Are you the fittest gym rat?
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[Are You The Fittest Gym Rat? | FiveThirtyEight]

This problem is much simpler in the case where there are many gym members. I set up a system where every member corresponds to number ranging from 0 to N-1, where the higher attending members have higher numbers. The probability a given member is at the gym at a given time is then:

$$p(x)=\frac{x}{N-1}$$

For large N, this is simply x/N:

$$p(x)=\lim_{N\to\infty}\frac{x}{N-1}=\frac{x}{N}$$

Then I calculate the expected number of members at the gym at a given time:

$$E=\int_0^N p(x)\cdot dx=\int_0^N \frac{x}{N}\cdot dx=\left[\frac{x^2}{2N}\right]_0^N=\frac{N}{2}$$

The probability that a given member will be the first person I see is the probability that member is at the gym (x/N) times the probability I will see them first (1/E = 2/N). I add the probabilities for members N/2 to N who have higher probabilities than me of being at the gym (i.e. greater than ½).

$$P=\int_{N/2}^N p(x)\cdot\frac{1}{N/2}dx\int_{N/2}^N p(x)\cdot\frac{2}{N}dx=\int_{N/2}^N\frac{x}{N}\frac{2}{N}\cdot dx=\left[\frac{x^2}{N^2}\right]_{N/2}^N=\frac{N}{2}=1-\frac{1}{4}=\frac{3}{4}$$

As a sanity check, notice that the integral over the range 0 to N equals 1, as it must since somebody must be the first person I see.

To find the probability for smaller numbers of members, I must consider two cases: Even number of members and odd number of members (not including myself). For both cases, the expected number of members present at any given time is still N/2.

For even N, the attendance rate of the N/2 member is greater than 50% (N/2/(N-1)>50%). So the probability the first member I see attends more than me is:

$$P=\sum_{i=N/2}^{N-1}\frac{i}{N-1}\cdot\frac{2}{N}$$

For N=2 to 10, the probabilities are:

$$\frac{2}{1\cdot 2},\frac{10}{3\cdot 4},\frac{24}{5\cdot 6},\frac{44}{7\cdot 8},\frac{70}{9\cdot 10}$$

I searched for the sequence 2,10,24,44,70 in the Online Encyclopedia of Integer Sequences and found it here: [A049450 - OEIS] , “Pentagonal numbers multiplied by 2: a(n) = n*(3*n-1).” “n” in this formula is “N/2.”

Inserting this formula into the equation for the probability, I get:

$$P=\frac{\frac{N}{2}\cdot\left(3\cdot\frac{N}{2}-1\right)}{(N-1)\cdot N}=\frac{3\cdot N-2}{4\cdot N-4}$$

For odd N, attendance percentage for member N/2-1/2 is 50%.

$$p\left(\frac{N}{2}-\frac{1}{2}\right)=\frac{\frac{N}{2}-\frac{1}{2}}{N-1}=\frac{N-1}{2\cdot(N-1)}=\frac{1}{2}$$

So the first member who has greater attendance than me is member N/2+1/2. The probability the first member I see has greater attendance than me is given by this sum:

$$P=\sum_{i=\frac{N}{2}+\frac{1}{2}}^{N-1}\frac{i}{N-1}\cdot\frac{2}{N}$$

For N=3 to 11, the probabilities are:

$$\frac{4}{2\cdot 3},\frac{14}{4\cdot 5},\frac{30}{6\cdot 7},\frac{52}{8\cdot 9},\frac{80}{10\cdot 11}$$

Again I searched the OEIS, this time for sequence 4, 14, 30, 52, 80. I found this result: A049451 - OEIS "Twice second pentagonal number." This sequence has formula: a(n) = n*(3*n+1), where n=N/2-1/2.

Inserting this formula into the equation for the probability, I get:

$$P=\frac{\left(\frac{N}{2}-\frac{1}{2}\right)\cdot\left(3\cdot\left(\frac{N}{2}-\frac{1}{2}\right)+1\right)}{(N-1)\cdot N}=\frac{3\cdot N-1}{4\cdot N}$$

Here's a graph showing the probability jumping between (3N-2)/(4N-4) for even N and (3N-1)/4N for odd N. As expected, the probability converges to 75% as N increases.