Predictive Analysis Problem Set 2

Linear Regression

Atrijo Roy - 736

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1. Problem to demonstrate that the population regression line is fixed, but least square regression line varies

Suppose the population regression line is given by $Y = 2 + 3x$, while the data comes from the model $y = 2 + 3x + \epsilon$.

Step 1: For x in the range [5,10] graph the population regression line.

Step 2: Generate $x_i (i = 1, 2, .., n)$ from $Uniform(5, 10)$ and $\epsilon_i (i = 1, 2, .., n)$ from $N(0, 4^2)$. Hence, compute $y_1, y_2, .., y_n$.

Step 3: On the basis of the data $(x_i, y_i)(i = 1, 2, .., n)$ generated in Step 2, report the least squares regression line.

Step 4: Repeat steps 2-3 five times. Graph the 5 least squares regression lines over the population regression line obtained in Step 1.

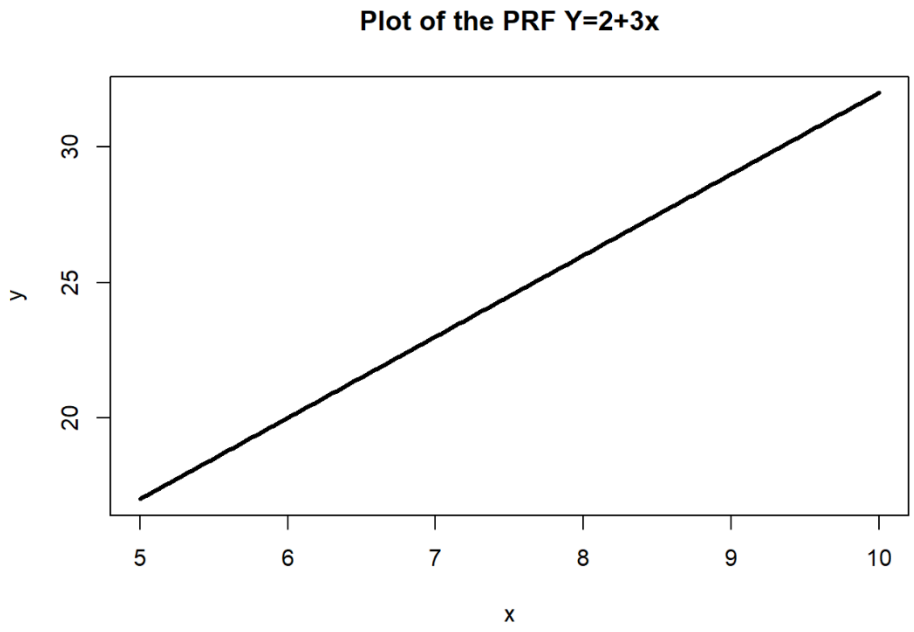
Interpret the findings.

Take $n = 50$. Set the seed as seed=123.

Model:

$$y = 2 + 3x + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

```
rm(list=ls())

#Step 1: Take 200 equidistant points in the range [5:10]
x=seq(5,10,length.out=200)
y=2+3*x
plot(x,y,type='l',lwd=3,main="Plot of the PRF Y=2+3x")
```

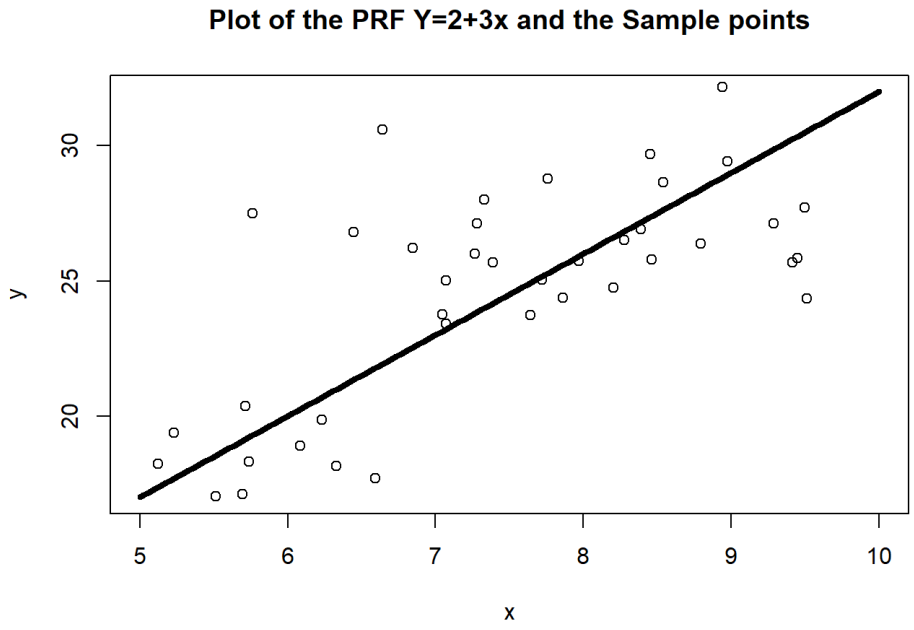


Interpretation: The graph shows a perfect positive linear relationship between x and y.

```
#Step 2: Generate xi from U(5,10) and ei from N(0,16)
n=50
set.seed(123)
xi=runif(n,5,10)
ei=rnorm(n,0,4)
#SRF y=2+3x+e
yi=2+3*xi+ei
#Overlaying the the points on the PRF
plot(x,y,type='l',lwd=4,main="Plot of the PRF Y=2+3x and the Sample points")
points(xi,yi)
```

We generate random samples of size n=50.

$$y_i \sim U(0,1), \qquad \varepsilon_i \sim \mathcal{N}(0, 4^2), \qquad \forall\, i = 1,2, \dots, n$$



Interpretation: The sample observations are scattered around the PRF, showing random deviations due to the error term while maintaining a positive linear relationship.

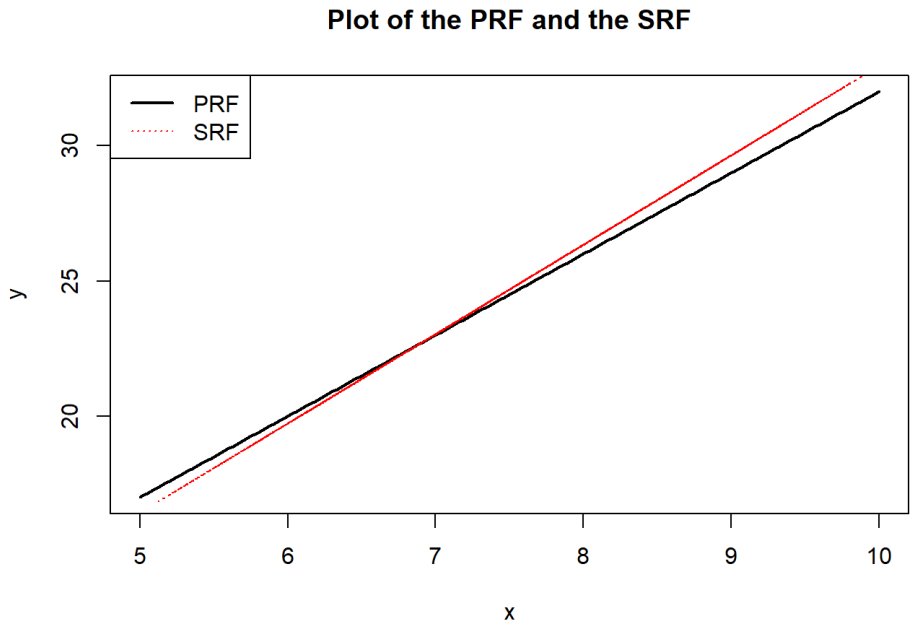
```
#Step 3: Least Square Regression line
model1=lm(yi~xi)
summary(model1)

##
## Call:
## lm(formula = yi ~ xi)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.0231 -2.2314 -0.2627  2.1970  8.7445
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.09639    2.82610  -0.034    0.973
## xi           3.30540    0.36519   9.051 5.96e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Residual standard error: 3.761 on 48 degrees of freedom
## Multiple R-squared:  0.6306, Adjusted R-squared:  0.6229
## F-statistic: 81.93 on 1 and 48 DF,  p-value: 5.962e-12
```

After applying the method of least squares, the Sample Regression Function is then given by,

$$\hat{y}_i = -0.09639 + 3.30540\, x_i$$

```
#Overlaying the PRF and SRF
y.hat = -0.09639 + 3.30540*xi
plot(x,y,type='l',lwd=2,main="Plot of the PRF and the SRF")
lines(xi,y.hat,type='l',col="red",lty="dotted")
legend("topleft",col=c("black","red"),lwd=c(2,1),legend=c("PRF","SRF"),
       lty=c("solid","dotted"))
```



Interpretation: The SRF closely approximates the PRF but does not coincide exactly because it is estimated from sample data.

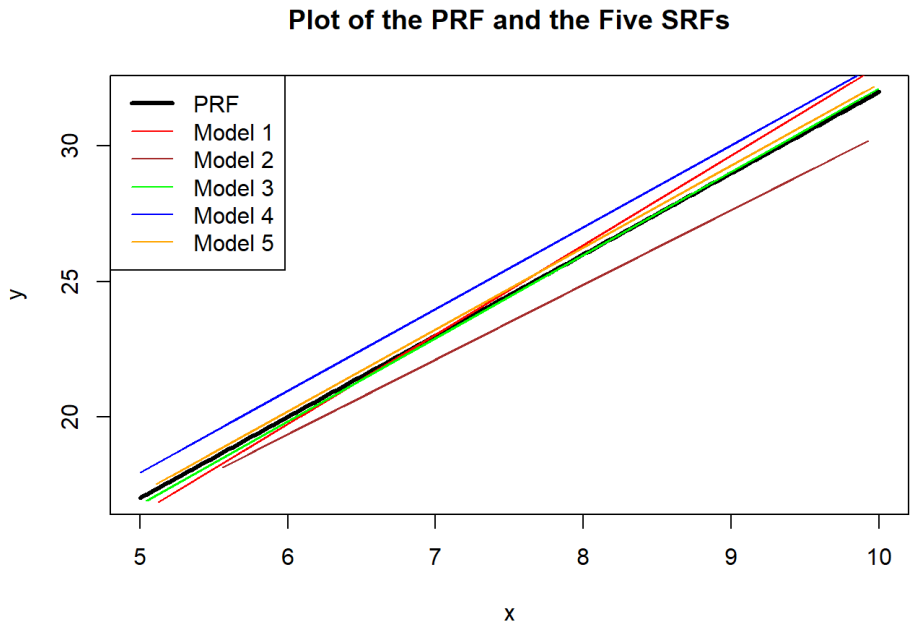
```
#Step 4: Repeat step 2 and 3 for 5 times.
set.seed(123)
x1=runif(n,5,10)
e1=rnorm(n,0,4)
y1=2+3*x1+e1
fit1=lm(y1~x1)
summary(fit1)

##
## Call:
## lm(formula = y1 ~ x1)
```

```
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.0231 -2.2314 -0.2627  2.1970  8.7445
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.09639    2.82610  -0.034    0.973
## x1           3.30540    0.36519   9.051 5.96e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.761 on 48 degrees of freedom
## Multiple R-squared:  0.6306, Adjusted R-squared:  0.6229
## F-statistic: 81.93 on 1 and 48 DF,  p-value: 5.962e-12
```

```
x2=runif(n,5,10)
e2=rnorm(n,0,4)
y2=2+3*x2+e2
fit2=lm(y2~x2)

x3=runif(n,5,10)
e3=rnorm(n,0,4)
y3=2+3*x3+e3
fit3=lm(y3~x3)

x4=runif(n,5,10)
e4=rnorm(n,0,4)
y4=2+3*x4+e4
fit4=lm(y4~x4)

x5=runif(n,5,10)
e5=rnorm(n,0,4)
y5=2+3*x5+e5
fit5=lm(y5~x5)

#Data frame containing the five models' coefficients
coeff=data.frame(coef(fit1),
                 coef(fit2),
                 coef(fit3),
                 coef(fit4),
                 coef(fit5))


coeff
```

```
##             coef.fit1. coef.fit2. coef.fit3. coef.fit4. coef.fit5.
## (Intercept) -0.09638929   2.792188   1.392997   2.823089   2.032506
## x1           3.30539569   2.761042   3.073267   3.023608   3.028097
```

```r
#Plot of the PRF and the Five SRFs

plot(x,y,type='l',lwd=3,main="Plot of the PRF and the Five SRFs")
lines(x1,predict(fit1),type='l',col="red")
lines(x2,predict(fit2),type='l',col="brown")
lines(x3,predict(fit3),type='l',col="green")
lines(x4,predict(fit4),type='l',col="blue")
lines(x5,predict(fit5),type='l',col="orange")
legend("topleft",legend=c("PRF","Model 1","Model 2","Model 3",
                          "Model 4","Model 5"),
       col=c("black","red","brown","green","blue","orange"),
       lwd=c(3,1,1,1,1,1))
```



Interpretation: Different samples produce different SRFs, all centered around the PRF. Hence the PRF is fixed while the SRF varies with respect to the samples.

```r
#Interpretation: PRF is fixed but SRF varies
```

---

2. Problem to demonstrate that β^0 and β^ minimises RSS

Step 1: Generate x_i from Uniform(5, 10) and mean centre the values. Generate ϵ_i from $N(0,1)$. Calculate $y_i = 2 + 3x_i + \epsilon_i$, i = 1,2,.., n. Take n=50 and seed=123.

Step 2: Now imagine that you only have the data on $(x_i, y_i), i = 1, 2, .., n$, without knowing the mechanism that was used to generate the data in step 1. Assuming a linear regression of the type $y_i = \beta_0 + \beta x_i + \epsilon_i$, and based on these data $(x_i, y_i), i = 1, 2, .., n$, obtain the least squares estimates of β_0 and β.

Step 3: Take a large number of grid values of (β_0, β) that also include the least squares estimates obtained from step 2. Compute the RSS for each parametric choice of (β_0, β), where $RSS = (y_1 - \beta_0 - \beta x_1)^2 + (y_2 - \beta_0 - \beta x_2)^2 + ...(y_n - \beta_0 - \beta x_n)^2$. Find out for which combination of (β_0, β), RSS is minimum.

Model

$$y = 2 + 3x + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

Here

$$\varepsilon_i \sim \mathcal{N}(0,1), \qquad \forall\, i = 1,2, \dots, n$$

```
rm(list=ls())
#step 1: Generate xi~U(5,10) of size 50, and ei~N(0,1)
#y=2+3x+e
n=50
set.seed(123)
x=runif(n,5,10)
xm=x-mean(x)
e=rnorm(n)
y=2+3*xm+e
fit=lm(y~xm)
summary(fit)

##
## Call:
## lm(formula = y ~ xm)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.25578 -0.55786 -0.06567  0.54926  2.18613
##
## Coefficients:
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2.0562     0.1330   15.46   <2e-16 ***
## xm            3.0764     0.0913   33.70   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9404 on 48 degrees of freedom
## Multiple R-squared:  0.9594, Adjusted R-squared:  0.9586
## F-statistic:  1135 on 1 and 48 DF,  p-value: < 2.2e-16
```

After applying the method of least squares we get,

$$\hat{\beta}_0 = 2.0562, \qquad \hat{\beta} = 3.0764$$

Now we calculate the Residual Sum of Squares (RSS) for a set of values of Beta

$$\text{RSS} = (y_1 - \beta_0 - \beta x_1)^2 + (y_2 - \beta_0 - \beta x_2)^2 + \cdots + (y_n - \beta_0 - \beta x_n)^2$$

OR

$$\text{RSS} = \sum_{i=1}^{n} (y_i - \beta_0 - \beta x_i)^2$$

```
beta0=2.0562
beta=3.0764
#Creating the beta and beta0 grid values
beta0.grid=seq(-1,1,length.out=101)+beta0
beta.grid=seq(-1,1,length.out=101)+beta
#Computing RSS
RSS=c()
for(i in 1:length(beta.grid))
{
  RSS[i]=sum((y-beta0.grid[i]-beta.grid[i]*xm)^2)
}
RSS
```

```
##   [1] 198.52541 192.34441 186.28828 180.35703 174.55065 168.86915
163.31252
##   [8] 157.88076 152.57388 147.39187 142.33473 137.40247 132.59508
127.91256
##  [15] 123.35492 118.92215 114.61425 110.43123 106.37308 102.43981
98.63140
##  [22]  94.94788  91.38922  87.95544  84.64653  81.46249  78.40333
75.46905
##  [29]  72.65963  69.97509  67.41542  64.98063  62.67071  60.48566
58.42549
##  [36]  56.49019  54.67976  52.99421  51.43353  49.99772  48.68679
```

```
47.50073
##  [43]  46.43954  45.50323  44.69179  44.00522  43.44353  43.00671
42.69477
##  [50]  42.50769  42.44550  42.50817  42.69572  43.00814  43.44544
44.00761
##  [57]  44.69465  45.50656  46.44335  47.50502  48.69155  50.00296
51.43925
##  [64]  53.00040  54.68643  56.49734  58.43311  60.49376  62.67929
64.98968
##  [71]  67.42495  69.98510  72.67012  75.48001  78.41477  81.47441
84.65892
##  [78]  87.96831  91.40257  94.96170  98.64570 102.45458 106.38833
110.44696
##  [85] 114.63046 118.93883 123.37208 127.93020 132.61319 137.42106
142.35380
##  [92] 147.41141 152.59390 157.90126 163.33349 168.89060 174.57258
180.37943
##  [99] 186.31116 192.36776 198.54924

df=data.frame(beta0.grid,beta.grid,RSS)
head(df) #For viewing purpose

##   beta0.grid beta.grid      RSS
## 1     1.0562    2.0764 198.5254
## 2     1.0762    2.0964 192.3444
## 3     1.0962    2.1164 186.2883
## 4     1.1162    2.1364 180.3570
## 5     1.1362    2.1564 174.5507
## 6     1.1562    2.1764 168.8692

df[which.min(RSS),]

##    beta0.grid beta.grid     RSS
## 51     2.0562    3.0764 42.4455

#We can verify the LSE minimises the RSS
```

RSS is minimum at $\beta_0 = 2.0562,\ \beta = 3.0764,$ with RSS $= 42.4455,$ which are the least squares estimates.

---

3. Problem to demonstrate that least square estimators are unbiased

Step 1: Generate $x_i(i = 1, 2, .., n)$ from $Uniform(0, 1)$, $\epsilon_i(i = 1, 2, .., n)$ from $N(0, 1)$ and hence generate y using $y_i = \beta_0 + \beta x_i + \epsilon_i$. (Take $\beta_0 = 2, \beta = 3$).

Step 2: On the basis of the data $(x_i, y_i)(i = 1, 2, .., n)$ generated in Step 1, obtain the least square estimates of β_0 and β.

Repeat Steps 1-2, $R = 1000$ times. In each simulation obtain $\hat{\beta}_0$ and $\hat{\beta}$. Finally, the least-square estimates will be given by the average of these estimated values. Compare these with the true β_0 and β and comment.

Take $n = 50$ and seed=123.

```
rm(list=ls())
#Step 1: Generate xi~U(5,10) of size 50, and ei~N(0,1)
n=50
set.seed(123)
x=runif(n,0,1)
e=rnorm(n)
y=2+3*x+e
#Step 2: obtain LSE
fit=lm(y~x)
summary(fit)

##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.25578 -0.55786 -0.06567  0.54926  2.18613
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   1.8576     0.2721   6.827 1.36e-08 ***
## x             3.3817     0.4565   7.408 1.75e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9404 on 48 degrees of freedom
## Multiple R-squared:  0.5334, Adjusted R-squared:  0.5237
## F-statistic: 54.88 on 1 and 48 DF,  p-value: 1.745e-09
```

$$\hat{\beta}_0 = 1.8576, \qquad \hat{\beta} = 3.3817$$

```r
beta0.hat=1.8576
beta.hat=3.3817
#Repeat R=1000 times
set.seed(123)
R=10000
beta0=numeric(R)
beta=numeric(R)
n=50
for(i in 1:R) {
  x=runif(n,0,1)
  e=rnorm(n,0,1)
  y=2+3*x+e
  fit=lm(y~x)
  beta0[i]=coef(fit)[1]
  beta[i]=coef(fit)[2]
}

list(Expected_beta0=mean(beta0),Expected_beta=mean(beta))
```

```
## $Expected_beta0
## [1] 2.005203
##
## $Expected_beta
## [1] 2.991338
```

```r
#comment: the long run means of beta0.hat and beta hat converges to true beta and beta0
```

$$\text{As } n \to \infty, \quad \mathbb{E}(\hat{\beta}_0) \to \beta_0 \quad \text{and} \quad \mathbb{E}(\hat{\beta}) \to \beta$$

```r
list(var(beta0),var(beta))
```

```
## [[1]]
## [1] 0.08292927
##
## [[2]]
## [1] 0.2495031
```

```r
#Comment: var approaches 0 as R goes up. Hence LSE is consistent.
```

$$\text{As } n \to \infty, \quad \text{Var}(\hat{\beta}_0) \to 0 \quad \text{and} \quad \text{Var}(\hat{\beta}) \to 0.$$

Hence, by the sufficient condition for consistency, the least squares estimators $\hat{\beta}_0$ and $\hat{\beta}$ are consistent.

---

4. Comparing several simple linear regressions

Attach "Boston" data from MASS library in R. Select median value of owner-occupied homes, as the response and per capita crime rate, nitrogen oxides concentration, proportion of blacks and percentage of lower status of the population as predictors.

(a) Selecting the predictors one by one, run four separate linear regressions to the data. Present the output in a single table.

(b) Which model gives the best fit?

(c) Compare the coefficients of the predictors from each model and comment on the usefulness of the predictors.

```
rm(list=ls())
library(MASS)

## Warning: package 'MASS' was built under R version 4.5.2

attach(Boston)
```

$$\begin{aligned}
&\text{Model 1:} \quad \text{medv}_i = \beta_0 + \beta_1\, \text{lstat}_i + \varepsilon_i,\\
&\text{Model 2:} \quad \text{medv}_i = \beta_0 + \beta_1\, \text{crim}_i + \varepsilon_i,\\
&\text{Model 3:} \quad \text{medv}_i = \beta_0 + \beta_1\, \text{nox}_i + \varepsilon_i,\\
&\text{Model 4:} \quad \text{medv}_i = \beta_0 + \beta_1\, \text{black}_i + \varepsilon_i,
\end{aligned}$$

$$\varepsilon_i \sim \mathcal{N}(0, \sigma^2), \qquad i = 1,2, \dots, n$$

β_0 andβ_1 in each case are unknown parameters and are estimated using the method of least squares.

```
fit1=lm(medv~lstat)
fit2=lm(medv~crim)
fit3=lm(medv~nox)
fit4=lm(medv~black)
```

(a) Selecting the predictors one by one, run four separate linear regressions to the data. Present the output in a single table.

```
library(stargazer)

## Warning: package 'stargazer' was built under R version 4.5.2

##
## Please cite as:
```

```
##  Hlavac, Marek (2022). stargazer: Well-Formatted Regression and Summary
Statistics Tables.

##  R package version 5.2.3. https://CRAN.R-project.org/package=stargazer

stargazer(fit1,fit2,fit3,fit4,type="text",out="f.txt")

##
## ===============================================================================
##                                         Dependent variable:
##                                 ------------------------------------------
##                                                 medv
##                                    (1)        (2)       (3)        (4)
## -------------------------------------------------------------------------------
## lstat                           -0.950***
##                                  (0.039)
##
## crim                                        -0.415***
##                                              (0.044)
##
## nox                                                   -33.916***
##                                                        (3.196)
##
## black                                                              0.034***
##                                                                     (0.004)
##
## Constant                        34.554***  24.033*** 41.346***  10.551***
##                                  (0.563)    (0.409)   (1.811)    (1.557)
##
## -------------------------------------------------------------------------------
## Observations                       506        506       506        506
## R2                                0.544      0.151     0.183      0.111
## Adjusted R2                       0.543      0.149     0.181      0.109
## Residual Std. Error (df = 504)    6.216      8.484     8.323      8.679
## F Statistic (df = 1; 504)      601.618*** 89.486*** 112.591*** 63.054***
## ===============================================================================
## Note:                                        *p<0.1; **p<0.05; ***p<0.01
```

(b) Which model gives the best fit?

Model 1 gives the best fit among the four simple linear regression models since it has the highest coefficient of determination.

The R^2 values for the models are:

$$R_1^2 = 0.544, \quad R_2^2 = 0.151, \quad R_3^2 = 0.183, \quad R_4^2 = 0.111.$$

Since R^2 measures the variability in the dependent variable explained by the predictor, the higher R^2 value for Model 1 indicates that lstat is the best explanatory variable among the four.

Hence, Model 1 provides the best fit to the data.

(c) Comparison and usefulness of predictors:

From the four simple linear regression models with medv as the dependent variable, all predictors are statistically significant at the 1% level, indicating that each variable has a meaningful linear association with median house value.

The coefficient of lstat is -0.950, implying that an increase in the percentage of lower-status population is associated with a substantial decrease in medv. This model also has the highest R^2 value (0.544), suggesting that lstat is the most useful predictor among the four in explaining variability in medv.

The coefficient of crim is -0.415, indicating that higher crime rates are associated with lower median house values. However, its explanatory power is relatively weak, with an R^2 of 0.151, making it less useful compared to lstat.

The coefficient of nox is -33.916, showing a strong negative effect of air pollution on medv. Although the magnitude of the coefficient is large due to the scale of nox, the R^2 value (0.183) suggests moderate usefulness in explaining variation in house values.

The coefficient of black is 0.034, indicating a positive association with medv. Despite being statistically significant, this predictor has the lowest explanatory power with an R^2 of 0.111, making it the least useful among the four.

In each of the four simple linear regression models, the respective predictors (lstat, crim, nox, and black) are statistically significant at the 1 percent level of significance, as indicated by the *** notation. Hence, each predictor has a significant linear effect on medv when considered separately.