Riddler Classic for September 17, 2021
https://fivethirtyeight.com/features/can-you-bake-the-radish-pie/

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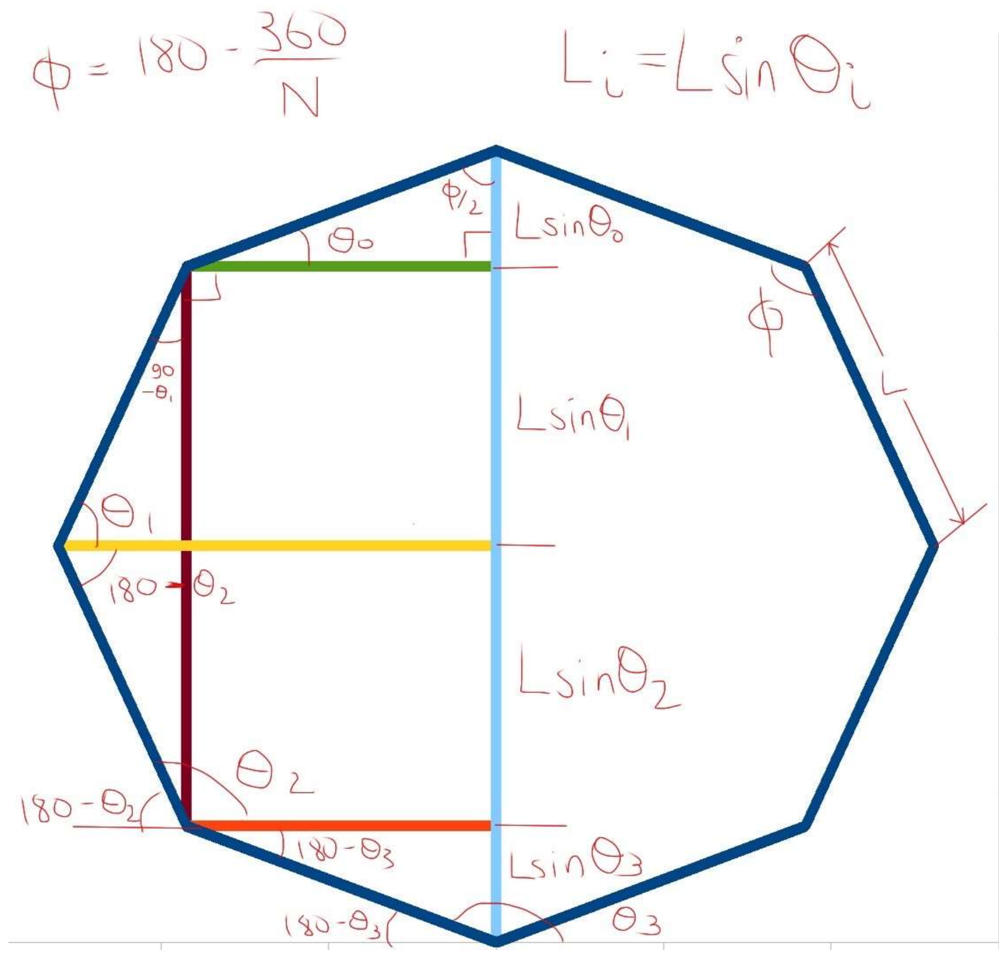
Consider a regular polygon with an even number of sides (N) of length L. The interior angle of each vertex is ϕ given by this equation:

$$\phi = 180 - \frac{360}{N}$$

I find the height of each line segment by finding the angles between the perimeter of the polygon and the horizontal line at the bottom of the segment. For each segment:

$$L_i = L \sin \theta_i$$

I find the angles θ_i as shown belong, illustrating the process for an octagon because the angles are too hard to see on a polygon with 1000 sides.



In the right triangle at the top of the polygon, angle θ_0 is the opposite angle from $\phi/2$.

$$\theta_0 = 90 - \frac{\phi}{2} = 90 - \frac{180 - \frac{360}{N}}{2} = \frac{180}{N}$$

At the first vertex down from the top, θ_0 , a right angle, and 90 - θ_1 must add to ϕ. This lets me calculate θ_1 :

$$\phi=\theta_0+90+(90-\theta_1)$$

$$\theta_1=\theta_0+180-\phi=\theta_0+180-(180-\frac{360}{N})=\theta_0+\frac{360}{N}$$

By the same logic, this same relation holds between further angles θ_i and θ_{i+1} as long as θ_{i+1} is acute:

$$\theta_{i+1}=\theta_i+\frac{360}{N}$$

For θ_{i+1} > 90 (such as θ_2 and θ_3 in the diagram of an octagon):

$$\phi=\theta_i+180-\theta_{i+1}$$

$$\theta_{i+1}=\theta_i+180-\phi=\theta_i+180-(180-\frac{360}{N})=\theta_i+\frac{360}{N}$$

So whether the angles is acute or not, the length of the ith line segment is given by this equation:

$$L_i=L\sin(\frac{180}{N}+i\frac{360}{N})$$

And the product of the line segments is:

$$P=\prod_{i=0}^{i<\frac{N}{2}} L\sin(\frac{180}{N}+i\frac{360}{N})$$

With length 2, as it happens, for every even number of sides N, this product multiplies out to exact 2.

As a check on this answer, I calculated the sum of segment lengths as well as the product. This sum should equal the diameter of the circle formed by the N vertices. This checks out since for all even N:

$$D=\sum_{i=0}^{i<\frac{N}{2}} L\sin(\frac{180}{N}+i\frac{360}{N})=\frac{L}{\sin(\frac{180}{N})}$$

Extra credit:

When N is odd, the product is very similar:

$$P=\prod_{i=0}^{i<integer(\frac{N}{2})} L\sin(\frac{180}{N}+i\frac{360}{N})$$

For N=1001 and L=2, this product is 31.64.

As with the even case, I also calculated the sum of the segments. As expected this sum equals the average of the diameters for the inscribed and circumscribed cirles.