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The isothermal MHD shock conditions and a 'new' type of isothermal MHD shocks under (dynamical) MHD energy conservation

Bo-Yuan Liu[1]*
[1] *Department of Physics, Tsinghua University, Beijing, China(PRC)*

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1 WHEN THE EFFECTIVE ENTROPY IS CONSERVED

This section shows how the equations for MHD shock conditions reduce from their general forms in a (sphericall symmetric) general polytropic magneto-fluid (under self-gravity) of Wang & Lou (2008) to those in an isothermal magneto-fluid of the present paper when $\gamma \to 1$.

The self-similar transformation in Wang & Lou (2008) (formulae (7)) is

$$
\begin{aligned}
& r = k^{\frac{1}{2}} t^n x \ , \quad u = k^{\frac{1}{2}} t^{n-1} v \ , \quad \rho = \frac{\alpha}{4\pi G t^2} \ , \\
& p = \frac{k t^{2n-4}}{4\pi G}\beta \ , \quad M = \frac{k^{\frac{3}{4}} t^{3n-2} m}{(3n-2)G} \ , \quad \langle B_t^2 \rangle = \frac{k t^{2n-4}}{G}\omega \ ,
\end{aligned} \tag{1}
$$

where the definitions of u, ρ, p and M are the same with my work and the B_t here is just the $B_\parallel$ in my draft. We further know from the MHD PDEs/ODEs that (equations (11) and (12) in Wang & Lou (2008))

$$
\omega = h\alpha^2 x^2 \ , \quad \beta = \alpha^\gamma m^q \ , \tag{2}
$$

where γ is the polytropic index and $q = 2(n+\gamma-2)/(3n-2)$. Since for the polytropic MHD model in Wang & Lou (2008), the effective entropy is conserved, which is only possible in our isothermal MHD model with $\gamma \to 1$. Therefore, as $\gamma \to 1$, $n \to 1$, $q \to 0$, $k = a^2$, and $h = \lambda$, the general polytropic MHD model just reduces to our isothermal MHD model, where a is the isothermal speed of sound and λ is the dimensionless parameter that characterises the strength of the magnetic field in our isothermal MHD model.

The shock conditions expressed by physical quantities are (expressions (17)-(20) in Wang & Lou (2008)[1])

$$
[\rho\,(u_s - u)]_1^2 = 0 \ , \tag{3}
$$

$$
\left[p + \rho\,(u_s - u)^2 + \frac{\langle B_t^2 \rangle}{8\pi}\right]_1^2 = 0 \ , \tag{4}
$$

$$
\left[\frac{\rho\,(u_s - u)^3}{2} + \frac{\gamma p\,(u_s - u)}{\gamma - 1} + \frac{\langle B_t^2 \rangle}{4\pi}\,(u_s - u)\right]_1^2 = 0 \ , \tag{5}
$$

$$
\left[(u_s - u)^2 \langle B_t^2 \rangle\right]_1^2 = 0 \ , \tag{6}
$$

of which euqation (5) for (dynamical) MHD energy conservation[2] is not well formulated in the isothermal model, since the second term $\gamma p\,(u_s - u)/(\gamma - 1)$ diverges as $\gamma \to 1$, and there is no relevant equation for MHD/HD energy conservation in the isothermal models of Lou & Shi (2014), Yu et al. (2006) and my work. So what is the condition of MHD/HD energy conservation in the isothermal MHD/HD model? The following derivations will show that in the isothermal MHD model with $\gamma \to 1$, MHD energy is conserved (dynamically) only when both sides of the shock have the same temperature.

By formulae (1) and (2), from equations (3)-(6), we obtain the shock conditions in self-similar variables (formulae (64) in Wang & Lou (2008)):

$$
\alpha_1 \Gamma_1 = \alpha_2 \Gamma_2 \ , \tag{7}
$$

$$
\alpha_1^{2-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + \alpha_1 \Gamma_1^2 + \frac{h\alpha_1^2}{2} = left(\alpha_2, x_2, \Gamma_2) \ , \tag{8}
$$

$$
\Gamma_1^2 + \frac{2\gamma}{\gamma - 1}\alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + 2h\alpha_1 = left(\alpha_2, x_2, \Gamma_2) \ , \tag{9}
$$

[1] It here uses a pair of square brackets outside each expression enclosed to denote the difference between the upstream (marked by sub-script '1') and downstream (marked by subscript '2') quantities, as has been done conventionally for shock analyses.
[2] Actually, equation (5) is based on the presupposition that gas around the shock layer experience an 'isentropic' process, i.e. $\partial s/\partial t + \mathbf{u} \cdot \nabla \mathbf{s}$, in which there is no heat flux, and that is why we use the qualifier 'dynamical'.

* E-mail: liu-by13@mails.tsinghua.edu.cn

where $left(\alpha_2, x_2, \Gamma_2)$ denotes the formula on the left side of the equal sign with α_1, x_1 and Γ_1 replaced by α_2, x_2 and Γ_2, $\Gamma_i = n - v_i/x_i$, and $\tau = \sqrt{k_2/k_1} = x_1/x_2$, since $u_s = dr_s/dt = nk_i^{1/2}t^{n-1}x_i$ $(i = 1, 2)$. And it is necessay to point out that equations (7)-(9) correspond to equations (3)-(5), respectively, while equation (6) has been combined with equation (3) to give $h_1 = h_2 = h$. We can further eliminate α_2 in equations (8) and (9) by equation (7) to give (equations (65) in Wang & Lou (2008))

$$\alpha_1^{2-n+\frac{3nq}{2}} x_1^{3q-2}\Gamma_1^q + \alpha_1\Gamma_1^2 + \frac{h\alpha_1^2}{2}$$
$$= \frac{(\alpha_1\Gamma_1)^{2-n+\frac{3nq}{2}}}{\Gamma_2^{2-n+\frac{(3n-2)q}{2}}} x_2^{3q-2} + \alpha_1\Gamma_1\Gamma_2 + \frac{h\alpha_1^2\Gamma_1^2}{2\Gamma_2^2} \ , \quad (10)$$
$$\frac{2\gamma}{(\gamma-1)}\alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2}\Gamma_1^q + \Gamma_1^2 + 2h\alpha_1$$
$$= \frac{2\gamma}{(\gamma-1)} \frac{(\alpha_1\Gamma_1)^{1-n+\frac{3nq}{2}}}{\Gamma_2^{1-n+\frac{(3n-2)q}{2}}} x_2^{3q-2} + \Gamma_2^2 + 2h\frac{\alpha_1\Gamma_1}{\Gamma_2} \ , \quad (11)$$

of which equation (11) can be regarded to embody (dynamical) MHD energy conservation since it is directly related to equation (5) which, as mentioned above, cannot be included in the $\gamma \to 1$ isothermal model in a straight forward manner, while equation (6) corresponds to momentum conservation.

Now, recall the cubic equation (40) and the quadric equation (34) (whose real solutions correspond to possible shocks) of my draft for two-temperature and one-temperature isothermal MHD shock conditions

$$X^3 + \tau^2X^2\beta_1 - \left(1 + \beta_1 + \beta_1M_1^2\right)X + M_1^2\beta_1 = 0 \ , \quad (12)$$
$$X^2 + (\beta_1 + 1)X - \beta_1M_1^2 = 0 \ , \quad (13)$$

where

$$\tau = a_j/a_i = x_{si}/x_{sj} \ , \ M_1 = v_i - x_{si} \ , \ \beta_1 = 2/(\lambda x_{si}^2\alpha_i) \ ,$$
$$X = \alpha_j/\alpha_i = (v_i - x_{si})/[\tau(v_j - x_{sj})] \ . \quad (14)$$

Our goal is to find the relations among equations (10)-(11) and equations (12)-(13). For the relations among the variables of these two groups of equations, when we set $(i, j) = (1, 2)$[3] in formulae (14), as $\gamma \to 1$, $n \to 1$, $q \to 0$, $k = a^2$, and $h = \lambda$, it turns out that

$$\beta_1 = \frac{2}{hx_1^2\alpha_1} \ , \quad M_1 = -x_1\Gamma_1 \ , \quad X = \frac{\Gamma_1}{\Gamma_2} \ . \quad (15)$$

Then through simple algebra and equations (15), the reduced (isothermal) version of equation (10)

$$\alpha_1x_1^{-2} + \alpha_1\Gamma_1^2 + \frac{h\alpha_1^2}{2}$$
$$-\frac{\alpha_1\Gamma_1}{\Gamma_2}\left(\frac{x_1}{\tau}\right)^{-2} - \alpha_1\Gamma_1\Gamma_2 - \frac{h\alpha_1^2\Gamma_1^2}{2\Gamma_2^2} = 0 \quad (16)$$

can be easily put into the form

$$\alpha_1x_1^{-2}\{1 + (x_1\Gamma_1)^2 + \frac{h\alpha_1x_1^2}{2}$$
$$-\tau^2\frac{\Gamma_1}{\Gamma_2} - (x_1\Gamma_1)^2\frac{\Gamma_2}{\Gamma_1} - \frac{h\alpha_1x_1^2}{2}\left(\frac{\Gamma_1}{\Gamma_2}\right)^2\} = 0 \ , i.e.$$
$$1 + M_1^2 + \frac{1}{\beta_1} - \tau^2X - \frac{M_1^2}{X} - \frac{1}{\beta_1}X^2 = 0 \ , \quad (17)$$

which is exactly equation (12) for two-temperature isothermal MHD shock conditions divided by $-X\beta_1$. **Here we do not consider equation (11) for (dynamical) MHD energy conservation at all. So the conclusion is that for two-temperature isothermal MHD shocks, the MHD energy does not conserve.** Then what about one-temperature MHD shocks?

Actually, we can eliminate x_2 in equations (10) and (11) to give (equations (66) and (67) in Wang & Lou (2008))

$$\{\frac{(\gamma+1)}{2\gamma}\Gamma_2^2 - \left(\alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2}\Gamma_1^{q-1} + \frac{\gamma-1}{2\gamma}\Gamma_1 + \frac{h\alpha_1}{2\Gamma_1}\right)\Gamma_2$$
$$-\frac{2-\gamma}{2\gamma}h\alpha_1\}(\Gamma_2 - \Gamma_1) = 0 \ , \quad (18)$$

which can be reduced to the isothermal version

$$\left[\Gamma_2^2 - \left(x_1^{-2}\Gamma_1^{-1} + \frac{h\alpha_1}{2\Gamma_1}\right)\Gamma_2 - \frac{h\alpha_1}{2}\right](\Gamma_2 - \Gamma_1) = 0 \ . \quad (19)$$

By equations (15), we can write this equation (19) as

$$\left[(\Gamma_1x_1)^2\left(\frac{\Gamma_2}{\Gamma_1}\right)^2 - \left(1 + \frac{h\alpha_1x_1^2}{2}\right)\frac{\Gamma_2}{\Gamma_1} - \frac{h\alpha_1x_1^2}{2}\right]$$
$$\times\left(1 - \frac{\Gamma_1}{\Gamma_2}\right)\Gamma_2x_1^{-2} = 0 \ , i.e.$$
$$\left[M_1^2\frac{1}{X^2} - \left(1 + \frac{1}{\beta_1}\right)\frac{1}{X} - \frac{1}{\beta_1}\right](1 - X) = 0 \ , \quad (20)$$

which is exactly equation (13) multiplied by a factor $(X - 1)/\left(X^2\beta_1\right)$, and we can throw away the factor $(1 - X)$, since $X = 1$ gives trivial results. This outcome indicates that **if we take into account (dynamical) MHD energy conservation by combining equations (10) and (11) to attain equation (18), the shock condition in the polytropic model will reduce to the one-temperature shock condition in the isothermal model under $\gamma \to$ 1.** We may further conclude that equation (5) for (dynamical) MHD energy conservation in the polytropic model is consistent with the very restriction that $a_1 = a_2$ (i.e. $(k_BT_u/\mu)^{1/2} = a_u = a_d = (k_BT_d/\mu)^{1/2}$, $T_u = T_d$: both sides of the shock have the same temperature) in the $\gamma \to 1$ isothermal model.

2 WHEN THE EFFECTIVE ENTROPY IS NOT CONSERVED: A 'NEW' ISOTHERMAL MHD MODEL

The problem of non-conservation of (dynamical)[4] MHD energy comes from the condition that $\gamma \to 1$ for an isothermal gas. So why shall $\gamma \to 1$ for an isothermal gas, if we regard

[3] To be consistent, we also have $x_{s1} = x_1$ and $x_{s2} = x_2$.

[4] The qualifier 'dynamical' here denotes that there is no heat flux.

it as the ratio of the isobaric heat capacity and the isovolumetric heat capacity which can be larger than 1 (e.g. 4/3 for the relativistic perfect gas and 5/3 for the non-relativistic perfect gas)? The answer is to conserve the effective entropy.

In the polytropic model mentioned above, the equation of effective entropy conservation (equation (6) in Wang & Lou (2008)) is

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\left(\ln\frac{p}{\rho^{\gamma}}\right)=0\ , \tag{21}$$

from which and other MHD ODEs, we can obtain the relation $\beta=\alpha^{\gamma}m^{q}$ (see equations (2)). If meanwhile the gas is isothermal, i.e. $p=a^2\rho$, the self-similar transformation (1) reduces to that (expressions (6) in my work)[5]

$$\begin{aligned}\rho(r,t)&=\frac{\alpha(x)}{4\pi Gt^2}, & M(r,t)&=\frac{a^3t}{G}m(x)\ ,\\ u(r,t)&=av(x), & B_{||}(r,\ t)&=\frac{ab(x)}{\sqrt{G}t}\ .\end{aligned} \tag{22}$$

Therefore, we must have $n=1$, $k=A^2=\gamma a^2$ (, where A is the adiabatic sound speed), $h=\lambda$, $\omega=b$, and $\alpha^{\gamma}m^{q}=\alpha/\gamma$, which means that $\gamma\to 1$ and $q=0$, then we arrive at the case of the last section that the effective entropy is conserved.

Actually, in my work as well as Lou & Shi (2014), Yu & Lou (2005) and Yu et al. (2006), equation (21) does not show up, but the effective entropy is still conserved when $\gamma\to 1$ **for the model itself**, since $s=\ln\left(p/\rho^{\gamma}\right)=2\ln a=$ *const*.. But for two-temperature shocks, because $a_u\neq a_d$, the effective entropy varies across the shock front.

If we discard the conservation of the effective entropy $\ln\left(p/\rho^{\gamma}\right)$ **for the model itself** and regard $\gamma>1$ as an adjustable parameter of the model, equation (5) is no longer badly formulated, and we can include it into our model to derive the isothermal MHD shock condition that conserves MHD energy (dynamically), but this leads to a completely different type of isothermmal shocks, which is beyond the range of our present paper . By the self-similar transformation (22), we can wirte equation (5) for (dynamical) MHD energy conservation in self-similar variables

$$\begin{aligned}&a_i^2\left[\frac{\left(v_i-x_{si}\right)^2}{2}+\frac{\gamma}{\gamma-1}+\lambda\alpha_i x_{si}^2\right]\\ &\quad=a_j^2\left[\frac{\left(v_j-x_{sj}\right)^2}{2}+\frac{\gamma}{\gamma-1}+\lambda\alpha_j x_{sj}^2\right]\ ,\end{aligned} \tag{23}$$

where all the variables involved follow the same definitions of my draft and we have used equation (3) to simplify the formula. This equation is shown by formulae (14) to be equivalent to

$$X^3+\left[\frac{\left(\tau^2-1\right)\gamma}{2(\gamma-1)}\beta_1-1-\frac{M_1^2\beta_1}{4}\right]X^2+\frac{M_1^2\beta_1}{4}=0\ , \tag{24}$$

which can be combined with equation (12) for momentum conservation to eliminate τ and give an equation for the complete isothermal shock condition with (dynamical) MHD energy conservation, that is independent of the strength of the shock τ, which is similar to what has been done in Wang & Lou (2008) for the polytropic model. Finally we obtain a quadratic equation

$$\begin{aligned}f(X)=&\frac{2-\gamma}{(\gamma-1)\beta_1}X^2+\left[\frac{M_1^2}{2}+\frac{\gamma}{\gamma-1}\left(1+\frac{1}{\beta_1}\right)\right]X\\ &-\frac{(\gamma+1)M_1^2}{2(\gamma-1)}=0\ ,\end{aligned} \tag{25}$$

for which we have thrown away the trivial factor $X-1$. When $1<\gamma<2$, the coefficient of the X^2 term $(2-\gamma)/\left[(\gamma-1)\beta_1\right]$ is positive, and we have

$$f(X=0)=-\frac{(\gamma+1)M_1^2}{2(\gamma-1)}<0\ , \tag{26}$$

$$f(X=1)=\frac{1}{\gamma-1}\left(\gamma+\frac{2}{\beta_1}-M_1^2\right)\ . \tag{27}$$

Therefore, quadratic equation (25) always has one positive root X_+ and one negative root X_-, and only the positive one is physical. When the upstream is given, i.e. $(i,\ j)=(u,\ d)$, we require that $X_+>1$, which means that $u_u-u_s>u_d-u_s$, where u_u-u_s and u_d-u_s are the upstream velocity and the downstream flow velocity relative to the shock front (Landau & Lifshitz 1987), so we must have

$$f(X=1)>0\ ,\textit{i.e.}\quad \gamma+\frac{2}{\beta_1}-M_1^2<0\ , \tag{28}$$

which exactly corresponds to the physical requirement that the upstream flow speed relative to the shock speed must exceed the upstream fast magneto-sonic speed, i.e. $a_u^2\left(v_u-x_{su}\right)^2>A_u^2+v_{Au}^2$, where $v_{Au}=\sqrt{\lambda\alpha_u}\cdot x_{su}a_u$, and $A_u=a_u\gamma^{1/2}$ is the upstream adiabatic sound speed. Once attain the positive root $X=X_+>1$, we can calculate τ (which is no longer an adjustable parameter now) by

$$\tau^2=\frac{X-1}{X}\left(\frac{M_1^2}{X}-\frac{1+X}{\beta_1}\right)+\frac{1}{X}\ , \tag{29}$$

the right side of which must be larger than 1 due to the second law of thermodynamics, given the upstream.

When the properties of the downstream are given, i.e. $(i,\ j)=(d,\ u)$, we require that $X=X_+<1$, and $\tau<1$, thus, $f(X=1)>0$, and the right hand side of formula (29) should be positive and smaller than 1. We can further prove that $a_u^2\left(v_u-x_{su}\right)^2>A_u^2+v_{Au}^2$ is still valid, which is now equivalent to $\gamma+2/\beta_2<M_2^2$, where the definitions of M_2 and β_2 are similar to those of M_1 and β_1, i.e. $M_2\equiv u_2/a_2=M_1/(X\tau)$ and $\beta_2\equiv 8\pi p_2/\left(B_2^2\right)=\beta_1\tau^2/X$. By these definitions and the expression of τ^2 (29), we have

$$\begin{aligned}&\left(\gamma+\frac{2}{\beta_2}-M_2^2\right)\tau^2=\tau^2\gamma+\frac{2X}{\beta_1}-\frac{M_1^2}{X^2}\\ &\quad=\left[\frac{\gamma\left(X-1\right)-1}{X^2}\right]M_1^2+\frac{1}{\beta_1}\left[(2-\gamma)X+\frac{\gamma}{X}\right]+\frac{\gamma}{X}\ .\end{aligned} \tag{30}$$

Since $X<1$, and $1<\gamma<2$, from equation (25), we note down an inequality

$$\begin{aligned}&\frac{(\gamma+1)M_1^2}{2X^2}-\frac{(\gamma-1)M_1^2}{2X}-\frac{\gamma}{X}\\ &\quad=\left[(2-\gamma)+\frac{\gamma}{X}\right]\frac{1}{\beta_1}>\left[(2-\gamma)X+\frac{\gamma}{X}\right]\frac{1}{\beta_1}\ ,\end{aligned} \tag{31}$$

which is substituted to formula (30) to eliminate the second

[5] $r=atx$.

term of the right side and gives

$$\left(\gamma+\frac{2}{\beta_2}-M_2^2\right)\tau^2<\frac{(\gamma+1)(X-1)M_1^2}{2X^2}<0\ ,i.e.$$
$$\gamma+\frac{2}{\beta_2}-M_2^2<0\ . \tag{32}$$

The above analysis implies that equation (25) is the correct equation for the isothermal MHD shock condition with (dynamical) MHD energy conservation, from which we obtained the prefered 'one to one' relation between the upstream and the downstream.

Besides, if we have $\gamma\to 1$, equation (25) can be put into the form

$$(\gamma-1)f(X)\beta_1$$
$$=(2-\gamma)X^2+\left[\frac{(\gamma-1)M_1^2\beta_1}{2}+\gamma\left(1+\beta_1\right)\right]X$$
$$-\frac{(\gamma+1)\beta_1M_1^2}{2}=0\ ,i.e.$$
$$X^2+\left(1+\beta_1\right)X-\beta_1M_1^2=0\ , \tag{33}$$

which is exactly the quadratic equation (13) for the one-temperature MHD shock condition. By this equation and equation (12), we can further show that

$$\frac{X^2-1}{\beta_1}+X\tau^2-1+\frac{1-X}{X}M_1^2$$
$$=\frac{X^2-1}{\beta_1}+X\tau^2-1+(1-X)+\frac{1-X^2}{\beta_1}$$
$$=X\left(\tau^2-1\right)=0\ ,i.e.$$
$$\tau=1\ , \tag{34}$$

where the first line is just the left hand side of equation (13) divided by β_1X, and for the second line we have used equation (33) multiplied by $(1/X-1)/\beta_1$. This outcome supports our argument that (dynamical) MHD energy conservation for isotherm MHD shocks under $\gamma\to 1$ is consistent with the very restriction that $a_1=a_2$ (i.e. $(k_BT_u/\mu)^{1/2}=a_u=a_d=(k_BT_d/\mu)^{1/2}$, $T_u=T_d$: both sides of the shock have the same temperature).

I have already written and tested the codes for this new type of isothermal MHD shocks under (dynamical) MHD energy conservation. By now, I have verified that the shock condition (25) is indeed reversible, but it would be more difficult to construct relevant shock solutions, in that τ is not adjustable here.

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