Signal Processing Project - Team 17

Sampling Signals with Finite Rate of Innovation

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1. Abstract

We define rate of innovation as the number of degrees of freedom that a class of signal has, such as a stream of Diracs, nonuniform splines, and piecewise polynomials. In this paper, we show that we can uniformly sample these signals at the rate of innovation, or even higher, by using an appropriate kernel, allowing us to perfectly reconstruct these signals even though they are not bandlimited. We basically sample and reconstruct periodic and finite-length stream of Diracs, nonuniform splines, and piecewise polynomials, using sinc and Gaussian kernels. For infinite-length signals, we define a *local* finite rate of innovation, which we sample and reconstruct based on spline kernels. We identify the innovative part of a signal, such as the time instants and the weight of Diracs, using an annihilating filter or a locator filter.

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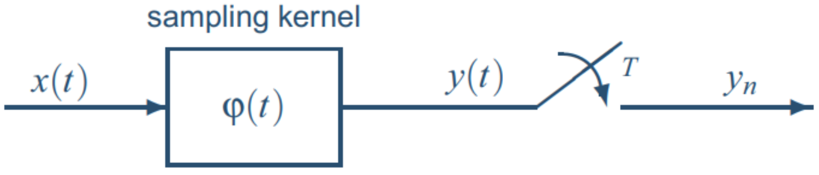
2. General Sampling Setup

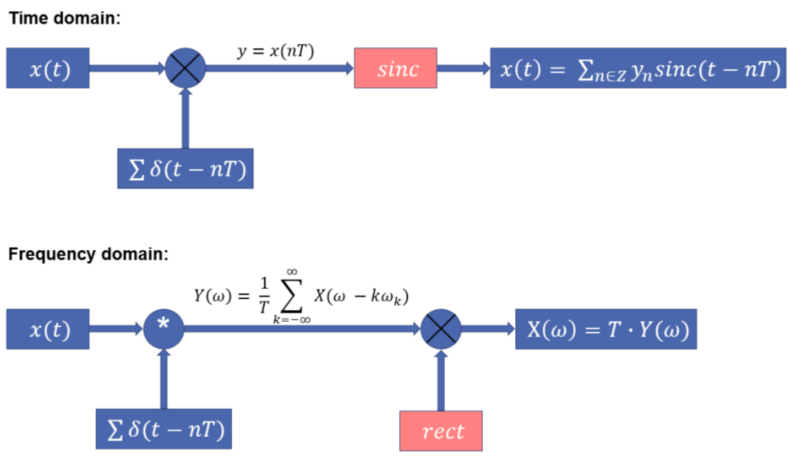
The general sampling setup involves convolving the original signal $x(t)$ with a sampling kernel $\varphi(t)$ to get a filtered signal $y(t)$ which is further sampled at intervals T to get the samples $y_n = y(nT)$.



If there is no sampling kernel, we simply get $y(nT) = x(nT)$.

2.1 Shannon Reconstruction for Bandlimited Signals

For Band-limited signals



If $x(t)$ is bandlimited, or $X(\omega) = 0 \; \forall \; \omega > \omega_m$, then samples $x(nT)$ with $\omega_s \geq 2\omega_m \implies T \leq \pi/\omega_m$ (Nyquist Condition) are sufficient to reconstruct $x(t)$.

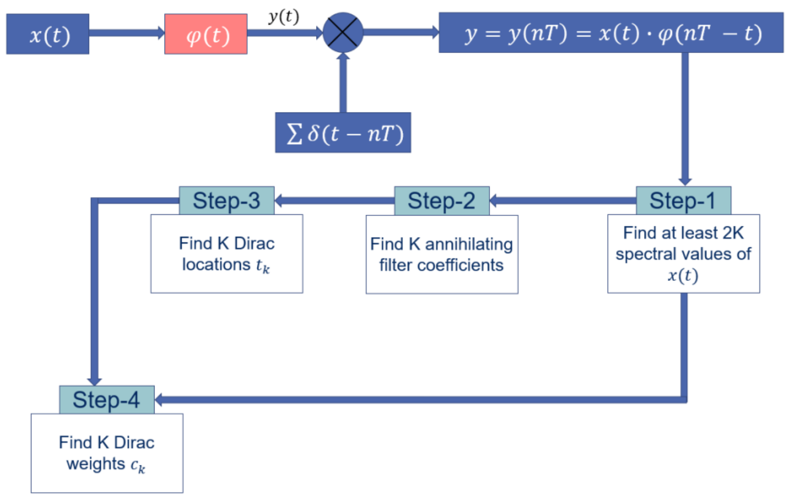
Calling $B = 2\omega_m/2\pi$, which is the bandwidth of $x(t)$ in cycles per second, we see that $B = 1/T$ sample are a sufficient representation of $x(t)$. The reconstruction formula is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t}{T} - n\right)$$

2.2 Non-Bandlimited Case

To generalize the Shannon Reconstruction to non-bandlimited signals we consider signals with finite rate of innovation.

Sampling and Reconstruction of periodic stream of Diracs



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3. Signals with Finite Rate of Innovation

Definition: A signal with a finite rate of innovation is a signal with parametric representation as

$$x(t) = \sum_{n\in\mathbb{Z}} \sum_{r=0}^{R} c_{nr}\varphi_r\left(\frac{t-t_n}{T}\right)$$

for a set of functions $\{\varphi_r(t)\}_{r=0,\ldots R}$ with a finite rate of innovation ρ defined as

$$\rho = \lim_{\tau\to\inf} \frac{1}{\tau} C_x(\frac{-\tau}{2}, \frac{\tau}{2})$$

- If we consider finite length signals, or periodic signals with length τ then degrees of freedom is finite, and then rate of innovation will be $\frac{1}{\tau}C_x(0,\tau)$.
- For infinite length signals, we define *local* rate of innovation with a moving window of size τ as

$$\rho_T(t) = \frac{1}{\tau} C_x(t-\frac{\tau}{2}, t+\frac{\tau}{2})$$

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4. Periodic Continuous-Time case

We represent such signals as

$$x(t) = \sum_{m\in\mathbb{Z}} X[m] e^{i(2\pi mt/\tau)}$$

4.1 Stream of Diracs

We take a stream of Diracs with a period of τ, and a rate of innovation as

$$\rho = \frac{2K}{\tau}$$

The stream of Diracs can be written as

$$\begin{aligned} x(t) &= \sum_{m\in\mathbb{Z}} c_m \delta(t-t_n) \\ &= \sum_{m\in\mathbb{Z}} \underbrace{\frac{1}{\tau}\left(\sum_{k=0}^{K-1} c_k e^{-i(2\pi m t_k/\tau)}\right)}_{X[m]} e^{i(2\pi mt/\tau)} \end{aligned}$$

where X[m] is the Fourier series coefficients. Taking another Fourier series $A[m], m \in \{0, 1\ldots, K\}$ with z-transform,

$$A(z) = \sum_{m=0}^{K} A[m] z^{-m}$$

and taking the zeros of the Fourier transform at $u_k = e^{(-i(2\pi t_k/\tau)}$ gives us

$$A(z) = \prod_{k=0}^{K-1} \left(1 - e^{(-i(2\pi t_k/\tau)} z^{-1}\right)$$

Since $X[m]$ is the sum of K exponentials, and each of them become zero by one of the roots of $A[m]$, we can deduce that

$$\boxed{A[m] * X[m] = 0}$$

Here, $A[m]$ is an _annihilating filter_. We obtain the time-domain function $x(t)$ by performing inverse Fourier-Transform.

Let us define a sinc function of bandwidth $[-B\pi, B\pi], B \in \mathbb{R}_+$ and name it as

$$h_B(t) = B \cdot sinc(Bt)$$

Theorem 1: Consider $x(t)$ as a periodic stream of Diracs, with a period of τ, with K Diracs of weight $\{c_k\}_{k=0}^{K-1}$ and at location $\{t_k\}_{k=0}^{K-1}$. Taking sampling kernel as the sinc function stated above, where $B \geq \rho$, and sampling $(h_B * x)(t)$ at N uniformly-distributed locations $t = nT, n \in \{0, 1, \ldots N-1\}, N \geq 2M+1, M = \lfloor \frac{B_\tau}{2} \rfloor$, then the samples that we collect, i.e.

$$y_n = \langle h_B(t - nT), x(t) \rangle, \ n = 0, \ldots, N-1$$

are sufficient to reconstruct the original signal $x(t)$

Proof: We follow the steps as follows:

1. We first find $X[m], |m| \leq M$ from our samples y_n. We plug in the general form of $x(t)$ in the above equation, giving us

$$y_n = \sum_{m=-M}^{M} X[m] \cdot e^{i(2\pi mnT/\tau)}$$

- When $\tau = \lambda T$, i.e. when τ is divisible by T, we observe that this simply is the IDTFT of $X[m]$.

2. We then find the annihilating coefficients for $X[m]$ by solving the equation $A[m] * X[m] = 0$. We write an equivalent form of this as

$$\underbrace{\begin{bmatrix} X[0] & X[-1] & \cdots & X[-K+1] \\ X[1] & X[0] & \cdots & X[-K+2] \\ \vdots & \vdots & \ddots & \vdots \\ X[K-1] & X[K-2] & \cdots & X[0] \end{bmatrix} \cdot \begin{bmatrix} A[1] \\ A[2] \\ \vdots \\ A[K] \end{bmatrix} = - \begin{bmatrix} X[1] \\ X[2] \\ \vdots \\ X[K] \end{bmatrix}}_{\text{Yule-Walker system}}$$

- Here, we obtain a unique solution for the case when there are K distinct Diracs in $x(t)$.

3. We factorize $A(z)$ into its roots as

$$A(z) = \prod_{k=0}^{K-1} \left(1 - u_k z^{-1}\right)$$

where $u_k = e^{(-i(2\pi t_k/\tau)}$, which gives us the locations of $\{t_k\}_{k=0}^{K-1}$

4. We then find the weights of the Diracs c_k by using the value of u_k as follows, by using $X[m] = \frac{1}{\tau} \sum\limits_{k=0}^{K-1} c_k \cdot e^{-i(2\pi m t_k/\tau)}$:

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[K-1] \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_0 & u_1 & \cdots & u_{K-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{K-1} & u_1^{K-1} & \cdots & u_{K-1}^{K-1} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{bmatrix}}_{\text{Vandermonde System}}$$

- Here, we obtain a unique solution for the case when all the t_k's are different.

4.2 Nonuniform Splines

A signal $x(t)$ is a periodic nonuniform spline of degree R with discontinuities at $\{t_k\}_{k=0}^{K-1} \in [0, \tau]$ *iff* its $(R+1)^{th}$ derivative is a periodic stream of K weighted Diracs, i.e.

$$x^{(R+1)}(t) = \sum_{n \in \mathbb{Z}} c_n \delta(t - t_n),\ t_{n+K} = t_n + \tau,\ c_{n+K} = c_n \forall\ n \in \mathbb{Z}$$

If we differentiate the standard form of the signal $x(t)$ $R+1$ times, we see that the Fourier series coefficients are

$$X^{R+1}[m] = \left(\frac{i2\pi m}{\tau}\right)^{R+1} \cdot X[m],\ m \in \mathbb{Z}$$

From the previously stated theorem, we can obtain the stream of K Diracs from the above coefficients, and thus reconstruct the periodic nonuniform spline.

Theorem 2: If we take a periodic nonuniform spline $x(t)$ periodic with period τ and containing K pieces of maximum degree R, and take samples of $(h_B * x)(t)$ at N uniform locations $t = nT,\ n = 0, \dots, N-1,\ N \geq 2M+1,\ M = \lfloor B_\tau/2 \rfloor$. Then we can say that the samples

$$y_n = \langle h_B(t - nT), x(t) \rangle$$

are sufficient to reconstruct the original signal $x(t)$.

4.3 Derivative of Diracs

We take the r^{th} derivative of the Dirac function, which has the property as follows:

$$\int f(t) \cdot \delta^{(r)}(t - t_0) dt = (-1)^r \cdot f^{(r)}(t_0)$$

here, $f(t)$ is r times continuously differentiable. We take a periodic stream of differentiated Diracs

$$x(t) = \sum_{n \in \mathbb{Z}} \sum_{r=0}^{R_n - 1} c_{nr} \delta^{(r)}(t - t_n)$$

here the degrees of freedom is different than usual. Here we have K locations, but our weights are different this time, i.e. our weights are $\tilde{K} = \sum_{k=0}^{K-1} R_k$, which gives us degrees of freedom as $K + \tilde{K}$.

We find the corresponding Fourier series coefficients as

$$X[m] = \underbrace{\underbrace{\frac{1}{\tau}} \sum_{k=0}^{K-1} \sum_{r=0}^{R_k - 1} c_{kr} \underbrace{\left(\frac{i2\pi m}{\tau}\right)}}_{\tilde{c}_{kr}} e^{-i(2\pi m t_k/\tau)}$$

We see that the filter

$$A(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1})^{R_k}$$

has R_k poles, at $z = u_k$ and annihilates $X[m]$. We find the locations of t_k in the same method as previously, stated, but here the weights $\tilde{c}_{kr}$ are found by solving the above Fourier series coefficients for $m = 0, \dots, \tilde{K} - 1$.

Theorem 3: If we take a stream of differentiated Diracs $x(t)$ periodic with period τ, and take samples of $(h_B * x)(t)$ at N uniform locations, with the same parameters as Theorems 1 and 2, then the samples

$$y_n = \langle h_B(t - nT), x(t) \rangle,\ n = 0, \dots, N-1$$

is sufficient to reconstruct $x(t)$

4.4 Piecewise Polynomials

We define piecewise polynomials in a fashion similar to the periodic nonuniform splines, where its $(R+1)^{th}$ derivative must be a stream of differentiated Diracs. Here, d.o.f. will be K from the locations, and $\tilde{K} = (R+1)K$, thus we get

$$\rho = \frac{(R+2)K}{\tau}$$

Theorem 4: If we take a piecewise polynomial $x(t)$ periodic with period τ, having K pieces of maximum degree R and take samples of $(h_B * x)(t)$ at N uniform locations, with the same parameters as Theorems 1 and 2, then the samples

$$y_n = \langle h_B(t - nT), x(t) \rangle, \ n = 0, \ldots, N-1$$

is sufficient to reconstruct $x(t)$.

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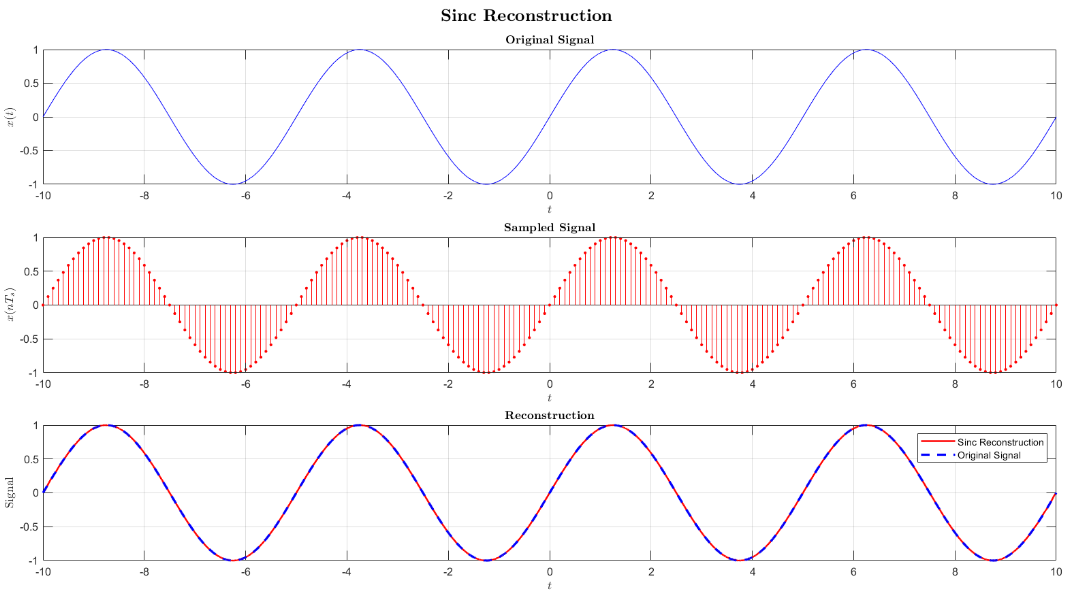
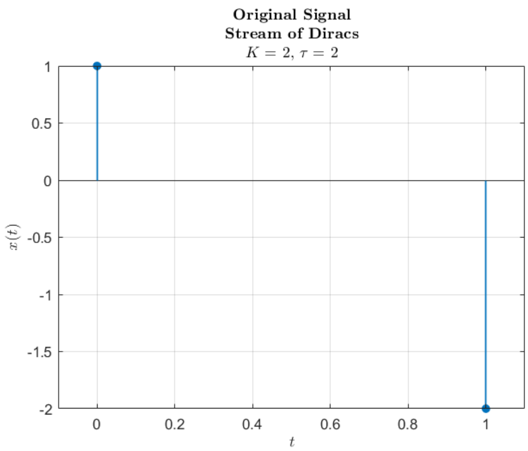
5. Results and Inferences

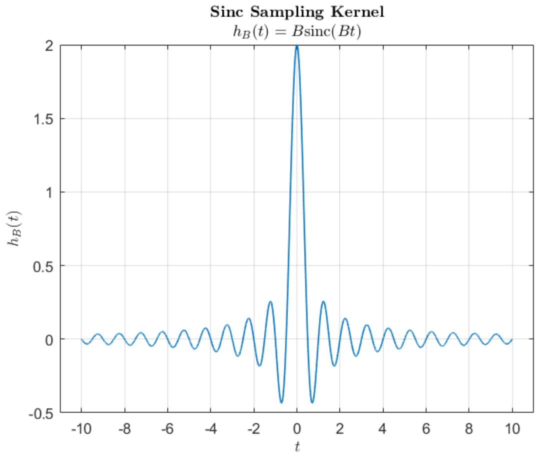
Bandlimited case



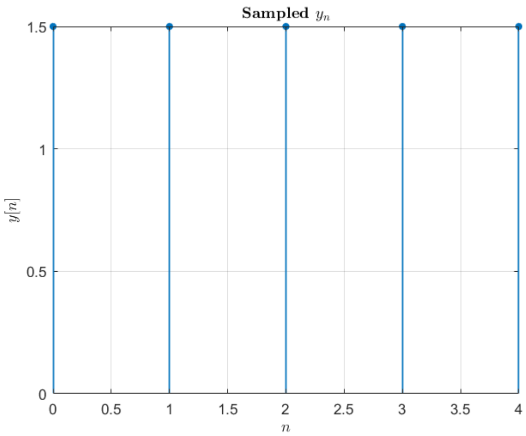
Non-bandlimited case



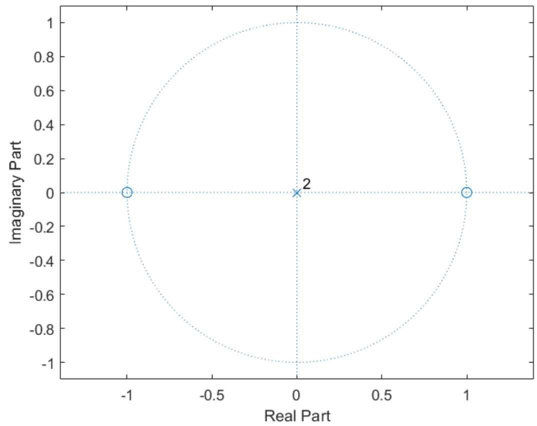
Original Signal $K = 8, \tau = 1024$



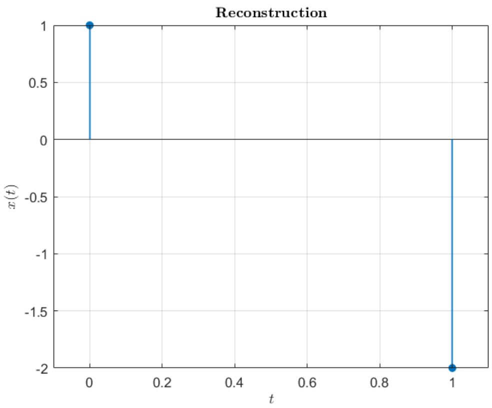
The sampling kernel



The sampled signal $y(nT)$



The zeros of the annihilating filter



Reconstructed Signal

We see that we get a perfect reconstruction of the original signal.

6. References

1. ‘Sampling Signals With Finite Rate of Innovation’, Martin Vetterli, Pina Marziliano, Thierry Blu, IEEE.
2. ‘Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon Meets Strang–Fix’, Pier Luigi Dragotti, Martin Vetterli, Thierry Blu, IEEE.
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4. ‘Sampling curves with finite rate of innovation’, Hanjie Pan, Thierry Blu, Pier-Luigi Dragotti.