DATA SCIENCE. INSTITUTE by Fabian Rappert

Statistics for Data Science

Dr. Harald Stein

Aug 2024

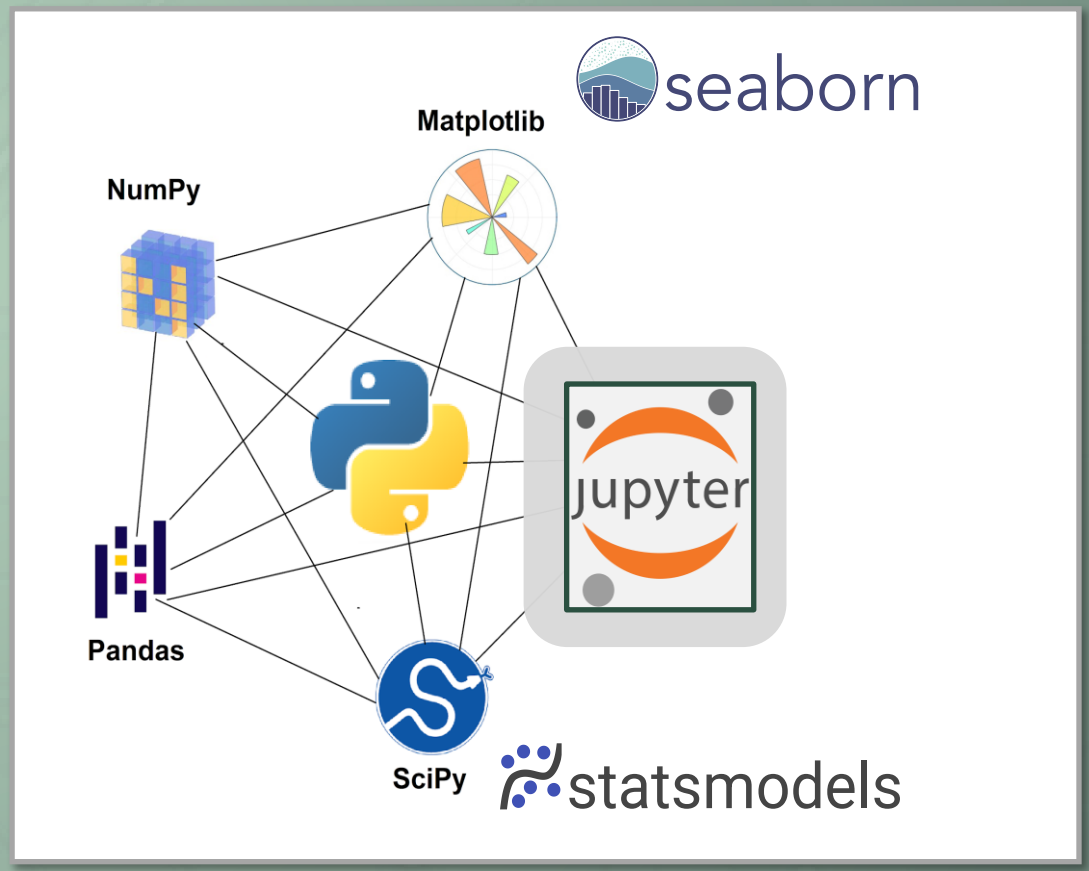
Content

- Technical Prerequisites
- Probability, Statistics and Information Theory
- Descriptive Statistics
- Combinatorics
- Joint and Conditional Probability
- Probability Distributions
- Structured Probabilistic Models
- Sampling, Monte Carlo
- Hypothesis Testing
- Model Estimation

Technical Prerequisites

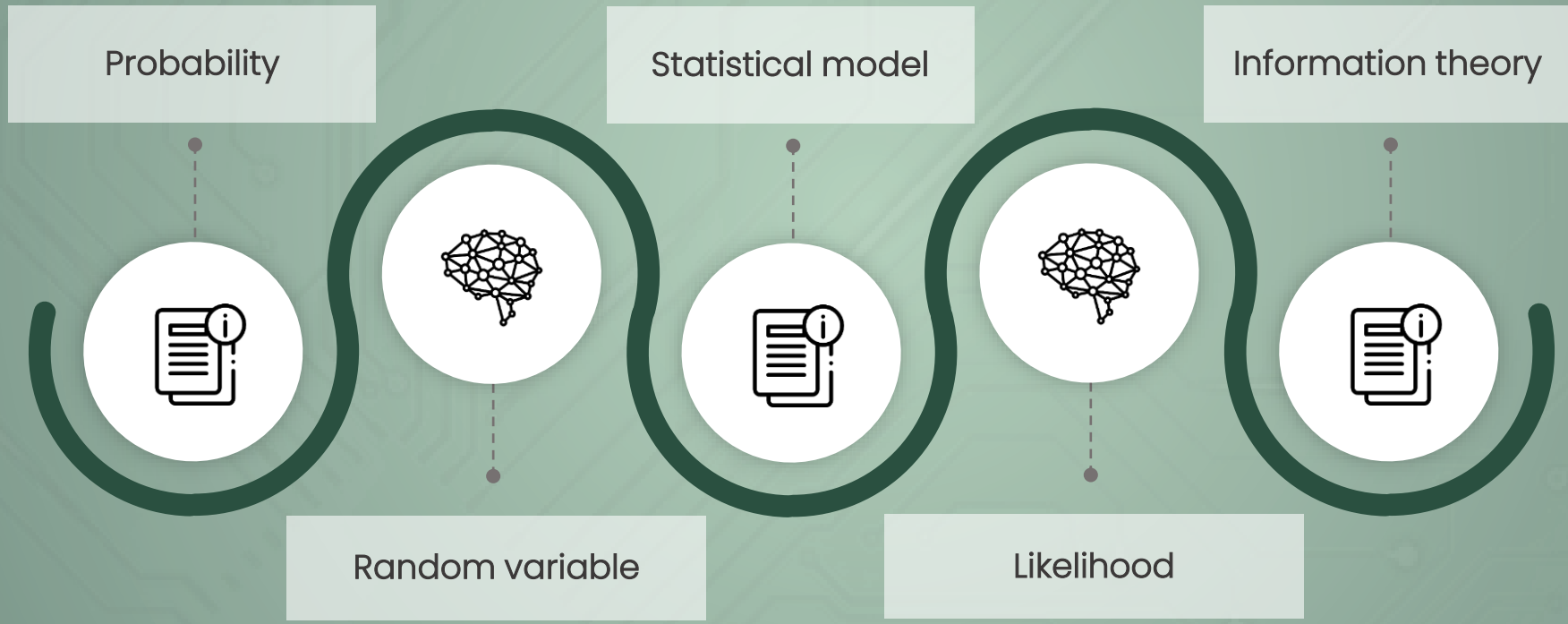
… means installation of Python environment with Jupyter

- GitHub Link:
  https://github.com/MathforDataScience/DSR_Statistics1/
- List of packages:
  requirements.txt
- How to install:
  readme.txt
- Start Jupyter Notebook



Probability, Statistical Models and Information Theory

Probability provides foundation for statistical models, which use probability to analyze data, make predictions, while information theory quantifies uncertainty and information content within those models.



What is Probability?

Probability is measure quantifying the chance that random events will occur. It is expressed as number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty.

Example

- Tossing a fair (unbiased) coin:
  - Two possible outcomes: "heads" and "tails“
  - Probability of "heads" = Probability of "tails" = 0.5 or 50%
  - Random event

Application Areas

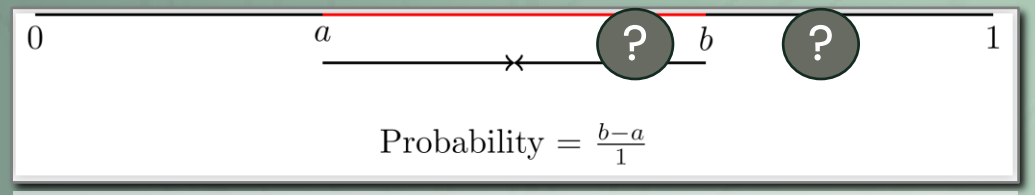
- Mathematics, Statistics, Finance, Gambling
- Science (including Physics)
- Artificial Intelligence/Machine Learning
- Computer Science, Game Theory, Philosophy

Purpose of Probability

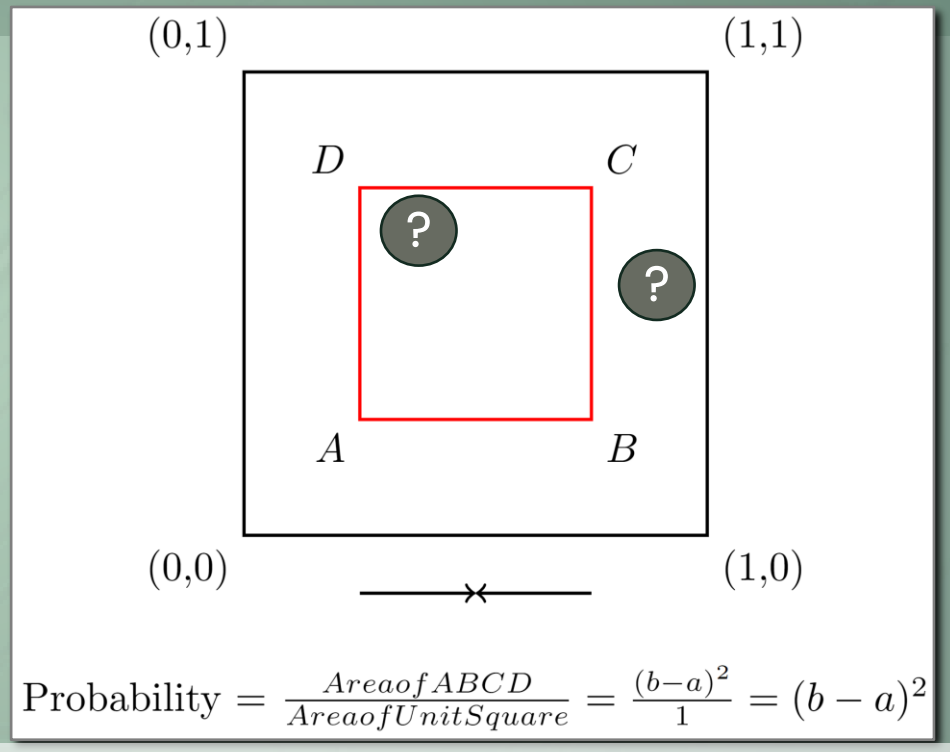
- Used to infer the expected frequency of events.
- Describes the underlying mechanics and regularities of complex systems.

Geometric Interpretation of Probability

… visualizes probabilities as areas (or volumes) within geometric space. Probability is proportional to size of favorable region relative to total possible region.



- Consider a point chosen at random on a line segment of length 1
- The probability that the point falls within the sub-segment is $\frac{b-a}{1}$



- Suppose a point is randomly selected within a unit square.
- The probability that the point lies within specific region (e.g., smaller square or circle) is equal to area of that region divided by area of the unit square (1)

Random Variable

… is variable that takes on numerical values determined by the outcome of random experiment. It maps outcomes of random process to numbers.

Aspect	Discrete Random Variable	Continuous Random Variable
Definition	Takes on a countable number of distinct values.	Takes on an infinite number of possible values within a given range.
Examples	Number of heads in 10 coin flips, number of students in a class.	Height of students, time to run a marathon.
Possible Values	Finite or countably infinite set (e.g., {0, 1, 2, …}).	Any value within a continuous range (e.g., [0, ∞), (−∞, ∞)).
Probability Distribution	Described by a probability mass function (PMF).	Described by a probability density function (PDF).
Cumulative Distribution	Step function with jumps at each possible value.	Smooth curve, the integral of the PDF.
Sum of Probabilities	Sum of probabilities for all possible values equals 1.	The area under the PDF curve equals 1.
Graphical Representation	Histogram or bar chart	Continuous curve (e.g., bell curve for normal distribution).
Real-World Analogy	Counting the number of occurrences (e.g., number of cars passing).	Measuring continuous quantities (e.g., measuring the exact weight).

Examples of Random Variables

Discrete vs. continuous

Type of Random Variable	Example	Description
Discrete Random Variable	Number of Students in a Classroom	Can only take specific integer values (e.g., 20, 21, 22).
	Number of Cars Passing Through a Toll Booth	Count of cars, taking on values like 0, 1, 2, etc.
	Number of Emails Received in a Day	Exact count of emails received (e.g., 5 emails).
	Number of Defective Items in a Batch	Number of defective items, counted in whole numbers (e.g., 0, 1, 2)
Continuous Random Variable	Height of Students in a Class	Can take any value within a range, measured in units like centimeters
	Time Taken to Run a Marathon	Measured in hours, minutes, and seconds with infinite possible values.
	Temperature Throughout the Day	Can be any value within a temperature range, measured in degrees (e.g., 25.3°C).
	Amount of Milk in a Bottle	Volume can be measured to any degree of precision (e.g., 1.5 liters).

Statistical Model

… is mathematical framework that represents relationships between variables in data and allows for analysis, inference, and prediction

Variables:

- Dependent Variable: The outcome or response being predicted or explained.
- Independent Variables: The predictors or inputs that influence the dependent variable.

Parameters

- Constants estimated from the data that define the model's specific form.
- Through estimated parameters the statistical model becomes generalization rule for the data sample referring to population

Probability Distributions

Assumptions about how data is distributed (e.g., normal distribution).

Example: Linear Regression $Y = \beta_0 + \beta_1 X + \varepsilon$

- Y: Outcome, dependent variable
- X: Predictor, independent variable
- β_0, β_1: Parameters that are estimated
- ε: Random error, unpredictable part of model

Likelihood vs Probability

Likelihood measures how well a set of parameters explains observed data.
Probability measures the chance of a specific event occurring.

Aspect	Probability	Likelihood
Definition	The measure of the chance that a specific event will occur.	The measure of how likely a particular set of parameters is, given the observed data.
Context	Used to describe the probability of future events based on known conditions.	Used in statistical models to estimate parameters based on observed data.
Focus	Focuses on predicting the occurrence of outcomes.	Focuses on finding the best parameters that explain the observed data.
Example	Probability of rolling a 6 on a fair die is 1/6.	Likelihood of a die being biased towards 6 given repeated observations of rolling a 6.
Mathematical Form	Expressed as $P(A)$, where A is the event.	Expressed as $L(\theta \mid X)$, where θ are parameters and X is data.
Use in Inference	Probability helps to infer about future events or outcomes.	Likelihood is used to estimate the parameters of a statistical model.

→ Statistical model is good predictor, i.e. generalization of data, if parameters are approximated in a way that likelihood is maximized.

https://www.youtube.com/watch?v=pYxNSUDSFH4

Information Theory

… is branch of applied mathematics focused on quantifying information in signals.
Main idea: Learning about unlikely event is more informative than learning about likely event.

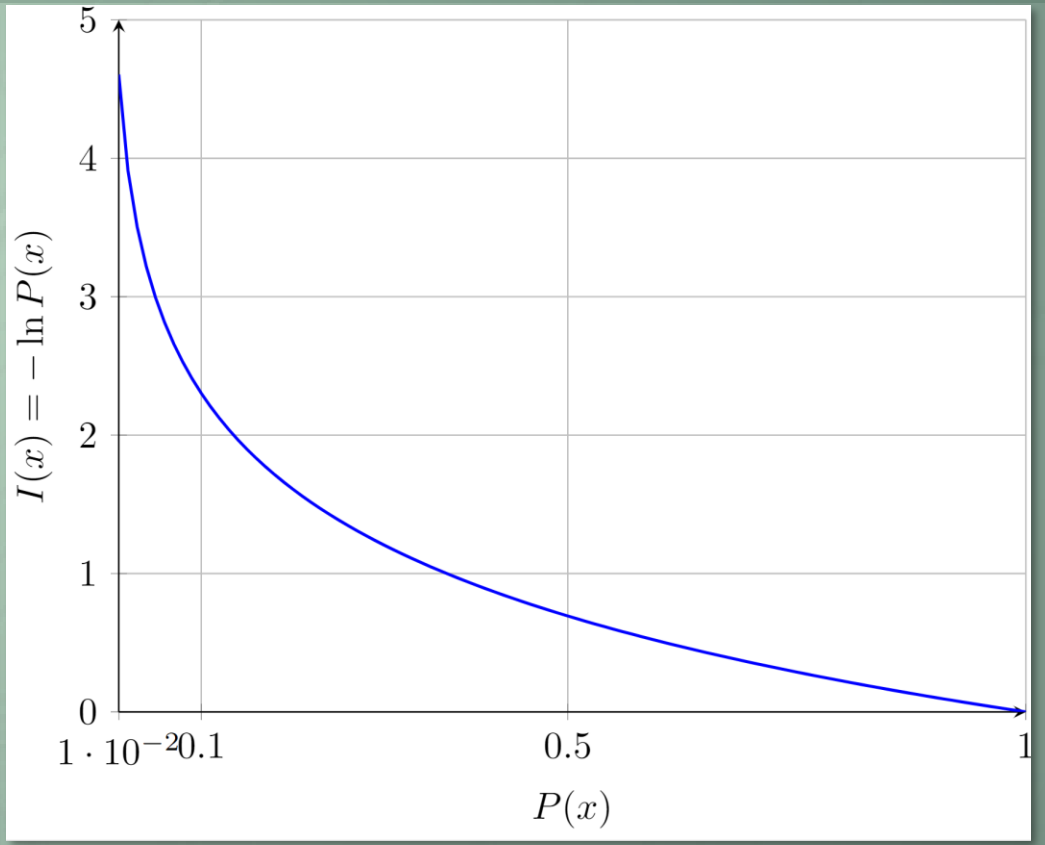
Key Properties of Information:

- Low frequency events: Have high information content.
- High frequency events: Have low information content.
- Independent events: Information should be additive.

Self-Information Formula

- mathematical way to quantify how much information is gained from observing outcome of random event.
- Rare events are more informative, surprising than common ones

$$I(x) = -\ln P(x)$$



https://www.deeplearningbook.org/contents/prob.html

Shannon Entropy

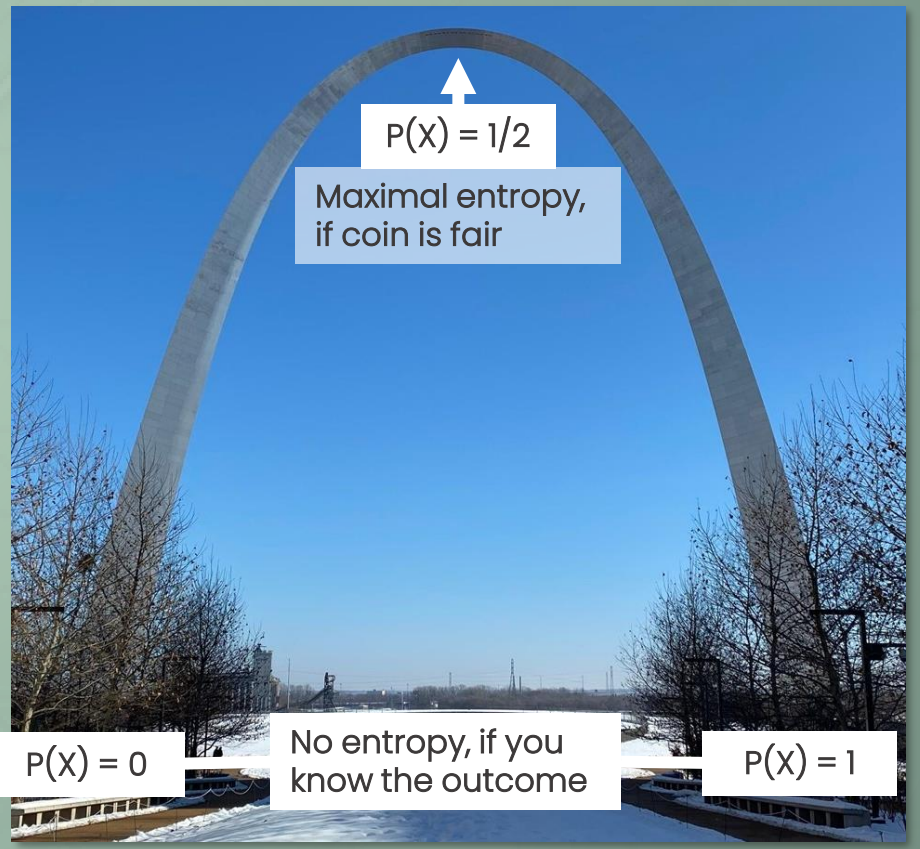
… is measure of randomness in set of possible outcomes. It quantifies expected amount of information/surprise, that event drawn from a probability distribution will provide.

- Weighted average of self-information, i.e. $-\ln P(x)$ of all possible outcomes.
- Represents level of uncertainty/surprise inherent in outcome, examples:
  - High Entropy:
    Fair six-sided die has higher entropy than biased die where one number appears with very high probability.
  - Low Entropy:
    If die is heavily biased towards one particular number, entropy is low.

Formula

$$H(X) = -\sum_{i=1}^{n} P(x_i) \ln P(x_i)$$

→ If any of the terms $P(x_i)$, $\ln P(x_i)$ become 0, then $H(X)$ becomes 0



Gateway Arch, St. Louis, USA

Shannon Entropy and Outliers

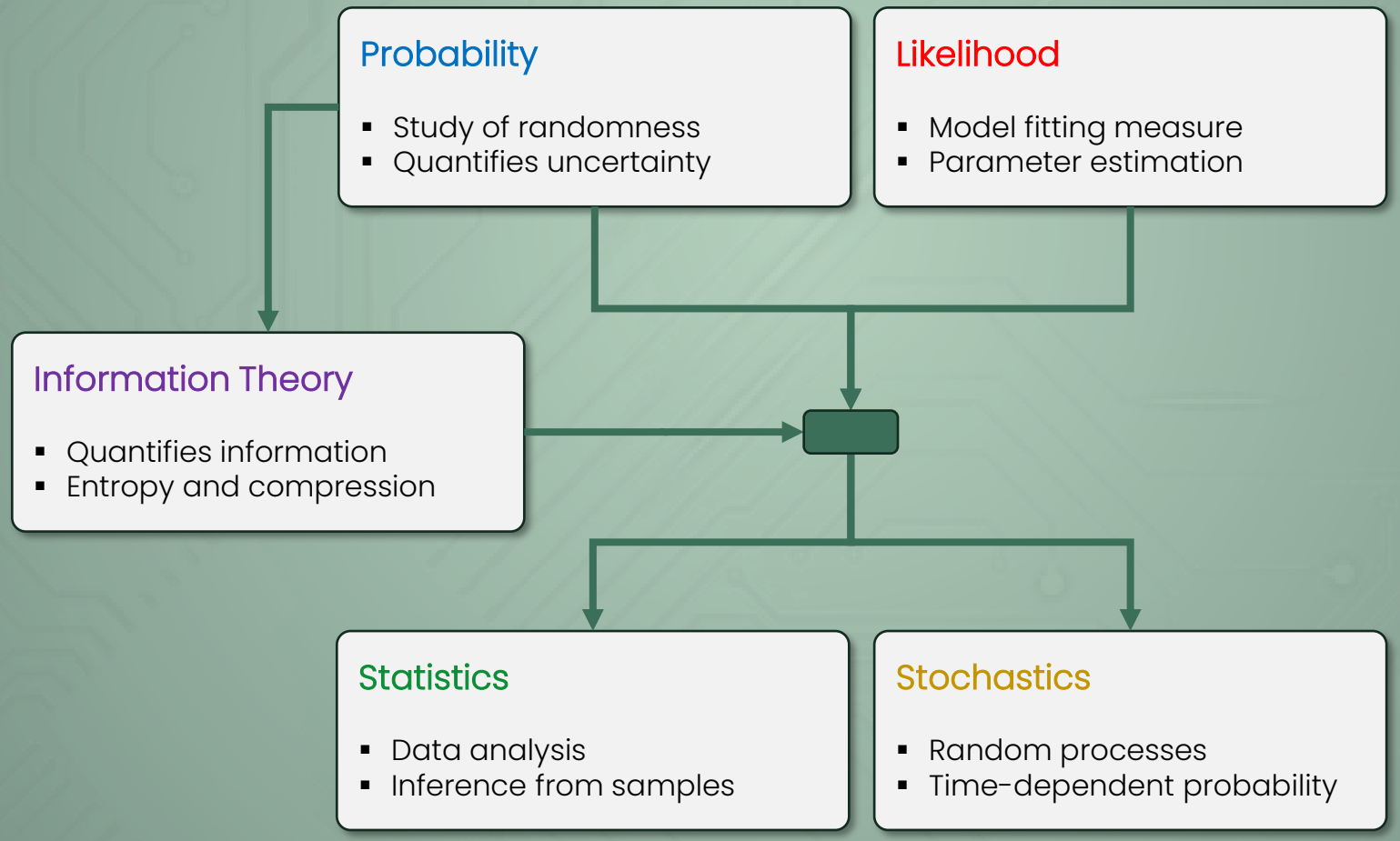
Outliers can increase the entropy if they represent rare but possible outcome, adding to the distribution's unpredictability.

Black Swan: Symbol for rare event
Book: The Black Swan, Taleb, 2007

"I have seen horses vomiting in front of pharmacy"
Translation of German phrase

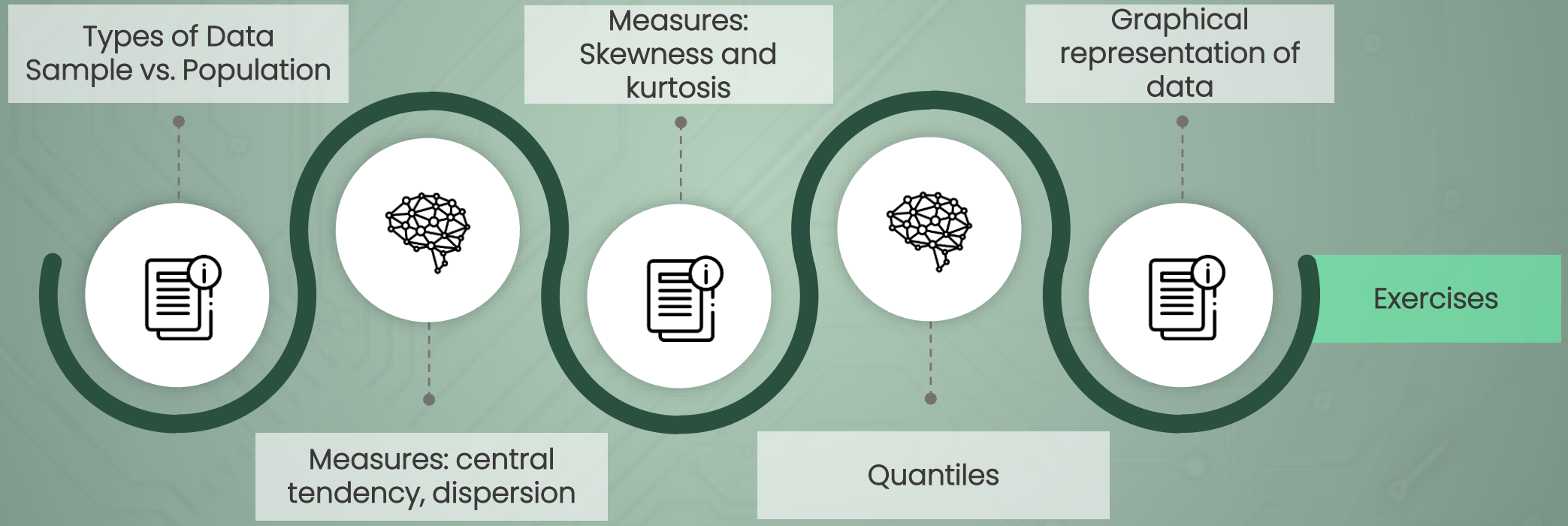
Probability, Likelihood, Information, Statistics, Stochastics

Probability quantifies uncertainty, Statistics analyzes data, Stochastics applies both to time-dependent processes, Likelihood assesses model fit, Information theory measures data content



Descriptive statistics

… involves the organization, summarization, and visualization of data. It provides simple summaries about the sample and the measures



Types of Data

Definition and examples of Qualitative and Quantitative data

Type of Data	Subtype of data	Example
Quantitative / Numerical: • can be measured • or counted	**Discrete:** • can only take certain values	• number of students in a class
	Continuous: • can take any value within certain range	• Age, Height, Weight, Temperature, Income
	Interval	• Temperature • IQ scores • Years
	Ratio	• Probability
Qualitative / Categorical: • non-numerical • often relates to subjective qualities, characteristics, or descriptions	**Nominal**: • No order, hierarchy or sequence	• Colors • Types of cuisine • Genders • Blood types
	Ordinal: • can be arranged in a particular order	• Survey responses • Educational level • Military rank

... other types of data: binary, time-series, geodata, textual, multimedia, etc.

Sample vs Population

A well-chosen sample should be representative of the population, allowing statisticians to draw accurate conclusions about the population based on sample data

Population

- Entire group of individuals or objects of interest
- Often too large to study completely
- Represented by N (size)

Sample

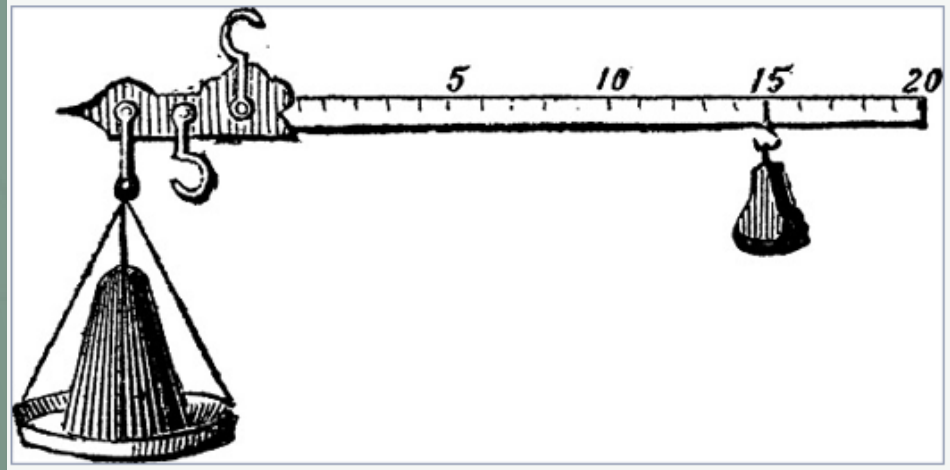
- Subset of the population
- Used to make inferences about the population
- Represented by n (size)

Measures of Central Tendency

… calculate typical central points that describe or represent dataset

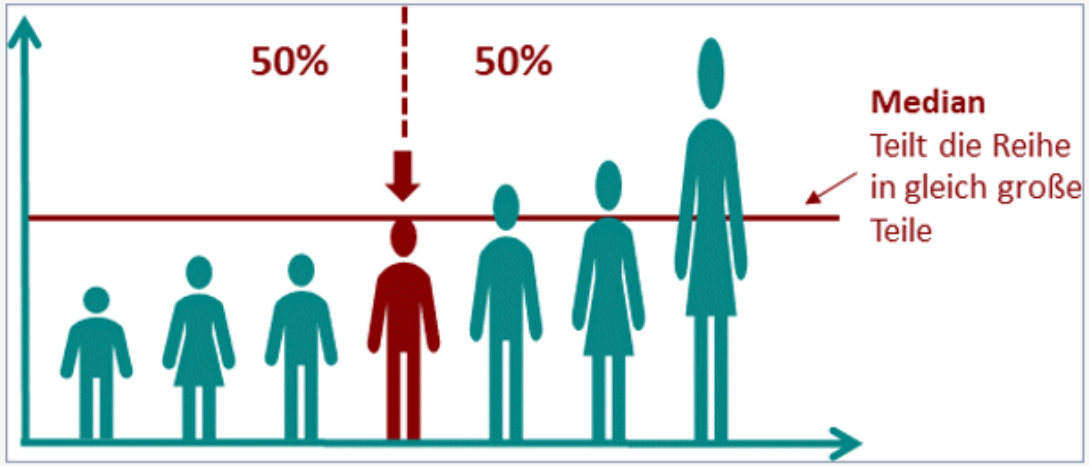
Arithmeric Mean = Average

- Sum of all data points divided by number of data points
- E.g.: (1+2+3)/3 = 2
- $\overline{X}$ = Sample Mean; μ = Population Mean



Median

- middle value in an ordered dataset
- E.g.: For {1, 2, 3}, the median is 2
- **Interesting because of outlier resistance!**



… Other measures of centrality:

- Geometric mean: e.g. population growth, investment return
- Harmonic mean: e.g. F-score (Machine Learning, binary classification, evaluation of results)

Expectation or Expected Value

Random variable's expected value represents arithmetic mean / average of large number of independent realizations of the random variable

- X is a random variable
- with finite number of outcomes ($x_1, x_2, \ldots, x_k$)
- occurring with probabilities ($p_1, p_2, \ldots, p_k$).
- For discrete variables:
  - $E[X] = \sum_{i=1}^{k} x_i p_i = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$
- For continues variables:
  - we compute it with an integral.

Example: Rolling a Fair Six-Sided Die

Let X be the random variable representing the outcome of rolling a fair six-sided die.

Outcomes (x_i): 1, 2, 3, 4, 5, 6

Probabilities (p_i): 1/6 for each outcome (since it's a fair die)

To calculate the expected value E[X], we'll use the formula:

$E[X] = \sum_{i=1}^{k} x_i p_i = x_1 p_1 + x_2 p_2 + \ldots + x_k p_k$

E[X] = 1 * (1/6) + 2 * (1/6) + 3 * (1/6) + 4 * (1/6) + 5 * (1/6) + 6 * (1/6)

= (1 + 2 + 3 + 4 + 5 + 6) / 6

= 21 / 6

= 3.5

Therefore, the expected value of rolling a fair six-sided die is 3.5.

Measures of Dispersion

… provide information about how data values are distributed around central tendency

Measure of Dispersion	Definition	Example
Range	The difference between the highest and lowest values in a dataset	For {1, 2, 3, 4}, the range is 4 - 1 = 3
Sum of Squares	$SS = \sum(x - \mu)^2$	For {1, 2, 3}, SS: (1+0+1) = 2
Variance (population)	$\sigma^2 = \frac{\sum(x-\mu)^2}{n}$	For {1, 2, 3}, pop. variance: (1+0+1)/3 = 0.67
Variance (sample)	$s^2 = \frac{\sum(x-\mu)^2}{n-1}$	For {1, 2, 3}, sample variance: (1+0+1)/2 = 1
Standard Deviation	The square root of the variance	The square root of 0.67 is approximately 0.82

Variance and Standard Deviation

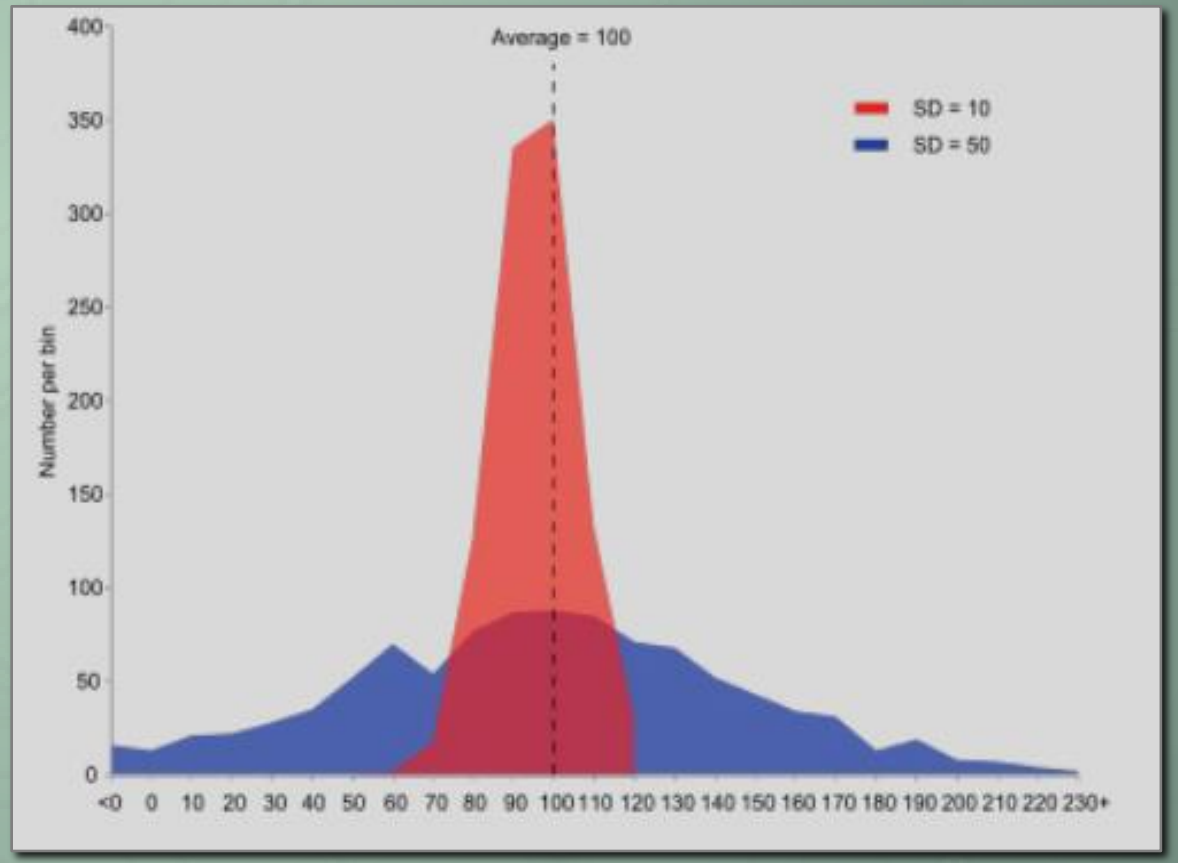
Variance $VAR(X) = E[(X - \mu)^2]$ of standard X (function) is expected value of squared deviation from mean X; $\mu = E[X]$

Example with Standard Deviation = 10

- Heights of certain species of plants
- normally distributed with mean height of 50 cm
- standard deviation of 10 cm.
- Heights are spread around mean
- Most plants' heights ranging 40 - 60 cm.

Example with Standard Deviation = 50:

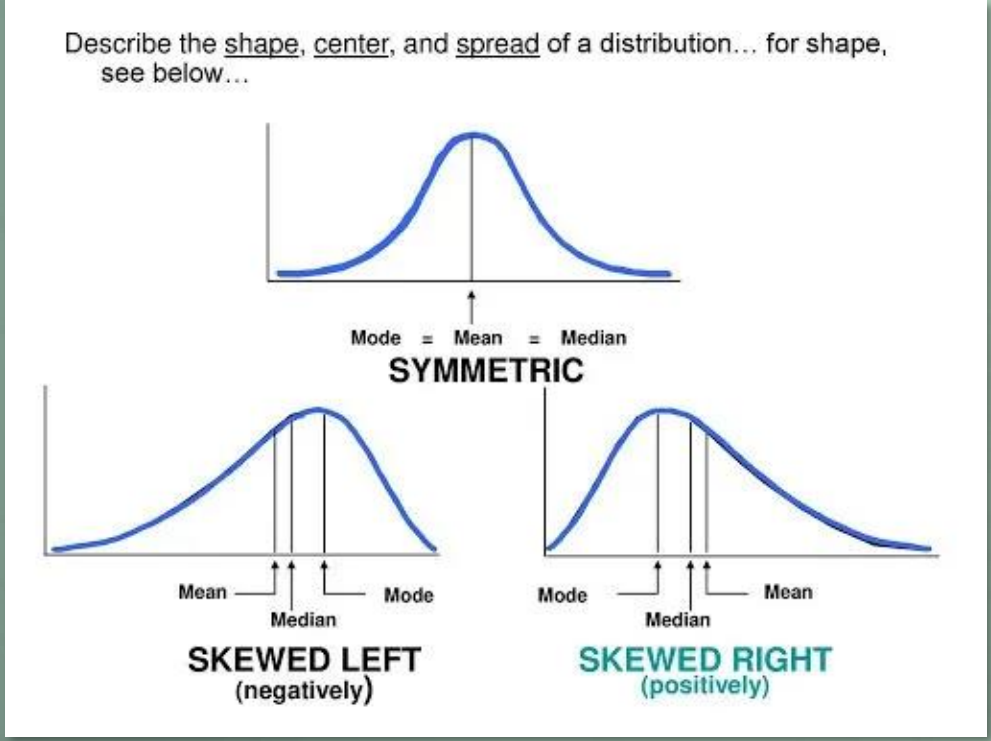
- Annual income of group of people in certain profession
- normally distributed with mean income of $200,000
- standard deviation of $50,000.
- Incomes vary widely
- most individuals earn $150,000 - $250,000.



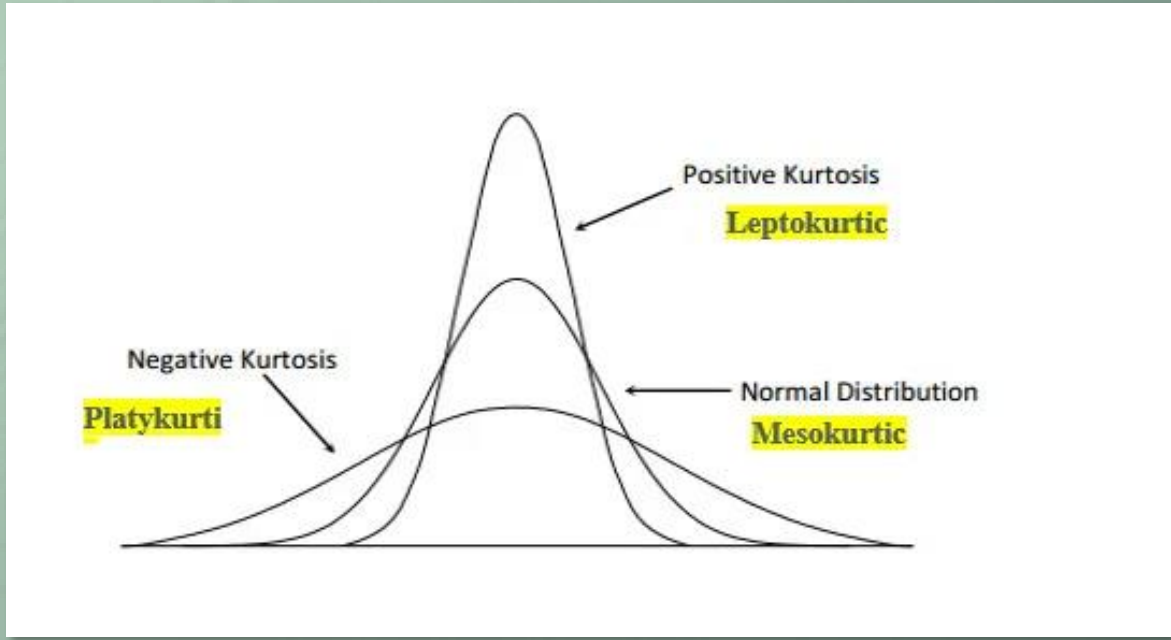
Skewness and Kurtosis

… describe asymmetry of distributions diviation from normal distribution

Skew
- Measure of asymmetry of the probability distribution
- of a real-valued random variable about its mean
- Positive (negative): Long tail on the right (left)

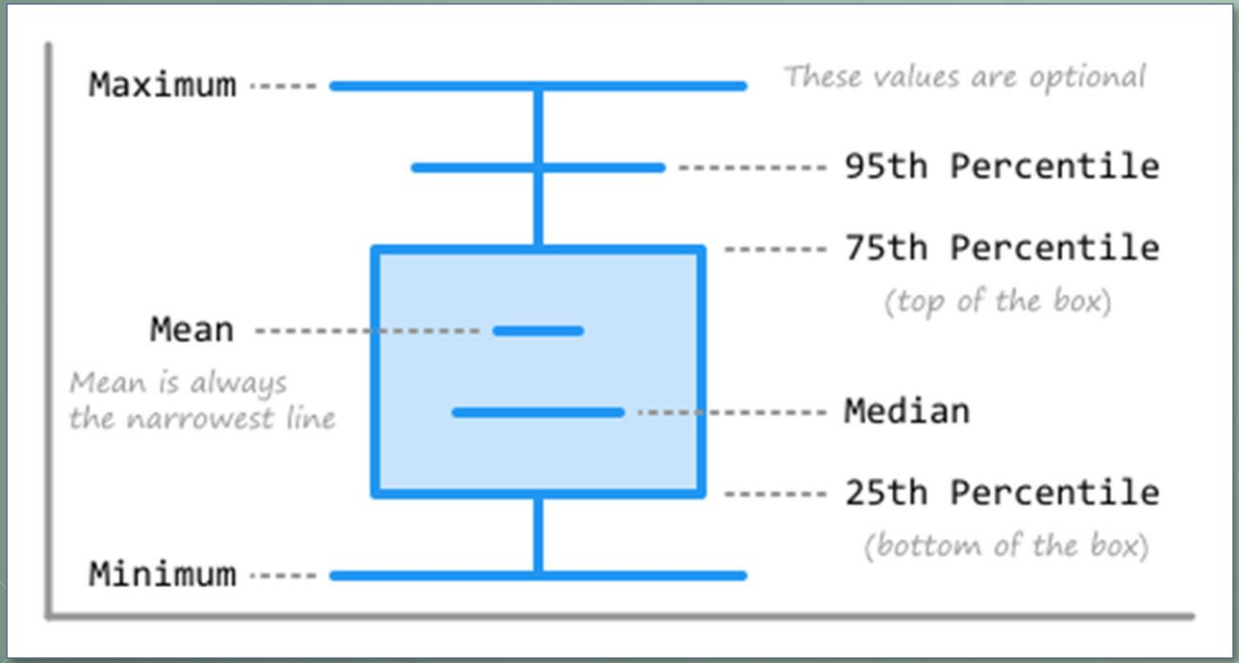


Kurtosis
- defines how heavily the tails of a distribution differ from the tails of a normal distribution
- Outliers may facilitate positive kurtosis



Quantiles

… statistical measures that divide a dataset into intervals of equal probability



Quantiles …

- are used to understand data distribution and identify outliers
- partition data into several subsets containing equal number of observations.

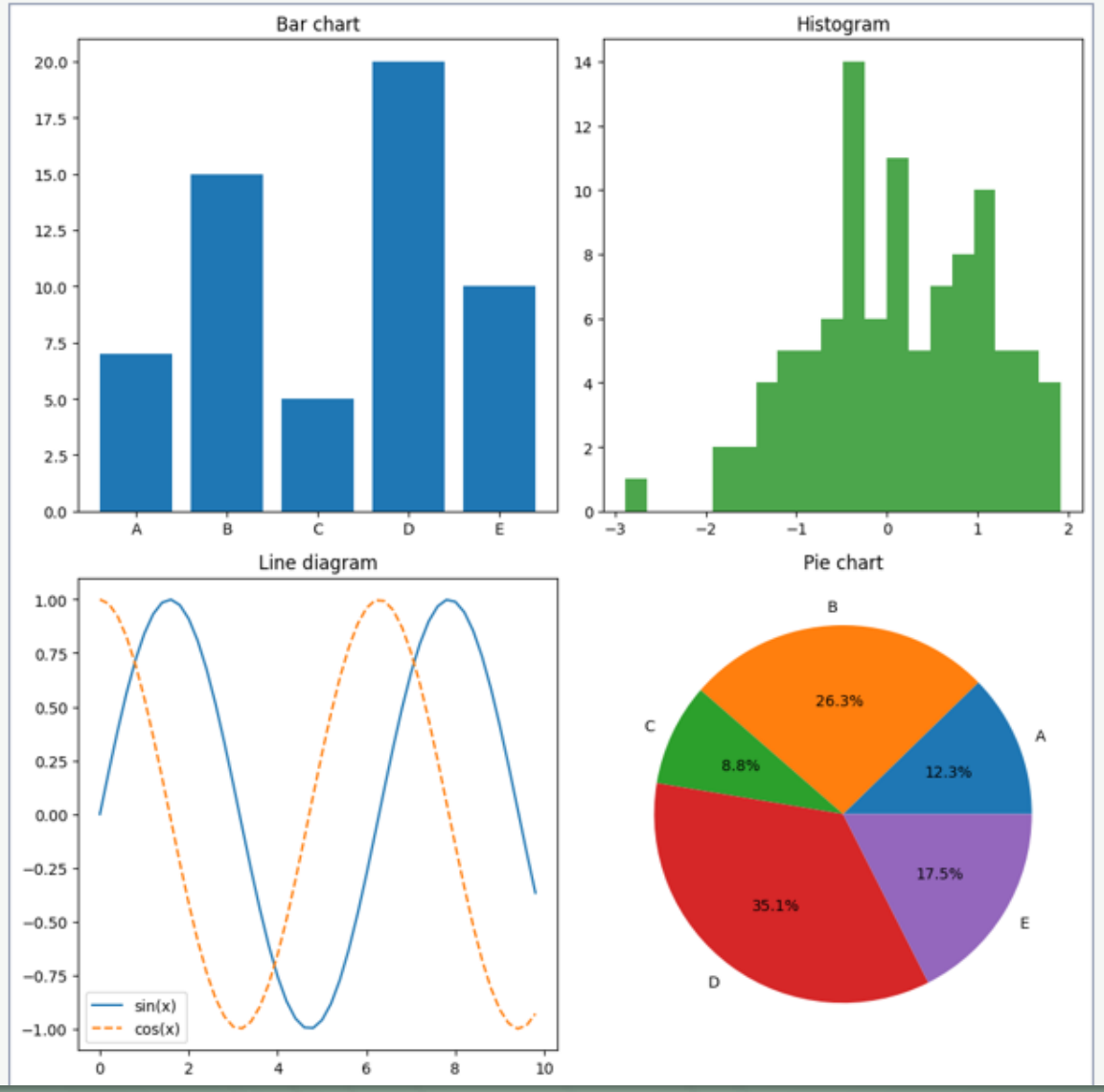
Several types of quantiles:

- Quartiles: divide the data into four equal parts.
- Deciles: These divide the data into ten equal parts.
- Percentiles: These divide the data into 100 equal parts.
- Median: separates data into two halves, with 50% of the data falling below and 50% above the median.

Graphical Representation of Data

Input:
- Categorical
- Nominal
- unordered

Output:
- Numerical
- Continuous
- Absolute



Input:
- Categorical
- Nominal
- ordered

Output:
- Numerical
- Continuous
- Absolute

Input:
- Numerical
- continuous

Output:
- Numerical
- continuous

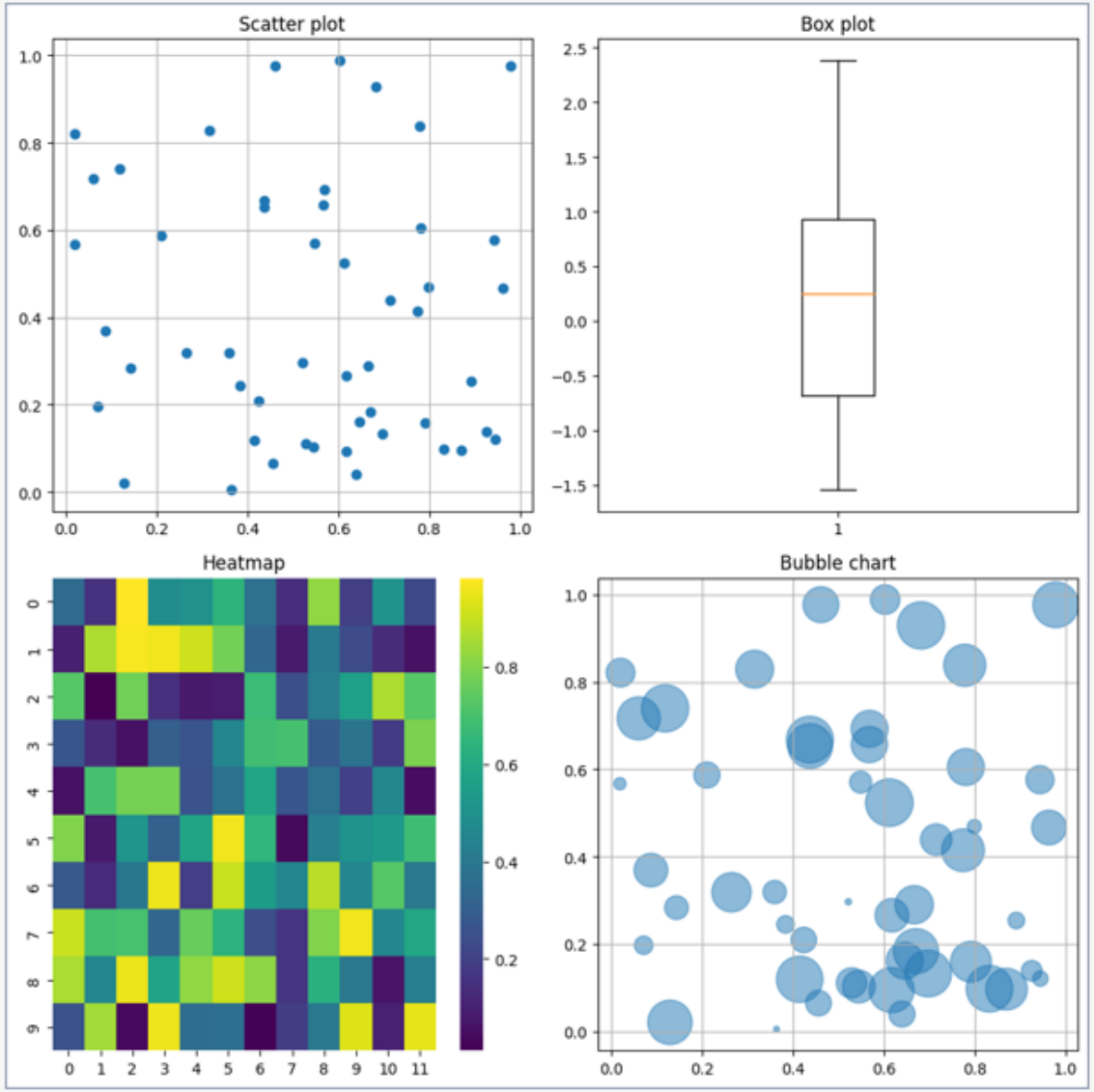
Input:
- Categorical
- Nominal
- Unordered or ordered

Output:
- Numerical
- Continuous
- relative

Graphical Representation of Data

Input:
- 2 Features
- Both numerical

Output:
- Boolean (1;0)



Input:
- 1 feature
- numerical

Output:
- At least ordered
- Or numerical

Input:
- 2 Features
- Both categorical

Output:
- Numerical

Input (like scatter plot):
- 2 Features
- Both numerical

Output:
- Numerical

Exercises

Exercise 1:
Question: Given the array numbers = np.array([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]), compute the mean of the numbers.

Exercise 2:
Question: Given the array numbers from Exercise 1, compute the median of the numbers.

Exercise 3:
Question: Compute the variance of the numbers.
a) Use population variance b) Use sample variance
ages = [21, 23, 25, 27, 29, 31, 33]
b) Let's say we take a sample of 5 ages from the original group: sample_ages = [21, 23, 27, 31, 33]

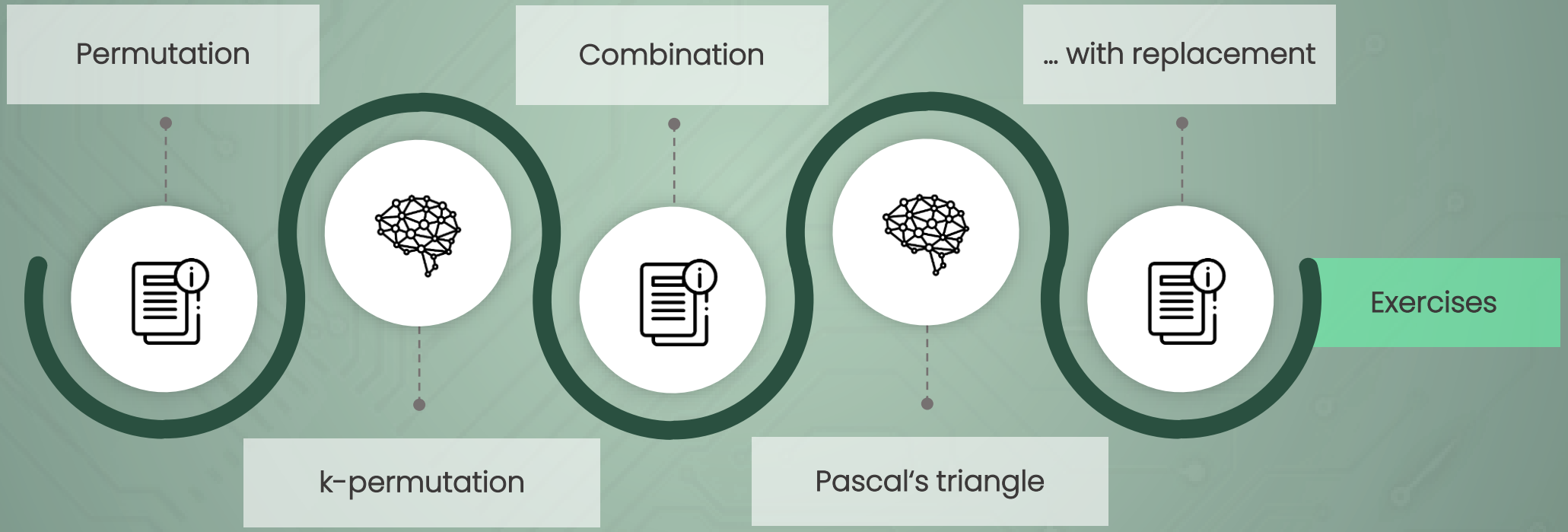
Exercise 4:
Question: Given the array numbers from Exercise 1, compute the standard deviation of the numbers.

Exercise 5:
Question: Given the array numbers from Exercise 1, compute the first quartile (25th percentile), the second quartile (50th percentile, or median), and the third quartile (75th percentile) of the numbers.

Exercise 6:
show with python plots in a 2x2 subfigure matrix for …
bar chart, histogram, line diagram, pie diagram
Scatter plot, Box plot (Box-and-whisker plot), Heatmap, bubble chart

Combinatorics

... is concerned with the study of countable, discrete structures and includes the arrangement, combination, and permutation of sets.



Permutation

… is an arrangement of objects in a specific order.

Table 5-1 Possible Rearrangements for Four People Sitting in a Straight Line

Rearrangement #	*Listing*	*Rearrangement #*	*Listing*
1	Tim, Syd, Elena, Mark	13	Elena, Tim, Syd, Mark
2	Tim, Syd, Mark, Elena	14	Elena, Tim, Mark, Syd
3	Tim, Elena, Syd, Mark	15	Elena, Syd, Tim, Mark
4	Tim, Elena, Mark, Syd	16	Elena, Syd, Mark, Tim
5	Tim, Mark, Syd, Elena	17	Elena, Mark, Tim, Syd
6	Tim, Mark, Elena, Syd	18	Elena, Mark, Syd, Tim
7	Syd, Tim, Elena, Mark	19	Mark, Tim, Syd, Elena
8	Syd, Tim, Mark, Elena	20	Mark, Tim, Elena, Syd
9	Syd, Elena, Tim, Mark	21	Mark, Syd, Elena, Tim
10	Syd, Elena, Mark, Tim	22	Mark, Syd, Tim, Elena
11	Syd, Mark, Tim, Elena	23	Mark, Elena, Tim, Syd
12	Syd, Mark, Elena, Tim	24	Mark, Elena, Syd, Tim

The number of permutations

- without repetition
- of a set of n objects is given by:

$$P(n) = n!$$

For $n = 4$ there are

$$P(4) = 4! = 4 * 3 * 2 * 1 = 24$$

possibilities.

Here: No replacement

Permutation

… is an arrangement of objects in a specific order.

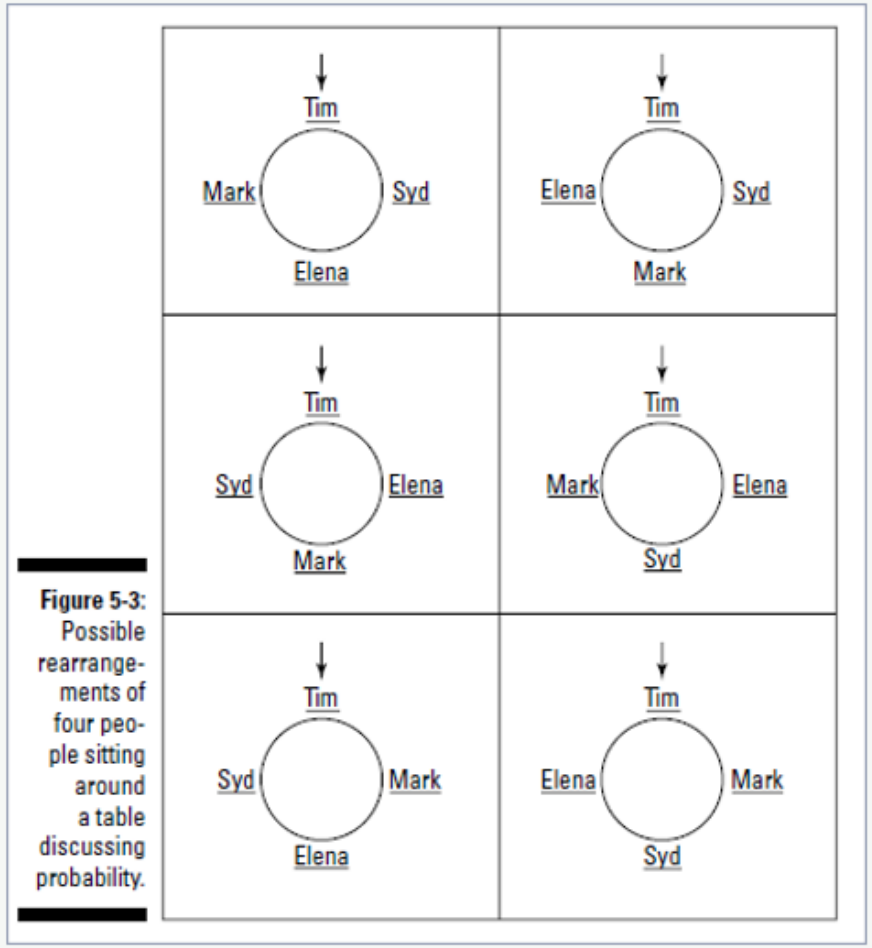


Figure 5-3: Possible rearrangements of four people sitting around a table discussing probability.

Example

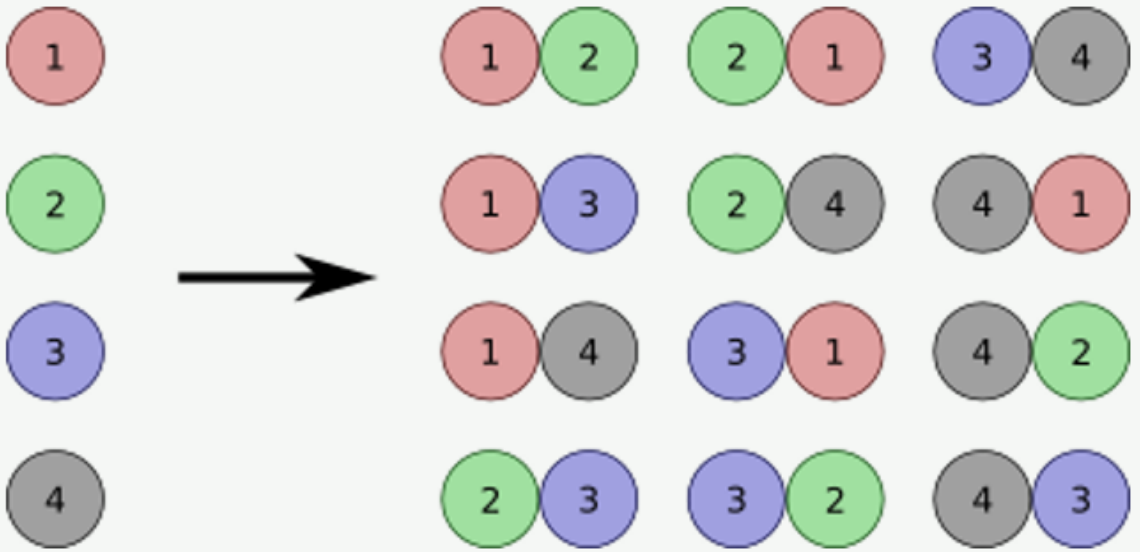
- 4 people sit at a table
- Restriction: The position of one of them is fixed.
- How many possible arrangements are there?

$$P(3) = 3! = 3 * 2 * 1 = 6$$

Here: No replacement

K-permutation

… is an ordered arrangement of k elements selected from that set. In German it is called “Variation”



File:Kpermutation.png - Wikimedia Commons

Here: No replacement

The number of k-permutations

- without repetition
- of a set of n objects and k selections is given by

$$P(n,k) = \frac{n!}{(n-k)!}$$

For $n = 4$ and $k = 2$ there are

$$P(4,2) = \frac{4*3*2}{2} = \frac{24}{3*2} = 12$$

possibilities.

Other k-permutations:

$$P(4,1) = \frac{4*3*2*1}{3*2*1} = \frac{24}{3*2} = 4$$

$$P(4,3) = \frac{4*3*2}{1} = \frac{24}{1} = 24$$

$$P(4,4) = \frac{24}{0!} = 24$$

Combination

… is a selection of items from a larger set where the order of selection does not matter.

Here: No replacement

The number of combinations

- without repetition
- of a set of n objects and k selections is given by

$$P(n,k) = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

In the example of German lottery there are

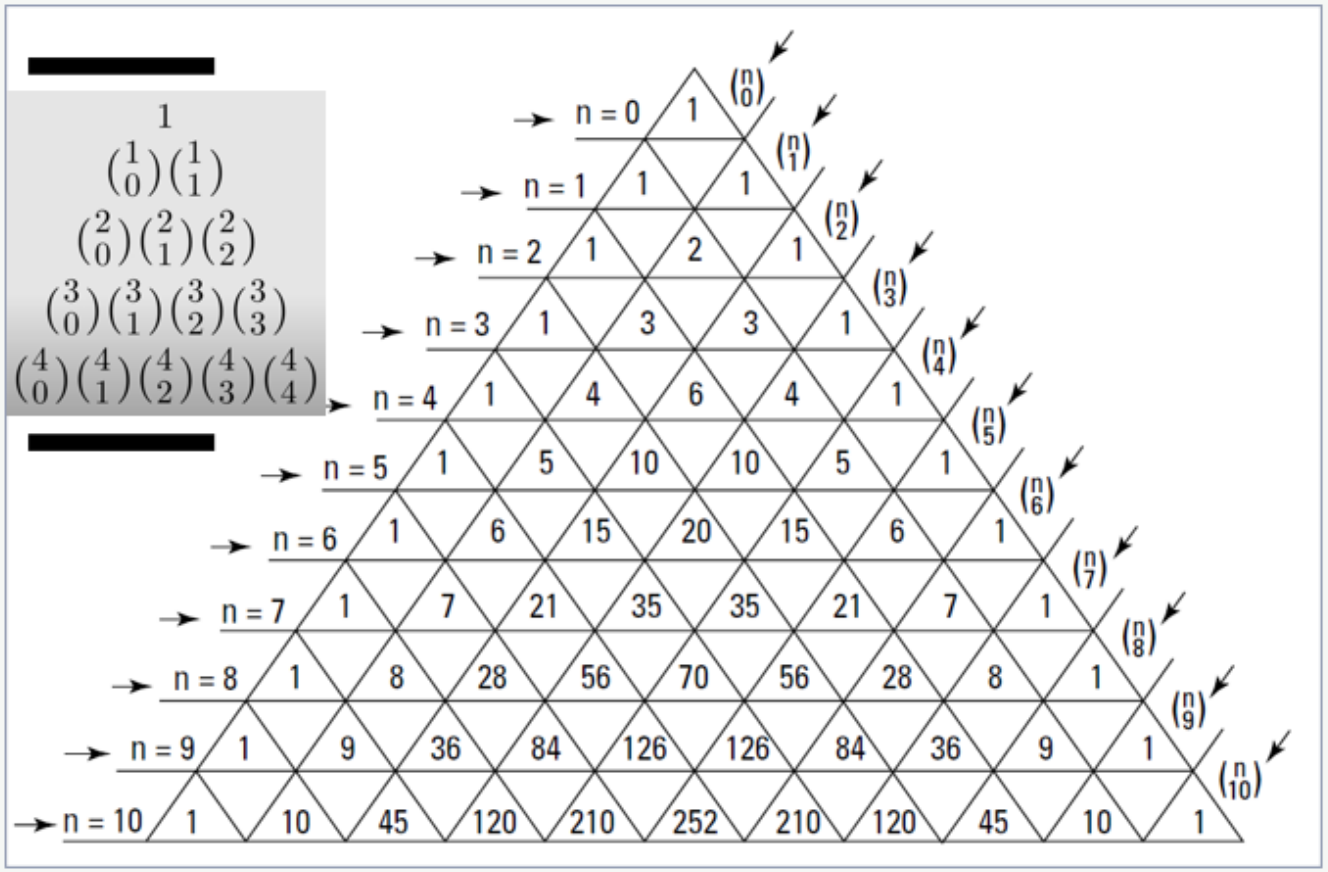
- 49 balls
- 6 selected balls

i.e. there are

$$P(49,6) = \binom{49}{6} = 13.983.816 \text{ possibilities.}$$

Pascal‘s triangle

… is a triangular array of numbers in which each number is the sum of the two directly above it. It is used for binomial expansions, combinatorics, probability calculations, etc.



Binomial theorem

- Coefficients from binomial expansion:

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

- Sum of rows → 2^n

Coming in next chapter:

- $\binom{n}{k}$: crucial for explaining distributions like binomial distribution

Combinatorics with Replacement

The number of ways to choose k items from n distinct items, where each item can be chosen more than once and the order doesn't matter.

Concept ...with Replacement	Definition	Formula	Example
Permutation	▪ Arrangement of items where order matters ▪ each item can be selected more than once.	$\frac{n!}{k_1!;\cdots;k_s!;}$	▪ Choosing 3-digit PIN from 10 digits (0-9). ▪ Possible permutations: 10^3 = 1000 combinations.
k-Permutation	▪ Subset of k items chosen ▪ arranged from n items ▪ allowing repetition.	n^k	▪ Selecting, arranging 2 letters from the alphabet with repetition. ▪ Example: 26^2 = 676 possibilities.
Combination	▪ Selection of items where order does not matter ▪ each item can be selected more than once.	$\binom{n+k-1}{k}$	▪ Choosing 3 fruits from basket of apples, oranges, and bananas ▪ where repetition is allowed ▪ 10 combinations.

Exercises

Exercise 1: Count the permutations
Write a function that returns the number of all possible permutations of a list.
def count_permutations(lst): # Your code here
print(count_permutations(['A', 'B', 'C']))
Additionally: Try also lists with other lengths

Exercise 2: Generate all permutations of a list: l1 = ['A', 'B', 'C']

Exercise 3: Count the k-permutations
Write a function that returns the number of all possible k-permutations of a list.

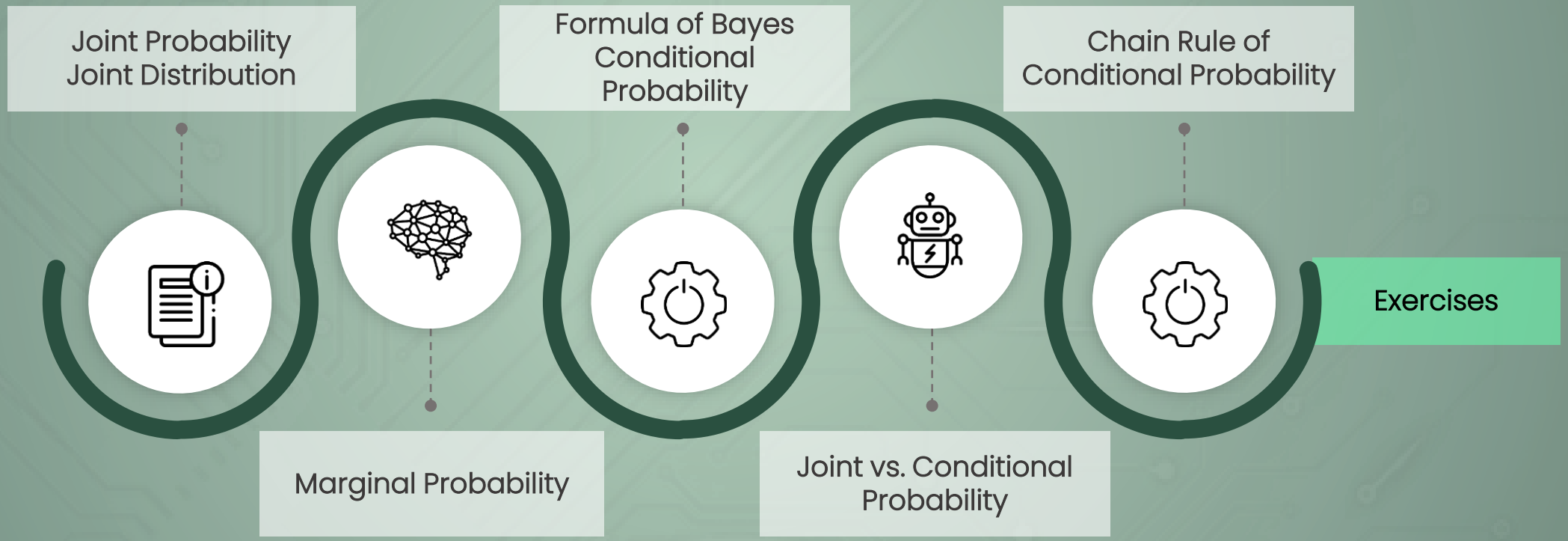
Exercise 4: Generate k-permutations of a list
In this exercise, your task is to write a function that generates all possible k-permutations of a given list using itertools.

Exercise 5: Generate all combinations of size k
Write a function that generates all possible combinations of size k from a given list of size n=8.
['apple','banana','pear','grapes','orange','mango','blueberry','strawberry']

Exercise 6: Calcucate the number of combinations
n = 8 k = 2

Joint and Conditional probability

Bridging Events: How Joint Occurrences and Informed Probabilities interrelate



Joint Probability

is probability of two or more events occurring simultaneously.
It is denoted as P(A and B) or P(A ∩ B) for events A and B.

Example: Two rolling dice (independent events)

```
P(First die = 6 and Second die = 6)
= P(First die = 6) * P(Second die = 6)
= (1/6) * (1/6)
= 1/36
```

Formula

*P(A and B) = P(A) * P(B|A)*

Where:

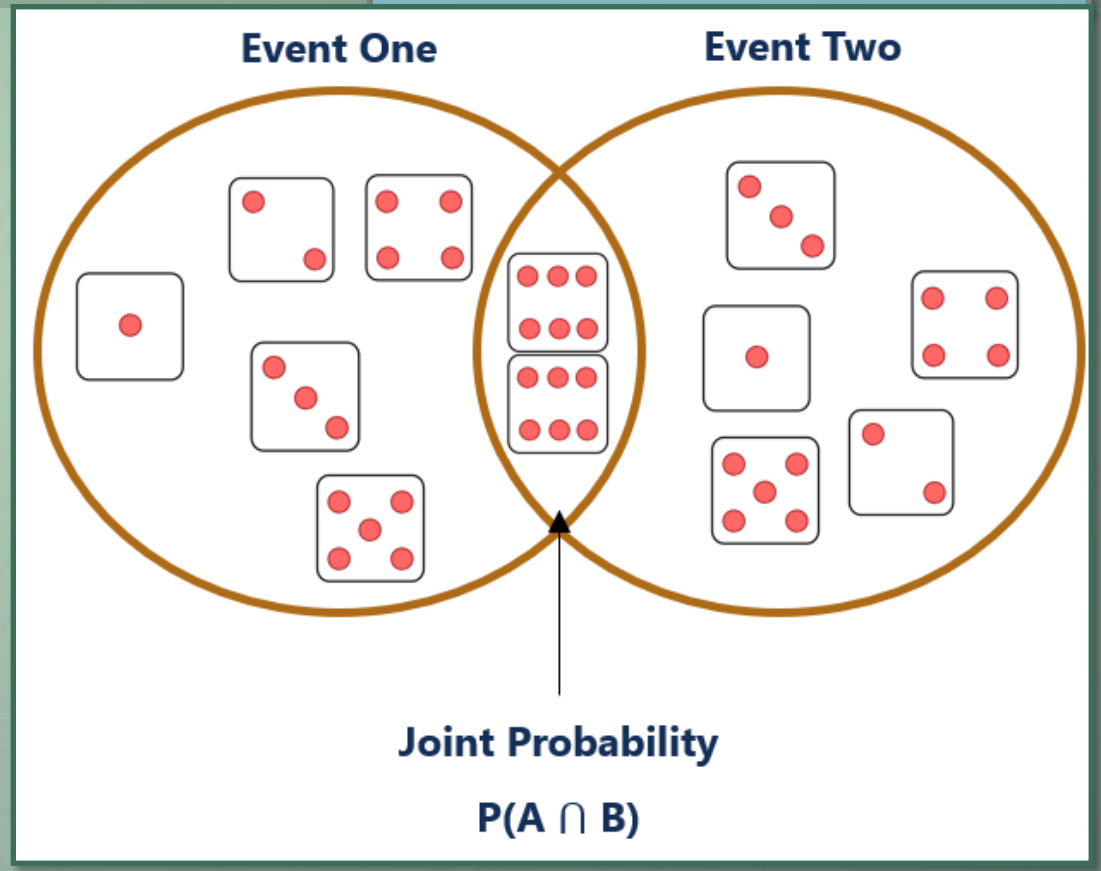
- P(A) is the probability of event A
- P(B|A) is the conditional probability of B given A

(In-) dependence

- If A and B are independent, P(A and B) = P(A) * P(B)
- Else: P(A and B) ≠ P(A) * P(B)
- Mutually Exclusive: If A and B cannot occur together, P(A and B) = 0

Joint Probability Distribution

- Describes probabilities of all possible combinations



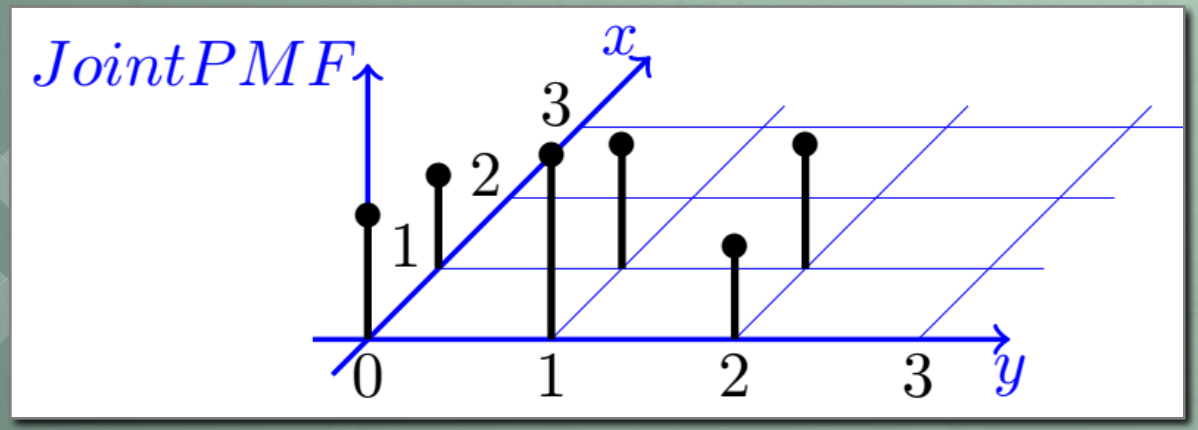
https://corporatefinanceinstitute.com/resources/data-science/joint-probability/

Joint Probability Distribution

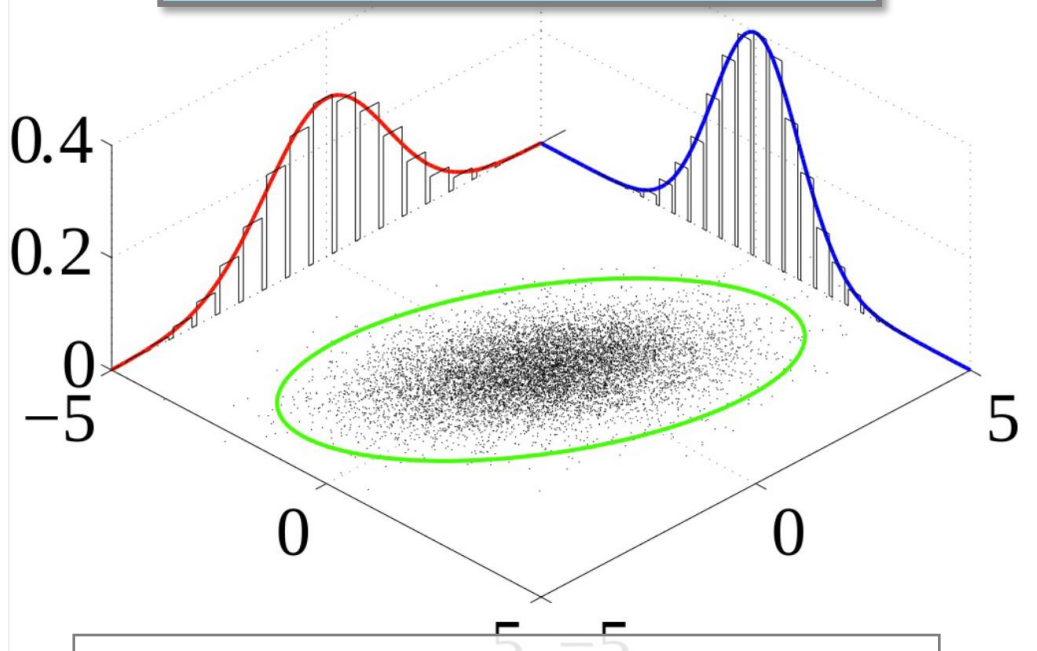
… is probability distribution that gives probability of two or more random variables occurring together. Results of random events are not added or averaged.

Discrete random variables

	Y=0	Y=1	Y=2
X=0	$^1/_6$	$^1/_4$	$^1/_8$
X=1	$^1/_8$	$^1/_6$	$^1/_6$



Continuous random variables



f(x, y) - joint probability density function

Marginal Probability

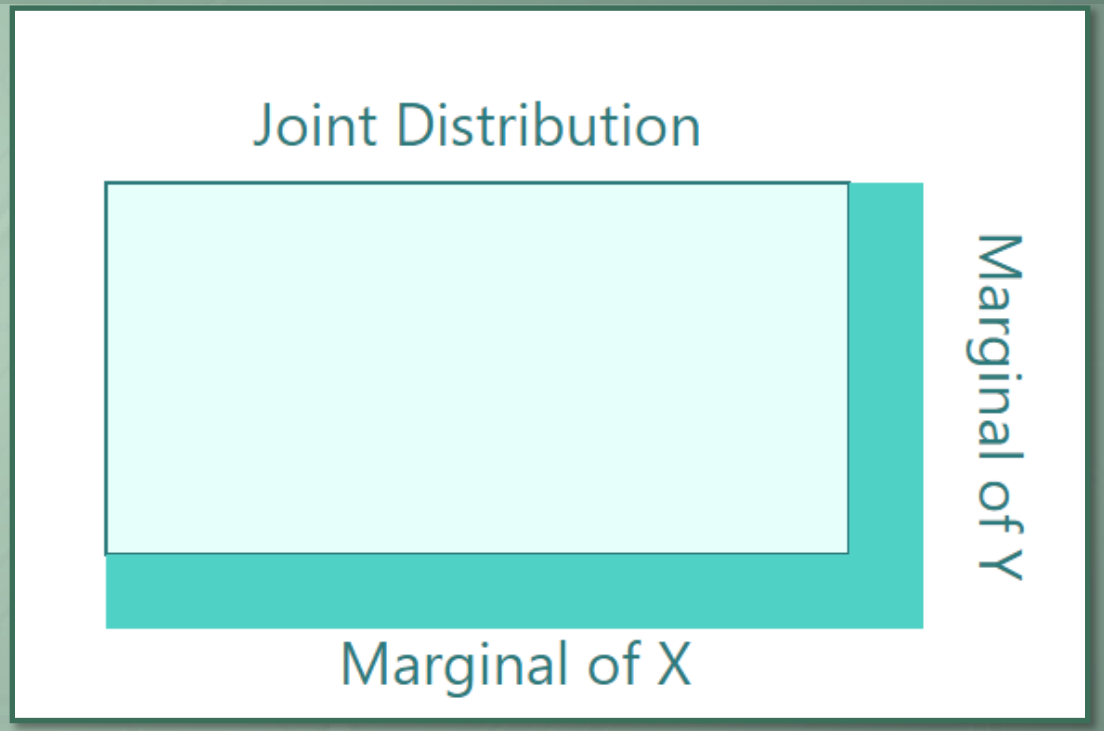
… is probability distribution of subset of variables from larger set of variables, ignoring the other variables.

Key Points

- Derived from joint distribution
- Represents distribution of (summarized) single variable, regardless of others
- Useful for analyzing individual variables in multivariate scenarios
- Can be used to check for independence between variables

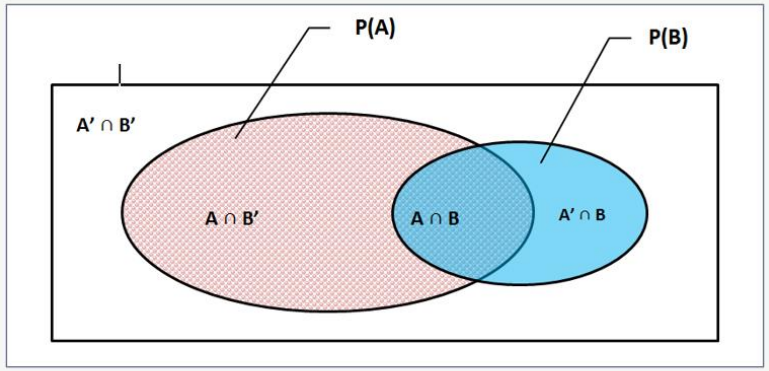
Calculation

- For discrete variables X and Y:
  - Sum over all possible values of Y
  - $P(X = x) = \Sigma P(X = x, Y = y)$
- For continuous variables:
  - Integrate over all possible values of Y
  - $f_X(x) = \int f(x, y)\, dy$



Bayes‘ Theorem

… describes how to update probabilities of hypotheses when given evidence. It’s is used to calculate the conditional probability, i.e., the probability of event based on prior knowledge.



From conditional probabilities to Bayesian theorem:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$$

$$P(A \text{ and } B) = P(B \text{ and } A)$$

Thus: $P(A|B)P(B) = P(B|A)P(A)$

Or:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

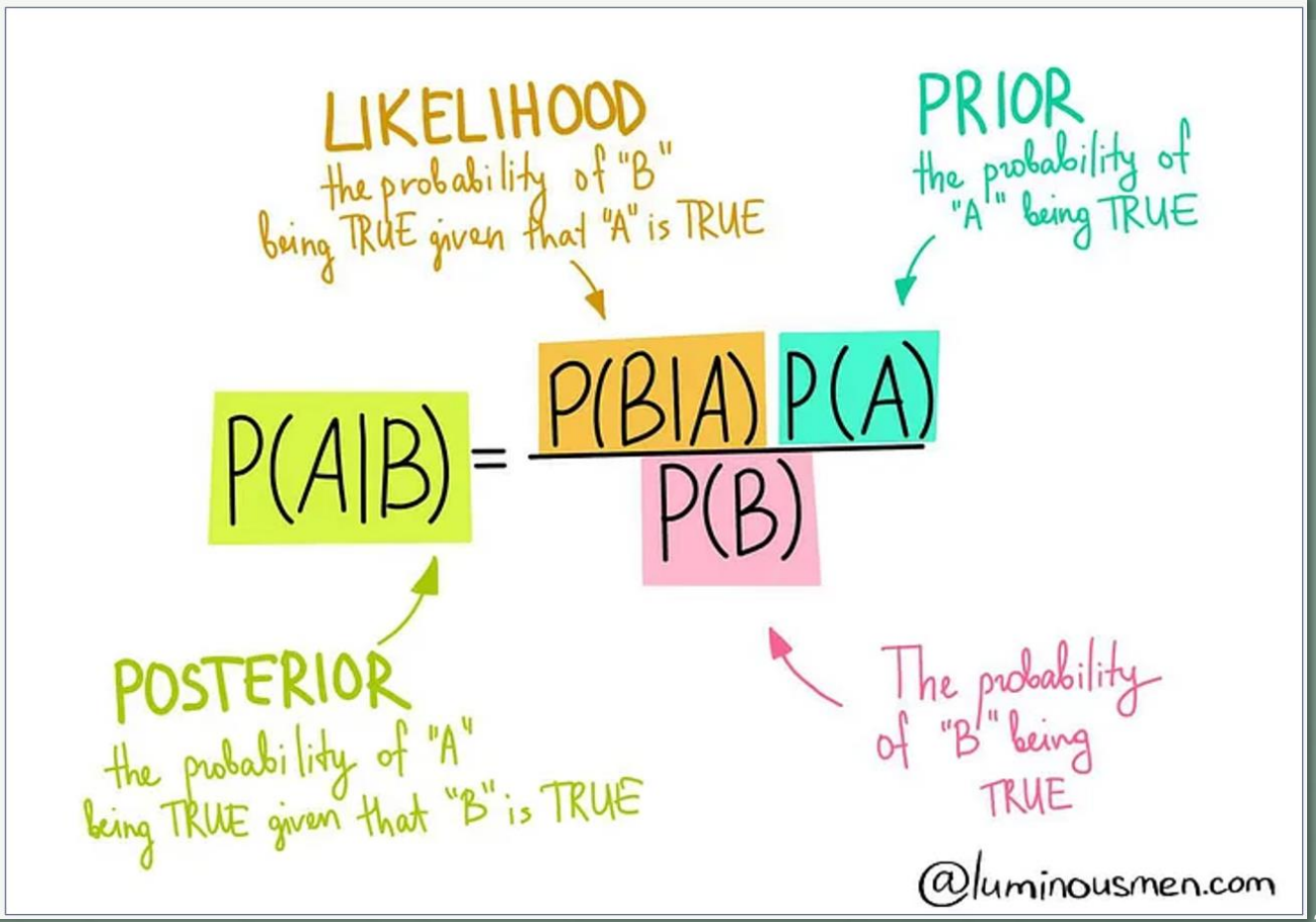


Table with conditional probabilities

… Use Case: How many visitors make a purchase depending on whether they visit website A or B

	Purchase	No Purchase	Total (MP)
Website A	80 (A1) JP: 4 % **CP: 8 %**	920 (A2) JP: 46 % CP: 92 %	$Total\_A$ 1000 $P(A)$ =50%
Website B	90 (B1) JP: 4.5 % CP: 9 %	910 (B2) JP: 45.5 % CP: 91 %	$Total\_B$ 1000 $P(B)$ =50%
Total (MP)	Purchase: 170 (T1) P(Purchase) = 8.5%	Non-purchase: 1830 (T2) P(Non-Purchase) = → 91.5 %	$Total = Total\_A + Total\_B$ 2000 $P(Total) = 100\%$

$$P(Purchase\ given\ Website\ A) = \frac{P(Website\ A\ given\ Purchase) * P(Purchase)}{P(Website\ A)}$$

$$0.08 = \frac{{}^{80}/_{170} * 0.085}{0.5}$$

Joint probabilities (JP):

- P(A and Purchase) = A1; T = 80 / 2000 = 0.04
- P(B and Purchase) = B1; T = 90 / 2000 = 0.045

Conditional probabilities (CP):

- P(Purchase | A) = A1; TA = 80 / **1000** = 0.08
  (probability of a purchase given the user saw website A)
- P(Purchase | B) = B1; TB = 90 / **1000** = 0.09
  (probability of a purchase given the user saw website B)

Marginal probabilities (MP):

- P(A) = TA; T = 1000 / 2000 = 0.5
  (probability a visitor sees website A)
- P(B) = TB; T = 1000 / 2000 = 0.5
  (probability a visitor sees website B)
- P(Purchase) = T1; T = 170 / 2000 = 0.085
  (probability a visitor makes a purchase)

Conditional vs. Joint Probability

Bridging Events: Conditional Probability is based on Joint Probability, however considers effect of new information like “Event B has taken place”

Conditional Probability

P(A|B)
Probability of A given B has occurred

- Represents updated probability based on new information
- Calculates likelihood of one event considering another event's occurrence
- Formula: $P(A|B) = P(A \cap B) / P(B)$

Joint Probability

$P(A \cap B)$
Probability of both A and B occurring together

- Represents simultaneous occurrence of multiple events
- Calculates likelihood of all specified events happening
- Formula: $P(A \cap B) = P(A|B) * P(B) = P(B|A) * P(A)$

Key Points

- Conditional probability can be derived from joint probability: $P(A|B) = P(A \cap B) / P(B)$
- For independent events: $P(A \cap B) = P(A) * P(B)$, and $P(A|B) = P(A)$
- Bayes' Theorem connects these concepts: $P(A|B) = P(B|A) * P(A) / P(B)$

Reference to Combinatorics

- Sampling without Replacement: Connected to Conditional Probability
- Sampling with Replacement: Connected to Joint Probability

Chain Rule of Conditional Probability

… allows to express joint probability of multiple events as product of conditional probabilities.

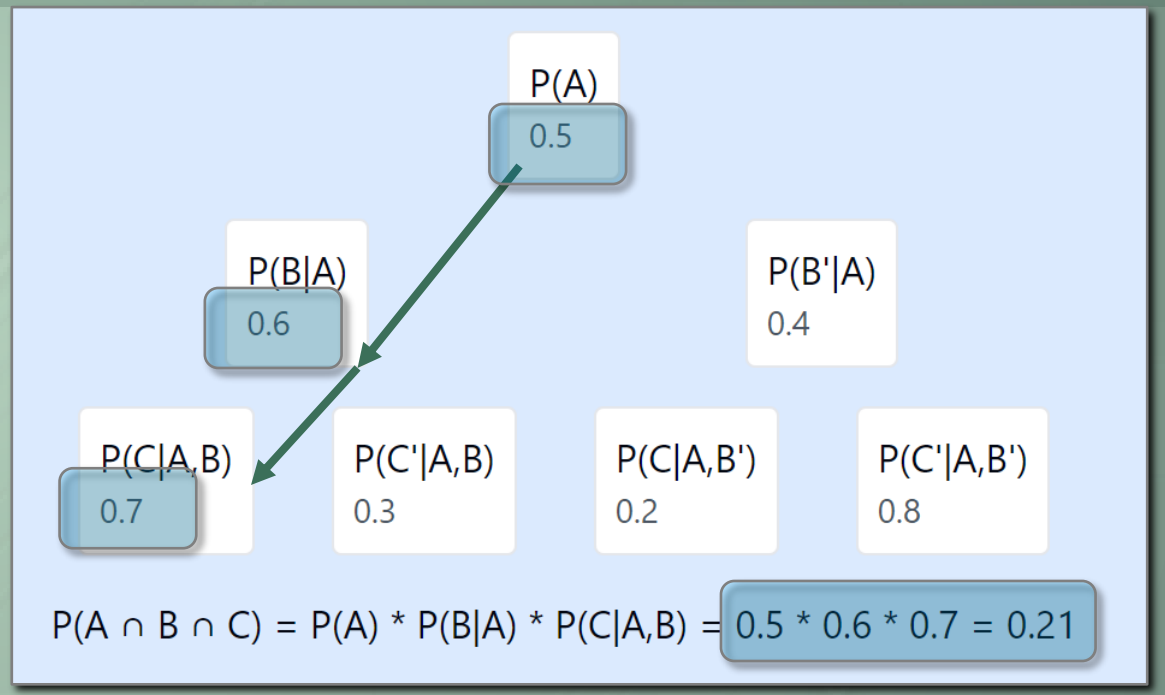
Key Points

- Breaks down complex joint probabilities
- Useful when direct calculation of joint probability is difficult
- Order of events can affect calculation complexity

Calculation

- For two events: A & B
  - $P(A \cap B) = P(A|B) * P(B)$
- Find an example?
- More than two events $A_1, \ldots, A_n$:
  - $P(A_n \cap \ldots \cap A_1) = P(A_n|A_{n-1} \cap \ldots \cap A_1) * P(A_{n-1} \cap \ldots \cap A_1)$
  - Also:

$$P(A_n \cap \ldots \cap A_1) = \prod_{k=1}^{n} P\left(A_k \middle| \cap_{j=1}^{k-1} A_j\right)$$



Exercises

Exercise 1:
Consider the following frequency table for students' preferences for Programming Language and whether they prefer Dark Theme or not

Programming Language	Dark Theme	Count
Python	Yes	90
Python	No	30
Java	Yes	50
Java	No	40
JavaScript	Yes	60
JavaScript	No	30

Represent this table in Python using a list of dictionaries and calculate the conditional probability of a student preferring a Dark Theme given that they prefer Python, Java, JavaScript.

Probability Distributions

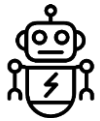
They describe the probabilities of occurrences of different possible outcomes in an experiment

PMF/PDF, CDF of Discrete and continuous distributions

Distributions Overview

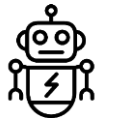
Binomial, Hypergeometric, Poisson, normal distribution

Students' t-distribution



χ^2- distribution Exponential, Gamma Distribution

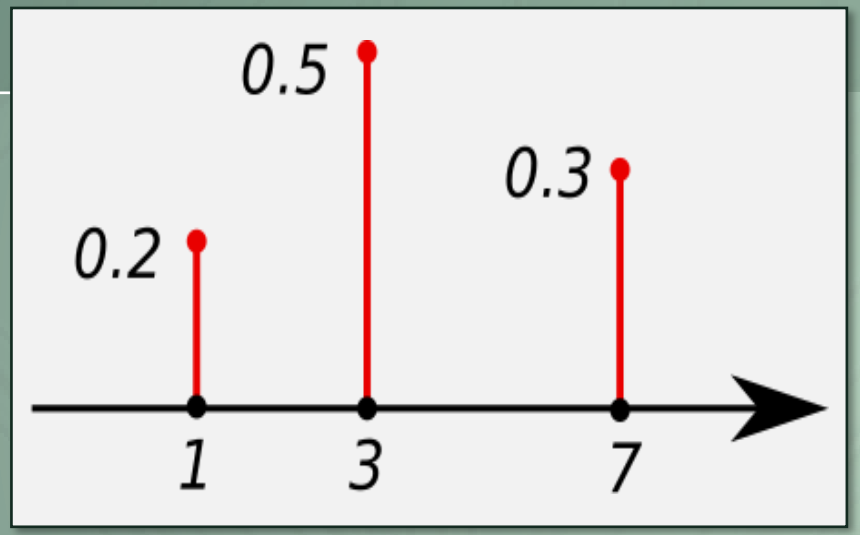
F-Distribution



Exercises

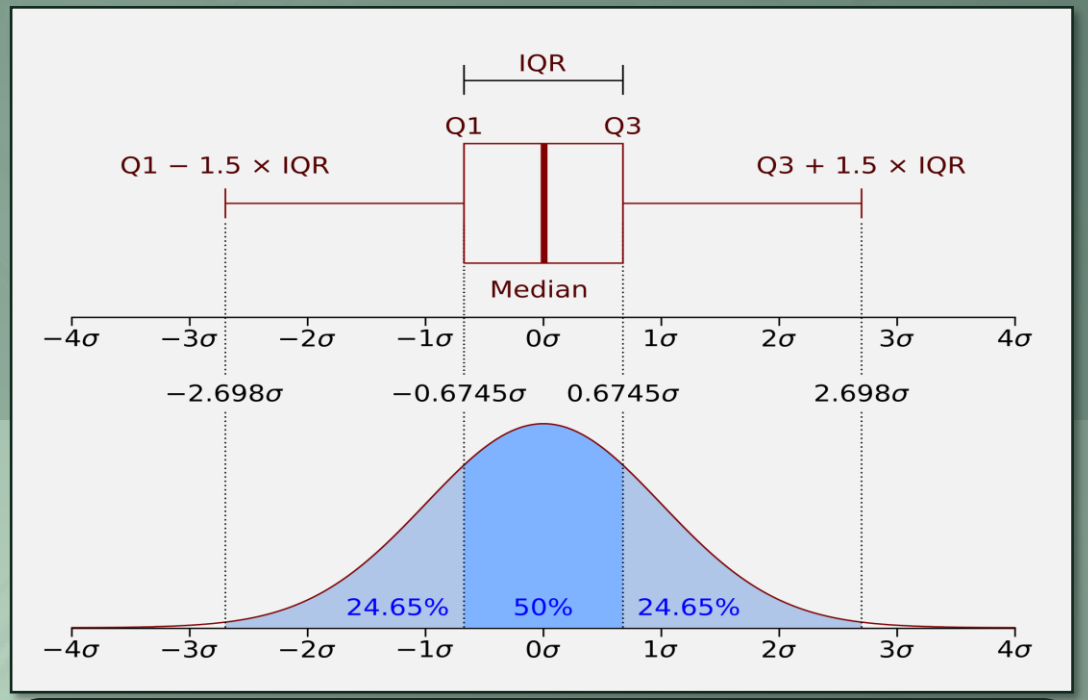
Probability Distributions

… are discrete or continuous



Discrete Probability Distribution

- Represented by vertical lines (lollipop chart)
- Each point represents a specific outcome
- Height of line indicates probability
- Example shown: {1: 0.2, 3: 0.5, 7: 0.3}
- Sum of all probabilities equals 1



Continuous Probability Distribution

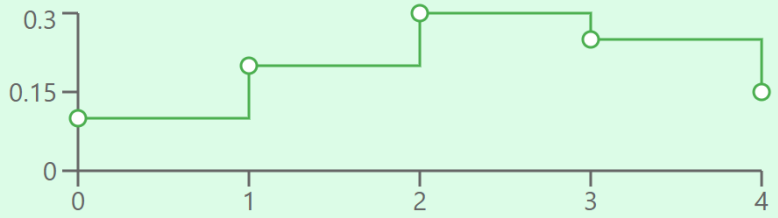
- Represented by a smooth curve (bell curve shown)
- Area under the curve represents probability
- Total area under curve equals 1
- Example: Normal (Gaussian) distribution
- Shows
  - median, quartiles, and standard deviations
  - Interquartile Range (IQR): Distance between Q1 and Q3
  - Standard Deviation: Measure of spread 1σ, 2σ, 3σ

Probability Distributions

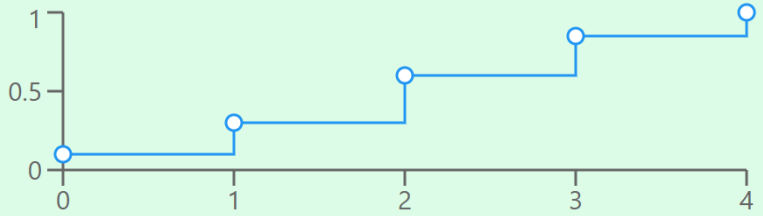
Comparing Probability Mass/Density and Cumulative Distribution Functions

Discrete Distributions

Probability Mass Function (PMF)

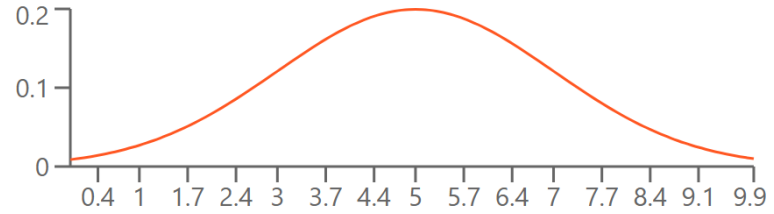


Cumulative Distribution Function (CDF)

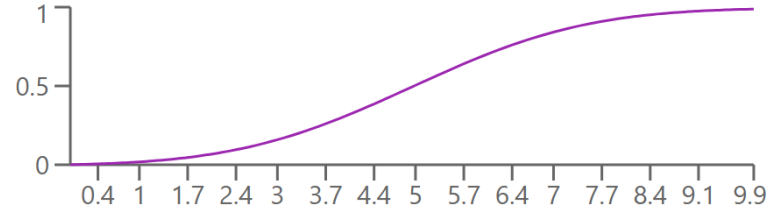


Continuous Distributions

Probability Density Function (PDF)



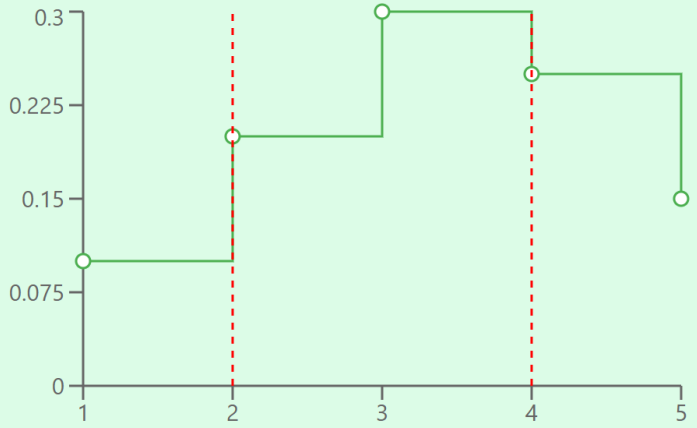
Cumulative Distribution Function (CDF)



Measuring Probability on Distribution Intervals

From Sums to Integrals: Calculating Probabilities Across Distribution Types

Discrete Distributions

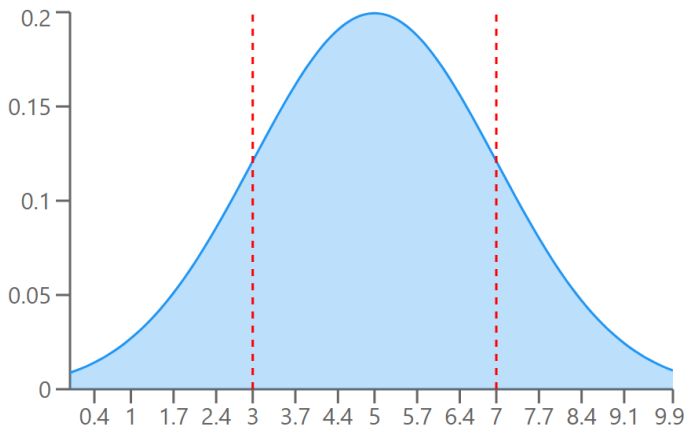


Probability = Sum of probabilities for each point in the interval

Example: P(2 ≤ X ≤ 4) = 0.2 + 0.3 + 0.25 = 0.75

Given the shown distribution …
What is the probability that the guest has drunk 2,3 or 4 beer?

Continuous Distributions



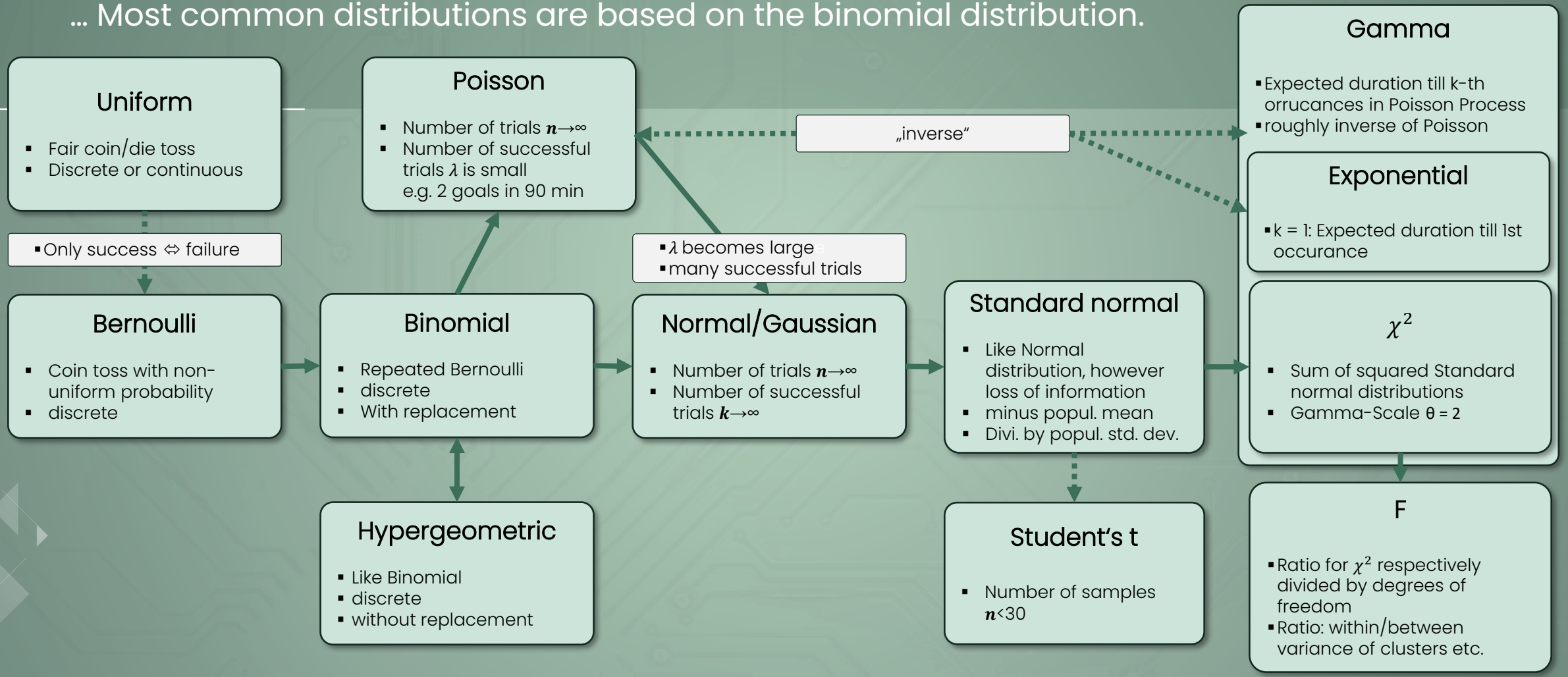
Probability = Area under the curve between interval points

Example: P(3 ≤ X ≤ 7) = ∫[3 to 7] f(x) dx

Given the shown distribution …
What is the probability that the person's income is between 3k and 7k € a month?

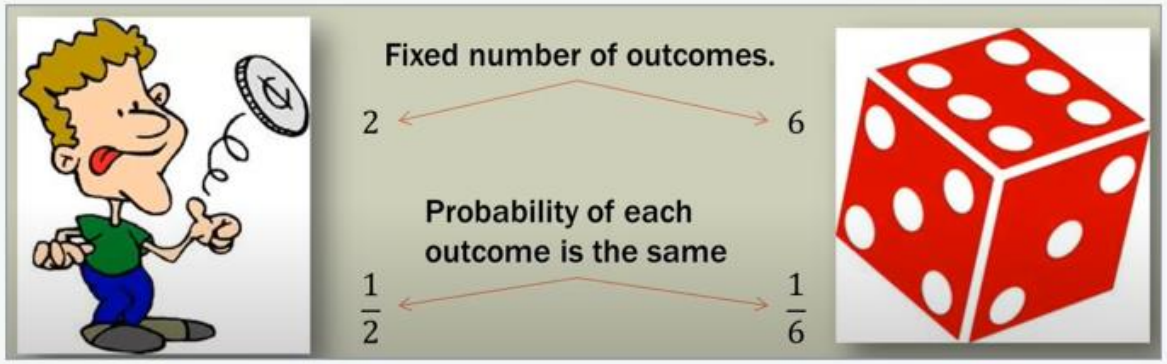
Overview of important distributions

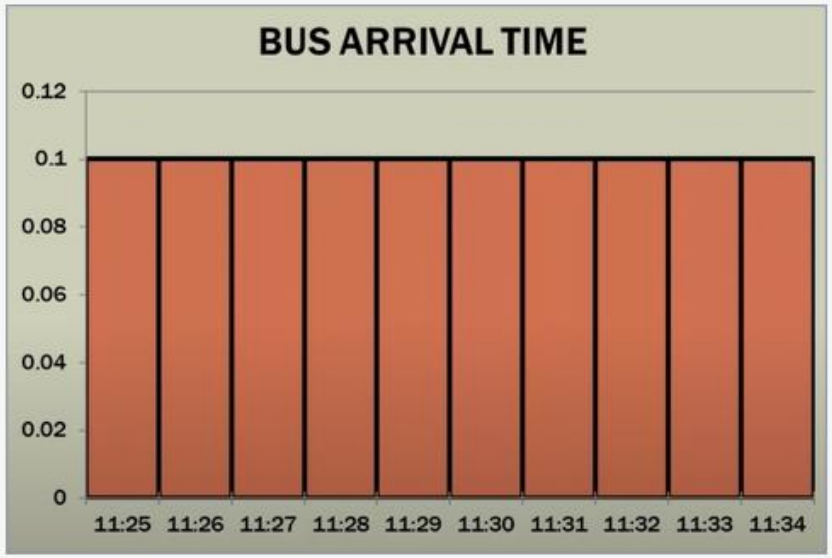
… Most common distributions are based on the binomial distribution.



Uniform distribution

... is a probability distribution that describes the number of successes in a fixed number of independent binary experiments, each with the same probability of success.





Expected value:

- Arithmetic mean

Use Cases:

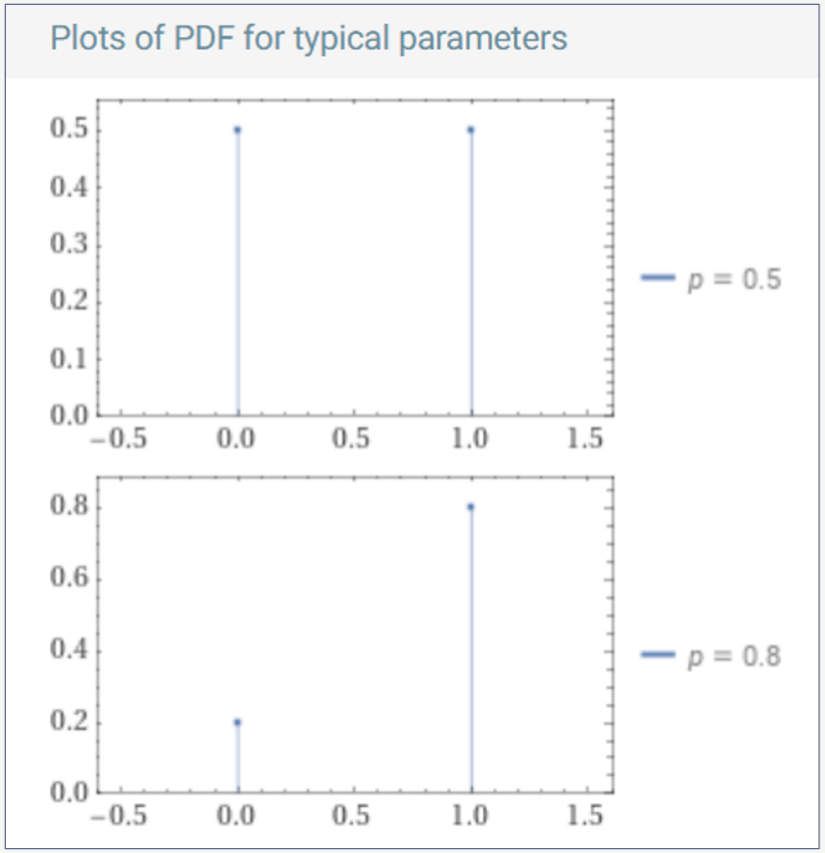
- Generating Random Numbers:
  e.g.: fair dice, fair coin tosses
- Quality Control and Manufacturing:
  e.g.: check the uniformity of products

Be careful:

- Sum of 2 dice is not uniformly distributed!
  (it is multinomial distribution, out of scope)

Bernoulli distribution

… is a discrete probability distribution for a random variable which can take a binary outcome: one (success) or zero (failure); example: toss of coin (that is not necessarily fair)



Formula:

$$P(x;p) = p^x * (1-p)^{1-x} \; for \; x \epsilon [0;1]$$

p: probability of success. E.g. if the coin is fair, the probability of getting a head (considered a success) would be 0.5

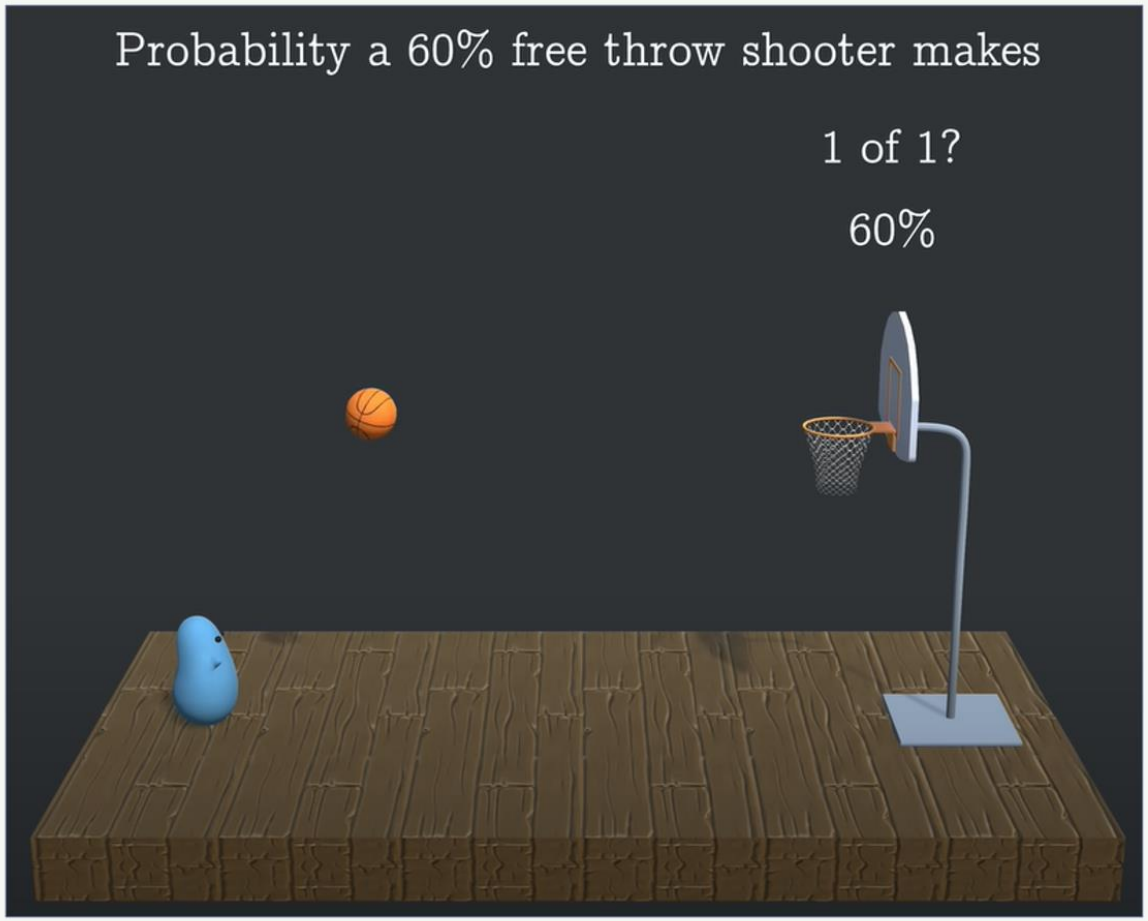
x: outcome of Bernoulli trial; it can take only two values: 1(success) or 0(failure)
Attention: $x \epsilon [0;1]$ is not an interval!

$P(x;p)$: probability of outcome x, given probability p of success.

Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10



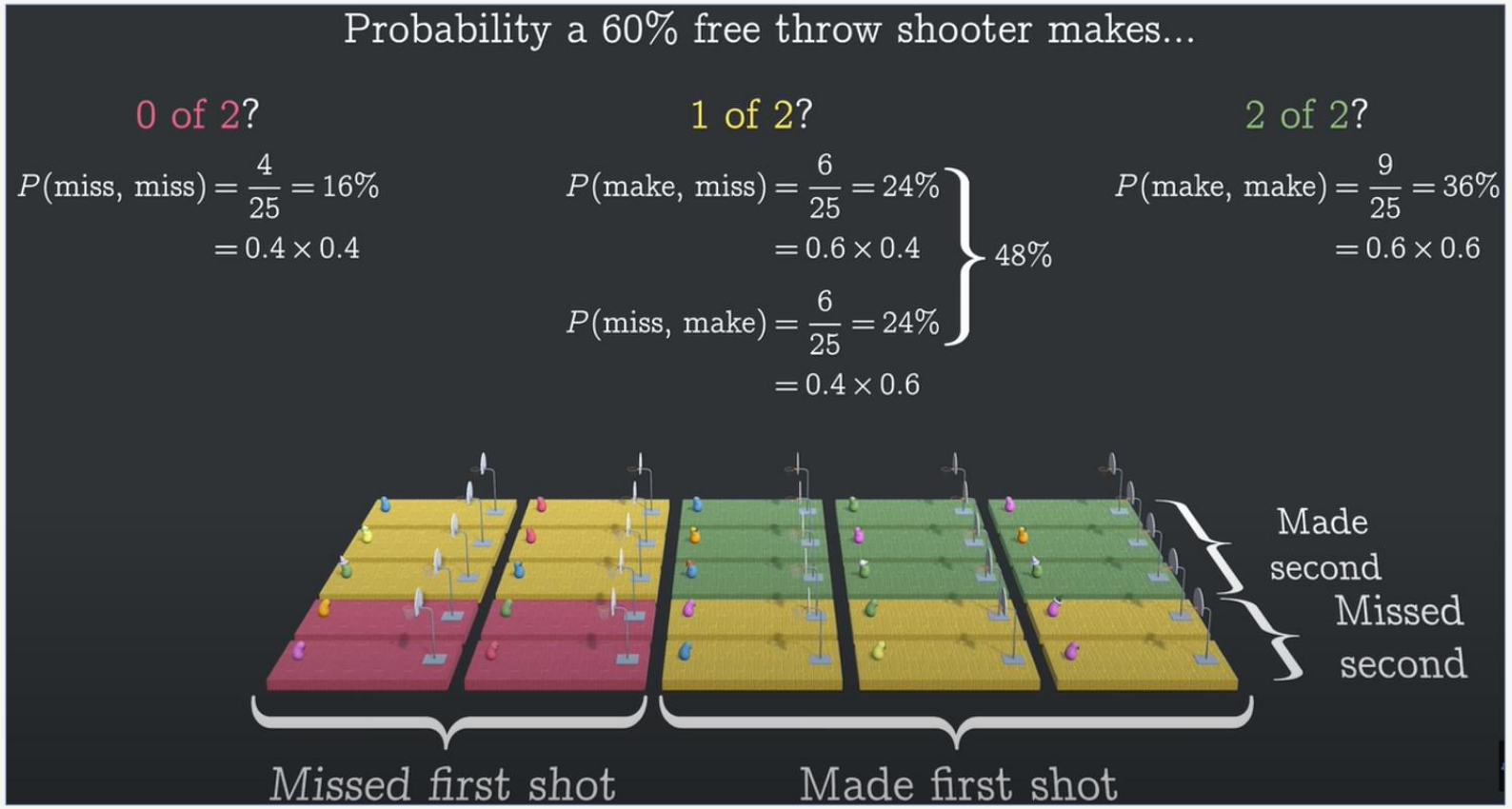
Step 1:

- P(Make the shot) = 60 %
- P(Miss the shot) = 40 %
  = 1 – 60 %

Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10



Step 2:

- P(miss, miss) = 16 %
- P(miss, make) = 48 %
- P(make, make) = 36 %

Assumption:
Independence of shots

Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10

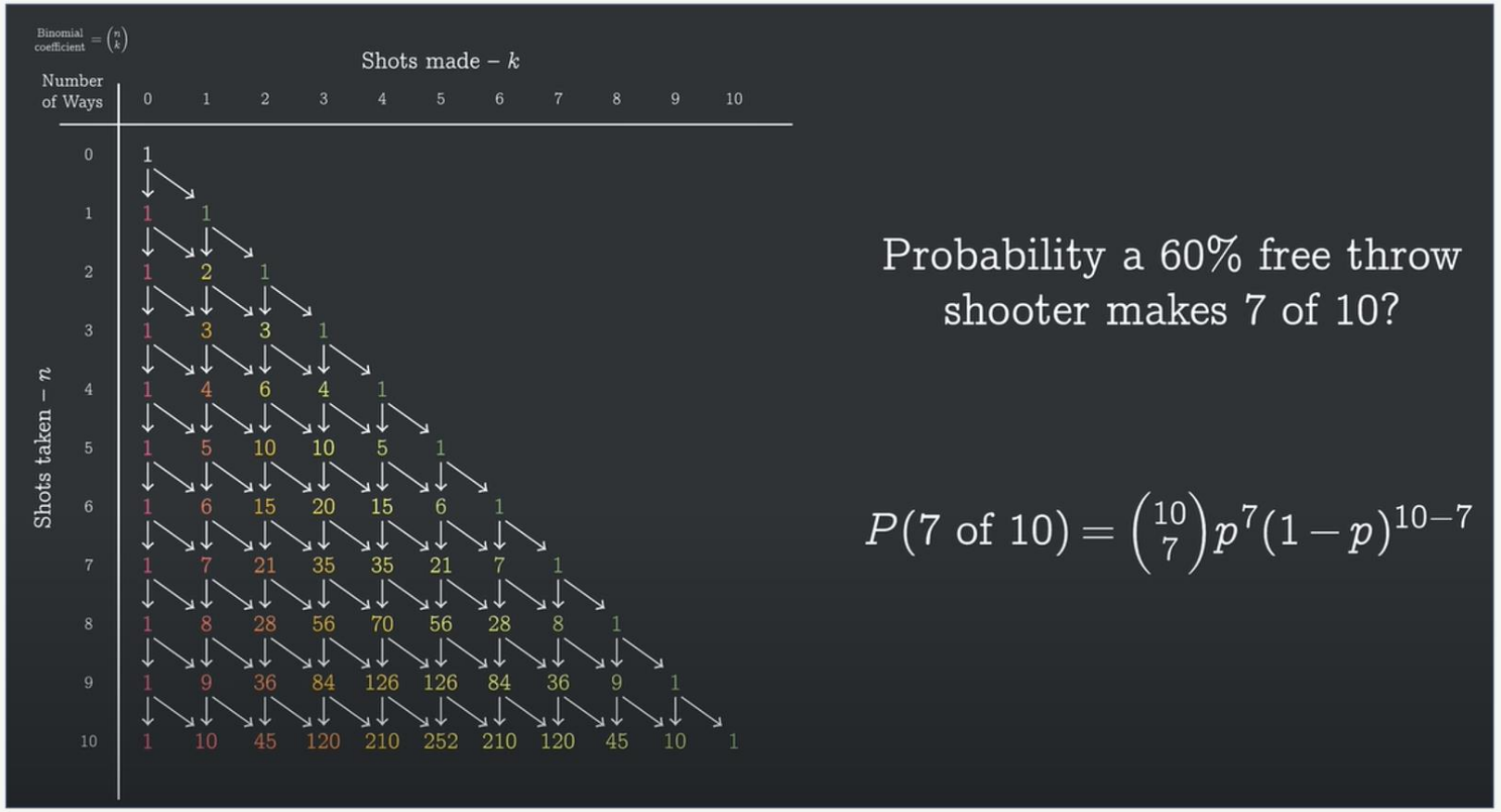
Step 3:

- P(miss, miss, miss) = 8 %
- P(miss, miss, make) = 28.8 %
- P(miss, make, make) = 43.2 %
- P(make, make, make) = 21.6 %

Binomial distribution

Example for probability calculation:

Basketball player that is known for 60% prob of making free shots makes 7 of 10

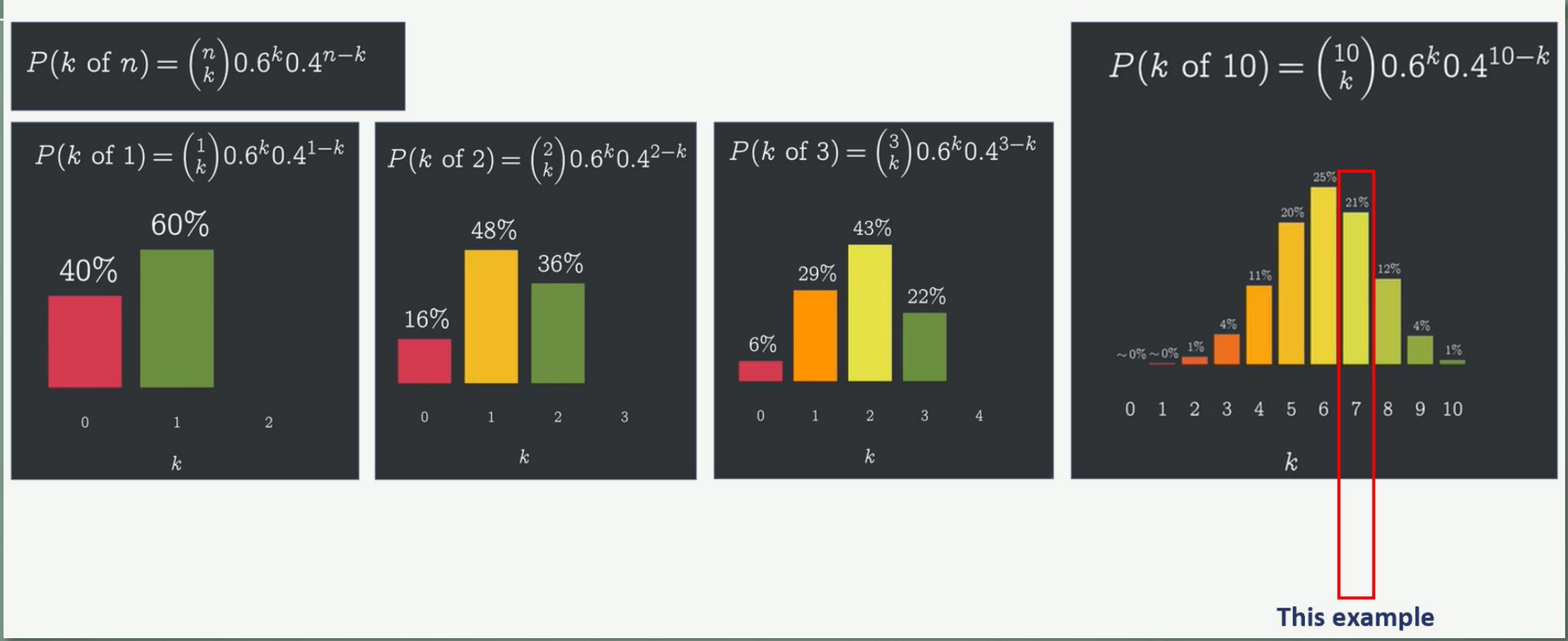


$$P(7\ of\ 10) = 120 \times 0.6^7 \times 0.4^3 \approx 21.5\%$$

Binomial distribution

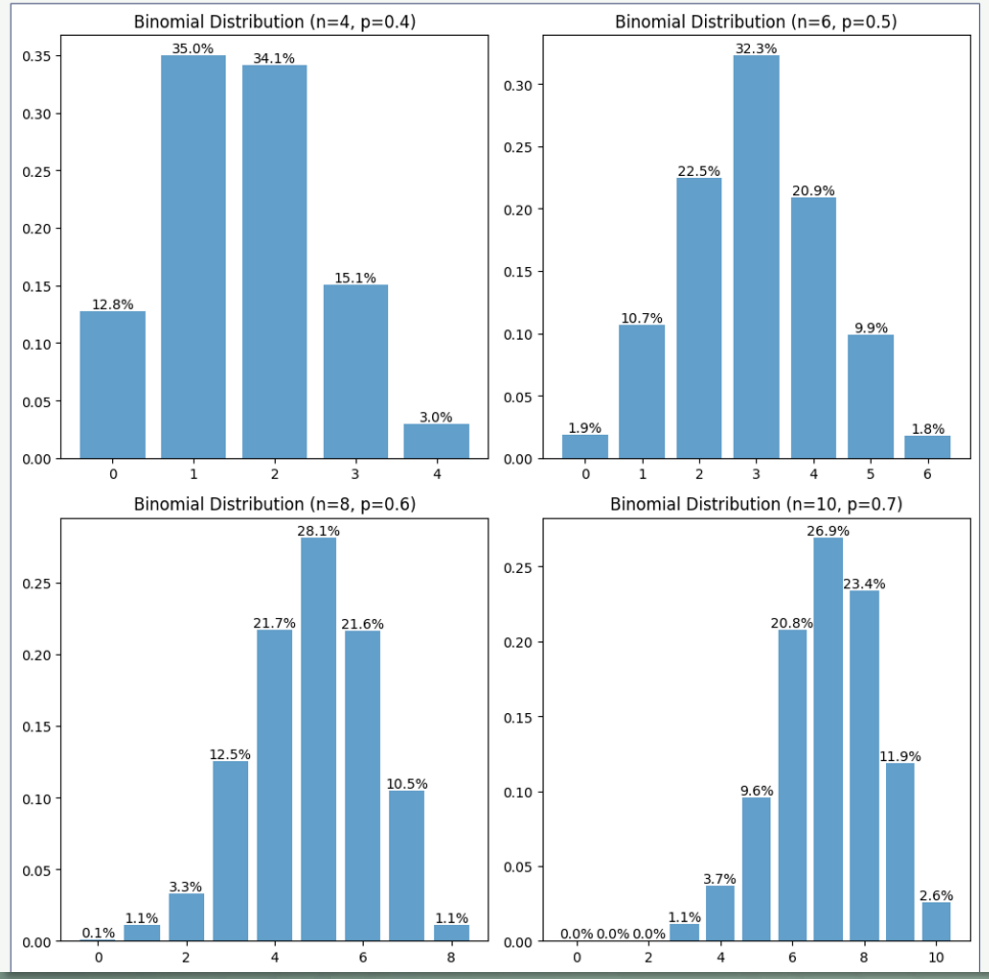
Example for probability calculation - Generalization:

Basketball player that is known for 60% prob of making free shots makes k of n



Binomial distribution

… is repeated Bernoulli distribution, i.e. probability distribution that describes number of successes in fixed number of independent binary experiments, each with same probability of success.



Formula:

$$P(k\ of\ n) = \binom{n}{k} p^k (1-p)^{n-k}$$

Parameters:

- n: number of trials
- p: probability of success on each trial
- k (input factor on x-axis): number of actual positive outcomes

Use cases:

- Quality Control: In a manufacturing company the probability of producing a defective item is 0.05, and it manufactures 200 items per day
- Spam email classifier

Connection between Joint and Binomial Probability

Binomial probability is sum of joint probabilities
for all possible ways to achieve k successes in n trials.

Joint Probability

For n independent trials:

$P(X_1=x_1, X_2=x_2, \ldots, X_n=x_n) = p^k * (1-p)^{(n-k)}$

Where:

- p = probability of success on each trial
- k = number of successes
- n = total number of trials

Binomial Probability

Probability of exactly k successes in n trials:

$P(X = k) = C(n,k) * p^k * (1-p)^{(n-k)}$

Where:

- C(n,k) = number of ways to choose k items from n
- p, k, and n are as defined for joint probability

Example: Coin Flips

For 3 coin flips (H = heads, T = tails):

Joint Probability: $P(H,H,T) = (1/2)^2 * (1/2)^1 = 1/8$

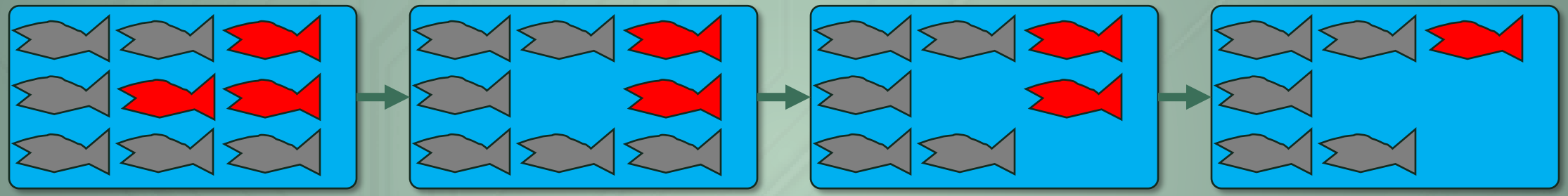
Binomial Probability: $P(2 \text{ heads in } 3 \text{ flips}) = C(3,2) * (1/2)^2 * (1/2)^1 = 3 * 1/8 = 3/8$

The binomial probability includes all sequences with 2 heads: HHT, HTH, THH

Hypergeometric distribution

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Example: In a lake there are 6 gray fishes and 3 red fishes.
What is the probability of selecting k red fish?



- Total number of fish (N):
  There are 9 fish in total
  → population size N = 9
- Number of red fish (K):
  3 red fish in the pool
  → "success" cases K = 3
- Number of grey fish:
  6 grey fish, no parameter,
  just N - K.
- Number of draws (n):
  Depends on how many fish are selected
- Number of successful draws (k)

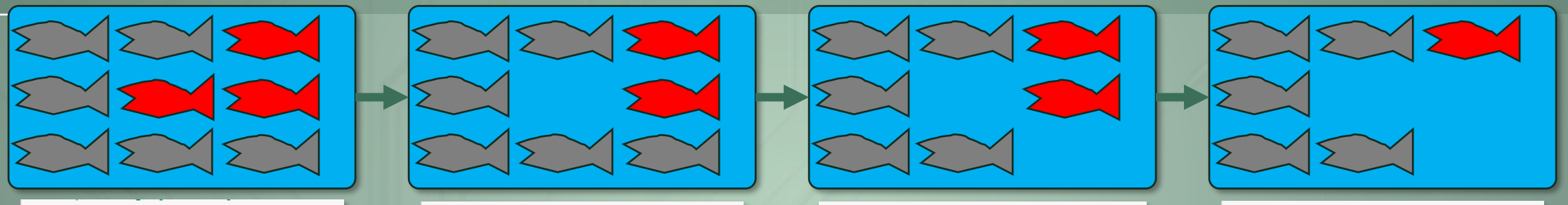
- Total number of fish (N):
  N = 8
- Number of red fish (K):
  K = 2
- Number of grey fish:
  N – K = 6

- Total number of fish (N):
  N = 7
- Number of red fish (K):
  K = 2
- Number of grey fish:
  N – K = 5

- Total number of fish (N):
  N = 6
- Number of red fish (K):
  K = 1
- Number of grey fish:
  N – K = 5

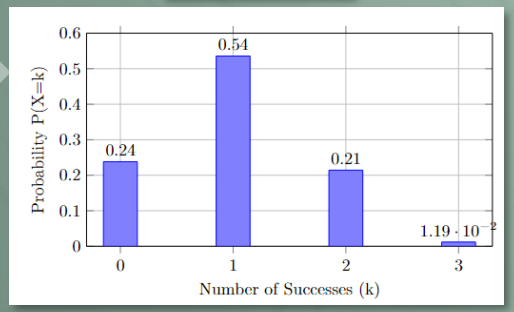
Hypergeometric distribution

Example: In a lake there are 6 gray fishes and 3 red fishes. Depending on the order of catching gray or red fish, probability distribution for next rounds changes



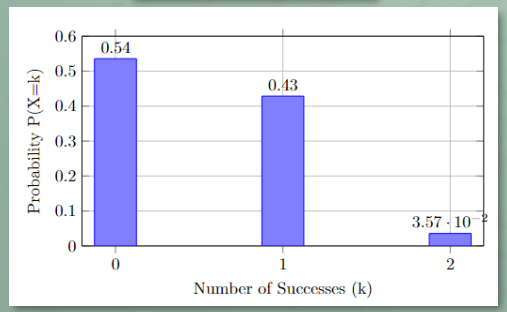
```
N = 9  # Total number of fish
K = 3  # Number of red fish
n = 3  # Number of draws (assumption)
```

k	$P(X = k)$
0	0.2381
1	0.5357
2	0.2143
3	0.0119



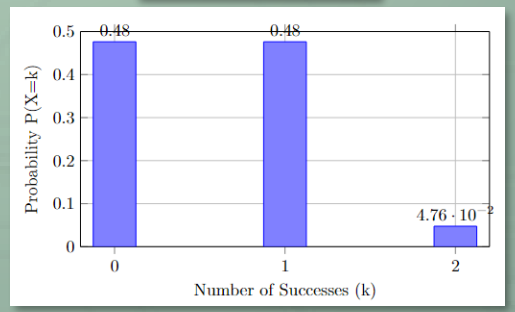

```
N = 8  # Total number of fish
K = 2  # Number of red fish
n = 2  # Number of draws (assumption)
```

k	$P(X = k)$
0	0.5357
1	0.4286
2	0.0357



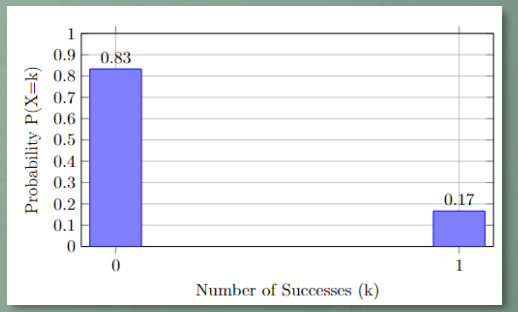

```
N = 7  # Total number of fish
K = 2  # Number of red fish
n = 2  # Number of draws (assumption)
```

k	$P(X = k)$
0	0.4762
1	0.4762
2	0.0476




```
N = 6  # Total number of fish
K = 1  # Number of red fish
n = 1  # Number of draws (assumption)
```

k	$P(X = k)$
0	0.8333
1	0.1667



Hypergeometric distribution

… models number of successes in sample drawn without replacement from finite population.

Probability mass function

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Where:

- N is the population size
- K is the number of success states in the population
- n is the number of draws
- k is the number of observed successes
- $\binom{a}{b}$ represents the binomial coefficient (combinations)

Connection to Conditional Probability

- Each draw changes the probability for subsequent draws:
- P(success on 2nd draw | success on 1st draw) = (K-1)/(N-1)
- This sequence of conditional probabilities forms the hypergeometric distribution.

Sampling Without Replacement

- Probability changes after each draw
- As N → ∞, approaches the binomial distribution

Use Cases

- Card game:
  - Consider deck of 52 cards (without jokers)
  - which includes 4 aces.
  - If you draw 5 cards without replacement, what is probability that exactly 2 of them are aces?
- Environmental Studies:
  - Ecologist is studying forest of 1,000 trees
  - 300 of which are of an endangered species.
  - Ecologist randomly samples 50 trees.
  - What is the probability that 15 of sampled trees are from endangered species?
  - What is the probability if new sample is collected that 20 more trees are from endangered species?

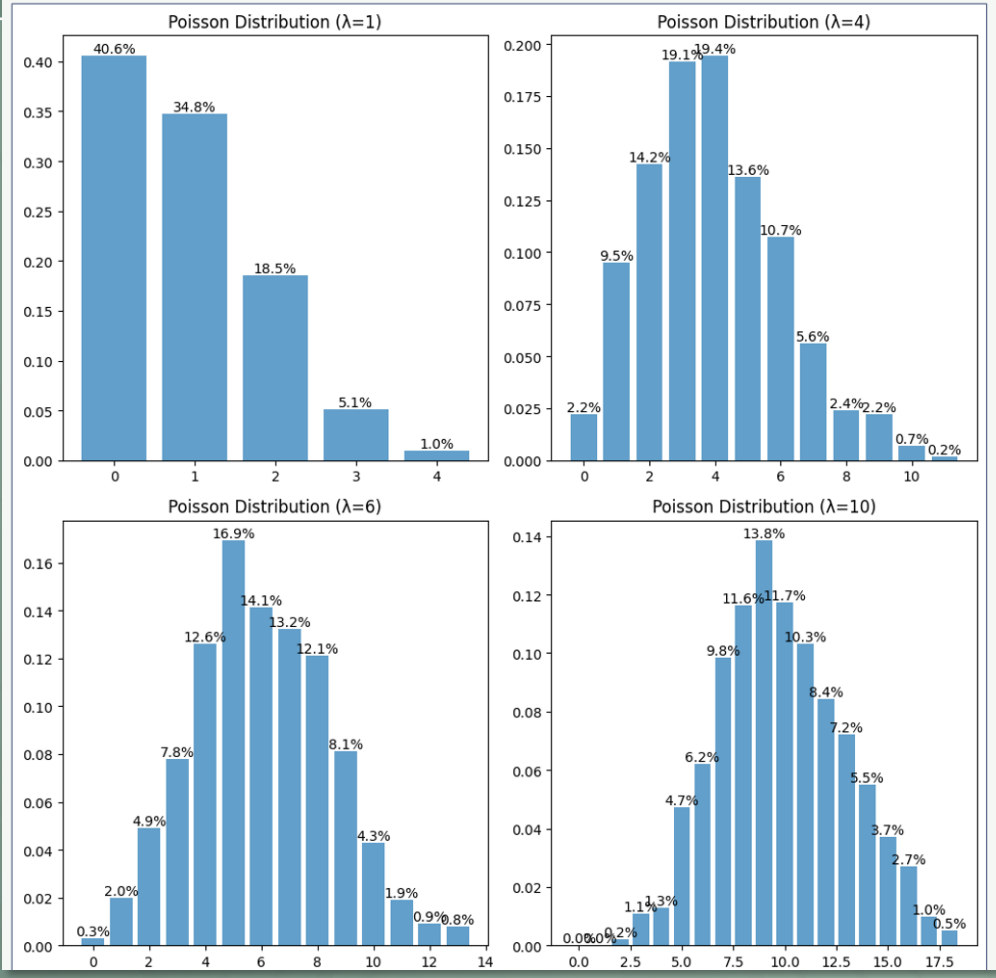
Hypergeometric vs. Binomial Distribution

As population size increases, hypergeometric distribution approaches binomial distribution. For large populations with small samples, binomial can approximate hypergeometric

Characteristic	Hypergeometric Distribution	Binomial Distribution
Sampling Method	Without replacement	With replacement
Population Size	Finite	Infinite or very large
Probability of Success	Changes after each draw	Constant for each trial
Probability Mass Function	P(X=k) = [C(K,k) * C(N-K,n-k)] / C(N,n)	P(X=k) = C(n,k) * p^k * (1-p)^(n-k)
Mean	n * (K/N)	n * p
Variance	n * (K/N) * ((N-K)/N) * ((N-n)/(N-1))	n * p * (1-p)
Trial Independence	Dependent	Independent

Poisson distribution

… probability distribution of a given number of independent events occurring in a fixed interval of time or space, given fixed average rate of occurrence.



Events occur with a known constant mean rate and independently of the time since the last event.

Relation to binomial distribution:

1. The number of trials (n) is very large, i.e. continuous.
2. The probability of success (p) is small.
3. The average number of successes (np) is moderate.

Parameter:

$\lambda = np$ (the mean of the distribution)

Use cases:

- Number of calls received in a given hour, sales per hour, etc.
- emails arriving in inbox per hour, given average arrival rate
- Very few defect items per hour

AI use cases:

- Sport bets: goals/baskets per match
  (Prediction of football, basketball results)

Poisson Process

... is stochastic process that models occurrence of random events over time or space, where events occur continuously and independently at constant average rate

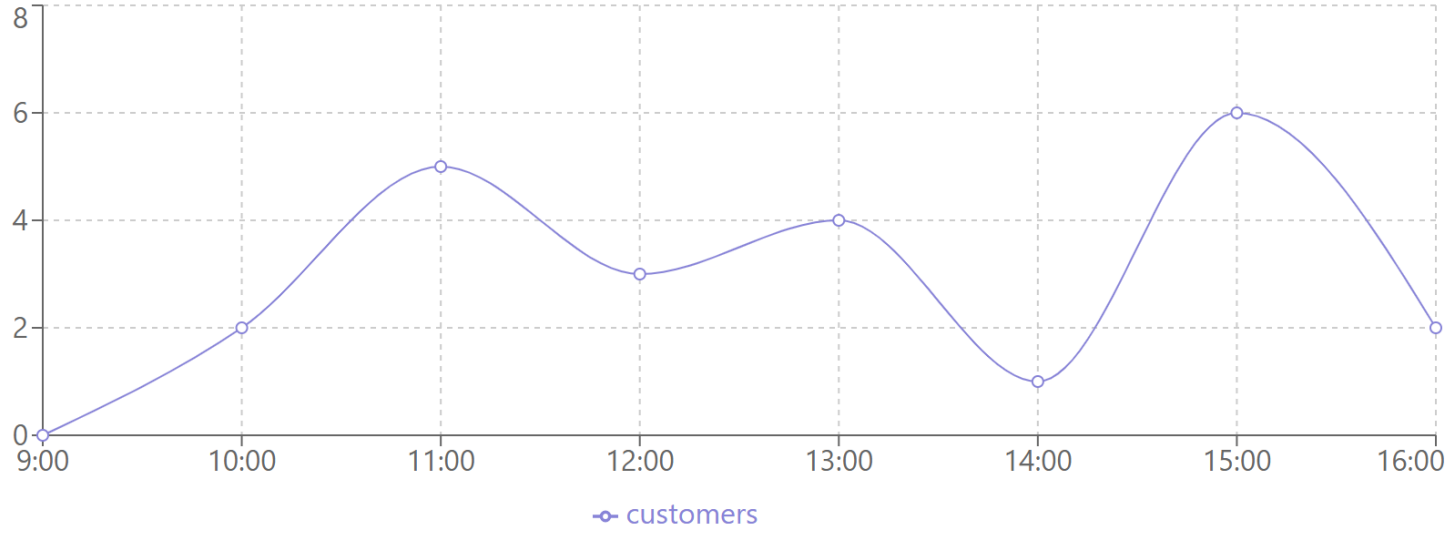
Poisson Process Characteristics:

- Random arrivals over time
- Customers arrive independently
- Average arrival rate (λ) is constant
- No simultaneous arrivals

Car Wash Application:

- Predict customer flow
- Optimize staff scheduling
- Manage queue lengths
- Plan for peak hours

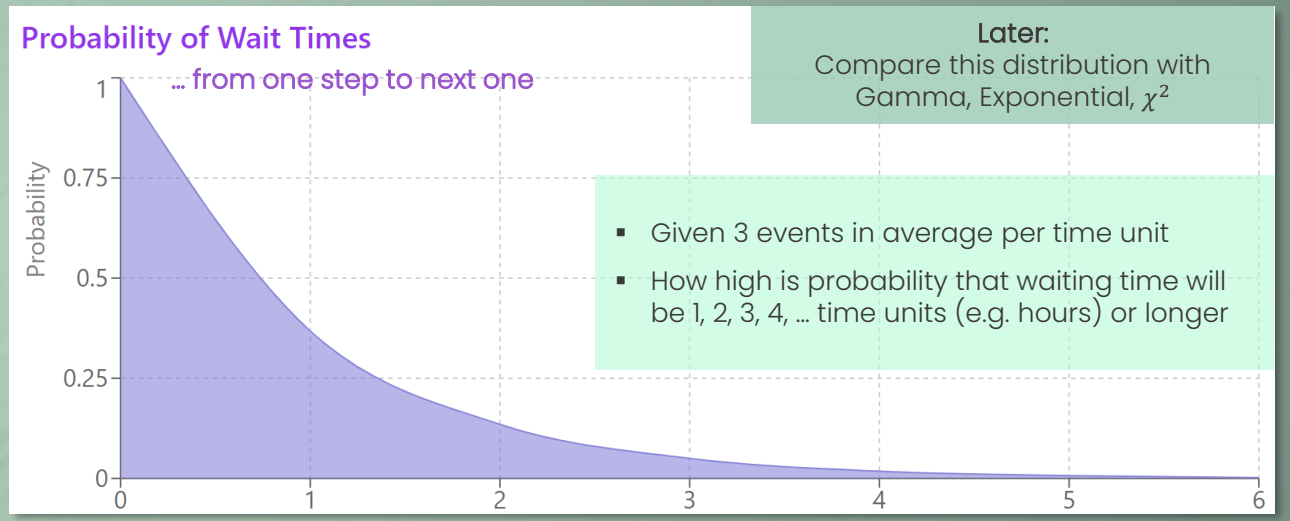
Customer Arrivals Throughout the Day



Note: This chart shows a sample of customer arrivals at a car wash following a Poisson process.

Poisson Processes: Inter-arrival Times and Variance

… is stochastic process that models occurrence of random events over time or space, where events occur continuously and independently at constant average rate



- **Varying Wait Times:**
  Can range from very short to very long
- **Short vs. Long Waits:**
  Short waits more common, long waits possible, but rate
- **Average Wait Time:**
  1 / (average arrival rate)
- **Variance:**
  Equal to square of average wait time

Normal distribution

… is continuous probability distribution characterized by symmetric bell-shaped curve, defined by its mean (average) and standard deviation, also known as the Gaussian distribution

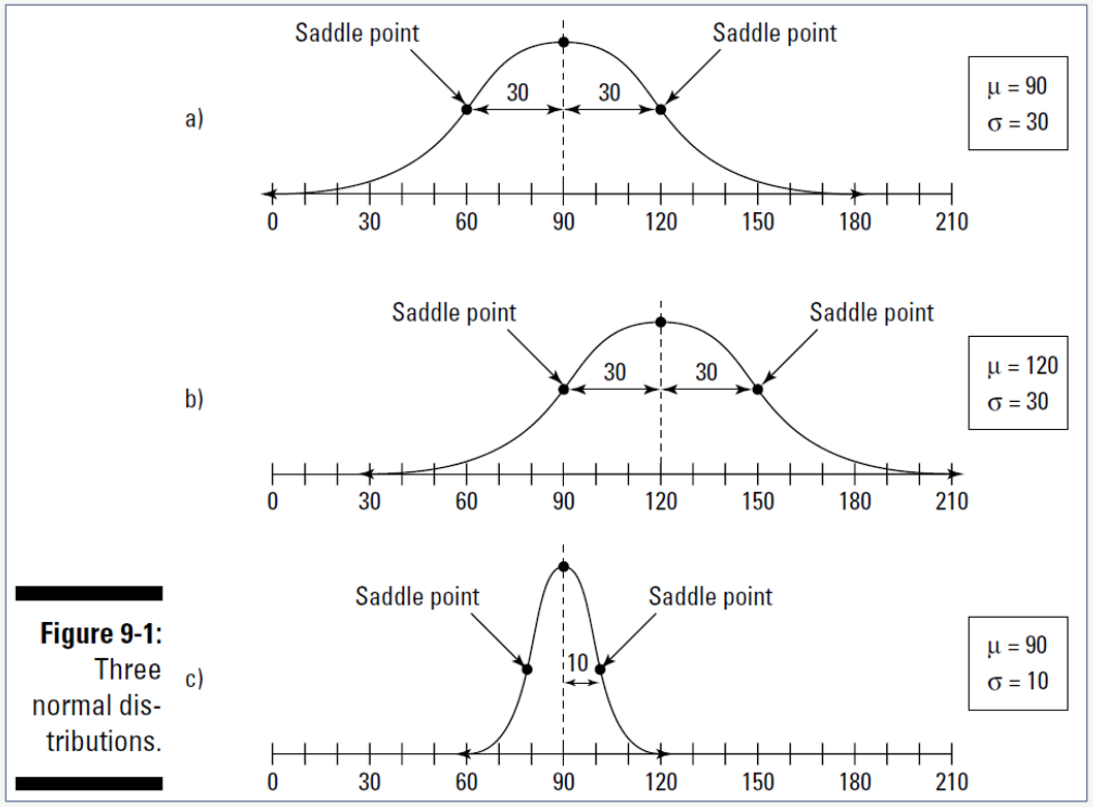


Figure 9-1: Three normal distributions.

Parameters:

- Mean: where the peak of the bell occurs
- Standard deviation: determines the width of the bell

Use cases:

- Human Heights: most individuals have heights around the average, very tall or small people are much less frequent
- Test Scores: most students will score around average, fewer students have very high/low scores.

Standard normal distribution

… is a special case of the normal distribution with a mean of zero and standard deviation of one.

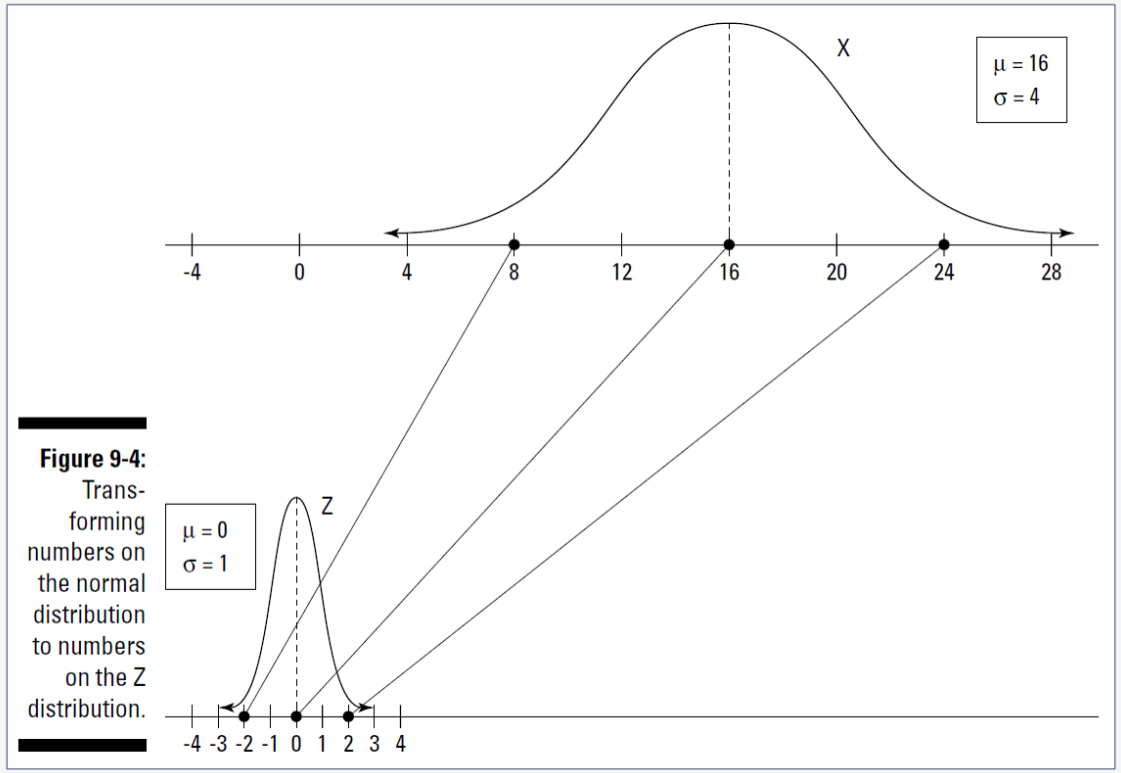


Figure 9-4: Transforming numbers on the normal distribution to numbers on the Z distribution.

z-scoring" or standardization

- subtracting the mean score
- and dividing by the standard deviation

Properties:

- mean (μ): 0
- standard deviation (σ): 1
- symmetric, bell-shaped

Why is it useful?

- This makes it easier to compare scores from different tests or datasets.

AI use case:

- Supervised learning: Prediction errors (differences between actual and predicted values) are often standard normally distributed, e.g. a house price predictor based area, number of rooms, etc.

Students‘ t-distribution

… is used when sample size is small or when population standard deviation is unknown. It is similar to normal distribution, but with heavier tails, e.g. more flat.

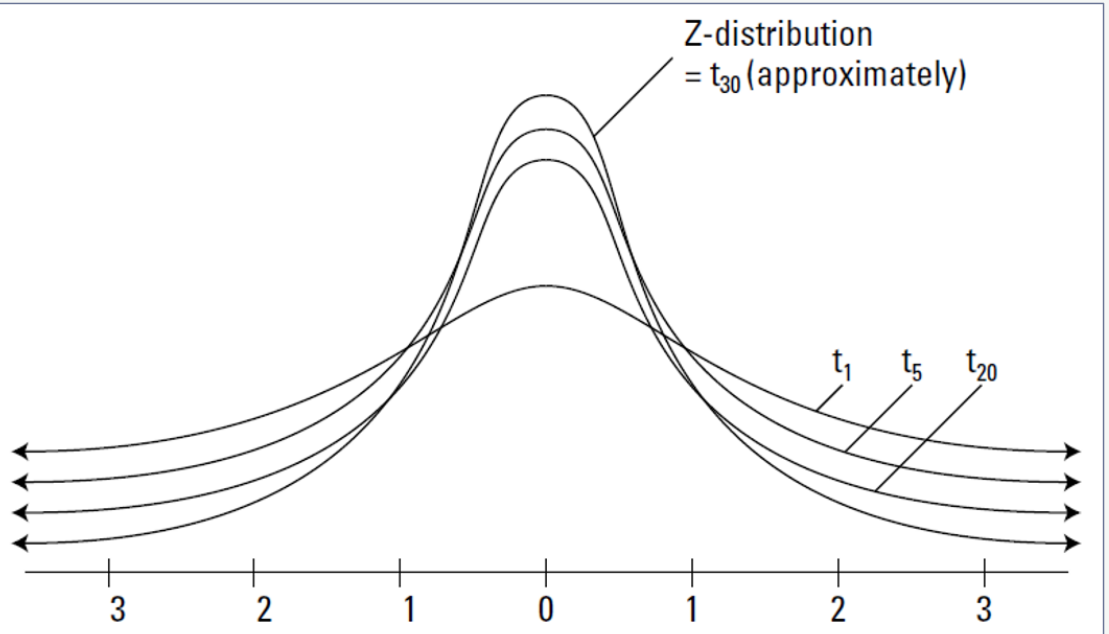


Figure 9-1: *t*-distributions for different sample sizes

If $n < 30$ (number of samples), t-distribution gives more accurate confidence intervals and hypothesis tests compared to normal distribution.

Use case:

- new drug is tested on small (say, 15 patients). Then, population standard deviation is unknown, and sample that is used for estimation is small.

AI use case:

- Any models with very small sample, e.g. for healthcare predictions.

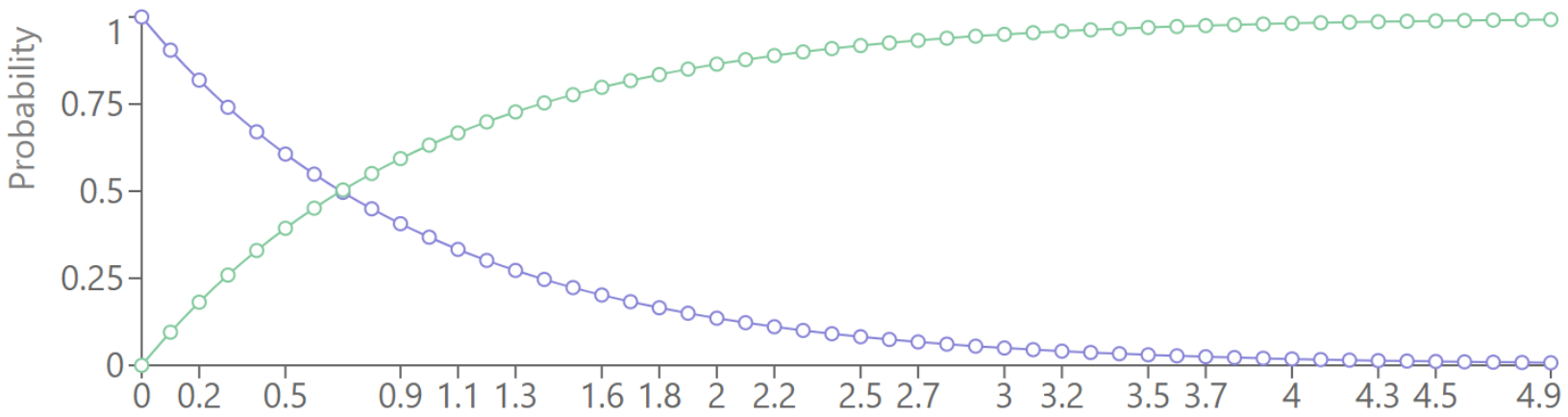
Exponential Distribution

Continuous probability distribution that describes time between events in Poisson process.

Use Cases

- Memoryless property: $P(X > s + t \mid X > s) = P(X > t)$
- Only parameter, constant hazard rate: λ
- Relationship with Poisson distribution

Probability Density Function (PDF) and Cumulative Distribution Function (CDF)



Example: $\lambda = 1$

Exponential vs Poisson Distribution

Exponential distribution models time between events in Poisson process, while Poisson distribution models number of events in fixed interval.

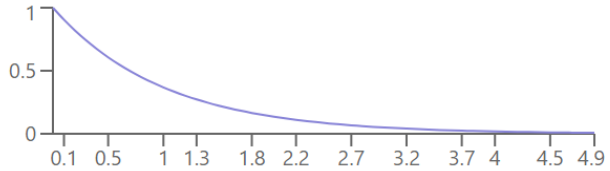
Example: Customer arrivals at a store

- **Poisson**: Number of customers arriving in one hour ($\lambda = 3$ customers/hour)
- **Exponential**: Time between customer arrivals ($\lambda = 1/20$ customers/minute or 3 customers/hour)
- If customer arrivals follow a Poisson process with rate λ, then the time between arrivals follows an Exponential distribution with rate λ

Exponential Distribution

Models time between events in a Poisson process

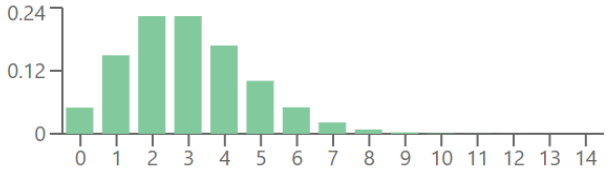
- PDF: $f(x) = \lambda e^{(-\lambda x)}$ for $x \geq 0$
- Mean: $1/\lambda$
- Continuous distribution



Poisson Distribution

Models number of events in a fixed interval

- PMF: $P(X = k) = (\lambda^k * e^{(-\lambda)}) / k!$
- Mean: λ
- Discrete distribution

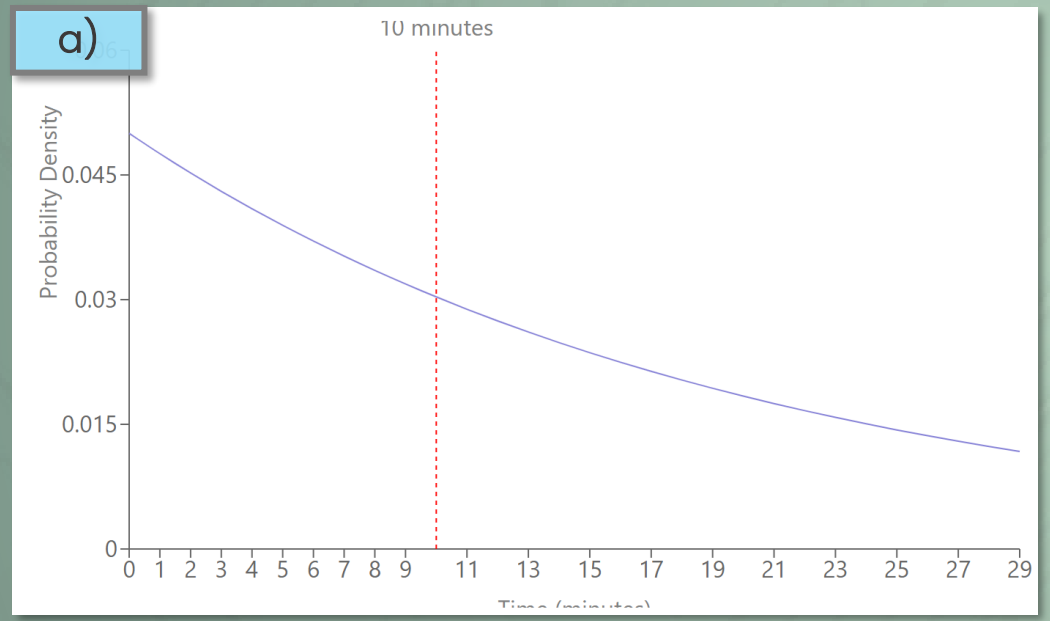


Exponential Distribution: Probability Calculation

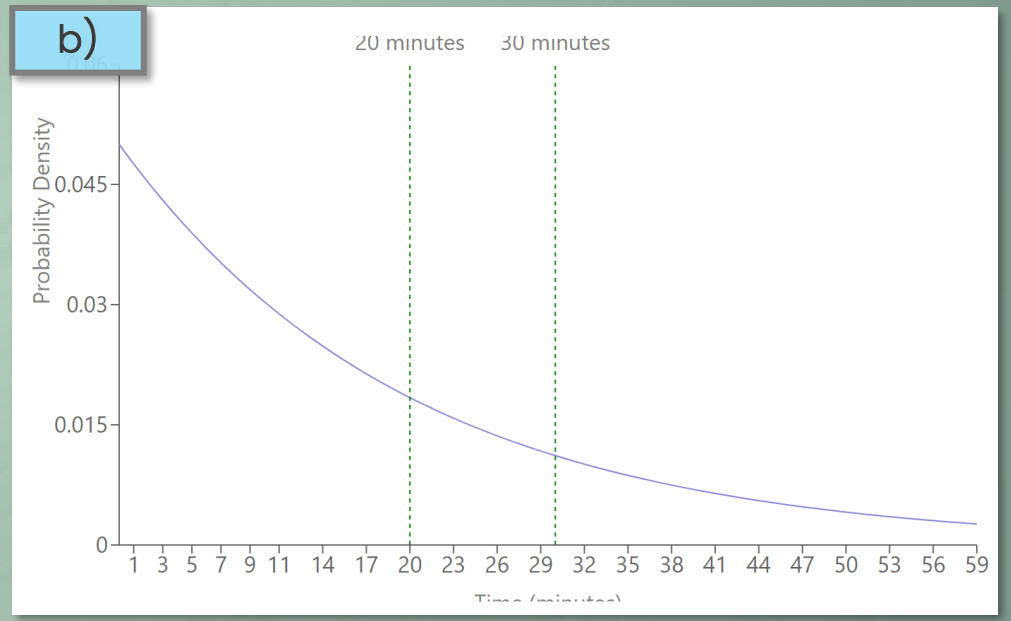
λ = 1/20 customers/minute or 3 customers/hour

Case a) Probability of customer arrival within first 10 minutes

Case b) Probability of customer arrival between 20 and 30 minutes



- Shaded area under curve represents probability of arrival within 10 minutes.
- Exponential curve shows how probability density decreases over time.
- Approximately 39.35% chance of a customer arriving within the first 10 minutes.



- Shaded area between 20 and 30 minutes represents the probability of arrival in this interval
- Calculation uses difference of CDF: F(30) - F(20)
- Approximately 14.47% chance of customer arriving between 20 and 30 minutes.

Gamma Distribution

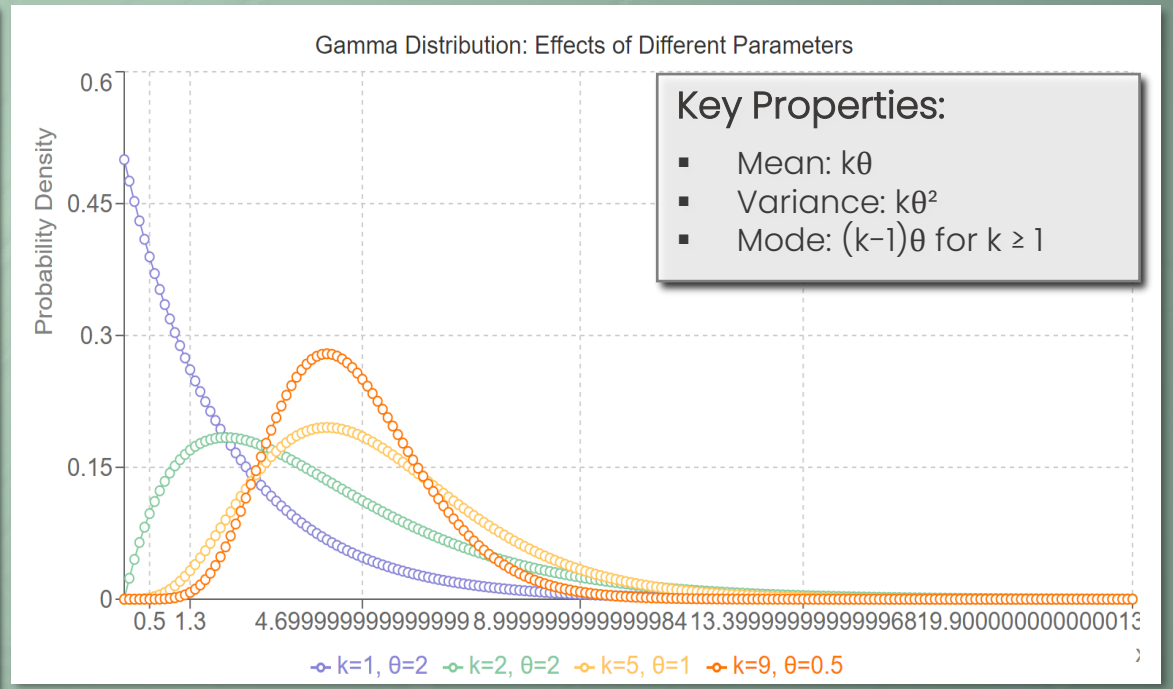
… is continuous, characterized by two parameters: shape (k) and scale (θ),modeling waiting times for k independent events in Poisson process, generalizing exponential distribution.

Parameters

- k (shape): Determines the basic shape of the distribution
- θ (scale): Stretches or shrinks the distribution along the x-axis

Examples and Interpretations

- k=1, θ=2 (Blue):
  Equivalent to an exponential distribution. Could model time between events in a Poisson process.
- k=2, θ=2 (Green):
  More bell-shaped. Might represent time until the second event in a process.
- k=5, θ=1 (Yellow):
  More symmetrical. Could model total processing time for 5 independent tasks.
- k=9, θ=0.5 (Orange):
  Nearly symmetrical, approaching normal distribution. Might represent a sum of many small, independent random variables.



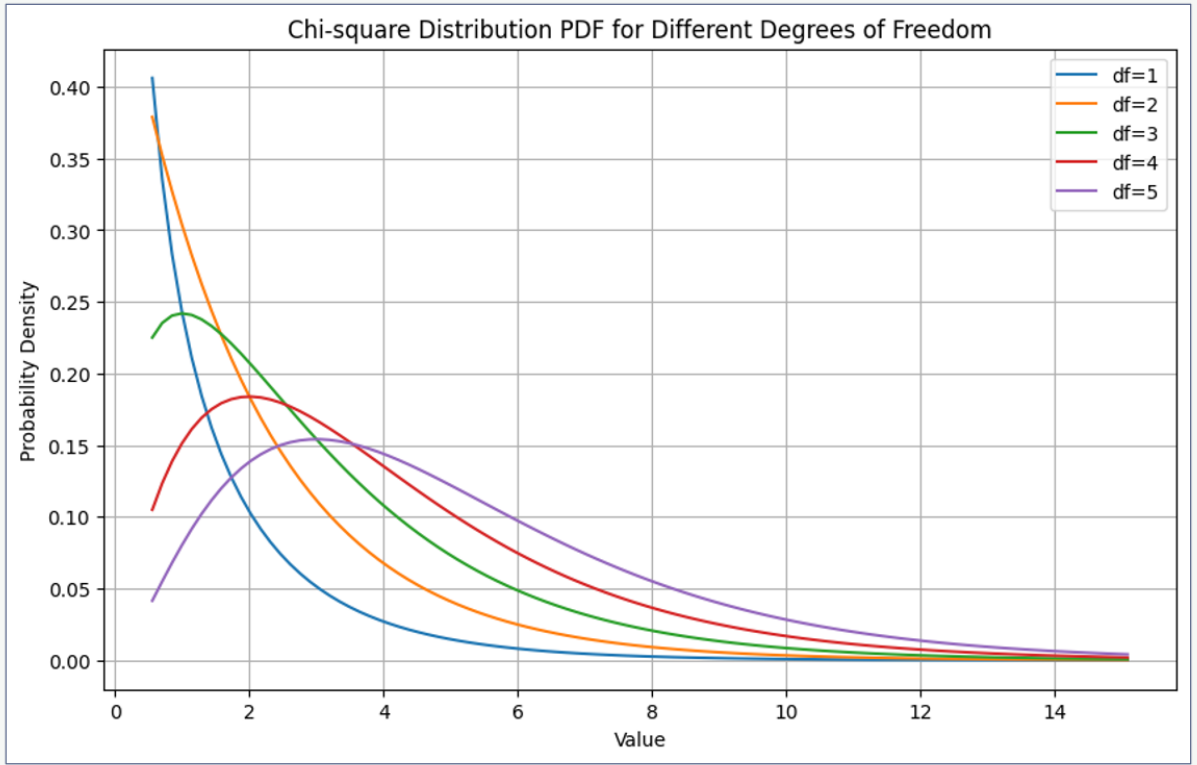
Connection of Poisson, Exponential, Gamma Distributions

Poisson counts events, Exponential measures time between events, and Gamma extends this to time until the k-th event, collectively modeling random processes.

Characteristic	Poisson	Exponential	Gamma
What it models	Number of events in a fixed interval	▪ Time between events ▪ Special case of Gamma: k=1	Time until k-th event
Parameters	λ (rate)	λ (rate)	k (shape), θ (scale) or λ (rate = $1/\theta$)
Support	Non-negative integers	Positive real numbers	Positive real numbers
Mean	λ	$1/\lambda$	$k\theta$
Variance	λ	$1/\lambda^2$	$k\theta^2$
Relationship to others	Inter-event times are Exponential(λ)	Special case of Gamma where k=1	Sum of k Exponential(λ) ~ Gamma(k,$1/\lambda$)
Key property	Sum of independent Poisson is Poisson	Memoryless property	Reproductive property
Example application	Number of calls received in an hour	Time until next call	Time until 5th call

χ^2 distribution

… is a probability distribution of the sum of squares of independent standard normal random variables and is important particularly in the context of variance



https://en.wikipedia.org/wiki/Chi-squared_distribution

It's the distribution of a sum of the squares of k independent standard normal random variables:

$Q = \sum_{i=1}^{k} Z_i^2$

Parameter:

k (mean): independent standard normal random variables, square each and sum them up

Use cases:

- χ^2-tests of independence: They use χ^2 -distribution.
  Is preference for having cat or dog is independent from city?
- Several other hypothesis tests, particularly connected with variance

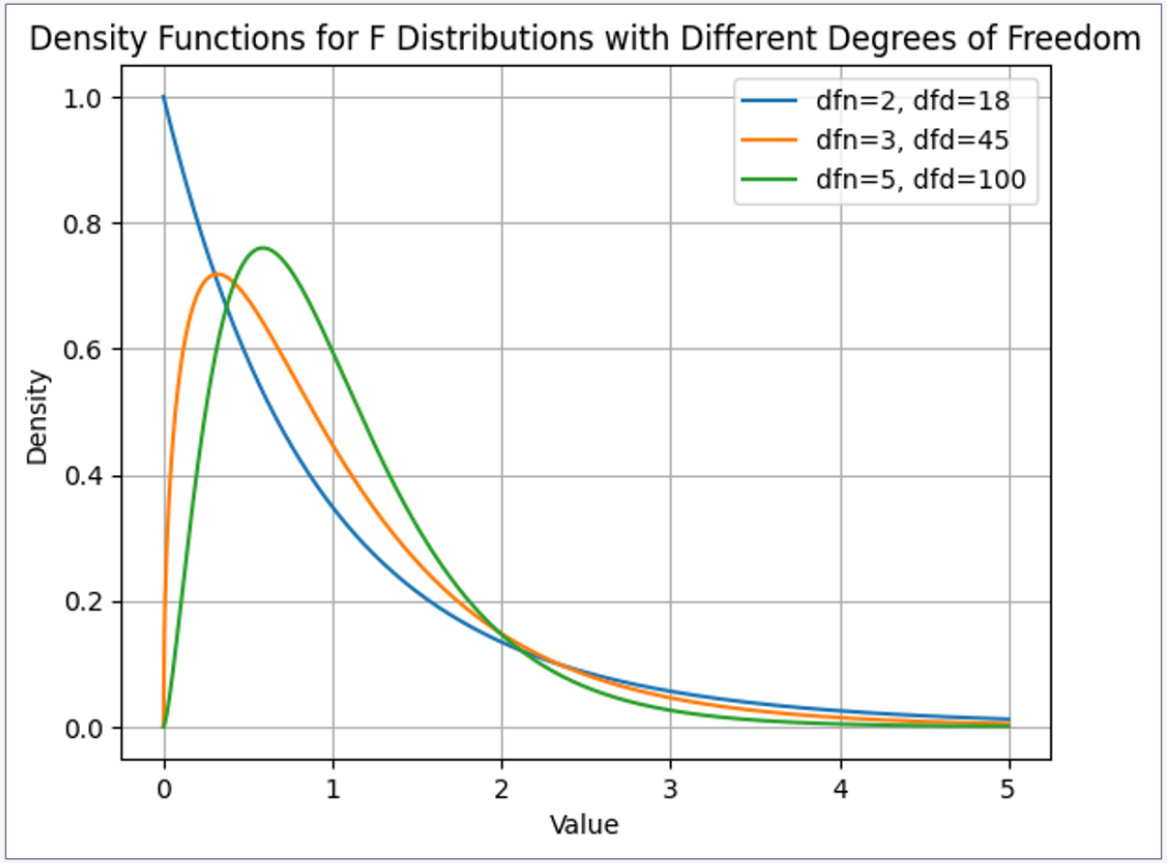
Connection of χ^2 and Gamma Distributions

$\chi^2(k)$ is equivalent to Gamma(k/2, 2), where k is degrees of freedom, i.e. number of occurences

Feature	Chi-Squared (χ^2)	Gamma
Parameters	k (degrees of freedom)	α (shape), θ (scale)
Relationship	▪ $\chi^2(k)$ = Gamma(k/2, 2) ▪ Special case of Gamma with scale = 2	
Support	[0, ∞)	[0, ∞)
Mean	k	αθ
Variance	2k	$\alpha\theta^2$
Skewness	√(8/k)	2/√α
Common Applications	Hypothesis testing, goodness-of-fit tests	Modeling waiting times, rainfall amounts
Flexibility	Less flexible, shape determined by k	More flexible, can model various shapes

F-distribution

… is ratio of 2 independent χ^2-distributions divided by their respective degrees of freedom



$$X = \frac{S_1/df_1}{S_2/df_2}$$

Degrees of freedom (df):

- dfn (df of numerator):
  number of groups being compared minus 1
  between group comparison
- dfd (df of denominator):
  total number of observations minus the number of groups
  within group comparison

Use Cases

- is used in Analysis of Variance (ANOVA)
- comparing variances of ≥ 3 samples
- Evaluation of regression results, overall model significance

https://en.wikipedia.org/wiki/F-distribution

Exercises

Exercise 1:
Problem: You are about to flip a coin five times. The probability of getting heads (success) is 0.5. What is the probability of getting 3 heads?

Exercise 2:
Problem: A manufacturer knows that 10% of his products are defective. He sells products in boxes of 20. What is the probability that a box will contain exactly 2 defective products?

Exercise 3:
Problem: A multiple-choice quiz contains 10 questions. Each question has four possible answers, of which one is correct. If a student guesses the answer to each question at random, what is the probability that the student will answer exactly 4 questions correctly?

Exercise 4:
Problem: A fast food restaurant serves an average of 10 customers every 15 minutes. What is the probability of serving exactly 7 customers in a 15 minute interval?

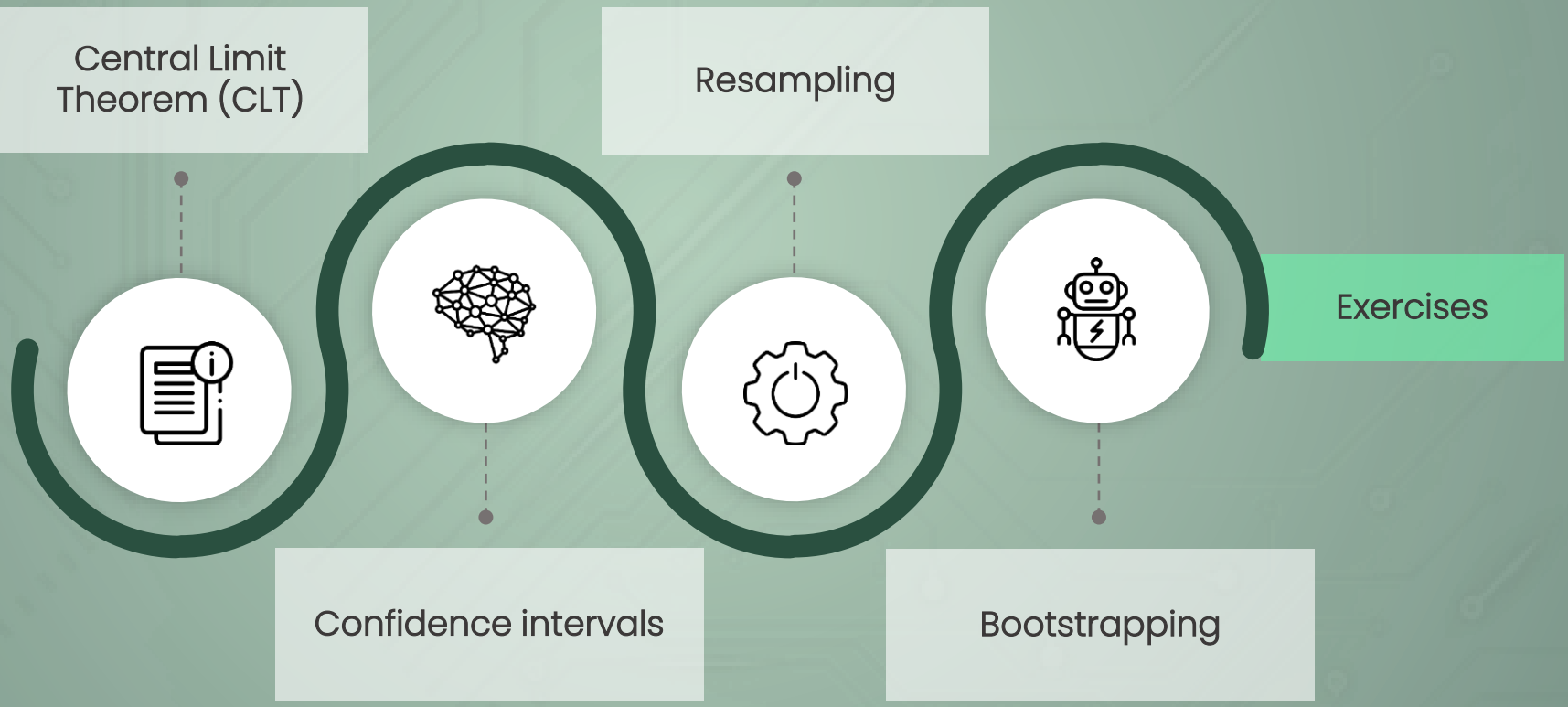
Exercise 5:
Problem: An IQ test is scored such that the mean score is 100 and the standard deviation is 15. What is the probability of a person scoring higher than 130?

Exercise 6:
Problem: A sample of 20 students' test scores has a mean of 76 and a standard deviation of 10. What is the probability of a student scoring less than 70?

Extra-exercises: Relevant use cases are in context of hypotheses and model estimation
Exercise 7:
Suppose the variance of a sample of 20 observations is 5. What is the probability that the sample variance is greater than 6?
Exercise 8:
If two groups of data are sampled from normal distributions with the same variance, the ratio of their sample variances will follow an F-distribution. Suppose you have two samples of sizes 15 and 20 with variances 4 and 2, respectively. What is the probability that the ratio of these sample variances is less than 1?

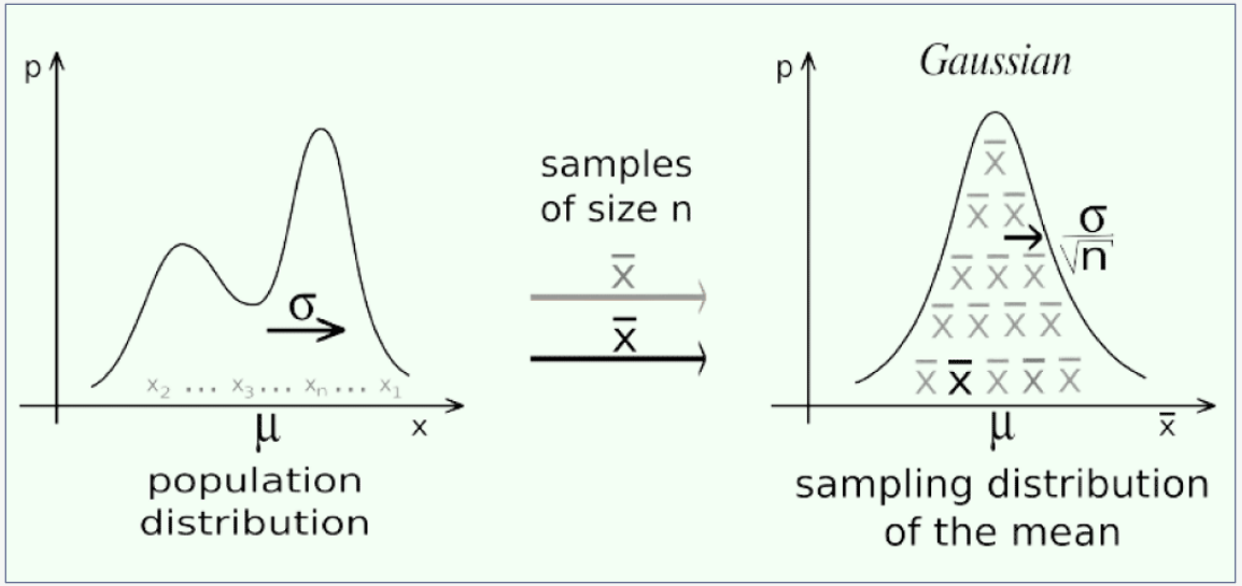
Sampling

… is the process of selecting a subset of individuals from a statistical population to estimate characteristics of the whole population



Central limit theorem

The distribution of sample means approximates a normal distribution as the sample size becomes larger, regardless of the population's distribution.



Definition:

- For population with mean μ and standard deviation σ
- take sufficiently large random samples from the population
- with replacement (independent)
- Independent and identically distributed (i.i.d) random variables
- Sample size is large enough (typically n > 30)

→ distribution of sample means is appr. normally distributed, regardless of population distribution

Why is it important?

- This principle enables us to make statistical inferences about populations based on sample data
- foundation for hypothesis testing, confidence intervals, etc.

Confidence intervals

… is range of values that is likely to contain an unknown population parameter with certain level of confidence, widely used e.g. for hypothesis testing

For large sample size

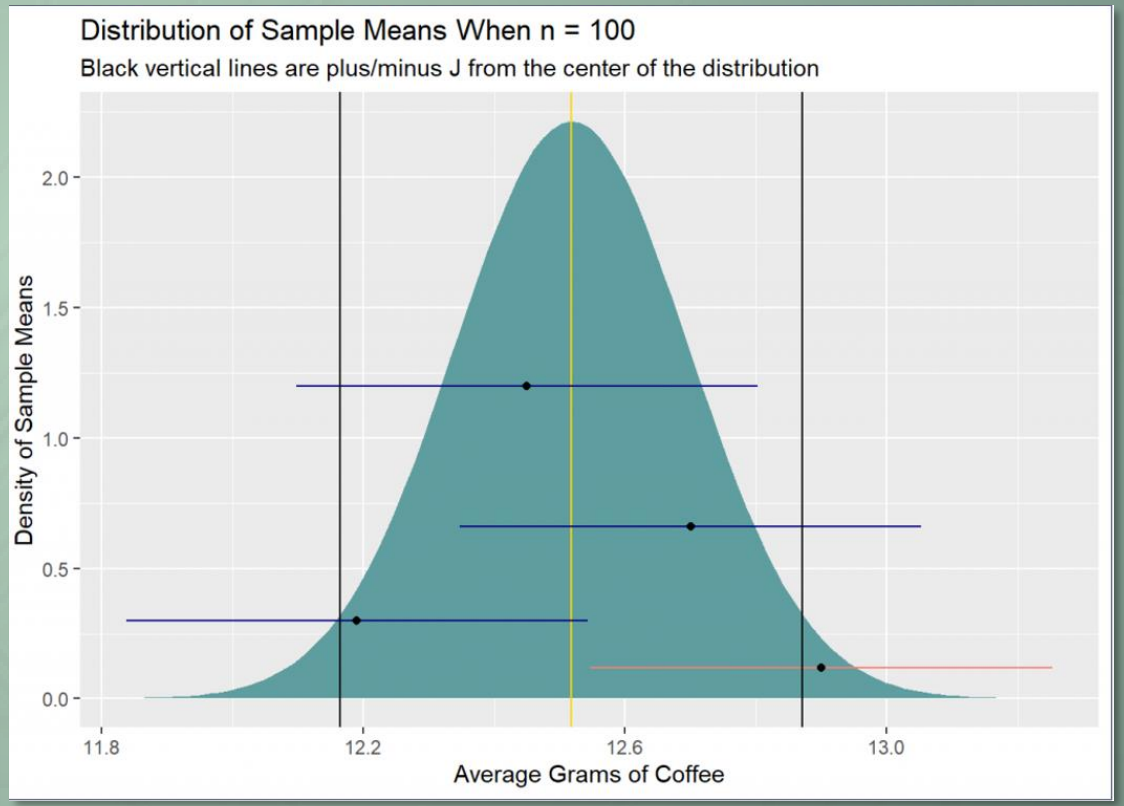
$$\mathrm{CI} = \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Where:

- $\bar{x}$ is the sample mean
- $z_{\alpha/2}$ is the z-score for the chosen confidence level
- σ is the population standard deviation
- n is the sample size

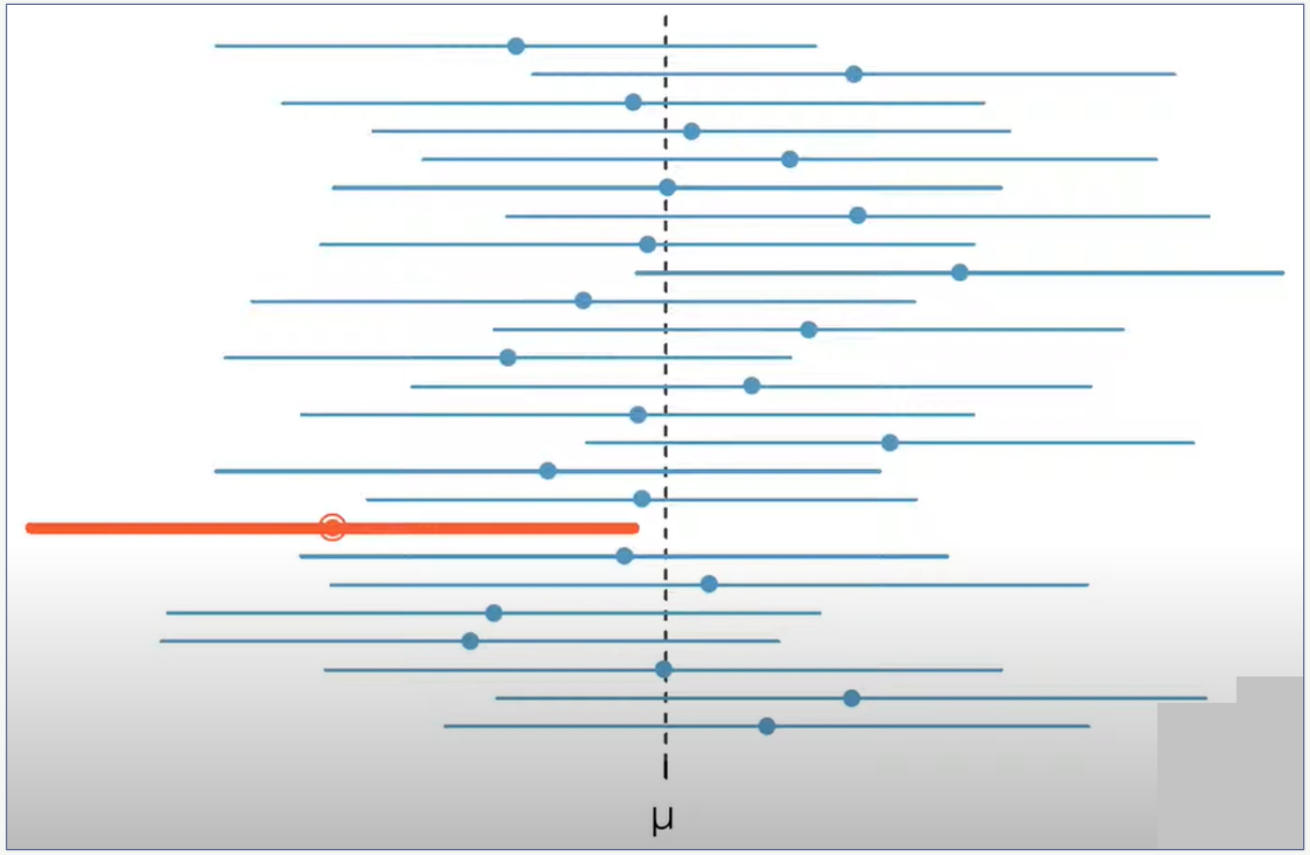
For small sample size

- Similar formula is used
- however not based on standard normal distribution but t-distribution



Confidence intervals

… need careful interpretation. Assumption: α=5%



Assumption:
population parameter μ (average of population) is fixed

Wrong interpretation:
- With a chance of 95% the confidence interval contains μ

Correct interpretation:
- If sampling process is repeated many times or infinitely
- confidence intervals are generated each time
- 95% of these confidence intervals contain μ

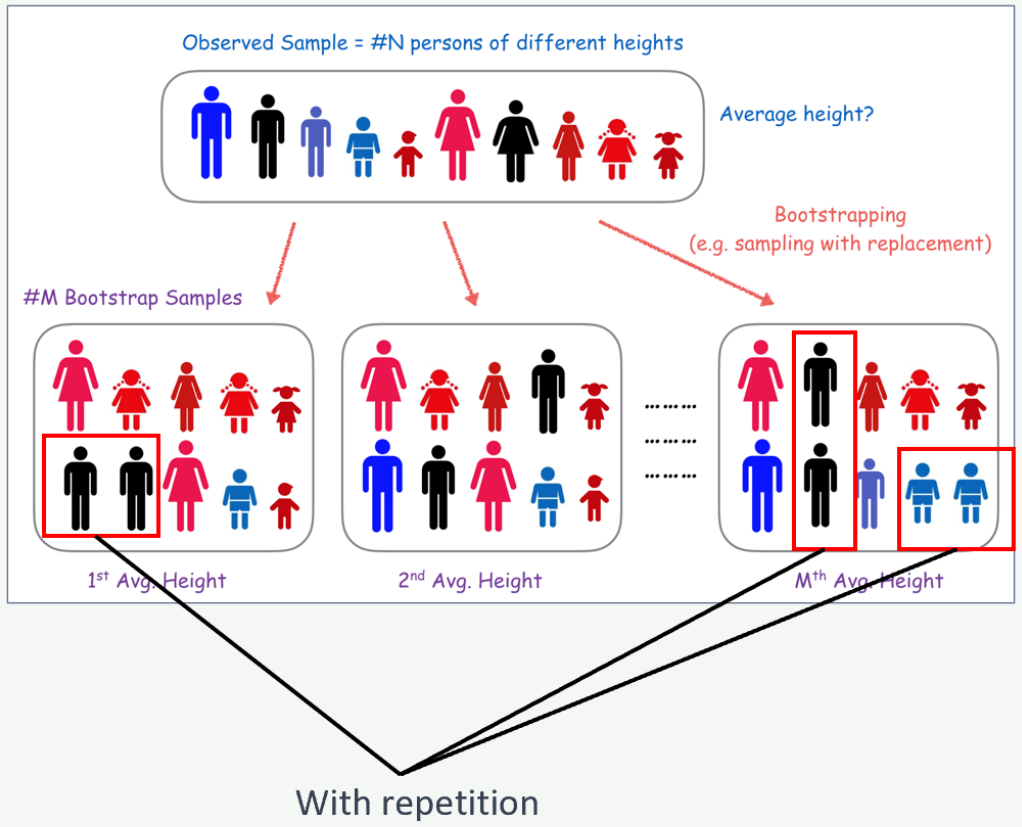
Resampling

… Is set of methods of statistical inference that involve drawing repeated samples from original data samples

What is it good for?	• Versatile method to make statistical inferences about a dataset • It helps to validate models by using random subsets (bootstrap) or sequences (cross-validation) dataset • It extends possibilities for hypothesis testing and estimation
Example: Bootstrap	• Many resamples (with replacement) of the observed dataset are generated • Arbitrary distribution is transformed into normal distribution • convenient for estimation and hypothesis testing.
Example: Cross-validation	• a form of resampling used to prevent overfitting in machine learning models • dataset is divided into 'k' subsets; used for training the model • respective residual data is used as test set

Bootstrapping

… is a resampling method that involves drawing repeated samples from the original data set, with replacement. Each sample drawn is the same size as the original dataset.



Procedure:

- Draw sample from the original dataset with replacement
- Often of same size like original dataset
- Compute desired statistic from this resample
- Repeat the process multiple times (often thousands or tens of thousands of times), each time storing the statistic.
- Use distribution of these bootstrapped statistics to make inferences about the population.

Why does it make sense?

- It provides method to estimate variability or confidence intervals of statistic when underlying distribution is unknown or complex.
- It's a simple and powerful method requiring very few assumptions

AI Use case:

- Random forests (like XGBoost):
  Classification or numeric prediction
  based on many resampled pruned decision trees

Exercises

Exercise 1
Implement a python function called central_limit_theorem that accepts three arguments: population_data, sample_size, and num_of_samples. Your function should simulate the central limit theorem. The function should draw the specified number of samples of the given size from the population data, calculate the means of these samples, and return a list of these means.

Test this function using a randomly generated population data. Plot the distribution of sample means and the mean of the population.

Exercise 2
Implement a python function called confidence_interval that accepts two arguments: data and confidence. Your function should calculate and return the confidence interval of population mean for the given data at the specified confidence level.
Test your function on a randomly generated data.

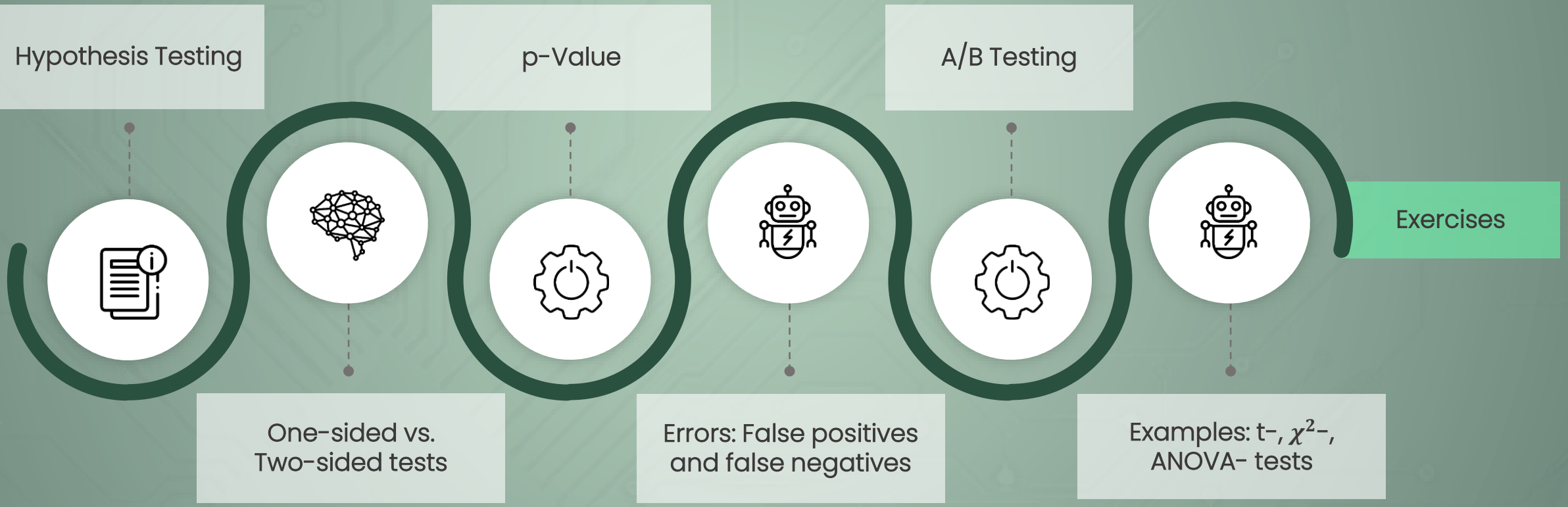
Exercise 3
How does the confidence interval of variance look like?

Exercise 4
Implement a python function called bootstrap that accepts three arguments: data, num_samples, and statistic. Your function should perform bootstrap sampling on the data (resampling with replacement) the specified number of times, applying the provided statistical function to each sample, and return an array of the calculated statistic for each sample.

Test this function using a randomly generated data and numpy's mean as the statistic. Calculate and print the mean and standard deviation of the bootstrap samples' statistic.

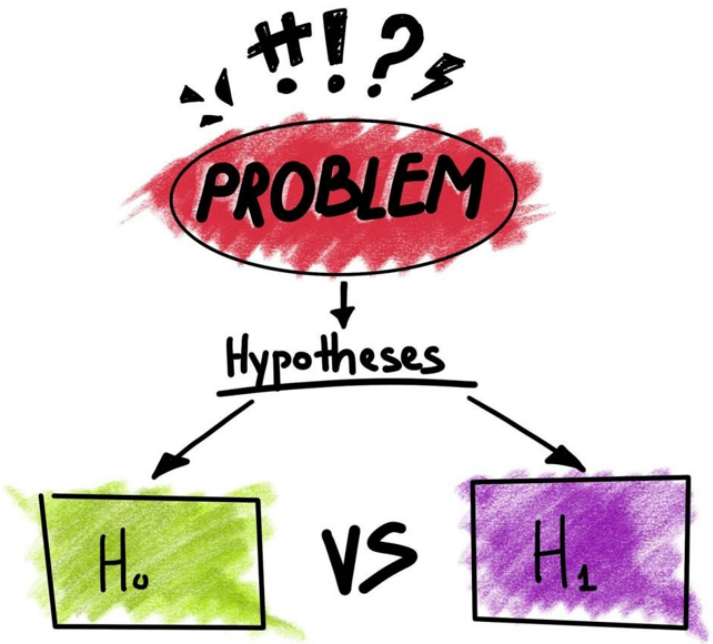
Hypothesis testing

… is statistical method that uses sample data to evaluate the plausibility of null hypothesis ("there is no effect"). Is there enough evidence In the data in order to reject the null?



Hypothesis testing

… is statistical method that uses sample data to evaluate the plausibility of null hypothesis ("there is no effect"). Is there enough evidence In the data in order to reject the null?

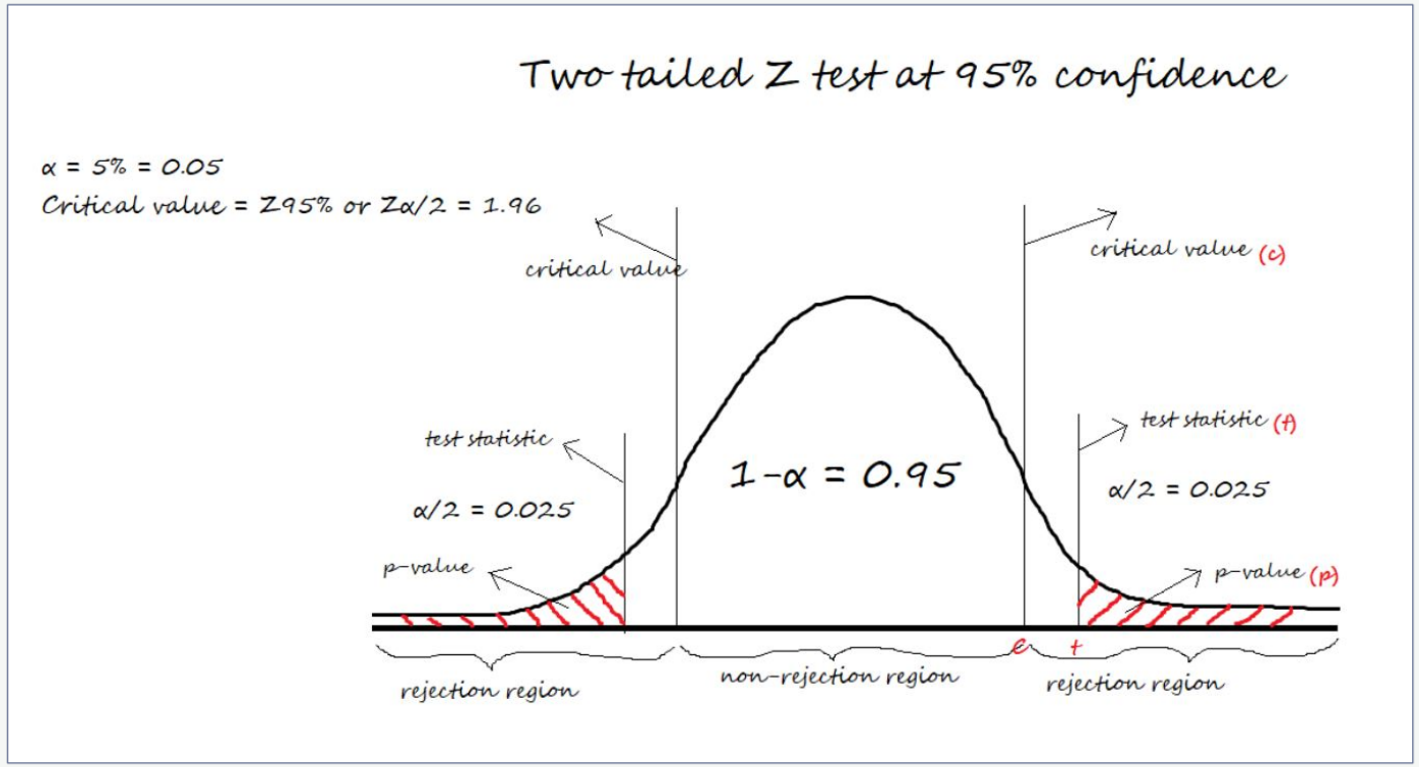


It involves formulating two competing hypotheses

- the null hypothesis (H_0): initial claim, status quo, which usually means: No effect/difference
- and the alternative hypothesis (H_1); we want evidence for the alternative hypothesis
- We assess evidence from the data, in order to determine whether to reject or fail to reject the null hypothesis.

Hypothesis testing

… involves specifying significance level, analyzing sample data using statistical test rejecting or not rejecting null hypothesis



Workflow:

1. **State Hypotheses:**
   Null Hypothesis (H_0)
   and Alternative Hypothesis (H_1)

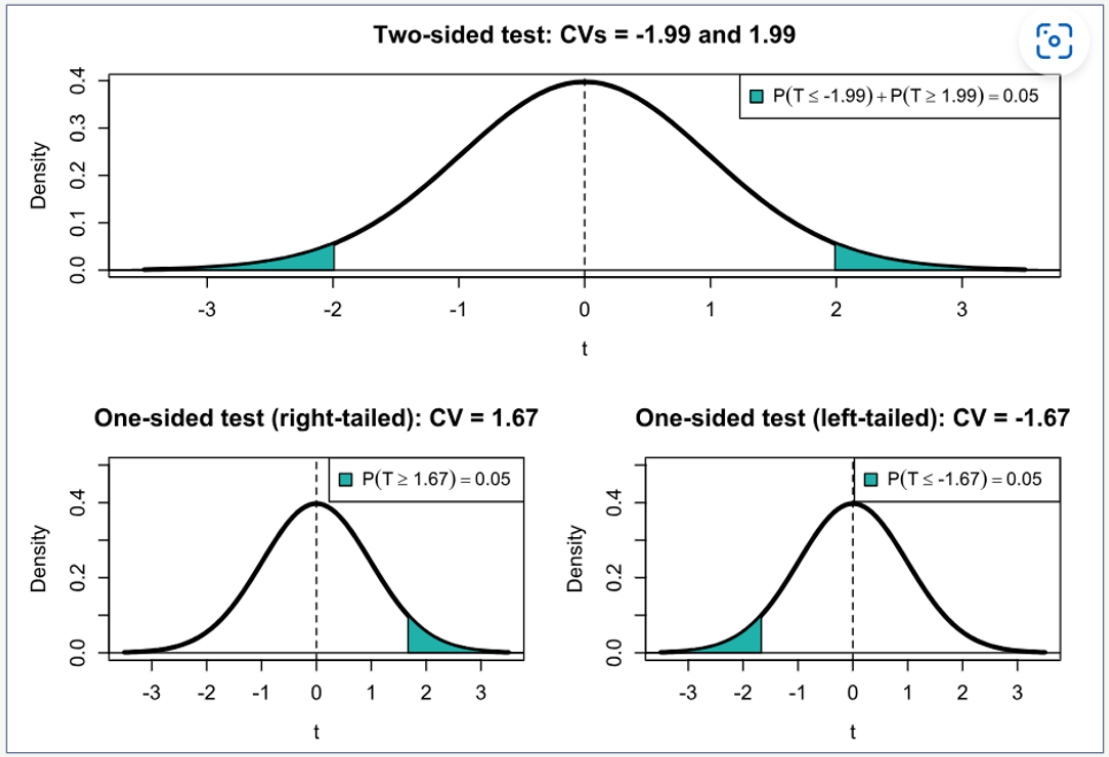
2. **Specify critical value α:**
   usually 0.05 for 2-sided tests

3. **Calculate test statistic t with:**
   - t-test
   - χ^2-test
   - z-test
   - ANOVA-test, and many more

4. **Interpret results comparing t and α :**
   - Reject null hypothesis (support H_1) or not (support H_0)?

One-sided vs. Two-sided tests

One-sided hypothesis test checks if parameter is greater/less than a certain value.
Two-sided test checks if parameter is not equal to a certain value, regardless of direction.



Two-sided (or Two-tailed) Tests

- … assess if a parameter is either greater or less than the null hypothesis value.
- **Example:**
  Company is testing if a website redesign affects,
  i.e. either increases or decreases user time on site.
  - H_0: Website redesign has no effect on user time on site.
  - H_1: Website redesign affects user time on site.

One-sided (or One-tailed) Tests

- … check if parameter is either greater or less than null hypothesis value, but not both.
- **Example:**
  Pharmaceutical company is testing a new drug and wants to know if it reduces disease recovery time (not interested if it increases).
- H_0: Drug does not reduce recovery time.
- H_1: Drug reduces recovery time.

Deciding to reject H_0 or not

… is standardized procedure like an “API”

- Initial assumption: “There is no effect”, i.e. we don’t reject H_0 yet
- Samples from population vary around the population mean μ
- Theoretical probability of samples
  - around μ (or further away, the integral) is high
  - far away from μ (or even further, the integral) is low
- Now you take the real sample and focus on distance from μ

If it is **very close**

sample is **not extreme**

theoretical probability of getting more extreme sample is **high**

we say: “There is no effect, there is **not enough evidence** to reject H_0”

simpler: “**We choose H_0**”

If it is **very far**

sample is **extreme**

theoretical probability of getting more extreme sample is **low**

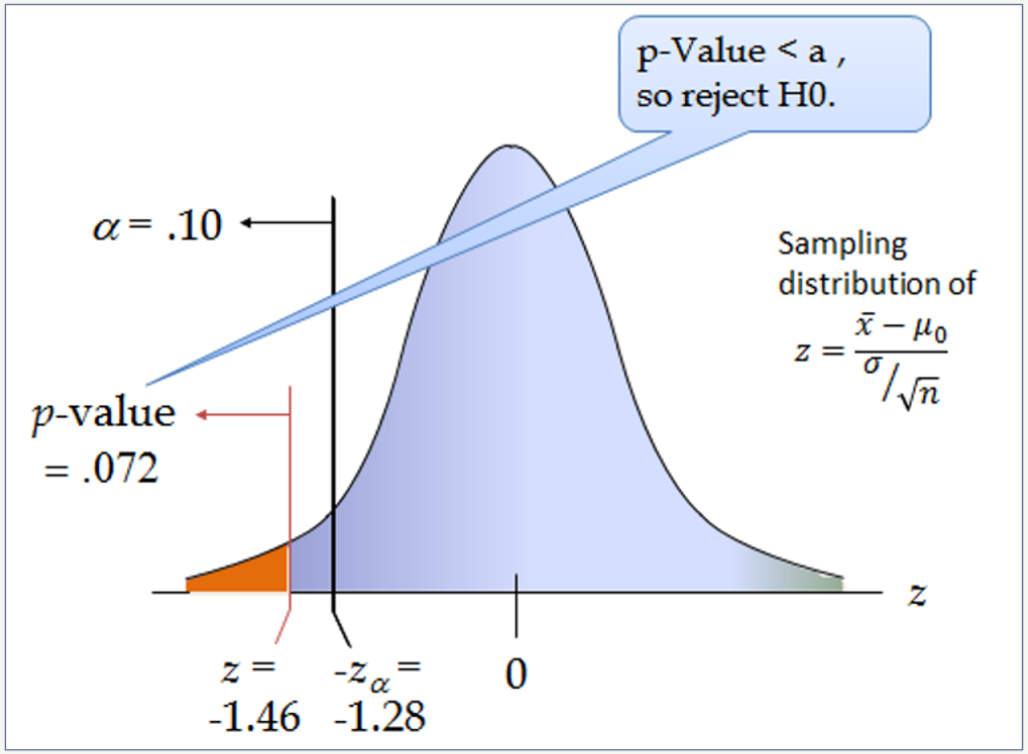
we say: “There is an effect, there is **enough evidence** to reject H_0”

simpler: “**We choose H_1**”

We decide based on this integral. How is it called?

p-value

… is a tool for deciding whether to reject or not reject the null hypothesis



Example for rejecting H_0, the 2 groups are different

Interpreting the P-value

- Small p-value (typically ≤ $\alpha = 0.05$)
  suggests strong evidence against the null hypothesis,
  "we reject the null", "we support H_1"
- Large p-value $\alpha = 0.05$)
  suggests weak evidence against the null hypothesis,
  "we fail to reject the null", "we support H_0"
- If we reject the null, there is typically significant difference between the 2 groups

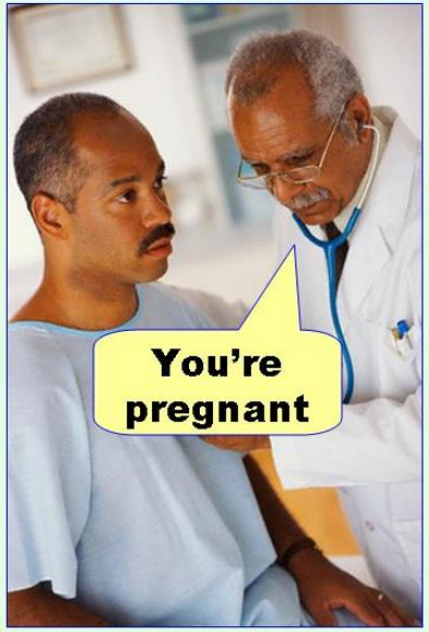
Properties

- p-values can be affected by the size of the sample.
- Figure: in 0.72 of 100 times you should have selected H_0.

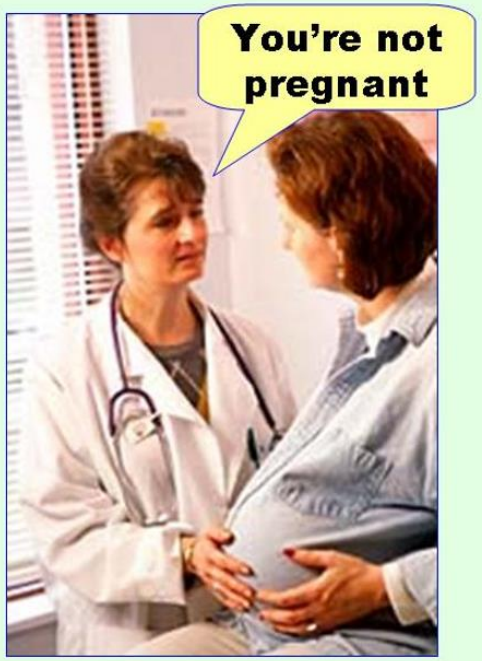
Errors: False positives and false negatives

… matter, as they represent incorrect conclusions that can impact decisions based on test results

Type I error
(false positive)



Type II error
(false negative)



- Selecting H_1 is the "positive" statement
- Selecting H_0 is the "negative" statement

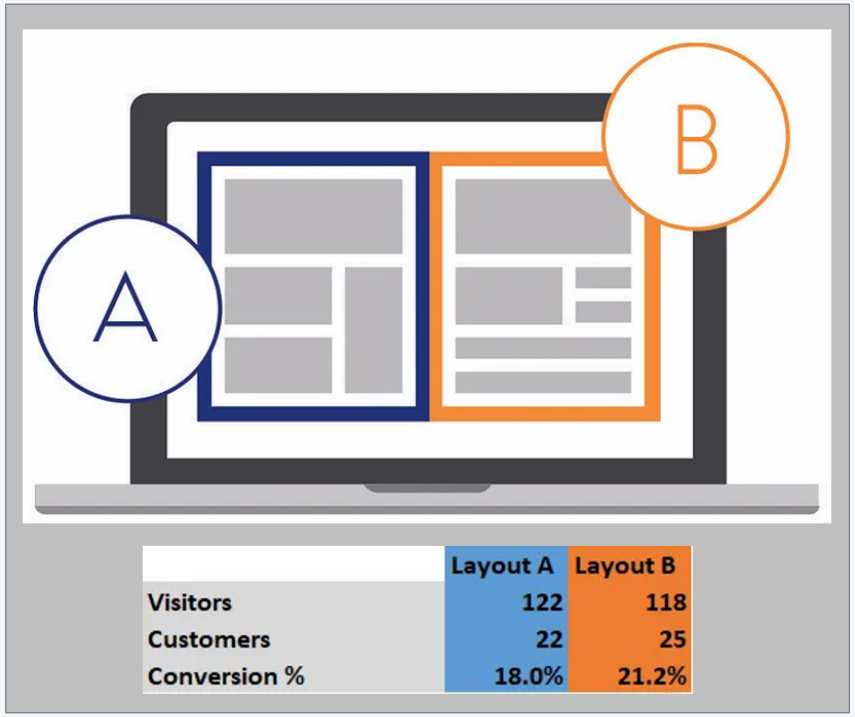
Type I error means "false positive":
- When H_0 is true and therefore "selecting H_1 is incorrect"
- precise formulation: H_0 is incorrectly rejected

Type II error means false negative:
- When the H_0 is false, but "we select it"
- precise formulation: we fail to reject it

Connection to significance level α:
- The maximal probability of making Type I error is defined by α, such as 0.05 or 0.01.

What is A/B-Testing?

… also known as split testing, is method of comparing two versions of webpage, email, or other user experience to determine which one performs better.



	Layout A	Layout B
Visitors	122	118
Customers	22	25
Conversion %	18.0%	21.2%

https://medium.com/towards-data-science/data-science-you-need-to-know-a-b-testing-f2f12aff619a

- **Identify goal:**
  Your goal might be to increase website conversions, improve email click-through rates, or boost the number of sign-ups.
- **Create Variants:**
  Develop two versions of element you want to test - the control (A) and the variant (B). The versions should be identical except for one change.
- **Split audience:**
  Randomly divide audience into two equal groups.
  One sees version A, the other sees B.
- **Conduct test:**
  Release both versions at the same time and collect data on how each version performs in relation to your goal.
- **Analyze the Results:**
  Use statistical analysis to determine which version performed better.
- **Implement Changes:**
  If B performs significantly better, consider replacing version A with B.

A/B Testing: Choosing the Right Test Type

Tests types are nuanced, slight differences in hypothesis determine selection

Test Type	Scipy method	Use Case, compare:	Examples
1. Two sample t	ttest_ind	• means of • continuous data between • two groups of independent samples	• Company tests new website B and compares it with old one A • Visitors are divided into 2 groups that only see one of them • Visit durations are measured → Does the subgroup stay longer on new website B?
2. Paired t	ttest_rel	• means of • continuous data for the • same group at different times	• Like 1 • But: The same group is tested • Visit durations are measured, first with website A later with B → Does the same group stay longer on new website B?
3. χ^2 for independence	chi2_contingency or chi2.sf	• Variance of • categorical data between • two or more groups	• A company has employees in 4 experience levels • On each level a part of them received promotion (yes/no) → Does experience level have impact on promotion?
4. One factor ANOVA	f_oneway	• means of • continuous data between • ≥ three groups of independent samples	• Education program in company, • 3 learning styles: self-paced online courses, instructor led, mixed • Employees are divided in 3 subgroups → Does the learning style have impact on learning success?

Two sample t-tests

… are statistical tests used to compare the means of continuous data between two groups and determine if they are significantly different from each other.

The formula for the t-statistic in a two-sample t-test is:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

With:

- $\bar{X}_1$ and $\bar{X}_2$ are the sample means of the two groups
- s_1^2 and s_2^2 are the sample variances of the two groups.
- n_1 and n_2 are the sample sizes of the two groups
- **Degrees of freedom:** $\mathbf{df = n_1 + n_2 - 2}$

Variant A: No significant difference

```
# Randomly generating test scores for Group A and Group B
np.random.seed(0)  # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(80, 10, 30)
```

→ df = 58

Variant B: Significant difference

```
# Randomly generating test scores for Group A and Group B
np.random.seed(0)  # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(90, 10, 30)
```

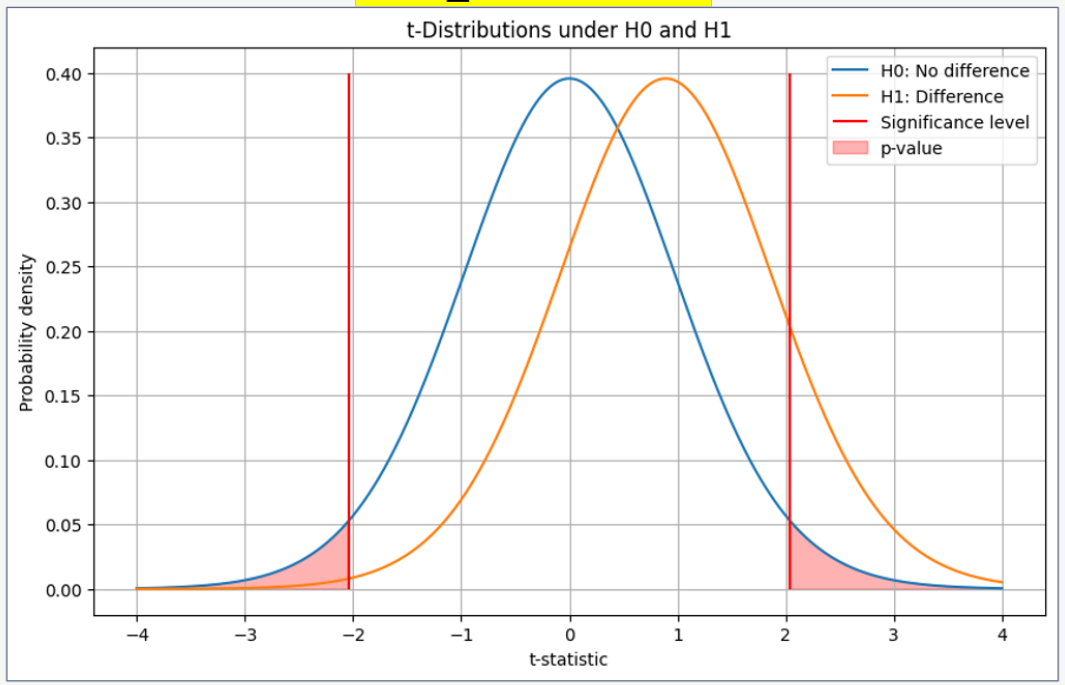
→ df = 58

Two sample t-tests

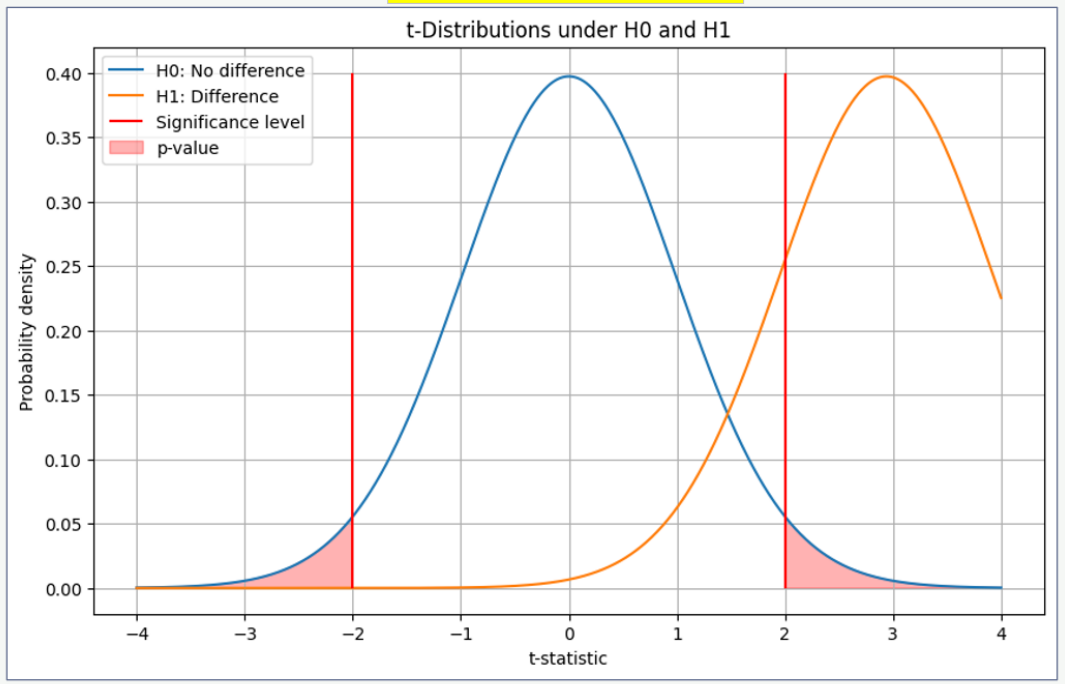
… are statistical tests used to compare the means of continuous data between two groups and determine if they are significantly different from each other.

Critical value: 2.001 (2-sided!)

ttest_stat = 0.8897



ttest_stat = 2.9389



H0:
There is no evidence that subgroup B stays longer on the website …

H1:
There is evidence that subgroup B stays longer on the website …

Paired t-tests

… are used when you want to compare the means of continuous data for the same group at two different times or under two different conditions

The formula for the t-statistic in a paired t-test is:

$$t = \frac{\bar{D}}{s_D / \sqrt{n}}$$

With:

- $\bar{D}$ is the mean of the differences between the paired observations.
- s_D is the standard deviation of the differences between the paired observations.
- n is the number of pairs.
- **Degrees of freedom:** $\mathbf{df = n - 1}$

Variant A: No significant difference

```
# Randomly generating test scores for Group A and Group B
np.random.seed(0)  # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(80, 10, 30)
→ df = 29
```

Variant B: Significant difference

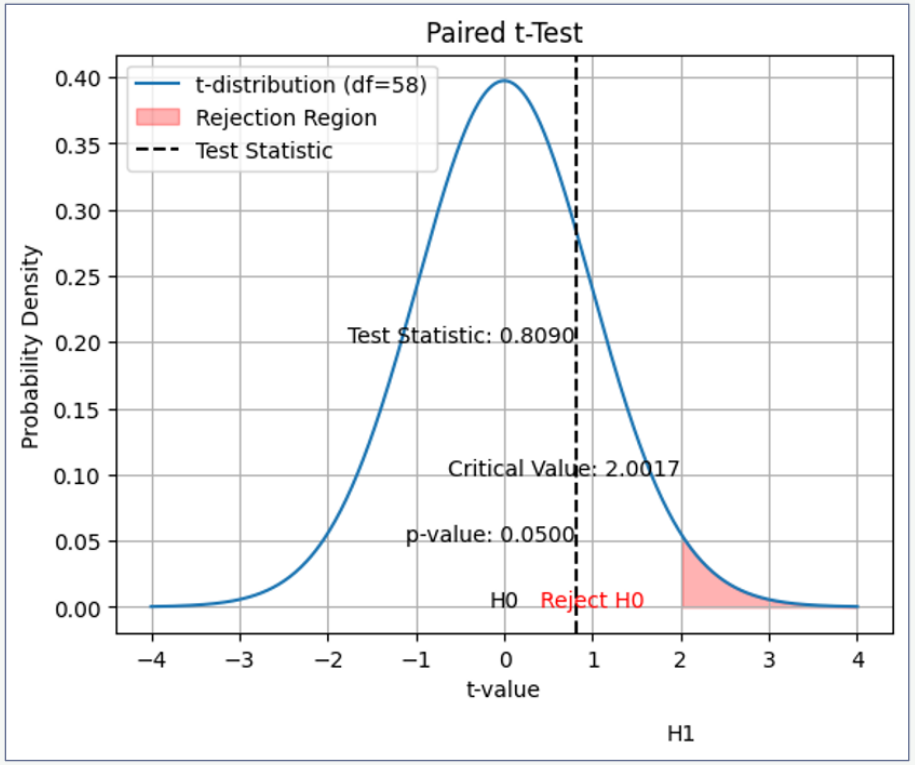
```
# Randomly generating test scores for Group A and Group B
np.random.seed(0)  # for reproducibility
group_A_scores = np.random.normal(75, 10, 30)
group_B_scores = np.random.normal(100, 10, 30)
→ df = 29
```

Paired t-tests

… are used when you want to compare the means of continuous data for the same group at two different times or under two different conditions

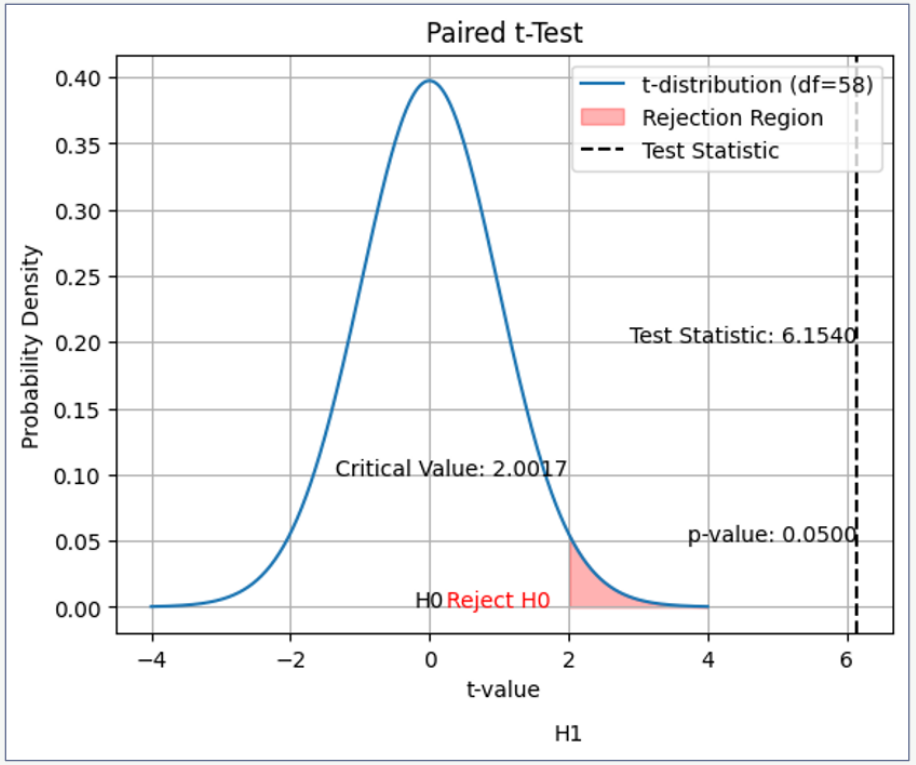
Critical value: 2.0452 (2-sided!)

paired_t_stat = 0.8090



H0:
There is no evidence that group stays longer on the new website …

paired_t_stat = 6.1540



H1:
There is evidence that group stays longer on the new website …

χ^2 independence test

… statistical tests used to determine if there's a significant association between two categorical variables in a sample.

Observed	2007	2008	2009	2010	2011	Total	
Freshman	560	495	553	547	512	2667	
Sophomore	369	385	358	361	393	1866	
Junior	209	226	248	268	285	1236	
Senior	267	277	304	328	340	1516	
Unclassified	64	70	93	77	126	430	
Total	**1469**	**1453**	**1556**	**1581**	**1656**	**7715**	**7715**

Observed	2007	2008	2009	2010	2011	Total	
Freshman	507,818924	502,287881	537,893973	546,536228	572,462994	2667	
Sophomore	355,301879	351,432016	376,344264	382,390927	400,530914	1866	
Junior	235,344653	232,781335	249,282696	253,287881	265,303435	1236	
Senior	288,658976	285,514971	305,754504	310,667012	325,404537	1516	
Unclassified	81,8755671	80,9837978	86,7245625	88,117952	92,2981205	430	
Total	**1469**	**1453**	**1556**	**1581**	**1656**	**7715**	**7715**

alpha	0,05
df	16
Chi^2 Critical value	26,2962276

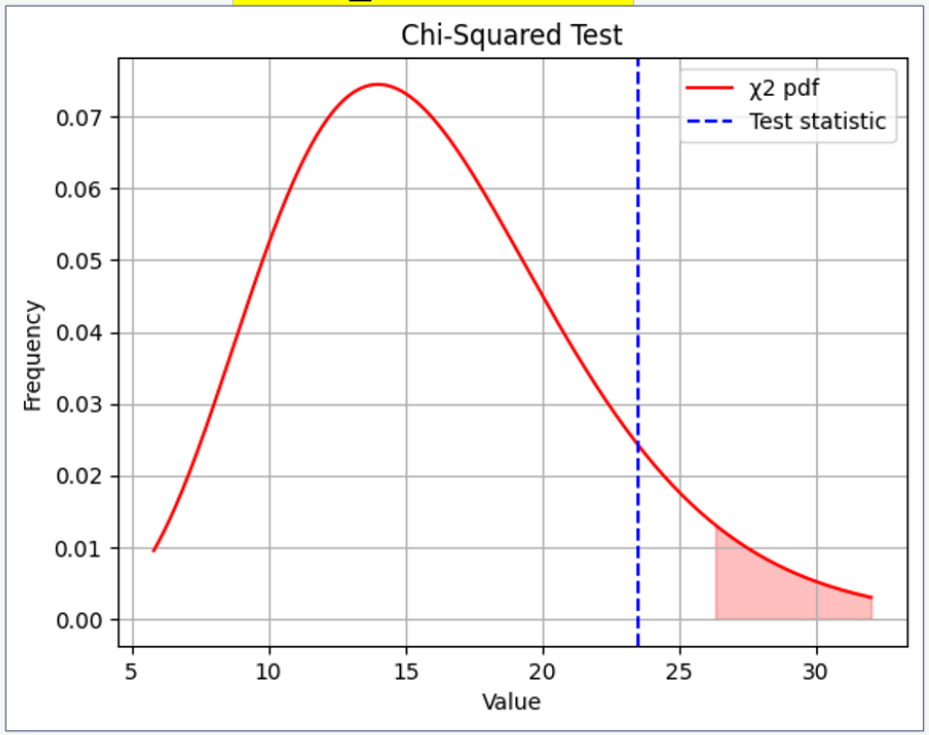
Ch^2-Computation				
Observed	**Expected**	**O - E**	**(O - E)^2**	**(O - E)^2/E**
560	507,82	52,18	2.722,86	5,36188107
369	355,30	13,70	187,64	0,52811009
209	235,34	-26,34	694,04	2,94903983
267	288,66	-21,66	469,11	1,62513998
64	81,88	-17,88	319,54	3,9027015
495	502,29	-7,29	53,11	0,10574256
385	351,43	33,57	1.126,81	3,2063373
226	232,78	-6,78	45,99	0,19755237
277	285,51	-8,51	72,50	0,2539437
70	80,98	-10,98	120,64	1,48972779
553	537,89	15,11	228,19	0,42423241
358	376,34	-18,34	336,51	0,89416013
248	249,28	-1,28	1,65	0,00660017
304	305,75	-1,75	3,08	0,01006783
93	86,72	6,28	39,38	0,45409414
547	546,54	0,46	0,22	0,00039354
361	382,39	-21,39	457,57	1,19660723
268	253,29	14,71	216,45	0,85454721
328	310,67	17,33	300,43	0,96705621
77	88,12	-11,12	123,61	1,40276589
512	572,46	-60,46	3.655,77	6,38604364
393	400,53	-7,53	56,71	0,14159872
285	265,30	19,70	387,95	1,46230552
340	325,40	14,60	213,03	0,65465452
126	92,30	33,70	1.135,82	12,3059567
			Chi^2-val	46,7812601

χ^2 independence test

… statistical tests used to determine if there's a significant association between two categorical variables in a sample.

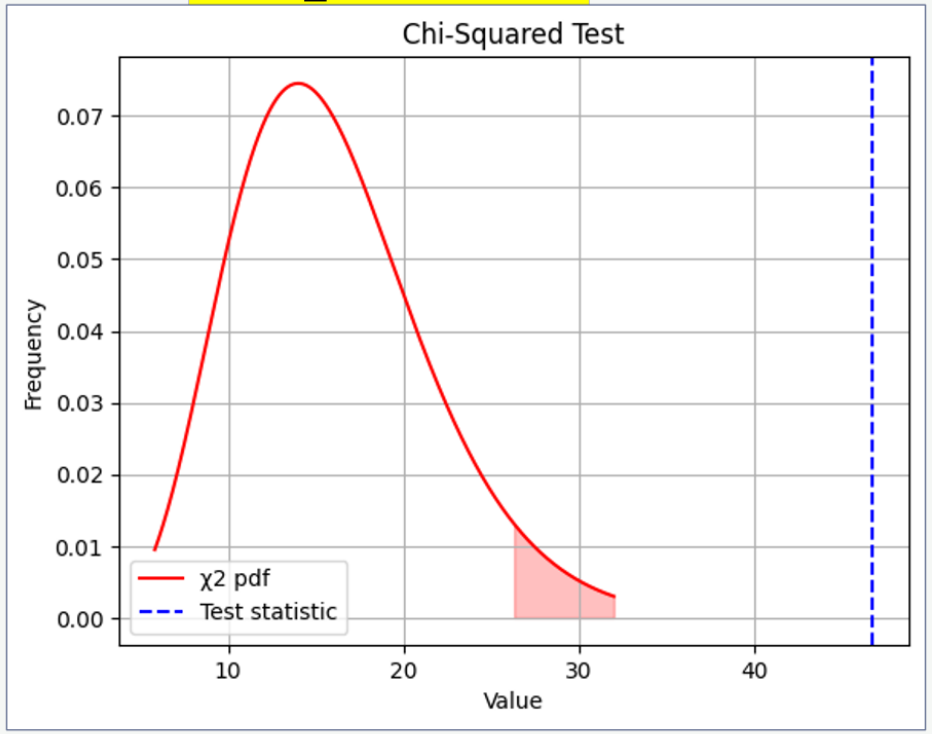
Critical value: 26.2962

Chi^2_stat = 23.5172



H0:
Experience level does not have significant impact on promotion…

Chi^2_stat = 46.7812



H1:
Experience level has significant impact on promotion…

Oneway – analysis of variance test

… statistical tests used to compare the means of three or more groups and determine if they are significantly different from each other.

Group 1	Group 2	Group 3	
82	71	64	
93	62	73	
61	85	87	
74	94	91	
69	78	56	
70	66	78	
53	71	87	
Mean	Mean	Mean	Overall mean
71,71	75,29	76,57	74,52

- The larger the differences between the group samples the larger F_stat
- Exceeding F_crit, we "reject the null" (H1)

N_Total	21
Num_Groups	3
N_per_Group	7
N_per_Group - 1	6
alpha	5%

	df	SS	MS	F_stat
Between (SSC)	2	88,66667	44,33333333	0,283726128
Within (SSE)	18	2812,571	156,2539683	
Total (SST)	20	2901,238		

F_crit(5%, 2, 18)	3,554557146
F_stat	0,283726128

- **F_stat < F_crit**

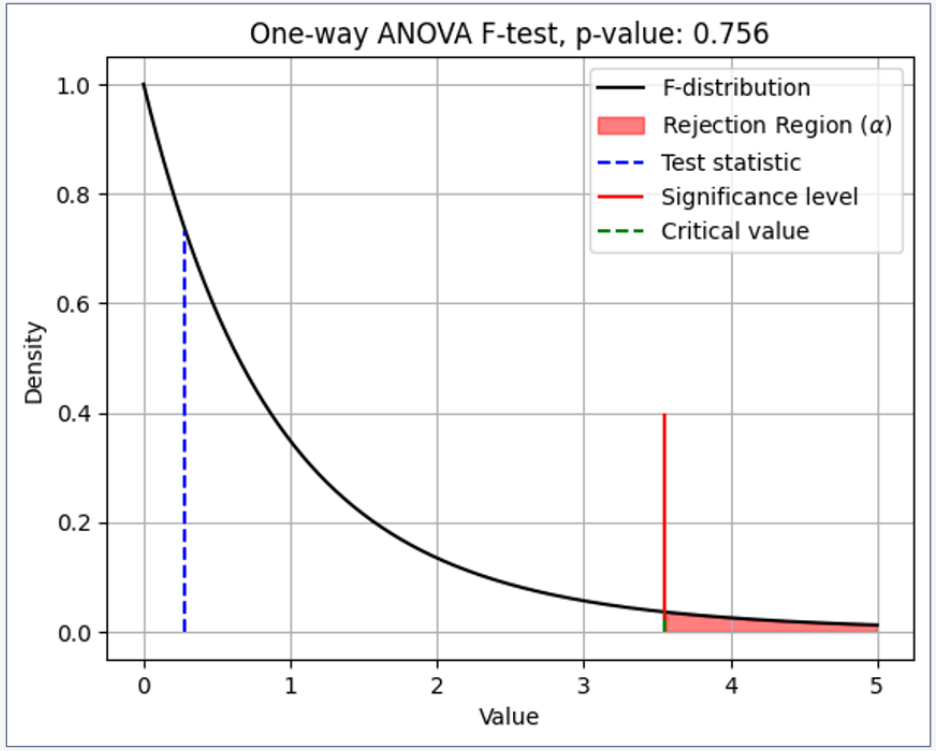
→ **H0 is not rejected ("H0")**

Oneway – analysis of variance test

… statistical tests used to compare the means of three or more groups and determine if they are significantly different from each other.

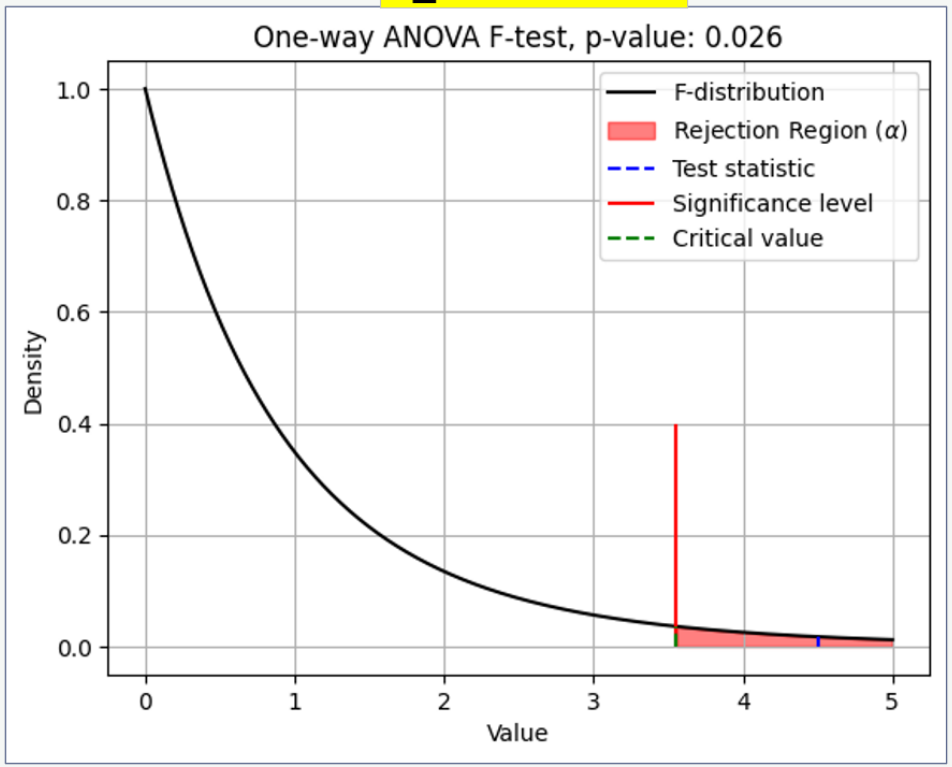
F_stat = 0.2837

Critical value: 3.5545



H0:
The learning style has no significant impact on learning success …

F_stat = 4.4822



H1:
The learning style has significant impact on learning success …

Exercises

Exercise for ttest_ind:
Based on the ttest_ind example, try new data points and find data point configurations, where H0 is very close of being rejected or is just rejected.
Generate figure according to the given figures.

Exercise for ttest_rel:
Based on the ttest_ind example, try new data points and find data point configurations, where H0 is very close of being rejected or is just rejected.
Generate figure according to the given figures.

Exercise for chi^2:
→ Next slide

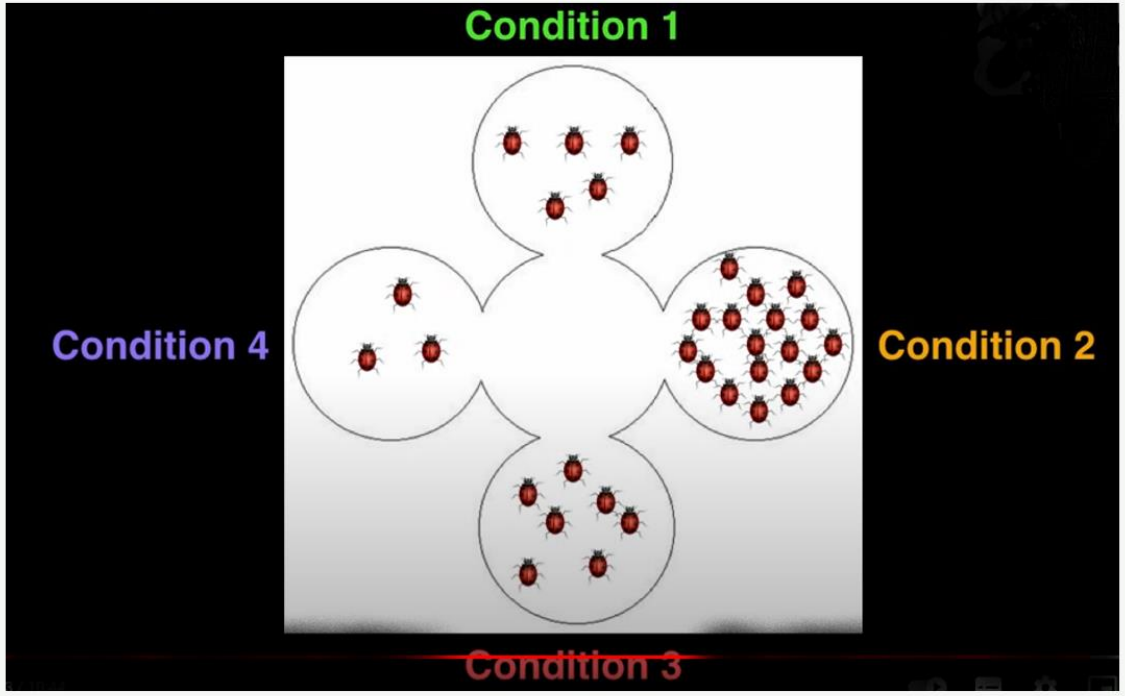
Exercise for F:
Based on the F example, try new data points and find data point configurations, where H0 is very close of being rejected or is just rejected.
Generate figure according to the given figures.

Additional exercise for chi^2:
Based on the chi^2 example, try new data points and find data point configurations, where H0 is very close of being rejected or is just rejected.
Generate figure according to the given figures.

Exercise – Chi squared

H_0: Bugs have no preference of moving to one or the other bins

H_1 : Or do we have enough evidence to reject H_0?

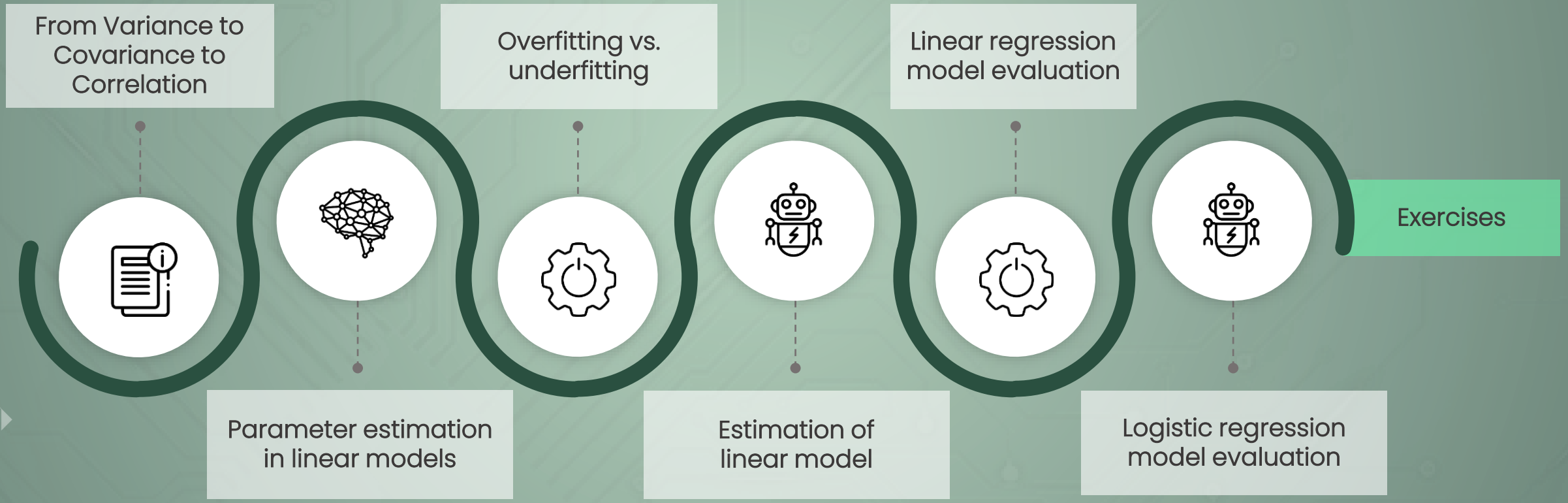


Nr	Obs
1	5
2	17
3	7
4	3

Source:
https://www.youtube.com/watch?v=qYOMO83Z1WU&t=552s

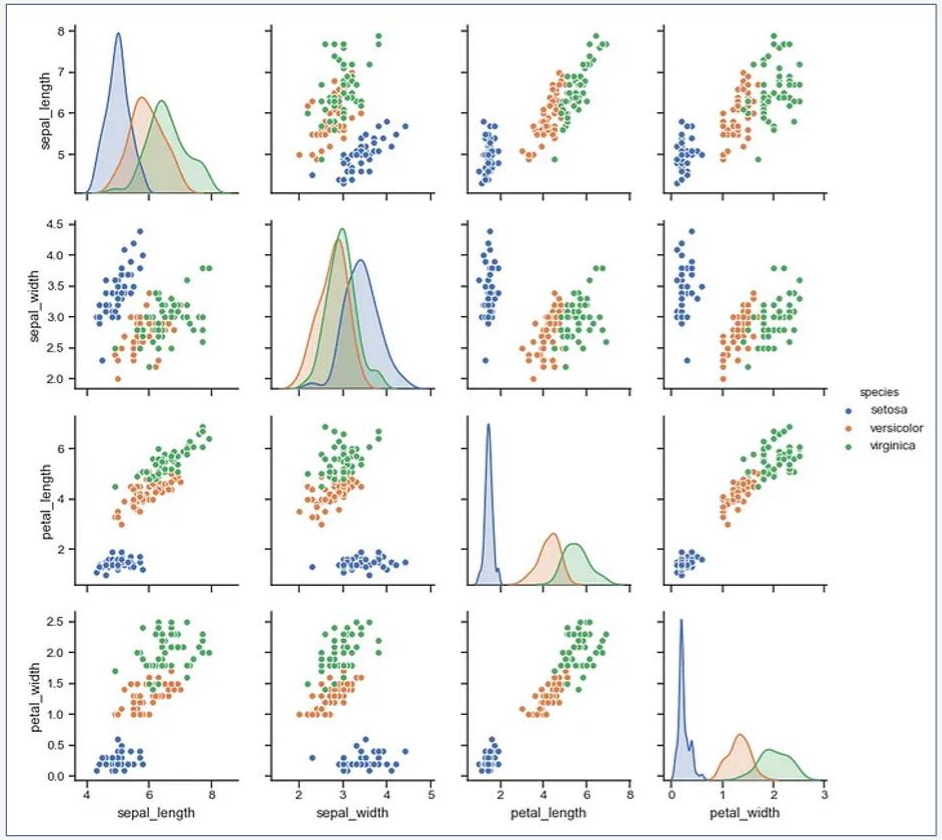
Model estimation

… is the process of determining the parameters of statistical model that best fit observed data



From Variance to Covariance to Correlation

Covariance is a generalization of variance to multiple dimensions or variables.
Correlation is a normalized form of covariance to have values between -1 and +1.



Variance (σ^2):
- Measures how far a set of numbers spread out from mean.
- In the figure: The diagonal

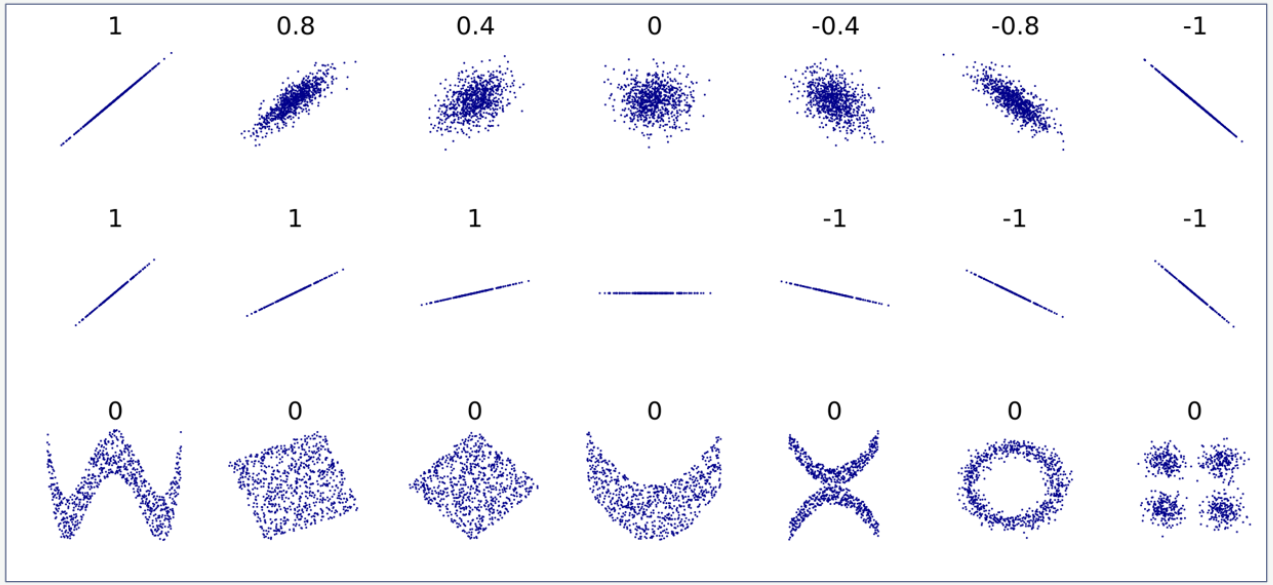
Covariance:
- Measures how much two random variables vary together.
- In the figure: The triangles above and below the diagonal

Correlation:
- Standardized measure of the relationship between two variables
- ranging from -1 (perfect negative correlation) to +1 (perfect positive correlation), e.g. it is the normalized covariance.

https://medium.com/@raghavan99o/scatter-matrix-covariance-and-correlation-explained-14921741ca56

Correlation

… is the standardized measure of the relationship between two variables



Source:
https://en.wikipedia.org/wiki/Correlation

Positive Correlation:

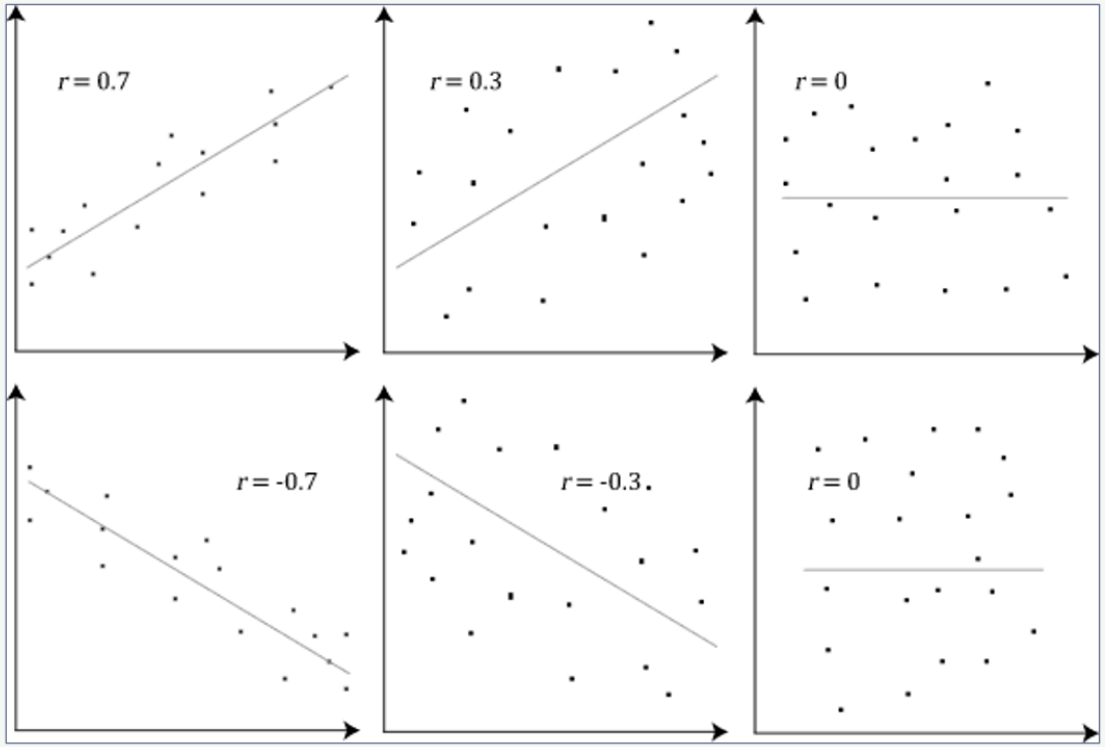
- When two variables increase or decrease together.
- For example, the more time spent studying (variable 1), the higher the grades (variable 2). Correlation value ranges from 0 to +1.

Negative Correlation:

- When one variable increases as the other decreases.
- For example, the more time spent watching TV (variable 1), the lower the grades (variable 2). Correlation value ranges from 0 to -1.

Pearson's Correlation Coefficient - Test

Tests whether two samples have a linear relationship.



Source:
https://machinelearningmastery.com/statistical-hypothesis-tests-in-python-cheat-sheet/

Assumptions

- Observations in each sample are independent and identically distributed (iid).
- Observations in each sample are normally distributed.
- Observations in each sample have the same variance.

Interpretation

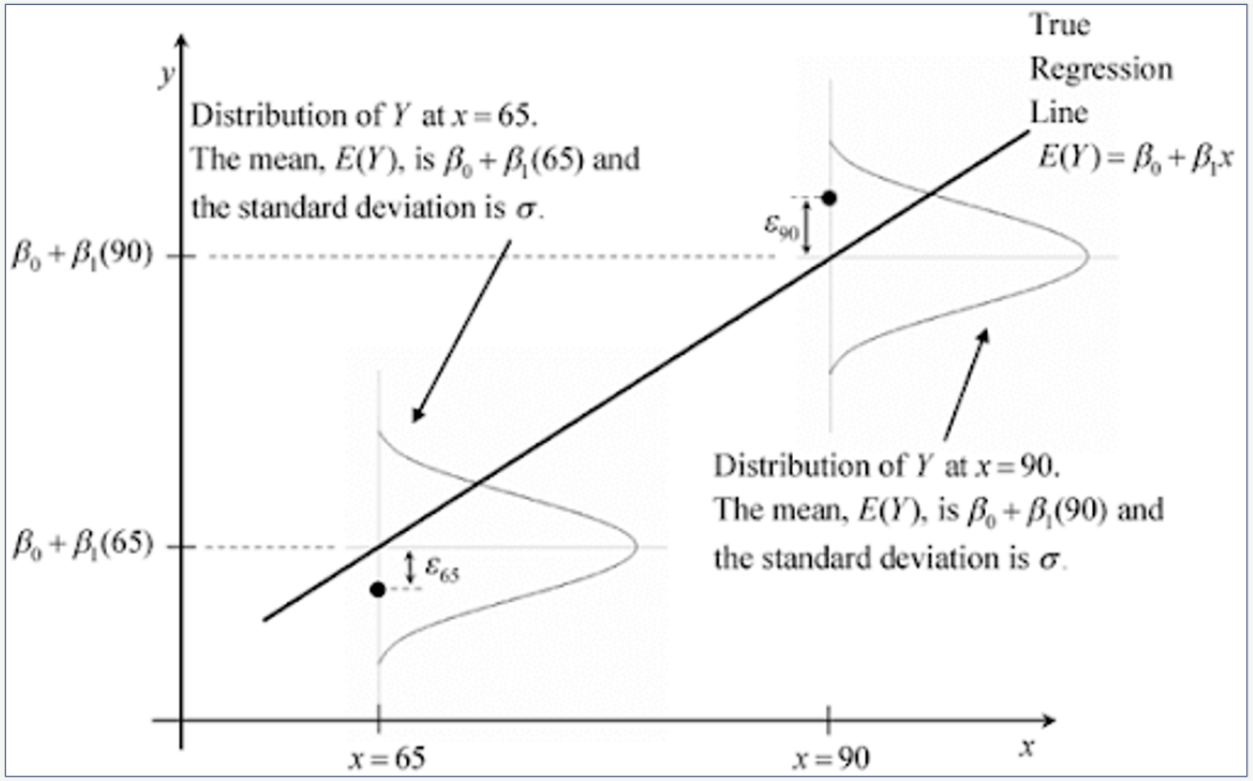
- H0: the two samples are independent.
- H1: there is a dependency between the samples.

```
# Example of the Pearson's Correlation test
from scipy.stats import pearsonr
data1 = [0.873, 2.817, 0.121, -0.945, -0.055, -1.436, 0.360, -1.478, -1.637, -1.869]
data2 = [0.353, 3.517, 0.125, -7.545, -0.555, -1.536, 3.350, -1.578, -3.537, -1.579]

stat, p = pearsonr(data1, data2)
print(f'stat={stat:.3f}, p={p:.3f}')

if p > 0.05:
    print('Probably independent')
else:
    print('Probably dependent')
```

```
stat=0.688, p=0.028
Probably dependent
```

Parameter estimation in linear models

... involves finding the coefficients that best fit the observed data, i.e. represent the relationship between two correlated variables (X and Y)



https://vitalflux.com/generalized-linear-models-explained-with-examples/

Simple linear regression model:

$Y = \beta_0 + \beta_1 * X + \varepsilon$

β_0 (intercept) and β_1 (slope) are the parameters to be estimated.

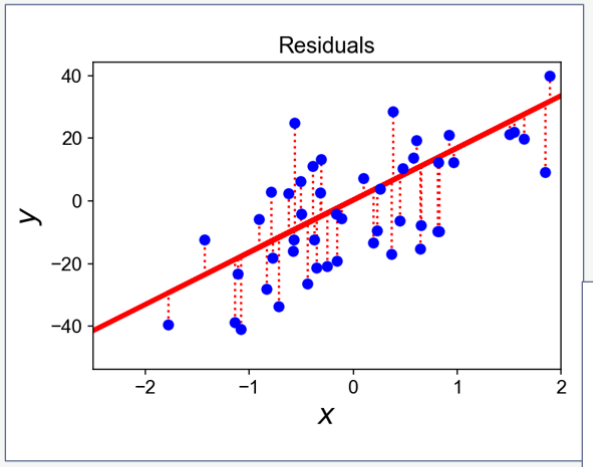
- β_0 : Value of Y when X = 0.
- β_1 : Change in Y for each one-unit change in X.

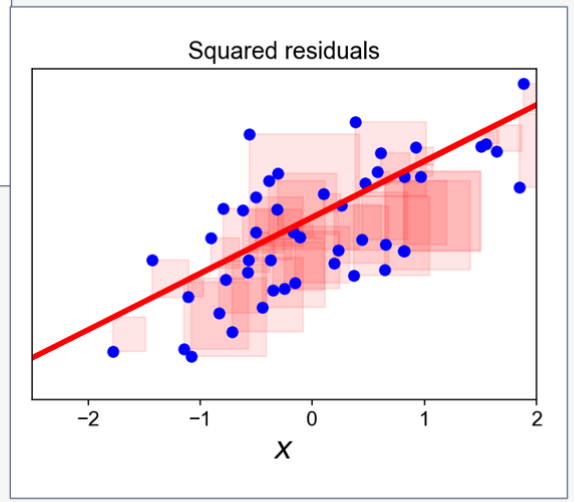
Use Cases:

- Generalization of sample data to overall population data
- Forecasting like prediction of house prices based on independent features

Ordinary Least Squares (OLS) estimation

… represents a relationship between two correlated variables (X and Y) and is used in linear regression analysis





Ordinary Least Squares (OLS) estimation:

- Method used in linear regression to estimate the model's parameters
- by minimizing the sum of the squares of the differences between the observed and predicted values of the dependent variable.
- Separation systematic relations from white noise

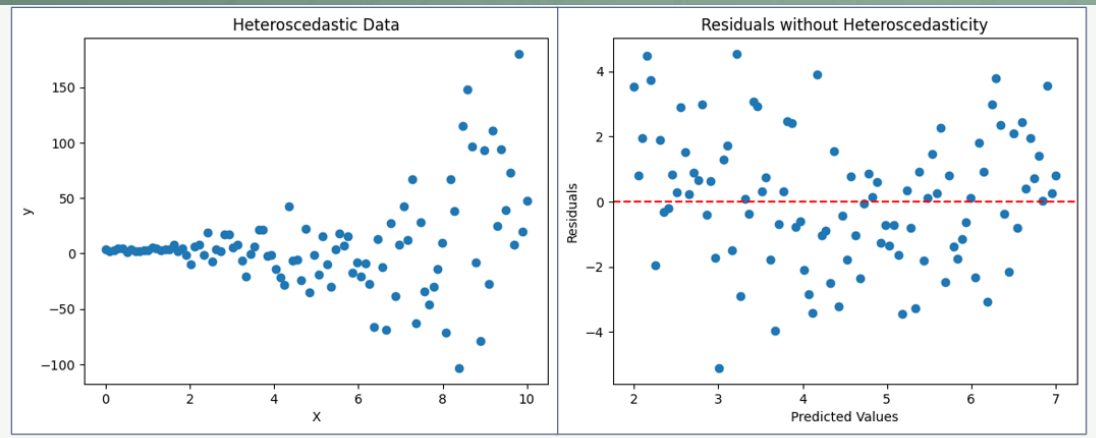
IID & BLUE:

If the residuals are identically and independently distributed (iid) the OLS estimator is used, because it provides best linear unbiased (BLUE) estimation.

https://gregorygundersen.com/blog/2020/01/04/ols/

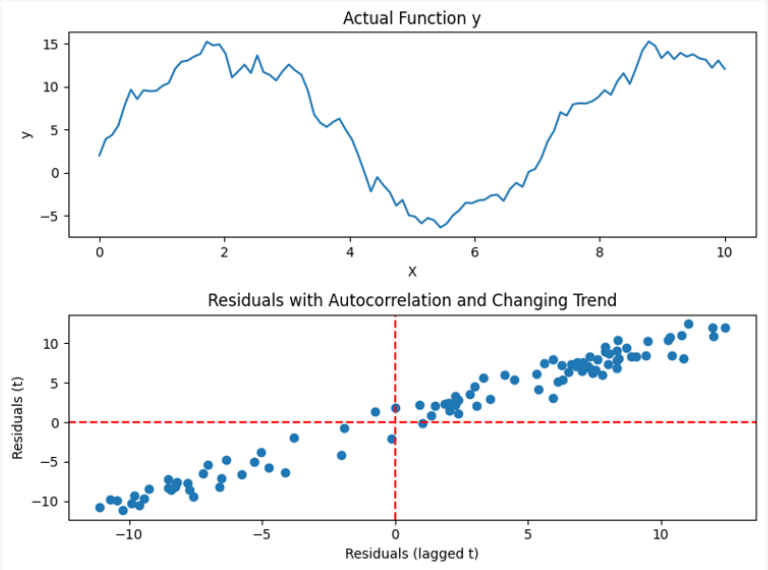
OLS - What if IID-ness of residuals is violated?

Here it gets interesting ☺



Heteroscedasticity

- Variance is not identical
- Breusch-Pagan-Test: H_1 means that there is enough evidence from residuals to assume heteroscedasticity



Autocorrelation (typical example: time series)

- Residuals depend on precedessors
- Durbin-Watson-Test:

$$d = \frac{\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2}$$

- $d \approx 2$: No autocorrelation
- $0 < d < 2$: Positive autocorrelation
- $2 < d < 4$: Negative autocorrelation

Transformations can be used in order to make OLS estimation usable...

Overfitting

… is a modeling error where a model is trained too well on the training data

Problems associated with Overfitting:

Poor Generalizability:

Overfit models perform well on training data but poorly on new, unseen data, which limits their predictive accuracy.

Complexity:

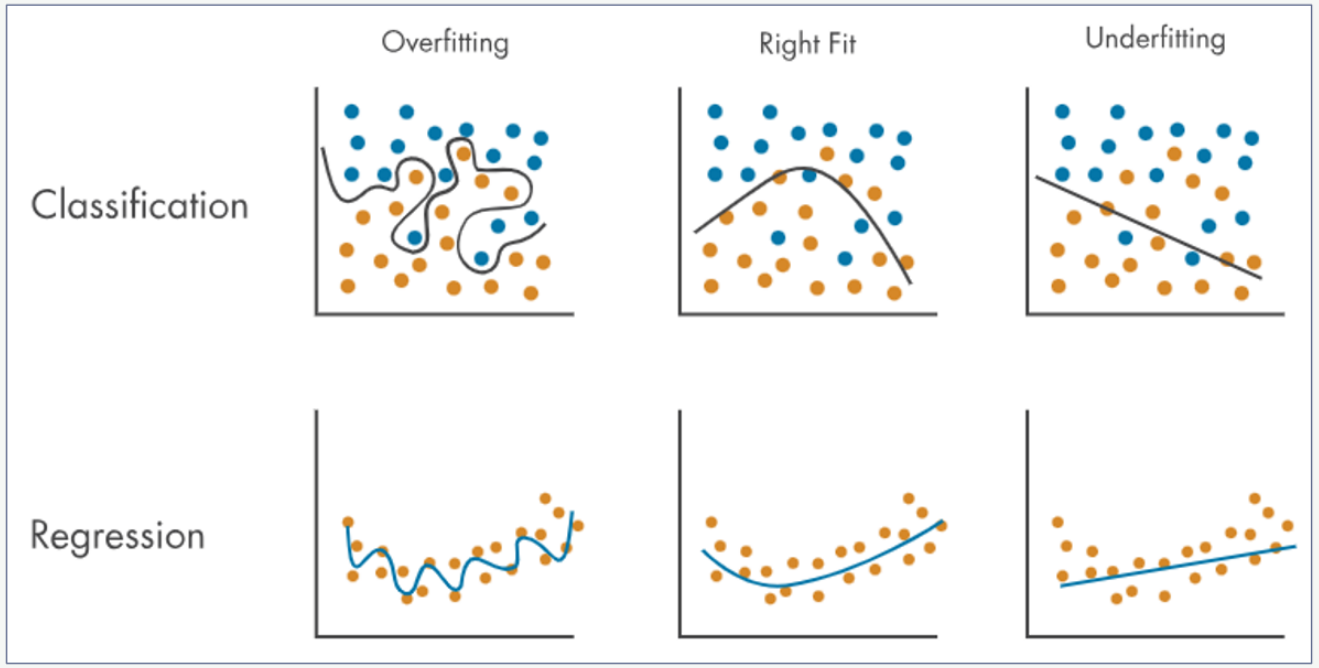
Overfitting often results in overly complex models that have too many parameters.

Unreliable Predictions:

Overfit models are sensitive to minor fluctuations in data, leading to unreliable, inconsistent predictions over time.

Overfitting vs. underfitting

In opposite to overfitting, underfitting is when the model fails to capture important trends in the training data, resulting in poor performance on both the training and unseen data.



Underfitting:

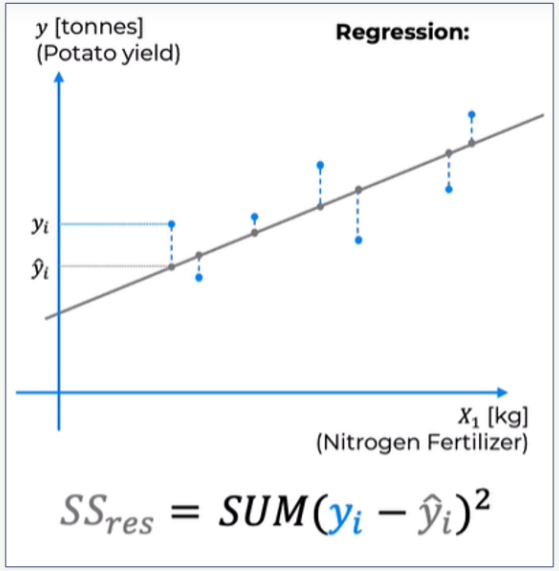
Occurs when a model cannot adequately capture the underlying structure of the data. It's oversimplified, has poor predictive performance on both training and unseen data due to high bias.

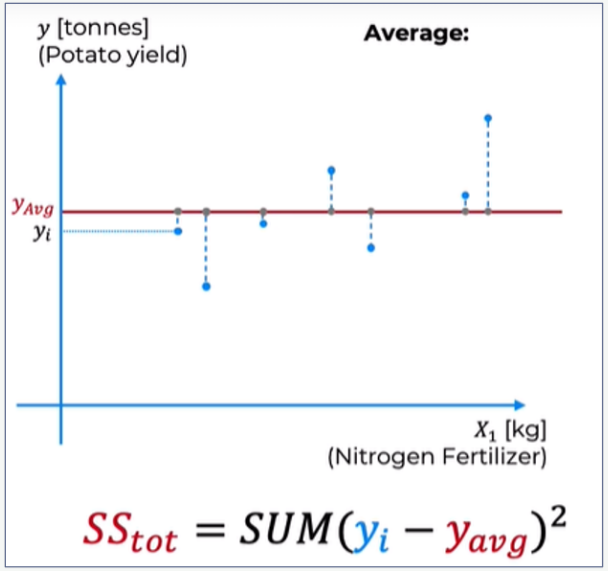
What is worse?

Both overfitting and underfitting lead to poor predictions and indicate that the model may not be the right choice or needs fine-tuning.

R-squared: Goodness of fit

… is a statistical measure in regression models that represents the proportion of the variance in the dependent variable that is predictable from the independent variables





$$R^2 = 1 - \frac{SSR}{SST}$$

Properties:

- Ranges from 0 to 1.
- Value of 1 → independent variable(s) perfectly predict dependent variable.
- Value of 0 → no predictive power.

- Higher R-Squared values → better fit, however very high R-Squared indicates overfitting.

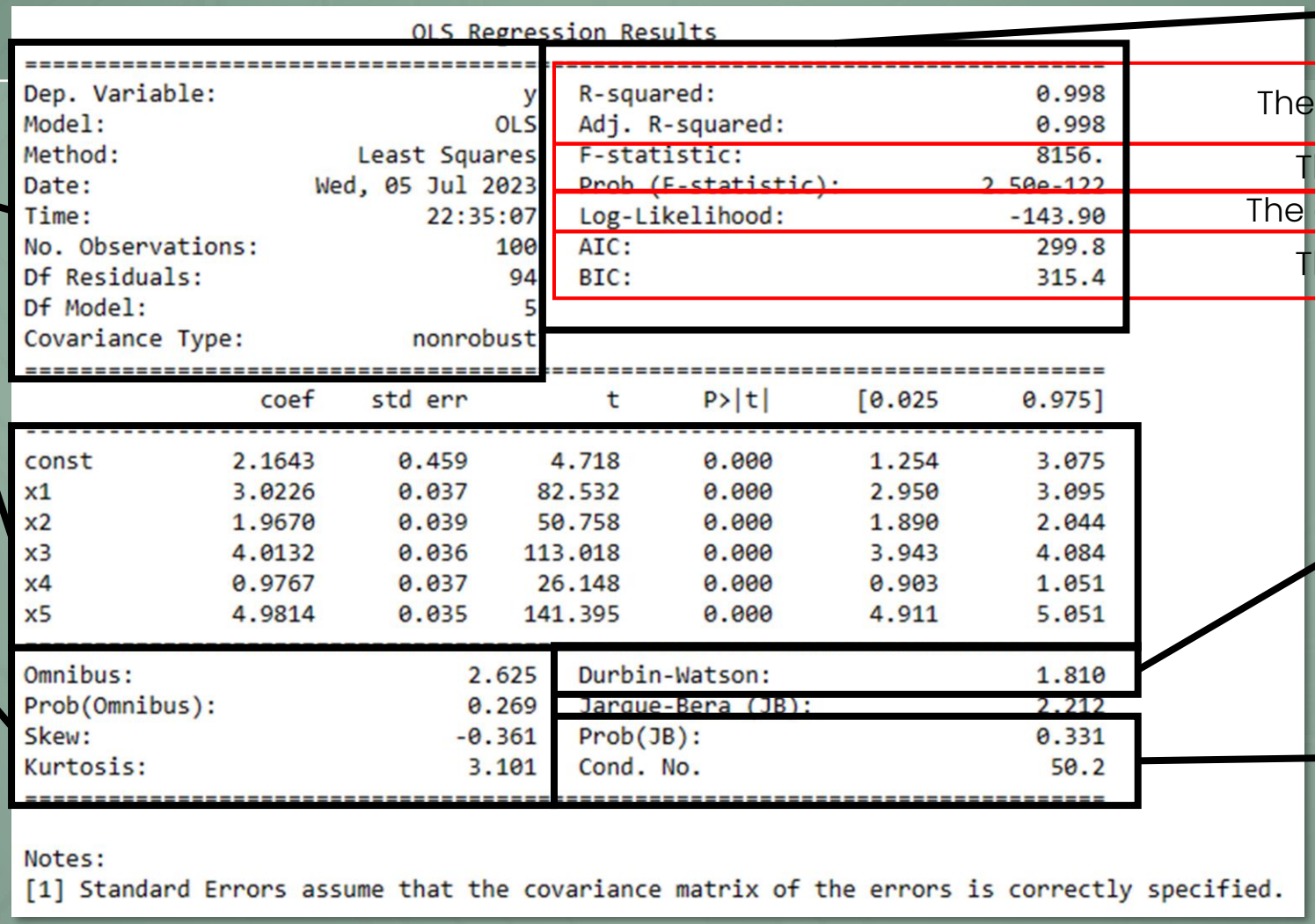
- Only interpretable in the context of the data and domain. Low R-Squared could be acceptable in some fields where prediction is difficult.

Adjusted R-squared:

- Problem to be solved: Many input factors may improve fit, but lead to multicoliniarity, overfitting etc.
- In order to incentivate leaner models, number of input factors is punished

Statsmodels: Summary for OLS

… provides a comprehensive summary of the results from fitted OLS model



OLS Regression Results

Dep. Variable:	y	R-squared:	0.998
Model:	OLS	Adj. R-squared:	0.998
Method:	Least Squares	F-statistic:	8156.
Date:	Wed, 05 Jul 2023	Prob (F-statistic):	2.50e-122
Time:	22:35:07	Log-Likelihood:	-143.90
No. Observations:	100	AIC:	299.8
Df Residuals:	94	BIC:	315.4
Df Model:	5		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	2.1643	0.459	4.718	0.000	1.254	3.075
x1	3.0226	0.037	82.532	0.000	2.950	3.095
x2	1.9670	0.039	50.758	0.000	1.890	2.044
x3	4.0132	0.036	113.018	0.000	3.943	4.084
x4	0.9767	0.037	26.148	0.000	0.903	1.051
x5	4.9814	0.035	141.395	0.000	4.911	5.051

Omnibus:	2.625	Durbin-Watson:	1.810
Prob(Omnibus):	0.269	Jarque-Bera (JB):	2.212
Skew:	-0.361	Prob(JB):	0.331
Kurtosis:	3.101	Cond. No.	50.2

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Model Info

Model Info

The closer to 1 the better

The smaller the better

The closer to 0 the better

The smaller the better

Coefficients Table

Autocorrelation, e.g. basis of time series

Skewness and kurtosis

Multicollinearity

Statsmodels: Summary – Coefficients table

… provides detailed information about each parameter in the model

	coef	std err	t	P>|t|	[0.025	0.975]
const	-3.2002	0.257	-12.458	0.000	-3.708	-2.693
x1	0.7529	0.044	17.296	0.000	0.667	0.839

coef:
- Estimates of the parameters

1.std err:
- Standard error of the estimate of the coefficient
- not standard deviation!
- Smaller values indicate a more precise estimate.

t:
- t-statistic value
- Ratio: coefficient / std. err
- measure of how statistically significant the coefficient is.

P>|t|:
- p-value associated with the t-statistic.
- Smaller p-value (<0.05, traditionally) suggests that null hypothesis can be rejected, i.e. independent variable **is a significant predictor** of dep. variable.

[0.025 0.975]
- 95% confidence intervals for the coefficient.
- If zero is not in this interval, that suggests the variable is a significant predictor at the 0.05 level.

Logistic regression

… is statistical model to predict a binary outcome (0 or 1, true or false) based on one or more predictor variables, using logistic function to model the probability of binary outcome.

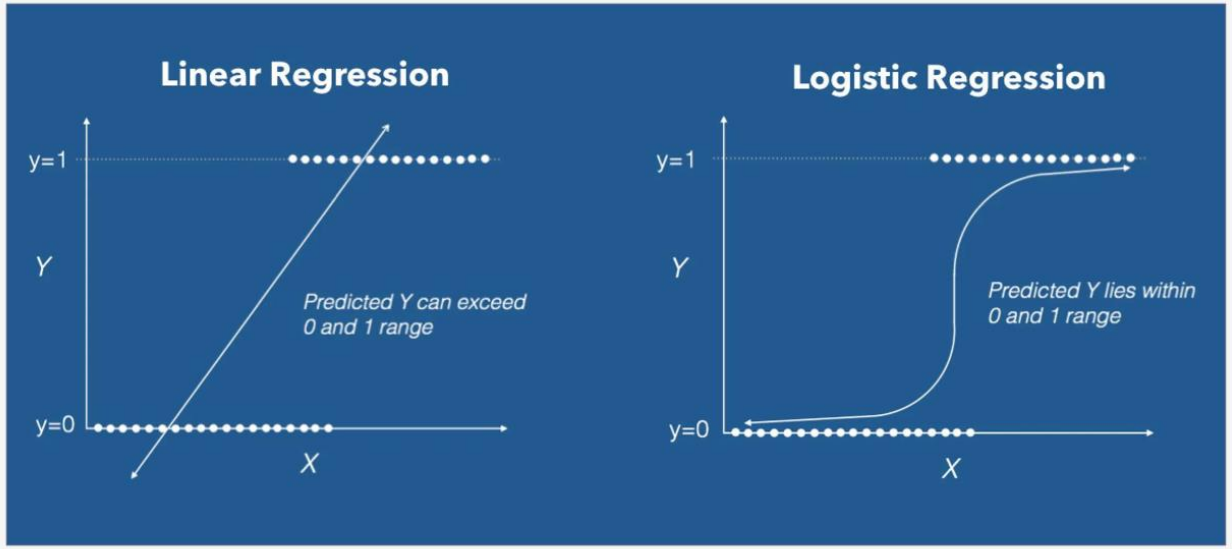


Fig 12: Typical logistic regression curve

Logistic function (= sigmoid function):

$P(z) = \frac{1}{1+e^{-z}}$ with $z = \beta_0 + \beta_1 x$

- It converts any input into a value between 0 and 1, which can be interpreted as a probability.
- For given input 'x', the output represents probability that 'x' belongs to class 1 (usually coded as positive outcome).
- Simple rounding can be used for predicting 0 or 1

It is based on …

Logged odds:

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x$$

→ Linear model is projected on S-curve

https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79

Statistics perspective: Summary for Logistic regression

… provides a comprehensive summary of the results from fitted logistic regression model

```
# Print the summary statistics of the regression model
print(result.summary())
```

```
                           Logit Regression Results
==============================================================================
Dep. Variable:                default   No. Observations:                24000
Model:                          Logit   Df Residuals:                    23975
Method:                           MLE   Df Model:                           24
Date:                Wed, 05 Jul 2023   Pseudo R-squ.:                 0.1211
Time:                        21:36:53   Log-Likelihood:                -11162.
converged:                       True   LL-Null:                       -12700.
Covariance Type:            nonrobust   LLR p-value:                    0.000
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -1.4592      0.018    -79.195      0.000      -1.495      -1.423
x1            -0.0039      0.017     -0.233      0.815      -0.037       0.029
x2            -0.1064      0.023     -4.677      0.000      -0.151      -0.062
x3            -0.0561      0.017     -3.344      0.001      -0.089      -0.023
x4            -0.0774      0.018     -4.194      0.000      -0.114      -0.041
x5            -0.0791      0.018     -4.290      0.000      -0.115      -0.043
x6             0.0768      0.018      4.203      0.000       0.041       0.113
x7             0.6476      0.022     29.092      0.000       0.604       0.691
x8             0.1086      0.027      4.008      0.000       0.055       0.162
x9             0.0776      0.030      2.548      0.011       0.018       0.137
x10            0.0532      0.033      1.633      0.103      -0.011       0.117
x11            0.0205      0.034      0.605      0.546      -0.046       0.087
x12            0.0163      0.028      0.579      0.562      -0.039       0.072
x13           -0.3759      0.092     -4.087      0.000      -0.556      -0.196
x14            0.1462      0.118      1.240      0.215      -0.085       0.377
x15            0.1065      0.101      1.050      0.294      -0.092       0.305
```

The closer to 1 the better

The closer to 0 the better

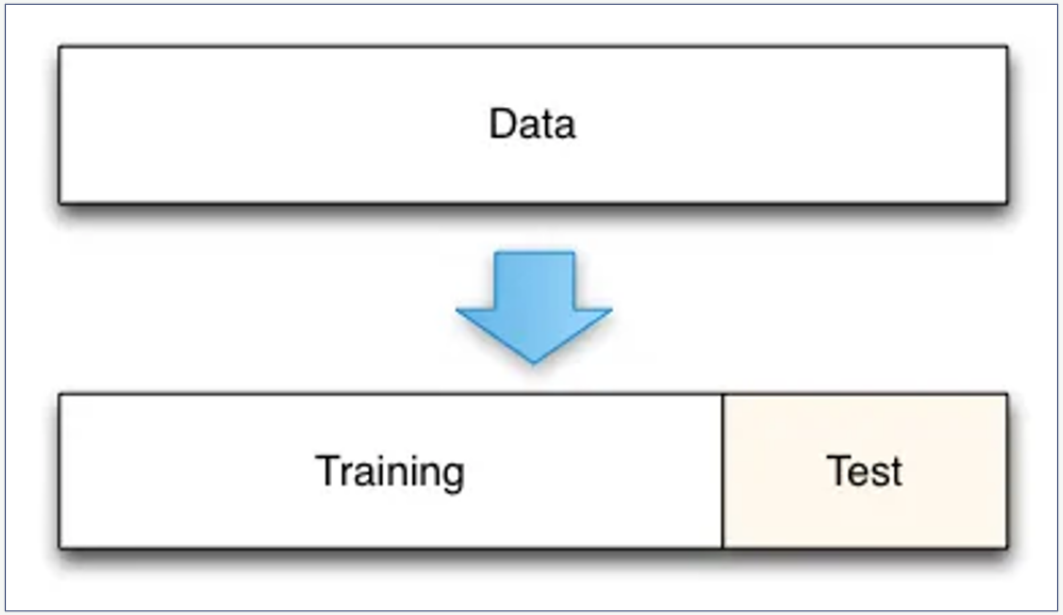
If LLR p-value is small (typically less than a pre-specified significance level, such as 0.05), full model provides significantly better fit to the data compared to reduced model.

coef:

- Input factors that have impact and should be used

ML perspective: Train test split

… is a method where the dataset is divided into subsets: the training set, used to train the model, and the testing set, used to evaluate the model's performance on unseen data



https://medium.com/@rathinavel.mph/how-the-train-and-test-samples-are-split-d7e46a8e2361

Training models vs. analytical solution

- Logistic regression does not have analytical solution in opposite to linear regression.
- Therefore squared error minimizing parameters are approximated, e.g. with Newton's approximation method
- However: Danger of overfitting

Training Set:

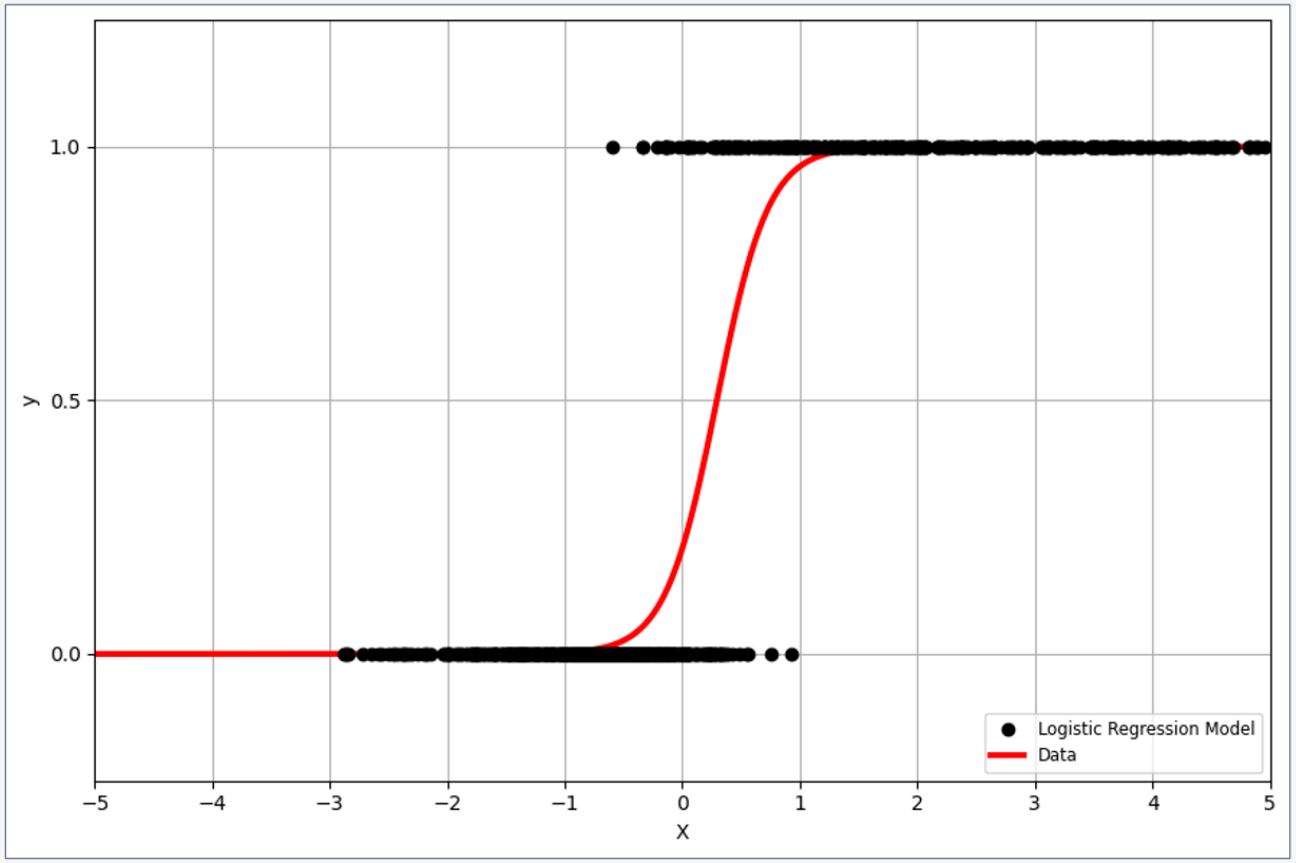
- Portion of data used to create the model, i.e. to learn/approximate the parameters
- Usually 70-80% of the entire dataset

Testing Set:

- Subset of data that the model has not seen during learning phase
- Used to evaluate performance of the model on unseen data and check for overfitting. Typically 20-30% of entire dataset

Logistic regression

… is a statistical model to predict a binary outcome (0 or 1, true or false) based on one or more predictor variables, using the logistic function to model the probability of the binary outcome.



Logistic function (= sigmoid function):

$P(z) = \frac{1}{1+e^{-z}}$ with $z = \beta_0 + \beta_1 x$

- It converts any input into a value between 0 and 1, which can be interpreted as probability.
- For given input 'x',
  output represents probability that 'x' belongs to 1
- Simple rounding can be used for predicting 0 or 1

It is based on …

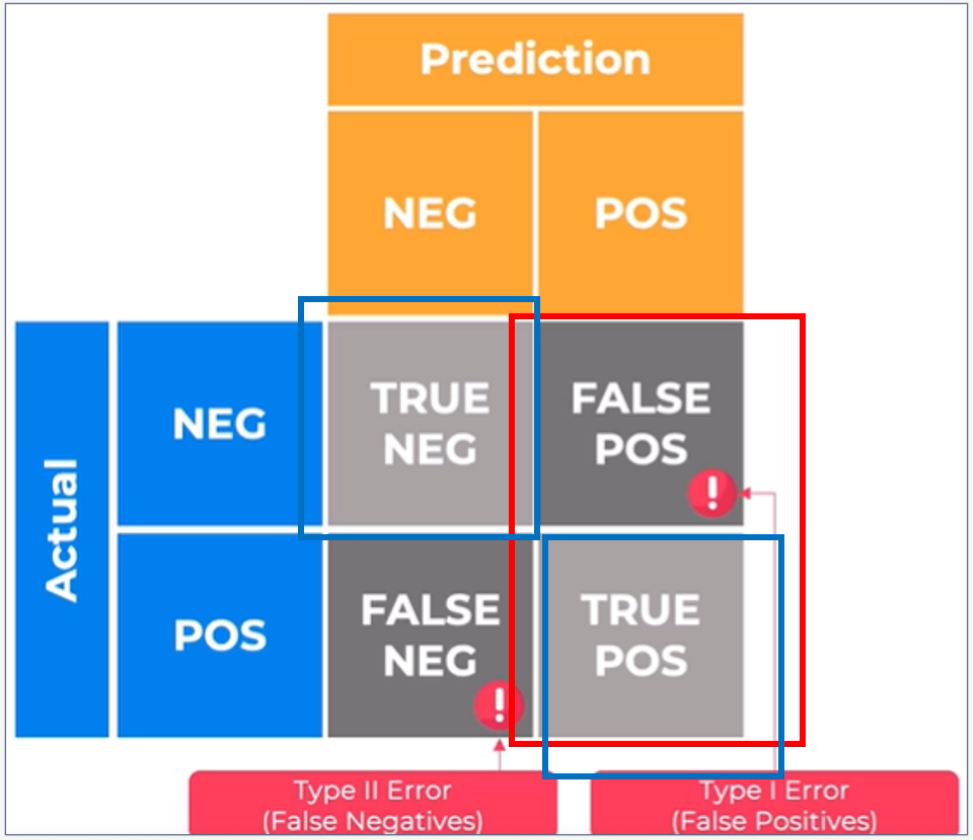
Logged odds:

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x$$

→ Linear model is projected on S-curve

Confusion matrix

… is a tabular representation to visualize the performance of a predictive model on a set of test data for which the true values are known.



1. **True Positives (TP):** These are cases in which we predicted yes (they have the condition), and they do have the condition.
2. **True Negatives (TN):** We predicted no, and they don't have the condition.
3. **False Positives (FP):** We predicted yes, but they don't actually have the condition. (Also known as a "Type I error.")
4. **False Negatives (FN):** We predicted no, but they actually do have the condition. (Also known as a "Type II error.")

Accuracy:

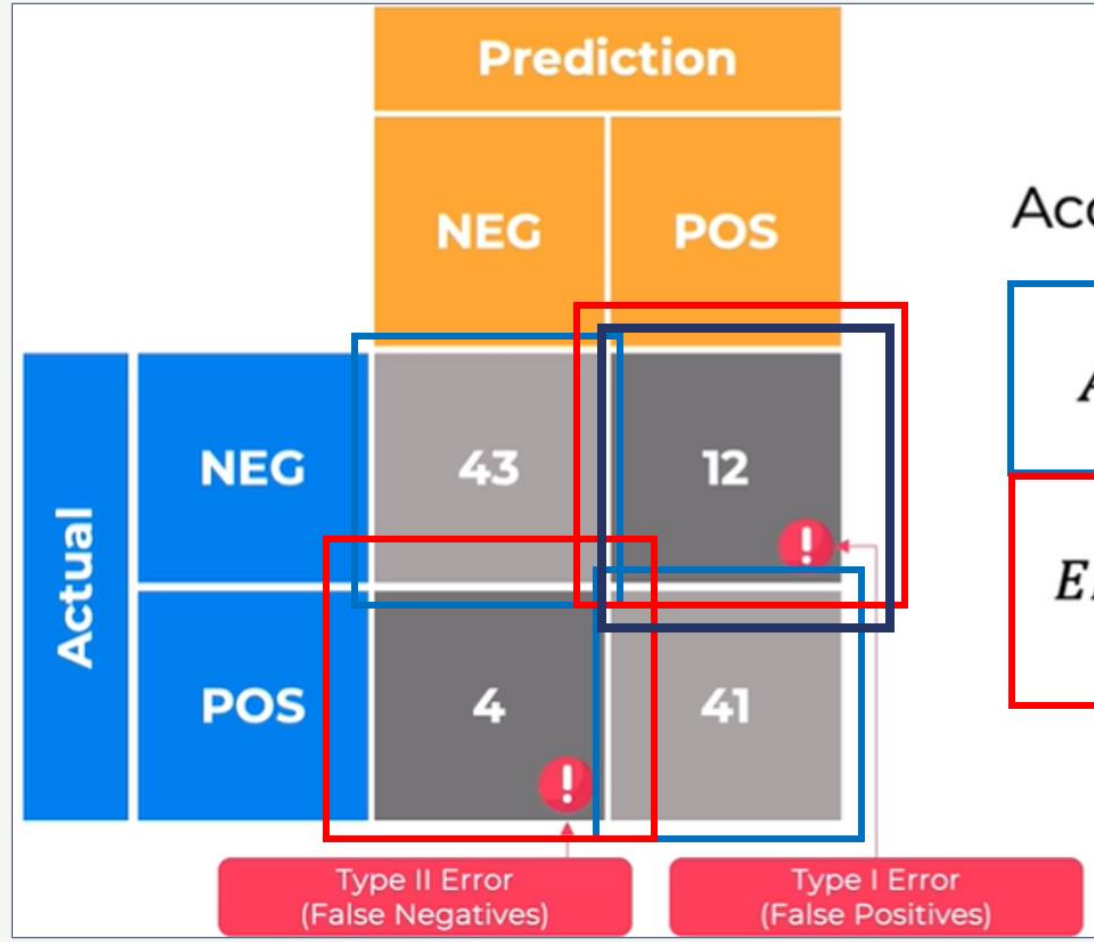
- Ratio of correctly predicted observation to the total observations
- Accuracy = (TP + TN) / (TP + FP + FN + TN)
- However misleading in case of imbalanced classes: “Sun in July in Greece”, highly accurate prediction, but no indicator of good prediction model

Precision:

- What proportion of positive identifications was actually correct?
- Precision = TP / (TP + FP)
- Better, because **false positives** should be reduced!

Accuracy rate and error rate

Try to improve your model by reducing false positives (Error Type I)



Accuracy Rate and Error Rate:

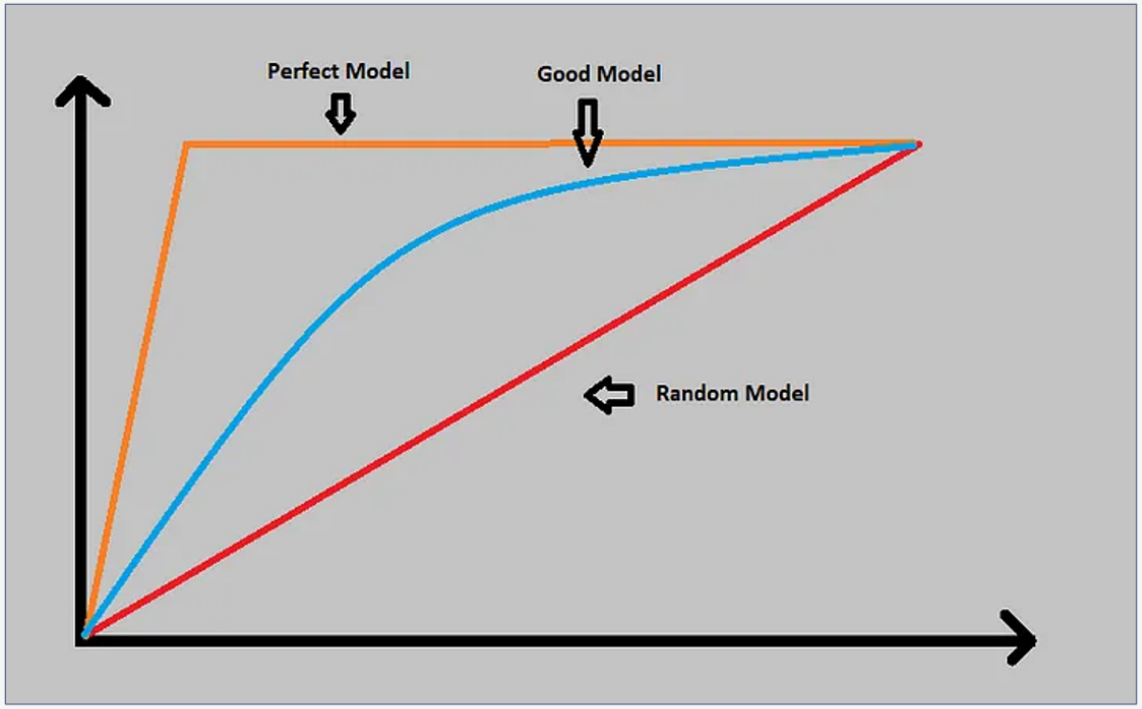
$$AR = \frac{Correct}{Total} = \frac{TN + TP}{Total} = \frac{84}{100} = 84\%$$

$$ER = \frac{Incorrect}{Total} = \frac{FP + FN}{Total} = \frac{16}{100} = 16\%$$

Cumulative Accuracy Profile (CAP) curve

… illustrates the cumulative number of positive outcomes along different thresholds of classification

•Y-axis: Cumulative percentage of positive samples.



X-axis: Cumulative percentage of total samples.

Key Components

- Random Model line: Represents a random guess model.
- Perfect Model line: Achieves 100% identification of true positives with no false positives.
- Logistic Regression Model line: Represents the actual model's cumulative accuracy profile.

50% Rule

- y < 60%: model might not be beneficial.
- y is between 60% and 70%: model is good.
- y > 70%: model possibly too good, indication of overfitting

Exercises

Exercise for logistic regression:

- Take the dataset “default of credit card clients”
- Develop model for binary classification by generalizing the dataset
- Discuss summary, confusion matrix

Outlook – Machine learning

This lecture is reference point for different machine learning topics

Supervised Learning, i.e. generalization:
- numeric regression
- Classification
- time-series prediction

Logistic regression:
- ROC-AUC-Curve, F-Score, etc.
- Application: Deep Learning

Reinforcement learning, i.e. exploratory controlling/steering:
- Different A-B-testing approach

Statistical tables

… let you decide whether to reject H_0 or not by hand

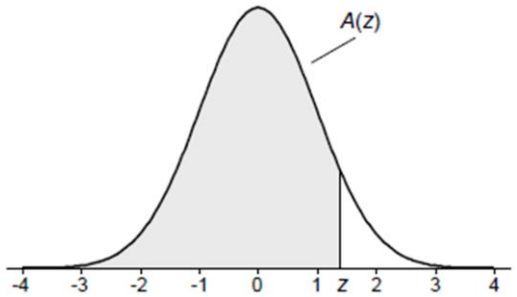
t Distribution: Critical Values of t

Degrees of freedom	*Two-tailed test:* / *One-tailed test:*	*Significance level* 10% / 5%	5% / 2.5%	2% / 1%	1% / 0.5%	0.2% / 0.1%	0.1% / 0.05%
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.894	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587
11		1.796	2.201	2.718	3.106	4.025	4.437
12		1.782	2.179	2.681	3.055	3.930	4.318
13		1.771	2.160	2.650	3.012	3.852	4.221
14		1.761	2.145	2.624	2.977	3.787	4.140
15		1.753	2.131	2.602	2.947	3.733	4.073
16		1.746	2.120	2.583	2.921	3.686	4.015
17		1.740	2.110	2.567	2.898	3.646	3.965
18		1.734	2.101	2.552	2.878	3.610	3.922
19		1.729	2.093	2.539	2.861	3.579	3.883
20		1.725	2.086	2.528	2.845	3.552	3.850
21		1.721	2.080	2.518	2.831	3.527	3.819
22		1.717	2.074	2.508	2.819	3.505	3.792
23		1.714	2.069	2.500	2.807	3.485	3.768
24		1.711	2.064	2.492	2.797	3.467	3.745
25		1.708	2.060	2.485	2.787	3.450	3.725
26		1.706	2.056	2.479	2.779	3.435	3.707
27		1.703	2.052	2.473	2.771	3.421	3.690
28		1.701	2.048	2.467	2.763	3.408	3.674
29		1.699	2.045	2.462	2.756	3.396	3.659
30		1.697	2.042	2.457	2.750	3.385	3.646
32		1.694	2.037	2.449	2.738	3.365	3.622
34		1.691	2.032	2.441	2.728	3.348	3.601
36		1.688	2.028	2.434	2.719	3.333	3.582
38		1.686	2.024	2.429	2.712	3.319	3.566
40		1.684	2.021	2.423	2.704	3.307	3.551

Statistical tables

… let you decide whether to reject H_0 or not by hand

Cumulative Standardized Normal Distribution



$A(z)$ is the integral of the standardized normal distribution from $-\infty$ to z (in other words, the area under the curve to the left of z). It gives the probability of a normal random variable not being more than z standard deviations above its mean. Values of z of particular importance:

z	$A(z)$	
1.645	0.9500	Lower limit of right 5% tail
1.960	0.9750	Lower limit of right 2.5% tail
2.326	0.9900	Lower limit of right 1% tail
2.576	0.9950	Lower limit of right 0.5% tail
3.090	0.9990	Lower limit of right 0.1% tail
3.291	0.9995	Lower limit of right 0.05% tail

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Statistical tables

... let you decide whether to reject H_0 or not by hand

χ^2 (Chi-Squared) Distribution: Critical Values of χ^2

	Significance level		
Degrees of freedom	5%	1%	0.1%
1	3.841	6.635	10.828
2	5.991	9.210	13.816
3	7.815	11.345	16.266
4	9.488	13.277	18.467
5	11.070	15.086	20.515
6	12.592	16.812	22.458
7	14.067	18.475	24.322
8	15.507	20.090	26.124
9	16.919	21.666	27.877
10	18.307	23.209	29.588

Statistical tables

… let you decide whether to reject H_0 or not by hand

TABLE A.3

F Distribution: Critical Values of *F* (5% significance level)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.36	246.46	247.32	248.01
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.42	19.43	19.44	19.45
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.71	8.69	8.67	8.66
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.87	5.84	5.82	5.80
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.64	4.60	4.58	4.56
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.96	3.92	3.90	3.87
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.53	3.49	3.47	3.44
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.24	3.20	3.17	3.15
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.03	2.99	2.96	2.94
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.86	2.83	2.80	2.77
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.74	2.70	2.67	2.65
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.64	2.60	2.57	2.54
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.55	2.51	2.48	2.46
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.48	2.44	2.41	2.39
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.42	2.38	2.35	2.33
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.37	2.33	2.30	2.28
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.33	2.29	2.26	2.23
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.29	2.25	2.22	2.19
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.26	2.21	2.18	2.16
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.22	2.18	2.15	2.12
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.20	2.16	2.12	2.10
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.17	2.13	2.10	2.07
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.15	2.11	2.08	2.05
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.13	2.09	2.05	2.03
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.11	2.07	2.04	2.01
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.09	2.05	2.02	1.99
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.08	2.04	2.00	1.97
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.06	2.02	1.99	1.96
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.05	2.01	1.97	1.94
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.04	1.99	1.96	1.93
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.99	1.94	1.91	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.95	1.90	1.87	1.84
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.89	1.85	1.81	1.78
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.86	1.82	1.78	1.75
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.84	1.79	1.75	1.72
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.82	1.77	1.73	1.70
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.80	1.76	1.72	1.69
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.79	1.75	1.71	1.68
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.78	1.73	1.69	1.66
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.76	1.71	1.67	1.64

Links

i.e. sources for self-learning

	Title	Link
Descriptive statistics	Variance	https://www.youtube.com/watch?v=JgMFhKi6f6Y
	Quantiles	https://docs.eggplantsoftware.com/epp/9.4.0/analyzer/analyzer-understanding-charts-events.htm
	Python Data Visualization Tutorial	https://www.youtube.com/watch?v=Nt84_TzRkbo

	Title	Link
Combinatorics	Introduction to Permutations and Combinations	https://www.youtube.com/watch?v=gAnKvHmrJ0g
	Miracles of Pascal's triangle	https://www.youtube.com/watch?v=J0l1NuxUcpQ

Links

i.e. sources for self-learning

	Title	Link
Distributions	Uniform probability distributions	https://www.youtube.com/watch?v=aCW8wm6nrRw
	Binomial distribution	https://www.youtube.com/watch?v=6YzrVUVO9M0
	Chi^2	https://www.youtube.com/watch?v=JCOeytq7F0E

	Title	Link
Conditional probability	Bayes' Theorem of Probability With Tree Diagrams & Venn Diagrams	https://www.youtube.com/watch?v=OByl4RJxnKA
	What is the Bayes' Theorem?	https://medium.com/mlearning-ai/what-is-the-bayes-theorem-545a2ef0b91c
	Data Science. Bayes theorem	https://luminousmen.com/post/data-science-bayes-theorem
	Bayesian Machine Learning in Python: A/B Testing	https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing

Links

i.e. sources for self-learning

	Title	Link
Sampling	Central limit theorem	https://www.geeksforgeeks.org/python-central-limit-theorem
	Confidence intervals	https://data.library.virginia.edu/the-intuition-behind-confidence-intervals/
	Resampling	https://www.linkedin.com/pulse/resampling-methods-pranshu-jaryal
	Bootstrapping	https://www.youtube.com/watch?v=9IFcRTyhd5Y
		https://dgarcia-eu.github.io/SocialDataScience/2_SocialDynamics/025_Bootstrapping/Bootstrapping.html
		https://janhove.github.io/teaching/2016/12/20/bootstrapping

	Title	Link
Hypothesis testing	Bayesian Machine Learning in Python: A/B Testing	https://www.udemy.com/course/bayesian-machine-learning-in-python-ab-testing
	Introduction to the Chi-square Test	https://www.youtube.com/watch?v=SvKv375sacA
	F-ratio Test for Two Equal Variances	https://www.youtube.com/watch?v=UWQO4gX7-IE
	17 Statistical Hypothesis Tests in Python (Cheat Sheet)	https://machinelearningmastery.com/statistical-hypothesis-tests-in-python-cheat-sheet/
	Chi Squared Testing	https://www.youtube.com/watch?v=qYOMO83Z1WU

Links

i.e. sources for self-learning

	Title	Link
Model estimation	Scatter matrix , Covariance and Correlation Explained	https://medium.com/@raghavan99o/scatter-matrix-covariance-and-correlation-explained-14921741ca56
	Logistic regression sigmoid function and threshold	https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79
	Introduction to logistic regression	https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.
	Linear Regression Diagnostics	https://www.youtube.com/watch?v=HLAg1yBfNk8&list=PLlbbWgBRF8EePgK40-i7aGU2_ky1yujgL&index=11
	Machine Learning Classifier evaluation using ROC and CAP Curves	https://towardsdatascience.com/machine-learning-classifier-evaluation-using-roc-and-cap-curves-7db60fe6b716
	Classification Model Performance Evaluation using AUC-ROC and CAP Curves	https://medium.com/geekculture/classification-model-performance-evaluation-using-auc-roc-and-cap-curves-66a1b3fc0480
	Deep Learning Prerequisites: Logistic Regression in Python	https://www.udemy.com/course/data-science-logistic-regression-in-python/
		https://en.wikipedia.org/wiki/Correlation
	Ordinary Least Squares	https://gregorygundersen.com/blog/2020/01/04/ols/
	Scatter matrix , Covariance and Correlation Explained	https://medium.com/@raghavan99o/scatter-matrix-covariance-and-correlation-explained-14921741ca56
	Logistic regression sigmoid function and threshold	https://medium.com/@cmukesh8688/logistic-regression-sigmoid-function-and-threshold-b37b82a4cd79
	Introduction to logistic regression	https://towardsdatascience.com/introduction-to-logistic-regression-66248243c148#:~:text=Logistic%20regression%20transforms%20its%20output,to%20return%20a%20probability%20value.
	Linear Regression Diagnostics	https://www.youtube.com/watch?v=HLAg1yBfNk8&list=PLlbbWgBRF8EePgK40-i7aGU2_ky1yujgL&index=11

DATA SCIENCE.
INSTITUTE
by Fabian Rappert

About me

Dr. Harald Stein

- Data Scientist ~ 8 years experience
- Algotrader ~ 4 years experience
- Ph.D. in Economics, Game Theory

- LinkedIn: https://www.linkedin.com/in/harald-stein-phd-1648b51a
- ResearchGate: https://www.researchgate.net/profile/Harald-Stein