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Math 189

HW5

1) a) prove $\left\|x_i - \sum_{j=1}^{k} z_{ij} v_i\right\|^2 = x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j$

we have $\left\|x_i - \sum_{j=1}^{k} z_{ij} v_i\right\|^2 = \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)^T \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)$

$$= \left(x_i^T - \left(\sum_{j=1}^{k} z_{ij} v_j\right)^T\right)\left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)$$

$$= x_i^T x_i - x_i^T \sum_{j=1}^{k} z_{ij} v_j - x_i \left(\sum_{j=1}^{k} z_{ij} v_j\right)^T + \left(\sum_{j=1}^{k} z_{ij} v_j\right)^T \left(\sum_{j=1}^{k} z_{ij} v_j\right)$$

$$= x_i^T x_i - \sum_{j=1}^{k} z_{ij} x_i^T v_j - \sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} v_j^T z_{ij}^T z_{ij} v_j$$

Since $x_i^T v_j = v_j^T x_i = \langle x_i, v_j \rangle$

$$= x_i^T x_i - 2\sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} v_j^T z_{ij}^T z_{ij} v_j$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} x_i^T v_j v_j^T x_j + \sum_{j=1}^{k} v_j^T v_j^T x_i x_i^T v_j v_j$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} v_j x_i^T x_i v_j^T + \sum_{j=1}^{k} v_j^T x_i v_j^T v_j x_i^T v_j$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} v_j^T x_i x_i^T v_j + \sum_{j=1}^{k} v_j^T x_i x_i^T v_j \quad (\text{since } v_i^T v_j = \delta_{ij})$$

$$= x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$

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b) prove $J_k = \frac{1}{n}\sum_{i=1}^{n}\left(x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j\right) = \frac{1}{n}\sum_{i=1}^{n} x^T x_i - \sum_{j=1}^{k}\lambda_j$

$$J_k = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} v_j^T\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i^T\right) v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} v_j^T \Sigma v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k}\lambda_j$$

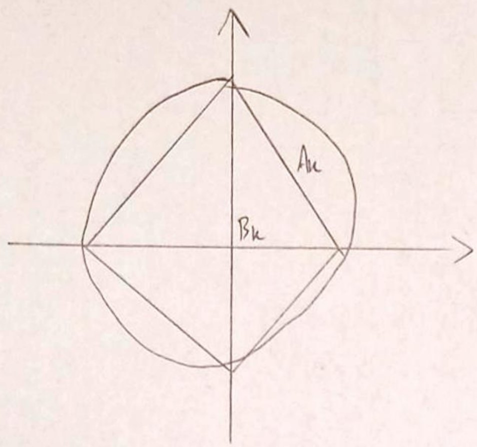
c) show that $J_k = \sum_{j=k+1}^{d}\lambda_j$

$$J_k = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k}\lambda_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{d}\lambda_j + \sum_{j=k+1}^{d}\lambda_j$$

$$= \sum_{j=1}^{d}\lambda_j - \sum_{j=1}^{d}\lambda_j + \sum_{j=k+1}^{d}\lambda_j = \sum_{j=k+1}^{d}\lambda_j$$

2)



prove that

$$\min f(x) \quad \text{s.t. } \|x\|_p \le k$$

is equal to

$$\min f(x) + \lambda \|x\|_p$$

we know $\min f(x)$ s.t. $\|x\|_p \le k$ is equal to $\inf_x \sup_{\lambda \ge 0} \mathcal{L}(x,\lambda) = \inf_x \sup_{\lambda \ge 0} f(x) + \lambda(\|x\|_p - k)$

which is equal to $\sup_{\lambda \ge 0} \inf_x f(x) + \lambda(\|x\|_p - k) = \sup_{\lambda \ge 0} g(\lambda)$

since $\min f(x) + \lambda(\|x\|_p - k)$ over x is equal to

$\min f(x) + \lambda \|x\|_p$

we have $\min f(x) + \lambda \|x\|_p$