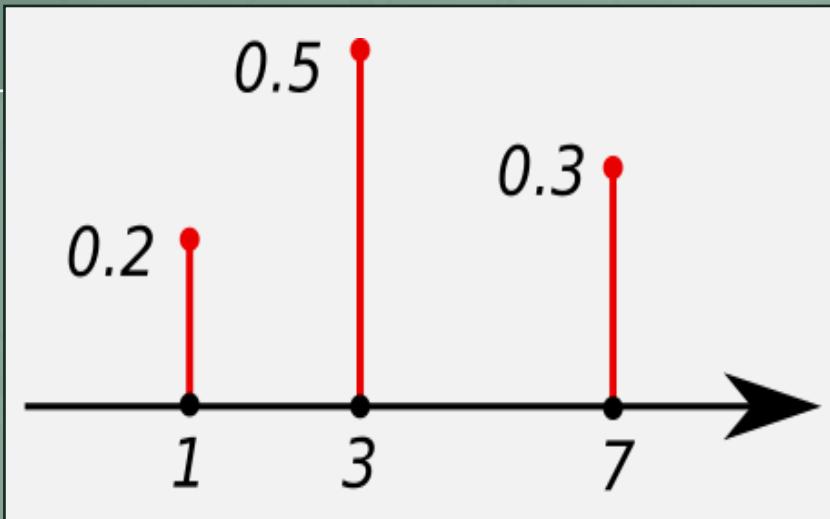
# Probability Distributions

They describe the probabilities of occurrences of different possible outcomes in an experiment



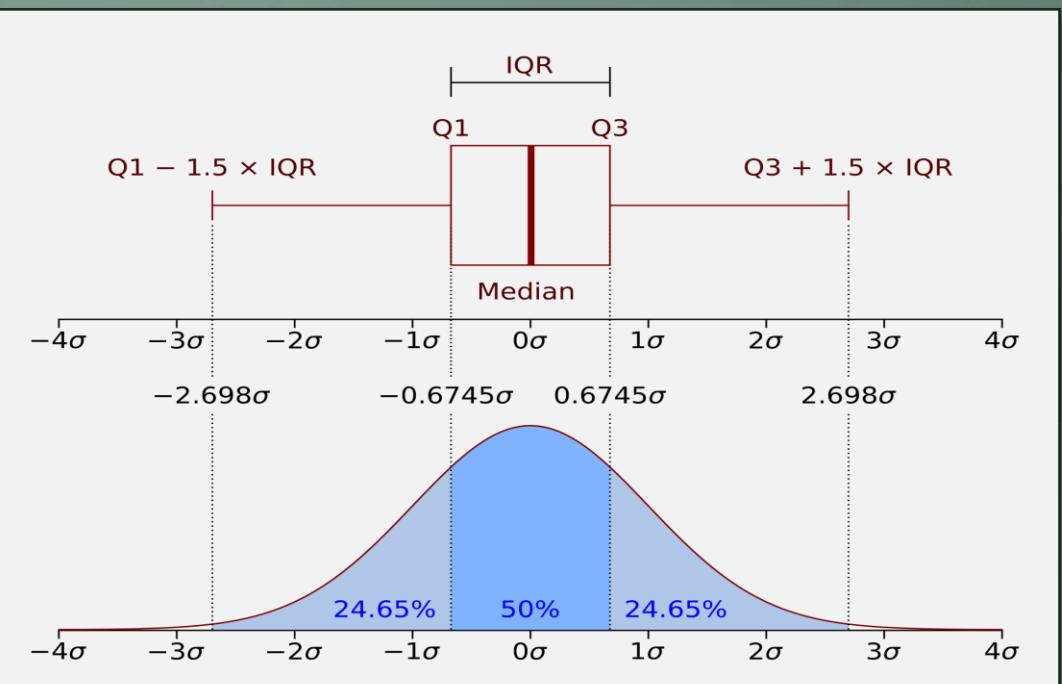
# Probability Distributions

... are discrete or continuous



## Discrete Probability Distribution

- Represented by vertical lines (lollipop chart)
- Each point represents a specific outcome
- Height of line indicates probability
- Example shown: {1: 0.2, 3: 0.5, 7: 0.3}
- Sum of all probabilities equals 1



## Continuous Probability Distribution

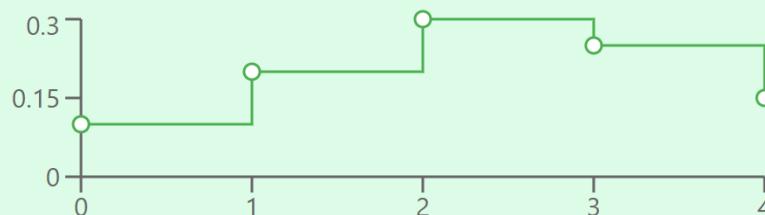
- Represented by a smooth curve (bell curve shown)
- Area under the curve represents probability
- Total area under curve equals 1
- Example: Normal (Gaussian) distribution
- Shows
  - median, quartiles, and standard deviations
  - Interquartile Range (IQR): Distance between Q1 and Q3
  - Standard Deviation: Measure of spread  $1\sigma$ ,  $2\sigma$ ,  $3\sigma$

# Probability Distributions

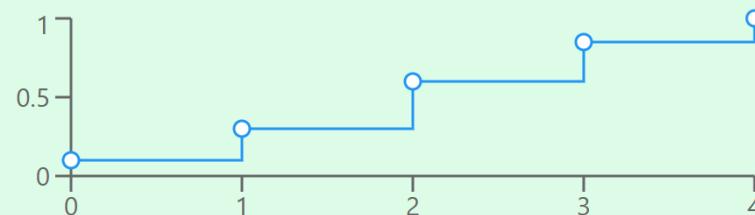
Comparing Probability Mass/Density and Cumulative Distribution Functions

## Discrete Distributions

Probability Mass Function (PMF)

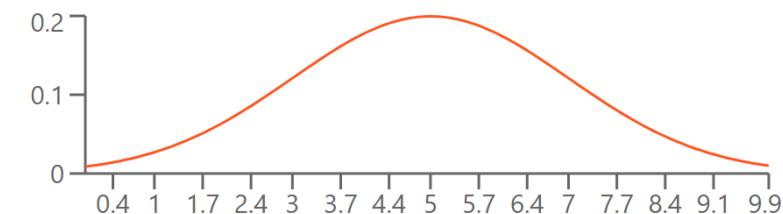


Cumulative Distribution Function (CDF)

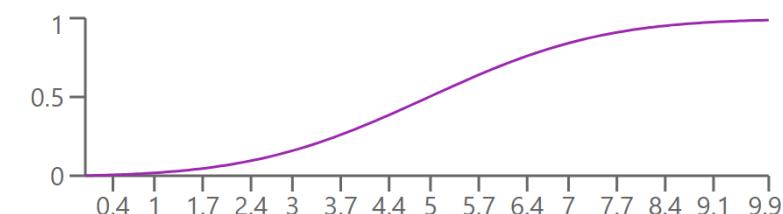


## Continuous Distributions

Probability Density Function (PDF)



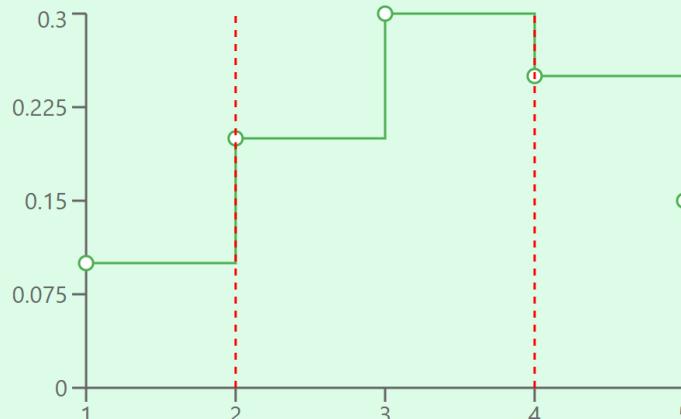
Cumulative Distribution Function (CDF)



# Measuring Probability on Distribution Intervals

From Sums to Integrals: Calculating Probabilities Across Distribution Types

## Discrete Distributions



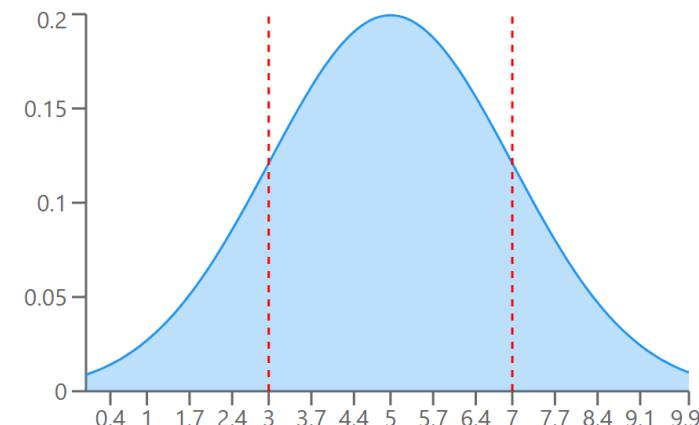
Probability = Sum of probabilities for each point in the interval

$$\text{Example: } P(2 \leq X \leq 4) = 0.2 + 0.3 + 0.25 = 0.75$$

Given the shown distribution ...

What is the probability that the guest has drunk 2,3 or 4 beer?

## Continuous Distributions



Probability = Area under the curve between interval points

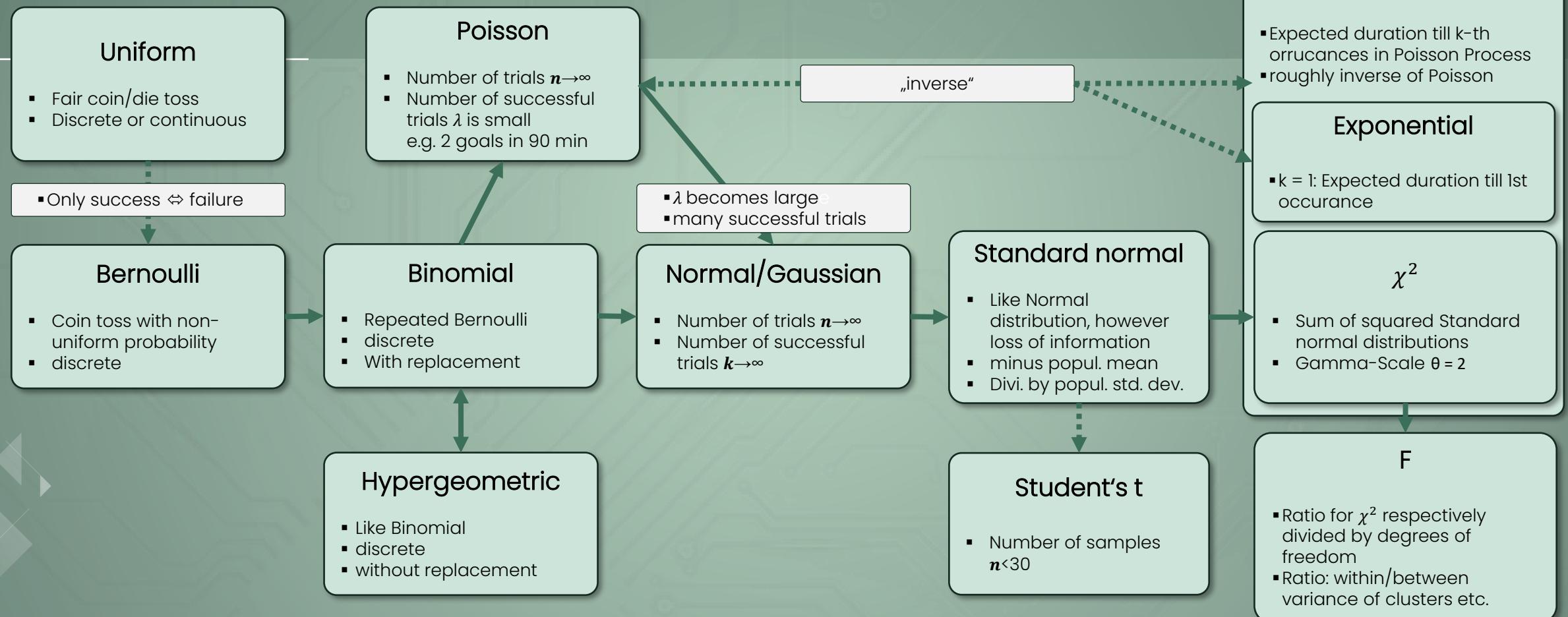
$$\text{Example: } P(3 \leq X \leq 7) = \int[3 \text{ to } 7] f(x) dx$$

Given the shown distribution ...

What is the probability that the person's income is between 3k and 7k € a month?

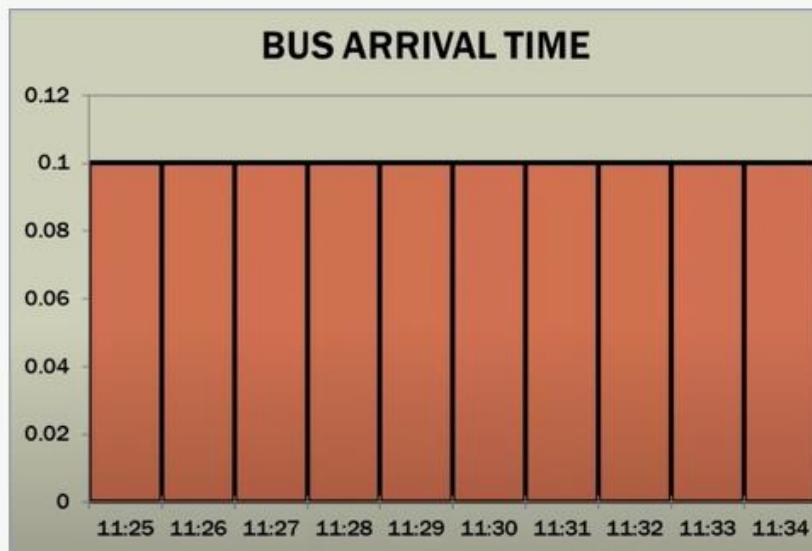
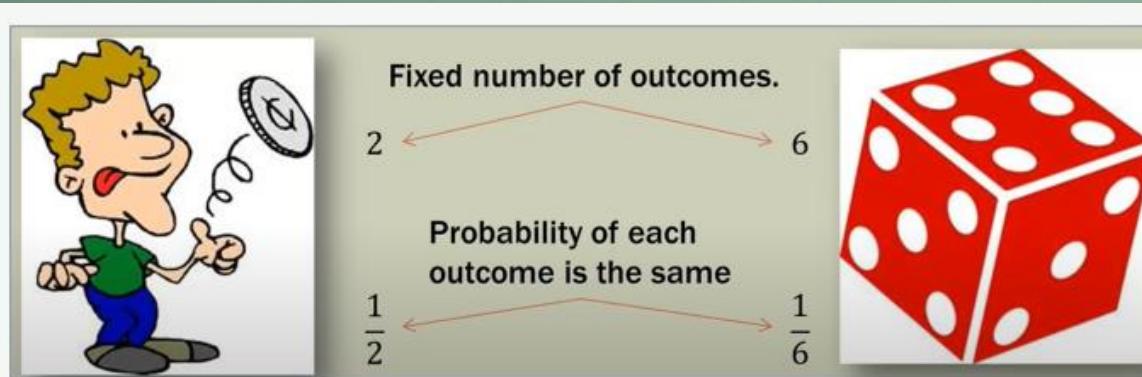
# Overview of important distributions

... Most common distributions are based on the binomial distribution.



# Uniform distribution

... is a probability distribution that describes the number of successes in a fixed number of independent binary experiments, each with the same probability of success.



## Expected value:

- Arithmetic mean

## Use Cases:

- Generating Random Numbers:  
e.g.: fair dice, fair coin tosses
- Quality Control and Manufacturing:  
e.g.: check the uniformity of products

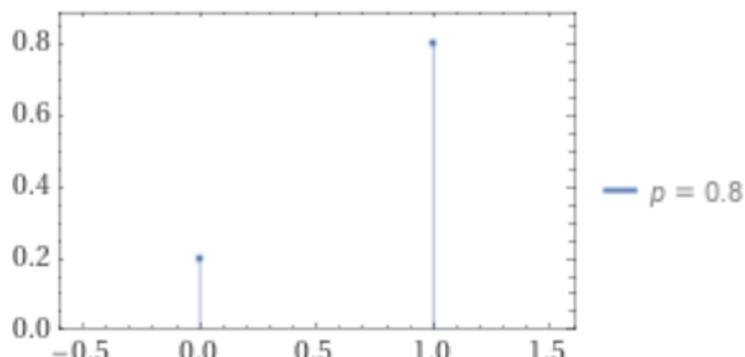
## Be careful:

- Sum of 2 dice is not uniformly distributed!  
(it is multinomial distribution, out of scope)

# Bernoulli distribution

... is a discrete probability distribution for a random variable which can take a binary outcome: one (success) or zero (failure); example: toss of coin (that is not necessarily fair)

Plots of PDF for typical parameters



## Formula:

$$P(x; p) = p^x * (1 - p)^{1-x} \text{ for } x \in [0; 1]$$

$p$ : probability of success. E.g. if the coin is fair, the probability of getting a head (considered a success) would be 0.5

$x$ : outcome of Bernoulli trial; it can take only two values:  
1(success) or 0(failure)

Attention:  $x \in [0; 1]$  is not an interval!

$P(x; p)$  : probability of outcome  $x$ , given probability  $p$  of success.



# Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10

Probability a 60% free throw shooter makes

1 of 1?

60%



## Step 1:

- $P(\text{Make the shot}) = 60\%$
- $P(\text{Miss the shot}) = 40\%$   
 $= 1 - 60\%$

# Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10

Probability a 60% free throw shooter makes...

0 of 2?

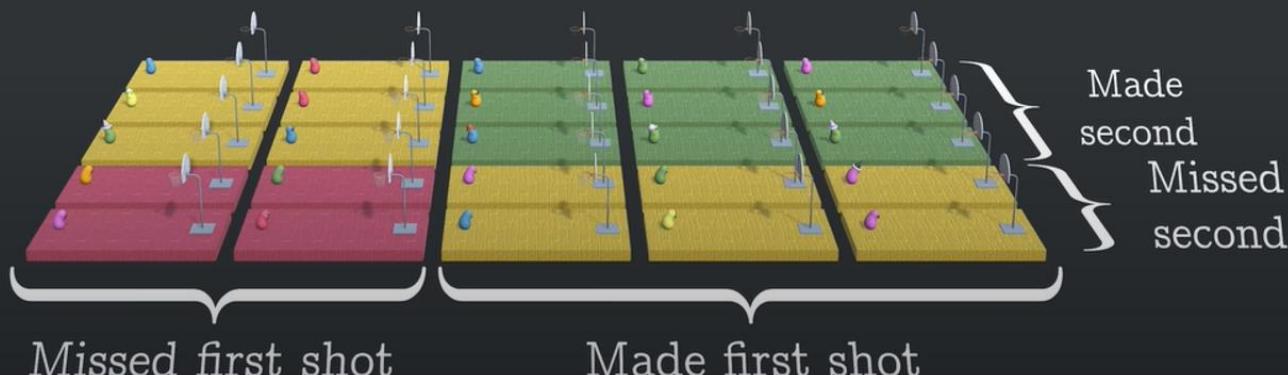
$$P(\text{miss, miss}) = \frac{4}{25} = 16\% \\ = 0.4 \times 0.4$$

1 of 2?

$$P(\text{make, miss}) = \frac{6}{25} = 24\% \\ = 0.6 \times 0.4 \\ P(\text{miss, make}) = \frac{6}{25} = 24\% \\ = 0.4 \times 0.6$$

2 of 2?

$$P(\text{make, make}) = \frac{9}{25} = 36\% \\ = 0.6 \times 0.6$$



**Step 2:**

- $P(\text{miss, miss}) = 16\%$
- $P(\text{miss, make}) = 48\%$
- $P(\text{make, make}) = 36\%$

**Assumption:**  
Independence of shots

# Binomial distribution

Example for probability calculation:

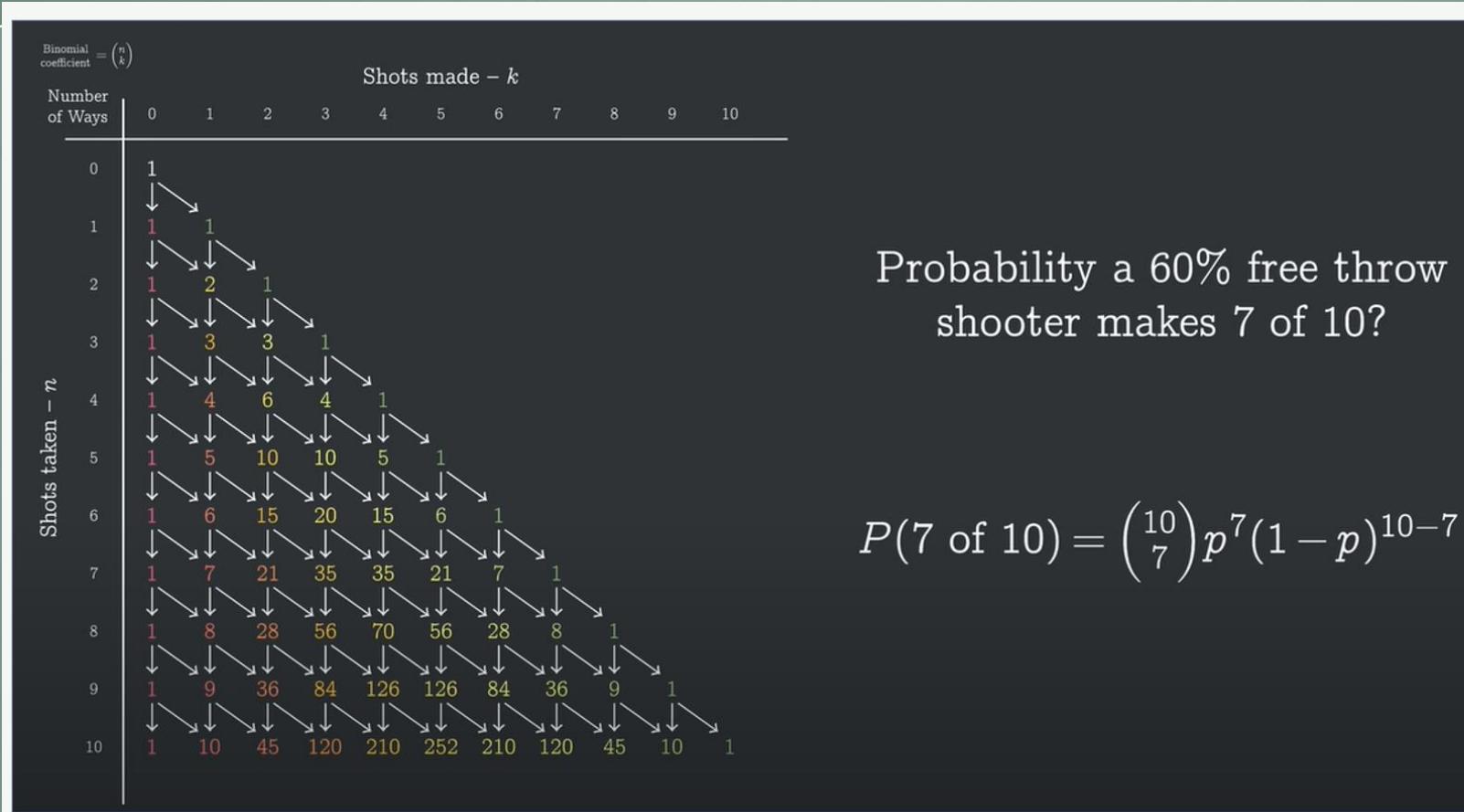
Basketball player that is known for 60% prob of making free shots makes 7 of 10



# Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10



Probability a 60% free throw shooter makes 7 of 10?

$$P(7 \text{ of } 10) = \binom{10}{7} p^7 (1-p)^{10-7}$$

$$\begin{aligned} P(7 \text{ of } 10) &= 120 \times 0.6^7 \times 0.4^3 \\ &\approx 21.5\% \end{aligned}$$

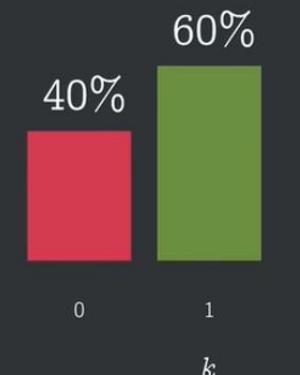
# Binomial distribution

Example for probability calculation – Generalization:

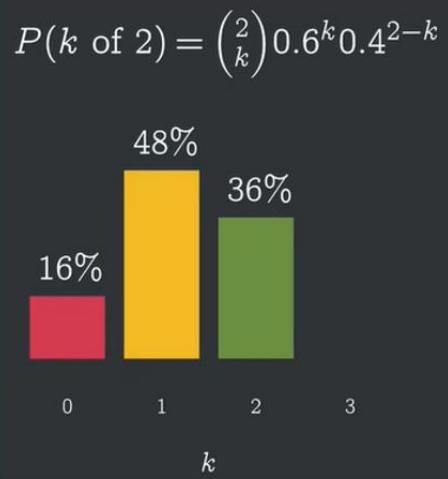
Basketball player that is known for 60% prob of making free shots makes k of n

$$P(k \text{ of } n) = \binom{n}{k} 0.6^k 0.4^{n-k}$$

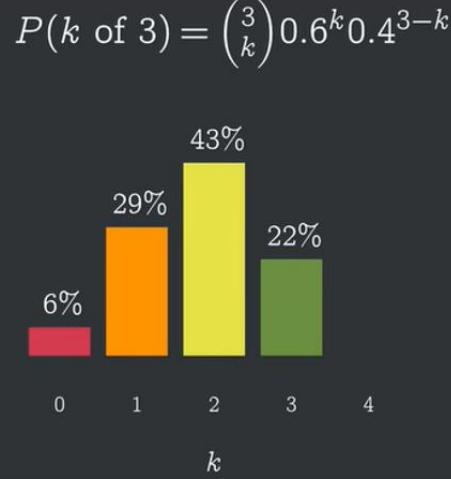
$$P(k \text{ of } 1) = \binom{1}{k} 0.6^k 0.4^{1-k}$$



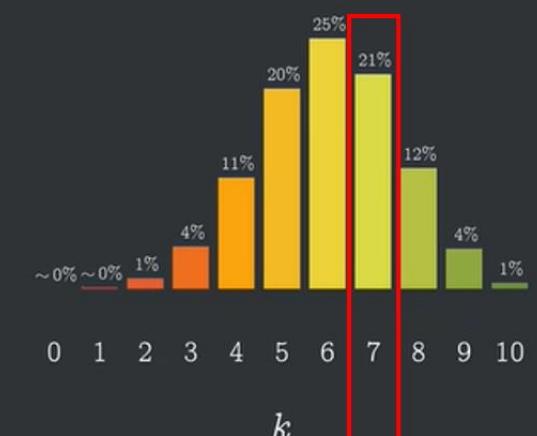
$$P(k \text{ of } 2) = \binom{2}{k} 0.6^k 0.4^{2-k}$$



$$P(k \text{ of } 3) = \binom{3}{k} 0.6^k 0.4^{3-k}$$



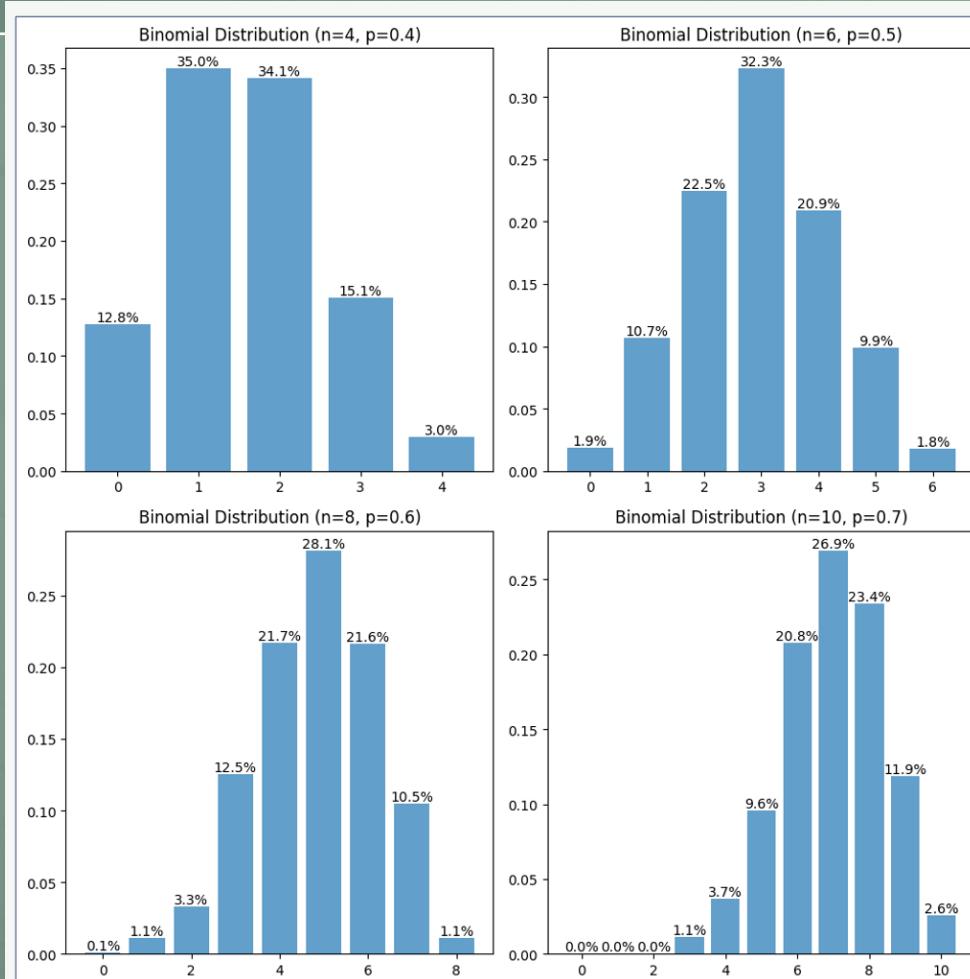
$$P(k \text{ of } 10) = \binom{10}{k} 0.6^k 0.4^{10-k}$$



This example

# Binomial distribution

... is repeated Bernoulli distribution, i.e. probability distribution that describes number of successes in fixed number of independent binary experiments, each with same probability of success.



## Formula:

$$P(k \text{ of } n) = \binom{n}{k} p^k (1-p)^{n-k}$$

## Parameters:

- n: number of trials
- p: probability of success on each trial
- k (input factor on x-axis): number of actual positive outcomes

## Use cases:

- Quality Control: In a manufacturing company the probability of producing a defective item is 0.05, and it manufactures 200 items per day
- Spam email classifier

# Connection between Joint and Binomial Probability

Binomial probability is sum of joint probabilities for all possible ways to achieve k successes in n trials.

## Joint Probability

For n independent trials:

$$P(X_1=x_1, X_2=x_2, \dots, X_n=x_n) = p^k * (1-p)^{n-k}$$

Where:

- p = probability of success on each trial
- k = number of successes
- n = total number of trials

## Binomial Probability

Probability of exactly k successes in n trials:

$$P(X = k) = C(n,k) * p^k * (1-p)^{n-k}$$

Where:

- $C(n,k)$  = number of ways to choose k items from n
- p, k, and n are as defined for joint probability

## Example: Coin Flips

For 3 coin flips (H = heads, T = tails):

**Joint Probability:**  $P(H,H,T) = (1/2)^2 * (1/2)^1 = 1/8$

**Binomial Probability:**  $P(2 \text{ heads in } 3 \text{ flips}) = C(3,2) * (1/2)^2 * (1/2)^1 = 3 * 1/8 = 3/8$

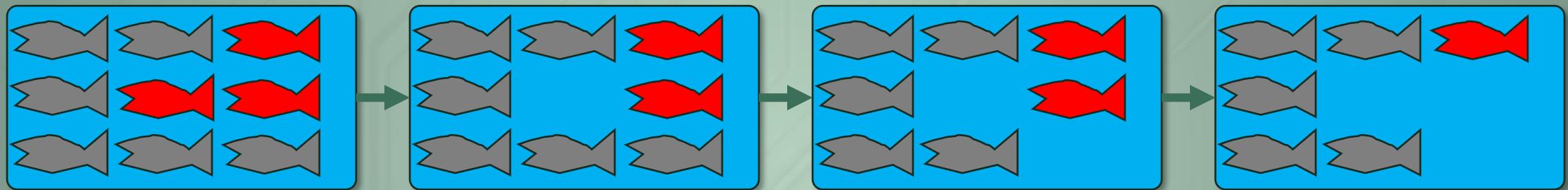
*The binomial probability includes all sequences with 2 heads: HHT, HTH, THH*

# Hypergeometric distribution

Example: In a lake there are 6 gray fishes and 3 red fishes.

What is the probability of selecting k red fish?

$$P(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$$



- Total number of fish (N):  
There are 9 fish in total  
→ population size N = 9
- Number of red fish (K):  
3 red fish in the pool  
→ "success" cases K = 3
- Number of grey fish:  
6 grey fish, no parameter,  
just N - K.
- Number of draws (n):  
Depends on how many fish are selected
- Number of successful draws (k)

- Total number of fish (N):  
N = 8
- Number of red fish (K):  
K = 2
- Number of grey fish:  
N - K = 6

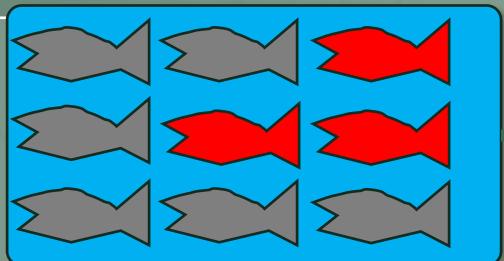
- Total number of fish (N):  
N = 7
- Number of red fish (K):  
K = 2
- Number of grey fish:  
N - K = 5

- Total number of fish (N):  
N = 6
- Number of red fish (K):  
K = 1
- Number of grey fish:  
N - K = 5



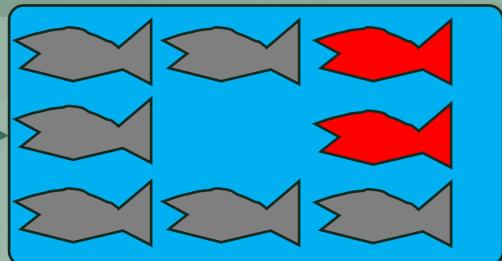
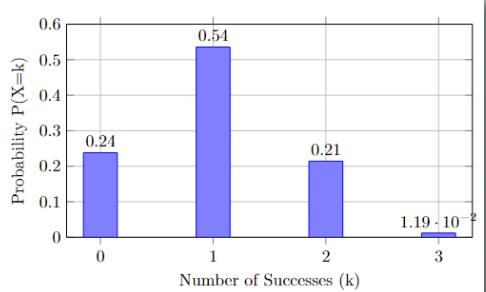
# Hypergeometric distribution

Example: In a lake there are 6 gray fishes and 3 red fishes. Depending on the order of catching gray or red fish, probability distribution for next rounds changes



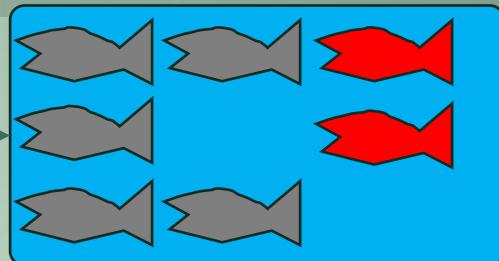
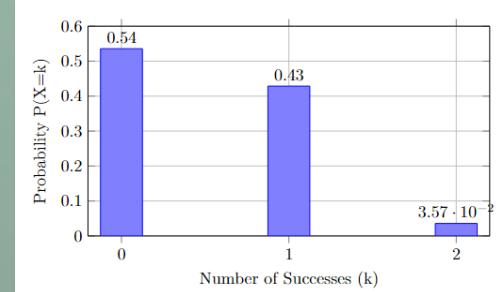
```
N = 9 # Total number of fish
K = 3 # Number of red fish
n = 3 # Number of draws (assumption)
```

| k | P(X = k) |
|---|----------|
| 0 | 0.2381   |
| 1 | 0.5357   |
| 2 | 0.2143   |
| 3 | 0.0119   |



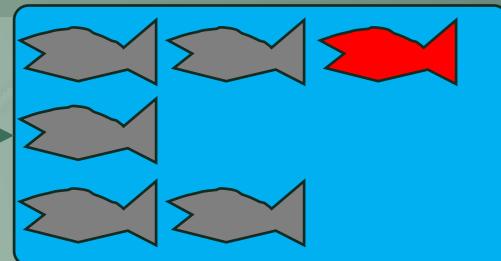
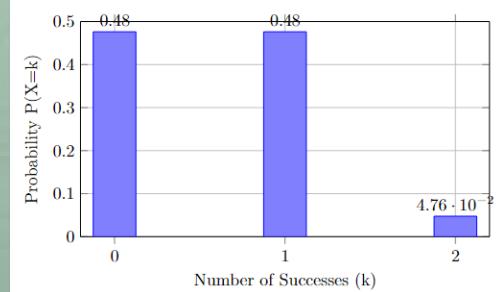
```
N = 8 # Total number of fish
K = 3 # Number of red fish
n = 2 # Number of draws (assumption)
```

| k | P(X = k) |
|---|----------|
| 0 | 0.5357   |
| 1 | 0.4286   |
| 2 | 0.0357   |



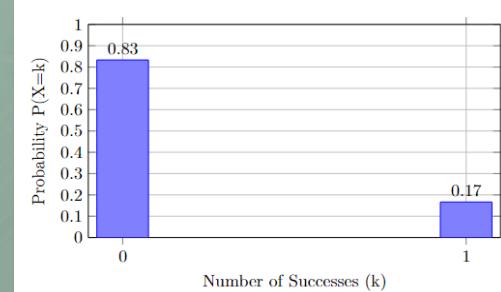
```
N = 7 # Total number of fish
K = 3 # Number of red fish
n = 2 # Number of draws (assumption)
```

| k | P(X = k) |
|---|----------|
| 0 | 0.4762   |
| 1 | 0.4762   |
| 2 | 0.0476   |



```
N = 6 # Total number of fish
K = 3 # Number of red fish
n = 1 # Number of draws (assumption)
```

| k | P(X = k) |
|---|----------|
| 0 | 0.8333   |
| 1 | 0.1667   |



# Hypergeometric distribution

... models number of successes in sample drawn without replacement from finite population.

## Probability mass function

$$P(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$$

Where:

- $N$  is the population size
- $K$  is the number of success states in the population
- $n$  is the number of draws
- $k$  is the number of observed successes
- $\binom{a}{b}$  represents the binomial coefficient (combinations)

## Connection to Conditional Probability

- Each draw changes the probability for subsequent draws:
- $P(\text{success on 2nd draw} | \text{success on 1st draw}) = (K-1)/(N-1)$
- This sequence of conditional probabilities forms the hypergeometric distribution.

## Sampling Without Replacement

- Probability changes after each draw
- As  $N \rightarrow \infty$ , approaches the binomial distribution

## Use Cases

- **Card game:**
  - Consider deck of 52 cards (without jokers)
  - which includes 4 aces.
  - If you draw 5 cards without replacement, what is probability that exactly 2 of them are aces?
- **Environmental Studies:**
  - Ecologist is studying forest of 1,000 trees
  - 300 of which are of an endangered species.
  - Ecologist randomly samples 50 trees.
  - What is the probability that 15 of sampled trees are from endangered species?
  - What is the probability if new sample is collected that 20 more trees are from endangered species?



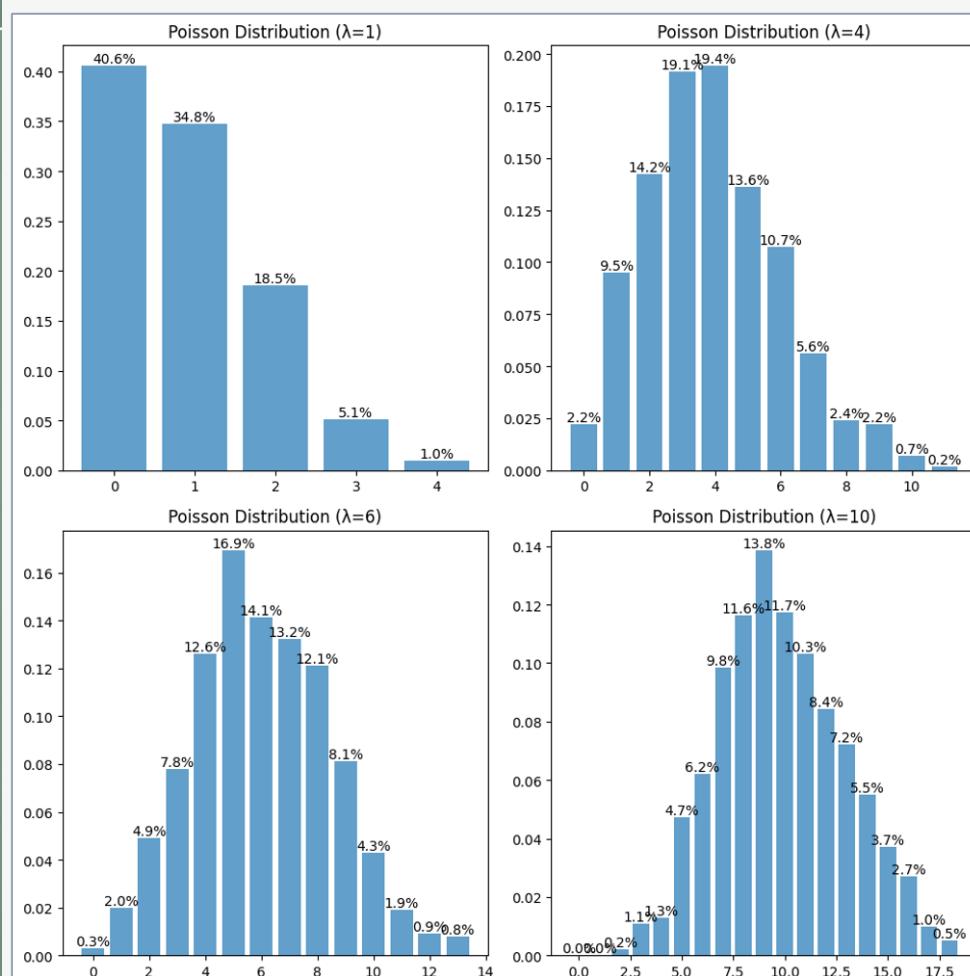
# Hypergeometric vs. Binomial Distribution

As population size increases, hypergeometric distribution approaches binomial distribution.  
For large populations with small samples, binomial can approximate hypergeometric

| Characteristic            | Hypergeometric Distribution               | Binomial Distribution                 |
|---------------------------|---|---------------------------------------|
| Sampling Method           | Without replacement                       | With replacement                      |
| Population Size           | Finite                                    | Infinite or very large                |
| Probability of Success    | Changes after each draw                   | Constant for each trial               |
| Probability Mass Function | $P(X=k) = [C(K,k) * C(N-K,n-k)] / C(N,n)$ | $P(X=k) = C(n,k) * p^k * (1-p)^{n-k}$ |
| Mean                      | $n * (K/N)$                               | $n * p$                               |
| Variance                  | $n * (K/N) * ((N-K)/N) * ((N-n)/(N-1))$   | $n * p * (1-p)$                       |
| Trial Independence        | Dependent                                 | Independent                           |

# Poisson distribution

... probability distribution of a given number of independent events occurring in a fixed interval of time or space, given fixed average rate of occurrence.



Events occur with a known constant mean rate and independently of the time since the last event.

## Relation to binomial distribution:

1. The number of trials ( $n$ ) is very large, i.e. continuous.
2. The probability of success ( $p$ ) is small.
3. The average number of successes ( $np$ ) is moderate.

## Parameter:

$\lambda = np$  (the mean of the distribution)

## Use cases:

- Number of calls received in a given hour, sales per hour, etc.
- emails arriving in inbox per hour, given average arrival rate
- Very few defect items per hour

## AI use cases:

- Sport bets: goals/baskets per match  
(Prediction of football, basketball results)

# Poisson Process

... is stochastic process that models occurrence of random events over time or space, where events occur continuously and independently at constant average rate

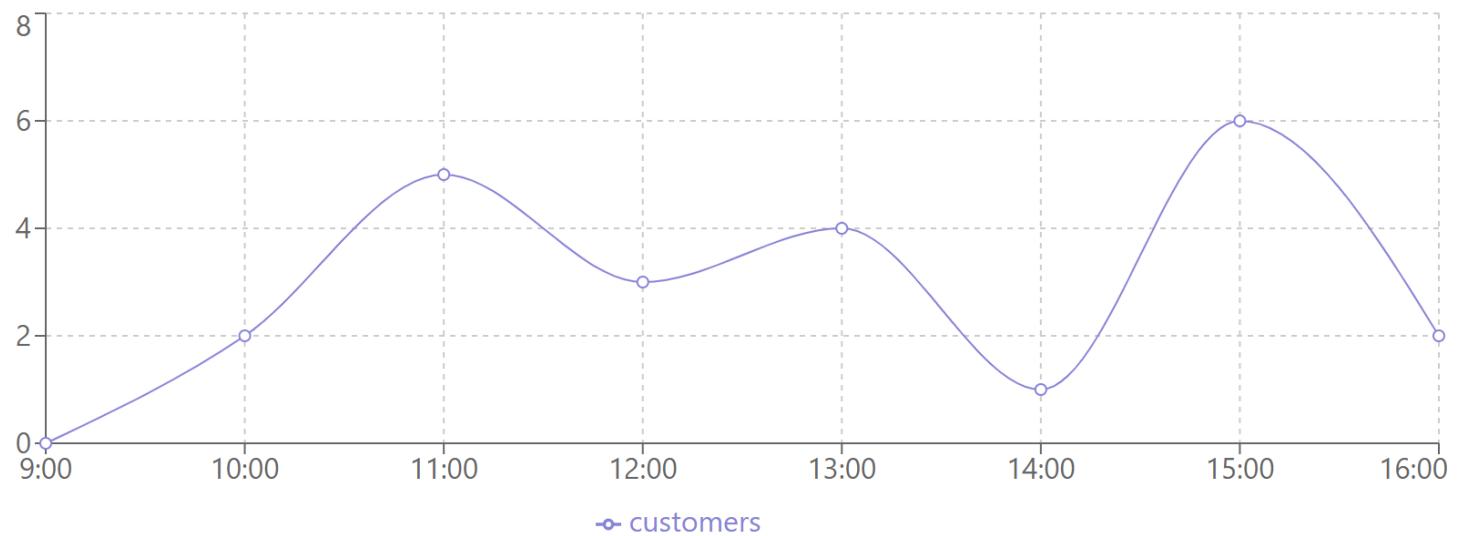
## Poisson Process Characteristics:

- Random arrivals over time
- Customers arrive independently
- Average arrival rate ( $\lambda$ ) is constant
- No simultaneous arrivals

## Car Wash Application:

- Predict customer flow
- Optimize staff scheduling
- Manage queue lengths
- Plan for peak hours

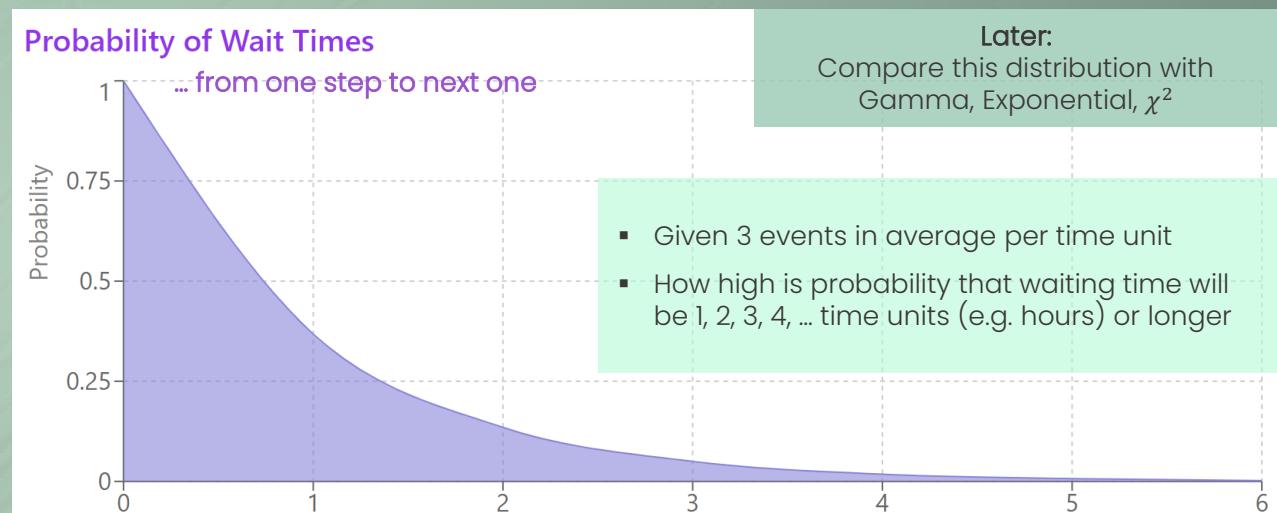
## Customer Arrivals Throughout the Day



Note: This chart shows a sample of customer arrivals at a car wash following a Poisson process.

# Poisson Processes: Inter-arrival Times and Variance

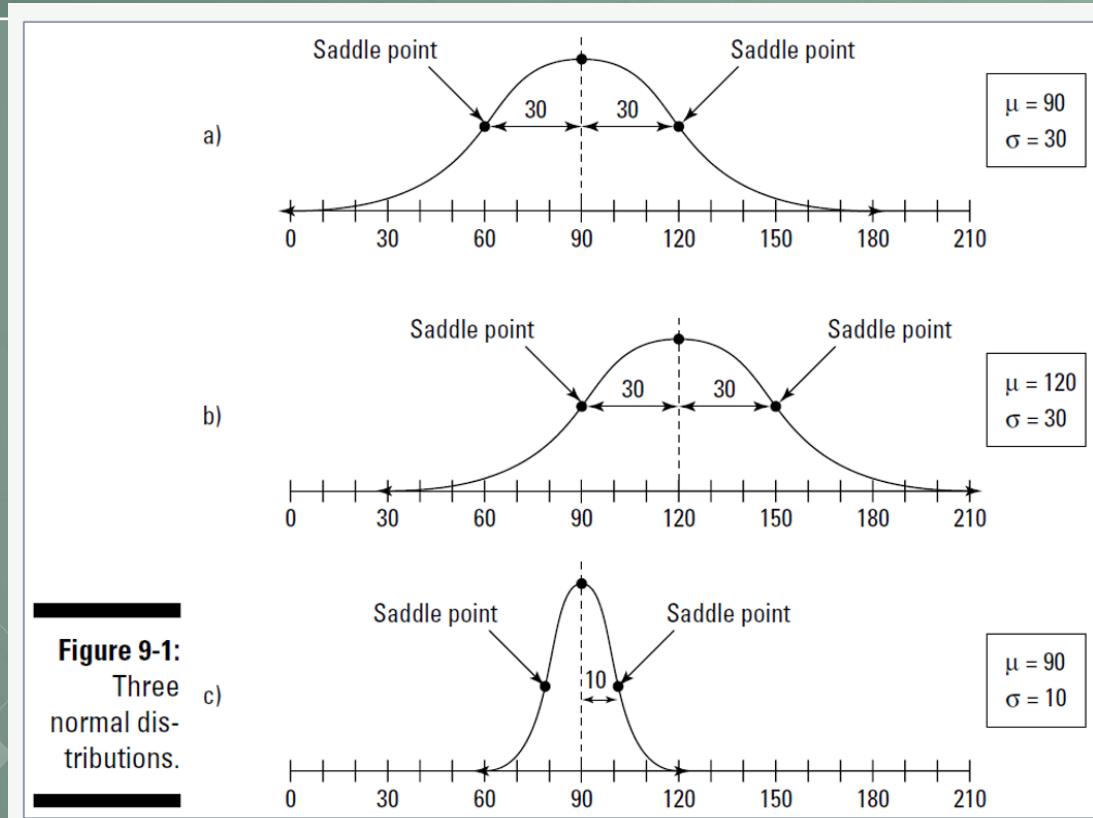
... is stochastic process that models occurrence of random events over time or space, where events occur continuously and independently at constant average rate



- **Varying Wait Times:**  
Can range from very short to very long
- **Short vs. Long Waits:**  
Short waits more common, long waits possible, but rare
- **Average Wait Time:**  
 $1 / (\text{average arrival rate})$
- **Variance:**  
Equal to square of average wait time

# Normal distribution

... is continuous probability distribution characterized by symmetric bell-shaped curve, defined by its mean (average) and standard deviation, also known as the Gaussian distribution



## Parameters:

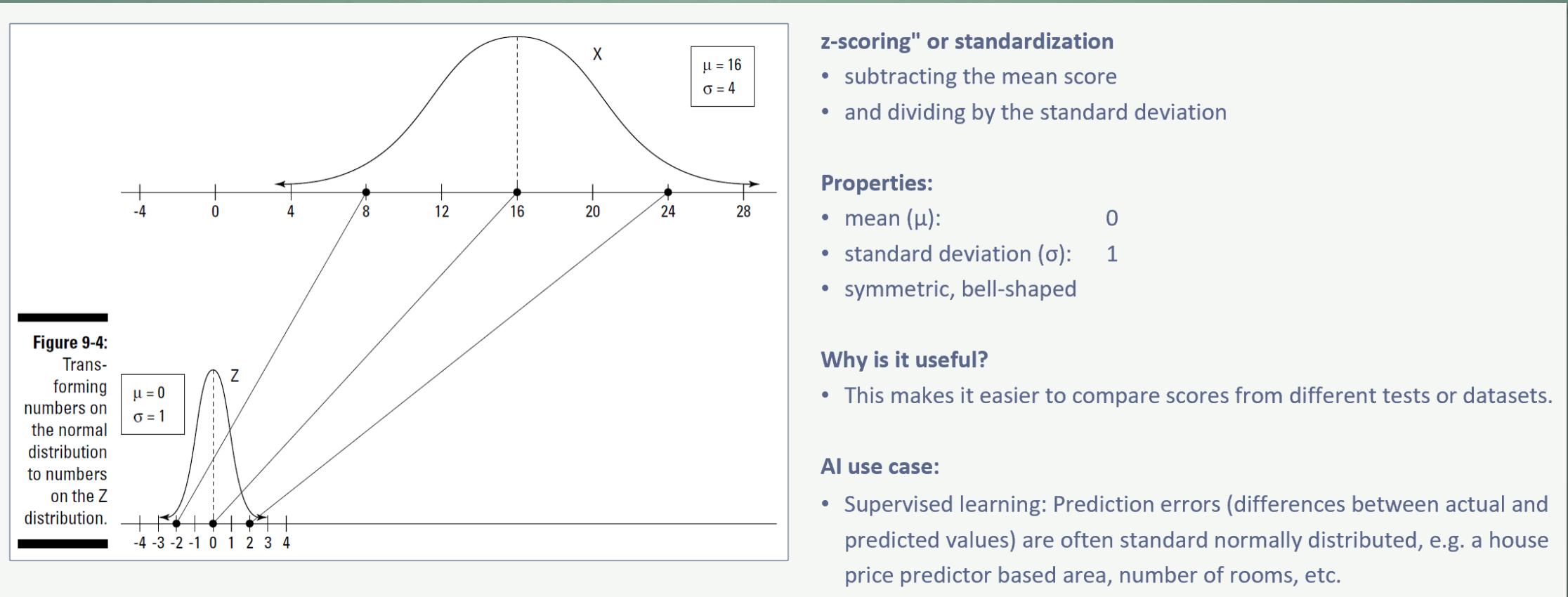
- Mean: where the peak of the bell occurs
- Standard deviation: determines the width of the bell

## Use cases:

- Human Heights: most individuals have heights around the average, very tall or small people are much less frequent
- Test Scores: most students will score around average, fewer students have very high/low scores.

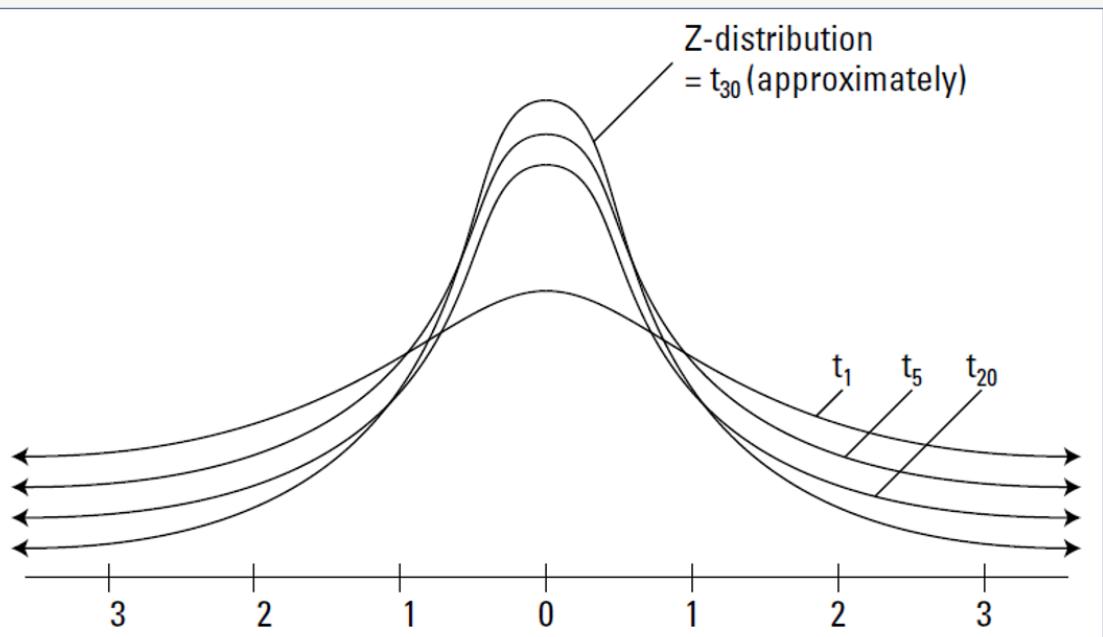
# Standard normal distribution

... is a special case of the normal distribution with a mean of zero and standard deviation of one.



# Students' t-distribution

... is used when sample size is small or when population standard deviation is unknown. It is similar to normal distribution, but with heavier tails, e.g. more flat.



**Figure 9-1:**  $t$ -distributions for different sample sizes

If  $n < 30$  (number of samples), t-distribution gives more accurate confidence intervals and hypothesis tests compared to normal distribution.

#### Use case:

- new drug is tested on small (say, 15 patients). Then, population standard deviation is unknown, and sample that is used for estimation is small.

#### AI use case:

- Any models with very small sample, e.g. for healthcare predictions.

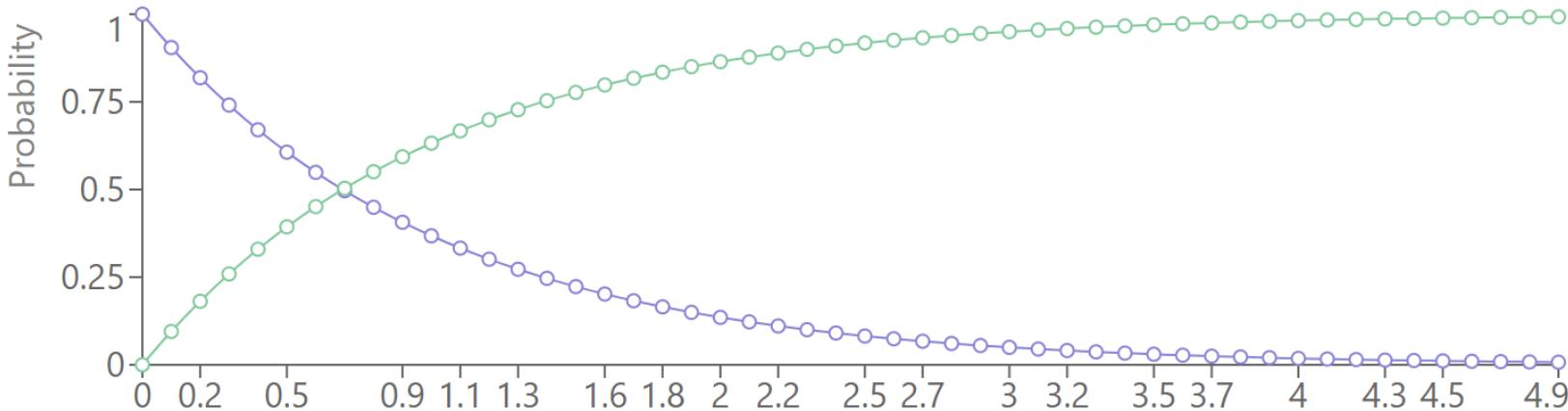
# Exponential Distribution

Continuous probability distribution that describes time between events in Poisson process.

## Use Cases

- Memoryless property:  $P(X > s + t | X > s) = P(X > t)$
- Only parameter, constant hazard rate:  $\lambda$
- Relationship with Poisson distribution

## Probability Density Function (PDF) and Cumulative Distribution Function (CDF)



Example:  $\lambda = 1$

# Exponential vs Poisson Distribution

Exponential distribution models time between events in Poisson process, while Poisson distribution models number of events in fixed interval.

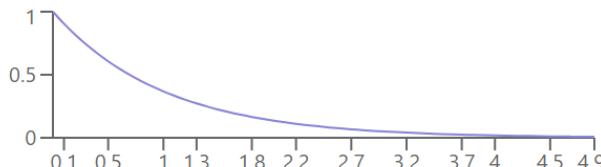
## Example: Customer arrivals at a store

- **Poisson:** Number of customers arriving in one hour ( $\lambda = 3$  customers/hour)
- **Exponential:** Time between customer arrivals ( $\lambda = 1/20$  customers/minute or 3 customers/hour)
- If customer arrivals follow a Poisson process with rate  $\lambda$ , then the time between arrivals follows an Exponential distribution with rate  $\lambda$

### Exponential Distribution

Models time between events in a Poisson process

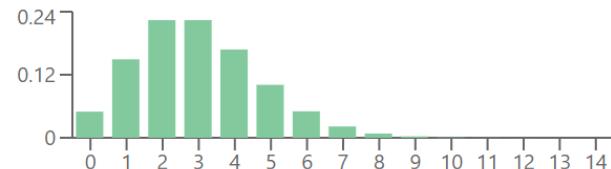
- PDF:  $f(x) = \lambda e^{-\lambda x}$  for  $x \geq 0$
- Mean:  $1/\lambda$
- Continuous distribution



### Poisson Distribution

Models number of events in a fixed interval

- PMF:  $P(X = k) = (\lambda^k * e^{-\lambda}) / k!$
- Mean:  $\lambda$
- Discrete distribution



# Exponential Distribution: Probability Calculation

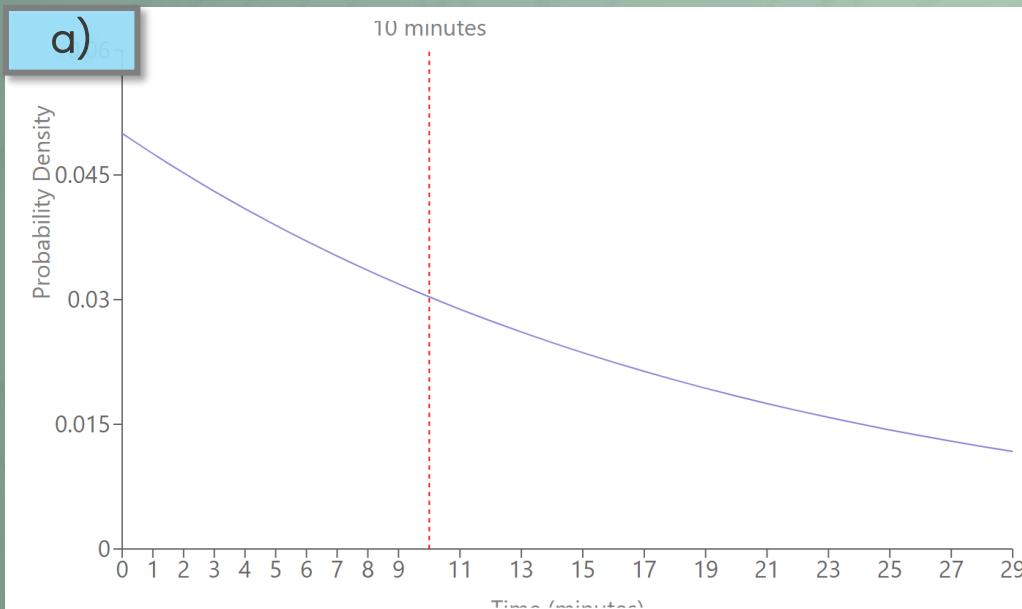
$\lambda = 1/20$  customers/minute or 3 customers/hour

Case a)

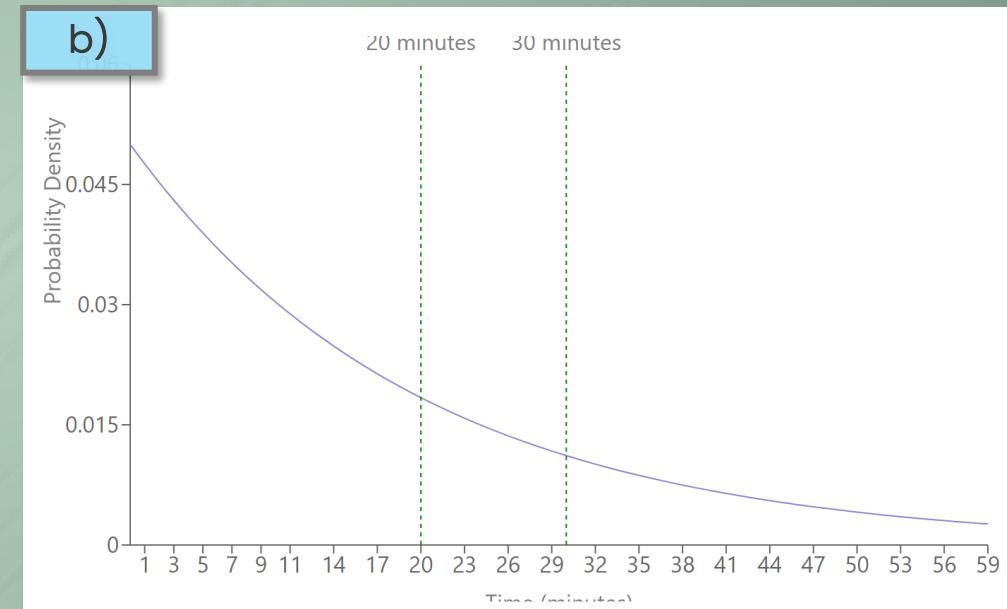
Probability of customer arrival within first 10 minutes

Case b)

Probability of customer arrival between 20 and 30 minutes



- Shaded area under curve represents probability of arrival within 10 minutes.
- Exponential curve shows how probability density decreases over time.
- Approximately 39.35% chance of a customer arriving within the first 10 minutes.



- Shaded area between 20 and 30 minutes represents the probability of arrival in this interval
- Calculation uses difference of CDF:  $F(30) - F(20)$
- Approximately 14.47% chance of customer arriving between 20 and 30 minutes.

# Gamma Distribution

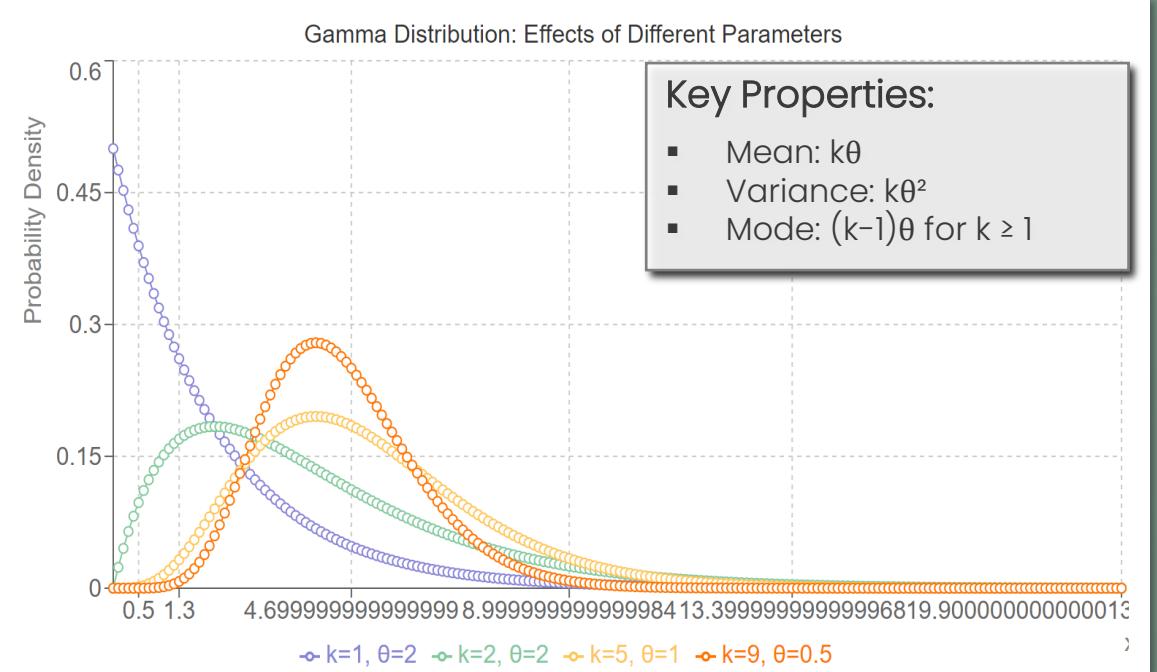
... is continuous, characterized by two parameters: shape ( $k$ ) and scale ( $\theta$ ), modeling waiting times for  $k$  independent events in Poisson process, generalizing exponential distribution.

## Parameters

- $k$  (shape): Determines the basic shape of the distribution
- $\theta$  (scale): Stretches or shrinks the distribution along the x-axis

## Examples and Interpretations

- **$k=1, \theta=2$  (Blue):**  
Equivalent to an exponential distribution. Could model time between events in a Poisson process.
- **$k=2, \theta=2$  (Green):**  
More bell-shaped. Might represent time until the second event in a process.
- **$k=5, \theta=1$  (Yellow):**  
More symmetrical. Could model total processing time for 5 independent tasks.
- **$k=9, \theta=0.5$  (Orange):**  
Nearly symmetrical, approaching normal distribution. Might represent a sum of many small, independent random variables.



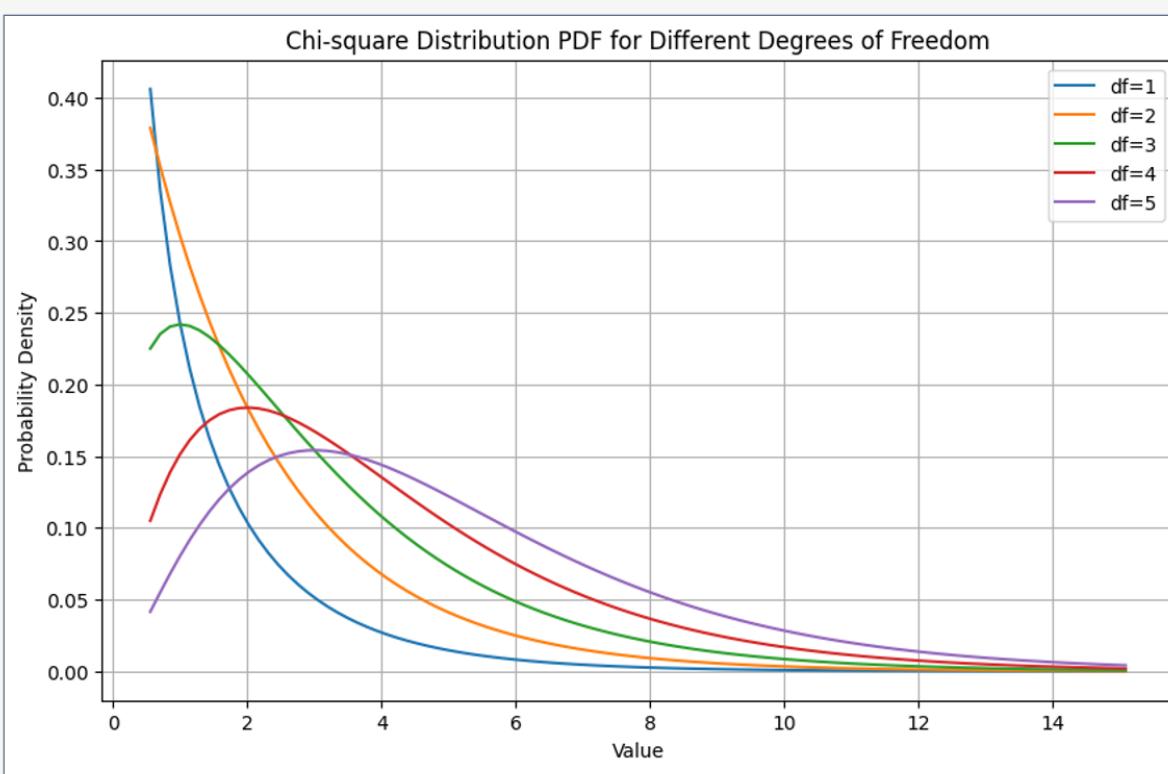
# Connection of Poisson, Exponential, Gamma Distributions

Poisson counts events, Exponential measures time between events, and Gamma extends this to time until the k-th event, collectively modeling random processes.

| Characteristic         | Poisson  | Exponential  | Gamma   |
|------------------------|--|--|---|
| What it models         | Number of events in a fixed interval           | <ul style="list-style-type: none"><li>Time between events</li><li>Special case of Gamma: k=1</li></ul> | Time until k-th event   |
| Parameters             | $\lambda$ (rate)                               | $\lambda$ (rate)   | k (shape), $\theta$ (scale) or $\lambda$ (rate = $1/\theta$ ) |
| Support                | Non-negative integers                          | Positive real numbers  | Positive real numbers   |
| Mean                   | $\lambda$                                      | $1/\lambda$  | $k\theta$   |
| Variance               | $\lambda$                                      | $1/\lambda^2$  | $k\theta^2$   |
| Relationship to others | Inter-event times are Exponential( $\lambda$ ) | Special case of Gamma where k=1  | Sum of k Exponential( $\lambda$ ) ~ Gamma( $k, 1/\lambda$ )   |
| Key property           | Sum of independent Poisson is Poisson          | Memoryless property  | Reproductive property   |
| Example application    | Number of calls received in an hour            | Time until next call   | Time until 5th call   |

# $\chi^2$ distribution

... is a probability distribution of the sum of squares of independent standard normal random variables and is important particularly in the context of variance



[https://en.wikipedia.org/wiki/Chi-squared\\_distribution](https://en.wikipedia.org/wiki/Chi-squared_distribution)

It's the distribution of a sum of the squares of k independent standard normal random variables:

$$Q = \sum_{i=1}^k Z_i^2$$

#### Parameter:

k (mean): independent standard normal random variables, square each and sum them up

#### Use cases:

- $\chi^2$ -tests of independence: They use  $\chi^2$  -distribution.  
Is preference for having cat or dog is independent from city?
- Several other hypothesis tests, particularly connected with variance

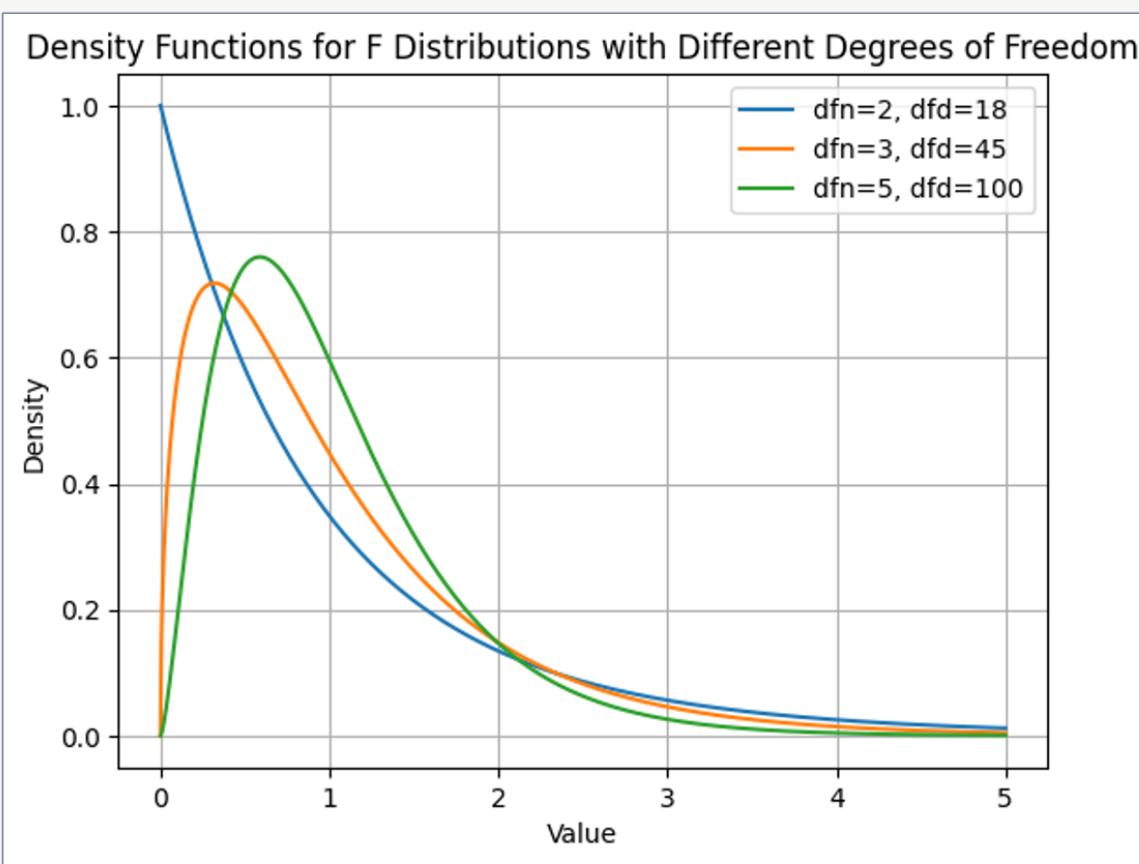
# Connection of $\chi^2$ and Gamma Distributions

$\chi^2(k)$  is equivalent to  $\text{Gamma}(k/2, 2)$ , where  $k$  is degrees of freedom,  
i.e. number of occurrences

| Feature             | Chi-Squared ( $\chi^2$ )   | Gamma                                    |
|---------------------|--|--|
| Parameters          | $k$ (degrees of freedom)   | $\alpha$ (shape), $\theta$ (scale)       |
| Relationship        | <ul style="list-style-type: none"><li><math>\chi^2(k) = \text{Gamma}(k/2, 2)</math></li><li>Special case of Gamma with scale = 2</li></ul> |  |
| Support             | $[0, \infty)$  | $[0, \infty)$                            |
| Mean                | $k$  | $\alpha\theta$                           |
| Variance            | $2k$   | $\alpha\theta^2$                         |
| Skewness            | $\sqrt{(8/k)}$   | $2/\sqrt{\alpha}$                        |
| Common Applications | Hypothesis testing, goodness-of-fit tests  | Modeling waiting times, rainfall amounts |
| Flexibility         | Less flexible, shape determined by $k$   | More flexible, can model various shapes  |

# F-distribution

... is ratio of 2 independent  $\chi^2$ -distributions divided by their respective degrees of freedom



$$X = \frac{s_1^2 / df_1}{s_2^2 / df_2}$$

## Degrees of freedom (df):

- dfn (df of numerator):  
number of groups being compared minus 1  
between group comparison
- dfd (df of denominator):  
total number of observations minus the number of groups  
within group comparison

## Use Cases

- is used in Analysis of Variance (ANOVA)
- comparing variances of  $\geq 3$  samples
- Evaluation of regression results, overall model significance

<https://en.wikipedia.org/wiki/F-distribution>

# Exercises

**Exercise 1:**

Problem: You are about to flip a coin five times. The probability of getting heads (success) is 0.5. What is the probability of getting 3 heads?

**Exercise 2:**

Problem: A manufacturer knows that 10% of his products are defective. He sells products in boxes of 20. What is the probability that a box will contain exactly 2 defective products?

**Exercise 3:**

Problem: A multiple-choice quiz contains 10 questions. Each question has four possible answers, of which one is correct. If a student guesses the answer to each question at random, what is the probability that the student will answer exactly 4 questions correctly?

**Exercise 4:**

Problem: A fast food restaurant serves an average of 10 customers every 15 minutes. What is the probability of serving exactly 7 customers in a 15 minute interval?

**Exercise 5:**

Problem: An IQ test is scored such that the mean score is 100 and the standard deviation is 15. What is the probability of a person scoring higher than 130?

**Exercise 6:**

Problem: A sample of 20 students' test scores has a mean of 76 and a standard deviation of 10. What is the probability of a student scoring less than 70?

**Extra-exercises: Relevant use cases are in context of hypotheses and model estimation****Exercise 7:**

Suppose the variance of a sample of 20 observations is 5. What is the probability that the sample variance is greater than 6?

**Exercise 8:**

If two groups of data are sampled from normal distributions with the same variance, the ratio of their sample variances will follow an F-distribution. Suppose you have two samples of sizes 15 and 20 with variances 4 and 2, respectively. What is the probability that the ratio of these sample variances is less than 1?

# Structured Probabilistic Models (Graphical Models)

... are powerful mathematical frameworks that use graphs to represent, visualize probabilistic relationships among set of variables, enabling efficient reasoning, inference in complex systems

Directed vs.  
Undirected  
Graphical Models



Markov Models



Directed Models



Undirected Models,  
Markov Fields

Exercises

# Structured Probabilistic Models

... use graph theory to intuitively and efficiently represent and reason about uncertainty, capturing complex variable dependencies better than traditional probabilistic methods.

## Directed Models

- Also known as: Bayesian Networks
- Represent asymmetric relationships
- Use arrows to show direction of influence
- Model causal and temporal relationships
- Example: Disease → Symptom

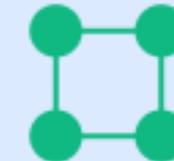
Capture causal or hierarchical relationships



## Undirected Models

- Also known as: Markov Random Fields
- Represent symmetric relationships
- Use lines without arrows
- Model mutual dependencies
- Example: Protein interactions

Represent mutual or symmetric interactions

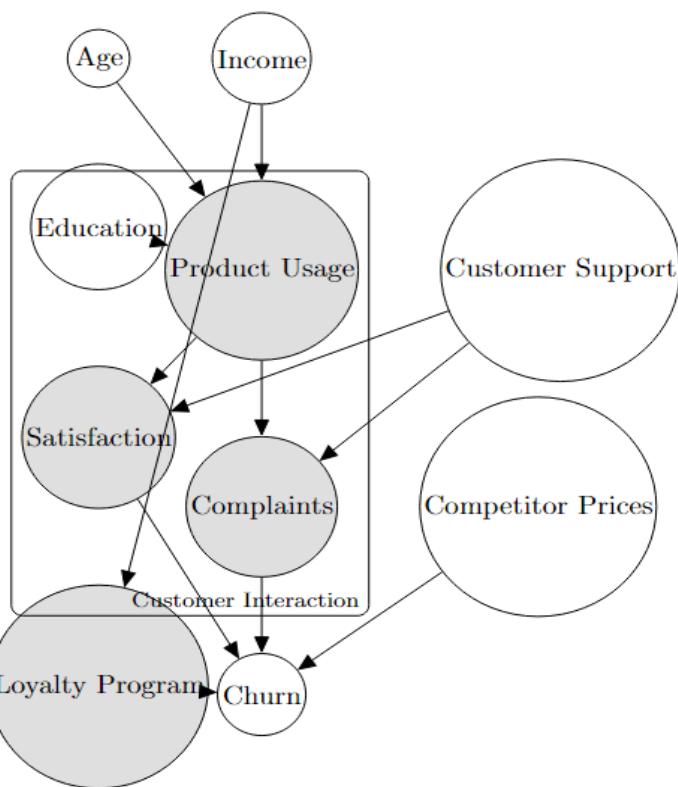


# Directed vs. Undirected Graphical Models

Directed models excel in causal inference and decision-making, while undirected models are preferred for symmetric relationships like in image processing or social networks

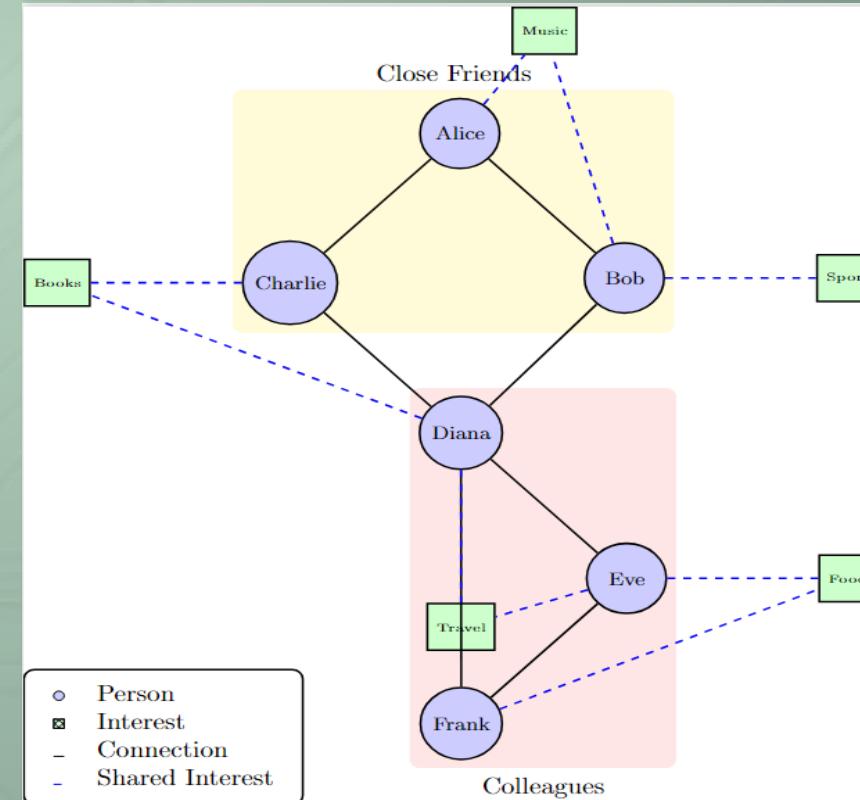
## Directed Graphs:

- Bayesian Networks
- Use Case: Customer Churn Prediction



## Undirected Graphs:

- Markov Random Fields
- Use Case: Social Network



# Directed Graphical Model: Home security alarm

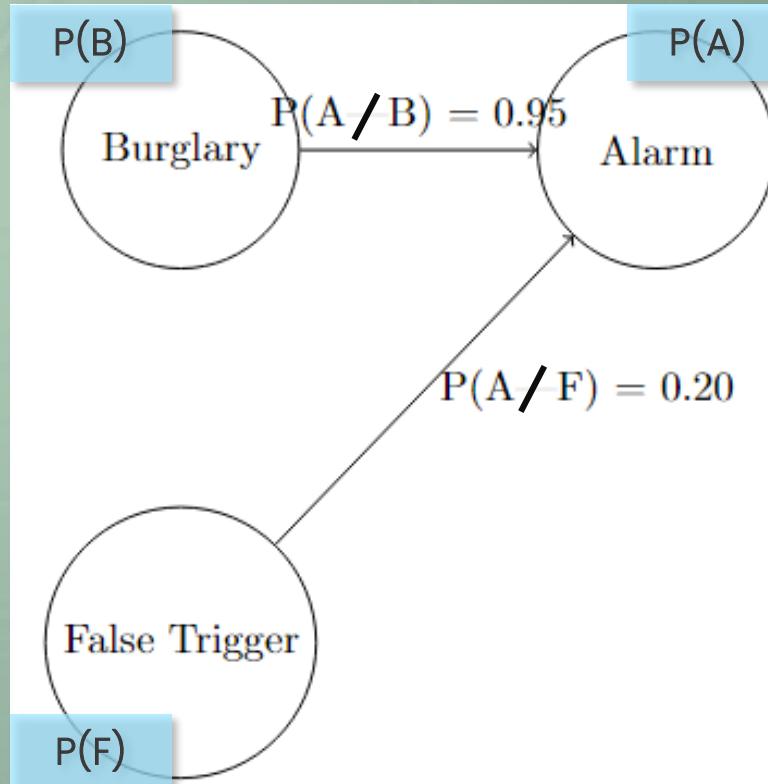
... Use Case: Home security alarm can be triggered by burglaries or false alarms.  
What's the probability of an actual burglary when the alarm is triggered?

What is  
 $P(B/A)$ ?  
0.1919

Conditional Probabilities

| Event   | Probability |
|---|-------------|
| $P(B)$ - Prior probability of burglary              | 0.001       |
| $P(F)$ - Probability of factors causing false alarm | 0.02        |
| $P(A/B)$ - Probability of alarm given burglary      | 0.95        |
| $P(A/F)$ - Probability of alarm given false trigger | 0.20        |

Directed Graphical Model



Answer

$$P(A/B)*P(B) \quad P(A/F)*P(F)$$

$$\begin{aligned} P(A) &= (0.95*0.001)+(0.20*0.02) \\ &= 0.00095+0.004 \\ &= 0.00495 \end{aligned}$$

$$P(B|A) = \frac{0.95 * 0.001}{0.00495} \approx 0.1919$$

P(A)

# Directed Graph Model: Use Cases

They are commonly used in scenarios where causality and conditional dependencies between variables are essential

## Web Page Ranking

Model hyperlinks between web pages for search engine algorithms

## Flow Networks

Represent material flow in supply chains or traffic flow in transportation

## Causal Inference

Represent cause-effect relationships in various domains

## Dependency Analysis

Model software package dependencies or project task dependencies

## Citation Networks

Represent citations between academic papers or patents

## Recommendation Systems

Model user preferences and item relationships for personalized recommendations

## Gene Regulatory Networks

Represent interactions and regulatory relationships between genes

## Social Influence Analysis

Model information propagation and influence in social networks

# Markov Models

... are mathematical system that transitions between states with probabilities that depend only on current state

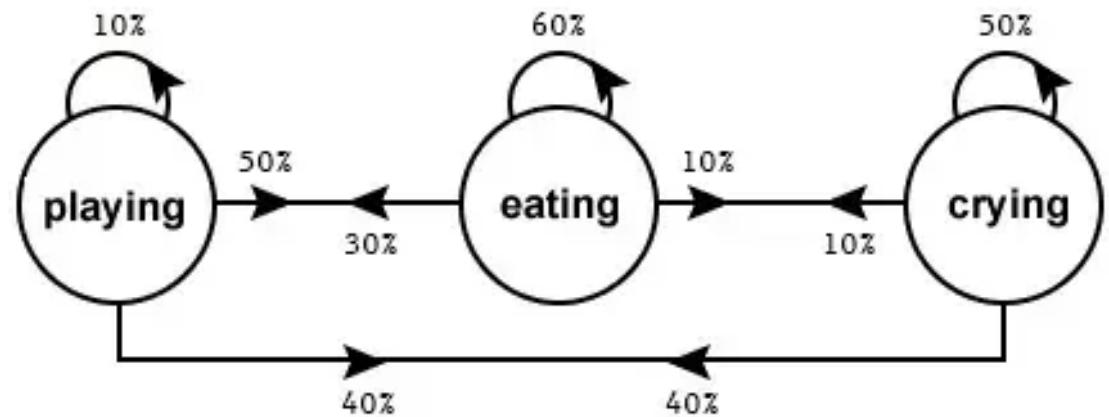
## Key Characteristics

- Models random processes
- Exhibits the Markov property (memoryless)
- Named after Andrey Markov (1856 – 1922, Russia)
- Consists of states and transition probabilities

## Markov Property Explained

The probability of transitioning to any particular state depends solely on the current state and time elapsed, not on the sequence of states that preceded it.

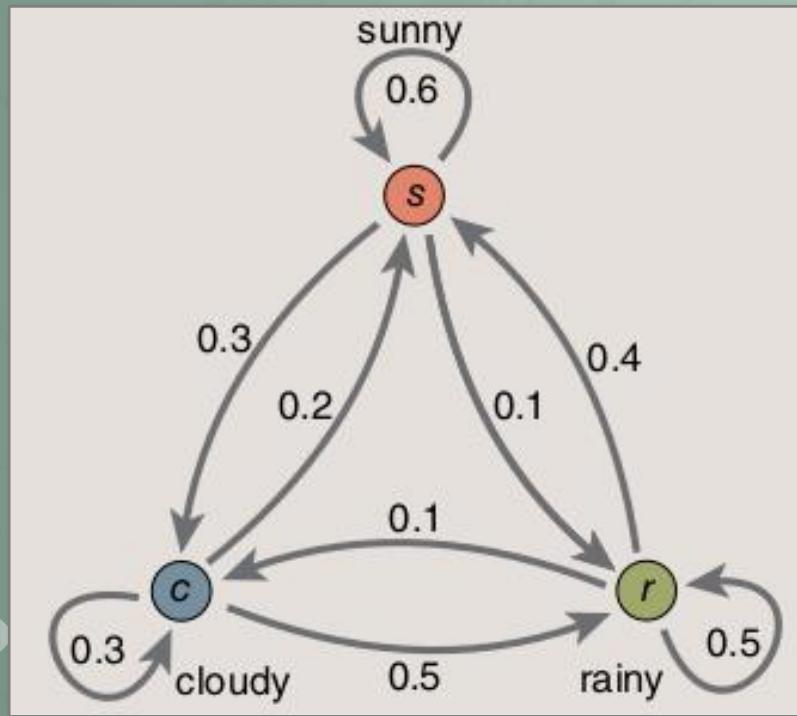
**Markov state diagram of a child behaviour**



# Markov Models

... Use Case: Weather prediction

How probable is rain in two days after cloudy day? Attention: Not 2 days of rain in a row!



probability matrix,  $P$

weather tomorrow

|               |   | s          | c          | r          |
|---------------|---|------------|------------|------------|
| weather today | s | 0.6<br>1,1 | 0.3<br>1,2 | 0.1<br>1,3 |
|               | c | 0.2<br>2,1 | 0.3<br>2,2 | 0.5<br>2,3 |
|               | r | 0.4<br>3,1 | 0.1<br>3,2 | 0.5<br>3,3 |

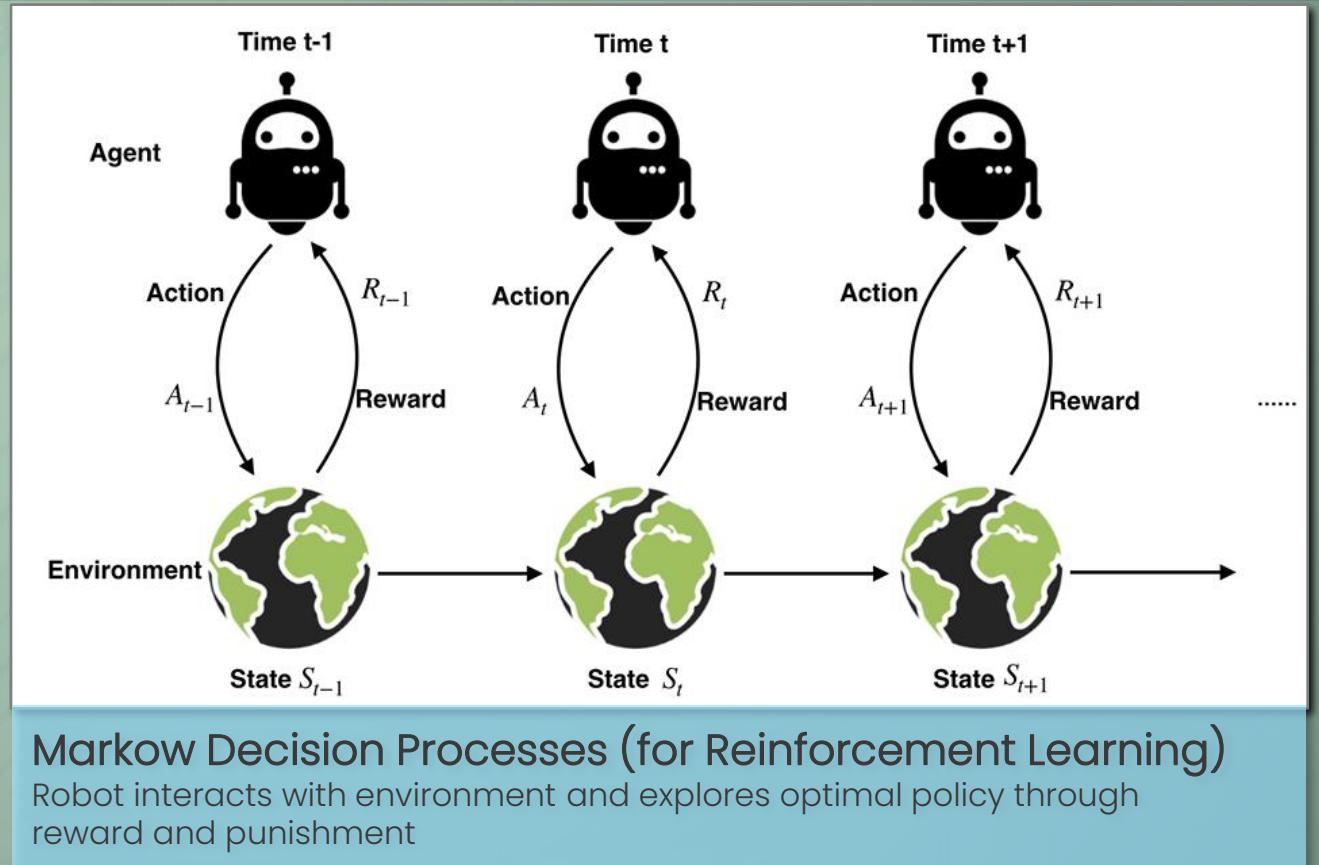
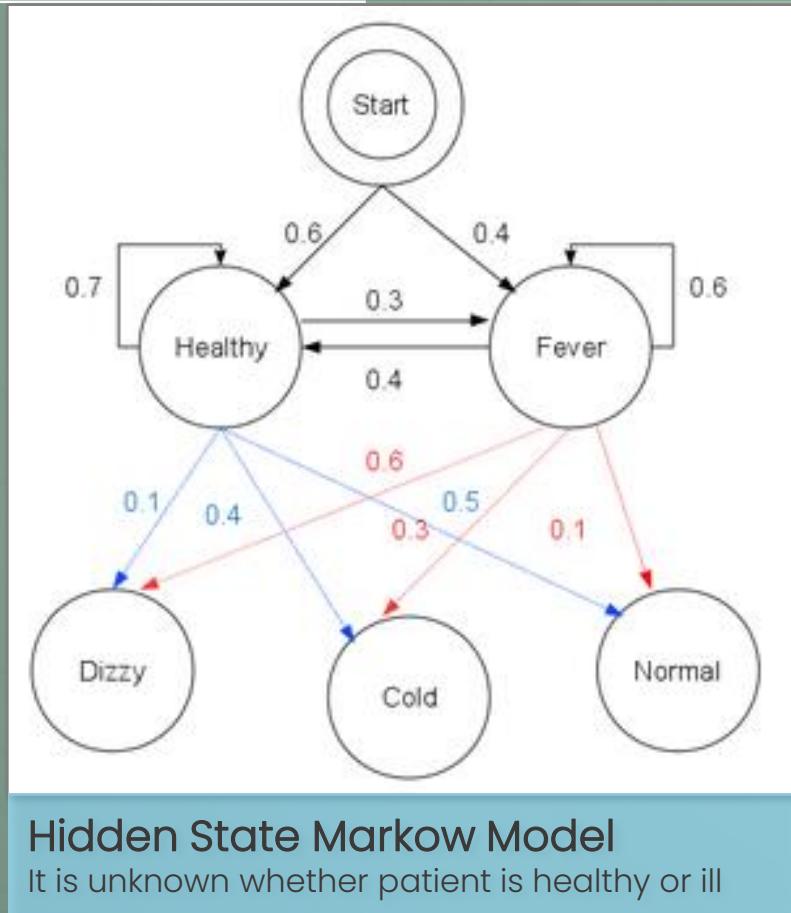
probability of rain in two days if it's cloudy today

$$(P_{2,3})^2 = \begin{matrix} c \\ 0.2 \\ 2,1 \end{matrix} \times \begin{matrix} 0.1 & 0.3 & 0.5 \\ 1,3 & 2,2 & 2,3 \end{matrix} = 0.1 \times 0.2 + 0.3 \times 0.5 + 0.5 \times 0.5 = 0.42$$


# Advanced Markov Model Approaches

Hidden State Markow Model: have unobservable state. Only some outputs can be observed.

Markow Decision Processes: Optimal policy that maximized rewards needs to be found



# (Hidden) Markov Models, Markov Decision Processes

## Comparison

| Characteristic  | MarkovModel (MM)   | Hidden MarkovModel (HMM)  | Markov Decision Process (MDP)   |
|-----------------|--|---|---|
| States          | Observable   | Hidden  | Observable  |
| Observations    | Same as states   | Emissions from states   | Same as states  |
| Transitions     | Probabilistic  | Probabilistic   | <ul style="list-style-type: none"><li>▪ Probabilistic</li><li>▪ action-dependent</li></ul>  |
| Decision Making | No   | No  | Yes   |
| Key Components  | <ul style="list-style-type: none"><li>▪ State space</li><li>▪ Transition probabilities</li></ul> | <ul style="list-style-type: none"><li>▪ Hidden state space</li><li>▪ Observable space</li><li>▪ Transition probabilities</li><li>▪ Emission probabilities</li></ul> | <ul style="list-style-type: none"><li>▪ State space</li><li>▪ Action space</li><li>▪ Transition probabilities</li><li>▪ Reward function</li></ul> |
| Main Purpose    | State prediction   | State inference   | Decision optimization   |
| Example         | Weather forecasting  | Speech recognition  | Robot navigation  |

# Undirected Graphical Model

... are mathematical structures used to represent relationships between objects where connections have no inherent direction. Causality is mutual

Example: Travel connections of airline



```
graph TD; SF((San Francisco)) --- DL((Denver)); SF --- CH((Chicago)); DL --- CH; CH --- NY((New York)); CH --- W((Washington)); NY --- W
```

The diagram illustrates an undirected graphical model representing travel connections between six cities: San Francisco, Denver, Chicago, Detroit, New York, and Los Angeles. The cities are represented by nodes (circles), and their connections are represented by edges (lines). Bidirectional arrows indicate that travel can occur in either direction between connected cities. The connections are as follows: San Francisco is connected to Denver and Chicago; Denver is connected to Chicago; Chicago is connected to Detroit, New York, and Washington; and New York is connected to Washington.

- Edges are bidirectional, allowing travel in both directions.
- Represented as  $G = (V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of edges.
- Properties include connectivity, degree (number of edges connected to one vertex), cycles.



# Undirected Graph Model: Use Cases

... are used when modeling relationships or connections that are inherently symmetric or bidirectional, where the direction of the relationship is not important or meaningful

## Collaborative Filtering

Power recommendation systems based on user-item interactions

## Social Network Analysis

Model friendships, collaborations, and information flow in social groups

## Collaboration Graphs

Represent joint authorship in academic publications

## Computer Networks

Model network topology and connectivity between devices

## Financial Markets

Model correlations between different financial instruments

## Ecological Networks

Represent species interactions in ecosystems



## Image Segmentation

Partition images into meaningful regions for computer vision tasks

## Power Grids

Model electricity distribution networks

## Transportation Networks

Model road networks, flight connections, or public transit systems

## Molecular Structures

Represent chemical compounds and protein interactions

# Typical Calculations in Undirected Graphs



## Shortest Path

Find the path with minimum total edge weight between two nodes



## Connectivity

Determine if the graph is connected or find connected components



## Centrality Measures

Identify important nodes (e.g., degree, betweenness, closeness centrality)



## Minimum Spanning Tree

Find a tree that connects all nodes with minimum total edge weight



## Graph Clustering

Identify groups of densely connected nodes



## Maximum Flow

Calculate the maximum flow between source and sink nodes

# Exercises

## Exercise 1:

Maritime Logistics Probabilistic Model Consider a directed graph model for predicting successful cargo delivery in a maritime logistics network:

W: Weather conditions (1: Favorable, 2: Moderate, 3: Severe)

P: Port congestion (1: Low, 2: Medium, 3: High)

S: Ship maintenance status (1: Excellent, 2: Average, 3: Poor)

D: Delivery outcome (1: On-time, 2: Delayed, 3: Significantly delayed)

The directed graph structure is as follows:

W → P

W → D

P → D

S → D

Given the following probabilities:

$P(W = 1) = 0.5, P(W = 2) = 0.3, P(W = 3) = 0.2$

$P(S = 1) = 0.6, P(S = 2) = 0.3, P(S = 3) = 0.1$

$P(P = 1 | W = 1) = 0.7, P(P = 2 | W = 1) = 0.2, P(P = 3 | W = 1) = 0.1$

$P(P = 1 | W = 2) = 0.3, P(P = 2 | W = 2) = 0.5, P(P = 3 | W = 2) = 0.2$

$P(P = 1 | W = 3) = 0.1, P(P = 2 | W = 3) = 0.3, P(P = 3 | W = 3) = 0.6$

$P(D = 1 | W = 1, P = 1, S = 1) = 0.9, P(D = 2 | W = 1, P = 1, S = 1) = 0.1, P(D = 3 | W = 1, P = 1, S = 1) = 0.0$

$P(D = 1 | W = 2, P = 2, S = 2) = 0.5, P(D = 2 | W = 2, P = 2, S = 2) = 0.4, P(D = 3 | W = 2, P = 2, S = 2) = 0.1$

$P(D = 1 | W = 3, P = 3, S = 3) = 0.1, P(D = 2 | W = 3, P = 3, S = 3) = 0.3, P(D = 3 | W = 3, P = 3, S = 3) = 0.6$

Tasks:

Visualize the directed graph for this maritime logistics model.

Calculate  $P(D = 1 | W = 1, S = 1)$ , the probability of on-time delivery given favorable weather and excellent ship maintenance.

Discuss how this model could be used to improve maritime logistics planning.

# Exercises

## Exercise 2:

Consider a simple weather model for a particular location with three possible states:

1. Sunny (S)\
2. Cloudy (C)\
3. Rainy (R)

The weather follows a Markov chain model with the following transition matrix:

|       |   |                  |    |
|-------|---|------------------|----|
| To:   | S | C                | R\ |
| From: | S | [0.7, 0.2, 0.1]\ |    |
|       | C | [0.3, 0.4, 0.3]\ |    |
|       | R | [0.2, 0.3, 0.5]  |    |

For example, if it's Sunny today, there's a 70% chance it will be Sunny tomorrow, 20% chance it will be Cloudy, and 10% chance it will be Rainy.

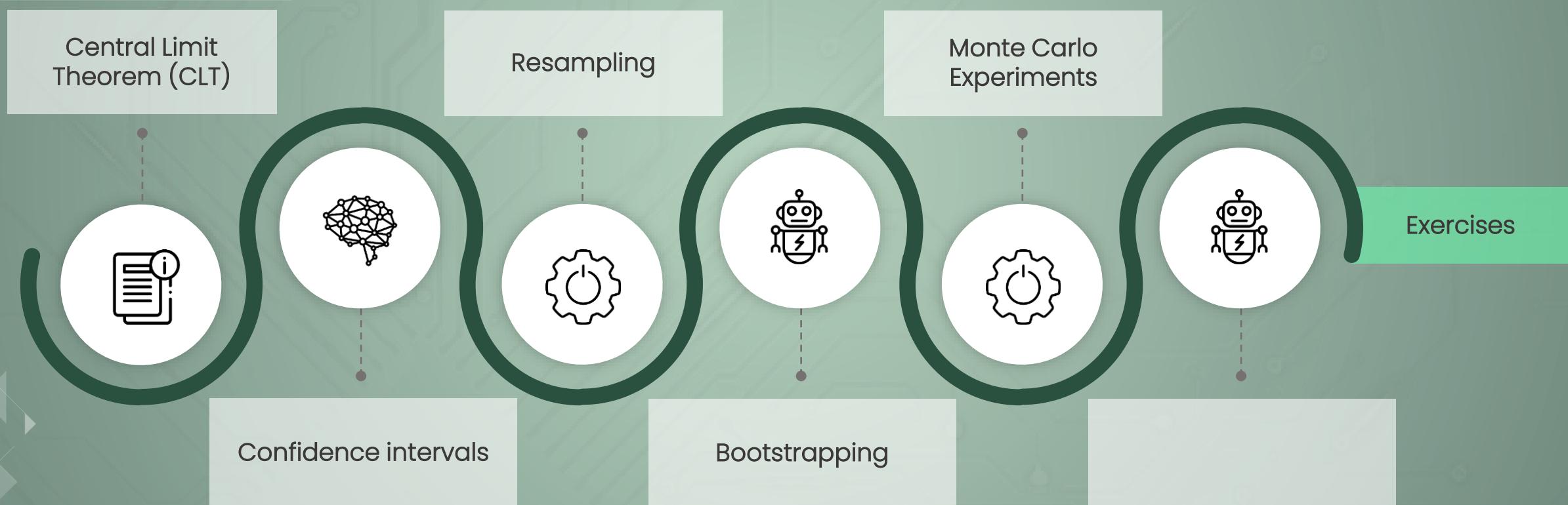
Tasks:\

1. Visualize the Markov chain as a directed graph.\
2. If it's Sunny today, what's the probability it will be Rainy two days from now?\
3. Calculate the steady-state probabilities for each weather state.\
4. Simulate the weather for the next 30 days, starting from a Sunny day.

Not 2 days of rain in a row!

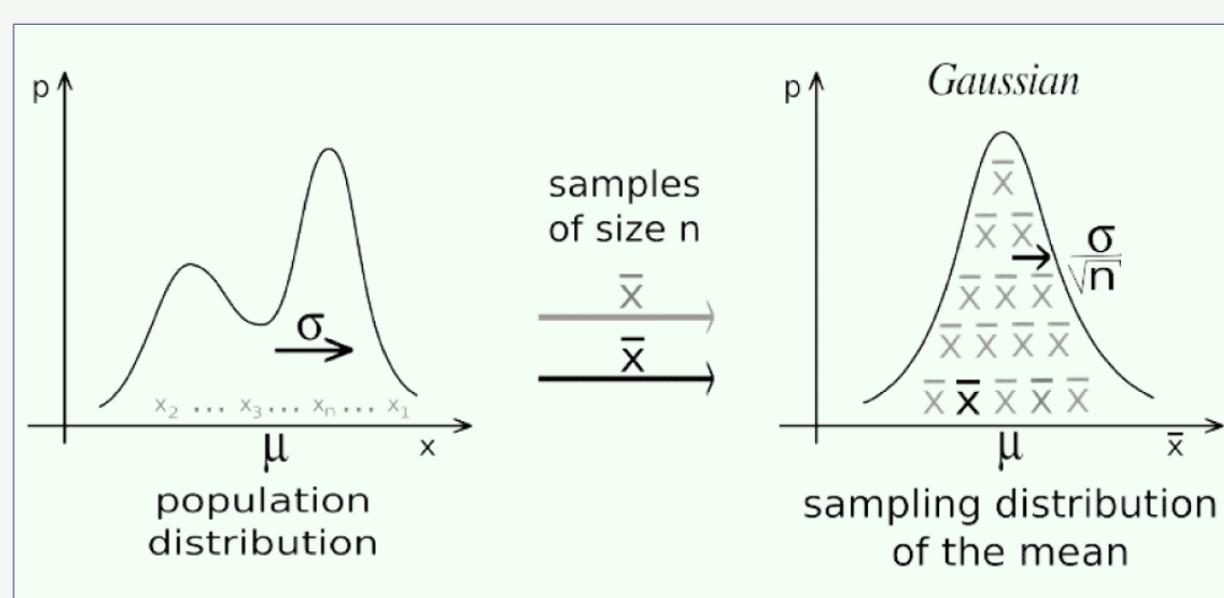
# Sampling, Monte Carlo Experiments

... is the process of selecting a subset of individuals from a statistical population to estimate characteristics of the whole population



# Central limit theorem

The distribution of sample means approximates a normal distribution as the sample size becomes larger, regardless of the population's distribution.



## Definition:

- For population with mean  $\mu$  and standard deviation  $\sigma$
  - take sufficiently large random samples from the population
  - with replacement (independent)
  - Independent and identically distributed (i.i.d.) random variables
  - Sample size is large enough (typically  $n > 30$ )
- distribution of sample means is appr. normally distributed, regardless of population distribution

## Why is it important?

- This principle enables us to make statistical inferences about populations based on sample data
- foundation for hypothesis testing, confidence intervals, etc.

# Confidence intervals

... is range of values that is likely to contain an unknown population parameter with certain level of confidence, widely used e.g. for hypothesis testing

For large sample size

$$CI = \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Where:

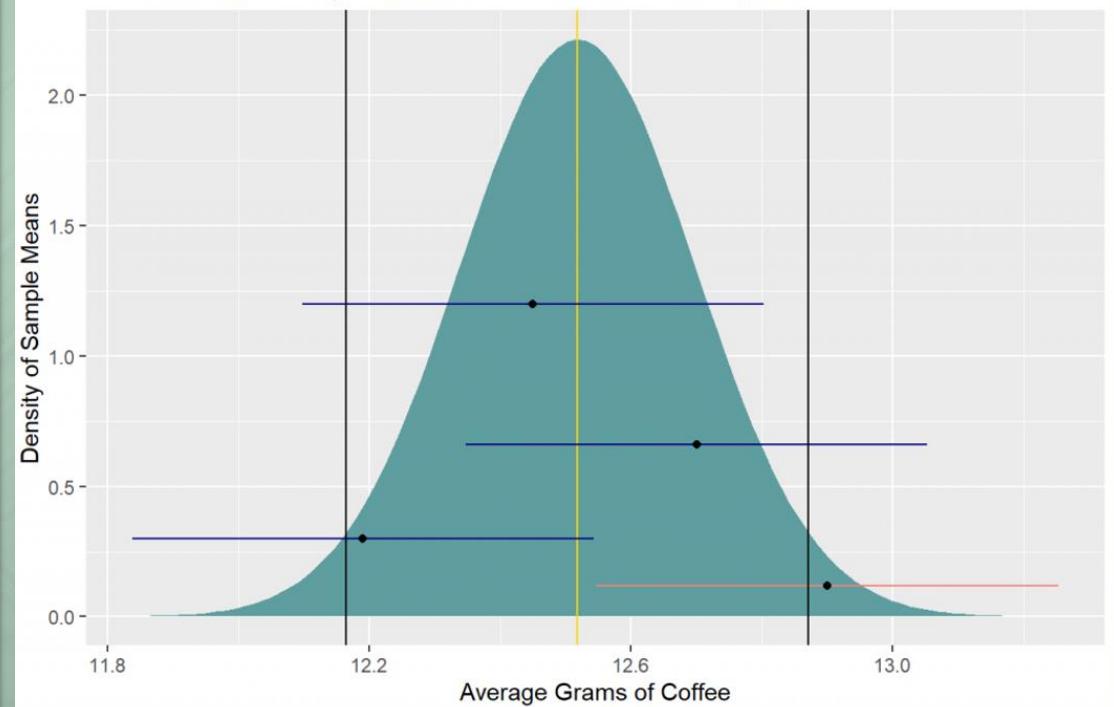
- $\bar{x}$  is the sample mean
- $z_{\alpha/2}$  is the z-score for the chosen confidence level
- $\sigma$  is the population standard deviation
- $n$  is the sample size

For small sample size

- Similar formula is used
- however not based on standard normal distribution but t-distribution

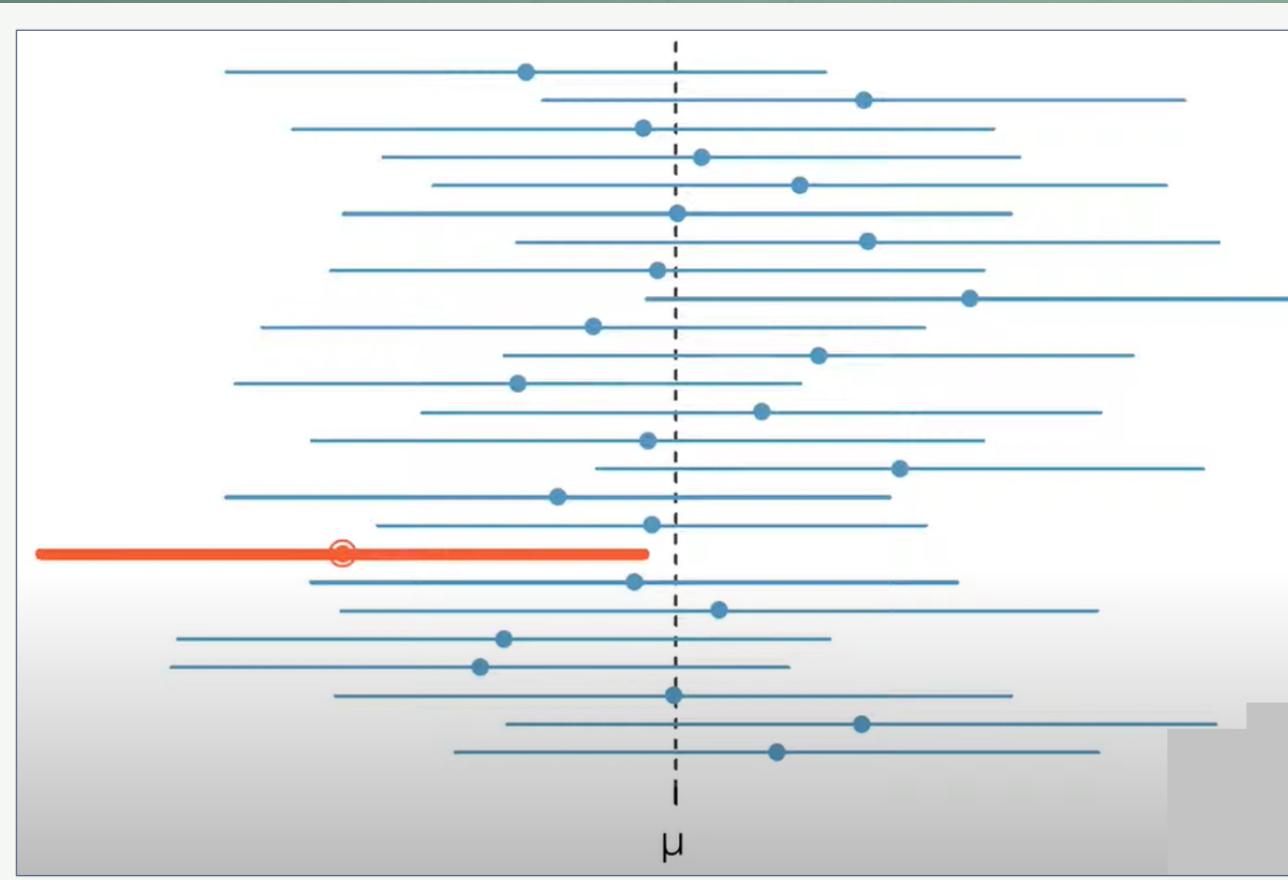
Distribution of Sample Means When  $n = 100$

Black vertical lines are plus/minus  $J$  from the center of the distribution



# Confidence intervals

... need careful interpretation. Assumption:  $\alpha=5\%$



## Assumption:

population parameter  $\mu$  (average of population) is fixed

## Wrong interpretation:

- With a chance of 95% the confidence interval contains  $\mu$

## Correct interpretation:

- If sampling process is repeated many times or infinitely
- confidence intervals are generated each time
- 95% of these confidence intervals contain  $\mu$

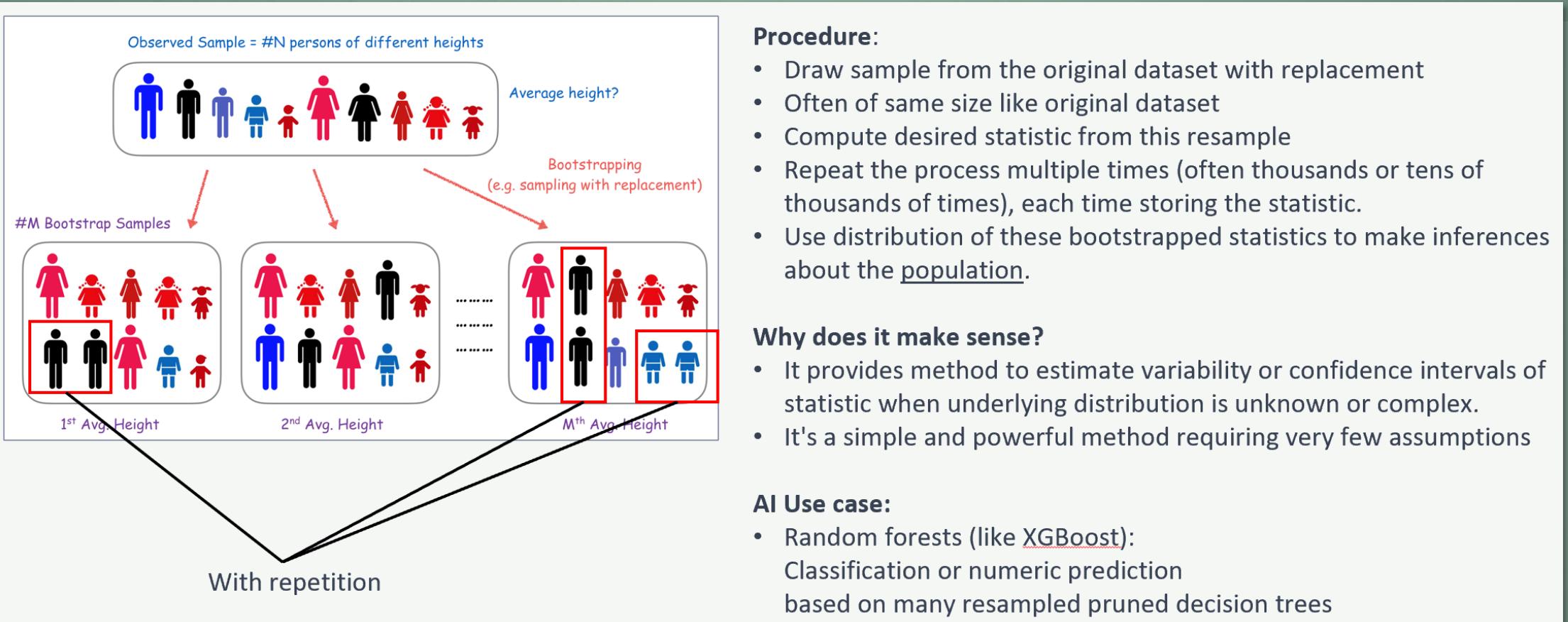
# Resampling

... Is set of methods of statistical inference that involve drawing repeated samples from original data samples

|                                      |  |
|--------------------------------------|--|
| <b>What is it good for?</b>          | <ul style="list-style-type: none"><li>• Versatile method to make statistical inferences about a dataset</li><li>• It helps to validate models by using random subsets (bootstrap) or sequences (cross-validation) dataset</li><li>• It extends possibilities for hypothesis testing and estimation</li></ul> |
| <b>Example:<br/>Bootstrap</b>        | <ul style="list-style-type: none"><li>• Many resamples (with replacement) of the observed dataset are generated</li><li>• Arbitrary distribution is transformed into normal distribution</li><li>• convenient for estimation and hypothesis testing.</li></ul>   |
| <b>Example:<br/>Cross-validation</b> | <ul style="list-style-type: none"><li>• a form of resampling used to prevent overfitting in machine learning models</li><li>• dataset is divided into 'k' subsets; used for training the model</li><li>• respective residual data is used as test set</li></ul>  |

# Bootstrapping

... is a resampling method that involves drawing repeated samples from the original data set, with replacement. Each sample drawn is the same size as the original dataset.



# Monte Carlo Simulations

... are statistical technique using repeated random sampling to obtain numerical results

## Process

1. Define input domain
2. Generate random inputs
3. Perform computations
4. Aggregate results

## Applications

Risk analysis, optimization, integration, physics simulations, finance

## Advantages

Can handle complex systems with many variables and uncertainties

## Limitations

Computationally intensive; accuracy depends on number of simulations

## Process

1. Define input domain
2. Generate random inputs
3. Perform computations
4. Aggregate results



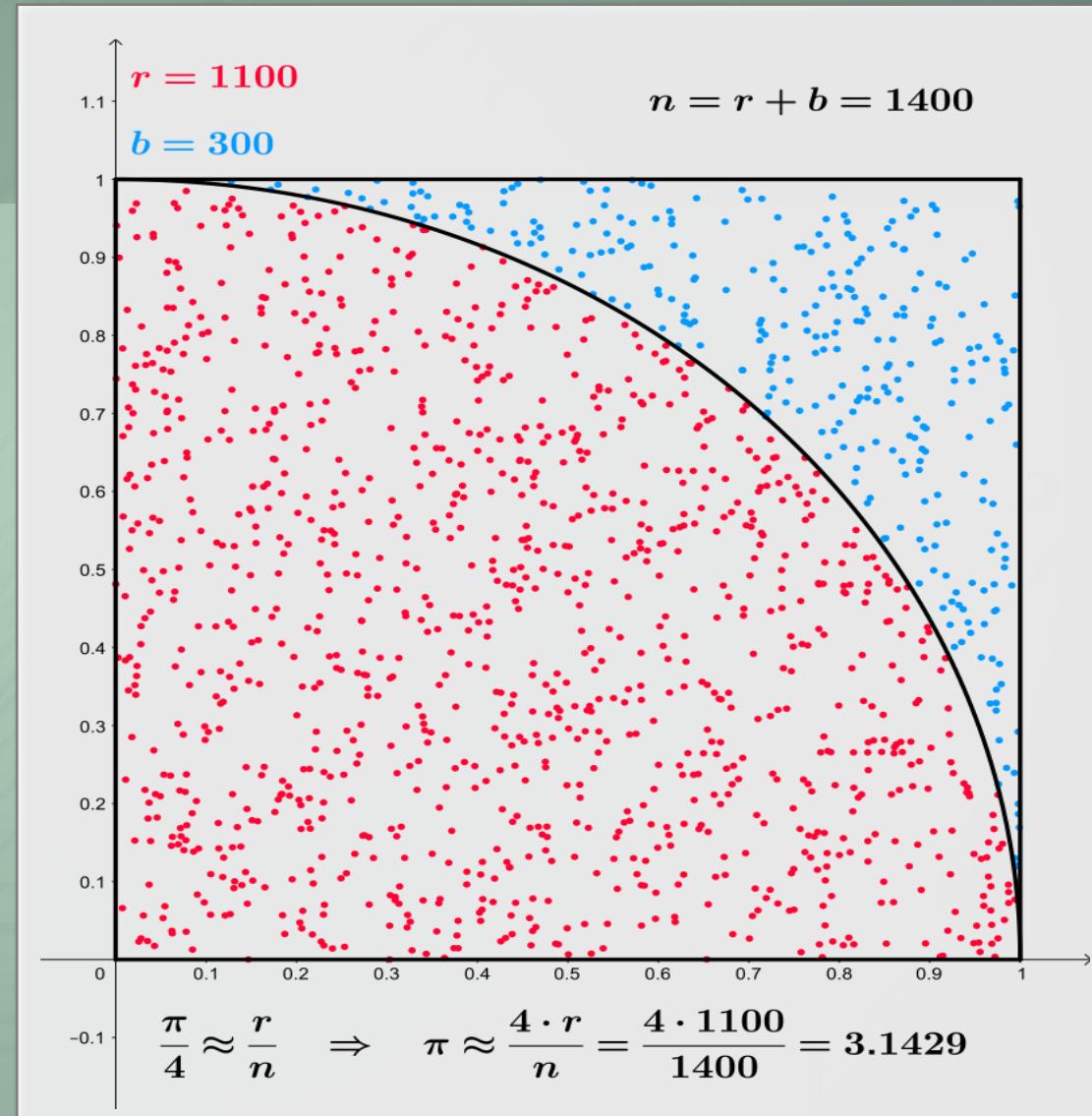
100s of dogs do random walk in Monte Carlo



# Monte Carlo Simulations: Example

Estimate  $\pi$  using random points in square with an inscribed quarter circle

- Concept: Use random sampling to estimate  $\pi$
- Setup: 1x1 square with inscribed quarter circle
- Process:
  - Generate random points in square
  - Count points inside quarter circle
  - Calculate:  $\pi \approx 4 \times (\text{points in circle} \div \text{total points})$
- Key Points:
  - More points increase accuracy
  - Simple but computationally intensive for high precision



# Monte Carlo vs. Bootstrapping

Monte Carlo simulations generate new data from assumed distributions to model future scenarios, bootstrapping resamples existing data to estimate statistical properties of current sample

| Aspect                  | Monte Carlo   | Bootstrapping  |
|-------------------------|---|--|
| Core Concept            | Generate new data from known or assumed distribution  | Resample with replacement from existing dataset  |
| Data Source             | Theoretical model or distribution   | Observed data  |
| Sample Size             | Can generate large samples  | Limited by original dataset size   |
| Primary Use             | <ul style="list-style-type: none"><li>▪ Simulating future scenarios</li><li>▪ risk assessment</li></ul> | <ul style="list-style-type: none"><li>▪ Estimating sampling distributions</li><li>▪ confidence intervals</li></ul> |
| Assumptions             | Requires specifying underlying distribution   | Assumes sample is representative of population   |
| Computational Intensity | Can be very high for complex models   | Generally less intensive   |
| Flexibility             | Highly flexible for various scenarios   | Limited to inference about observed data   |



# Exercises

## Exercise 1

Implement a python function called central\_limit\_theorem that accepts three arguments: population\_data, sample\_size, and num\_of\_samples. Your function should simulate the central limit theorem. The function should draw the specified number of samples of the given size from the population data, calculate the means of these samples, and return a list of these means.

Test this function using a randomly generated population data. Plot the distribution of sample means and the mean of the population.

## Exercise 2

Implement a python function called confidence\_interval that accepts two arguments: data and confidence. Your function should calculate and return the confidence interval of population mean for the given data at the specified confidence level.

Test your function on a randomly generated data.

## Exercise 3

How does the confidence interval of variance look like?

## Exercise 4

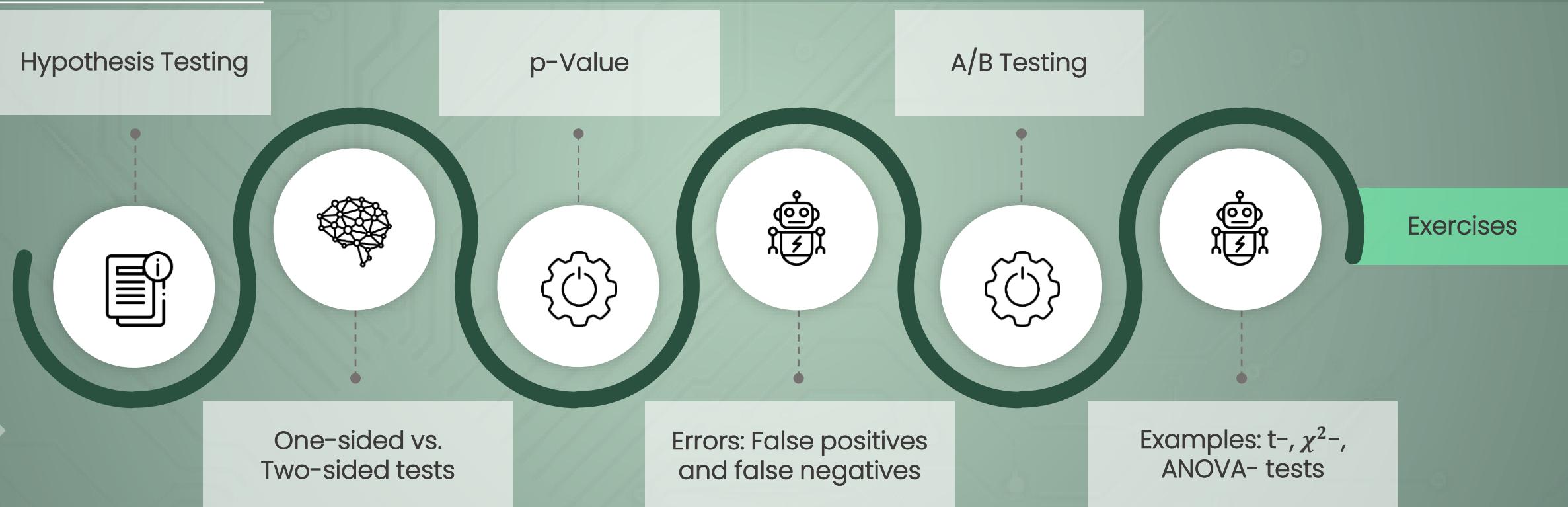
Implement a python function called bootstrap that accepts three arguments: data, num\_samples, and statistic. Your function should perform bootstrap sampling on the data (resampling with replacement) the specified number of times, applying the provided statistical function to each sample, and return an array of the calculated statistic for each sample.

Test this function using a randomly generated data and numpy's mean as the statistic. Calculate and print the mean and standard deviation of the bootstrap samples' statistic.



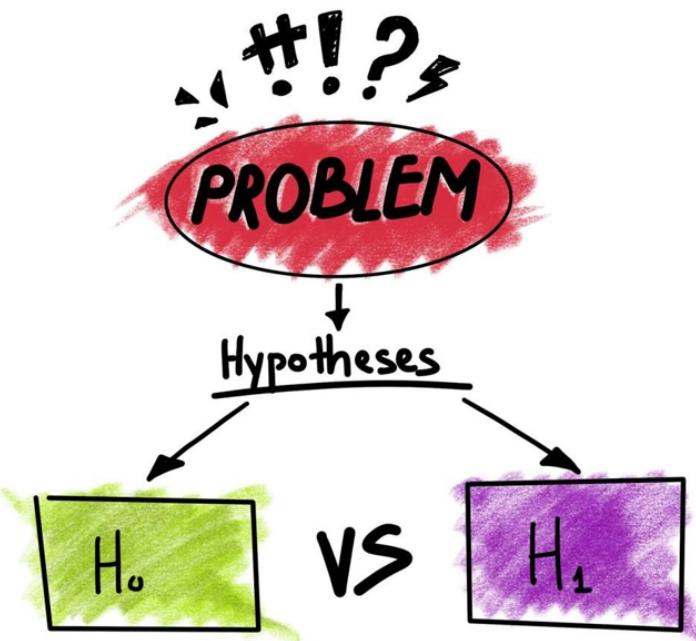
# Hypothesis testing

... is statistical method that uses sample data to evaluate the plausibility of null hypothesis ("there is no effect"). Is there enough evidence in the data in order to reject the null?



# Hypothesis testing

... is statistical method that uses sample data to evaluate the plausibility of null hypothesis ("there is no effect"). Is there enough evidence in the data in order to reject the null?

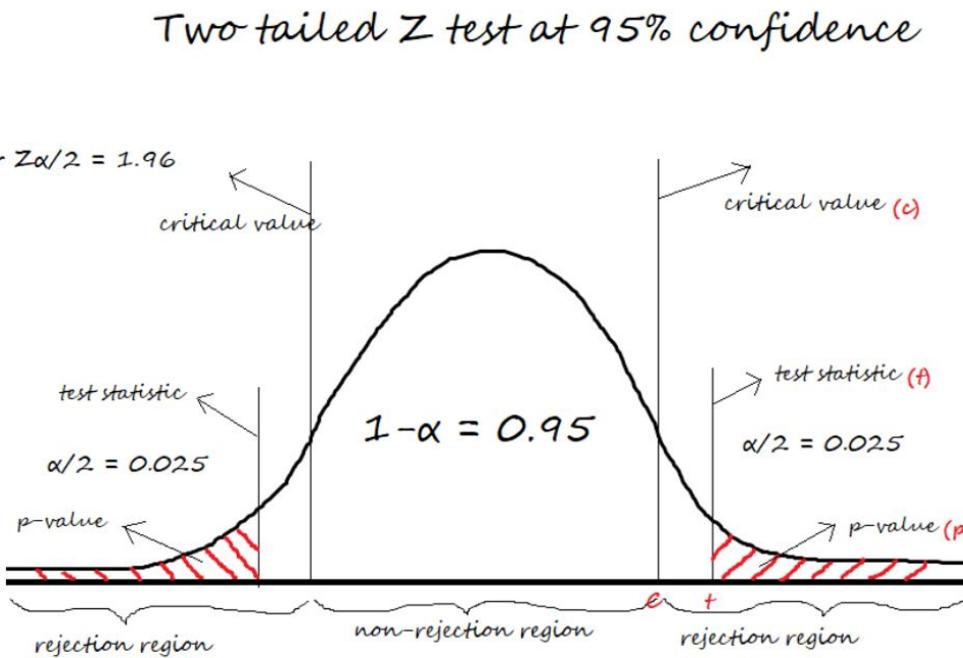


**It involves formulating two competing hypotheses**

- the null hypothesis ( $H_0$ ): initial claim, status quo, which usually means: No effect/difference
- and the alternative hypothesis ( $H_1$ ); we want evidence for the alternative hypothesis
- We assess evidence from the data, in order to determine whether to reject or fail to reject the null hypothesis.

# Hypothesis testing

... involves specifying significance level, analyzing sample data using statistical test rejecting or not rejecting null hypothesis



## Workflow:

### 1. State Hypotheses:

Null Hypothesis ( $H_0$ )  
and Alternative Hypothesis ( $H_1$ )

### 2. Specify critical value $\alpha$ :

usually 0.05 for 2-sided tests

### 3. Calculate test statistic $t$ with:

- t-test
- $\chi^2$ -test
- z-test
- ANOVA-test, and many more

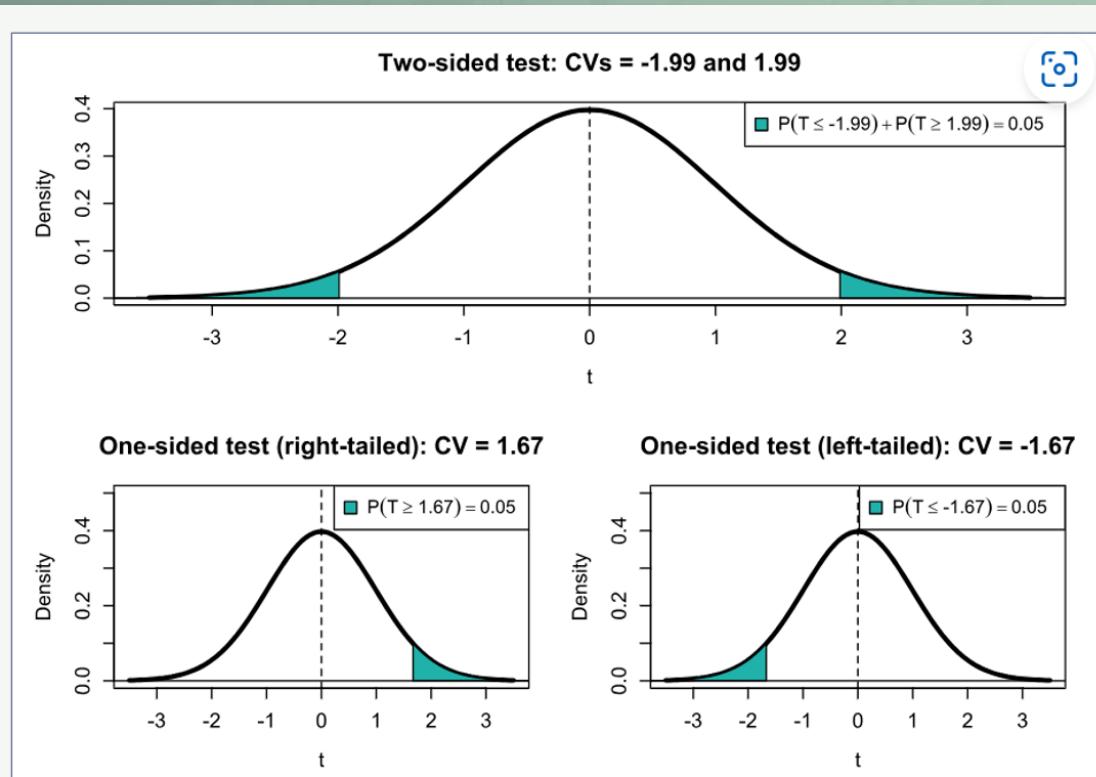
### 4. Interpret results comparing $t$ and $\alpha$ :

- Reject null hypothesis (support  $H_1$ ) or not (support  $H_0$ )?

# One-sided vs. Two-sided tests

One-sided hypothesis test checks if parameter is greater/less than a certain value.

Two-sided test checks if parameter is not equal to a certain value, regardless of direction.



## Two-sided (or Two-tailed) Tests

- ... assess if a parameter is either greater or less than the null hypothesis value.
- Example:**  
Company is testing if a website redesign affects, i.e. either increases or decreases user time on site.
  - $H_0$ : Website redesign has no effect on user time on site.
  - $H_1$ : Website redesign affects user time on site.

## One-sided (or One-tailed) Tests

- ... check if parameter is either greater or less than null hypothesis value, but not both.
- Example:**  
Pharmaceutical company is testing a new drug and wants to know if it reduces disease recovery time (not interested if it increases).
  - $H_0$ : Drug does not reduce recovery time.
  - $H_1$ : Drug reduces recovery time.

# Deciding to reject $H_0$ or not

... is standardized procedure like an “API”

- Initial assumption: “There is no effect”, i.e. we don’t reject  $H_0$  yet
- Samples from population vary around the population mean  $\mu$
- Theoretical probability of samples
  - around  $\mu$  (or further away, the integral) is high
  - far away from  $\mu$  (or even further, the integral) is low
- Now you take the real sample and focus on distance from  $\mu$

If it is **very close**

sample is **not extreme**

theoretical probability of getting more extreme sample is **high**

we say: “There is no effect, there is **not enough evidence** to reject  $H_0$ ”

simpler: “**We choose  $H_0$** ”

If it is **very far**

sample is **extreme**

theoretical probability of getting more extreme sample is **low**

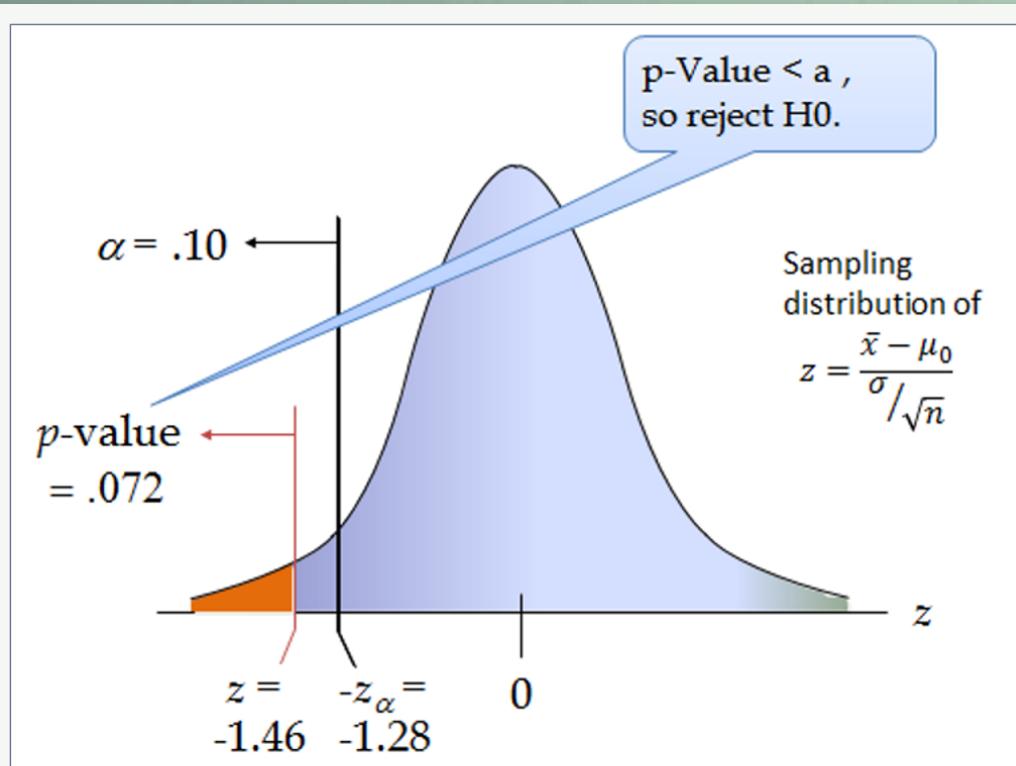
we say: “There is an effect, there is **enough evidence** to reject  $H_0$ ”

simpler: “**We choose  $H_1$** ”

We decide based on this integral. How is it called?

# p-value

... is a tool for deciding whether to reject or not reject the null hypothesis



Example for rejecting  $H_0$ , the 2 groups are different

## Interpreting the P-value

- Small p-value (typically  $\leq \alpha = 0.05$ ) suggests strong evidence against the null hypothesis, "we reject the null", "we support  $H_1$ "
- Large p-value  $\alpha = 0.05$ ) suggests weak evidence against the null hypothesis, "we fail to reject the null", "we support  $H_0$ "
- If we reject the null, there is typically significant difference between the 2 groups

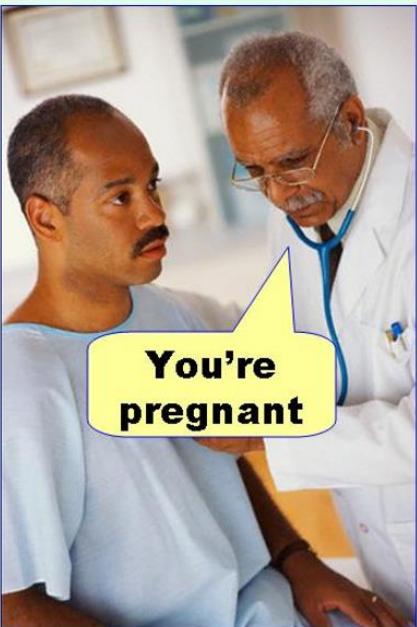
## Properties

- p-values can be affected by the size of the sample.
- Figure: in 0.72 of 100 times you should have selected  $H_0$ .

# Errors: False positives and false negatives

... matter, as they represent incorrect conclusions that can impact decisions based on test results

**Type I error**  
(false positive)



**Type II error**  
(false negative)



- Selecting  $H_1$  is the “positive” statement
- Selecting  $H_0$  is the “negative” statement

**Type I error means “false positive”:**

- When  $H_0$  is true and therefore “selecting  $H_1$  is incorrect”
- precise formulation:  $H_0$  is incorrectly rejected

**Type II error means false negative:**

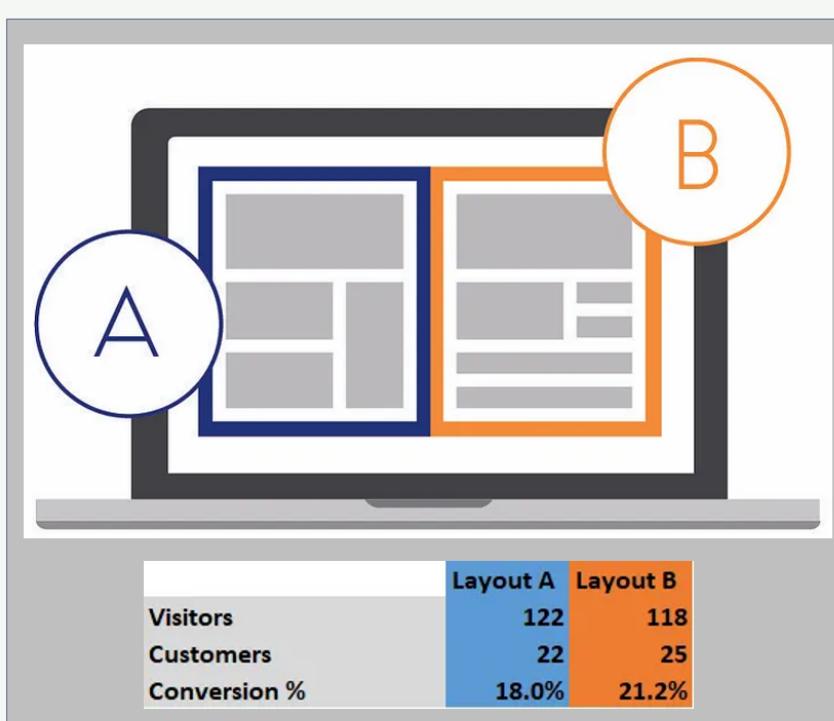
- When the  $H_0$  is false, but “we select it”
- precise formulation: we fail to reject it

**Connection to significance level  $\alpha$ :**

- The maximal probability of making Type I error is defined by  $\alpha$ , such as 0.05 or 0.01.

# What is A/B-Testing?

... also known as split testing, is method of comparing two versions of webpage, email, or other user experience to determine which one performs better.



<https://medium.com/towards-data-science/data-science-you-need-to-know-a-b-testing-f2f12aff619a>

- **Identify goal:**

Your goal might be to increase website conversions, improve email click-through rates, or boost the number of sign-ups.

- **Create Variants:**

Develop two versions of element you want to test - the control (A) and the variant (B). The versions should be identical except for one change.

- **Split audience:**

Randomly divide audience into two equal groups.  
One sees version A, the other sees B.

- **Conduct test:**

Release both versions at the same time and collect data on how each version performs in relation to your goal.

- **Analyze the Results:**

Use statistical analysis to determine which version performed better.

- **Implement Changes:**

If B performs significantly better, consider replacing version A with B.

# A/B Testing: Choosing the Right Test Type

Tests types are nuanced, slight differences in hypothesis determine selection

| Test Type                    | Scipy method                | Use Case, compare:   | Examples  |
|------------------------------|-----------------------------|--|---|
| 1. Two sample t              | ttest_ind                   | <ul style="list-style-type: none"><li>• means of</li><li>• continuous data between</li><li>• two groups of independent samples</li></ul>     | <ul style="list-style-type: none"><li>• Company tests new website B and compares it with old one A</li><li>• Visitors are divided into 2 groups that only see one of them</li><li>• Visit durations are measured</li></ul> <p>→ Does the subgroup stay longer on new website B?</p> |
| 2. Paired t                  | ttest_rel                   | <ul style="list-style-type: none"><li>• means of</li><li>• continuous data for the same group at different times</li></ul>                   | <ul style="list-style-type: none"><li>• Like 1</li><li>• But: The same group is tested</li><li>• Visit durations are measured, first with website A later with B</li></ul> <p>→ Does the same group stay longer on new website B?</p>   |
| 3. $\chi^2$ for independence | chi2_contingency or chi2.sf | <ul style="list-style-type: none"><li>• Variance of</li><li>• categorical data between</li><li>• two or more groups</li></ul>                | <ul style="list-style-type: none"><li>• A company has employees in 4 experience levels</li><li>• On each level a part of them received promotion (yes/no)</li></ul> <p>→ Does experience level have impact on promotion?</p>  |
| 4. One factor ANOVA          | f_oneway                    | <ul style="list-style-type: none"><li>• means of</li><li>• continuous data between</li><li>• ≥ three groups of independent samples</li></ul> | <ul style="list-style-type: none"><li>• Education program in company,</li><li>• 3 learning styles: self-paced online courses, instructor led, mixed</li><li>• Employees are divided in 3 subgroups</li></ul> <p>→ Does the learning style have impact on learning success?</p>      |

# Two sample t-tests

... are statistical tests used to compare the means of continuous data between two groups and determine if they are significantly different from each other.

The formula for the t-statistic in a two-sample t-test is:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

With:

- $\bar{X}_1$  and  $\bar{X}_2$  are the sample means of the two groups
- $s_1^2$  and  $s_2^2$  are the sample variances of the two groups.
- $n_1$  and  $n_2$  are the sample sizes of the two groups
- Degrees of freedom:  $df = n_1 + n_2 - 2$

## Variant A: No significant difference

```
# Randomly generating test scores for Group A and Group B
np.random.seed(0) # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(80, 10, 30)
→ df = 58
```

## Variant B: Significant difference

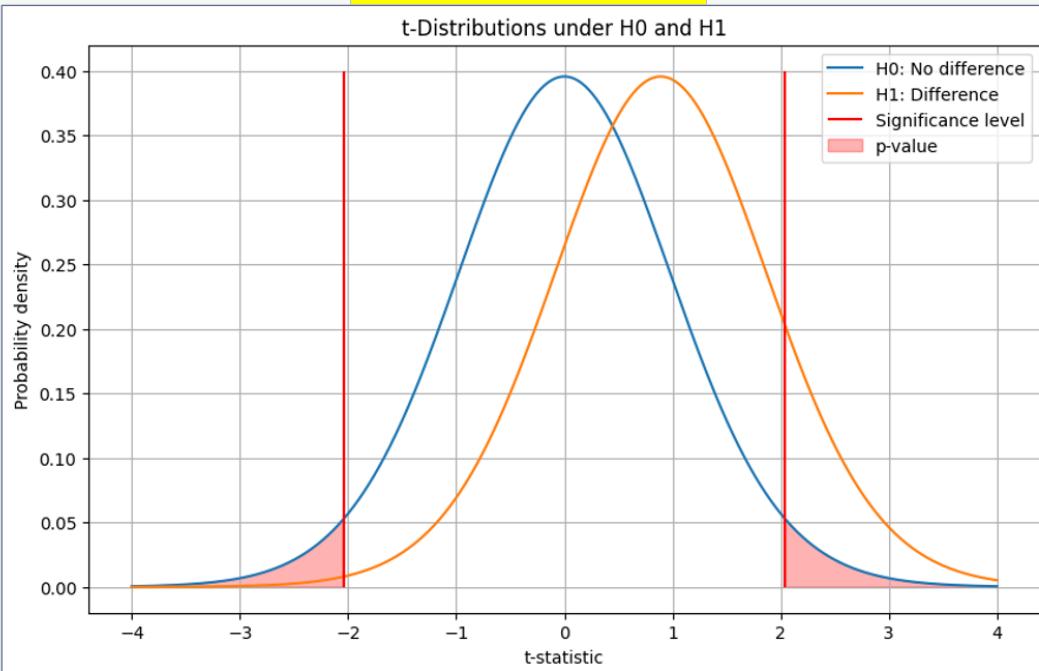
```
# Randomly generating test scores for Group A and Group B
np.random.seed(0) # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(90, 10, 30)
→ df = 58
```

# Two sample t-tests

... are statistical tests used to compare the means of continuous data between two groups and determine if they are significantly different from each other.

**ttest\_stat = 0.8897**

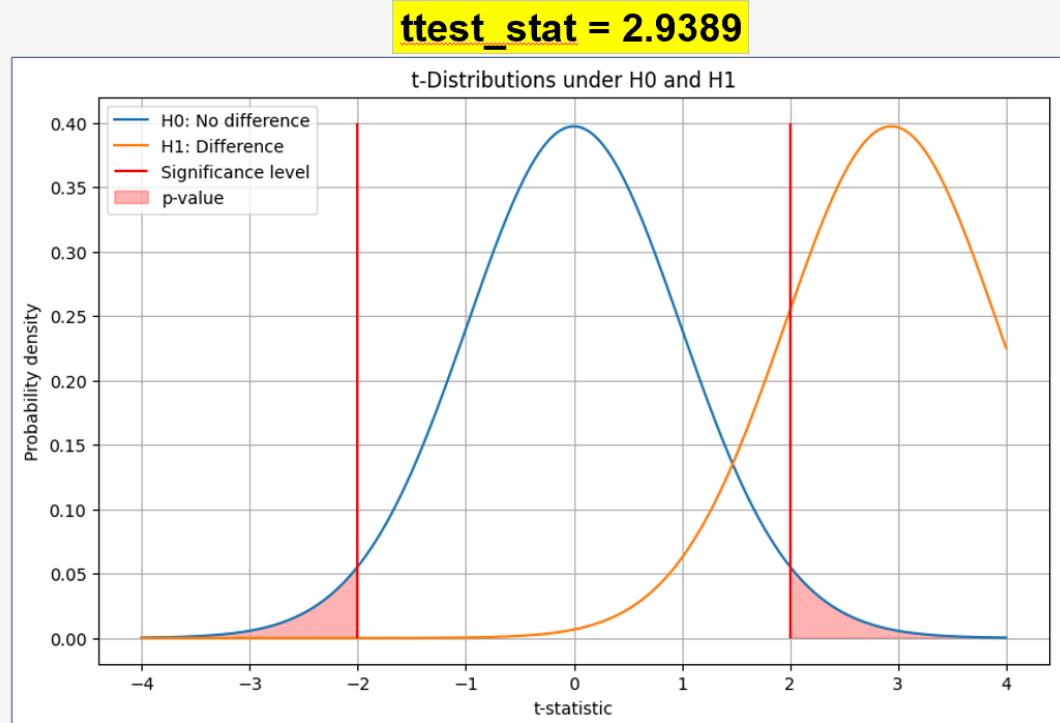
**Critical value: 2.001 (2-sided!)**



**H0:**

There is no evidence that subgroup B stays longer on the website ...

**ttest\_stat = 2.9389**



**H1:**

There is evidence that subgroup B stays longer on the website ...

# Paired t-tests

... are used when you want to compare the means of continuous data for the same group at two different times or under two different conditions

The formula for the t-statistic in a paired t-test is:

$$t = \frac{\bar{D}}{s_D / \sqrt{n}}$$

With:

- $\bar{D}$  is the mean of the differences between the paired observations.
- $s_D$  is the standard deviation of the differences between the paired observations.
- $n$  is the number of pairs.
- **Degrees of freedom:  $df = n - 1$**

## Variant A: No significant difference

```
# Randomly generating test scores for Group A and Group B
np.random.seed(0) # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(80, 10, 30)
→ df = 29
```

## Variant B: Significant difference

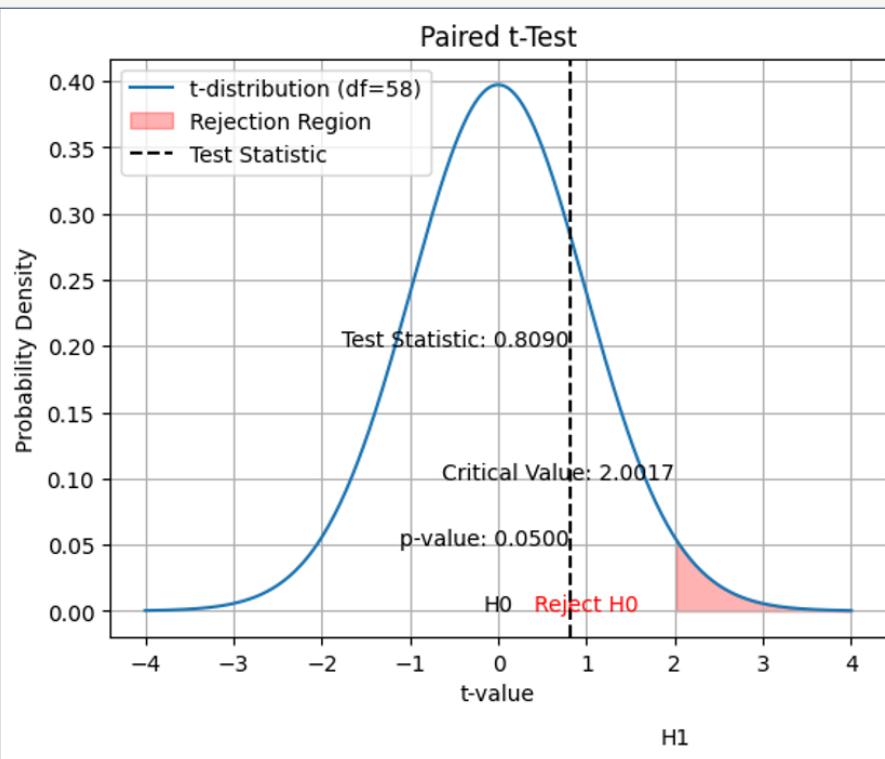
```
# Randomly generating test scores for Group A and Group B
np.random.seed(0) # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(100, 10, 30)
→ df = 29
```

# Paired t-tests

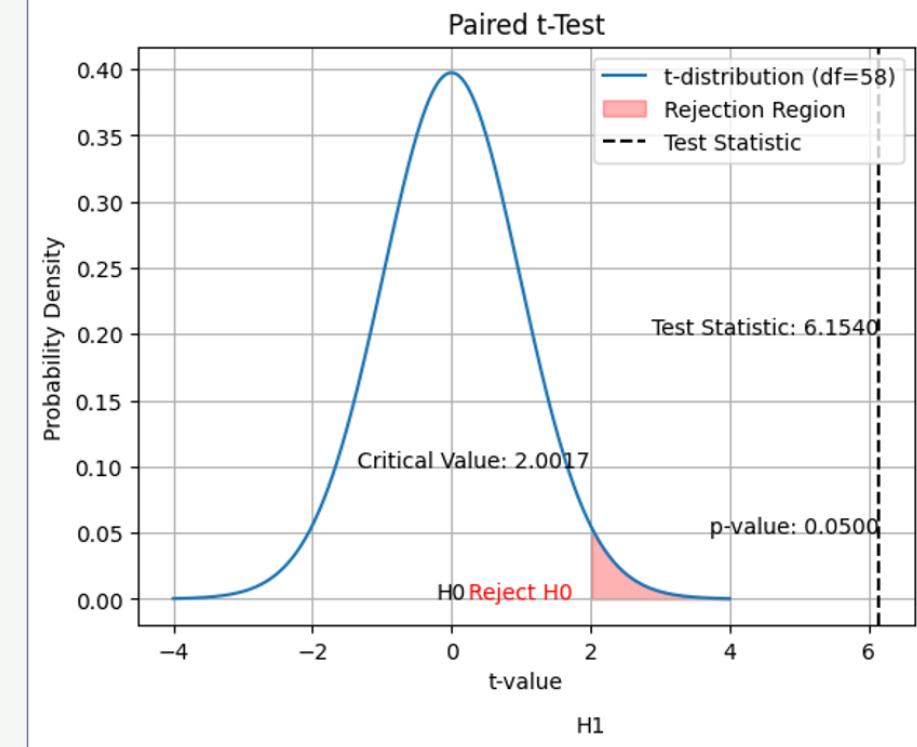
... are used when you want to compare the means of continuous data for the same group at two different times or under two different conditions

paired\_t\_stat = 0.8090

Critical value: 2.0452 (2-sided!)



paired\_t\_stat = 6.1540



H0:

There is no evidence that group stays longer on the new website ...

H1:

There is evidence that group stays longer on the new website ...

# $\chi^2$ independence test

... statistical tests used to determine if there's a significant association between two categorical variables in a sample.

| Observed     | 2007        | 2008        | 2009        | 2010        | 2011        | Total       |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Freshman     | 560         | 495         | 553         | 547         | 512         | 2667        |
| Sophomore    | 369         | 385         | 358         | 361         | 393         | 1866        |
| Junior       | 209         | 226         | 248         | 268         | 285         | 1236        |
| Senior       | 267         | 277         | 304         | 328         | 340         | 1516        |
| Unclassified | 64          | 70          | 93          | 77          | 126         | 430         |
| <b>Total</b> | <b>1469</b> | <b>1453</b> | <b>1556</b> | <b>1581</b> | <b>1656</b> | <b>7715</b> |

| Observed     | 2007        | 2008        | 2009        | 2010        | 2011        | Total       |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Freshman     | 507,818924  | 502,287881  | 537,893973  | 546,536228  | 572,462994  | 2667        |
| Sophomore    | 355,301879  | 351,432016  | 376,344264  | 382,390927  | 400,530914  | 1866        |
| Junior       | 235,344653  | 232,781335  | 249,282696  | 253,287881  | 265,303435  | 1236        |
| Senior       | 288,658976  | 285,514971  | 305,754504  | 310,667012  | 325,404537  | 1516        |
| Unclassified | 81,8755671  | 80,9837978  | 86,7245625  | 88,117952   | 92,2981205  | 430         |
| <b>Total</b> | <b>1469</b> | <b>1453</b> | <b>1556</b> | <b>1581</b> | <b>1656</b> | <b>7715</b> |

|                             |                   |
|-----------------------------|-------------------|
| alpha                       | 0,05              |
| df                          | 16                |
| <b>Chi^2 Critical value</b> | <b>26,2962276</b> |

| Chi^2-Computation |          |        |           |             |
|-------------------|----------|--------|-----------|-------------|
| Observed          | Expected | O - E  | (O - E)^2 | (O - E)^2/E |
| 560               | 507,82   | 52,18  | 2.722,86  | 5,36188107  |
| 369               | 355,30   | 13,70  | 187,64    | 0,52811009  |
| 209               | 235,34   | -26,34 | 694,04    | 2,94903983  |
| 267               | 288,66   | -21,66 | 469,11    | 1,62513998  |
| 64                | 81,88    | -17,88 | 319,54    | 3,9027015   |
| 495               | 502,29   | -7,29  | 53,11     | 0,10574256  |
| 385               | 351,43   | 33,57  | 1.126,81  | 3,2063373   |
| 226               | 232,78   | -6,78  | 45,99     | 0,19755237  |
| 277               | 285,51   | -8,51  | 72,50     | 0,2539437   |
| 70                | 80,98    | -10,98 | 120,64    | 1,48972779  |
| 553               | 537,89   | 15,11  | 228,19    | 0,42423241  |
| 358               | 376,34   | -18,34 | 336,51    | 0,89416013  |
| 248               | 249,28   | -1,28  | 1,65      | 0,00660017  |
| 304               | 305,75   | -1,75  | 3,08      | 0,01006783  |
| 93                | 86,72    | 6,28   | 39,38     | 0,45409414  |
| 547               | 546,54   | 0,46   | 0,22      | 0,00039354  |
| 361               | 382,39   | -21,39 | 457,57    | 1,19660723  |
| 268               | 253,29   | 14,71  | 216,45    | 0,85454721  |
| 328               | 310,67   | 17,33  | 300,43    | 0,96705621  |
| 77                | 88,12    | -11,12 | 123,61    | 1,40276589  |
| 512               | 572,46   | -60,46 | 3.655,77  | 6,38604364  |
| 393               | 400,53   | -7,53  | 56,71     | 0,14159872  |
| 285               | 265,30   | 19,70  | 387,95    | 1,46230552  |
| 340               | 325,40   | 14,60  | 213,03    | 0,65465452  |
| 126               | 92,30    | 33,70  | 1.135,82  | 12,3059567  |

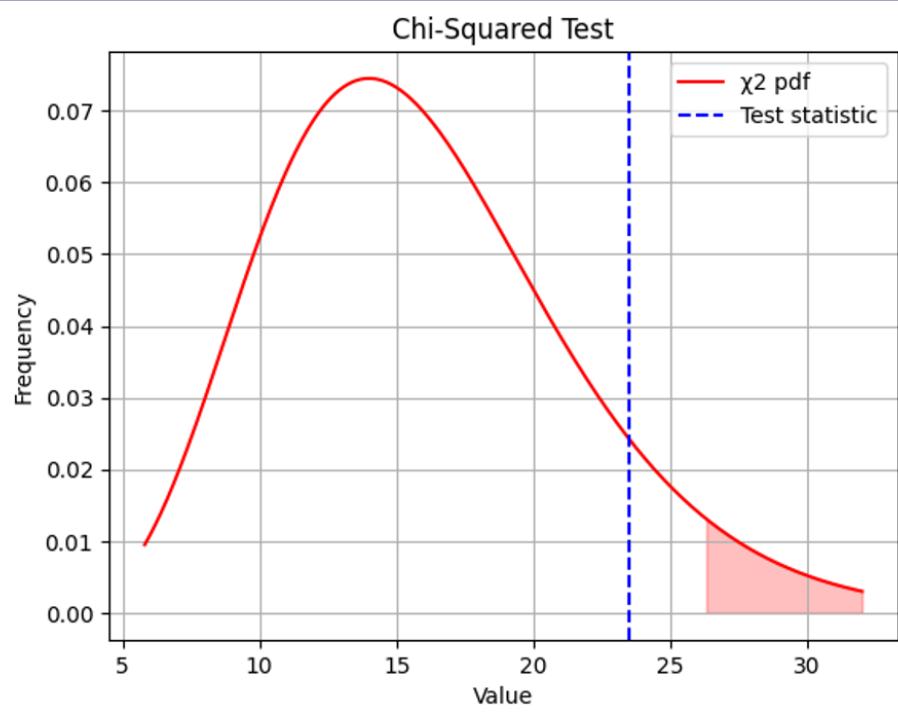
Chi^2-val **46,7812601**

# $\chi^2$ independence test

... statistical tests used to determine if there's a significant association between two categorical variables in a sample.

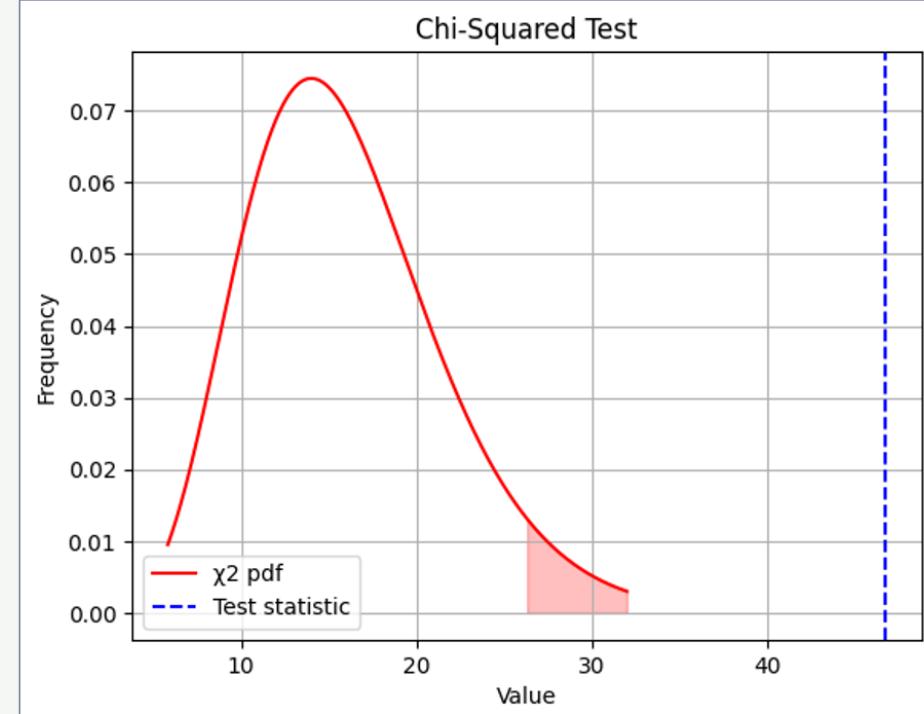
**Chi<sup>2</sup>\_stat = 23.5172**

**Critical value: 26.2962**



**H0:**  
Experience level does not have significant impact on promotion...

**Chi<sup>2</sup>\_stat = 46.7812**



**H1:**  
Experience level has significant impact on promotion...

# Oneway – analysis of variance test

... statistical tests used to compare the means of three or more groups and determine if they are significantly different from each other.

| Group 1 | Group 2 | Group 3 |              |
|---------|---------|---------|--------------|
| 82      | 71      | 64      |              |
| 93      | 62      | 73      |              |
| 61      | 85      | 87      |              |
| 74      | 94      | 91      |              |
| 69      | 78      | 56      |              |
| 70      | 66      | 78      |              |
| 53      | 71      | 87      |              |
| Mean    | Mean    | Mean    | Overall mean |
| 71,71   | 75,29   | 76,57   | 74,52        |

- The larger the differences between the group samples the larger  $F_{\text{stat}}$
- Exceeding  $F_{\text{crit}}$ , we “reject the null” ( $H_1$ )

|                 |    |
|-----------------|----|
| N_Total         | 21 |
| Num_Groups      | 3  |
| N_per_Group     | 7  |
| N_per_Group - 1 | 6  |
| alpha           | 5% |

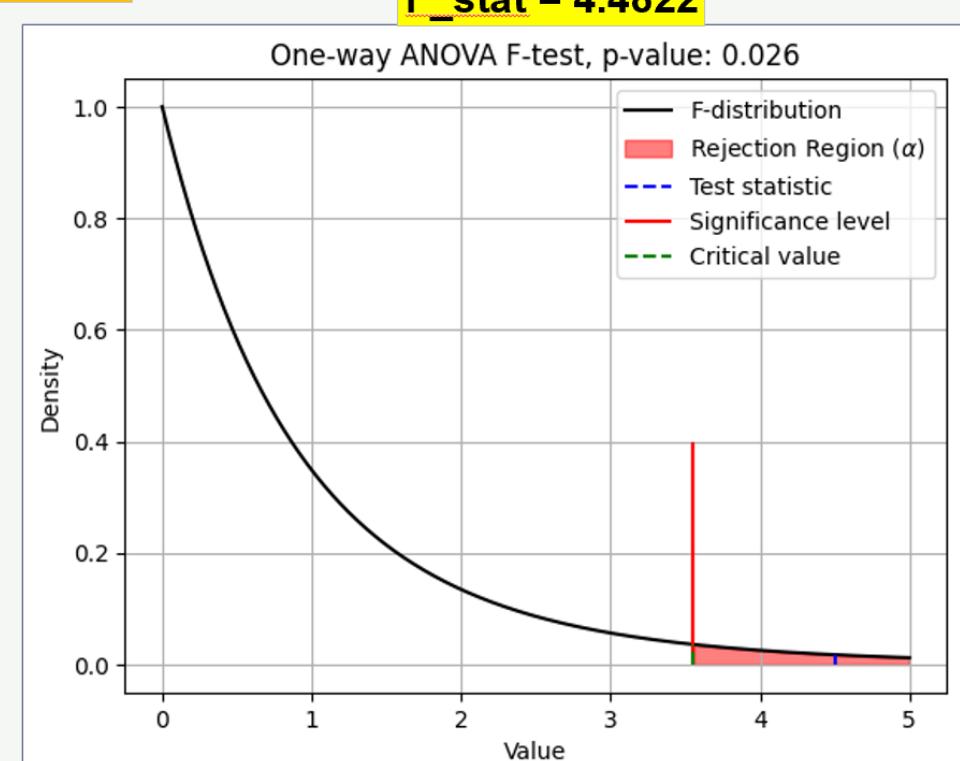
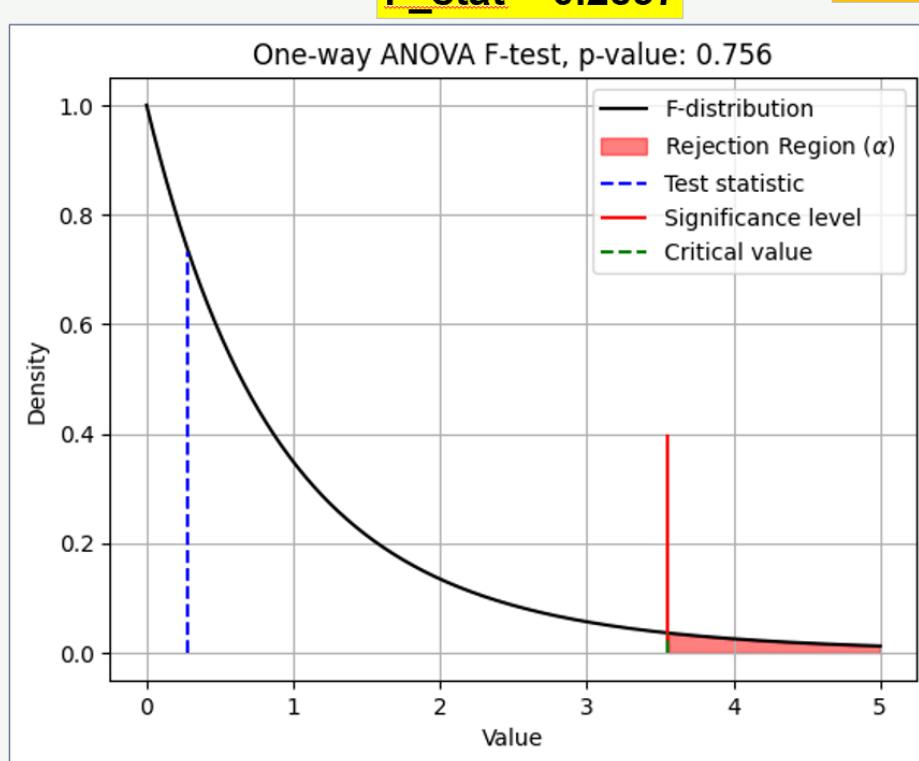
|               | df | SS       | MS          | F_stat      |
|---------------|----|----------|-------------|-------------|
| Between (SSC) | 2  | 88,66667 | 44,33333333 | 0,283726128 |
| Within (SSE)  | 18 | 2812,571 | 156,2539683 |             |
| Total (SST)   | 20 | 2901,238 |             |             |

|                   |             |
|-------------------|-------------|
| F_crit(5%, 2, 18) | 3,554557146 |
| F_stat            | 0,283726128 |

- $F_{\text{stat}} < F_{\text{crit}}$   
→  $H_0$  is not rejected (“ $H_0$ ”)

# Oneway – analysis of variance test

... statistical tests used to compare the means of three or more groups and determine if they are significantly different from each other.



**H0:**

The learning style has no significant impact on learning success ...

**H1:**

The learning style has significant impact on learning success ...

# Exercises

## Exercise for ttest\_ind:

Based on the ttest\_ind example, try new data points and find data point configurations, where  $H_0$  is very close of being rejected or is just rejected.

Generate figure according to the given figures.

## Exercise for ttest\_rel:

Based on the ttest\_ind example, try new data points and find data point configurations, where  $H_0$  is very close of being rejected or is just rejected.

Generate figure according to the given figures.

## Exercise for chi^2:

→ Next slide

## Exercise for F:

Based on the F example, try new data points and find data point configurations, where  $H_0$  is very close of being rejected or is just rejected.

Generate figure according to the given figures.

## Additional exercise for chi^2:

Based on the chi^2 example, try new data points and find data point configurations, where  $H_0$  is very close of being rejected or is just rejected.

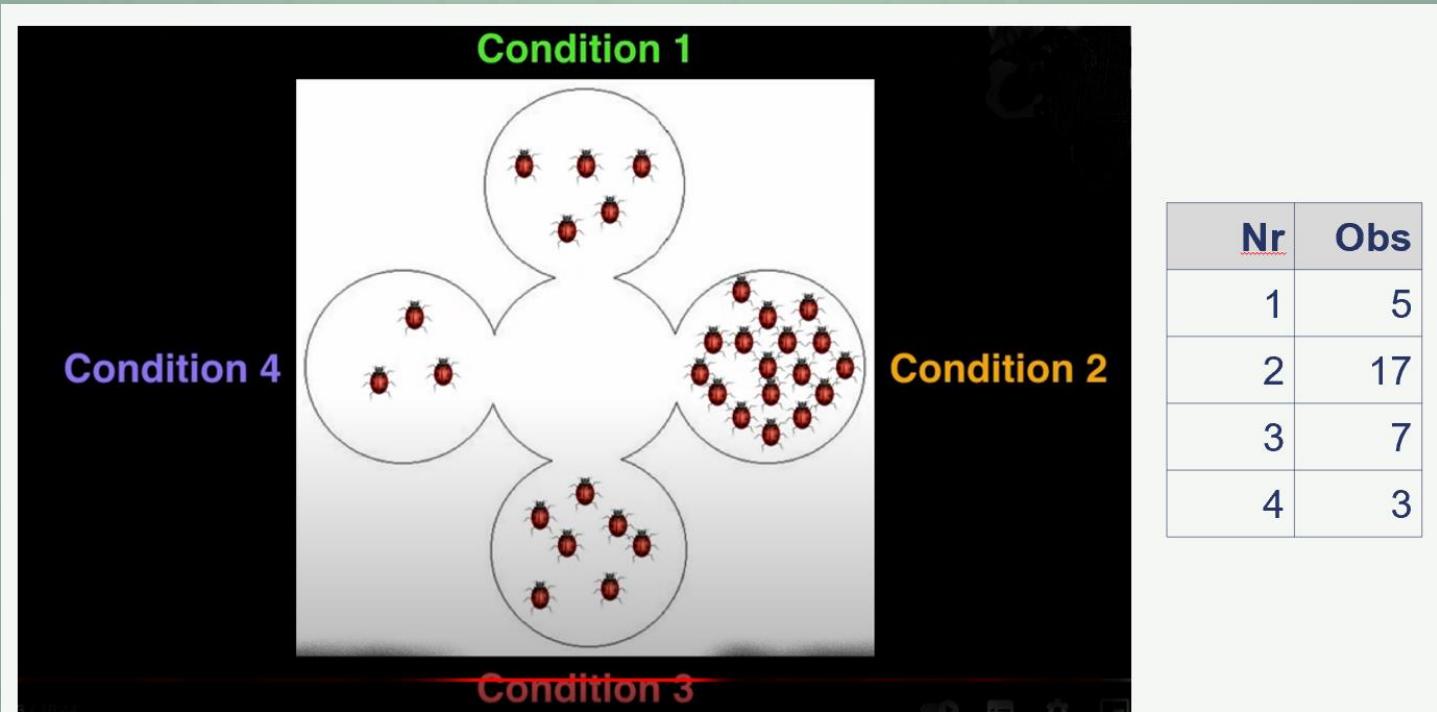
Generate figure according to the given figures.



# Exercise – Chi squared

$H_0$ : Bugs have no preference of moving to one or the other bins

$H_1$  : Or do we have enough evidence to reject  $H_0$ ?

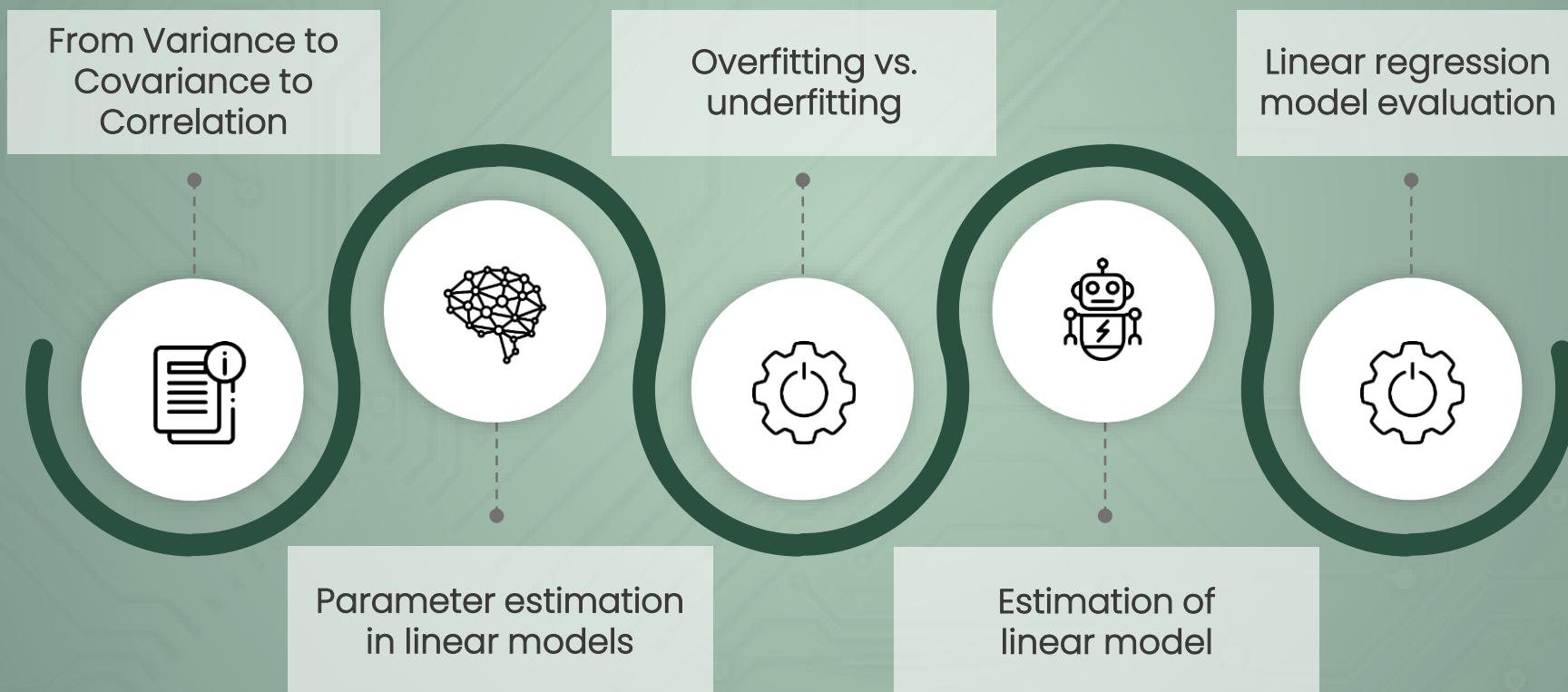


Source:

<https://www.youtube.com/watch?v=qYOMO83Z1WU&t=552s>

# Model estimation: Linear Regression

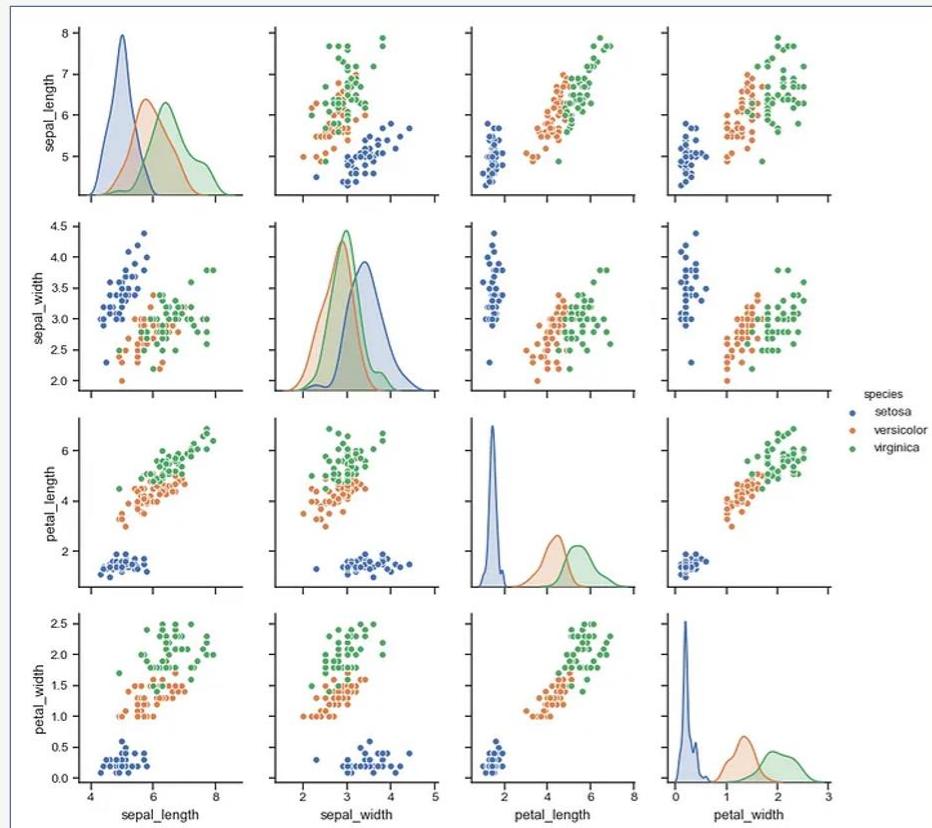
Model estimation is the process of determining the parameters of statistical model that best fit observed data. Linear regression models relationships using straight line.



# From Variance to Covariance to Correlation

Covariance is a generalization of variance to multiple dimensions or variables.

Correlation is a normalized form of covariance to have values between -1 and +1.



## Variance ( $\sigma^2$ ):

- Measures how far a set of numbers spread out from mean.
- In the figure: The diagonal

## Covariance:

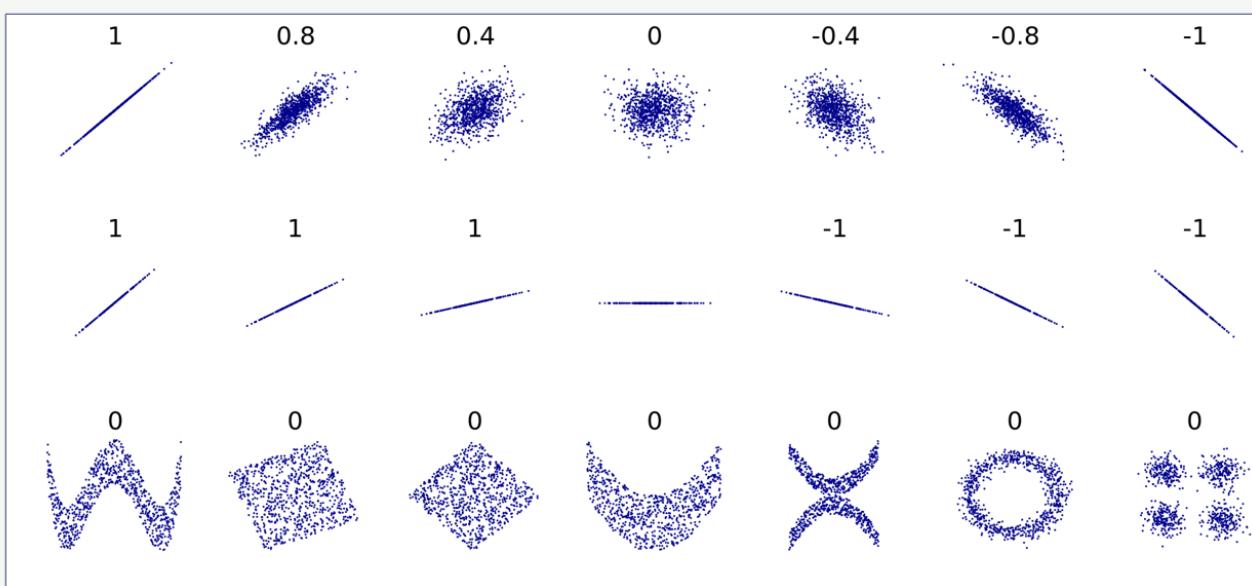
- Measures how much two random variables vary together.
- In the figure: The triangles above and below the diagonal

## Correlation:

- Standardized measure of the relationship between two variables
- ranging from -1 (perfect negative correlation) to +1 (perfect positive correlation), e.g. it is the normalized covariance.

# Correlation

... is the standardized measure of the relationship between two variables



## Positive Correlation:

- When two variables increase or decrease together.
- For example, the more time spent studying (variable 1), the higher the grades (variable 2). Correlation value ranges from 0 to +1.

## Negative Correlation:

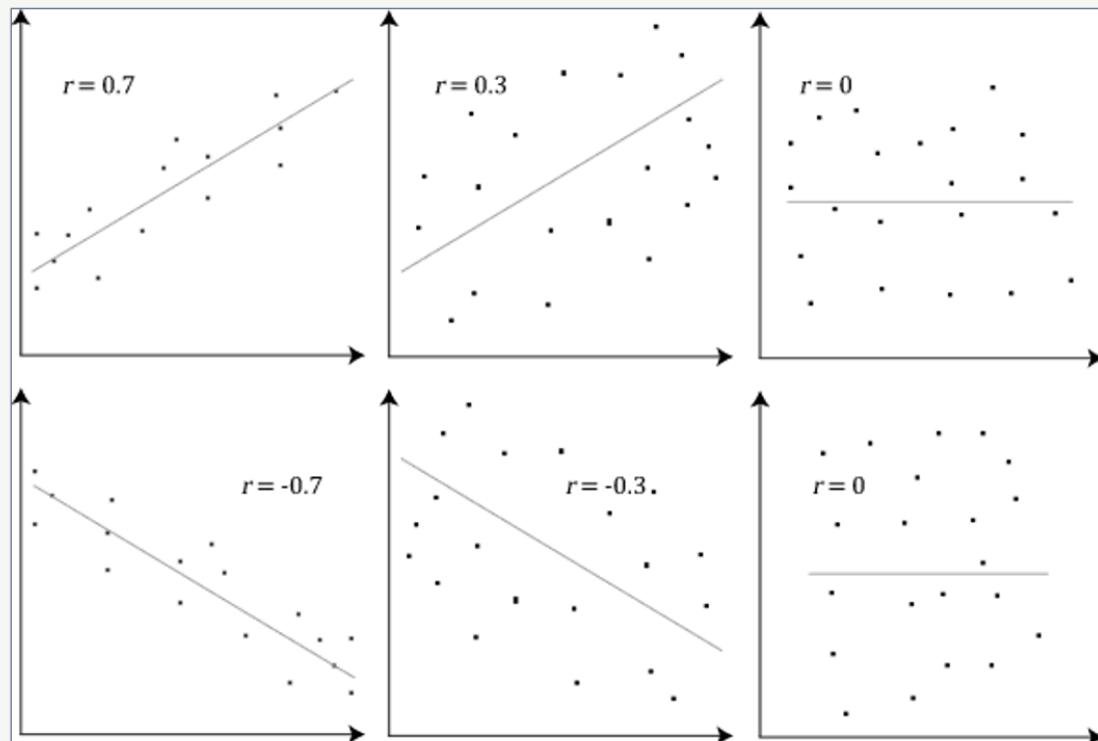
- When one variable increases as the other decreases.
- For example, the more time spent watching TV (variable 1), the lower the grades (variable 2). Correlation value ranges from 0 to -1.

Source:

<https://en.wikipedia.org/wiki/Correlation>

# Pearson's Correlation Coefficient - Test

Tests whether two samples have a linear relationship.



Source:

<https://machinelearningmastery.com/statistical-hypothesis-tests-in-python-cheat-sheet/>

## Assumptions

- Observations in each sample are independent and identically distributed (iid).
- Observations in each sample are normally distributed.
- Observations in each sample have the same variance.

## Interpretation

- H<sub>0</sub>: the two samples are independent.
- H<sub>1</sub>: there is a dependency between the samples.

```
# Example of the Pearson's Correlation test
from scipy.stats import pearsonr
data1 = [0.873, 2.817, 0.121, -0.945, -0.055, -1.436, 0.360, -1.478, -1.637, -1.869]
data2 = [0.353, 3.517, 0.125, -7.545, -0.555, -1.536, 3.350, -1.578, -3.537, -1.579]

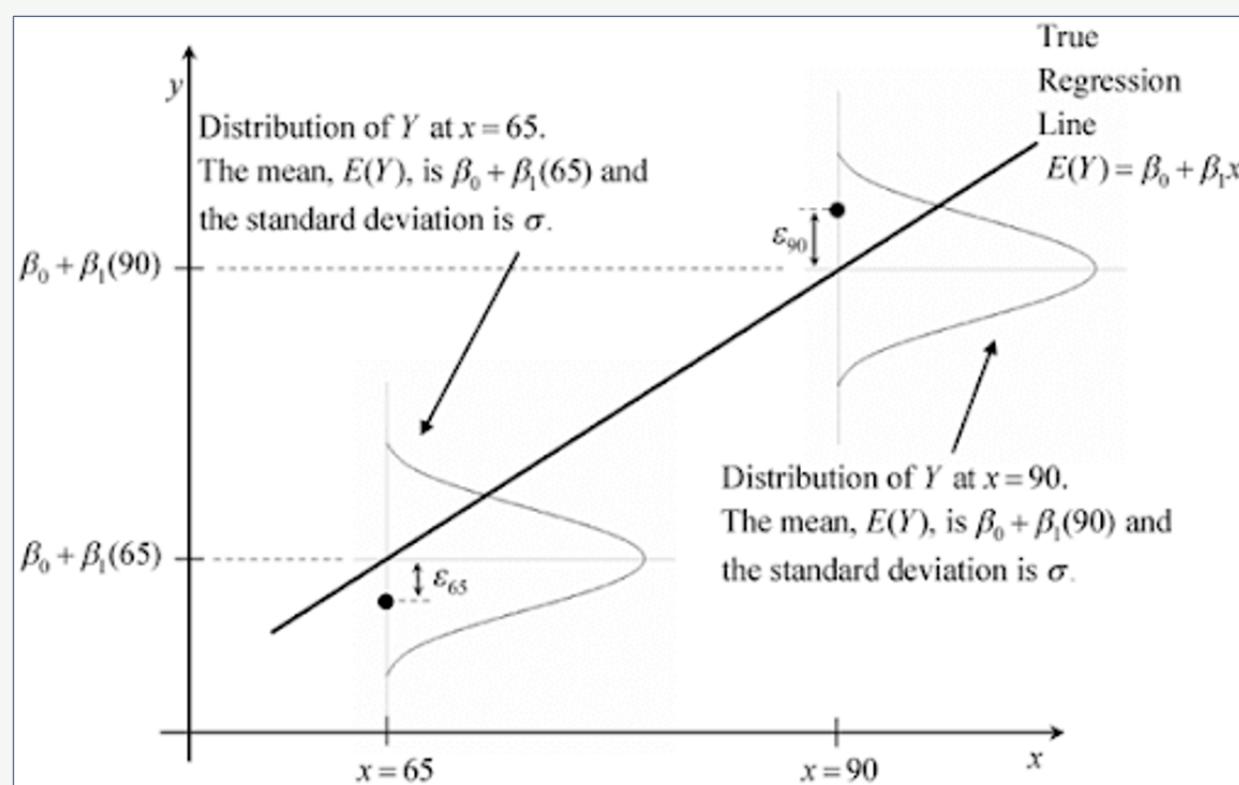
stat, p = pearsonr(data1, data2)
print(f'stat={stat:.3f}, p={p:.3f}')

if p > 0.05:
    print('Probably independent')
else:
    print('Probably dependent')

stat=0.688, p=0.028
Probably dependent
```

# Parameter estimation in linear models

... involves finding the coefficients that best fit the observed data, i.e. represent the relationship between two correlated variables (X and Y)



## Simple linear regression model:

$$Y = \beta_0 + \beta_1 * X + \varepsilon$$

$\beta_0$  (intercept) and  $\beta_1$  (slope) are the parameters to be estimated.

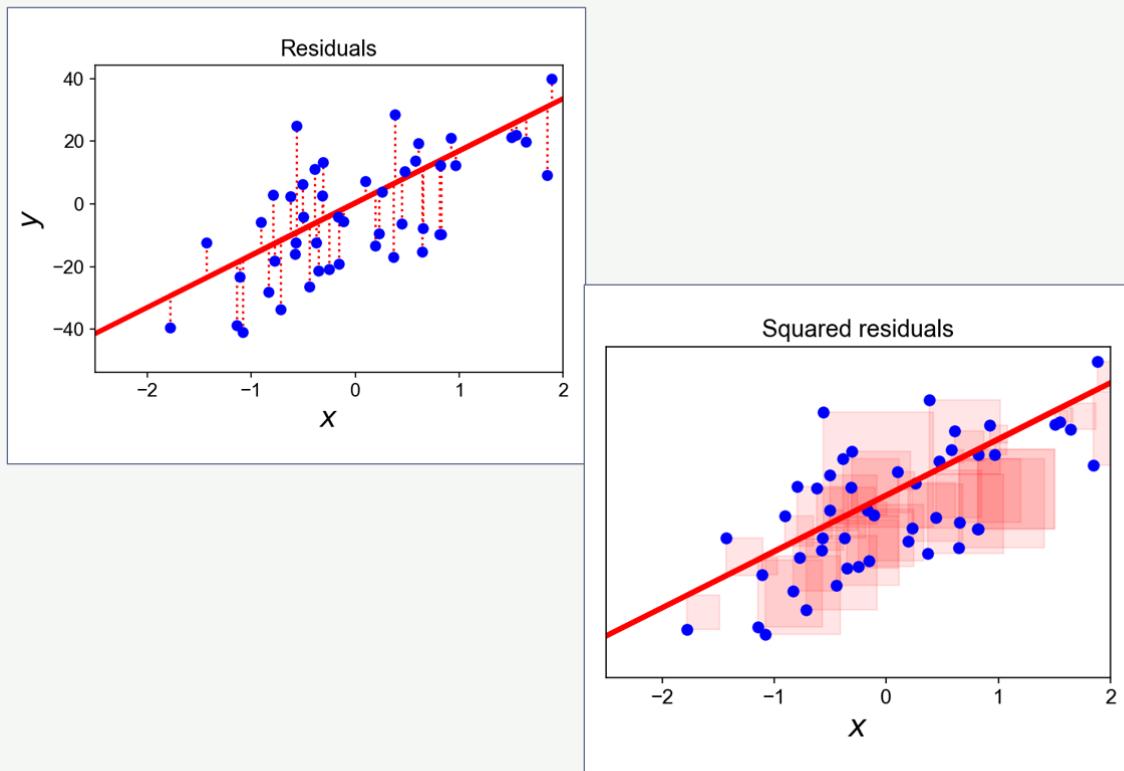
- $\beta_0$  : Value of Y when X = 0.
- $\beta_1$  : Change in Y for each one-unit change in X.

## Use Cases:

- Generalization of sample data to overall population data
- Forecasting like prediction of house prices based on independent features

# Ordinary Least Squares (OLS) estimation

... represents a relationship between two correlated variables ( $X$  and  $Y$ ) and is used in linear regression analysis



## Ordinary Least Squares (OLS) estimation:

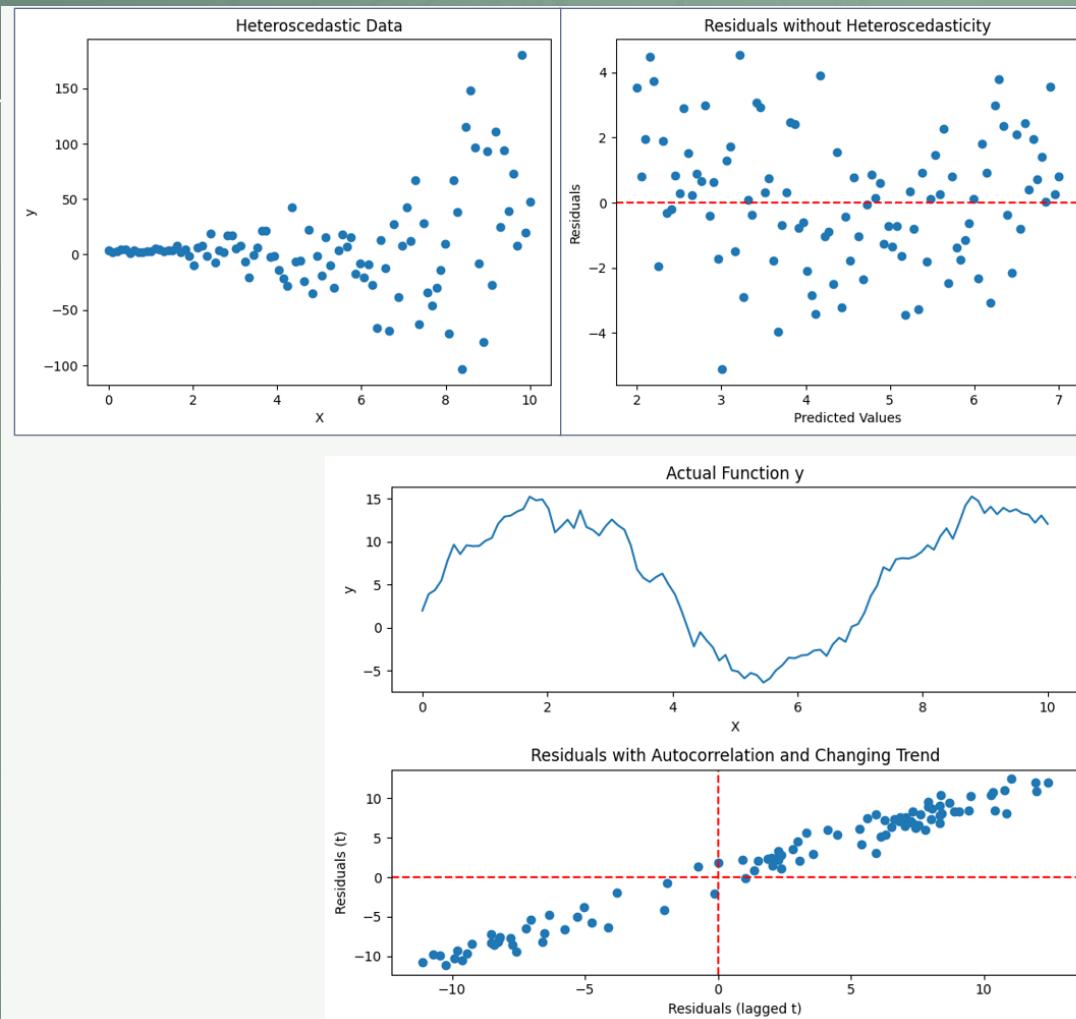
- Method used in linear regression to estimate the model's parameters
- by minimizing the sum of the squares of the differences between the observed and predicted values of the dependent variable.
- Separation systematic relations from white noise

## IID & BLUE:

If the residuals are identically and independently distributed (iid) the OLS estimator is used, because it provides best linear unbiased (BLUE) estimation.

# OLS - What if IID-ness of residuals is violated?

Here it gets interesting 😊



## Heteroscedasticity

- Variance is not identical
- Breusch-Pagan-Test:  $H_1$  means that there is enough evidence from residuals to assume heteroscedasticity

## Autocorrelation (typical example: time series)

- Residuals depend on preceders
- Durbin-Watson-Test:

$$d = \frac{\sum_{t=2}^T (e_t - e_{t-1})^2}{\sum_{t=1}^T e_t^2}$$

- $d \approx 2$ : No autocorrelation
- $0 < d < 2$ : Positive autocorrelation
- $2 < d < 4$ : Negative autocorrelation

Transformations can be used in order to make OLS estimation usable...

# Overfitting

... is a modeling error where a model is trained too well on the training data

---

Problems associated with Overfitting:

Poor Generalizability:

Overfit models perform well on training data but poorly on new, unseen data, which limits their predictive accuracy.

Complexity:

Overfitting often results in overly complex models that have too many parameters.

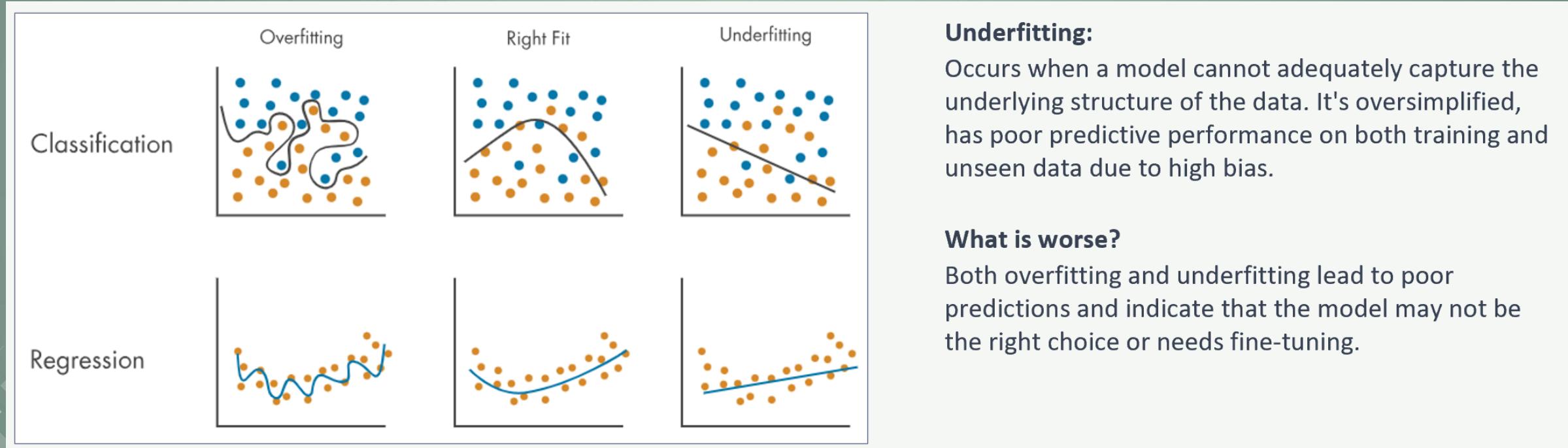
Unreliable Predictions:

Overfit models are sensitive to minor fluctuations in data, leading to unreliable, inconsistent predictions over time.



# Overfitting vs. underfitting

In opposite to overfitting, underfitting is when the model fails to capture important trends in the training data, resulting in poor performance on both the training and unseen data.



## Underfitting:

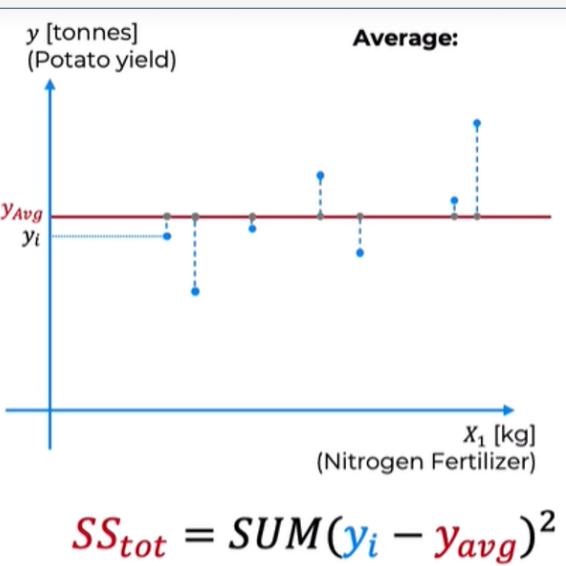
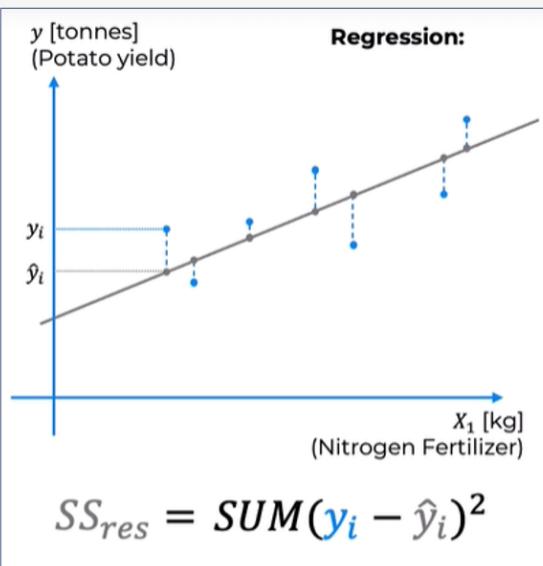
Occurs when a model cannot adequately capture the underlying structure of the data. It's oversimplified, has poor predictive performance on both training and unseen data due to high bias.

## What is worse?

Both overfitting and underfitting lead to poor predictions and indicate that the model may not be the right choice or needs fine-tuning.

# R-squared: Goodness of fit

... is a statistical measure in regression models that represents the proportion of the variance in the dependent variable that is predictable from the independent variables



## Properties:

- Ranges from 0 to 1.
- Value of 1 → independent variable(s) perfectly predict dependent variable.
- Value of 0 → no predictive power.
- Higher R-Squared values → better fit, however very high R-Squared indicates overfitting.
- Only interpretable in the context of the data and domain. Low R-Squared could be acceptable in some fields where prediction is difficult.

## Adjusted R-squared:

- Problem to be solved: Many input factors may improve fit, but lead to multicollinearity, overfitting etc.
- In order to incentivize leaner models, number of input factors is punished

# Statsmodels: Summary for OLS

... provides a comprehensive summary of the results from fitted OLS model

| OLS Regression Results  |           |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
|---|-----------|---|---------|-------|--------|--------|--|--|--|------------|-------|----------------|-------|------|--------|--------|-------|-----------------|-------|-------------------|-------|-------|-------|----|--------|--------------|--------|-----------|-------|-------|----|--------|-------|---------------------|-----------|-----------|-------|----|--------|-------|---------|-----------------|---------|-------|----|--------|-------|--------|-------|-------|-------|----|--------|-------|---------|-------|-------|-------|-------|--|--|--|--|--|--|
| Model Info  |           | <pre>Dep. Variable: y<br/>Model: OLS<br/>Method: Least Squares<br/>Date: Wed, 05 Jul 2023<br/>Time: 22:35:07<br/>No. Observations: 100<br/>Df Residuals: 94<br/>Df Model: 5<br/>Covariance Type: nonrobust</pre>  |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Model Info  |           | <table><tr><td>R-squared:</td><td>0.998</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>Adj. R-squared:</td><td>0.998</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>F-statistic:</td><td>8156.</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>Prob (F-statistic):</td><td>2.50e-122</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>Log-Likelihood:</td><td>-143.90</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>AIC:</td><td>299.8</td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>BIC:</td><td>315.4</td><td></td><td></td><td></td><td></td><td></td><td></td></tr></table>  |         |       |        |        |  |  |  | R-squared: | 0.998 |                |       |      |        |        |       | Adj. R-squared: | 0.998 |                   |       |       |       |    |        | F-statistic: | 8156.  |           |       |       |    |        |       | Prob (F-statistic): | 2.50e-122 |           |       |    |        |       |         | Log-Likelihood: | -143.90 |       |    |        |       |        |       | AIC:  | 299.8 |    |        |       |         |       |       | BIC:  | 315.4 |  |  |  |  |  |  |
| R-squared:  | 0.998     |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Adj. R-squared:   | 0.998     |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| F-statistic:  | 8156.     |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Prob (F-statistic):   | 2.50e-122 |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Log-Likelihood:   | -143.90   |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| AIC:  | 299.8     |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| BIC:  | 315.4     |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Coefficients Table  |           | <table><thead><tr><th></th><th>coef</th><th>std err</th><th>t</th><th>P&gt; t </th><th>[0.025</th><th>0.975]</th></tr></thead><tbody><tr><td>const</td><td>2.1643</td><td>0.459</td><td>4.718</td><td>0.000</td><td>1.254</td><td>3.075</td></tr><tr><td>x1</td><td>3.0226</td><td>0.037</td><td>82.532</td><td>0.000</td><td>2.950</td><td>3.095</td></tr><tr><td>x2</td><td>1.9670</td><td>0.039</td><td>50.758</td><td>0.000</td><td>1.890</td><td>2.044</td></tr><tr><td>x3</td><td>4.0132</td><td>0.036</td><td>113.018</td><td>0.000</td><td>3.943</td><td>4.084</td></tr><tr><td>x4</td><td>0.9767</td><td>0.037</td><td>26.148</td><td>0.000</td><td>0.903</td><td>1.051</td></tr><tr><td>x5</td><td>4.9814</td><td>0.035</td><td>141.395</td><td>0.000</td><td>4.911</td><td>5.051</td></tr></tbody></table> |         |       |        |        |  |  |  |            | coef  | std err        | t     | P> t | [0.025 | 0.975] | const | 2.1643          | 0.459 | 4.718             | 0.000 | 1.254 | 3.075 | x1 | 3.0226 | 0.037        | 82.532 | 0.000     | 2.950 | 3.095 | x2 | 1.9670 | 0.039 | 50.758              | 0.000     | 1.890     | 2.044 | x3 | 4.0132 | 0.036 | 113.018 | 0.000           | 3.943   | 4.084 | x4 | 0.9767 | 0.037 | 26.148 | 0.000 | 0.903 | 1.051 | x5 | 4.9814 | 0.035 | 141.395 | 0.000 | 4.911 | 5.051 |       |  |  |  |  |  |  |
|   | coef      | std err   | t       | P> t  | [0.025 | 0.975] |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| const   | 2.1643    | 0.459   | 4.718   | 0.000 | 1.254  | 3.075  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| x1  | 3.0226    | 0.037   | 82.532  | 0.000 | 2.950  | 3.095  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| x2  | 1.9670    | 0.039   | 50.758  | 0.000 | 1.890  | 2.044  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| x3  | 4.0132    | 0.036   | 113.018 | 0.000 | 3.943  | 4.084  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| x4  | 0.9767    | 0.037   | 26.148  | 0.000 | 0.903  | 1.051  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| x5  | 4.9814    | 0.035   | 141.395 | 0.000 | 4.911  | 5.051  |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Skewness and kurtosis   |           | <table><tr><td>Omnibus:</td><td>2.625</td><td>Durbin-Watson:</td><td>1.810</td><td></td><td></td><td></td><td></td></tr><tr><td>Prob(Omnibus):</td><td>0.269</td><td>Jarque-Bera (JB):</td><td>2.212</td><td></td><td></td><td></td><td></td></tr><tr><td>Skew:</td><td>-0.361</td><td>Prob(JB):</td><td>0.331</td><td></td><td></td><td></td><td></td></tr><tr><td>Kurtosis:</td><td>3.101</td><td>Cond. No.</td><td>50.2</td><td></td><td></td><td></td><td></td></tr></table>  |         |       |        |        |  |  |  | Omnibus:   | 2.625 | Durbin-Watson: | 1.810 |      |        |        |       | Prob(Omnibus):  | 0.269 | Jarque-Bera (JB): | 2.212 |       |       |    |        | Skew:        | -0.361 | Prob(JB): | 0.331 |       |    |        |       | Kurtosis:           | 3.101     | Cond. No. | 50.2  |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Omnibus:  | 2.625     | Durbin-Watson:  | 1.810   |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Prob(Omnibus):  | 0.269     | Jarque-Bera (JB):   | 2.212   |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Skew:   | -0.361    | Prob(JB):   | 0.331   |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Kurtosis:   | 3.101     | Cond. No.   | 50.2    |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |
| Notes:<br>[1] Standard Errors assume that the covariance matrix of the errors is correctly specified. |           |   |         |       |        |        |  |  |  |            |       |                |       |      |        |        |       |                 |       |                   |       |       |       |    |        |              |        |           |       |       |    |        |       |                     |           |           |       |    |        |       |         |                 |         |       |    |        |       |        |       |       |       |    |        |       |         |       |       |       |       |  |  |  |  |  |  |

Model Info

The closer to 1 the better  
The smaller the better  
The closer to 0 the better  
The smaller the better

Coefficients Table

Autocorrelation,  
e.g. basis of time  
series

Skewness and kurtosis

Multicollinearity

# Statsmodels: Summary – Coefficients table

... provides detailed information about each parameter in the model

|       | coef    | std err | t       | P> t  | [0.025 | 0.975] |
|-------|---------|---------|---------|-------|--------|--------|
| const | -3.2002 | 0.257   | -12.458 | 0.000 | -3.708 | -2.693 |
| x1    | 0.7529  | 0.044   | 17.296  | 0.000 | 0.667  | 0.839  |

## coef:

- Estimates of the parameters

## 1.std err:

- Standard error of the estimate of the coefficient
- not standard deviation!
- Smaller values indicate a more precise estimate.

## t:

- t-statistic value
- Ratio: coefficient / std. err
- measure of how statistically significant the coefficient is.

## [0.025 0.975]

- 95% confidence intervals for the coefficient.
- If zero is not in this interval, that suggests the variable is a significant predictor at the 0.05 level.

## P>|t|:

- p-value associated with the t-statistic.
- Smaller p-value (<0.05, traditionally) suggests that null hypothesis can be rejected, i.e. independent variable **is a significant predictor** of dep. variable.

# Model estimation: Logistic Regression

... is statistical method for modeling binary or categorical outcomes using logistic function

Binary Logistic  
Regression

Exercise:  
Credit Card Default  
Prediction



Confusion Matrix

# Logistic regression

... is statistical model to predict a binary outcome (0 or 1, true or false) based on one or more predictor variables, using logistic function to model the probability of binary outcome.

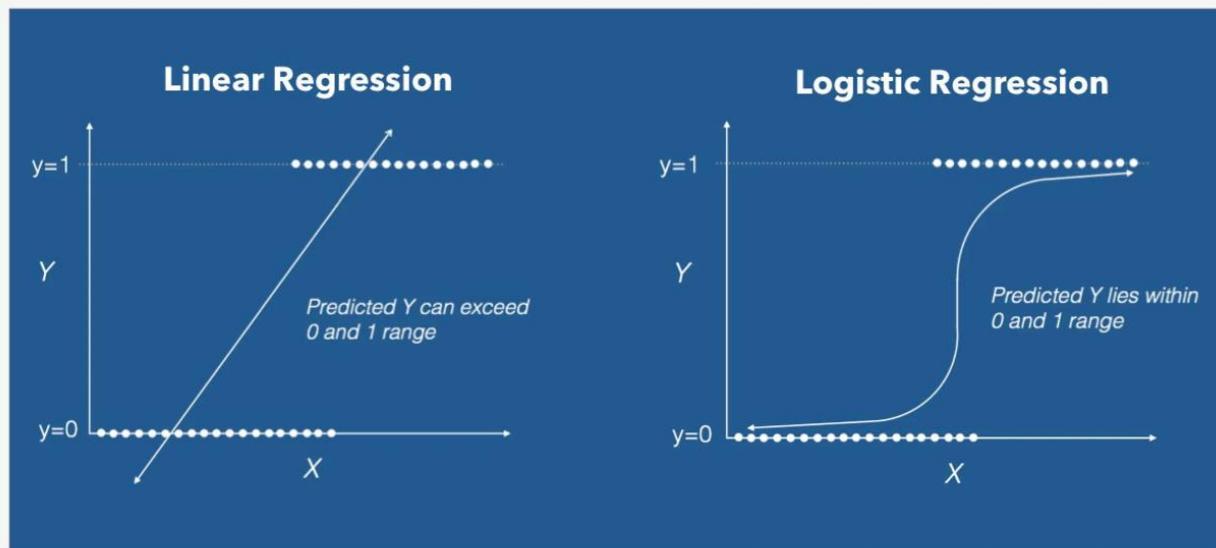


Fig 12: Typical logistic regression curve

**Logistic function (= sigmoid function):**

$$P(z) = \frac{1}{1+e^{-z}} \text{ with } z = \beta_0 + \beta_1 x$$

- It converts any input into a value between 0 and 1, which can be interpreted as a probability.
- For given input 'x', the output represents probability that 'x' belongs to class 1 (usually coded as positive outcome).
- Simple rounding can be used for predicting 0 or 1

It is based on ...

**Logged odds:**

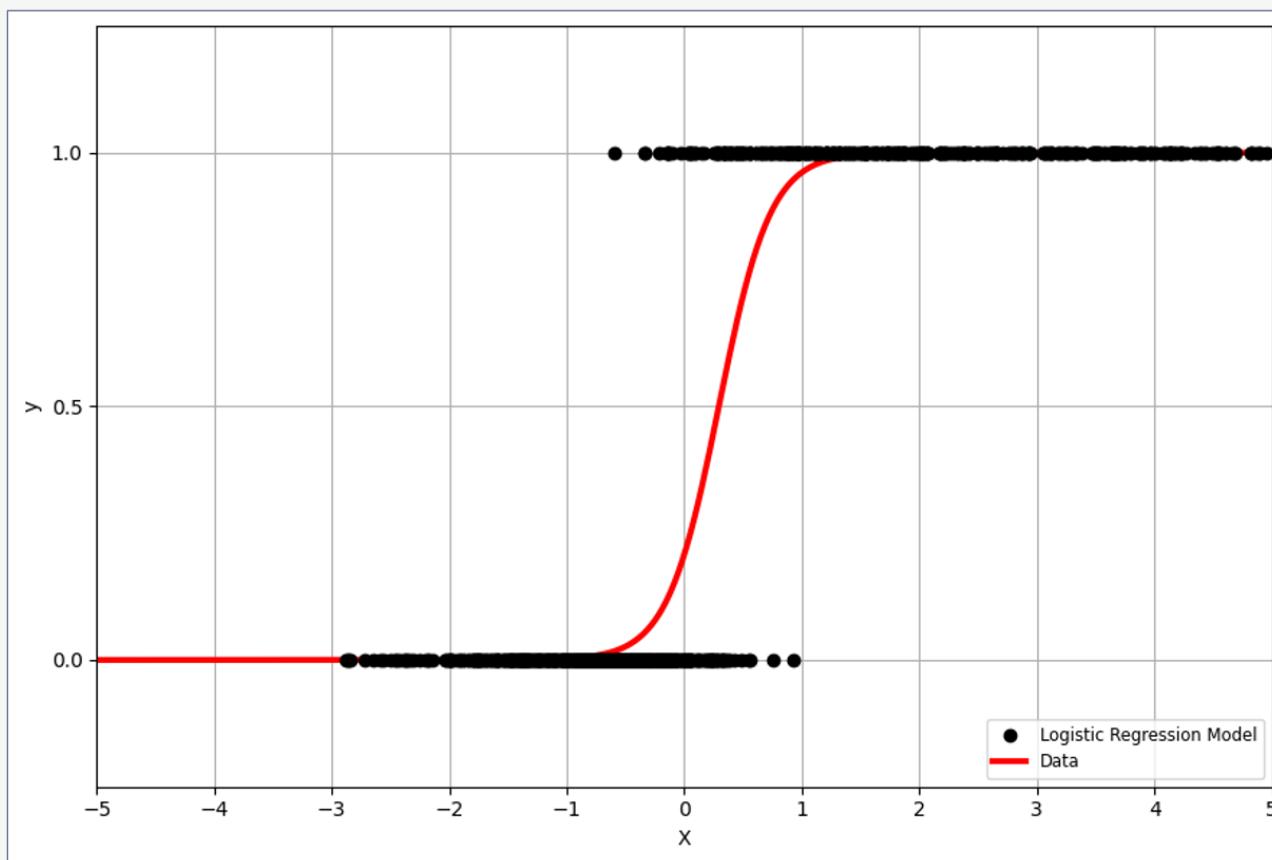
$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x$$

→ Linear model is projected on S-curve

<https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79>

# Logistic regression

... is a statistical model to predict a binary outcome (0 or 1, true or false) based on one or more predictor variables, using the logistic function to model the probability of the binary outcome.



**Logistic function (= sigmoid function):**

$$P(z) = \frac{1}{1+e^{-z}} \text{ with } z = \beta_0 + \beta_1 x$$

- It converts any input into a value between 0 and 1, which can be interpreted as probability.
- For given input 'x', output represents probability that 'x' belongs to 1
- Simple rounding can be used for predicting 0 or 1

It is based on ...

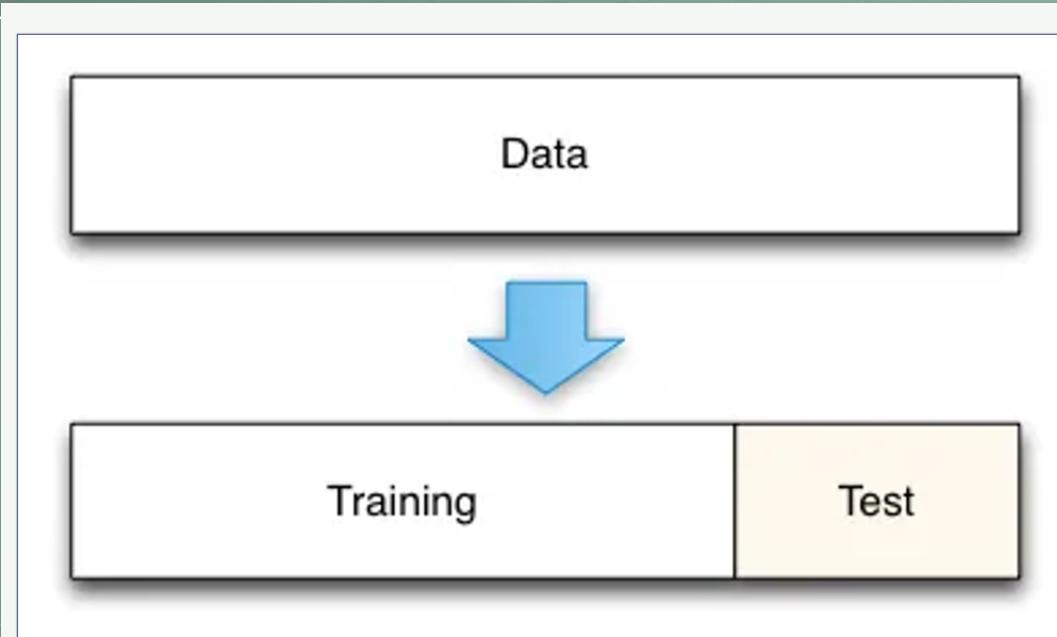
**Logged odds:**

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x$$

→ Linear model is projected on S-curve

# ML perspective: Train test split

... is a method where the dataset is divided into subsets: the training set, used to train the model, and the testing set, used to evaluate the model's performance on unseen data



## Training models vs. analytical solution

- Logistic regression does not have analytical solution in opposite to linear regression.
- Therefore squared error minimizing parameters are approximated, e.g. with Newton's approximation method
- However: Danger of overfitting

## Training Set:

- Portion of data used to create the model, i.e. to learn/approximate the parameters
- Usually 70-80% of the entire dataset

## Testing Set:

- Subset of data that the model has not seen during learning phase
- Used to evaluate performance of the model on unseen data and check for overfitting. Typically 20-30% of entire dataset

<https://medium.com/@rathinavel.mph/how-the-train-and-test-samples-are-split-d7e46a8e2361>

# Statistics perspective: Summary for Logistic regression

... provides a comprehensive summary of the results from fitted logistic regression model

| # Print the summary statistics of the regression model<br>print(result.summary()) |                  |                   |         |       |        |        |
|---|------------------|-------------------|---------|-------|--------|--------|
| Logit Regression Results  |                  |                   |         |       |        |        |
| Dep. Variable:  | default          | No. Observations: | 24000   |       |        |        |
| Model:  | Logit            | Df Residuals:     | 23975   |       |        |        |
| Method:   | MLE              | Df Model:         | 24      |       |        |        |
| Date:   | Wed, 05 Jul 2023 | Pseudo R-squ.:    | 0.1211  |       |        |        |
| Time:   | 21:36:53         | Log-Likelihood:   | -11162. |       |        |        |
| converged:  | True             | LL-Null:          | -12700. |       |        |        |
| Covariance Type:  | nonrobust        | LLR p-value:      | 0.000   |       |        |        |
|   |                  |                   |         |       |        |        |
|   | coef             | std err           | z       | P> z  | [0.025 | 0.975] |
| const   | -1.4592          | 0.018             | -79.195 | 0.000 | -1.495 | -1.423 |
| x1  | -0.0039          | 0.017             | -0.233  | 0.815 | -0.037 | 0.029  |
| x2  | -0.1064          | 0.023             | -4.677  | 0.000 | -0.151 | -0.062 |
| x3  | -0.0561          | 0.017             | -3.344  | 0.001 | -0.089 | -0.023 |
| x4  | -0.0774          | 0.018             | -4.194  | 0.000 | -0.114 | -0.041 |
| x5  | -0.0791          | 0.018             | -4.290  | 0.000 | -0.115 | -0.043 |
| x6  | 0.0768           | 0.018             | 4.203   | 0.000 | 0.041  | 0.113  |
| x7  | 0.6476           | 0.022             | 29.092  | 0.000 | 0.604  | 0.691  |
| x8  | 0.1086           | 0.027             | 4.008   | 0.000 | 0.055  | 0.162  |
| x9  | 0.0776           | 0.030             | 2.548   | 0.011 | 0.018  | 0.137  |
| x10   | 0.0532           | 0.033             | 1.633   | 0.103 | -0.011 | 0.117  |
| x11   | 0.0205           | 0.034             | 0.605   | 0.546 | -0.046 | 0.087  |
| x12   | 0.0163           | 0.028             | 0.579   | 0.562 | -0.039 | 0.072  |
| x13   | -0.3759          | 0.092             | -4.087  | 0.000 | -0.556 | -0.196 |
| x14   | 0.1462           | 0.118             | 1.240   | 0.215 | -0.085 | 0.377  |
| x15   | 0.1065           | 0.101             | 1.050   | 0.294 | -0.092 | 0.305  |

## coef:

- Input factors that have impact and should be used

The closer to 1 the better  
The closer to 0 the better

If LLR p-value is small (typically less than a pre-specified significance level, such as 0.05), full model provides significantly better fit to the data compared to reduced model.

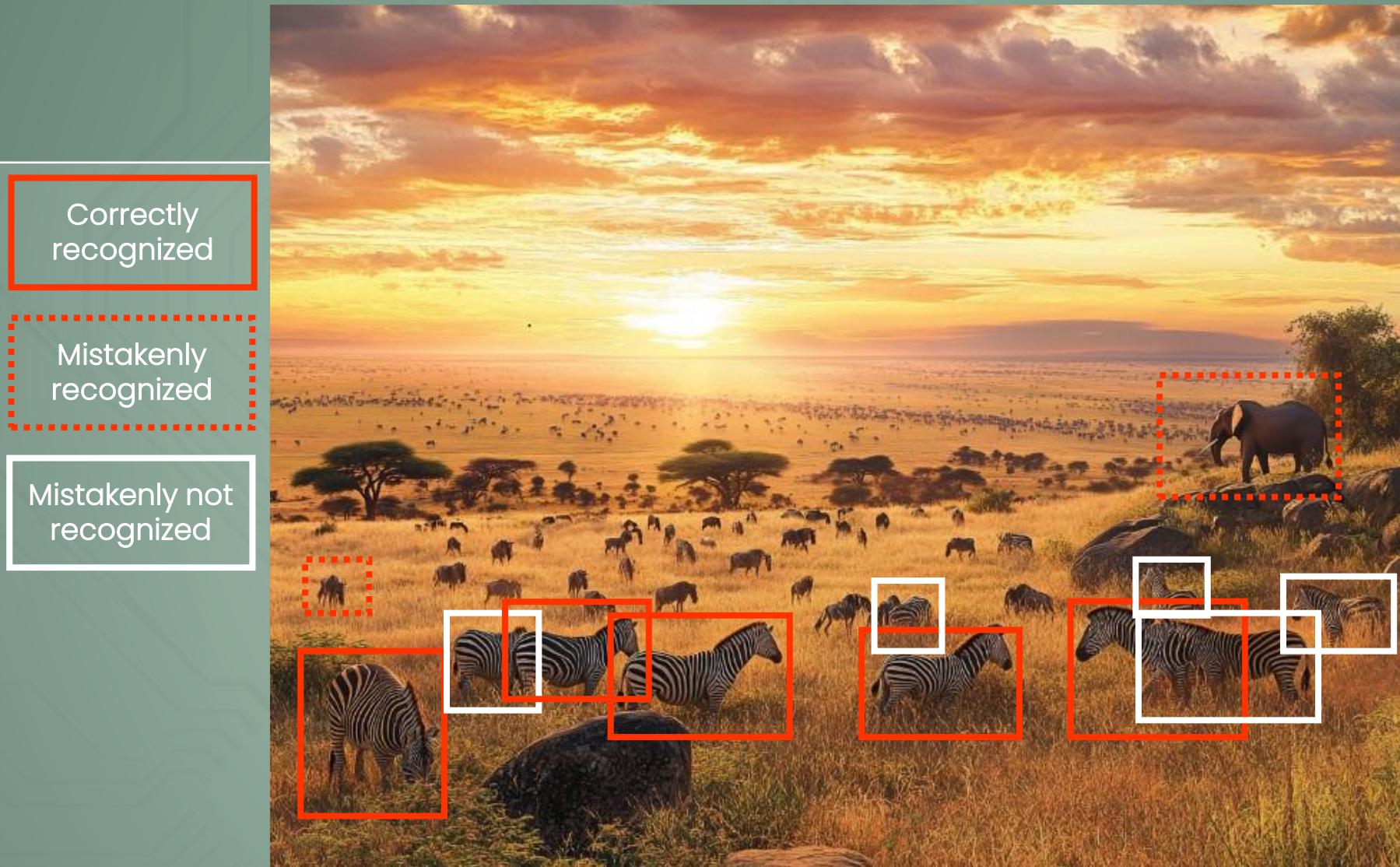
# How many zebras do you see?



# 10 out of 55 animals are zebras



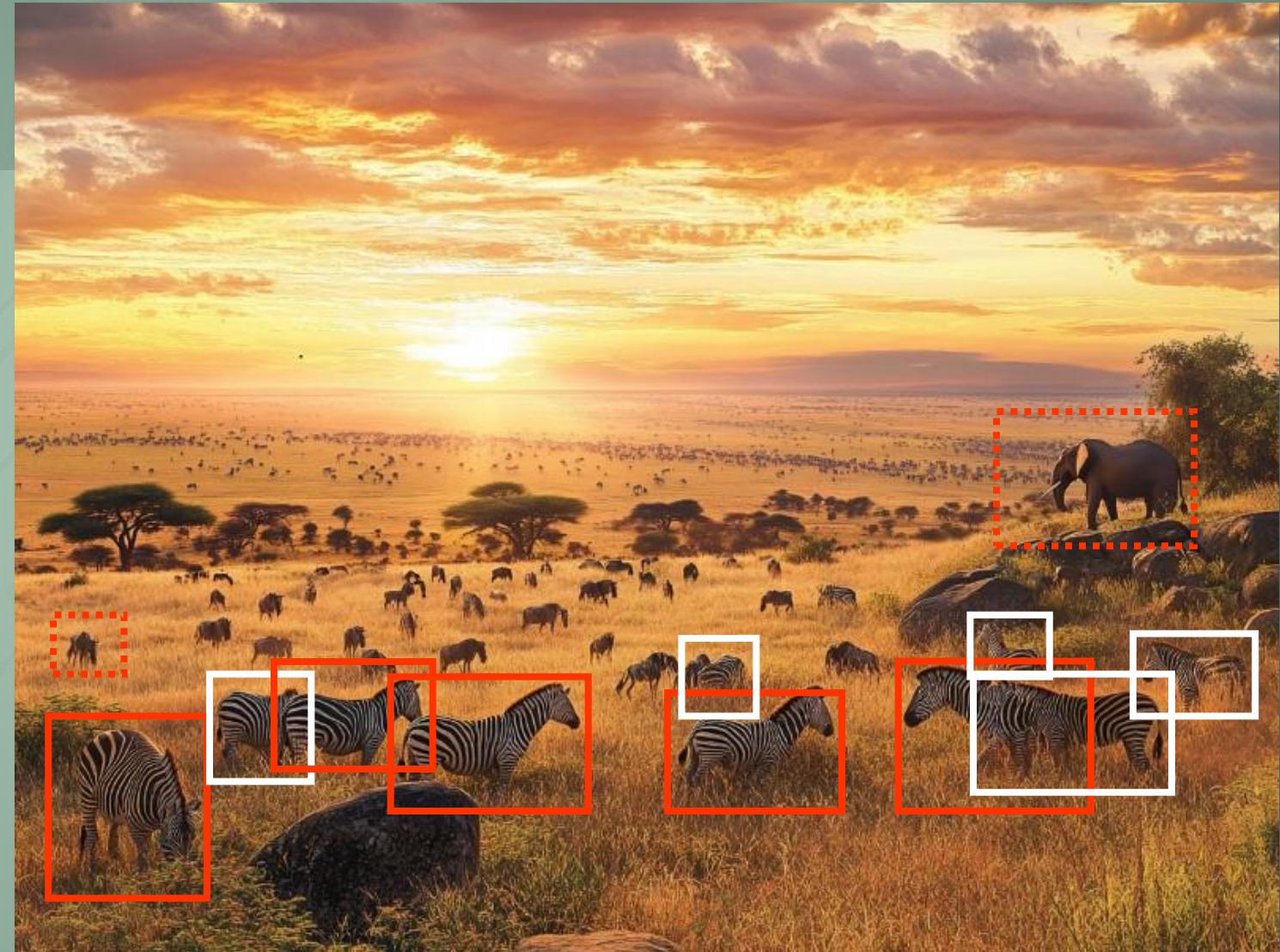
# Example: Model marks 7 animals as zebras, but makes mistakes



# Model marks 7 animals as zebras, but makes mistakes

Confusion Matrix:

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |



# Confusion Matrix

.. Is a table used to evaluate classification model performance by comparing predicted outcomes with actual results.

|           |          | Actual                      |                                |                               |
|-----------|----------|-----------------------------|--------------------------------|-------------------------------|
|           |          | Total                       | Positive                       | Negative                      |
| Predicted | Total    | Total number of items       | Total of actual Positive       | Total of actual Negative      |
|           | Positive | Total of predicted Positive | True Positive                  | False Positive (type I error) |
|           | Negative | Total of negative Positive  | False Negative (type II error) | True Negative                 |



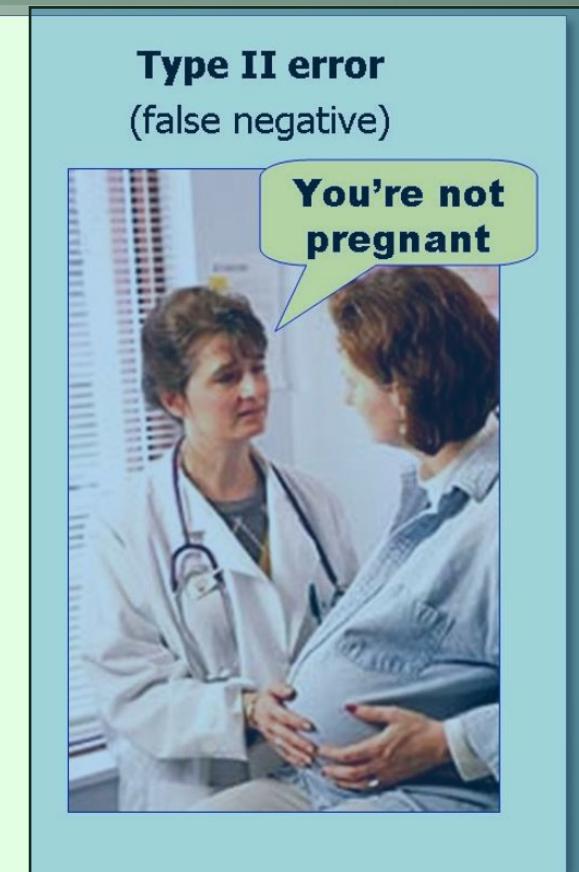
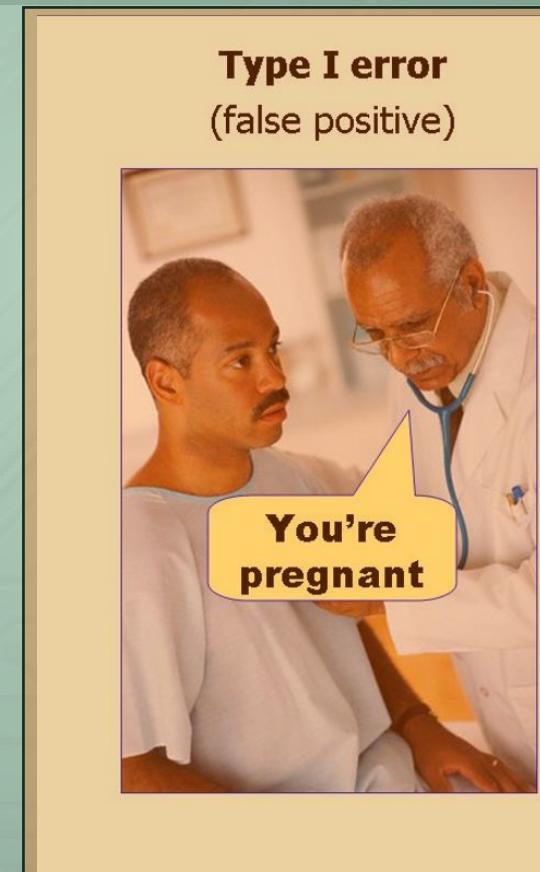
# Mistakes or errors: False positives and false negatives

## False positives and false negatives

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |

False positive

False negative



# Evaluation metrics for classification

... ensure fair and thorough comparison of trained classification models

## Key Metrics

### Accuracy:

Overall correctness of predictions in %

### Precision:

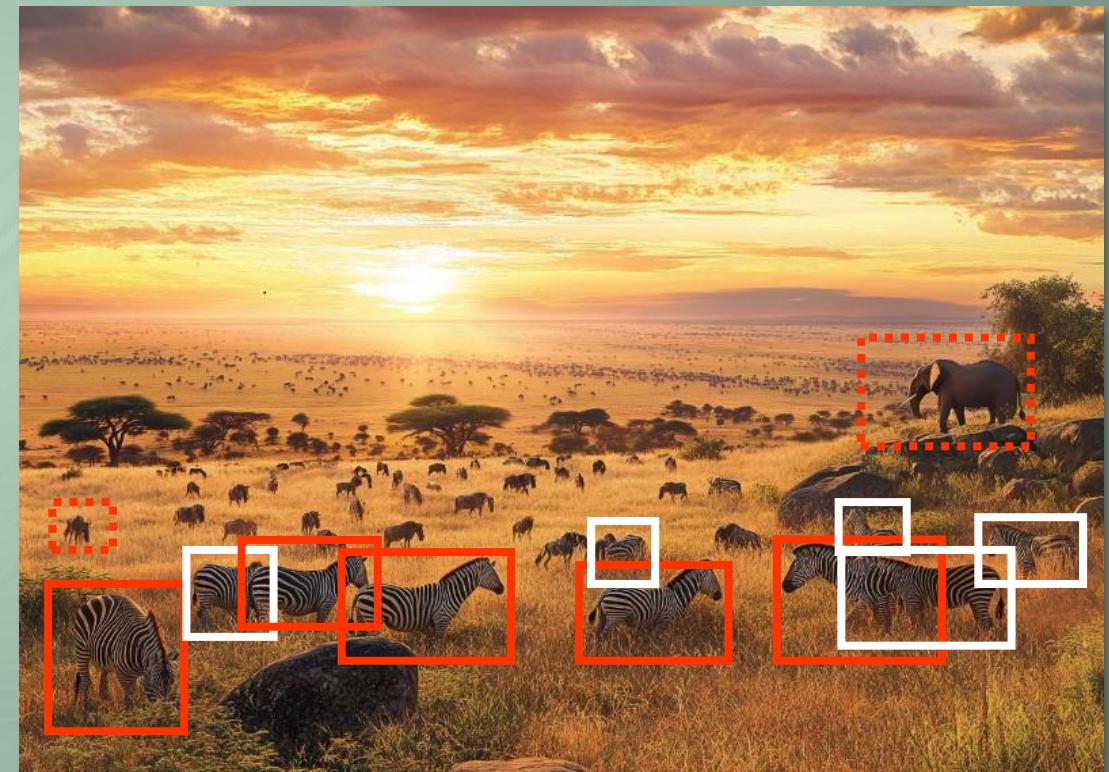
Correctly identified entities out of all predicted entities in %.  
From all predicted items, how many are correct?

### Recall Pass:

True positives identified out of all actual positives in %.  
From all positive items, how many have been identified as positive?

### F1-Score:

- Provides balance between precision and recall
- by calculating their harmonic mean



# Fallacy of “accuracy” as evaluation metric of classification

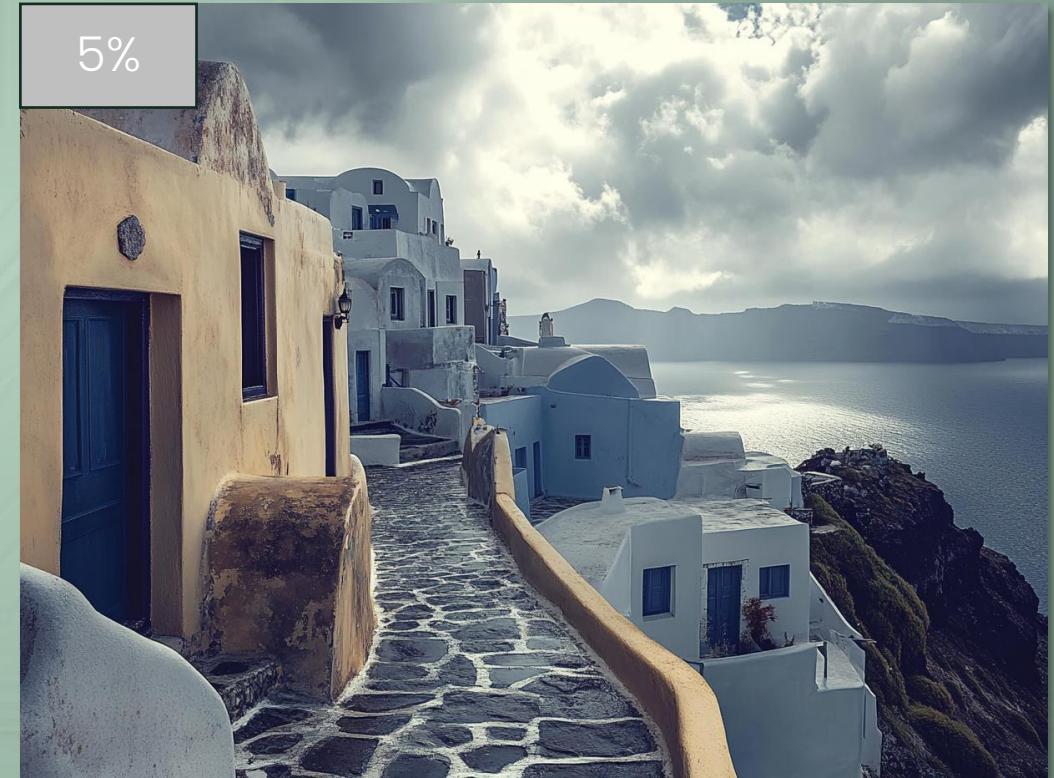
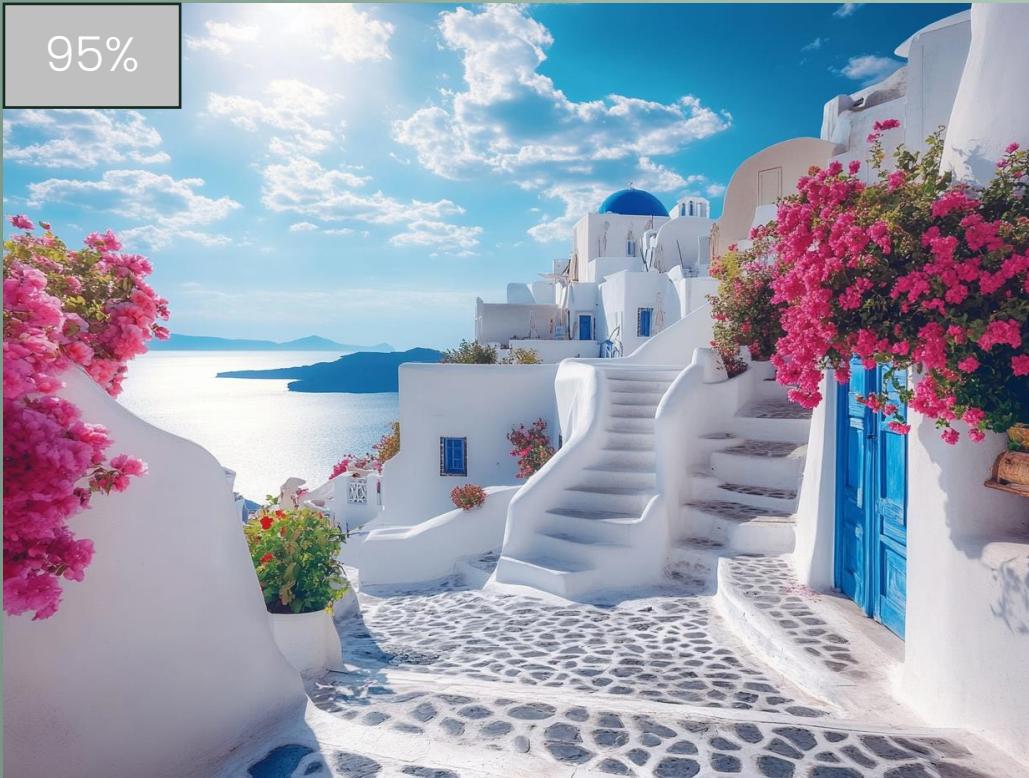
It's **July** in Greece, model predicts sun (often, 95%) or rain (rare, 5%) for next day.

It predicts sun (often). Next day this takes place.

It predicts rain (rare). Next day this takes place.

Was this prediction good or bad?

Was this prediction good or bad?



**Model that always predicts sun has accuracy = 95%.**  
→ Therefore, baseline for accuracy must be at least 95%

# Confusion matrix metric: Accuracy

How many items were classified correctly out of all items?

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |

$$\text{Accuracy} = \frac{\text{Correct Queries}}{\text{Total Queries}} \times 100$$



$$\text{Accuracy}(\%) = \frac{5 + 43}{55} = 87\%$$

# Confusion matrix metric: Precision

From all predicted items, how many are correct (in %)?

Or: From all as positive predicted items, how many are actually positive?

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |



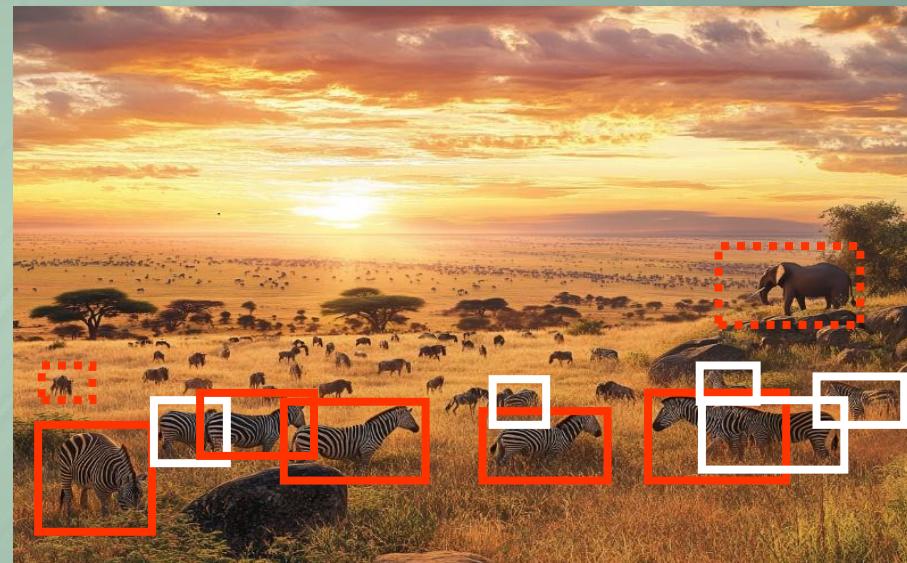
$$\text{Precision} = \frac{\text{True Positives}}{\text{True Positives} + \text{False Positives}}$$

$$\text{Precision}(\%) = \frac{5}{5+2} = 71\%$$

# Confusion matrix metric: Recall

Measures the proportion of actual positives correctly identified by the model, i.e. completeness of prediction. Out of all positive items, how many were predicted correctly (in %)?

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |



$$\text{Recall} = \frac{\text{True Positives}}{\text{True Positives} + \text{False Negatives}}$$

$$\text{Precision}(\%) = \frac{5}{5+5} = 50\%$$

**"Recall" is also called "Sensitivity", they are the same.**

# Confusion matrix metric: F1-Score

... is the harmonic mean of precision and recall

Balances the trade-off between precision and recall

|           |       | Actual |     |     |
|-----------|-------|--------|-----|-----|
|           |       | Total  | Pos | Neg |
| Predicted | Total | 55     | 10  | 45  |
|           | Pos   | 7      | 5   | 2   |
|           | Neg   | 48     | 5   | 43  |

$$F1 = 2 \times \frac{\text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}$$



$$F1 (\%) = \frac{2 \times 0.71 \times 0.5}{0.71 + 0.5} = \frac{0.71}{1.21} = 59\%$$

# Confusion matrix metric: Interpreting F1-Score

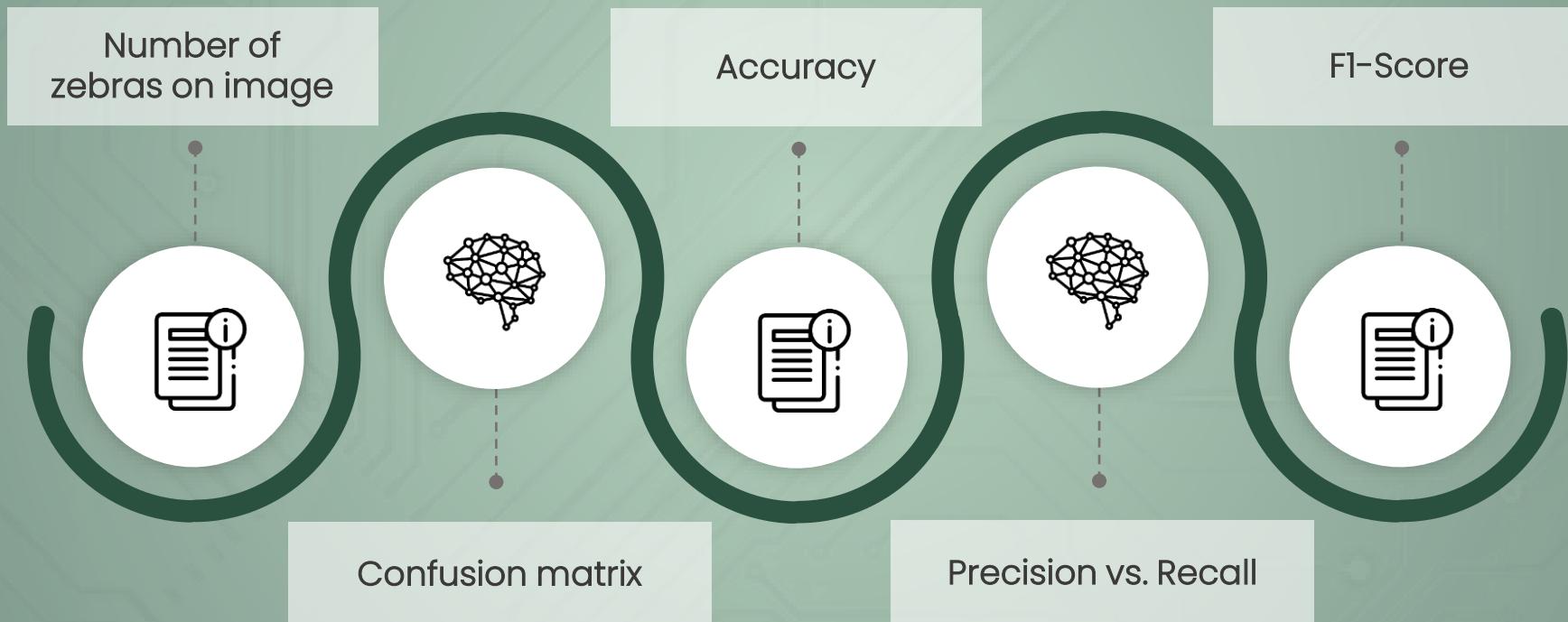
... Let's finds balance between precision and recall,  
especially useful for imbalanced datasets.

$$F1 = 2 \times \frac{\text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}$$

| Metric    | Definition  | Focus   | Strengths  | Weaknesses   |
|-----------|---|---|--|--|
| Precision | <ul style="list-style-type: none"><li>▪ Proportion of correctly predicted positives</li><li>▪ out of all predicted positives.</li></ul> | <ul style="list-style-type: none"><li>▪ Accuracy of positive predictions</li></ul>      | <ul style="list-style-type: none"><li>▪ Minimizes false positives</li><li>▪ useful in scenarios where false positives are costly</li><li>▪ e.g., spam detection</li></ul>            | <ul style="list-style-type: none"><li>▪ May miss true positives,</li><li>▪ leading to low recall.</li></ul>        |
| Recall    | <ul style="list-style-type: none"><li>▪ Proportion of actual positives</li><li>▪ correctly identified by the model.</li></ul>           | <ul style="list-style-type: none"><li>▪ Completeness of positive predictions.</li></ul> | <ul style="list-style-type: none"><li>▪ Minimizes false negatives</li><li>▪ critical in scenarios where missing true positives is costly</li><li>▪ e.g., disease diagnosis</li></ul> | <ul style="list-style-type: none"><li>▪ May increase false positives</li><li>▪ leading to low precision.</li></ul> |
| F1-Score  | <ul style="list-style-type: none"><li>▪ Harmonic mean of precision and recall;</li><li>▪ balances the two metrics.</li></ul>            | Balance between precision and recall.   | <ul style="list-style-type: none"><li>▪ Useful for imbalanced datasets</li><li>▪ or when both false positives and false negatives are important.</li></ul>                           | Does not distinguish between cases where precision or recall is more critical.                                     |

# Exercise: Default of credit card clients

Develop model for binary classification by generalizing the dataset



# Exercise: Predicting Credit Card Default

Expected Outcome: Trained logistic regression model capable of predicting credit card defaults with interpretable results

## Exercise Overview

- **Objective:** Build a logistic regression model to predict whether a client will default on their credit card payment (Yes = 1, No = 0)
- **Dataset:** Default of Credit Card Clients dataset (2005).

## Key Variables

- Response Variable (Y): Default payment status (1 = Yes, 0 = No).
- Explanatory Variables:
  - X1: Credit limit (NT dollars)
  - X2-X5: Demographics (Gender, Education, Marital Status, Age)
  - X6-X11: Past repayment history (-1 = on time, 1-9 = months delayed)
  - X12-X17: Bill statement amounts (April–September 2005)
  - X18-X23: Previous payments made (April–September 2005)



# Predicting Credit Card Default

Dataset:

| Credit limit<br>ID | Demographics       |               | Past repayment history |            |    |             |             |             | Bill statement amounts |              |              |                |                |                | Previous payments made |                |                |                |               |               | Response Variable |               |               |                                 |   |   |
|--------------------|--------------------|---------------|------------------------|------------|----|-------------|-------------|-------------|------------------------|--------------|--------------|----------------|----------------|----------------|------------------------|----------------|----------------|----------------|---------------|---------------|-------------------|---------------|---------------|---------------------------------|---|---|
|                    | X1<br>LIMIT_BAISEX | X2<br>EDUCATI | X3<br>MARRIA           | X4<br>CAGE | X5 | X6<br>PAY_0 | X7<br>PAY_2 | X8<br>PAY_3 | X9<br>PAY_4            | X10<br>PAY_5 | X11<br>PAY_6 | X12<br>BILL_AM | X13<br>BILL_AM | X14<br>BILL_AM | X15<br>BILL_AM         | X16<br>BILL_AM | X17<br>BILL_AM | X18<br>BILL_AM | X19<br>PAY_AM | X20<br>PAY_AM | X21<br>PAY_AM     | X22<br>PAY_AM | X23<br>PAY_AM | Y<br>default payment next month |   |   |
| 1                  | 20000              | 2             | 2                      | 1          | 24 | 2           | 2           | -1          | -1                     | -2           | -2           | 3913           | 3102           | 689            | 0                      | 0              | 0              | 0              | 689           | 0             | 0                 | 0             | 0             | 0                               | 1 |   |
| 2                  | 120000             | 2             | 2                      | 2          | 26 | -1          | 2           | 0           | 0                      | 0            | 2            | 2682           | 1725           | 2682           | 3272                   | 3455           | 3261           | 0              | 1000          | 1000          | 1000              | 0             | 2000          | 0                               | 1 |   |
| 3                  | 90000              | 2             | 2                      | 2          | 34 | 0           | 0           | 0           | 0                      | 0            | 0            | 29239          | 14027          | 13559          | 14331                  | 14948          | 15549          | 1518           | 1500          | 1000          | 1000              | 1000          | 5000          | 0                               | 0 |   |
| 4                  | 50000              | 2             | 2                      | 1          | 37 | 0           | 0           | 0           | 0                      | 0            | 0            | 46990          | 48233          | 49291          | 28314                  | 28959          | 29547          | 2000           | 2019          | 1200          | 1100              | 1069          | 1000          | 0                               | 0 |   |
| 5                  | 50000              | 1             | 2                      | 1          | 57 | -1          | 0           | -1          | 0                      | 0            | 0            | 8617           | 5670           | 35835          | 20940                  | 19146          | 19131          | 2000           | 36681         | 10000         | 9000              | 689           | 679           | 0                               | 0 |   |
| 6                  | 50000              | 1             | 1                      | 2          | 37 | 0           | 0           | 0           | 0                      | 0            | 0            | 64400          | 57069          | 57608          | 19394                  | 19619          | 20024          | 2500           | 1815          | 657           | 1000              | 1000          | 800           | 0                               | 0 |   |
| 7                  | 500000             | 1             | 1                      | 2          | 29 | 0           | 0           | 0           | 0                      | 0            | 0            | 367965         | 412023         | 445007         | 542653                 | 483003         | 473944         | 55000          | 40000         | 38000         | 20239             | 13750         | 13770         | 0                               | 0 |   |
| 8                  | 100000             | 2             | 2                      | 2          | 23 | 0           | -1          | -1          | 0                      | 0            | -1           | 11876          | 380            | 601            | 221                    | -159           | 567            | 380            | 601           | 0             | 581               | 1687          | 1542          | 0                               | 0 |   |
| 9                  | 140000             | 2             | 3                      | 1          | 28 | 0           | 0           | 2           | 0                      | 0            | 0            | 11285          | 14096          | 12108          | 12211                  | 11793          | 3719           | 3329           | 0             | 432           | 1000              | 1000          | 1000          | 0                               | 0 |   |
| 10                 | 20000              | 1             | 3                      | 2          | 35 | -2          | -2          | -2          | -1                     | -1           | 0            | 0              | 0              | 0              | 13007                  | 13912          | 0              | 0              | 0             | 13007         | 1122              | 0             | 0             | 0                               | 0 | 0 |
| 11                 | 200000             | 2             | 3                      | 2          | 34 | 0           | 0           | 2           | 0                      | 0            | -1           | 11073          | 9787           | 5535           | 2513                   | 1828           | 3731           | 2306           | 12            | 50            | 300               | 3738          | 66            | 0                               | 0 |   |
| 12                 | 260000             | 2             | 1                      | 2          | 51 | -1          | -1          | -1          | -1                     | -1           | 2            | 12261          | 21670          | 9966           | 8517                   | 22287          | 13668          | 21818          | 9966          | 8583          | 22301             | 0             | 3640          | 0                               | 0 |   |
| 13                 | 630000             | 2             | 2                      | 2          | 41 | -1          | 0           | -1          | -1                     | -1           | -1           | 12137          | 6500           | 6500           | 6500                   | 2870           | 1000           | 6500           | 6500          | 6500          | 2870              | 0             | 0             | 0                               | 0 |   |
| 14                 | 70000              | 1             | 2                      | 2          | 30 | 1           | 2           | 2           | 0                      | 0            | 2            | 65802          | 67369          | 65701          | 66782                  | 36137          | 36894          | 3200           | 0             | 3000          | 3000              | 1500          | 0             | 1                               | 0 |   |
| 15                 | 250000             | 1             | 1                      | 2          | 29 | 0           | 0           | 0           | 0                      | 0            | 0            | 70887          | 67060          | 63561          | 59696                  | 56875          | 55512          | 3000           | 3000          | 3000          | 3000              | 3000          | 3000          | 0                               | 0 |   |
| 16                 | 50000              | 2             | 3                      | 3          | 23 | 1           | 2           | 0           | 0                      | 0            | 0            | 50614          | 29173          | 28116          | 28771                  | 29531          | 30211          | 0              | 1500          | 1100          | 1200              | 1300          | 1100          | 0                               | 0 |   |
| 17                 | 20000              | 1             | 1                      | 2          | 24 | 0           | 0           | 2           | 2                      | 2            | 2            | 15376          | 18010          | 17428          | 18338                  | 17905          | 19104          | 3200           | 0             | 1500          | 0                 | 1650          | 0             | 1                               | 0 |   |
| 18                 | 320000             | 1             | 1                      | 1          | 49 | 0           | 0           | 0           | -1                     | -1           | -1           | 253286         | 246536         | 194663         | 70074                  | 5856           | 195599         | 10358          | 10000         | 75940         | 20000             | 195599        | 50000         | 0                               | 0 |   |
| 19                 | 360000             | 2             | 1                      | 1          | 49 | 1           | -2          | -2          | -2                     | -2           | 0            | 0              | 0              | 0              | 0                      | 0              | 0              | 0              | 0             | 0             | 0                 | 0             | 0             | 0                               |   |   |
| 20                 | 180000             | 2             | 1                      | 2          | 29 | 1           | -2          | -2          | -2                     | -2           | 0            | 0              | 0              | 0              | 0                      | 0              | 0              | 0              | 0             | 0             | 0                 | 0             | 0             | 0                               |   |   |
| 21                 | 130000             | 2             | 3                      | 2          | 39 | 0           | 0           | 0           | 0                      | 0            | -1           | 38358          | 27688          | 24489          | 20616                  | 11802          | 930            | 3000           | 1537          | 1000          | 2000              | 930           | 33764         | 0                               | 0 |   |
| 22                 | 120000             | 2             | 2                      | 1          | 39 | -1          | -1          | -1          | -1                     | -1           | -1           | 316            | 316            | 316            | 0                      | 632            | 316            | 316            | 0             | 632           | 316               | 0             | 1             | 0                               |   |   |
| 23                 | 70000              | 2             | 2                      | 2          | 26 | 2           | 0           | 0           | 2                      | 2            | 2            | 41087          | 42445          | 45020          | 44006                  | 46905          | 46012          | 2007           | 3582          | 0             | 3601              | 0             | 1820          | 1                               | 0 |   |
| 24                 | 450000             | 2             | 1                      | 1          | 40 | -2          | -2          | -2          | -2                     | -2           | -2           | 5512           | 19420          | 1473           | 560                    | 0              | 0              | 19428          | 1473          | 560           | 0                 | 0             | 1128          | 1                               | 0 |   |
| 25                 | 90000              | 1             | 1                      | 2          | 23 | 0           | 0           | 0           | -1                     | 0            | 0            | 4744           | 7070           | 0              | 5398                   | 6360           | 8292           | 5757           | 0             | 5398          | 1200              | 2045          | 2000          | 0                               | 0 |   |
| 26                 | 50000              | 1             | 3                      | 2          | 23 | 0           | 0           | 0           | 0                      | 0            | 0            | 47620          | 41810          | 36023          | 28967                  | 29829          | 30046          | 1973           | 1426          | 1001          | 1432              | 1062          | 997           | 0                               | 0 |   |
| 27                 | 60000              | 1             | 1                      | 2          | 27 | 1           | -2          | -1          | -1                     | -1           | -1           | -109           | -425           | 259            | -57                    | 127            | -189           | 0              | 1000          | 0             | 500               | 0             | 1000          | 1                               | 0 |   |
| 28                 | 50000              | 2             | 3                      | 2          | 30 | 0           | 0           | 0           | 0                      | 0            | 0            | 22541          | 16138          | 17163          | 17878                  | 18931          | 19617          | 1300           | 1300          | 1000          | 1500              | 1000          | 1012          | 0                               | 0 |   |
| 29                 | 50000              | 2             | 3                      | 1          | 47 | -1          | -1          | -1          | -1                     | -1           | -1           | 650            | 3415           | 3416           | 2040                   | 30430          | 257            | 3415           | 3421          | 2044          | 30430             | 257           | 0             | 0                               | 0 |   |
| 30                 | 50000              | 1             | 1                      | 2          | 26 | 0           | 0           | 0           | 0                      | 0            | 0            | 15329          | 16575          | 17496          | 17907                  | 18375          | 11400          | 1500           | 1500          | 1000          | 1600              | 0             | 0             | 0                               | 0 |   |
| 31                 | 230000             | 2             | 1                      | 2          | 27 | -1          | -1          | -1          | -1                     | -1           | -1           | 16646          | 17265          | 13266          | 15339                  | 14307          | 36923          | 17270          | 13281         | 15339         | 14307             | 37292         | 0             | 0                               | 0 |   |
| 32                 | 50000              | 1             | 2                      | 2          | 33 | 2           | 0           | 0           | 0                      | 0            | 0            | 30518          | 29618          | 22102          | 22734                  | 23217          | 23680          | 1718           | 1500          | 1000          | 1000              | 1000          | 716           | 1                               | 0 |   |
| 33                 | 100000             | 1             | 1                      | 2          | 32 | 0           | 0           | 0           | 0                      | 0            | 0            | 93036          | 84071          | 82880          | 80958                  | 78703          | 75589          | 3023           | 3511          | 3302          | 3204              | 3200          | 2504          | 0                               | 0 |   |
| 34                 | 500000             | 2             | 2                      | 1          | 54 | -2          | -2          | -2          | -2                     | -2           | -2           | 10929          | 4152           | 22722          | 7521                   | 71439          | 8981           | 4152           | 22827         | 7521          | 71439             | 981           | 51582         | 0                               | 0 |   |
| 35                 | 500000             | 1             | 1                      | 1          | 58 | -2          | -2          | -2          | -2                     | -2           | -2           | 13709          | 5006           | 31130          | 3180                   | 0              | 5293           | 5006           | 31178         | 3180          | 0                 | 5293          | 768           | 0                               | 0 |   |
| 36                 | 160000             | 1             | 1                      | 2          | 30 | -1          | -1          | -2          | -2                     | -2           | -1           | 30265          | -131           | -527           | -923                   | -1488          | -1884          | 131            | 396           | 396           | 565               | 792           | 0             | 0                               | 0 |   |

# Predicting Credit Card Default

Expected Outcome: Trained logistic regression model capable of predicting credit card defaults with interpretable results

## Steps to Complete the Exercise

### 1. Data Preprocessing:

Only if relevant:

Handle missing values, normalize numeric features, and encode categorical variables.

### 2. Model Training:

Use logistic regression to train model with explanatory variables.

### 3. Evaluation:

Assess model performance using metrics like

- Accuracy
- Precision & Recall
- F1-Score

### 4. Insights:

Identify key predictors of default and interpret coefficients for actionable insights.

```
# Train our custom logistic regression model
print("\nTraining our custom LogisticRegression model...")
# Using a smaller number of iterations for time constraints
my_model = MyLogisticRegression(learning_rate=0.1, num_iterations=10000)
my_model.fit(X_train_scaled, y_train)
```

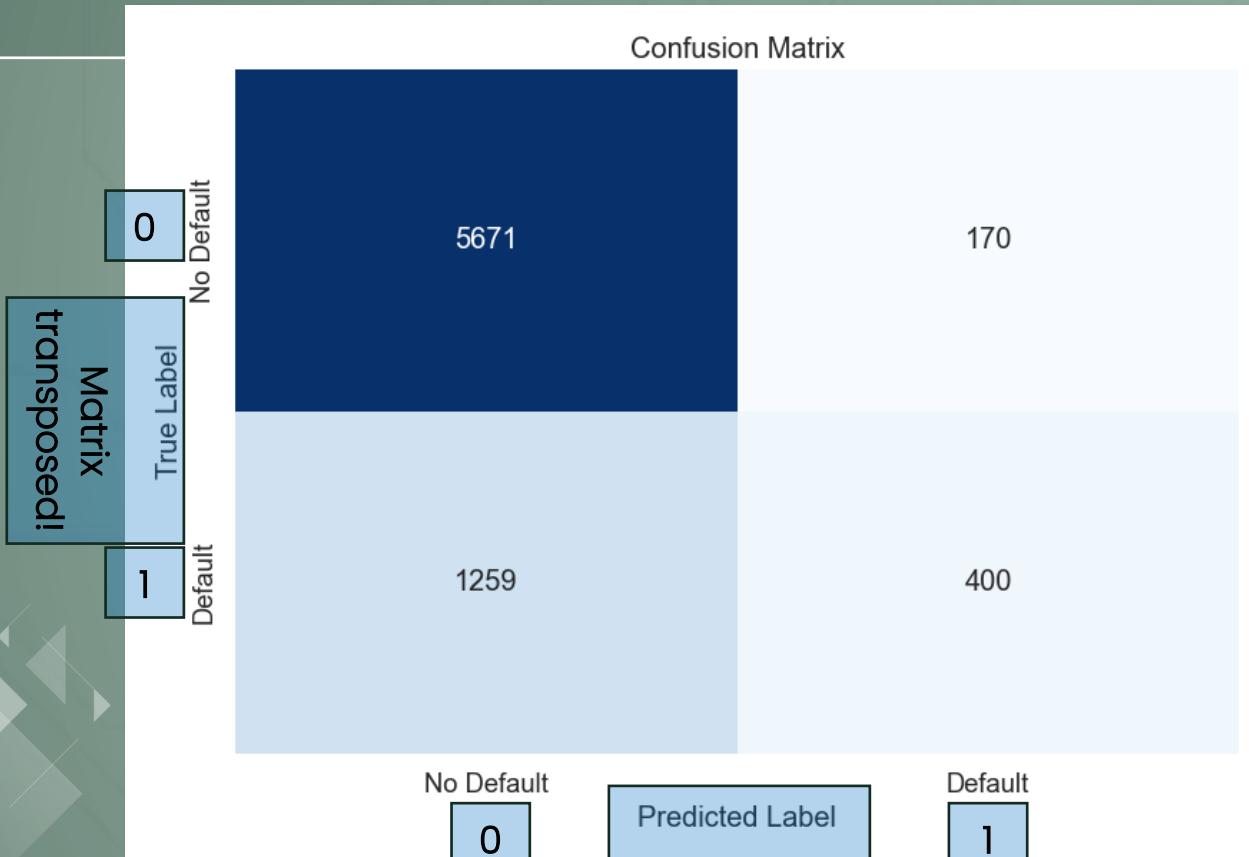
### 4. MODEL TRAINING: LOGISTIC REGRESSION

---

```
Training our custom LogisticRegression model...
Cost at iteration 500: 0.46402681019559955
Cost at iteration 1000: 0.46391625110907125
Cost at iteration 1500: 0.463881906608544
Cost at iteration 2000: 0.4638625289985032
Cost at iteration 2500: 0.46385059653172084
Cost at iteration 3000: 0.4638429819015393
Cost at iteration 3500: 0.4638379809509058
Cost at iteration 4000: 0.4638346091873176
Cost at iteration 4500: 0.46383228109929964
Cost at iteration 5000: 0.46383063939171343
Cost at iteration 5500: 0.463829460346282
Cost at iteration 6000: 0.4638286002434648
Cost at iteration 6500: 0.4638279644346782
Cost at iteration 7000: 0.4638274891188923
Cost at iteration 7500: 0.4638271303666043
Cost at iteration 8000: 0.4638268573547721
Cost at iteration 8500: 0.46382664809774654
Cost at iteration 9000: 0.463826486689604
Cost at iteration 9500: 0.4638263614823324
```

# Predicting Credit Card Default: Evaluation

... with Confusion Matrix, accuracy, precision, recall, F1-Score  
→ becomes relevant when comparing models



## 5. MODEL EVALUATION

---

Classification Report (sklearn model):

|              | precision | recall   | f1-score | support |
|--------------|-----------|----------|----------|---------|
| 0            | 0.82      | 0.97     | 0.89     | 5841    |
| 1            | 0.70      | 0.24     | 0.36     | 1659    |
| accuracy     | 400/570   | 400/1651 | 0.81     | 7500    |
| macro avg    | 0.76      | 0.61     | 0.62     | 7500    |
| weighted avg | 0.79      | 0.81     | 0.77     | 7500    |

### Detailed Metrics:

Accuracy: 0.8095

Precision: 0.7018

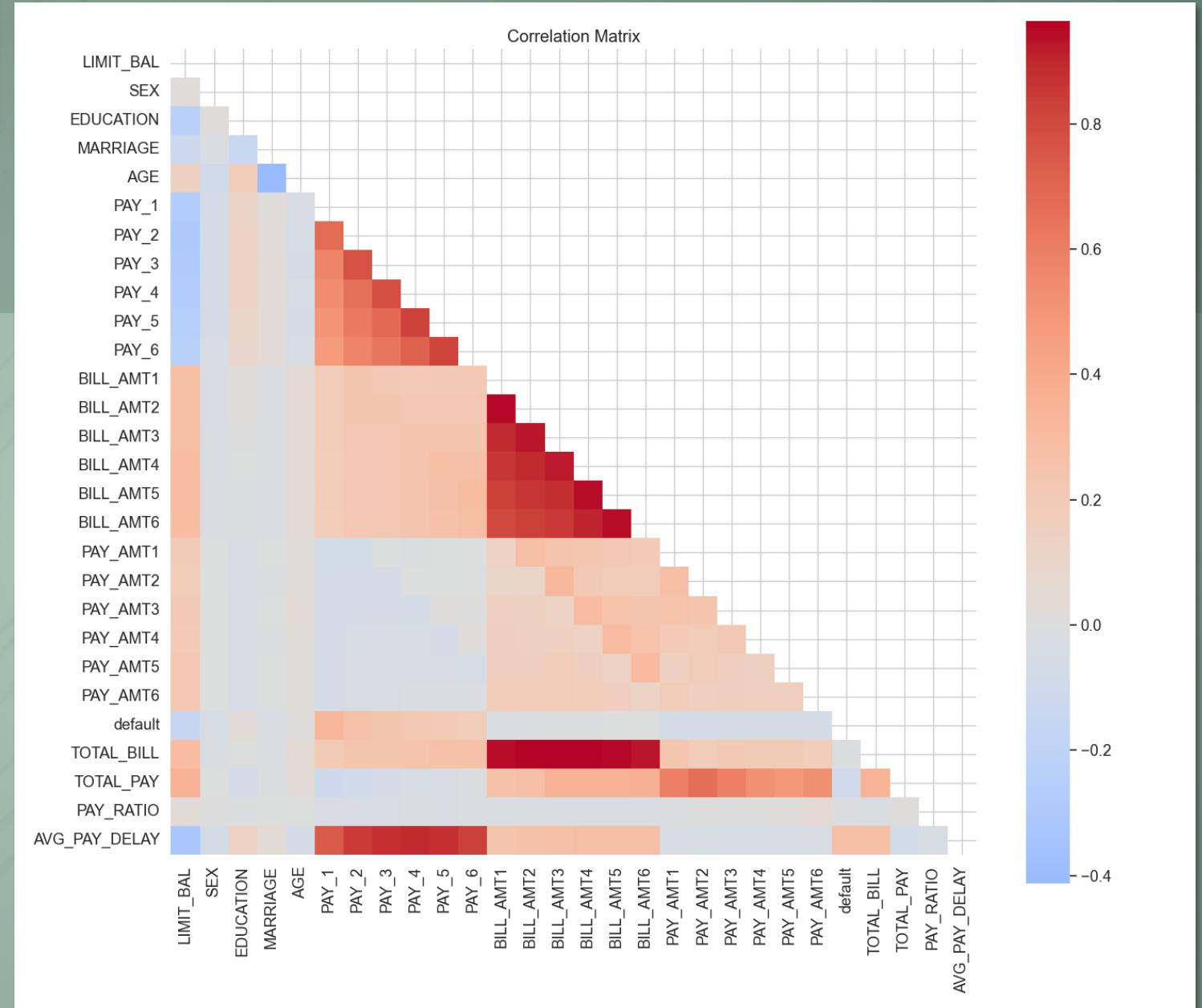
Recall: 0.2411

F1-Score: 0.3589

# Correlation Matrix

Influence factors might also influence each other

- For unbiasedness of estimation high correlation between factors should be avoided.
- Therefore, part of factors should be removed from model.



# Statistical tables

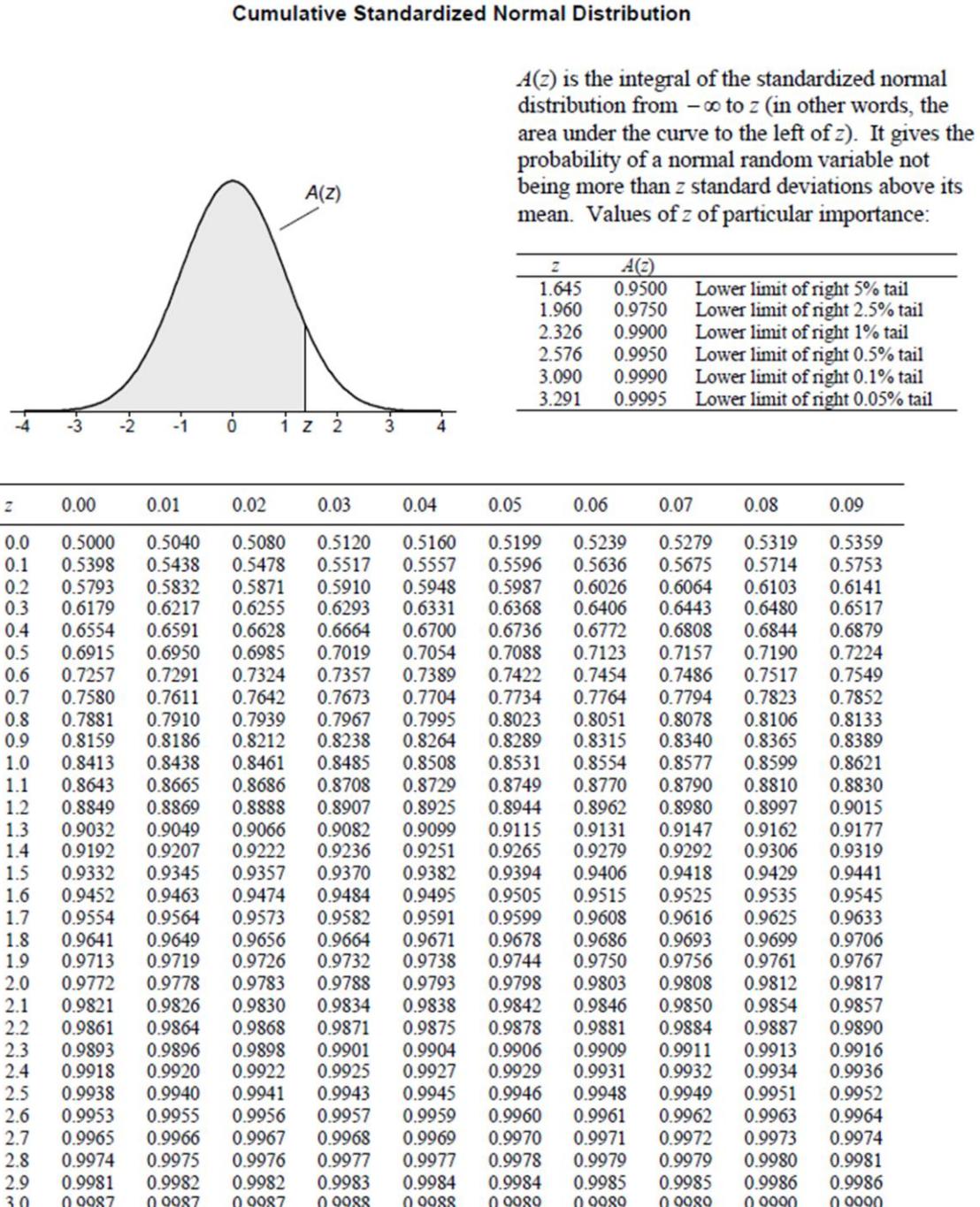
... let you decide whether to reject  
 $H_0$  or not by hand

t Distribution: Critical Values of t

| Degrees of freedom | Two-tailed test:<br>One-tailed test: | Significance level |        |        |        |         |         |      |
|--------------------|--------------------------------------|--------------------|--------|--------|--------|---------|---------|------|
|                    |                                      | 10%                | 5%     | 2.5%   | 2%     | 1%      | 0.2%    | 0.1% |
| 1                  |                                      | 6.314              | 12.706 | 31.821 | 63.657 | 318.309 | 636.619 |      |
| 2                  |                                      | 2.920              | 4.303  | 6.965  | 9.925  | 22.327  | 31.599  |      |
| 3                  |                                      | 2.353              | 3.182  | 4.541  | 5.841  | 10.215  | 12.924  |      |
| 4                  |                                      | 2.132              | 2.776  | 3.747  | 4.604  | 7.173   | 8.610   |      |
| 5                  |                                      | 2.015              | 2.571  | 3.365  | 4.032  | 5.893   | 6.869   |      |
| 6                  |                                      | 1.943              | 2.447  | 3.143  | 3.707  | 5.208   | 5.959   |      |
| 7                  |                                      | 1.894              | 2.365  | 2.998  | 3.499  | 4.785   | 5.408   |      |
| 8                  |                                      | 1.860              | 2.306  | 2.896  | 3.355  | 4.501   | 5.041   |      |
| 9                  |                                      | 1.833              | 2.262  | 2.821  | 3.250  | 4.297   | 4.781   |      |
| 10                 |                                      | 1.812              | 2.228  | 2.764  | 3.169  | 4.144   | 4.587   |      |
| 11                 |                                      | 1.796              | 2.201  | 2.718  | 3.106  | 4.025   | 4.437   |      |
| 12                 |                                      | 1.782              | 2.179  | 2.681  | 3.055  | 3.930   | 4.318   |      |
| 13                 |                                      | 1.771              | 2.160  | 2.650  | 3.012  | 3.852   | 4.221   |      |
| 14                 |                                      | 1.761              | 2.145  | 2.624  | 2.977  | 3.787   | 4.140   |      |
| 15                 |                                      | 1.753              | 2.131  | 2.602  | 2.947  | 3.733   | 4.073   |      |
| 16                 |                                      | 1.746              | 2.120  | 2.583  | 2.921  | 3.686   | 4.015   |      |
| 17                 |                                      | 1.740              | 2.110  | 2.567  | 2.898  | 3.646   | 3.965   |      |
| 18                 |                                      | 1.734              | 2.101  | 2.552  | 2.878  | 3.610   | 3.922   |      |
| 19                 |                                      | 1.729              | 2.093  | 2.539  | 2.861  | 3.579   | 3.883   |      |
| 20                 |                                      | 1.725              | 2.086  | 2.528  | 2.845  | 3.552   | 3.850   |      |
| 21                 |                                      | 1.721              | 2.080  | 2.518  | 2.831  | 3.527   | 3.819   |      |
| 22                 |                                      | 1.717              | 2.074  | 2.508  | 2.819  | 3.505   | 3.792   |      |
| 23                 |                                      | 1.714              | 2.069  | 2.500  | 2.807  | 3.485   | 3.768   |      |
| 24                 |                                      | 1.711              | 2.064  | 2.492  | 2.797  | 3.467   | 3.745   |      |
| 25                 |                                      | 1.708              | 2.060  | 2.485  | 2.787  | 3.450   | 3.725   |      |
| 26                 |                                      | 1.706              | 2.056  | 2.479  | 2.779  | 3.435   | 3.707   |      |
| 27                 |                                      | 1.703              | 2.052  | 2.473  | 2.771  | 3.421   | 3.690   |      |
| 28                 |                                      | 1.701              | 2.048  | 2.467  | 2.763  | 3.408   | 3.674   |      |
| 29                 |                                      | 1.699              | 2.045  | 2.462  | 2.756  | 3.396   | 3.659   |      |
| 30                 |                                      | 1.697              | 2.042  | 2.457  | 2.750  | 3.385   | 3.646   |      |
| 32                 |                                      | 1.694              | 2.037  | 2.449  | 2.738  | 3.365   | 3.622   |      |
| 34                 |                                      | 1.691              | 2.032  | 2.441  | 2.728  | 3.348   | 3.601   |      |
| 36                 |                                      | 1.688              | 2.028  | 2.434  | 2.719  | 3.333   | 3.582   |      |
| 38                 |                                      | 1.686              | 2.024  | 2.429  | 2.712  | 3.319   | 3.566   |      |
| 40                 |                                      | 1.684              | 2.021  | 2.423  | 2.704  | 3.307   | 3.551   |      |

# Statistical tables

... let you decide whether to reject  $H_0$  or not by hand



# Statistical tables

... let you decide whether to reject  $H_0$  or not by hand

## $\chi^2$ (Chi-Squared) Distribution: Critical Values of $\chi^2$

| <i>Degrees of freedom</i> | <i>Significance level</i> |        |        |
|---------------------------|---------------------------|--------|--------|
|                           | 5%                        | 1%     | 0.1%   |
| 1                         | 3.841                     | 6.635  | 10.828 |
| 2                         | 5.991                     | 9.210  | 13.816 |
| 3                         | 7.815                     | 11.345 | 16.266 |
| 4                         | 9.488                     | 13.277 | 18.467 |
| 5                         | 11.070                    | 15.086 | 20.515 |
| 6                         | 12.592                    | 16.812 | 22.458 |
| 7                         | 14.067                    | 18.475 | 24.322 |
| 8                         | 15.507                    | 20.090 | 26.124 |
| 9                         | 16.919                    | 21.666 | 27.877 |
| 10                        | 18.307                    | 23.209 | 29.588 |



# Statistical tables

... let you decide whether to reject  
 $H_0$  or not by hand

TABLE A.3

F Distribution: Critical Values of F (5% significance level)

| $v_1$ | 1     | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     | 12     | 14     | 16     | 18     | 20     |        |
|-------|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $v_2$ | 1     | 161.45 | 199.50 | 215.71 | 224.58 | 230.16 | 233.99 | 236.77 | 238.88 | 240.54 | 241.88 | 243.91 | 245.36 | 246.46 | 247.32 | 248.01 |
| 2     | 18.51 | 19.00  | 19.16  | 19.25  | 19.30  | 19.33  | 19.35  | 19.37  | 19.38  | 19.40  | 19.41  | 19.42  | 19.43  | 19.44  | 19.45  |        |
| 3     | 10.13 | 9.55   | 9.28   | 9.12   | 9.01   | 8.94   | 8.89   | 8.85   | 8.81   | 8.79   | 8.74   | 8.71   | 8.69   | 8.67   | 8.66   |        |
| 4     | 7.71  | 6.94   | 6.59   | 6.39   | 6.26   | 6.16   | 6.09   | 6.04   | 6.00   | 5.96   | 5.91   | 5.87   | 5.84   | 5.82   | 5.80   |        |
| 5     | 6.61  | 5.79   | 5.41   | 5.19   | 5.05   | 4.95   | 4.88   | 4.82   | 4.77   | 4.74   | 4.68   | 4.64   | 4.60   | 4.58   | 4.56   |        |
| 6     | 5.99  | 5.14   | 4.76   | 4.53   | 4.39   | 4.28   | 4.21   | 4.15   | 4.10   | 4.06   | 4.00   | 3.96   | 3.92   | 3.90   | 3.87   |        |
| 7     | 5.59  | 4.74   | 4.35   | 4.12   | 3.97   | 3.87   | 3.79   | 3.73   | 3.68   | 3.64   | 3.57   | 3.53   | 3.49   | 3.47   | 3.44   |        |
| 8     | 5.32  | 4.46   | 4.07   | 3.84   | 3.69   | 3.58   | 3.50   | 3.44   | 3.39   | 3.35   | 3.28   | 3.24   | 3.20   | 3.17   | 3.15   |        |
| 9     | 5.12  | 4.26   | 3.86   | 3.63   | 3.48   | 3.37   | 3.29   | 3.23   | 3.18   | 3.14   | 3.07   | 3.03   | 2.99   | 2.96   | 2.94   |        |
| 10    | 4.96  | 4.10   | 3.71   | 3.48   | 3.33   | 3.22   | 3.14   | 3.07   | 3.02   | 2.98   | 2.91   | 2.86   | 2.83   | 2.80   | 2.77   |        |
| 11    | 4.84  | 3.98   | 3.59   | 3.36   | 3.20   | 3.09   | 3.01   | 2.95   | 2.90   | 2.85   | 2.79   | 2.74   | 2.70   | 2.67   | 2.65   |        |
| 12    | 4.75  | 3.89   | 3.49   | 3.26   | 3.11   | 3.00   | 2.91   | 2.85   | 2.80   | 2.75   | 2.69   | 2.64   | 2.60   | 2.57   | 2.54   |        |
| 13    | 4.67  | 3.81   | 3.41   | 3.18   | 3.03   | 2.92   | 2.83   | 2.77   | 2.71   | 2.67   | 2.60   | 2.55   | 2.51   | 2.48   | 2.46   |        |
| 14    | 4.60  | 3.74   | 3.34   | 3.11   | 2.96   | 2.85   | 2.76   | 2.70   | 2.65   | 2.60   | 2.53   | 2.48   | 2.44   | 2.41   | 2.39   |        |
| 15    | 4.54  | 3.68   | 3.29   | 3.06   | 2.90   | 2.79   | 2.71   | 2.64   | 2.59   | 2.54   | 2.48   | 2.42   | 2.38   | 2.35   | 2.33   |        |
| 16    | 4.49  | 3.63   | 3.24   | 3.01   | 2.85   | 2.74   | 2.66   | 2.59   | 2.54   | 2.49   | 2.42   | 2.37   | 2.33   | 2.30   | 2.28   |        |
| 17    | 4.45  | 3.59   | 3.20   | 2.96   | 2.81   | 2.70   | 2.61   | 2.55   | 2.49   | 2.45   | 2.38   | 2.33   | 2.29   | 2.26   | 2.23   |        |
| 18    | 4.41  | 3.55   | 3.16   | 2.93   | 2.77   | 2.66   | 2.58   | 2.51   | 2.46   | 2.41   | 2.34   | 2.29   | 2.25   | 2.22   | 2.19   |        |
| 19    | 4.38  | 3.52   | 3.13   | 2.90   | 2.74   | 2.63   | 2.54   | 2.48   | 2.42   | 2.38   | 2.31   | 2.26   | 2.21   | 2.18   | 2.16   |        |
| 20    | 4.35  | 3.49   | 3.10   | 2.87   | 2.71   | 2.60   | 2.51   | 2.45   | 2.39   | 2.35   | 2.28   | 2.22   | 2.18   | 2.15   | 2.12   |        |
| 21    | 4.32  | 3.47   | 3.07   | 2.84   | 2.68   | 2.57   | 2.49   | 2.42   | 2.37   | 2.32   | 2.25   | 2.20   | 2.16   | 2.12   | 2.10   |        |
| 22    | 4.30  | 3.44   | 3.05   | 2.82   | 2.66   | 2.55   | 2.46   | 2.40   | 2.34   | 2.30   | 2.23   | 2.17   | 2.13   | 2.10   | 2.07   |        |
| 23    | 4.28  | 3.42   | 3.03   | 2.80   | 2.64   | 2.53   | 2.44   | 2.37   | 2.32   | 2.27   | 2.20   | 2.15   | 2.11   | 2.08   | 2.05   |        |
| 24    | 4.26  | 3.40   | 3.01   | 2.78   | 2.62   | 2.51   | 2.42   | 2.36   | 2.30   | 2.25   | 2.18   | 2.13   | 2.09   | 2.05   | 2.03   |        |
| 25    | 4.24  | 3.39   | 2.99   | 2.76   | 2.60   | 2.49   | 2.40   | 2.34   | 2.28   | 2.24   | 2.16   | 2.11   | 2.07   | 2.04   | 2.01   |        |
| 26    | 4.22  | 3.37   | 2.98   | 2.74   | 2.59   | 2.47   | 2.39   | 2.32   | 2.27   | 2.22   | 2.15   | 2.09   | 2.05   | 2.02   | 1.99   |        |
| 27    | 4.21  | 3.35   | 2.96   | 2.73   | 2.57   | 2.46   | 2.37   | 2.31   | 2.25   | 2.20   | 2.13   | 2.08   | 2.04   | 2.00   | 1.97   |        |
| 28    | 4.20  | 3.34   | 2.95   | 2.71   | 2.56   | 2.45   | 2.36   | 2.29   | 2.24   | 2.19   | 2.12   | 2.06   | 2.02   | 1.99   | 1.96   |        |
| 29    | 4.18  | 3.33   | 2.93   | 2.70   | 2.55   | 2.43   | 2.35   | 2.28   | 2.22   | 2.18   | 2.10   | 2.05   | 2.01   | 1.97   | 1.94   |        |
| 30    | 4.17  | 3.32   | 2.92   | 2.69   | 2.53   | 2.42   | 2.33   | 2.27   | 2.21   | 2.16   | 2.09   | 2.04   | 1.99   | 1.96   | 1.93   |        |
| 35    | 4.12  | 3.27   | 2.87   | 2.64   | 2.49   | 2.37   | 2.29   | 2.22   | 2.16   | 2.11   | 2.04   | 1.99   | 1.94   | 1.91   | 1.88   |        |
| 40    | 4.08  | 3.23   | 2.84   | 2.61   | 2.45   | 2.34   | 2.25   | 2.18   | 2.12   | 2.08   | 2.00   | 1.95   | 1.90   | 1.87   | 1.84   |        |
| 50    | 4.03  | 3.18   | 2.79   | 2.56   | 2.40   | 2.29   | 2.20   | 2.13   | 2.07   | 2.03   | 1.95   | 1.89   | 1.85   | 1.81   | 1.78   |        |
| 60    | 4.00  | 3.15   | 2.76   | 2.53   | 2.37   | 2.25   | 2.17   | 2.10   | 2.04   | 1.99   | 1.92   | 1.86   | 1.82   | 1.78   | 1.75   |        |
| 70    | 3.98  | 3.13   | 2.74   | 2.50   | 2.35   | 2.23   | 2.14   | 2.07   | 2.02   | 1.97   | 1.89   | 1.84   | 1.79   | 1.75   | 1.72   |        |
| 80    | 3.96  | 3.11   | 2.72   | 2.49   | 2.33   | 2.21   | 2.13   | 2.06   | 2.00   | 1.95   | 1.88   | 1.82   | 1.77   | 1.73   | 1.70   |        |
| 90    | 3.95  | 3.10   | 2.71   | 2.47   | 2.32   | 2.20   | 2.11   | 2.04   | 1.99   | 1.94   | 1.86   | 1.80   | 1.76   | 1.72   | 1.69   |        |
| 100   | 3.94  | 3.09   | 2.70   | 2.46   | 2.31   | 2.19   | 2.10   | 2.03   | 1.97   | 1.93   | 1.85   | 1.79   | 1.75   | 1.71   | 1.68   |        |
| 120   | 3.92  | 3.07   | 2.68   | 2.45   | 2.29   | 2.18   | 2.09   | 2.02   | 1.96   | 1.91   | 1.83   | 1.78   | 1.73   | 1.69   | 1.66   |        |
| 150   | 3.90  | 3.06   | 2.66   | 2.43   | 2.27   | 2.16   | 2.07   | 2.00   | 1.94   | 1.89   | 1.82   | 1.76   | 1.71   | 1.67   | 1.64   |        |

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|                        | Title                              | Link  |
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| Descriptive statistics | Variance                           | <a href="https://www.youtube.com/watch?v=JgMFhKi6f6Y">https://www.youtube.com/watch?v=JgMFhKi6f6Y</a>   |
|                        | Quantiles                          | <a href="https://docs.eggplantsoftware.com/epp/9.4.0/analyzer/analyzer-understanding-charts-events.htm">https://docs.eggplantsoftware.com/epp/9.4.0/analyzer/analyzer-understanding-charts-events.htm</a> |
|                        | Python Data Visualization Tutorial | <a href="https://www.youtube.com/watch?v=Nt84_TzRkbo">https://www.youtube.com/watch?v=Nt84_TzRkbo</a>   |

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|---------------|---|---|
| Combinatorics | Introduction to Permutations and Combinations | <a href="https://www.youtube.com/watch?v=gAnKvHmrJ0g">https://www.youtube.com/watch?v=gAnKvHmrJ0g</a> |
|               | Miracles of Pascal's triangle                 | <a href="https://www.youtube.com/watch?v=J0iINuxUcpQ">https://www.youtube.com/watch?v=J0iINuxUcpQ</a> |
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| Distributions | Uniform probability distributions | <a href="https://www.youtube.com/watch?v=aCW8wm6nrRw">https://www.youtube.com/watch?v=aCW8wm6nrRw</a> |
|               | Binomial distribution             | <a href="https://www.youtube.com/watch?v=6YzrVUV09M0">https://www.youtube.com/watch?v=6YzrVUV09M0</a> |
|               | Chi^2                             | <a href="https://www.youtube.com/watch?v=JCOeytq7F0E">https://www.youtube.com/watch?v=JCOeytq7F0E</a> |

|                         | Title  | Link  |
|-------------------------|--|---|
| Conditional probability | Bayes' Theorem of Probability With Tree Diagrams & Venn Diagrams | <a href="https://www.youtube.com/watch?v=OByl4RJxnKA">https://www.youtube.com/watch?v=OByl4RJxnKA</a>   |
|                         | What is the Bayes' Theorem?                                      | <a href="https://medium.com/mlearning-ai/what-is-the-bayes-theorem-545a2ef0b91c">https://medium.com/mlearning-ai/what-is-the-bayes-theorem-545a2ef0b91c</a>           |
|                         | Data Science. Bayes theorem                                      | <a href="https://luminousmen.com/post/data-science-bayes-theorem">https://luminousmen.com/post/data-science-bayes-theorem</a>   |
|                         | Bayesian Machine Learning in Python: A/B Testing                 | <a href="https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing">https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing</a> |

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| Sampling           | Central limit theorem                                   | <a href="https://www.geeksforgeeks.org/python-central-limit-theorem">https://www.geeksforgeeks.org/python-central-limit-theorem</a>   |
|                    | Confidence intervals                                    | <a href="https://data.library.virginia.edu/the-intuition-behind-confidence-intervals/">https://data.library.virginia.edu/the-intuition-behind-confidence-intervals/</a>   |
|                    | Resampling  | <a href="https://www.linkedin.com/pulse/resampling-methods-pranshu-jaryal">https://www.linkedin.com/pulse/resampling-methods-pranshu-jaryal</a>   |
|                    | Bootstrapping   | <a href="https://www.youtube.com/watch?v=9IFcRTyhd5Y">https://www.youtube.com/watch?v=9IFcRTyhd5Y</a>   |
|                    |   | <a href="https://dgarcia-eu.github.io/SocialDataScience/2_SocialDynamics/025_Bootstrapping/Bootstrapping.html">https://dgarcia-eu.github.io/SocialDataScience/2_SocialDynamics/025_Bootstrapping/Bootstrapping.html</a> |
|                    |   | <a href="https://janhove.github.io/teaching/2016/12/20/bootstrapping">https://janhove.github.io/teaching/2016/12/20/bootstrapping</a>   |
|                    | Title   | Link  |
| Hypothesis testing | Bayesian Machine Learning in Python: A/B Testing        | <a href="https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing">https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing</a>   |
|                    | Introduction to the Chi-square Test                     | <a href="https://www.youtube.com/watch?v=SvKv375sacA">https://www.youtube.com/watch?v=SvKv375sacA</a>   |
|                    | F-ratio Test for Two Equal Variances                    | <a href="https://www.youtube.com/watch?v=UWQO4gX7-IE">https://www.youtube.com/watch?v=UWQO4gX7-IE</a>   |
|                    | 17 Statistical Hypothesis Tests in Python (Cheat Sheet) | <a href="https://machinelearningmastery.com/statistical-hypothesis-tests-in-python-cheat-sheet/">https://machinelearningmastery.com/statistical-hypothesis-tests-in-python-cheat-sheet/</a>                             |
|                    | Chi Squared Testing                                     | <a href="https://www.youtube.com/watch?v=qYOM083ZIWU">https://www.youtube.com/watch?v=qYOM083ZIWU</a>   |
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| Model estimation   | Scatter matrix , Covariance and Correlation Explained                    | <a href="https://medium.com/@raghavan990/scatter-matrix-covariance-and-correlation-explained-14921741ca56">https://medium.com/@raghavan990/scatter-matrix-covariance-and-correlation-explained-14921741ca56</a>   |
|                    | Logistic regression sigmoid function and threshold                       | <a href="https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79">https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79</a>   |
|                    | Introduction to logistic regression                                      | <a href="https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.">https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.</a> |
|                    | Linear Regression Diagnostics  | <a href="https://www.youtube.com/watch?v=HLAglyBfNk8&amp;list=PLlbbWgBRF8EePgK40-i7aGU2_kylyujgL&amp;index=11">https://www.youtube.com/watch?v=HLAglyBfNk8&amp;list=PLlbbWgBRF8EePgK40-i7aGU2_kylyujgL&amp;index=11</a>   |
|                    | Machine Learning Classifier evaluation using ROC and CAP Curves          | <a href="https://towardsdatascience.com/machine-learning-classifier-evaluation-using-roc-and-cap-curves-7db60fe6b716">https://towardsdatascience.com/machine-learning-classifier-evaluation-using-roc-and-cap-curves-7db60fe6b716</a>   |
|                    | Classification Model Performance Evaluation using AUC-ROC and CAP Curves | <a href="https://medium.com/geekculture/classification-model-performance-evaluation-using-auc-roc-and-cap-curves-66a1b3fc0480">https://medium.com/geekculture/classification-model-performance-evaluation-using-auc-roc-and-cap-curves-66a1b3fc0480</a>   |
|                    | Deep Learning Prerequisites: Logistic Regression in Python               | <a href="https://www.udemy.com/course/data-science-logistic-regression-in-python/">https://www.udemy.com/course/data-science-logistic-regression-in-python/</a>   |
|                    |  | <a href="https://en.wikipedia.org/wiki/Correlation">https://en.wikipedia.org/wiki/Correlation</a>   |
|                    | Ordinary Least Squares   | <a href="https://gregorygundersen.com/blog/2020/01/04/ols/">https://gregorygundersen.com/blog/2020/01/04/ols/</a>   |
|                    | Scatter matrix , Covariance and Correlation Explained                    | <a href="https://medium.com/@raghavan990/scatter-matrix-covariance-and-correlation-explained-14921741ca56">https://medium.com/@raghavan990/scatter-matrix-covariance-and-correlation-explained-14921741ca56</a>   |
| Model selection    | Logistic regression sigmoid function and threshold                       | <a href="https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79">https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79</a>   |
|                    | Introduction to logistic regression                                      | <a href="https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.">https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.</a> |
| Model optimization | Linear Regression Diagnostics  | <a href="https://www.youtube.com/watch?v=HLAglyBfNk8&amp;list=PLlbbWgBRF8EePgK40-i7aGU2_kylyujgL&amp;index=11">https://www.youtube.com/watch?v=HLAglyBfNk8&amp;list=PLlbbWgBRF8EePgK40-i7aGU2_kylyujgL&amp;index=11</a>   |
|                    |  |   |



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