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Stability analysis of nonlinear digital systems under hardware overflow constraint for dealing with finite word-length effects of digital technologies

Muhammad Umair Amjad [a], Muhammad Rehan [a,*], Muhammad Tufail [a], Choon Ki Ahn [b], Haroon Ur Rashid [a]

[a] Department of Electrical Engineering, Pakistan Institute of Engineering and Applied Sciences (PIEAS), Islamabad, Pakistan
[b] School of Electrical Engineering, Korea University, Seoul 136-701, Korea

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ABSTRACT

The purpose of this paper is to examine stability and originate stability criteria for nonlinear digital systems under the influence of saturation overflow, both in the absence and presence of external interference. The developed approaches can be employed to analyse overflow oscillation-free implementation of a nonlinear digital system under saturation overflow nonlinearity, caused by the finite word-length limitation of a digital hardware, such as computer processor or micro-controller. Asymptotic stability is examined in the absence of disturbance, whereas in the presence of external interference, the form of stability ensured is uniformly ultimately bounded stability, in which the states trajectories converge to an ellipsoidal region around the origin. In most of the studies reported so far, the authors have performed the overflow stability analysis of linear systems but very little (if any) work has been reported on the overflow oscillation elimination (for nonlinear systems). In the present work, sector conditions derived from saturation constraint along with Lipschitz condition are used with a suitable Lyapunov function for the stability analysis of nonlinear digital systems under overflow. The validity and efficacy of these criteria are tested by using examples from real nonlinear physical systems, including Moon chaotic system's observer and recurrent neural network.

1. Introduction

An electrical signal, having specific magnitude and frequency, can be processed by a digital filter to suppress the noisy components. There are many applications of digital filters, such as speech signal processing, digital image processing, medical electronics, geophysics, seismographic data processing, thermal processes, heating water stream, communication, defense, and spectrum and vibration analyses [1–3] and [4–6]. Implementation of digital systems in hardware causes problems due to finite word-length limitation of the hardware that can result into overflow oscillations, limit cycles, instabilities and performance degradation. This highlights the significance of analysis and design of digital systems under the repercussions of overflow and quantization nonlinearities [7–12] (see also the works in [13–18]). It may be noted that decoupling of overflow and quantization effects can be achieved if the number of quantization steps is sufficiently large [13,14,19]. Moreover, several low order digital filter units are generally cascaded to realize a high order digital filter, prompting mutual interference problem, that may lead to performance degradation and even complete failure of the digital filter [1].

Many studies have discussed the elimination of overflow oscillations in linear digital filters and systems such as [20–30]. In [20], zero-input overflow oscillation elimination analysis has been reported for arbitrary order lossless digital integrators and differentiators, provided certain conditions are satisfied. Stability criteria development for the elimination of overflow oscillations for direct-form digital filters, in the presence of saturation overflow, has been dealt with, in the work [21], which proclaims the global asymptotic stability of the states of a filter. The criterion presented in [24] also deals with the global stability in the presence of saturation overflow nonlinearity. However, compared to [21], fixed-point state-space digital filters with zero-input have been considered and structural properties of multiple saturation nonlinearities are investigated in [24]. The paper [22] presents a criterion for $\mathcal{H}_\infty$

* Corresponding author.
E-mail addresses: mumairamjad91@gmail.com (M. Umair Amjad), rehanqau@gmail.com (M. Rehan), tufail@pieas.edu.pk (M. Tufail), hironaka@korea.ac.kr (C.K. Ahn), haroon@pieas.edu.pk (H. Ur Rashid).

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disturbance elimination, which guarantees exponential and robust stability for the linear digital systems with saturation arithmetic in the absence and presence of external interference. The results in [23,26] incorporate the impact of different combinations of overflow and quantization nonlinearities for the global asymptotic stability of fixed-point state-space digital filters.

The research paper in [25] presents a less conservative and multi-bound-dependent stability criterion that includes bound-dependent delay under overflow constraint for the fixed-point state-space digital systems. In the work [27], a new criteria for the externally interfered and saturation overflow affected direct-form digital filters to ensure input or output to state stability has been developed, which also guarantees output to state stability and asymptotic stability in the absence of disturbance. A dissipativity criterion has been established for the externally interfered fixed-point digital filters under overflow nonlinearity in [28], which introduces $\mathcal{H}_\infty$, passivity and mixed $\mathcal{H}_\infty$/passivity performances in a combined structure. Recently, in [30] one-dimensional (1-D) digital filter's local stability has been investigated, under overflow constraint, by developing LMI-based stability criteria while utilizing the local saturation arithmetic. One of these criteria is a local asymptotic stability criterion, considering zero external disturbance, and the other one is an $\mathcal{H}_\infty$ criterion for analyzing external interferences rejection performance. Global stability analysis has been treated as a special case of the presented local stability analysis in the same work.

Linear two-dimensional (2-D) systems [31] under overflow and external interferences form the subject matter of the works [32–36], with the noise assumed to be deterministic in [33,34]. For the local state-space (LSS) Fornasini–Marchesini modeled two-dimensional digital filters, new criteria for the global asymptotic overflow stability have been established in [32], which are less conservative than the previously cited works. The approach of [33] actually considers the effect of both quantization and overflow nonlinearities for the 2-D filter case. Another work on stability analysis of the digital systems under overflow function in [35] considers the Wiener-type stochastic external noise for the stability investigation, which is more realistic than the case of deterministic perturbations. In another work, [36], the proposed stability criterion to eliminate overflow oscillations, can be deployed for linear parameter varying time-variant digital filters, considering overflow effects, and disturbance in states and output of the system. The same criterion in [36], certifies both, the overflow stability in absence of input, and $\mathcal{H}_\infty$ performance. In [29], two new criteria have been established for input to state stability, by utilizing the input-output information, for fixed-point state-space and direct form digital filters by considering the effects of external interference.

It has been observed that most of the available literature, including the above-mentioned works, addresses the stability analysis of linear digital systems in the presence of overflow constraint. Even if stability analysis of nonlinear digital systems is discussed, the effects of overflow are not incorporated [37–41]. There are many examples of nonlinear digital systems implemented on a hardware like a computer processor, field-programmable gate array (FPGA), digital signal processing (DSP) kits or a micro-controller for signal processing, control and communication applications by considering the hardware constraints as observed in the literature [42–44]. For instance, observer or state filter for the nonlinear systems [45] are also nonlinear, which are implemented digitally. Similarly, digital controllers for nonlinear systems [46] used to regulate the behavior by taking advantages of the digital technology are also nonlinear. Neural networks, implemented in digital hardware, contain nonlinear components [47], which are employed for several filtering, control, image processing, and pattern recognition applications. Adaptive systems and controllers are also inherently nonlinear, therefore, their digital implementation constitutes a nonlinear digital system [48]. Another example of nonlinear digital system is controller for feedback linearization [49]. Recursive least square (RLS) algorithm can also be referred to as a nonlinear digital system which can be realized in a digital hardware [50]. Practically, digital nonlinear systems hold a great deal of importance, and therefore, stability analysis for these systems is an important task in the presence of overflow, which is inevitable in any digital hardware, and also in the presence of inescapable external interferences. However, this non-trivial problem of stability analysis of nonlinear digital systems subject to the hardware overflow constraints, having remarkable practical importance, is not evidently addressed in the previous methods.

This research work focuses on the development of overflow stability criteria for (fixed-point) Lipschitz nonlinear systems in the absence or presence of external perturbations to examine overflow oscillation-free implementation of the digital system on a computer processor, FPGA, DSP kits or a micro-controller. Stability conditions have been derived using the Lyapunov theory along with the sector condition attained from the saturation nonlinearity and the Lipschitz condition. These stability criteria have been presented in form of linear matrix inequalities (LMIs), which offers computational convenience for the stability analysis. In the absence of the external interferences, the proposed criterion can be used to analyse asymptotic stability of the nonlinear digital system under overflow. Furthermore, a condition for external interference attenuation analysis of the nonlinear digital system with the saturation-type overflow has been provided, which ensures convergence of the system's states into an ellipsoidal region.

In contrast to the previous works on the stability analysis of the digital systems (or filters) under overflow constraint (like [20–30,32–36]), the effects of both the overflow nonlinearity and the nonlinear dynamical digital component have been studied while acquiring the stability conditions for the nonlinear digital systems for consideration of hardware implementation. The present work addresses a fundamental and inaugural problem of the stability analysis of the digital systems with hardware constraints and inherent nonlinear complexity. Therefore, the proposed approach can handle the nonlinear filtering algorithms and can also be readily applied to the linear counterpart as a specific case. In addition, the resultant analysis conditions can be applied to the nonlinear digital systems under overflow nonlinearity in the presence of bounded external interferences to attain an idea of the external interferences attenuation level of a filter. The stability criteria have been applied to the overflow oscillation elimination analysis of two nonlinear physical systems, including state filter for the observer of Moon chaotic system and recurrent neural network (RNN) and the solutions attained are provided to depict validity of the proposed approaches.

The organization of the remaining paper is as follows: Section 2 presents the general structure of the concerned digital system and associated assumptions. Detailed stability analysis for the nonlinear digital systems, without disturbance, and representation of the final results in form of LMIs are provided in Section 3. The results are extended in Section 4 to consider the external disturbance. The proposed methods have been employed for overflow stability analysis of two physical nonlinear systems and the results are presented in Section 5. Finally, conclusions are drawn in Section 6.

Notations: The notation $\lambda_{\max}(\mathbf{R})$ means the maximum eigenvalue of a positive-definite matrix $\mathbf{R}$. $\mathbf{\Lambda}_1 < 0$ denotes that the matrix $\mathbf{\Lambda}_1$ is negative-definite. Similarly, a positive-definite matrix $\mathbf{P}$ is symbolized by $\mathbf{P} > 0$. $\|\cdot\|$ is the Euclidean norm operator. k and t are independent variables indicating sample number and time instant, respectively. The symbol $\mathbb{R}^n$ represents that the vector dimension is $n\times 1$, while $\mathbb{R}^{n\times m}$ shows that the matrix dimension is $n\times m$.

2. System description

The concerned digital system is expressed as

$$\mathbf{x}(k+1) = f(\mathbf{y}(k)) + \mathbf{B}_w\mathbf{w}(k), \tag{1}$$

$$\mathbf{y}(k) = \mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)), \tag{2}$$

where $\mathbf{x}(k) = [x_1(k)\ x_2(k)\ \cdots\ x_n(k)]^T \in \mathbb{R}^n$, denotes the state vector, and $\mathbf{w}(k) = [w_1(k)\ w_2(k)\ \cdots\ w_m(k)]^T \in \mathbb{R}^m$ represents external interference. The vector influenced by overflow is indicated by $\mathbf{y}(k) = [y_1(k)\ y_2(k)\ \cdots\ y_n(k)]^T \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the coefficient matrix and $\mathbf{B}_w \in \mathbb{R}^{n\times m}$ is a matrix of known constants associated with the disturbance vector. The important term in this system is $g(\mathbf{x}(k)) = [g(x_1(k))\ g(x_2(k))\ \cdots\ g(x_n(k))]^T \in \mathbb{R}^n$ compared to the existing works, which represents the nonlinear component of the system.

The function $f(\cdot)$ represents the saturation overflow nonlinearity satisfying

$$f(y_i(k)) = \begin{cases} 1 & \text{if } y_i(k) > 1, \\ y_i(k) & \text{if } -1 \le y_i(k) \le 1, \\ -1 & \text{if } y_i(k) < -1. \end{cases} \tag{3}$$

When a system is implemented in a digital hardware like a processor, DSP kits or a micro-controller, overflow of the states occur (due to finite word-length effect) may cause severe performance degradation. For the present case, the nonlinear digital system given by (1)–(2) is considered under the effect of saturation overflow. A large number of digital systems contain nonlinear components and examples of such systems include digital state-space filters, neural networks, adaptive systems, controllers and observers.

Assumption 1. The nonlinearity $g(\mathbf{x}(k))$ satisfies

$$\|g(\mathbf{x}(k)) - g(\bar{\mathbf{x}}(k))\| \le L\|\mathbf{x}(k) - \bar{\mathbf{x}}(k)\|, \tag{4}$$

where $\mathbf{x}(k),\ \bar{\mathbf{x}}(k) \in \mathbb{R}^n$ and L is the Lipschitz constant of a known finite value.

Remark 1. Assumption 1, well-known as the Lipschitz condition, is a property used to ensure continuity of a function $g(\mathbf{x}(k))$. This condition can be validated for all differentiable functions; therefore, a large number of the digital systems satisfies this condition. It is required to ensure well-posedness of a digital system like (1)–(2). If Assumption 1 is validated, a nonlinear digital system of type (1)–(2) can be smoothly implemented via digital technologies.

Assumption 2. The amplitude of the external interference is bounded, that is,

$$\mathbf{w}^T(k)\mathbf{w}(k) \le \delta. \tag{5}$$

The present study assumes that the value of tolerable bound on the amplitude of external interference, in the form of δ, is known a priori for the stability analysis. In most of the cases, a guess of the worst case amplitude of external perturbations is available, through which an idea of δ can be obtained. Based on the value of δ, our approach aims to provide an estimate of the region in which state of the nonlinear digital filter converges. If the value of δ is unknown, we can still obtain the stability information; however, estimate of the region cannot be obtained.

Remark 2. Most of the existing research work has been accomplished to devise stability criteria for the linear digital systems under overflow in the presence or absence of external interferences [20–24] (see also [25–30,32–36]). These existing works have ignored the presence of any nonlinearity in a digital system. In the present work, we have incorporated a nonlinear function g($\mathbf{x}$(k), which satisfies the Lipschitz condition in Assumption 1. Time-derivatives of many nonlinearities have bounded values, which satisfy the Lipschitz condition. Therefore, Assumption 1 is valid for a large number of digital systems since most of the nonlinearities are either globally or locally Lipschitz and satisfy the inequality (4).

The following sector condition is obtained from the saturation condition (3):

$$f^T(\mathbf{y}(k))f(\mathbf{y}(k)) \le \mathbf{y}^T(k)\mathbf{y}(k). \tag{6}$$

The condition (6) can be employed to derive the stability investigation criteria for the nonlinear system (1)–(2), under overflow. The aim of this research is to develop a criterion to study asymptotic stability for a nonlinear digital system under saturation overflow constraint in the absence of external interference. Another criterion will be developed for the system (1)–(2) to examine uniformly ultimately bounded stability in the presence of external disturbances.

3. Stability analysis for nonlinear digital system

In this section, a criterion for the asymptotic stability is established for nonlinear digital systems without external interferences under overflow nonlinearity.

Theorem 1. *Suppose there exist positive-definite matrices* $\mathbf{N}$ *(diagonal),* $\mathbf{P}$, *and* $\mathbf{Q}$, *and positive scalars* ε_1, ε_2, ε_3, *and* ε_4. *If the LMI* $\mathbf{\Lambda}_1 < 0$ *holds, where*

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \mathbf{\Omega}_{11} & \mathbf{A}^T\mathbf{N}^T & 0 & 0 & \varepsilon_1\mathbf{A}^T \\ \mathbf{NA} & \mathbf{\Omega}_{22} & 0 & \varepsilon_2\mathbf{A}^T & \mathbf{N} \\ 0 & 0 & \mathbf{Q}-\varepsilon_2\mathbf{I} & 0 & 0 \\ 0 & \varepsilon_2\mathbf{A} & 0 & \varepsilon_2\mathbf{I}-\varepsilon_4\mathbf{I} & 0 \\ \varepsilon_1\mathbf{A} & \mathbf{N}^T & 0 & 0 & \varepsilon_1\mathbf{I}-\varepsilon_3\mathbf{I} \end{bmatrix}, \tag{7}$$

where

$\mathbf{\Omega}_{11} = -\mathbf{P} + \varepsilon_1\mathbf{A}^T\mathbf{A} + \varepsilon_3L^2\mathbf{I}$,

$\mathbf{\Omega}_{22} = \mathbf{P} - \mathbf{Q} - 2\mathbf{N} - \varepsilon_1\mathbf{I} + \varepsilon_2\mathbf{A}^T\mathbf{A} + \varepsilon_4L^2\mathbf{I}$,

then asymptotic stability is ensured for the Lipschitz nonlinear digital system (1)–(2) under saturation overflow nonlinearity and zero external interference.

Proof. Consider, the overflow-constrained digital system (1)–(2) and let the Lyapunov function be

$$\begin{aligned} V(\mathbf{x}(k)) &= \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k) + f^T(\mathbf{A}\mathbf{x}(k) \\ &\quad + g(\mathbf{x}(k)))\mathbf{Q}f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))), \end{aligned} \tag{8}$$

where $\mathbf{P} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q} \in \mathbb{R}^{n\times n}$ are positive-definite symmetric matrices. Taking the first difference of $V(x(k))$, we obtain

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \mathbf{x}^T(k+1)\mathbf{P}\mathbf{x}(k+1) \\ &\quad + f^T(\mathbf{A}\mathbf{x}(k+1) + g(\mathbf{x}(k+1)))\mathbf{Q}f(\mathbf{A}\mathbf{x}(k+1) \\ &\quad + g(\mathbf{x}(k+1))) - \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k) + f^T(\mathbf{A}\mathbf{x}(k) \\ &\quad + g(\mathbf{x}(k)))\mathbf{Q}f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))). \end{aligned}$$

Substituting $\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))$ for $\mathbf{y}(k)$ and simplifying, we have

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= f^T(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)))[\mathbf{P}-\mathbf{Q}]f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)))\mathbf{P}\mathbf{B}_w\mathbf{w}(k) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{P}f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{P}\mathbf{B}_w\mathbf{w}(k) + f^T(\mathbf{A}f(\mathbf{y}(k)) + \mathbf{A}\mathbf{B}_w\mathbf{w}(k) \\ &\quad + g(f(\mathbf{y}(k)) + \mathbf{B}_w\mathbf{w}(k)))\mathbf{Q}f(\mathbf{A}f(\mathbf{y}(k)) + \mathbf{A}\mathbf{B}_w\mathbf{w}(k) \\ &\quad + g(f(\mathbf{y}(k)) + \mathbf{B}_w\mathbf{w}(k))) - \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k) \\ &\quad + 2f^T(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)))\mathbf{N}[\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)) \\ &\quad - f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)))] - 2f^T(\mathbf{y}(k))\mathbf{N}[\mathbf{y}(k) - f(\mathbf{y}(k))], \end{aligned} \tag{9}$$

for a positive-definite diagonal matrix $\mathbf{N}\in\mathbb{R}^{n\times n}$. The sector condition (6) yields

$$\begin{aligned} f^T(\mathbf{y}(k))f(\mathbf{y}(k)) &= f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\le (\mathbf{Ax}(k)+g(\mathbf{x}(k)))^T(\mathbf{Ax}(k)+g(\mathbf{x}(k))), \end{aligned} \tag{10}$$

$$\begin{aligned} & f^T(\mathbf{y}(k+1))f(\mathbf{y}(k+1)) \\ &= f^T(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))) \\ &\quad \times f(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))) \\ &\le (\mathbf{A}f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)))^T \\ &\quad \times (\mathbf{A}f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))). \end{aligned} \tag{11}$$

Rearranging (10) and multiplying it with a positive scalar ε_1, we get

$$\begin{aligned} \mathfrak{F}_1(k) &= \varepsilon_1[\mathbf{x}^T(k)\mathbf{A}^T\mathbf{Ax}(k)+\mathbf{x}^T(k)\mathbf{A}^Tg(\mathbf{x}(k)) \\ &\quad + g^T(\mathbf{x}(k))\mathbf{Ax}(k)+g^T(\mathbf{x}(k))g(\mathbf{x}(k)) \\ &\quad - f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))]\ge 0. \end{aligned} \tag{12}$$

Similarly, for (11), we have

$$\begin{aligned} \mathfrak{F}_2(k) &= \varepsilon_2[f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{A}^T\mathbf{A}f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{A}^T\mathbf{AB}_w\mathbf{w}(k) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{A}^Tg(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{A}^T\mathbf{A}f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{A}^T\mathbf{AB}_w\mathbf{w}(k)+\mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{A}^Tg(f(\mathbf{y}(k)) \\ &\quad + \mathbf{B}_w\mathbf{w}(k))+g^T(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))Af(\mathbf{Ax}(k) \\ &\quad + g(\mathbf{x}(k)))+g^T(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))\mathbf{AB}_w\mathbf{w}(k) \\ &\quad + g^T(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)) \\ &\quad - f^T(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))) \\ &\quad \times f(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)))] \\ &\ge 0. \end{aligned} \tag{13}$$

Now employing the inequalities (12) and (13) in (9) we obtain

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &\le f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\mathbf{P}-\mathbf{Q}]f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{PB}_w\mathbf{w}(k) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{P}f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))+\mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{PB}_w\mathbf{w}(k) \\ &\quad + f^T(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)))\mathbf{Q} \\ &\quad \times f(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))) \\ &\quad - \mathbf{x}^T(k)\mathbf{Px}(k)+2f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{N}[\mathbf{Ax}(k) \\ &\quad + g(\mathbf{x}(k))-f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))] \\ &\quad - 2f^T(\mathbf{y}(k))\mathbf{N}[\mathbf{y}(k)-f(\mathbf{y}(k))]+\mathfrak{F}_1(k)+\mathfrak{F}_2(k). \end{aligned} \tag{14}$$

For positive scalars ε_3 and ε_4, Assumption 1 implies

$$\begin{aligned} & g^T(\mathbf{x}(k))g(\mathbf{x}(k))\le \mathbf{x}^T(k)L^TL\mathbf{x}(k), \\ & \mathfrak{F}_3(k)=\varepsilon_3\mathbf{x}^T(k)L^2\mathbf{x}(k)-\varepsilon_3\, g^T(\mathbf{x}(k))g(\mathbf{x}(k))\ge 0, \\ & g^T(\mathbf{x}(k+1))g(\mathbf{x}(k+1))\le \mathbf{x}^T(k+1)L^TL\mathbf{x}(k+1), \end{aligned} \tag{15}$$

$$\begin{aligned} \mathfrak{F}_4(k) &= \varepsilon_4[f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))L^2f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))L^2\mathbf{B}_w\mathbf{w}(k) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^TL^2f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))+\mathbf{w}^T(k)\mathbf{B}_w^TL^2\mathbf{B}_w\mathbf{w}(k)] \\ &\quad - \varepsilon_4\, g^T(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))\ge 0. \end{aligned} \tag{16}$$

The conditions (15) and (16) are added to the inequality (14) to obtain an upper bound on $\Delta V(\mathbf{x}(k))$ as

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &\le f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\mathbf{P}-\mathbf{Q}]f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{PB}_w\mathbf{w}(k) \\ &\quad + \mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{P}f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))+\mathbf{w}^T(k)\mathbf{B}_w^T\mathbf{PB}_w\mathbf{w}(k) \\ &\quad + f^T(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k)))\mathbf{Q} \\ &\quad \times f(\mathbf{A}f(\mathbf{y}(k))+\mathbf{AB}_w\mathbf{w}(k)+g(f(\mathbf{y}(k))+\mathbf{B}_w\mathbf{w}(k))) \\ &\quad - \mathbf{x}^T(k)\mathbf{Px}(k)+2f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\mathbf{N}[\mathbf{Ax}(k) \\ &\quad + g(\mathbf{x}(k)) \\ &\quad - f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))]-2f^T(\mathbf{y}(k))\mathbf{N}[\mathbf{y}(k) \\ &\quad - f(\mathbf{y}(k))]+\mathfrak{F}_1(k)+\mathfrak{F}_2(k)+\mathfrak{F}_3(k)+\mathfrak{F}_4(k). \end{aligned} \tag{17}$$

Under zero external interferences, that is, $\mathbf{w}(k)=0$, and using the values of $\mathfrak{F}_1(k)$, $\mathfrak{F}_2(k)$, $\mathfrak{F}_3(k)$, and $\mathfrak{F}_4(k)$, we have

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &\le f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\mathbf{P}-\mathbf{Q}-2\mathbf{N}-\varepsilon_1\mathbf{I} \\ &\quad + \varepsilon_2\mathbf{A}^T\mathbf{A}+\varepsilon_4L^2\mathbf{I}]f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + \mathbf{x}^T(k)[-\mathbf{P}+\varepsilon_1\mathbf{A}^T\mathbf{A}+\varepsilon_3L^2\mathbf{I}]\mathbf{x}(k) \\ &\quad + g^T(\mathbf{x}(k))[\varepsilon_1\mathbf{I}-\varepsilon_3\mathbf{I}]g(\mathbf{x}(k))+f^T(\mathbf{A}f(\mathbf{y}(k)) \\ &\quad + g(f(\mathbf{y}(k))))[\mathbf{Q}-\varepsilon_2\mathbf{I}]f(\mathbf{A}f(\mathbf{y}(k))+g(f(\mathbf{y}(k)))) \\ &\quad + g^T(f(\mathbf{y}(k)))[\varepsilon_2\mathbf{I}-\varepsilon_4L^2\mathbf{I}]g(f(\mathbf{y}(k))) \\ &\quad + \mathbf{x}^T(k)[\varepsilon_1\mathbf{A}^T]g(\mathbf{x}(k))+g^T(\mathbf{x}(k))[\varepsilon_1\mathbf{A}]\mathbf{x}(k) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\varepsilon_2\mathbf{A}^T]g(f(\mathbf{y}(k))) \\ &\quad + g^T(f(\mathbf{y}(k)))[\varepsilon_2\mathbf{A}]f(\mathbf{Ax}(k)+g(\mathbf{x}(k))) \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\mathbf{NA}]\mathbf{x}(k) \\ &\quad + [\mathbf{x}^T(k)[\mathbf{A}^T\mathbf{N}^T]f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))]^T \\ &\quad + f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))[\mathbf{N}]g(\mathbf{x}(k)) \\ &\quad + [g^T(\mathbf{x}(k))[\mathbf{N}^T]f(\mathbf{Ax}(k)+g(\mathbf{x}(k)))]^T \\ &\quad - 2f^T(\mathbf{y}(k))\mathbf{N}[\mathbf{y}(k)-f(\mathbf{y}(k))], \end{aligned} \tag{18}$$

which is equivalent to

$$\Delta V(\mathbf{x}(k))\le \mathbf{\Theta}_1^T(k)\mathbf{\Lambda}_1\mathbf{\Theta}_1(k)+\mathbf{\Psi}(k), \tag{19}$$

where

$$\mathbf{\Psi}(k)=-2f^T(\mathbf{y}(k))\mathbf{N}[\mathbf{y}(k)-f(\mathbf{y}(k))], \tag{20}$$

$$\begin{aligned} \mathbf{\Theta}_1(k) &= [\mathbf{x}^T(k)\ \ f^T(\mathbf{Ax}(k)+g(\mathbf{x}(k)))\ \ f^T(\mathbf{A}f(\mathbf{y}(k))+g(f(\mathbf{y}(k)))) \\ &\quad g^T(f(\mathbf{y}(k)))\ \ g^T(\mathbf{x}(k))]. \end{aligned} \tag{21}$$

It can be easily seen that $\mathbf{\Psi}<0$, therefore, from inequality (19) we obtain

$$\Delta V(\mathbf{x}(k))<\mathbf{\Theta}_1^T(k)\mathbf{\Lambda}_1\mathbf{\Theta}_1(k). \tag{22}$$

Consequently, $\mathbf{\Lambda}_1<0$ implies $\Delta V(\mathbf{x}(k))<0$, which ensures that the state vector $\mathbf{x}(k)$ asymptotically converges to the origin. □

Remark 3. A large number of digital systems, like state filters, tunable filters, observers, estimators, controllers, adaptive systems and neural networks, are inherently nonlinear. The problem of stability analysis of these nonlinear systems under any form of overflow arithmetic is rarely addressed in literature. Therefore, the criteria developed in the previous papers, like [20–30,32–36], work only for the linear systems, and may not guarantee overflow consequences elimination under nonlinear dynamics. In contrast, the proposed criterion (7) in Theorem 1 investigates asymptotic stability for the nonlinear systems by considering the nonlinear dynamics under saturation overflow caused by finite word-length limitation of digital technologies and can be effectively employed for the consideration of implementation of complex digital systems and algorithms.

Remark 4. It should be noted that derivation of a stability condition for the nonlinear digital systems under an overflow nonlinearity, such as in Theorem 1, is a non-trivial research problem. Mainly, the digital system (1)–(2) contains two nonlinearities: an overflow nonlinearity and an inherently present dynamical nonlinearity. Further complexity arises since the dynamical nonlinearity $g(\mathbf{x}(k)$ is nested in $f(\mathbf{y}(k))$ as $\mathbf{y}(k) = \mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))$. These nested nonlinearities form a complex nonlinear function, which has been addressed in the present study for overflow oscillation elimination analysis for dealing hardware implementation issues.

4. Extension to external interference rejection

In this section, we address establishment of a criterion for the uniformly ultimately bounded stability of nonlinear digital systems in the presence of external interferences (caused by electromagnetic interference of cascade structures) and overflow nonlinearity (due to finite-word length issue of a digital hardware). Uniformly ultimately bounded stability applies to the systems with amplitude-bounded disturbances and provides a bound on the amplitude of the state of the filter. In most of the cases, the amplitudes of disturbances, noises and perturbations are bounded; therefore, the uniformly ultimately bounded stability can be a suitable choice for the dynamical systems.

Theorem 2. *Suppose there exist positive-definite matrices* $\mathbf{N}$ *(diagonal)* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, *and* $\mathbf{S}$, *and positive scalars* ε_1, ε_2, ε_3, *and* ε_4 *such that the LMI* $\mathbf{\Lambda}_2 < 0$, *holds, where*

$$\mathbf{\Lambda}_2 = \begin{bmatrix} \tilde{\mathbf{\Omega}}_{11} & \mathbf{A}^T\mathbf{N}^T & 0 & 0 & 0 & \varepsilon_1 A^T \\ \mathbf{NA} & \tilde{\mathbf{\Omega}}_{22} & 0 & \tilde{\mathbf{\Omega}}_{24} & \varepsilon_2\mathbf{A}^T & \mathbf{N} \\ 0 & 0 & \tilde{\mathbf{\Omega}}_{33} & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{\Omega}}_{42} & 0 & \tilde{\mathbf{\Omega}}_{44} & \varepsilon_2\mathbf{B}_w^T\mathbf{A}^T & 0 \\ 0 & \varepsilon_2\mathbf{A} & 0 & \varepsilon_2\mathbf{A}\mathbf{B}_w & \tilde{\mathbf{\Omega}}_{55} & 0 \\ \varepsilon_1\mathbf{A} & \mathbf{N}^T & 0 & 0 & 0 & \tilde{\mathbf{\Omega}}_{66} \end{bmatrix}, \tag{23}$$

$$\tilde{\mathbf{\Omega}}_{11} = -\mathbf{P} + \varepsilon_1\mathbf{A}^T\mathbf{A} + \mathbf{S} + \varepsilon_3 L^2\mathbf{I},$$
$$\tilde{\mathbf{\Omega}}_{22} = \mathbf{P} - \mathbf{Q} - 2\mathbf{N} - \varepsilon_1\mathbf{I} + \varepsilon_2\mathbf{A}^T\mathbf{A} + \varepsilon_4 L^2\mathbf{I},$$
$$\tilde{\mathbf{\Omega}}_{24} = \mathbf{P}\mathbf{B}_w + \varepsilon_2\mathbf{A}^T\mathbf{A}\mathbf{B}_w + \varepsilon_4 L^2\mathbf{I}\mathbf{B}_w,$$
$$\tilde{\mathbf{\Omega}}_{33} = \mathbf{Q} - \varepsilon_2\mathbf{I},$$
$$\tilde{\mathbf{\Omega}}_{42} = \mathbf{B}_w^T\mathbf{P} + \varepsilon_2\mathbf{B}_w^T\mathbf{A}^T\mathbf{A} + \varepsilon_4\mathbf{B}_w^T L^2\mathbf{I},$$
$$\tilde{\mathbf{\Omega}}_{44} = \mathbf{B}_w^T\mathbf{P}\mathbf{B}_w + \varepsilon_2\mathbf{B}_w^T\mathbf{A}^T\mathbf{A}\mathbf{B}_w - \mathbf{R}\mathbf{I} + \varepsilon_4\mathbf{B}_w^T L^2\mathbf{I}\mathbf{B}_w,$$
$$\tilde{\mathbf{\Omega}}_{55} = \varepsilon_2\mathbf{I} - \varepsilon_4\mathbf{I},$$
$$\tilde{\mathbf{\Omega}}_{66} = \varepsilon_1\mathbf{I} - \varepsilon_3\mathbf{I}.$$

Then uniformly ultimately bounded stability is ensured for the Lipschitz nonlinear digital system (1)–(2) to an ellipsoidal region of the form $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$ *for a positive scalar* η *under saturation overflow nonlinearity and external interferences.*

Proof. Adding and subtracting $\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k)$ and $\mathbf{w}^T(k)\mathbf{R}\mathbf{w}(k)$ terms to the inequality (17) for symmetric positive-definite matrices $\mathbf{S} \in \mathbb{R}^{n\times n}$ and $\mathbf{R} \in \mathbb{R}^{m\times m}$, we attain

$$\Delta V(\mathbf{x}(k)) \le \mathbf{\Theta}_2^T(k)\mathbf{\Lambda}_2\mathbf{\Theta}_2(k) + \mathbf{\Psi}(k) - \mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) + \mathbf{w}^T(k)\mathbf{R}\mathbf{w}(k), \tag{24}$$

where

$$\mathbf{\Theta}_2(k) = [\ \mathbf{x}^T(k) \quad f^T(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))) \quad f^T(\mathbf{A}f(y(k)) + \mathbf{A}\mathbf{B}_w\mathbf{w}(k) + g(f(\mathbf{y}(k)) + \mathbf{B}_w\mathbf{w}(k))) \quad \mathbf{w}^T(k) \quad g^T(f(\mathbf{y}(k)) + \mathbf{B}_w\mathbf{w}(k)) \quad g^T(\mathbf{x}(k))]. \tag{25}$$

Since $\mathbf{\Psi} < 0$, the inequality $\mathbf{\Lambda}_2 < 0$ implies that

$$\Delta V(\mathbf{x}(k)) < -\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) + \mathbf{w}^T(k)\mathbf{R}\mathbf{w}(k), \tag{26}$$

which further reduces to

$$\Delta V(\mathbf{x}(k)) < -\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) + \lambda_{max}(\mathbf{R})\delta, \tag{27}$$

under Assumption 2 and the fact $\lambda_{\max}(\mathbf{R})\mathbf{I} \ge \mathbf{R}$. The following two cases arise:

Case 1: If $-\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) + \lambda_{\max}(\mathbf{R})\delta < 0$, we have $\Delta V(\mathbf{x}(k)) < 0$ for the region $\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) > \lambda_{\max}(\mathbf{R})\delta$. Assigning $\lambda_{\max}(\mathbf{R})\delta = \eta^{-1}$, we obtain $\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) > \eta^{-1}$, that is, the filter states $\mathbf{x}(k)$ will converge to the ellipsoidal region $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$ in the steady-state.

Case 2: In the second case, we take $-\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) + \lambda_{\max}(\mathbf{R})\delta \ge 0$. It implies that the states $\mathbf{x}(k)$ of the filter are bounded in the ellipsoid $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$.

From this discussion, it can be concluded that the digital system (1)–(2) is stable and the states trajectory remain bounded in the steady-state. Hence, the proposed criterion, given by (23), ensures uniformly ultimately bounded stability for the digital system (1)–(2). This concludes the proof. □

Remark 5. Digital systems and algorithms are affected by disturbances and perturbations when implemented on a hardware. Many practical applications require multiple cascade stages of digital filters, which are influenced by interferences from different stages. These external perturbations may cause severe performance degradation of the digital systems in the presence of overflow and dynamical nonlinearities. The approach presented in Theorem 1 cannot be used to analyse overflow effects elimination of digital filters under external perturbations. Consequently, the proposed approach in Theorem 2 is derived to study the ability of a digital filter to deal with the disturbances in addition to overflow and filter nonlinearities. The later methodology ensures uniformly ultimately bounded stability, rather than asymptotic stability, of a filter subjected to unwanted signals.

Remark 6. The approach provided in Theorem 2 investigates uniformly ultimately bounded stability of a digital system (1)–(2) under external perturbations by examining the convergence of the filter state $\mathbf{x}(k)$ into the ellipsoidal region $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$. To estimate the actual effect of the external interference on the system's state, the ellipsoidal region defined by $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$ should be minimized. For this purpose, one approach can be to minimize the semi-major axis of the ellipsoid $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) = 1$, which can be achieved through minimization of $\lambda_{\max}(\mathbf{S}^{-1})$. Therefore, additional constraints of $\mathbf{S} \ge \zeta\mathbf{I}$ and $\zeta > 0$ can be introduced in Theorem 2 and ζ can be maximized through convex LMI routines, where ζ^{-1} gives a bound on $\lambda_{\max}(\mathbf{S}^{-1})$. Consequently, the matrices $\mathbf{R}$ and $\mathbf{S}$ can be employed to attain a small region $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$, in which the states of the filter converge. In addition, the values of $\mathbf{R}$ and $\mathbf{S}$ can be selected through convex routines to attain the feasibility of the constraints in Theorem 2.

When the conditions derived in Theorems 1 and 2 are not feasible, the stability of a nonlinear digital system can be checked through the diagonally dominant matrices based approach presented in [51]. For instance, the scalars ε_1, ε_2, ε_3, and ε_4 can be replaced with the diagonally dominant positive-definite matrices $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ to improve the results for feasibility. Let $l(\mathcal{P}_{i,vt})$ and $\mathcal{P}_{i,vz}$ represent the number of rows and entries of the matrices $\mathcal{P}_i$ for $i = 1, \cdots, 4.$. The corresponding extension is given below in Theorem 3.

Theorem 3. *Suppose there exist positive-definite matrices* $\mathbf{N}$ *(diagonal),* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, *and* $\mathbf{S}$, *and diagonally dominant positive-definite matrices* $\mathcal{P}_i = [\mathcal{P}_{i,vz}]$, *for* $i = 1, 2, 3, 4$, *such that the LMIs* $\mathbf{\Lambda}_3 < 0$,

$\mathcal{P}_{i,vv} \geq \sum_{t=1,t\neq v}^{l(\mathcal{P}_{i,vt})} \left(\mathcal{P}_{i,vt} + \mathcal{Q}_{i,vt}\right) > 0, i = 1, \cdots, 4, v = 1, \cdots, l(\mathcal{P}_{i,vt})$, *and* $\mathcal{Q}_{i,vt} \geq 0, \mathcal{P}_{i,vt} \geq 0, i = 1, \cdots, 4, t = 1, \cdots, l(\mathcal{P}_{i,vt}), v \neq t$, *hold, where*

$$\mathbf{\Lambda}_3 = \begin{bmatrix} \check{\mathbf{\Omega}}_{11} & \mathbf{A}^T\mathbf{N}^T & 0 & 0 & 0 & A^T\mathcal{P}_1 \\ \mathbf{NA} & \check{\mathbf{\Omega}}_{22} & 0 & \check{\mathbf{\Omega}}_{24} & \mathbf{A}^T\mathcal{P}_2 & \mathbf{N} \\ 0 & 0 & \check{\mathbf{\Omega}}_{33} & 0 & 0 & 0 \\ 0 & \check{\mathbf{\Omega}}_{42} & 0 & \check{\mathbf{\Omega}}_{44} & \mathbf{B}_w^T\mathbf{A}^T\mathcal{P}_2 & 0 \\ 0 & \mathcal{P}_2\mathbf{A} & 0 & \mathcal{P}_2\mathbf{AB}_w & \check{\mathbf{\Omega}}_{55} & 0 \\ \mathcal{P}_1\mathbf{A} & \mathbf{N}^T & 0 & 0 & 0 & \check{\mathbf{\Omega}}_{66} \end{bmatrix}, \quad (28)$$

$\check{\mathbf{\Omega}}_{11} = -\mathbf{P} + \mathbf{A}^T\mathcal{P}_1\mathbf{A} + \mathbf{S} + \mathcal{P}_3L^2$,
$\check{\mathbf{\Omega}}_{22} = \mathbf{P} - \mathbf{Q} - 2\mathbf{N} - \mathcal{P}_1 + \mathbf{A}^T\mathcal{P}_2\mathbf{A} + \mathcal{P}_4L^2$,
$\check{\mathbf{\Omega}}_{24} = \mathbf{PB}_w + \mathbf{A}^T\mathcal{P}_2\mathbf{AB}_w + \mathcal{P}_4L^2\mathbf{B}_w$,
$\check{\mathbf{\Omega}}_{33} = \mathbf{Q} - \mathcal{P}_2$,
$\check{\mathbf{\Omega}}_{42} = \mathbf{B}_w^T\mathbf{P} + \mathbf{B}_w^T\mathbf{A}^T\mathcal{P}_2\mathbf{A} + \mathbf{B}_w^TL^2\mathcal{P}_4$,
$\check{\mathbf{\Omega}}_{44} = \mathbf{B}_w^T\mathbf{PB}_w + \mathbf{B}_w^T\mathbf{A}^T\mathcal{P}_2\mathbf{AB}_w - \mathbf{R} + L^2\mathbf{B}_w^T\mathcal{P}_4\mathbf{B}_w$,
$\check{\mathbf{\Omega}}_{55} = \mathcal{P}_2 - \mathcal{P}_4$,
$\check{\mathbf{\Omega}}_{66} = \mathcal{P}_1 - \mathcal{P}_3$.
Then uniformly ultimately bounded stability is ensured for the Lipschitz nonlinear digital system (1)–(2) to an ellipsoidal region of the form $\eta\mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \leq 1$ *under saturation overflow nonlinearity and external interferences.*

The proof can be easily followed by application of the approach in [51] to the proposed results of Theorem 2; therefore, it is left for the readers. To ensure matrices $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ as diagonally dominant, the same approaches provided in [51] can be adopted by employing additional constraints. Compared to the approach in Theorem 2, the methodology developed in Theorem 3 has more computational complexity; however, it can be readily employed to address the infeasibility issue of Theorem 2 and can be used to attain less conservative performance estimates than Theorem 2.

The present study explored stability of the digital systems under inherent complexity and overflow nonlinearity by considering a nonlinear model with fixed parameters. Often, we encounter digital filters and systems with varying parameters due to adaptive, switching and fuzzy nature of the models as observed in the methods [52–57]. Consequently, future work is needed to investigate stability of the nonlinear digital systems of varying co-efficient matrices under external interferences and finite word-length effects.

5. Simulation results

This section provides numerical simulation results for overflow stability analysis of two nonlinear digital systems including an observer for a Moon system [58] and recurrent neural network (RNN) [59].

5.1. Example 1: observer for Moon system

Moon system is a physical system, comprising of a pendulum attached to a bracket, which has some flexibility. There is a metal ball attached at the end of the pendulum, and two magnets are fixed around the pendulum, which are at equal distance from it when the overall system of pendulum and bracket is not moving. A harmonic oscillatory movement is utilized, which has constant amplitude, to excite the bracket and drag the pendulum into motion [58]. Consider a Moon chaotic system, which has been discretized using Euler's method, as

$$x_{p1}(k+1) = x_{p1}(k) + Tx_{p2}(k), \quad (29)$$

$$x_{p2}(k+1) = \frac{T}{2}x_{p1}(k) + (1-mT)x_{p2}(k) - T(10x_{p1}(k) + x_{p2}(k))x_{p1}(k) + TI_s, \quad (30)$$

$$y_p(k) = 10x_{p1}(k) + x_{p2}(k), \quad (31)$$

where $x_{p1}(k)$ and $x_{p2}(k)$ are the states, $y_p(k)$ is the output, I_s is the stimulation applied to the system, m is the mass of the metal ball, and T is the sampling time. With $m = 0.15$, chaotic behavior is evinced by the system. In [58], an observer for the Moon chaotic system has been developed for the state filtering purpose. The equations for the observer of the system (29)–(31) are given by

$$\hat{x}_{p1}(k+1) = \hat{x}_{p1}(k) + T\hat{x}_{p2}(k) + 0.0998(y_p(k) - \hat{y}_p(k)), \quad (32)$$

$$\hat{x}_{p2}(k+1) = \frac{T}{2}\hat{x}_{p1}(k) + (1-mT)\hat{x}_{p2}(k) - T(10\hat{x}_{p1}(k) + \hat{x}_{p2}(k))\hat{x}_{p1}(k) + TI_s + 0.2051(y_p(k) - \hat{y}_p(k)), \quad (33)$$

$$\hat{y}_p(k) = 10\hat{x}_{p1}(k) + \hat{x}_{p2}(k). \quad (34)$$

By substituting $\hat{y}_p(k)$ from Eq. (34) into Eqs. (32) and (33), the state filtering system becomes

$$\hat{x}_{p1}(k+1) = \hat{x}_{p1}(k) + T\hat{x}_{p2}(k) + 0.0998y_p(k) - 0.0998(10\hat{x}_{p1}(k) + \hat{x}_{p2}(k)), \quad (35)$$

$$\hat{x}_{p2}(k+1) = \frac{T}{2}\hat{x}_{p1}(k) + (1-mT)\hat{x}_{p2}(k) - T(10\hat{x}_{p1}(k) + \hat{x}_{p2}(k))\hat{x}_{p1}(k) + TI_s + 0.2051y_p(k) - 0.2051(10\hat{x}_{p1}(k) + \hat{x}_{p2}(k)). \quad (36)$$

Setting $x_1(k) = \hat{x}_{p1}(k)$ and $x_2(k) = \hat{x}_{p2}(k)$, the Eqs. (35)–(36) can be represented as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)) + \mathbf{u}_F, \quad (37)$$

where

$$\mathbf{A} = \begin{bmatrix} 0.002 & T - 0.0998 \\ \frac{T}{2} - 2.051 & 1 - mT - 0.2051 \end{bmatrix},$$

$$g(\mathbf{x}(k)) = \begin{bmatrix} 0 \\ (10x_1 + x_2)^2x_1 \end{bmatrix}, \ \mathbf{u}_F = \begin{bmatrix} 0.0998y_p \\ 0.2051y_p + TI_s \end{bmatrix}.$$

In this work, the zero-input stability is being investigated for nonlinear systems without disturbance and with disturbance by selecting $T = 0.1$sec. Therefore, the input to the digital filtering system is considered to be zero, that is, $\mathbf{u}_F = 0$. The observer (35)–(36), for the system (29)–(31) is nonlinear and it can be implemented via a digital hardware. Eventually, this can inevitably cause the overflow $f(\cdot)$ and the external perturbation $\mathbf{w}(k)$ effects. Hence, the equation (37) with $\mathbf{u}_F = 0$, overflow nonlinearity and external perturbations becomes

$$\mathbf{x}(k+1) = f(\mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k))) + \mathbf{B}_w\mathbf{w}(k), \quad (38)$$

which is the same equation as for the digital system (1)–(2).

5.1.1. Without external interference

The stability criterion given by LMI (7) in Theorem 1 under $\mathbf{w}(k) = 0$ is applied to the observer of the Moon chaotic system , and the solution is found to be feasible with the matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{N}$ as

$$\mathbf{P} = \begin{bmatrix} 35.7189 & 0.5870 \\ 0.5870 & 1.8167 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 1.5525 & -0.1930 \\ -0.1930 & 2.9065 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 65.2845 & 0 \\ 0 & 1.24931 \end{bmatrix}.$$

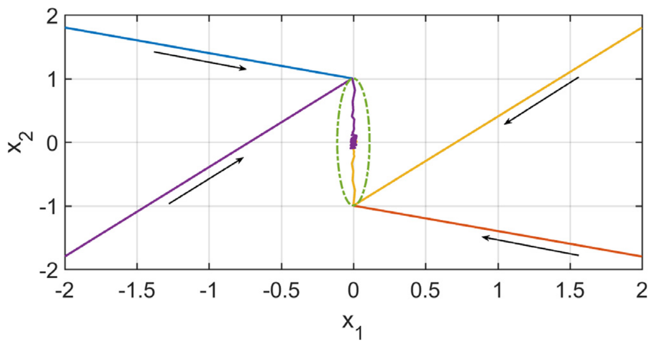

Fig. 1. Convergence of the state trajectories in the ellipse $\eta \mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \leq 1$.

The values of the positive scalars are given by

$$\varepsilon_1 = 0.6779,\ \varepsilon_2 = 3.0740,\ \varepsilon_3 = 17.1100,\ \varepsilon_4 = 83.8346.$$

The conventional approaches like [20–30] fail in this example due to presence of a nonlinearity in the observer; however, the solution to the LMI (7) is feasible. Therefore, it can be concluded that the proposed criterion, presented in Theorem 1, can be successfully used to analyze the asymptotic stability of a nonlinear digital system, in the absence of external interferences.

5.1.2. With external interference

Now, the stability criterion given by LMI (23) in Theorem 2 is applied to the Moon chaotic system observer (29), (30), and (31). The matrix $\mathbf{R}$ is chosen to be identity, and the solution is found to be feasible with the matrices

$$\mathbf{P} = \begin{bmatrix} 15.4875 & -0.2547 \\ -0.2547 & 0.7944 \end{bmatrix},$$

$$\mathbf{Q} = 10^{-4} \times \begin{bmatrix} 0.2160 & -0.0150 \\ -0.0150 & 0.4219 \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 7.9674 & 0.7353 \\ 0.7353 & 0.1005 \end{bmatrix},\ \mathbf{N} = \begin{bmatrix} 22.2580 & 0 \\ 0 & 0.5531 \end{bmatrix},$$

and the scalars

$$\varepsilon_1 = 0.2704,\ \varepsilon_2 = 4.2993 \times 10^{-5},\ \varepsilon_3 = 22.5250,$$
$$\varepsilon_4 = 5.1808 \times 10^{-4}.$$

As the solution to the LMI (23) is feasible, the filter is stable under external perturbations. Our methodology of Theorem 2 is convenient to examine the uniformly ultimately bounded stability of a nonlinear digital system subject to external interference. In another experiment, the filter's state $\mathbf{x}$ have been plotted for the observer of Moon chaotic system with four different initial states, under the disturbance which has been assumed to be uniformly distributed random vector, $\mathbf{w}(k) = [0.1w_1 \ \ -0.1w_2]^T$, where w_1, and w_2 are Gaussian random variables of mean zero and unity variance, as shown in Fig. 1 by considering the optimization stated in Remark 6. It is evident that all of the state trajectories are converging to the ellipsoidal region specified by $\eta \mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \leq 1$ around the origin as anticipated from the criterion presented in Theorem 2.

It is worth mentioning that the stability of the nonlinear digital systems under overflow, such as the digital state observer given by (35)–(36), can be analyzed through the proposed criteria (7) and (23), the existing methods [20–30] (see also [32–36]) fail to study stability for the nonlinear digital systems under overflow constraint in the absence or presence of external interferences.

5.2. Example 2: Recurrent neural network

Artificial neural networks (ANN), similar to the biological neurons, process information in parallel disseminated way [60]. However feed-forward neural networks (FNN) have limitation that these can work only for static systems. It highlights the need of another type of neural networks, called recurrent neural networks (RNNs), which have the ability to handle dynamic systems and, hence, RNNs perform more complicated computations for the complex systems [61,62]. A neural network which has one or more feedback loops, for reusing the output values of the network, including a global feedback loop throughout the network, and local feedback in a single neuron, is a called as RNN [63]. These neural networks have extensive number of applications in adaptive control, optimization, signal processing, and pattern recognition (see [59,64]). Most of these applications assume that global asymptotic or exponential stability is guaranteed for the recurrent neural networks, which, in reality, is not always true, as the concealed neural network can be rendered unstable by RNN [59]. In addition, when these networks are implemented via a digital hardware, the performance degradation and instability can occur owing to the finite word-length restriction. The general equation for continuous-time recurrent neural network is as follows:

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \mathbf{W}_0 \bar{f}(\mathbf{x}(t)) + \bar{\mathbf{B}}\mathbf{I}_e(t), \tag{39}$$

where $\bar{\mathbf{A}}$ is the diagonal system matrix, $\mathbf{W}_0$ is the connection weight matrix, $\bar{f}(\mathbf{x}(t)) = [\bar{f}(x_1(t)) \ \bar{f}(x_2(t)) \ \cdots \ \bar{f}(x_n(t))]^T \in \mathbb{R}^n$ is a nonlinear activation function, $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^T \in \mathbb{R}^n$ is the designated state vector for n neurons, and $\mathbf{I}_e(t) \in \mathbb{R}^p$ is the excitation input, which is taken zero because this research focuses on the stability criteria derivation for zero-input stability, that is, $\mathbf{I}_e(t) = 0$. Let us choose

$$\bar{\mathbf{A}} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -15 \end{bmatrix},\ \mathbf{W}_0 = \begin{bmatrix} 2 & -0.1 & -0.2 \\ -5 & 2 & 0 \\ 0.3 & 0.3 & -0.7 \end{bmatrix},$$

$$\bar{f}(\mathbf{x}(t)) = \begin{bmatrix} \tanh(x_1) \\ \tanh(x_2) \\ \tanh(x_3) \end{bmatrix}.$$

The nonlinearity $\bar{f}(\mathbf{x}(t))$ is globally Lipschitz with Lipschitz constant $L_f = 1$. Defining $\bar{g}(\mathbf{x}(t))$ as

$$\bar{g}(\mathbf{x}(t)) = \mathbf{W}_0 \bar{f}(\mathbf{x}(t)), \tag{40}$$

the RNN system leads to

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{g}(\mathbf{x}(t)). \tag{41}$$

The state-space representation (41) is in continuous-time; however, neural networks are implemented in digital hardware. Therefore, we converted it into discrete-time form using Euler's forward method. The resulting discrete-time system becomes

$$\mathbf{x}(k+1) = (T\bar{\mathbf{A}} + \mathbf{I})\mathbf{x}(k) + T\bar{g}(\mathbf{x}(k)), \tag{42}$$

where T is the sampling time. By defining $(T\bar{\mathbf{A}} + \mathbf{I}) = \mathbf{A}$ and $T\bar{g}(\mathbf{x}(k)) = g(\mathbf{x}(k))$, the discretized RNN system's state-space representation is given as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + g(\mathbf{x}(k)). \tag{43}$$

Neural networks are nonlinear; consequently, the previously developed stability criteria [20–30,32–36] fail for the stability analysis of neural networks under overflow condition. The model (43) becomes same as (1) and (2) by incorporating the overflow nonlinearity and external perturbations with Lipschitz constant $L = \|T\mathbf{W}_0\|$ (under $L_f = 1$).

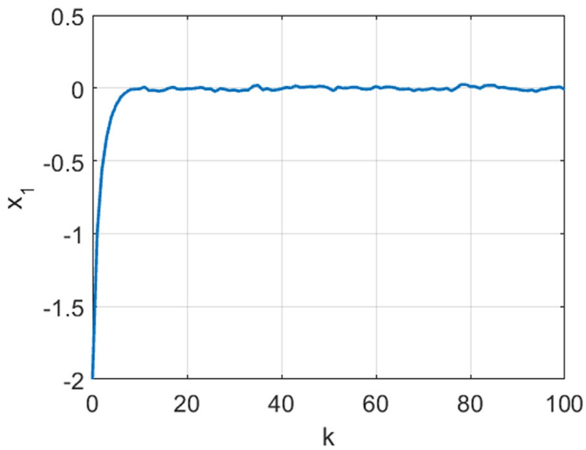

Fig. 2. Convergence of RNN state trajectory x_1.

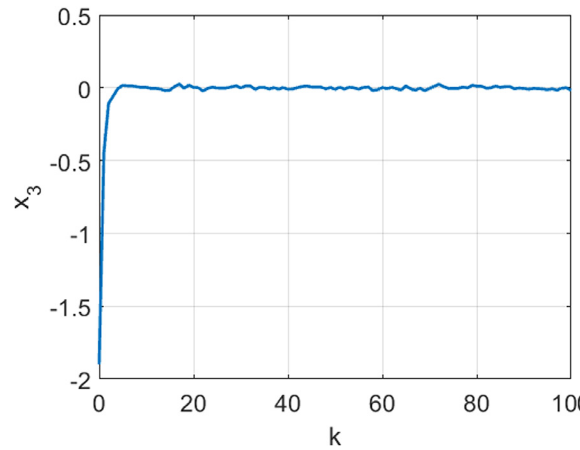

Fig. 4. Convergence of RNN state trajectory x_3.

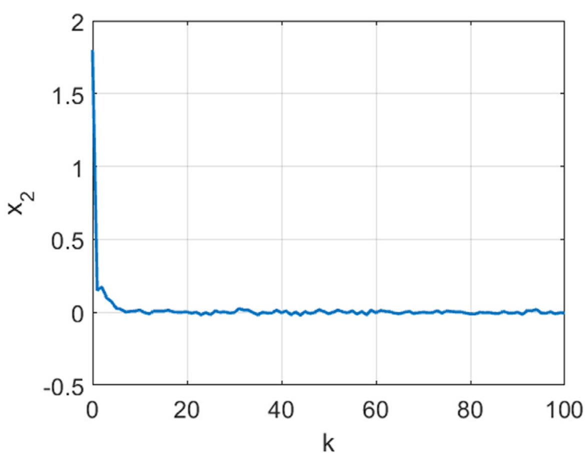

Fig. 3. Convergence of RNN state trajectory x_2.

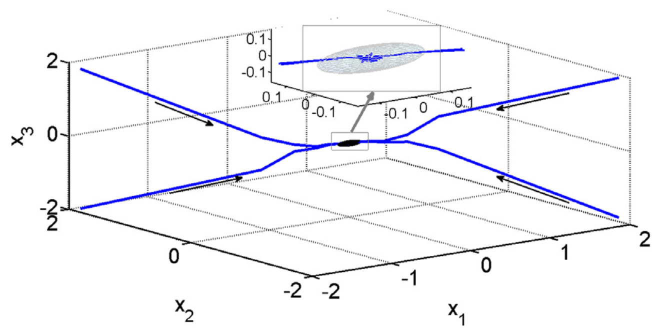

Fig. 5. Convergence of the state trajectories in the ellipsoidal region $\eta \mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$.

By application of Theorem 2, the solution is found to be feasible with the LMI variables as

$$\mathbf{P} = \begin{bmatrix} 22.8496 & 0 & 0 \\ 0 & 45.7176 & 0 \\ 0 & 0 & 34.2933 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0.0680 & 0 & 0 \\ 0 & 0.0494 & 0 \\ 0 & 0 & 0.0638 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 4.5754 & 0 & 0 \\ 0 & 38.8545 & 0 \\ 0 & 0 & 18.8706 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 23.1931 & 0 & 0 \\ 0 & 77.8241 & 0 \\ 0 & 0 & 45.7540 \end{bmatrix},$$

and

$$\varepsilon_1 = 6.4265, \ \varepsilon_2 = 0.0682, \ \varepsilon_3 = 84.2491, \ \varepsilon_4 = 0.1792.$$

Next the state trajectories for x_1, x_2, and x_3 are plotted in Figs. 2, 3, and 4, respectively. Initial state vector is considered to be $\mathbf{x}(0) = [-2 \ 1.8 \ -1.9]^T$. The disturbance is assumed to be uniformly distributed random vector, $\mathbf{w}(k) = [0.1w_1 \ -0.1w_2 \ -0.1w_3]^T$, where w_1, w_2, and w_3 are Gaussian random variables of mean zero and unity variance. All three states are converging near the origin from the initial conditions, and their steady-values remain within bounds.

The phase portraits have been plotted for the RNN system (43) as well, with four different initial conditions as seen in Fig. 5. It may be noted that all these state trajectories converge inside the ellipsoid defined by $\eta \mathbf{x}^T(k)\mathbf{S}\mathbf{x}(k) \le 1$, as expected from the proposed criterion in Theorem 2.

Consequently, in this section we have shown that the proposed methodology can be successfully employed to evaluate the stability of the nonlinear systems, like recurrent neural networks (RNN) and state filters. Stability analysis of such systems cannot be performed by using previously reported criteria [20–30,32–36] under overflow constraint caused by digital hardware limitations.

6. Conclusions

This paper established two criteria for assessment of the stability of digital nonlinear systems under the consequence of saturation overflow nonlinearity (which is the direct result of finite word-length), without and with consideration of the disturbance from environment (which is also inevitable in a digital hardware). The proposed approaches investigated the asymptotic stability and uniformly ultimately bounded stability in the absence and presence of external perturbations, respectively, for the nonlinear digital filter. Appropriate Lyapunov function along with sector conditions were employed while considering the fact that most of the nonlinear systems are either globally or locally Lipschitz. The outcomes of both of the criteria were presented in form of LMIs for an efficient stability test. The proposed approaches can handle inherent nonlinearity in the dynamics of a system as well as the saturation nonlinearity due to the digital hardware constraint for the stability analysis in contrast to the conventional methods. The resultant schemes were tested for two real-world physical systems, including an observer of a Moon chaotic system and a recurrent neural network. The state trajectories have been plotted and phase portraits are shown to converge to a bounded ellipsoidal region, ensuring the uniformly ultimately bounded stability under external interferences. The simulation results demonstrated the usefulness of the developed approaches, and the importance of these criteria was further enhanced as the previous research works lack in offering the analysis of nonlinear digital systems' stability under the overflow nonlinearity.

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