Neural Networks 61 (2015) 49–58

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Passivity analysis for memristor-based recurrent neural networks with discrete and distributed delays

Guodong Zhang [a], Yi Shen [b], Quan Yin [b,*], Junwei Sun [c]

[a] College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China
[b] School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China
[c] College of Electric and Information Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, China

ARTICLE INFO

Article history:
Received 17 September 2013
Received in revised form 18 July 2014
Accepted 3 October 2014
Available online 30 October 2014

Keywords:
Neural networks
Passivity
Time delay
Memristor
Filippov solution

ABSTRACT

In this paper, based on the knowledge of memristor and recurrent neural networks (RNNs), the model of the memristor-based RNNs with discrete and distributed delays is established. By constructing proper Lyapunov functionals and using inequality technique, several sufficient conditions are given to ensure the passivity of the memristor-based RNNs with discrete and distributed delays in the sense of Filippov solutions. The passivity conditions here are presented in terms of linear matrix inequalities, which can be easily solved by using Matlab Tools. In addition, the results of this paper complement and extend the earlier publications. Finally, numerical simulations are employed to illustrate the effectiveness of the obtained results.

1. Introduction

In both real and artificial neural systems, synapses are essential elements for neural computation and information storage. An artificial synapse needs to remember its past dynamical history, store a continuous set of states, and be "plastic" according to the pre-synaptic and post-synaptic neuronal activity, fortunately, all this can be accomplished by a memristor (Chua, 1971; Kim, Sah, Yang, Roska, & Chua, 2012; Sharifiy & Banadaki, 2010; Strukov, Snider, Stewart, & Williams, 2008; Wang et al., 2012). Memristor (as a contraction of memory and resistor) is a two-terminal passive device whose value (i.e., memristance) depends on the magnitude and polarity of the voltage applied to it and the length of time that the voltage has been applied. When the voltage is turned off, the memristor remembers its most recent value until next time it is turned on (Chua, 1971; Strukov et al., 2008). It is shown in Strukov et al. (2008) that the relation between current and voltage exhibits so called pinched-hysteresis loops as the signature of memristor, and the voltage–current characteristic can be seen in Fig. 1 (Strukov et al., 2008).

As we know, resistors are used to model connection weights to emulate the synapses while in analog implementation of neural networks. The synapses among neurons in biological neural networks are known as long-term memories, but the conventional resistors cannot implement the function of memory. Now, by utilizing the memristor which has memory and behavior more like biological synapses, we are able to develop memristor-based neural network models. Recently, some results of memristor-based neural networks have been proposed and studied in Bao and Zeng (2013); Guo, Wang, and Yan (2014); Pershin and Di Ventra (2010); Wen, Zeng, and Huang (2012); Wen, Zeng, Huang, and Chen (2013); Wu, Wen, and Zeng (2012); Wu and Zeng (2014); Zhang and Shen (2013); Zhang and Shen (2014) and Yang, Cao, and Yu (2014). Among these works, Pershin and Di Ventra (2010) obtained an important finding that the electronic (memristor) synapses and neurons can represent important functionalities of their biological counterparts. And so, in the understanding of neural processes using memory devices, memristor-based neural networks of the neuromorphic computing architecture will enjoy great potentials in developing high performance neural computing and information storing systems.

In the real world, because the stability of nonlinear systems and neural networks is a prerequisite for the applications, considerable attention has been paid to the research on the problem of stability analysis, e.g., see Hu and Wang (2002); Hu and Wang (2012); Hu and Zhang (2010); Liang and Wang (2000) and Zhang, Shen, and Chen (2014). On the other hand, time delays are frequently encountered in engineering (Huang, Li, Yu, & Chen, 2009). Due to the finite speed of information processing and the

* Corresponding author. Tel.: +86 27 87543630; fax: +86 27 87543130.
E-mail addresses: zgdhbnu@163.com (G. Zhang), yinquan534@163.com (Q. Yin).

http://dx.doi.org/10.1016/j.neunet.2014.10.004

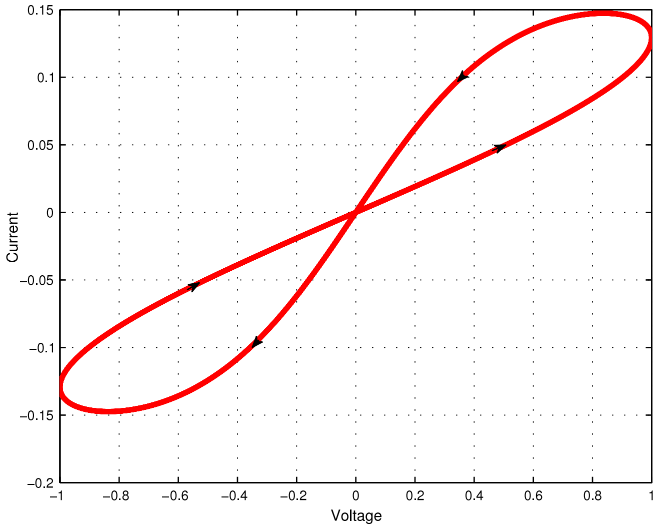

Fig. 1. The current–voltage characteristic curve of the memristor.

inherent communication time of neurons, the existence of time delays usually causes divergence, oscillation, or even instability of neural networks. Therefore, it is of both theoretical and practical importance to study the problem of stability analysis of neural networks with time delays. And these years appeared many works, e.g., see Cao and Song (2006); Chen and Zheng (2008); Chen and Zheng (2010a, 2010b); Chen and Zheng (2011); Forti, Nistri, and Papini (2005); Huang, Chan, Huang, and Cao (2007); Jiang and Teng (2004); Lou and Cui (2006); Wang and Chen (2012); Xu, Lam, and Ho (2006); Zeng and Wang (2006) and Zhang, Liu, Huang, and Wang (2010).

The passivity theory intimately related to circuit analysis methods has received a lot of attention from the control community since 1970s. Many nonlinear systems need to be passive in order to attenuate noises effectively, and the robustness measure such as robust stability or robust performance of a system often reduces to a subsystem or a modified system that is passive. Passivity analysis is a major tool for studying stability of nonlinear systems, and the essence of the passivity theory is that the passive properties of a system can keep the system internal stability. The passivity properties of dynamical neural networks has received a lot of attention and studied in Chua (1999); Li, Lam, and Cheung (2012); Li and Liao (2005); Lou and Cui (2007); Lozano, Brogliato, Egeland, and Maschke (2000); Song, Liang, and Wang (2009); Wu, Park, Su, and Chu (2012); Xu, Zheng, and Zou (2009) and Yu and Li (2007).

From a systems-theoretic point of view, a memristor-based neural network is a state-dependent nonlinear system family (Bao & Zeng, 2013; Guo et al., 2014; Wen et al., 2013; Wu, Wen et al., 2012; Wu & Zeng, 2014; Yang et al., 2014; Zhang & Shen, 2013, 2014). Such system family can reveal jumped, transient chaos of rich and complex nonlinear behaviors. In order to allow the memristors to be readily used in emerging technologies, the stability of such state-dependent nonlinear system family should be studied in the first position, as the above discussion, we know that the passivity theory provides a nice tool for analyzing the stability of memristor-based neural networks.

More recently, in Wen et al. (2013) and Wu and Zeng (2014) the authors studied the passivity of memristor-based recurrent neural networks, and some passivity conditions for memristor-based recurrent neural networks with or without time delays using linear matrix inequalities were also derived. On the other hand, neural networks often have a spatial extent because of the presence of an amount of parallel pathways of varying of axon size and lengths. Then, there may exist either a distribution of conduction velocities along these pathways or a distribution of propagation delays over a period of time in some cases, which may cause another type of time delays, namely, distributed time delays in neural networks, and these years also have appeared many good works, e.g., see Liao, Wong, and Yang (2003); Liu, Wang, and Liu (2006); Song and Wang (2008) and Wang, Liu, and Liu (2005), therefore, it is meaningful and important to study the passivity of memristor-based recurrent neural networks with discrete and distributed delays.

Motivated by the above discussions, in the present paper, based on the previous works (Bao & Zeng, 2013; Guo et al., 2014; Wen et al., 2012, 2013; Wu, Wen et al., 2012; Wu & Zeng, 2014; Yang et al., 2014; Zhang & Shen, 2013, 2014), we consider the memristor-based recurrent neural networks with both discrete delays and distributed delays as follows:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -d_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(x_i(t))f_j(x_j(t-\tau_j(t))) \\ &\quad + \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} k_{ij}(t-s)f_j(x_j(s))\mathrm{d}s \\ &\quad + u_i(t), \quad t \geq 0,\ i \in N,\end{aligned} \tag{1}$$

where

$$d_i(x_i(t)) = \frac{1}{\mathbf{C}_i}\left[\sum_{j=1}^{n}(\mathbf{M}_{ij} + \mathbf{W}_{ij} + \overline{\mathbf{R}^{***}}_{ij}) \times \delta_{ij} + \overline{\mathcal{R}}_i\right],$$

$$a_{ij}(x_i(t)) = \frac{\mathbf{M}_{ij}}{\mathbf{C}_i} \times \delta_{ij}, \qquad b_{ij}(x_i(t)) = \frac{\mathbf{W}_{ij}}{\mathbf{C}_i} \times \delta_{ij},$$

$$u_i(t) = \frac{I_i(t)}{\mathbf{C}_i}$$

where $\delta_{ij} = 1$, if $i \neq j$ holds, otherwise, -1. $\mathbf{M}_{ij}$ and $\mathbf{W}_{ij}$ denote the memductances of memristors $\mathbf{R}^{*}_{ij}$ and $\mathbf{R}^{**}_{ij}$, respectively. In addition, $\mathbf{R}^{*}_{ij}$ represents the memristor between the neuron activation function $f_j(x_j(t))$ and $x_i(t)$, $\mathbf{R}^{**}_{ij}$ represents the memristor between the neuron activation function $f_j(x_j(t-\tau_j(t)))$ and $x_i(t)$, $\mathbf{R}^{***}_{ij}$ represents the resistor between the neuron activation function $\int_{-\infty}^{t} k_{ij}(t-s)f_j(x_j(s))\mathrm{d}s$ and $x_i(t)$. $d_i(x_i(t))$ is the ith neuron self-inhibitions at time t. $a_{ij}(x_i(t))$, $b_{ij}(x_i(t))$ are memristors synaptic connection weights. c_{ij} is weight coefficients of the neurons. $f_j : \mathbb{R} \to \mathbb{R}$ denotes the neuron activation functions, $\tau_j(t)$ corresponds to the transmission delays, $I_i(t)$ is an external constant input, R_i and $\mathbf{C}_i$ are the resistor and capacitor, $\overline{\mathbf{R}^{***}}_{ij} = \frac{1}{\mathbf{R}^{***}_{ij}}$, $\overline{\mathcal{R}}_i = \frac{1}{R_i}$, $i, j \in N$, $N = 1, 2, \ldots, n$. The memristor-based recurrent neural network can be implemented by VLSI circuits as shown in Fig. 2.

It is shown in Itoh and Chua (2008) that memristor can be described by a nonlinear constitutive relation $V = M(Q)I$ or $I = W(\varphi)V$ between the device terminal voltage V and the terminal current I, where the flux $\varphi = \int V\mathrm{d}t$ and the charge $Q = \int I\mathrm{d}t$. The nonlinear function $M(Q)$ denotes the memristance, and it is defined by $M(Q) = \frac{\mathrm{d}\varphi(Q)}{\mathrm{d}Q}$. The nonlinear function $W(\varphi)$ denotes the memductance, and it is defined by $W(\varphi) = \frac{\mathrm{d}Q(\varphi)}{\mathrm{d}\varphi}$.

Memristor needs to exhibit only two sufficient distinct equilibrium states since digital computer applications requiring only two memory states (Chua, 2011). Based on the analysis in Chua (2011), Fig. 1 can be simplified as Fig. 3, which depicts the simplification of memductance of the memristor. Through the typical feature of the memristor in Fig. 3 and the previous works (Chua, 2011; Wen

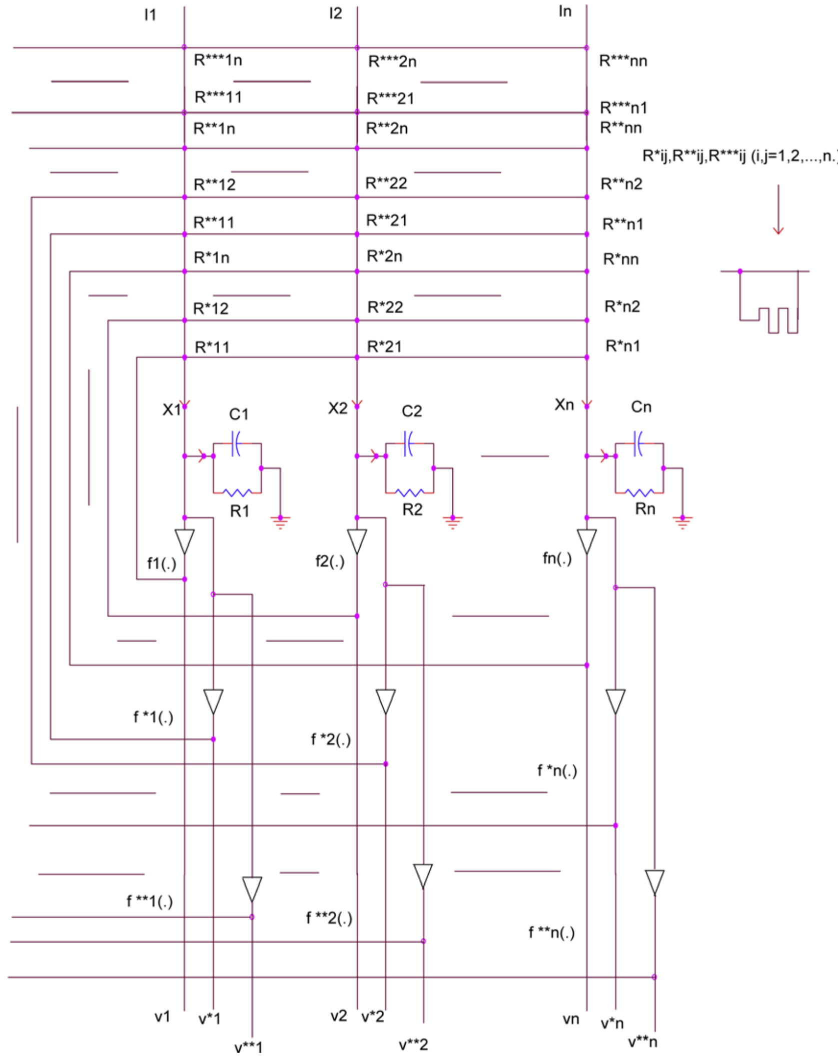

Fig. 2. Circuit of memristor-based recurrent neural network. R^{*}_{ij} is the memristor between the neuron activation function $f_j(x_j(t))$ and $x_i(t)$, R^{**}_{ij} is the memristor between the neuron activation function $f_j(x_j(t-\tau_j))$ ($f^{*}_j(\cdot)$ in the figure) and $x_i(t)$, R^{***}_{ij} is the resistor between the neuron activation function $\int_{-\infty}^{t} k_{ij}(t-s)f_j(x_j(s))\mathrm{d}s$ ($f^{**}_j(\cdot)$ in the figure) and $x_i(t)$, $I_i(t)$ is the external input, R_i and C_i are the resistor and capacitor, v_i, v^{*}_i, v^{**}_i are the outputs, $i, j = 1, 2, \ldots, n$.

et al., 2012, 2013), then

$$d_i(x_i(t)) = \begin{cases} d^{*}_i, & -\dfrac{\mathrm{d}f_i(x_i(t))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ d^{**}_i, & -\dfrac{\mathrm{d}f_i(x_i(t))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$$a_{ij}(x_i(t)) = \begin{cases} a^{*}_{ij}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ a^{**}_{ij}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$$b_{ij}(x_i(t)) = \begin{cases} b^{*}_{ij}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t-\tau_j(t)))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ b^{**}_{ij}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t-\tau_j(t)))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$d^{*}_i > 0$, $d^{**}_i > 0$, $a^{*}_{ij}, a^{**}_{ij}, b^{*}_{ij}, b^{**}_{ij}$, $i, j \in N$ are all constant numbers. Obviously, the memristor-based recurrent neural networks (1) is a state-dependent switched system, which is the generalization of those for conventional recurrent neural networks.

In this paper, our aim is dealing with the problem of passivity analysis of memristor-based recurrent neural networks with discrete and distributed delays (1). The main contribution of this paper lies in the following aspects. Firstly, in this paper, we adopt nonsmooth analysis and control theory to handle memristor-based neural networks with discontinuous right-hand side. Compared with the results on passivity of the neural networks with continuous right-hand side (Li et al., 2012; Li & Liao, 2005; Lou & Cui, 2007; Xu et al., 2009; Yu & Li, 2007), our results of the passivity are less

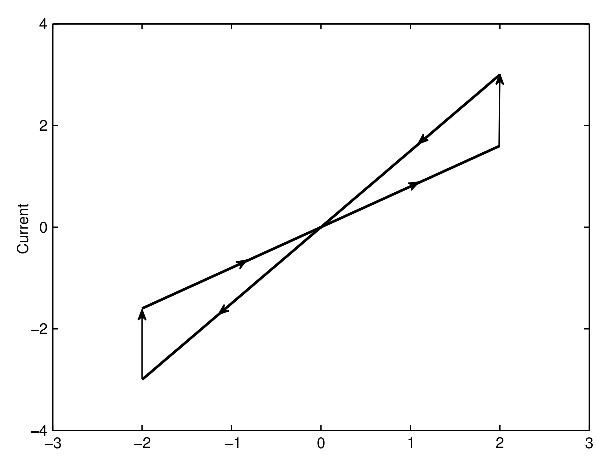

Fig. 3. Typical current–voltage characteristics of a memristor.

conservative, more general and achieve a valuable improvement. Secondly, the passivity analysis is extended to the memristor-based recurrent neural networks with discrete and distributed delays. Furthermore, several new sufficient conditions are derived to ensure the passivity of the memristor-based recurrent neural networks, which complement and extend the earlier publications.

The organization of this paper is as follows. In Section 2, some preliminaries are introduced. In Section 3, several new sufficient conditions for the passivity are derived by constructing suitable Lyapunov functionals and using inequality technique. Then, numerical simulations are given to demonstrate the effectiveness of the proposed approach in Section 4. Finally, this paper ends by a conclusion.

2. Preliminaries

For convenience, some notations are introduced. Solutions of all the systems considered throughout this paper are intended in Filippov's sense (Filippov, 1988). In Banach space, $\mathcal{C}((-\infty,0],\mathfrak{R}^n)$ denotes all continuous functions. For vector $v=(v_1,v_2,\ldots,v_n)^T\in\mathfrak{R}^n$, $\|v\|$ is said to be the Euclidean norm, i.e., $\|v\|=\sqrt{[\sum_{i=1}^n(v_i)^2]}$. $\mathrm{co}\{\ell,\jmath\}$ represents closure of the convex hull generated by real matrices ℓ and $\jmath$ or real numbers ℓ and $\jmath$. Let $\bar{d}_i=\max\{d_i^*,d_i^{**}\}$, $\underline{d}_i=\min\{d_i^*,d_i^{**}\}$, $\bar{a}_{ij}=\max\{a_{ij}^*,a_{ij}^{**}\}$, $\underline{a}_{ij}=\min\{a_{ij}^*,a_{ij}^{**}\}$, $\bar{b}_{ij}=\max\{b_{ij}^*,b_{ij}^{**}\}$, $\underline{b}_{ij}=\min\{b_{ij}^*,b_{ij}^{**}\}$, $A_{ij}=\max\{|\bar{a}_{ij}|,|\underline{a}_{ij}|\}$, $B_{ij}=\max\{|\bar{b}_{ij}|,|\underline{b}_{ij}|\}$. Let $M=(m_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ denote real square matrix, M^{-1}, M^T and $\lambda(M)$ denote the inverse, the transpose and the eigenvalues of the square matrix M, respectively. $M>0\,(M<0)$ means that M is a positive definite (negative definite) matrix. $\|M\|_\infty=\max_{1\le i\le n}\sum_{j=1}^n|m_{ij}|$, $\|M\|_1=\max_{1\le j\le n}\sum_{i=1}^n|m_{ij}|$. $[\Theta,\Delta]$ by the interval matrix, where $\Theta=(\theta_{ij})_{n\times n}$, $\Delta=(\vartheta_{ij})_{n\times n}$, it follows that $\Theta\ll\Delta$, which means $\Theta\ll\Sigma=(\sigma_{ij})_{n\times n}\ll\Delta$, we have $\theta_{ij}<\sigma_{ij}<\vartheta_{ij}, i,j=1,2,\ldots,n$, where $\Sigma\in[\Theta,\Delta]$. For a continuous functions $k(t):R\to R$, $D^+k(t)$ is called the upper right dini derivative and defined as $D^+k(t)=\overline{\lim}_{h\to0^+}\frac{1}{h}(k(t+h)-k(t))$. Now, we first introduce the following definitions about set-valued map (Aubin & Cellina, 1984; Clarke, Ledyaev, Stem, & Wolenski, 1998).

Definition 1. Let $E\subset R^n, x\mapsto F(x)$ be a set-valued map from $E\hookrightarrow R^n$, if to each point x of a set $E\subset R^n$, there corresponds a nonempty set $F(x)\subset R^n$. A set-valued map F with nonempty values is said to be upper-semi-continuous at $x_0\in E\subset R^n$ for, if any open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M)\subset N$. $F(x)$ is said to have a closed (convex, compact) image if for each $x\in E$, $F(x)$ is closed (convex, compact).

Now, we introduce the concept of Filippov solution (Filippov, 1988). Consider the following differential system in vector notation:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=g(t,x),\quad g:R_+\times R^n\mapsto R^n. \tag{2}$$

Definition 2. For the system (2), with discontinuous right-hand sides, a set-valued map is defined as

$$\Phi(t,x)=\bigcap_{\rho>0}\bigcap_{\mu(N)=0}\overline{\mathrm{co}}[g(t,B(x,\rho)\setminus N)],$$

where $\overline{\mathrm{co}}[\cdot]$ denotes the closure of the convex hull, $B(x,\rho)$ is the ball of center x and radius ρ, and $\mu(N)$ is Lebesgue measure of set N. A vector-value function $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T$ defined on a nondegenerate interval $I\subset R$ is called a Filippov solution of system (2), if $x(t)$ is an absolutely continuous function on any subinterval $[l_1,l_2]\subset I$, which satisfies the differential inclusion:

$$\frac{\mathrm{d}x}{\mathrm{d}t}\in\Phi(t,x),\quad \text{for a.a. } t\in I. \tag{3}$$

By applying the theories of set-valued maps and differential inclusions above, the memristor-based neural network (1) can be written as follows:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}&\in-\mathrm{co}[d_i(x_i(t))]x_i(t)+\sum_{j=1}^n\mathrm{co}[a_{ij}(x_i(t))]f_j(x_j(t))\\&+\sum_{j=1}^n\mathrm{co}[b_{ij}(x_i(t))]f_j(x_j(t-\tau_j(t)))\\&+\sum_{j=1}^n c_{ij}\int_{-\infty}^t k_{ij}(t-s)f_j(x_j(s))\mathrm{d}s+u_i(t),\\&\text{for a.a. } t\ge0,\ i\in N,\end{aligned} \tag{4}$$

where

$$\mathrm{co}[d_i(x_i(t))]=[\underline{d}_i,\bar{d}_i],\qquad \mathrm{co}[a_{ij}(x_i(t))]=[\underline{a}_{ij},\bar{a}_{ij}] \tag{5}$$

$$\mathrm{co}[b_{ij}(x_i(t))]=[\underline{b}_{ij},\bar{b}_{ij}]. \tag{6}$$

In this paper, we consider the output of memristor-based recurrent neural networks (4) as follows:

$$y(t)=f(x(t)), \tag{7}$$

where $y(t)=(y_i(t))^T, f(x(t))=(f_i(x_i(t)))^T,\ i=1,2,\ldots,n.$

Definition 3. A vector-value function (in Filippov's sense) $x^*(t)=(x_1^*(t),x_2^*(t),\ldots,x_n^*(t))^T$ is a solution of system (1), with the initial conditions $\phi(s)=(\phi_1(s),\phi_2(s),\ldots,\phi_n(s))^T\in\mathcal{C}((-\infty,0],R^n)$, if $x^*(t)$ is an absolutely continuous function and satisfies the differential inclusion (4).

Now we do the following assumptions for the system (1):

(A1) For $\forall s_1,s_2\in R, s_1\ne s_2$, the neuron activation functions $f_i\ (i=1,2,.,n)$ are bounded and satisfy

$$0\le\frac{f_i(s_1)-f_i(s_2)}{s_1-s_2}\le\rho_i,\quad f_i(0)=0, \tag{8}$$

where $\rho_i>0$.

(A2) Let $R_+=[0,+\infty)$, the delay kernel function $k_{ij}:R_+\mapsto R_+$ are real-valued nonnegative continuous functions that satisfy the following conditions as in Liao et al. (2003)

$$\int_0^{+\infty}k_{ij}(s)\mathrm{d}s=1,\qquad\int_0^{+\infty}se^{\gamma s}k_{ij}(s)\mathrm{d}s<+\infty,\quad\gamma>0.$$

The delay $\tau_j(t)$ is differential and bounded function with $0\le\tau_j(t)\le\tau,\ \dot{\tau}_j(t)\le\mu<1,\ i,j=1,2,\ldots,n.$

There are several different definitions about passivity. Here, we give a less restrictive definition of passivity introduced by Lozano et al. (2000) as follows:

Definition 4. System (1) is called passive if there exists a scalar $\eta>0$ such that

$$2\int_0^{t_p}y^T(s)u(s)\mathrm{d}s\ge-\eta\int_0^{t_p}u^T(s)u(s)\mathrm{d}s, \tag{9}$$

for all $t_p\ge0$ and for all solutions of the system (1) with $x(0)=0$, and here $y(t)=(y_1(t),y_2(t),\ldots,y_n(t))^T, u(t)=(u_1(t),u_2(t),\ldots,u_n(t))^T$.

Lemma 1. *If assumptions* (A1) *and* (A2) *hold, then for* $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in \mathcal{C}((-\infty, 0], R^n)$, *there is at least a local solution* $x^*(t)$ *of system* (1) *with initial condition* $x^*(s) = \phi(s),\ s \in (-\infty, 0]$, *and the local solution* $x^*(t)$ *can be extended to the interval* $[0, +\infty)$ *in the sense of Filippov.*

Lemma 2 (Poznyak & Sanchez, 1995)**.** *For any two real vectors* $Y \in R^{n\times 1}$ *and* $Z \in R^{n\times 1}$, *and a symmetrical positive definite matrix* $P \in R^{n\times n}$, *the following matrix inequality holds:*

$$Y^TZ + Z^TY \le Y^TP^{-1}Y + Z^TPZ.$$

In the next section, the paper aims to find some sufficient conditions to ensure the passivity of the memristor-based RNNs (1).

3. Passivity analysis of the memristor-based recurrent neural networks

Here, we are in a position to present the following results:

Theorem 1. *Under assumptions* (A1) *and* (A2), *if there exist a positive diagonal matrix* $H = \text{diag}(h_1, h_2, \ldots, h_n)$ *and a scalar* $\eta > 0$ *such that the following LMI holds:*

$$\Pi_1 = \begin{pmatrix} \Pi_{11} & HA - D^TH & HB & H \\ A^TH - DH & \Pi_{22} & HB & H - I \\ B^TH & B^TH & -2(1-\mu)I & 0 \\ H & H - I & 0 & -\eta I \end{pmatrix} < 0, \tag{10}$$

where $\Pi_{11} = -2HD + \|HC\|_\infty I$, $\Pi_{22} = HA + A^TH + (2 + 2\|HC\|_1 + \|HC\|_\infty)I$, $A = (A_{ij})_{n\times n}$, $B = (B_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = \text{diag}(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)$, $i, j = 1, 2, \ldots, n$, *then the memristor-based RNNs with discrete and distributed delays* (1) *is passive.*

Proof. We consider a Lyapunov functional as

$$\begin{aligned} V(t,x) &= x^T(t)Hx(t) + 2\sum_{i=1}^n h_i \int_0^{x_i(t)} f_i(s)\mathrm{d}s \\ &\quad + 2\sum_{j=1}^n \int_{t-\tau_j(t)}^t f_j^2(x_j(s))\mathrm{d}s + 2\sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}| \\ &\quad \times \int_0^{+\infty}\int_{t-s}^t k_{ij}(s)f_j^2(x_j(\xi))\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{11}$$

Under assumptions (A1) and (A2), by calculating the upper right derivation $D^+V(t)$ of $V(t)$ along the solution of system (4), we obtain

$$\begin{aligned} &D^+V - 2y^T(t)u(t) - \eta u^T(t)u(t) \\ &= D^+x^T(t)Hx(t) + [x^T(t)H + 2f^T(x(t))H]D^+x(t) \\ &\quad + 2\sum_{j=1}^n f_j^2(x_j(t)) - 2\sum_{j=1}^n (1-\dot{\tau}_j(t))f_j^2(x_j(t-\tau_j(t))) \\ &\quad + 2\sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|\int_0^{+\infty} k_{ij}(s)[f_j^2(x_j(t)) \\ &\quad - f_j^2(x_j(t-s))]\mathrm{d}s - 2y^T(t)u(t) - \eta u^T(t)u(t) \\ &\le -2x^T(t)HDx(t) + 2x^T(t)HAf(x(t)) \\ &\quad + 2x^T(t)HBf(x(t-\tau(t))) + 2\sum_{i=1}^n\sum_{j=1}^n h_ic_{ij} \\ &\quad \times \int_0^{+\infty} k_{ij}(s)x_i(t)f_j(x_j(t-s))\mathrm{d}s + 2x^T(t)Hu(t) \\ &\quad - 2f^T(x(t))HDx(t) + f^T(x(t))(HA + A^TH)f(x(t)) \\ &\quad + 2f^T(x(t))HBf(x(t-\tau(t))) + 2f^T(x(t))Hu(t) \\ &\quad + 2\sum_{i=1}^n\sum_{j=1}^n h_ic_{ij}\int_0^{+\infty} k_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))\mathrm{d}s \\ &\quad + 2f^T(x(t))f(x(t)) - 2(1-\mu)f^T(x(t-\tau(t))) \\ &\quad \times f(x(t-\tau(t))) + 2\sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|f_j^2(x_j(t)) \\ &\quad - 2\sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|\int_0^{+\infty} k_{ij}(s) \\ &\quad \times f_j^2(x_j(t-s))\mathrm{d}s - 2y^T(t)u(t) - \eta u^T(t)u(t). \end{aligned} \tag{12}$$

By using the inequality $\alpha^2 + \beta^2 \ge 2\alpha\beta$ for any $\alpha,\ \beta \in R$. Then, from (12), we can obtain that

$$\begin{aligned} &\sum_{i=1}^n\sum_{j=1}^n h_ic_{ij}\int_0^{+\infty} 2k_{ij}(s)x_i(t)f_j(x_j(t-s))\mathrm{d}s \\ &\le \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|\int_0^{+\infty} k_{ij}(s)[x_i^2(t) + f_j^2(x_j(t-s))]\mathrm{d}s \\ &= \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|x_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}| \\ &\quad \times \int_0^{+\infty} k_{ij}(s)f_j^2(x_j(t-s))\mathrm{d}s \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\sum_{i=1}^n\sum_{j=1}^n h_ic_{ij}\int_0^{+\infty} 2k_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))\mathrm{d}s \\ &\le \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|\int_0^{+\infty} k_{ij}(s)[f_i^2(x_i(t)) + f_j^2(x_j(t-s))]\mathrm{d}s \\ &= \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}|f_i^2(x_i(t)) + \sum_{i=1}^n\sum_{j=1}^n h_i|c_{ij}| \\ &\quad \times \int_0^{+\infty} k_{ij}(s)f_j^2(x_j(t-s))\mathrm{d}s. \end{aligned} \tag{14}$$

Now, from (8), (10), (13) and (14), we obtain the following estimate for the right-hand side of (12)

$$\begin{aligned} &D^+V - 2y^T(t)u(t) - \eta u^T(t)u(t) \\ &\le -2x^T(t)HDx(t) + 2x^T(t)HAf(x(t)) \\ &\quad + 2x^T(t)HBf(x(t-\tau(t))) + 2x^T(t)Hu(t) \\ &\quad - 2f^T(x(t))HDx(t) + f^T(x(t))(HA + A^TH)f(x(t)) \\ &\quad + 2f^T(x(t))HBf(x(t-\tau(t))) + \|HC\|_\infty x^T(t)x(t) \\ &\quad + 2f^T(x(t))Hu(t) + (2 + 2\|HC\|_1 + \|HC\|_\infty) \\ &\quad \times f^T(x(t))f(x(t)) - 2(1-\mu)f^T(x(t-\tau(t))) \\ &\quad \times f(x(t-t(t))) - 2y^T(t)u(t) - \eta u^T(t)u(t) \\ &= -x^T(t)[2HD - \|HC\|_\infty]x(t) + 2x^T(t)HAf(x(t)) \\ &\quad + 2x^T(t)HBf(x(t-\tau(t))) + 2x^T(t)Hu(t) - 2f^T(x(t)) \\ &\quad \times HDx(t) + f^T(x(t))[HA + A^TH + (2 + 2\|HC\|_1 \\ &\quad + \|HC\|_\infty)I\,]f(x(t)) + 2f^T(x(t))HBf(x(t-\tau(t))) \\ &\quad + 2f^T(x(t))Hu(t) - 2(1-\mu)f^T(x(t-\tau(t)))f(x(t-t(t))) \\ &\quad - 2y^T(t)u(t) - \eta u^T(t)u(t) \end{aligned}$$

$$\begin{aligned}
&= x^T(t)[-2HD + \|HC\|_\infty]x(t) + x^T(t)(HA - D^TH)f(x(t)) \\
&\quad + f^T(x(t))(A^TH - HD)x(t) + x^T(t)HBf(x(t-\tau(t))) \\
&\quad + f^T(x(t-\tau(t)))B^THx(t) + x^T(t)Hu(t) + u^T(t)Hx(t) \\
&\quad + f^T(x(t))[HA + A^TH + (2 + 2\|HC\|_1 + \|HC\|_\infty)I\,]f(x(t)) \\
&\quad + f^T(x(t))HBf(x(t-\tau(t))) + f^T(x(t-\tau(t)))B^THf(x(t)) \\
&\quad + f^T(x(t))(H-I)u(t) + u^T(t)(H-I)f(x(t)) \\
&\quad - 2(1-\mu)f^T(x(t-\tau(t)))f(x(t-t(t))) - \eta u^T(t)u(t) \\
&= \Omega^T(t)\begin{pmatrix} \Sigma_{11} & HA - D^TH & HB & H \\ A^TH - HD & \Sigma_{22} & HB & H - I \\ B^TH & B^TH & -2(1-\mu)I & 0 \\ H & H - I & 0 & -\eta I \end{pmatrix} \\
&\quad \times \Omega(t) \\
&= \Omega^T(t)\Pi_1\Omega(t) < 0 \qquad (15)
\end{aligned}$$

where $\Omega(t) = [x^T(t)\, f^T(x(t))\, f^T(x(t-\tau(t)))\, u^T(t)]^T$, Then, we can easily get

$$D^+V - 2y^T(t)u(t) - \eta u^T(t)u(t) < 0. \qquad (16)$$

From (11), we know when $x(0) = 0, V(0, x(0)) = 0$. Now, by integrating (16) with respect to t over the time period 0 to t_p, we have

$$2\int_0^{t_p} y^T(s)u(s)\mathrm{d}s \ge V(t_p, x(t_p)) - \eta\int_0^{t_p} u^T(s)u(s)\mathrm{d}s. \qquad (17)$$

Obviously, $V(t_p, x(t_p)) \ge 0$, so the inequality (9) holds, and then, we get the memristor-based RNNs with discrete and distributed delays (1) is passive. The proof is completed.

From the proof of Theorem 1, we can easily obtain the following results.

Corollary 1. *Under assumptions* (A1) *and* (A2), *if there exist a positive diagonal matrix* $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$ *and a scalar* $\eta > 0$ *such that the following LMI holds:*

$$\Pi_2 = \begin{pmatrix} \Pi_{11} & HA & HB & H \\ A^TH & \Pi_{22} & HB & H - I \\ B^TH & B^TH & -2(1-\mu)I & 0 \\ H & H - I & 0 & -\eta I \end{pmatrix} < 0, \qquad (18)$$

where $\Pi_{11} = -2HD + \|HC\|_\infty I$, $\Pi_{22} = HA + A^TH - 2HD\Gamma^{-1} + (2 + 2\|HC\|_1 + \|HC\|_\infty)I$, $\Gamma = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$, $A = (A_{ij})_{n\times n}$, $B = (B_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = \mathrm{diag}(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)$, $i, j = 1, 2, \ldots, n$, *then the memristor-based RNNs with discrete and distributed delays* (1) *is passive.*

Theorem 2. *Under assumptions* (A1) *and* (A2), *if there exist a positive diagonal matrix* $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$ *and a scalar* $\eta > 0$ *such that the following LMI holds:*

$$\Pi_3 = \begin{pmatrix} \Pi_{11} & HA & H \\ A^TH & \Pi_{22} & H - I \\ H & H - I & -\eta I \end{pmatrix} < 0, \qquad (19)$$

where $\Pi_{11} = -2HD + \|HC\|_\infty I + \frac{1}{1-\mu}HBB^TH$, $\Pi_{22} = HA + A^TH - 2HD\Gamma^{-1} + (2 + 2\|HC\|_1 + \|HC\|_\infty)I + \frac{1}{1-\mu}HBB^TH$, $\Gamma = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$, $A = (A_{ij})_{n\times n}$, $B = (B_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = \mathrm{diag}(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)$, $i, j = 1, 2, \ldots, n$, *then the memristor-based RNNs with discrete and distributed delays* (1) *is passive.*

Proof. From Lemma 2, here let $P = (1-\mu)I$, then, we get

$$\begin{aligned}
&2x^T(t)HBf(t-\tau(t)) \\
&\quad = (B^THx(t))^Tf(t-\tau(t)) + f^T(t-\tau(t))B^THx(t) \\
&\quad \le \frac{1}{1-\mu}x^T(t)HBB^THx(t) + (1-\mu)f^T(t-\tau(t)) \\
&\quad\quad \times f(t-\tau(t)) \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&2f^T(x(t))HBf(t-\tau(t)) \\
&\quad = (B^THf(x(t)))^Tf(t-\tau(t)) + f^T(t-\tau(t))B^THf(x(t)) \\
&\quad \le \frac{1}{1-\mu}f^T(x(t))HBB^THf(x(t)) + (1-\mu) \\
&\quad\quad \times f^T(t-\tau(t))f(t-\tau(t)). \qquad (21)
\end{aligned}$$

Now, from (15), (20) and (21), let $\Psi = [x^T(t)\, f^T(x(t))\, u^T(t)]^T$, we can obtain

$$\begin{aligned}
&D^+V|_{(4a)\text{ or }(4b)} - 2y^T(t)u(t) - \eta u^T(t)u(t) \\
&\quad \le -x^T(t)[2HD - \|HC\|_\infty]x(t) + 2x^T(t)HAf(x(t)) \\
&\quad\quad + 2x^T(t)HBf(x(t-\tau(t))) + 2x^T(t)Hu(t) \\
&\quad\quad - 2f^T(x(t))HD\Gamma^{-1}f(x(t)) + f^T(x(t))[HA + A^TH \\
&\quad\quad + (2 + 2\|HC\|_1 + \|HC\|_\infty)I]f(x(t)) + 2f^T(x(t)) \\
&\quad\quad \times HBf(x(t-\tau(t))) + 2f^T(x(t))Hu(t) - 2(1-\mu) \\
&\quad\quad \times f^T(x(t-\tau(t)))f(x(t-t(t))) - 2f^T(x(t))u(t) \\
&\quad\quad - \eta u^T(t)u(t) \\
&\quad \le x^T(t)\left[-2HD + \|HC\|_\infty + \frac{1}{1-\mu}HBB^TH\right]x(t) \\
&\quad\quad + x^T(t)HAf(x(t)) + f^T(x(t))A^THx(t) + x^T(t)Hu(t) \\
&\quad\quad + u^T(t)Hx(t) + f^T(x(t))\Big[HA + A^TH - 2HD\Gamma^{-1} \\
&\quad\quad + (2 + 2\|HC\|_1 + \|HC\|_\infty)I + \frac{1}{1-\mu}HBB^TH\Big]f(x(t)) \\
&\quad\quad + f^T(x(t))(H-I)u(t) + u^T(t)(H-I) \\
&\quad\quad \times f(x(t)) - \eta u^T(t)u(t) \\
&\quad = \Psi^T(t)\begin{pmatrix} \Pi_{11} & HA & H \\ A^TH & \Pi_{22} & H - I \\ H & H - I & -\eta I \end{pmatrix}\Psi(t) \\
&\quad = \Psi^T(t)\Pi_3\Psi(t) < 0. \qquad (22)
\end{aligned}$$

The following proof as Theorem 1, so we obtain the memristor-based RNNs with discrete and distributed delays (1) is passive. The proof is completed.

Combining the proof of Theorems 1 and 2, we can easily obtain the following results.

Corollary 2. *Under assumptions* (A1) *and* (A2), *if there exist a positive diagonal matrix* $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$ *and a scalar* $\eta > 0$ *such that the following LMI holds:*

$$\Pi_4 = \begin{pmatrix} \Pi_{11} & HA - D^TH & H \\ A^TH - HD & \Pi_{22} & H - I \\ H & H - I & -\eta I \end{pmatrix} < 0, \qquad (23)$$

where $\Pi_{11} = -2HD + \|HC\|_\infty I + \frac{1}{1-\mu}HBB^TH$, $\Pi_{22} = HA + A^TH + (2 + 2\|HC\|_1 + \|HC\|_\infty)I + \frac{1}{1-\mu}HBB^TH$, $A = (A_{ij})_{n\times n}$, $B = (B_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = \mathrm{diag}(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)$, $i, j = 1, 2, \ldots, n$, *then the memristor-based RNNs with discrete and distributed delays* (1) *is passive.*

Theorem 3. *Under assumptions* (A1) *and* (A2), *if* $C = (c_{ij})_{n\times n} \equiv 0$, *and there exist a positive diagonal matrix* $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$

and a scalar $\eta > 0$ with one of the LMI Π_l, $l = 1, 2, 3, 4$ holds, then, the memristor-based RNNs with time-varying delays (1) is passive.

Remark 1. In this paper, we adopt nonsmooth analysis and control theory to handle memristor-based neural networks with discontinuous right-hand side. Compared with the results on passivity of the neural networks with continuous right-hand side (Li et al., 2012; Li & Liao, 2005; Lou & Cui, 2007; Xu et al., 2009; Yu & Li, 2007), our results of the passivity are less conservative, more general and achieve a valuable improvement. Moreover, the research method here can be extend to study the stability and passivity problems for more complex situations, such as the fourth term of system (1) with memristor, the high-order type memristor-based neural networks with time delays, the system with hybrid jump and so on.

Remark 2. For the memristor-based neural networks, sufficient conditions were obtained for global exponential stability (Guo et al., 2014; Wen et al., 2012), multistability of periodic solutions (Bao & Zeng, 2013), exponential synchronization (Wu, Wen et al., 2012; Zhang & Shen, 2013, 2014). Compared with the above results, the main results of this paper are obtained for passivity, which complement the earlier publications. The main results of passivity about the systems of our paper are with discrete and distributed delays, which are different from the main results of passivity about the system without delay (Wu & Zeng, 2014) and without distributed delay (Wen et al., 2013). So, our results extend the earlier publications.

4. Numerical examples

Now, we perform some numerical simulations to illustrate our analysis.

Example 1. Consider two-dimensional memristor-based recurrent neural networks with discrete and distributed delays

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -d_1(x_1(t))x_1(t) + a_{11}(x_1(t))f_1(x_1(t)) \\ \quad + a_{12}(x_1(t))f_2(x_2(t)) + b_{11}(x_1(t))f_1(x_1(t-\tau_1(t))) \\ \quad + b_{12}(x_1(t))f_2(x_2(t-\tau_2(t))) + c_{11} \\ \quad \times \displaystyle\int_{-\infty}^{t} k_{11}(t-s)f_1(x_1(s))ds \\ \quad + c_{12}\displaystyle\int_{-\infty}^{t} k_{12}(t-s)f_2(x_2(s))ds + u_1(t), \\ \dfrac{dx_2(t)}{dt} = -d_2(x_2(t))x_1(t) + a_{21}(x_2(t))f_1(x_1(t)) \\ \quad + a_{22}(x_2(t))f_2(x_2(t)) + b_{21}(x_2(t))f_1(x_1(t-\tau_1(t))) \\ \quad + b_{22}(x_2(t))f_2(x_2(t-\tau_2(t))) + c_{21} \\ \quad \times \displaystyle\int_{-\infty}^{t} k_{21}(t-s)f_1(x_1(s))ds \\ \quad + c_{22}\displaystyle\int_{-\infty}^{t} k_{22}(t-s)f_2(x_2(s))ds + u_2(t), \end{cases} \tag{24}$$

where

$$d_1(x_1(t)) = \begin{cases} 5, & -\dfrac{df_1(x_1(t))}{dt} \le \dfrac{dx_1(t)}{dt}, \\ 5.5, & -\dfrac{df_1(x_1(t))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$d_2(x_2(t)) = \begin{cases} 5.5, & -\dfrac{df_2(x_2(t))}{dt} \le \dfrac{dx_2(t)}{dt}, \\ 5, & -\dfrac{df_2(x_2(t))}{dt} > \dfrac{dx_2(t)}{dt}, \end{cases}$$

$$a_{11}(x_1(t)) = \begin{cases} -0.3, & -\dfrac{df_1(x_1(t))}{dt} \le \dfrac{dx_1(t)}{dt}, \\ -0.28, & -\dfrac{df_1(x_1(t))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$a_{12}(x_1(t)) = \begin{cases} 0.18, & \dfrac{df_2(x_2(t))}{dt} \le \dfrac{dx_1(t)}{dt}, \\ 0.2, & \dfrac{df_2(x_2(t))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$b_{11}(x_1(t)) = \begin{cases} -0.5, & -\dfrac{df_1(x_1(t-\tau_1(t)))}{dt} \le \dfrac{dx_1(t)}{dt}, \\ -0.48, & -\dfrac{df_1(x_1(t-\tau_1(t)))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$b_{12}(x_1(t)) = \begin{cases} 0.68, & \dfrac{df_2(x_2(t-\tau_2(t)))}{dt} \le \dfrac{dx_1(t)}{dt}, \\ 0.7, & \dfrac{df_2(x_2(t-\tau_2(t)))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$a_{21}(x_2(t)) = \begin{cases} 0.38, & \dfrac{df_1(x_1(t))}{dt} \le \dfrac{dx_2(t)}{dt}, \\ 0.4, & \dfrac{df_1(x_1(t))}{dt} > \dfrac{dx_2(t)}{dt}, \end{cases}$$

$$a_{22}(x_2(t)) = \begin{cases} -0.1, & -\dfrac{df_2(x_2(t))}{dt} \le \dfrac{dx_2(t)}{dt}, \\ -0.09, & -\dfrac{df_2(x_2(t))}{dt} > \dfrac{dx_2(t)}{dt}, \end{cases}$$

$$b_{21}(x_2(t)) = \begin{cases} 0.68, & \dfrac{df_1(x_1(t-\tau_1(t)))}{dt} \le \dfrac{dx_2(t)}{dt}, \\ 0.7, & \dfrac{df_1(x_1(t-\tau_1(t)))}{dt} > \dfrac{dx_2(t)}{dt}, \end{cases}$$

$$b_{22}(x_2(t)) = \begin{cases} -0.4, & -\dfrac{df_2(x_2(t-\tau_2(t)))}{dt} \le \dfrac{dx_2(t)}{dt}, \\ -0.38, & -\dfrac{df_2(x_2(t-\tau_2(t)))}{dt} > \dfrac{dx_2(t)}{dt} \end{cases}$$

and

$$C = \begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 1 \end{pmatrix}, \qquad \tau_j(t) = 0.25,$$

$$f_i(x_i) = \frac{1}{2}(|x_i+1| - |x_i-1|), \qquad k_{ij}(r) = e^{-r}, \quad i, j = 1, 2. \tag{25}$$

Obviously, $f_i(x)$, $i = 1, 2$ are bounded and Lipschitz continuous functions with Lipschitz constants $\rho_1 = \rho_2 = 1$, and

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \qquad A = \begin{pmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.5 & 0.7 \\ 0.7 & 0.4 \end{pmatrix}.$$

For convenience, here, we choose $H = \text{diag}(1, 1)$, $\eta = 2$, Then, by using the Matlab tools, we can easily get matrix (18) holds, and six eigenvalues of the matrix (18) are $\lambda(\Pi_5) = (-8.8637, -7.2738, -4.3273, -1.8542, -1.8284, -1.3459)$. Then, all conditions of Theorem 2 hold. It follows from Theorem 2, we know that system (24) is passive. Choose randomly 100 initial conditions, Figs. 4–6 depict the passivity of the systems (24) with input $u(t) = (u_1(t), u_2(t)) = (0.25x_1(t), -0.5x_2(t))$. Choose randomly 50 initial conditions, Figs. 7 and 8 depict the passivity of the systems (24) with input $u(t) = (u_1(t), u_2(t)) = (3 - 2\cos(t), 2 - 1.5\sin(t))$.

Remark 3. In system (24), as $d_i(x_i(t))$, $a_{ij}(x_i(t))$ and $b_{ij}(x_i(t))$, $i, j = 1, 2, \ldots, n$ are discontinuous, the results on passivity of the neural networks with continuous right-hand side in Li et al. (2012); Li and Liao (2005); Lou and Cui (2007); Xu et al. (2009) and Yu and Li (2007) cannot be used here.

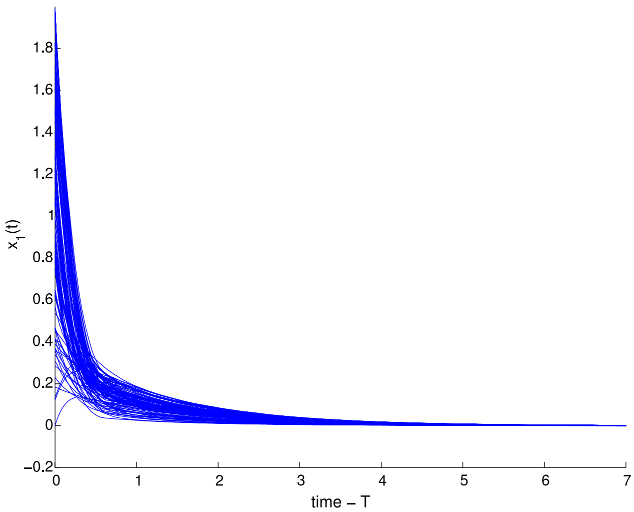

Fig. 4. Choose randomly 100 initial conditions, state trajectory $x_1(t)$ of system (24) with input $u(t) = (0.25x_1(t), -0.5x_2(t))$.

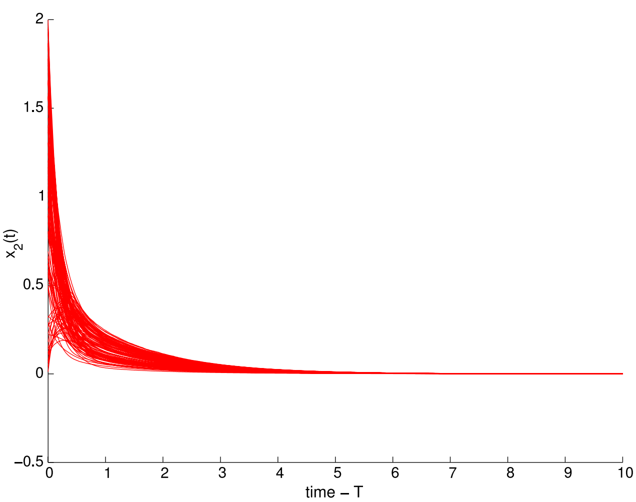

Fig. 5. Choose randomly 100 initial conditions, state trajectory $x_2(t)$ of system (24) with input $u(t) = (0.25x_1(t), -0.5x_2(t))$.

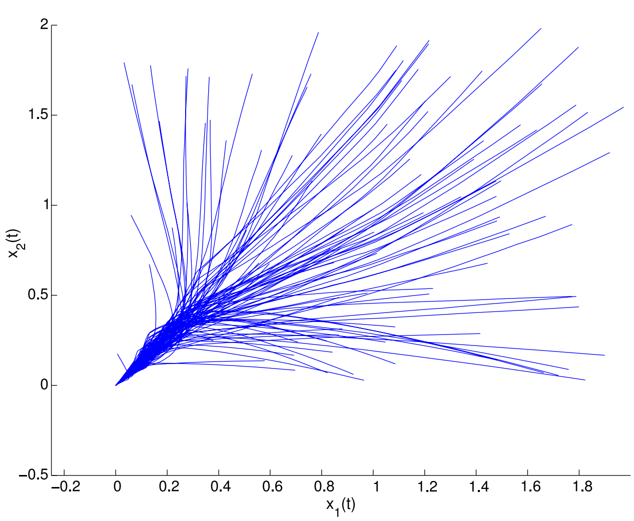

Fig. 6. Choose randomly 100 initial conditions, the phase plot of system (24) with input $u(t) = (0.25x_1(t), -0.5x_2(t))$.

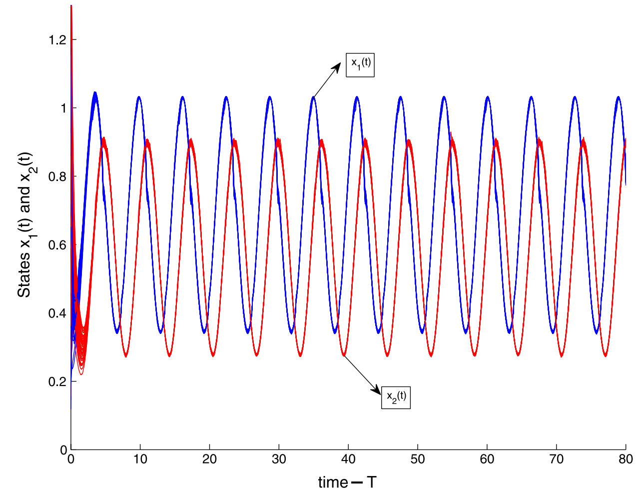

Fig. 7. Choose randomly 50 initial conditions, state trajectory of system (24) with input $u(t) = (3 - 2\cos(t), 2 - 1.5\sin(t))$.

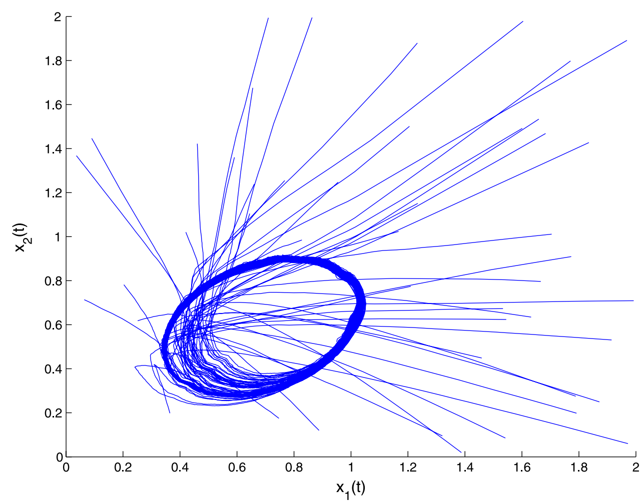

Fig. 8. Choose randomly 50 initial conditions, the phase plot of system (24) with input $u(t) = (3 - 2\cos(t), 2 - 1.5\sin(t))$.

Remark 4. In the above example, because of the time delays $\tau_{ij}(t) \neq 0$ and $c_{ij} \neq 0$, the results obtained in Wen et al. (2013) and Wu and Zeng (2014) cannot be used here. So, our results complement and extend the earlier publications.

Example 2. Consider two-dimensional memristor-based recurrent neural networks without delay

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -d_1(x_1(t))x_1(t) + a_{11}(x_1(t))f_1(x_1(t)) \\ \quad + a_{12}(x_1(t))f_2(x_2(t)) + u_1(t), \\ \dfrac{dx_2(t)}{dt} = -d_2(x_2(t))x_1(t) + a_{21}(x_2(t))f_1(x_1(t)) \\ \quad + a_{22}(x_2(t))f_2(x_2(t)) + u_2(t), \end{cases} \tag{26}$$

where

$$d_1(x_1(t)) = d_2(x_2(t)) = 1,$$

$$a_{11}(x_1(t)) = \begin{cases} -0.8, & -\dfrac{df_1(x_1(t))}{dt} \leq \dfrac{dx_1(t)}{dt}, \\ -0.75, & -\dfrac{df_1(x_1(t))}{dt} > \dfrac{dx_1(t)}{dt}, \end{cases}$$

$$a_{12}(x_1(t)) = a_{21}(x_2(t)) = 0,$$

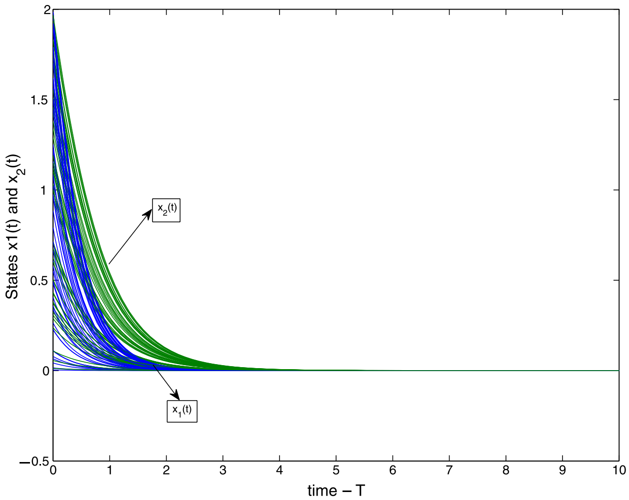

Fig. 9. Choose randomly 50 initial conditions, state trajectory of system (26) with input $u(t) = (-0.5x_1(t), 0.25x_2(t))$.

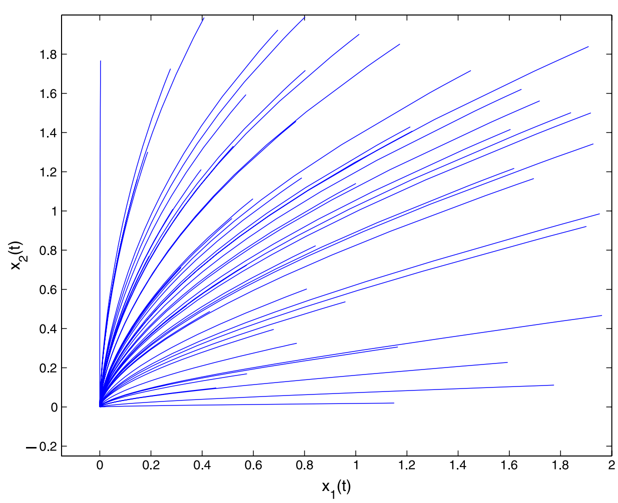

Fig. 10. Choose randomly 50 initial conditions, the phase plot of system (26) with input $u(t) = (-0.5x_1(t), 0.25x_2(t))$.

$$a_{22}(x_2(t)) = \begin{cases} -0.75, & -\dfrac{\mathrm{d}f_2(x_2(t))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t}, \\ -0.8, & -\dfrac{\mathrm{d}f_2(x_2(t))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t}, \end{cases}$$

and

$$B = C = 0, \qquad \tau_j(t) = 0, \qquad f_i(x_i) = \tanh(x_i), \qquad i = 1, 2. \qquad (27)$$

Obviously, $f_i(x)$, $i = 1, 2$ are bounded and Lipschitz continuous functions with Lipschitz constants $\rho_1 = \rho_2 = 1$, and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}.$$

By using the MATLAB LMI toolbox, the feasible solution was not found for LMI (10) in Wu and Zeng (2014). In fact, when $A = \mathrm{diag}(0.8, 0.8)$, we can easily get the LMI (10) in Wu and Zeng (2014) is not a negative definite matrix. So, the condition of Theorem 1 in Wu and Zeng (2014) is not satisfied, one cannot determine passivity of memristor-based neural network (26).

By using the MATLAB LMI toolbox, calculating the feasible solutions for LMI (18) in this paper, we have $H = \mathrm{diag}(32\,500, 32\,500) > 0$, $\eta = 1.247 \times 10^6 > 0$. Six eigenvalues of matrix (18) are $\lambda(\Pi_5) = (-1.2488, -1.2488, -0.0755, -0.0755, -0.0008, -0.0008) \times 10^6$. Then, all conditions of Theorem 3 hold. It follows from Theorem 3, we know that system (26) is passive. Choose randomly 50 initial conditions, Figs. 9 and 10 depict the passivity of the systems (26) with input $u(t) = (-0.5x_1(t), 0.25x_2(t))$. Some examples can also be given for other Theorems, here they are omitted.

Remark 5. From the above example, we can see the results of this paper are more effective than the earlier publications for passivity of the memristor-based neural networks.

5. Conclusions

In this paper, we adopt nonsmooth analysis and control theory to handle memristor-based neural networks with discontinuous right-hand side. Under the framework of Filippov's solution, and by building some useful Lyapunov functional and using the inequality technique, we got some sufficient conditions to ensure the passivity of the memristor-based RNNs with discrete and distributed delays. The proposed results here are different from the present works (Wen et al., 2013; Wu & Zeng, 2014), and they enrich the earlier publications. What is more, the research method here can be extended to study the stability and passivity problems for more complex situations, such as the fourth term of system (1) with memristor as follows:

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -d_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(x_i(t))f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} c_{ij}(x_i(t)) \\ &\quad \times \int_{t-r_j(t)}^{t} f_j(x_j(s))\mathrm{d}s + u_i(t), \quad t \ge 0,\ i \in N, \end{aligned} \qquad (28)$$

where

$$d_i(x_i(t)) = \frac{1}{\mathbf{C}_i}\left[\sum_{j=1}^{n}(\mathbf{M}_{ij} + \mathbf{W}_{ij} + \widehat{\mathbf{W}}_{ij}) \times \delta_{ij} + \overline{\mathcal{R}}_i\right],$$

$$a_{ij}(x_i(t)) = \frac{\mathbf{M}_{ij}}{\mathbf{C}_i} \times \delta_{ij}, \qquad b_{ij}(x_i(t)) = \frac{\mathbf{W}_{ij}}{\mathbf{C}_i} \times \delta_{ij},$$

$$c_{ij}(x_i(t)) = \frac{\widehat{\mathbf{W}}_{ij}}{\mathbf{C}_i} \times \delta_{ij}, \qquad u_i(t) = \frac{I_i(t)}{\mathbf{C}_i}.$$

According to the feature of the memristor and the current–voltage characteristic, see Fig. 3, then, we can get

$$d_i(x_i(t)) = \begin{cases} d_i^*, & -\dfrac{\mathrm{d}f_i(x_i(t))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ d_i^{**}, & -\dfrac{\mathrm{d}f_i(x_i(t))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$$a_{ij}(x_i(t)) = \begin{cases} a_{ij}^*, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ a_{ij}^{**}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$$b_{ij}(x_i(t)) = \begin{cases} b_{ij}^*, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t-\tau_j(t)))}{\mathrm{d}t} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ b_{ij}^{**}, & \delta_{ij}\dfrac{\mathrm{d}f_j(x_j(t-\tau_j(t)))}{\mathrm{d}t} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \end{cases}$$

$$c_{ij}(x_i(t)) = \begin{cases} c_{ij}^*, & \delta_{ij}\{f_j(x_j(t)) - f_j(x_j(t-r_j(t)))\} \le \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \\ c_{ij}^{**}, & \delta_{ij}\{f_j(x_j(t)) - f_j(x_j(t-r_j(t)))\} > \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t}. \end{cases}$$

So, the results in our paper provided fundamental results and the preparative work for further discussing more complicated delayed memristor-based neural networks. We will study the stability, passivity problems and other dynamical behaviors for more complex and reasonable memristor-based neural networks such as (28) in future research.

Acknowledgments

The authors gratefully acknowledge anonymous referees comments and patient work. This work is supported by the National Science Foundation of China (Grant Nos. 11271146 and 61374150), Science and Technology Program of Wuhan under Grant No. 2013010501010117, Key Program of National Natural Science Foundation of China under Grant No. 61134012, and the prior developing Field for the Doctoral Program of Higher Education of China under Grant No. 20130142130012.

References

Aubin, J. P., & Cellina, A. (1984). *Differential inclusions*. Germany, Berlin: Springer-Verlag.
Bao, G., & Zeng, Z. G. (2013). Multistability of periodic delayed recurrent neural network with memristors. *Neural Computing & Applications*, *23*, 1963–1967.
Cao, J., & Song, Q. K. (2006). Stability in Cohen–Grossberg-type bidirectional associative memory neural networks with time-varying delays. *Nonlinearity*, *19*, 1601–1617.
Chen, W. H., & Zheng, W. X. (2008). Improved delay-dependent asymptotic stability criteria for delayed neural networks. *IEEE Transactions on Neural Networks*, *19*, 2154–2161.
Chen, W. H., & Zheng, W. X. (2010a). Robust stability analysis for stochastic neural networks with time-varying delay. *IEEE Transactions on Neural Networks*, *21*, 508–514.
Chen, W. H., & Zheng, W. X. (2010b). A new method for complete stability analysis of cellular neural networks with time delay. *IEEE Transactions on Neural Networks*, *21*, 1126–1139.
Chen, Y., & Zheng, W. X. (2011). Stability and L2 performance analysis of stochastic delayed neural networks. *IEEE Transactions on Neural Networks*, *22*, 1662–1668.
Chua, L. O. (1971). Memristor-the missing circuit element. *IEEE Transactions on Circuit Theory*, 507–519.
Chua, L. O. (1999). Passivity and complexity. *IEEE Transactions on Circuits and Systems I*, *46*, 71–82.
Chua, L. O. (2011). Resistence switching memories are memristor. *Applied Physics A*, *102*, 765–783.
Clarke, F. H., Ledyaev, Y. S., Stem, R. J., & Wolenski, R. R. (1998). *Nonsmooth analysis and control theory*. New York: Springer.
Filippov, A. F. (1988). *Differential equations with discontinuous right-hand sides*. Dordrecht: Kluwer.
Forti, M., Nistri, P., & Papini, D. (2005). Global exponential stability and global convergence in finite time of delayed neural networks with infinite gain. *IEEE Transactions on Neural Networks*, *16*, 1449–1463.
Guo, Z. Y., Wang, J., & Yan, Z. (2014). Attractivity analysis of memristorbased cellular neural networks with time-varying delays. *IEEE Transactions on Neural Networks and Learning Systems*, *25*, 704–717.
Hu, S., & Wang, J. (2002). Global asymptotic stability and global exponential stability of continuous-time recurrent neural networks. *IEEE Transactions on Automatic Control*, *47*, 802–807.
Hu, X., & Wang, J. (2012). Solving the assignment problem using continuous-time and discrete-time improved dual networks. *IEEE Transactions on Neural Networks and Learning Systems*, *23*, 821–827.
Hu, X., & Zhang, B. (2010). A Gaussian attractor network for memory and recognition with experience-dependent Learning. *Neural Computation*, *22*, 1333–1357.
Huang, T., Chan, A., Huang, Y., & Cao, J. (2007). Stability of Cohen–Grossberg neural networks with time-varying delays. *Neural Networks*, *20*, 868–873.
Huang, T., Li, C., Yu, W., & Chen, G. (2009). Synchronization of delayed chaotic systems with parameter mismatches by using intermittent linear state feedback. *Nonlinearity*, *22*, 569–584.
Itoh, M., & Chua, L. O. (2008). Memristor oscillators. *International Journal of Bifurcation and Chaos*, *18*, 3183–3206.
Jiang, H. J., & Teng, Z. D. (2004). Global eponential stability of cellular neural networks with time-varying coefficients and delays. *Neural Networks*, *17*, 1415–1425.
Kim, H., Sah, M. P., Yang, C. J., Roska, T., & Chua, L. O. (2012). Neural synaptic weighting with a pulse-based memristor circuit. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *59*, 148–158.
Li, H. Y., Lam, J., & Cheung, K. C. (2012). Passivity criteria for continuous-time neural networks with mixed time-varying delays. *Applied Mathematics and Computation*, *218*, 11062–11074.
Li, C. G., & Liao, X. F. (2005). Passivity analysis of neural networks with time delay. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *52*, 471–475.
Liang, X., & Wang, J. (2000). Absolute exponential stability of neural networks with a general class of activation functions. *IEEE Transactions on Circuits and Systems I*, *47*, 1258–1263.
Liao, X. F., Wong, K., & Yang, S. Z. (2003). Convergence dynamics of hybrid bidirectional associative memory neural networks with distributed delays. *Physics Letters A*, *316*, 55–64.
Liu, Y., Wang, Z. D., & Liu, X. H. (2006). Global exponential stability of generalized recurrent neural networks with discrete and distributed delays. *Neural Networks*, *19*, 667–675.
Lou, X. Y., & Cui, B. T. (2006). New LMI conditions for delay-dependent asymptotic stability of delayed Hopfield neural networks. *Neurocomputing*, *69*, 2374–2378.
Lou, X. Y., & Cui, B. T. (2007). Passivity analysis of integro-differential neural networks with time-varying delays. *Neurocomputing*, *70*, 1071–1078.
Lozano, R., Brogliato, B., Egeland, O., & Maschke, B. (2000). *Dissipative systems analysis and control: theory and applications*. London: Springer.
Pershin, Y. V., & Di Ventra, M. (2010). Experimental demonstration of associative memory with memristive neural networks. *Neural Networks*, *23*, 881–886.
Poznyak, A. S., & Sanchez, E. N. (1995). Nonlinear systems approximation by neural networks: error stability analysis. *International Journal of Artificial Intelligence and Soft Computing*, *1*, 247–258.
Sharifiy, M. J., & Banadaki, Y. M. (2010). General spice models for memristor and application to circuit simulation of memristor-based synapses and memory cells. *Journal of Circuits, Systems and Computers*, *19*, 407–424.
Song, Q. K., Liang, J., & Wang, Z. (2009). Passivity analysis of discrete-time stochastic neural networks with time-varying delays. *Neurocomputing*, *72*, 1782–1788.
Song, Q. K., & Wang, Z. D. (2008). Neural networks with discrete and distributed time-varying delays: a general stability analysis. *Chaos, Solitons & Fractals*, *37*, 1538–1547.
Strukov, D. B., Snider, G. S., Stewart, G. R., & Williams, R. S. (2008). The missing memristor found. *Nature*, *453*, 80–83.
Wang, L. L., & Chen, T. P. (2012). Complete stability of cellular neural networks with unbounded time-varying delays. *Neural Networks*, *36*, 11–17.
Wang, Z. D., Liu, Y., & Liu, X. (2005). On global asymptotic stability of neural networks with discrete and distributed delays. *Physics Letters A*, *345*, 299–308.
Wang, Z. Q., et al. (2012). Synaptic learning and memory functions achieved using oxygen ion migration/diffusion in an amorphous in GaZnO memristor. *Advanced Functional Materials*, *22*, 2759–2765.
Wen, S., Zeng, Z., & Huang, T. (2012). Exponential stability analysis of memristor-besed recurrent neural networks with time-varying delays. *Neurocomputing*, *97*, 233–240.
Wen, S., Zeng, Z., Huang, T., & Chen, Y. R. (2013). Passivity analysis of memristor-based recurrent neural networks with time-varying delays. *Journal of the Franklin Institute*, *350*, 2354–2370.
Wu, Z. G., Park, J. H., Su, H. Y., & Chu, J. (2012). Passivity analysis of Markov jump neural networks with mixed time-delays and piecewise-constant transition rates. *Nonlinear Analysis: Real World Applications*, *13*, 2423–2431.
Wu, A. L., Wen, S., & Zeng, Z. (2012). Synchronization control of a class of memristor-based recurrent neural networks. *Information Sciences*, *183*, 106–116.
Wu, A. L., & Zeng, Z. G. (2014). Passivity analysis of memristive neural networks with different memductance functions. *Communications in Nonlinear Science and Numerical Simulation*, *19*, 274–285.
Xu, S. Y., Lam, J., & Ho, D. W. C. (2006). A new LMI condition for delay-dependent asymptotic stability of delayed Hopfield neural networks. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *53*, 230–234.
Xu, S. Y., Zheng, W. X., & Zou, Y. (2009). Passivity analysis of neural networks with time-varying delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *56*, 325–329.
Yang, X., Cao, J., & Yu, W. (2014). Exponential synchronization of memristive Cohen–Grossberg neural networks with mixed delays. *Cognitive Neurodynamics*, http://dx.doi.org/10.1007/s11571-013-9277-6.
Yu, W., & Li, X. (2007). Passivity analysis of dynamic neural networks with different time-scales. *Neural Processing Letters*, *25*, 143–155.
Zeng, Z., & Wang, J. (2006). Global exponential stability of recurrent neural networks with time-varying delays in the presence of strong external stimuli. *Neural Networks*, *19*, 1528–1537.
Zhang, H. G., Liu, Z., Huang, G., & Wang, Z. (2010). Novel weighting-delay based stability criteria for recurrent neural networks with time-varying delay. *IEEE Transactions on Neural Networks*, *21*, 91–106.
Zhang, G. D., & Shen, Y. (2013). New algebraic criteria for synchronization stability of chaotic memristive neural networks with time-varying delays. *IEEE Transactions on Neural Networks and Learning Systems*, *24*, 1701–1707.
Zhang, G. D., & Shen, Y. (2014). Exponential synchronization of delayed memristor-based chaotic neural networks via periodically intermittent control. *Neural Networks*, *55*, 1–10.
Zhang, G. D., Shen, Y., & Chen, B. S. (2014). Bifurcation analysis in a discrete differential–algebraic predator–prey system. *Applied Mathematical Modelling*, *38*, 4835–4848.