

# The isothermal MHD shock conditions under (dynamical) isentropic/isothermal MHD energy conservation

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## 1 WHEN THE EFFECTIVE ENTROPY IS CONSERVED

This section shows how the equations for MHD shock conditions reduce from their general forms in a (spherically symmetric) general polytropic magneto-fluid (under self-gravity) of Wang & Lou (2008) to those in an isothermal magneto-fluid of the present paper when  $\gamma \rightarrow 1$ .

The self-similar transformation in Wang & Lou (2008) (formulae (7)) is

$$\begin{aligned} r &= k^{\frac{1}{2}} t^n x, \quad u = k^{\frac{1}{2}} t^{n-1} v, \quad \rho = \frac{\alpha}{4\pi G t^2}, \\ p &= \frac{k t^{2n-4}}{4\pi G} \beta, \quad M = \frac{k^{\frac{3}{2}} t^{3n-2} m}{(3n-2)G}, \quad \langle B_t^2 \rangle = \frac{k t^{2n-4}}{G} \omega, \end{aligned} \quad (1)$$

where the definitions of  $u$ ,  $\rho$ ,  $p$  and  $M$  are the same with my work and the  $B_t$  here is just the  $B_{\parallel}$  in my draft. We further know from the MHD PDEs/ODEs that (equations (11) and (12) in Wang & Lou (2008))

$$\omega = h \alpha^2 x^2, \quad \beta = \alpha^\gamma m^q, \quad (2)$$

where  $\gamma$  is the polytropic index and  $q = 2(n+\gamma-2)/(3n-2)$ . Since for the polytropic MHD model in Wang & Lou (2008), the effective entropy is conserved, which is only possible in our isothermal MHD model with  $\gamma \rightarrow 1$ . Therefore, as  $\gamma \rightarrow 1$ ,  $n \rightarrow 1$ ,  $q \rightarrow 0$ ,  $k = a^2$ , and  $h = \lambda$ , the general polytropic MHD model just reduces to our isothermal MHD model, where  $a$  is the isothermal speed of sound and  $\lambda$  is the dimensionless parameter that characterises the strength of the magnetic field in our isothermal MHD model.

The shock conditions expressed by physical quantities

are (expressions (17)-(20) in Wang & Lou (2008)<sup>1</sup>)

$$[\rho(u_s - u)]_u^d = 0, \quad (3)$$

$$\left[ p + \rho(u_s - u)^2 + \frac{\langle B_t^2 \rangle}{8\pi} \right]_u^d = 0, \quad (4)$$

$$\left[ \frac{\rho(u_s - u)^3}{2} + \frac{\gamma p(u_s - u)}{\gamma - 1} + \frac{\langle B_t^2 \rangle}{4\pi}(u_s - u) \right]_u^d = 0, \quad (5)$$

$$[(u_s - u)^2 \langle B_t^2 \rangle]_u^d = 0, \quad (6)$$

of which equation (5) for (dynamical) MHD energy conservation<sup>2</sup> is not well formulated in the isothermal model, since the second term  $\gamma p(u_s - u)/(\gamma - 1)$  diverges as  $\gamma \rightarrow 1$ , and there is no relevant equation for MHD/HD energy conservation in the isothermal models of Lou & Shi (2014), Yu et al. (2006) and my work. So what is the condition of MHD/HD energy conservation in the isothermal MHD/HD model? The following derivations will show that in the isothermal MHD model with  $\gamma \rightarrow 1$ , MHD energy is conserved (dynamically) only when both sides of the shock have the same temperature.

By formulae (1) and (2), from equations (3)-(6), we obtain the shock conditions in self-similar variables (formulae (64) in Wang & Lou (2008)):

$$\alpha_1 \Gamma_1 = \alpha_2 \Gamma_2, \quad (7)$$

$$\alpha_1^{2-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + \alpha_1 \Gamma_1^2 + \frac{h \alpha_1^2}{2} = \text{left}(\alpha_2, x_2, \Gamma_2), \quad (8)$$

$$\Gamma_1^2 + \frac{2\gamma}{\gamma - 1} \alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + 2h \alpha_1 = \text{left}(\alpha_2, x_2, \Gamma_2), \quad (9)$$

where  $\text{left}(\alpha_2, x_2, \Gamma_2)$  denotes the formula on the left side

<sup>1</sup> It here uses a pair of square brackets outside each expression enclosed to denote the difference between the upstream (marked by sub-script ‘u’) and downstream (marked by subscript ‘d’) quantities, as has been done conventionally for shock analyses.

<sup>2</sup> Actually, equation (5) is based on the presupposition that gas around the shock layer experience an ‘isentropic’ process, i.e.  $\partial s/\partial t + \mathbf{u} \cdot \nabla \mathbf{s}$ , in which there is no heat flux, and that is why we use the qualifier ‘dynamical’.

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of the equal sign with  $\alpha_1$ ,  $x_1$  and  $\Gamma_1$  replaced by  $\alpha_2$ ,  $x_2$  and  $\Gamma_2$ ,  $\Gamma_i = n - v_i/x_i$ , and  $\tau = \sqrt{k_2/k_1} = x_1/x_2$ , since  $u_s = dr_s/dt = nk_i^{1/2}t^{n-1}x_i$  ( $i = 1, 2$ ). And it is necessary to point out that equations (7)-(9) correspond to equations (3)-(5), respectively, while equation (6) has been combined with equation (3) to give  $h_1 = h_2 = h$ . We can further eliminate  $\alpha_2$  in equations (8) and (9) by equation (7) to give equations (65) in Wang & Lou (2008))

$$\alpha_1^{2-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + \alpha_1 \Gamma_1^2 + \frac{h\alpha_1^2}{2} = \frac{(\alpha_1 \Gamma_1)^{2-n+\frac{3nq}{2}} x_2^{3q-2} + \alpha_1 \Gamma_1 \Gamma_2 + \frac{h\alpha_1^2 \Gamma_1^2}{2\Gamma_2^2}}{\Gamma_2^{2-n+\frac{(3n-2)q}{2}}}, \quad (10)$$

$$\frac{2\gamma}{(\gamma-1)} \alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^q + \Gamma_1^2 + 2h\alpha_1 = \frac{2\gamma}{(\gamma-1)} \frac{(\alpha_1 \Gamma_1)^{1-n+\frac{3nq}{2}} x_2^{3q-2} + \Gamma_2^2 + 2h\frac{\alpha_1 \Gamma_1}{\Gamma_2}}{\Gamma_2^{1-n+\frac{(3n-2)q}{2}}}, \quad (11)$$

of which equation (11) can be regarded to embody (dynamical) MHD energy conservation since it is directly related to equation (5) which, as mentioned above, cannot be included in the  $\gamma \rightarrow 1$  isothermal model in a straight forward manner, while equation (6) corresponds to momentum conservation.

Now, recall the cubic equation (40) and the quadric equation (34) (whose real solutions correspond to possible shocks) of my draft for two-temperature and one-temperature isothermal MHD shock conditions

$$X^3 + \tau^2 X^2 \beta_1 - (1 + \beta_1 + \beta_1 M_1^2) X + M_1^2 \beta_1 = 0, \quad (12)$$

$$X^2 + (\beta_1 + 1) X - \beta_1 M_1^2 = 0, \quad (13)$$

where

$$\tau = a_j/a_i = x_{si}/x_{sj}, \quad M_1 = v_i - x_{si}, \quad \beta_1 = 2/(\lambda x_{si}^2 \alpha_i), \quad X = \alpha_j/\alpha_i = (v_i - x_{si})/[\tau(v_j - x_{sj})]. \quad (14)$$

Our goal is to find the relations among equations (10)-(11) and equations (12)-(13). For the relations among the variables of these two groups of equations, when we set  $(i, j) = (1, 2)$ <sup>3</sup> in formulae (14), as  $\gamma \rightarrow 1$ ,  $n \rightarrow 1$ ,  $q \rightarrow 0$ ,  $k = a^2$ , and  $h = \lambda$ , it turns out that

$$\beta_1 = \frac{2}{hx_1^2 \alpha_1}, \quad M_1 = -x_1 \Gamma_1, \quad X = \frac{\Gamma_1}{\Gamma_2}. \quad (15)$$

Then through simple algebra and equations (15), the reduced (isothermal) version of equation (10)

$$\alpha_1 x_1^{-2} + \alpha_1 \Gamma_1^2 + \frac{h\alpha_1^2}{2} - \frac{\alpha_1 \Gamma_1}{\Gamma_2} \left(\frac{x_1}{\tau}\right)^{-2} - \alpha_1 \Gamma_1 \Gamma_2 - \frac{h\alpha_1^2 \Gamma_1^2}{2\Gamma_2^2} = 0 \quad (16)$$

can be easily put into the form

$$\alpha_1 x_1^{-2} \{1 + (x_1 \Gamma_1)^2 + \frac{h\alpha_1 x_1^2}{2} - \tau^2 \frac{\Gamma_1}{\Gamma_2} - (x_1 \Gamma_1)^2 \frac{\Gamma_2}{\Gamma_1} - \frac{h\alpha_1 x_1^2}{2} \left(\frac{\Gamma_1}{\Gamma_2}\right)^2\} = 0, \quad i.e. \quad 1 + M_1^2 + \frac{1}{\beta_1} - \tau^2 X - \frac{M_1^2}{X} - \frac{1}{\beta_1} X^2 = 0, \quad (17)$$

which is exactly equation (12) for two-temperature isothermal MHD shock conditions divided by  $-X\beta_1$ . **Here we do not consider equation (11) for (dynamical) MHD energy conservation at all. So the conclusion is that for two-temperature isothermal MHD shocks, the MHD energy does not conserve.** Then what about one-temperature MHD shocks?

Actually, we can eliminate  $x_2$  in equations (10) and (11) to give equations (66) and (67) in Wang & Lou (2008))

$$\left\{ \frac{(\gamma+1)}{2\gamma} \Gamma_2^2 - \left( \alpha_1^{1-n+\frac{3nq}{2}} x_1^{3q-2} \Gamma_1^{q-1} + \frac{\gamma-1}{2\gamma} \Gamma_1 + \frac{h\alpha_1}{2\Gamma_1} \right) \Gamma_2 - \frac{2-\gamma}{2\gamma} h\alpha_1 \right\} (\Gamma_2 - \Gamma_1) = 0, \quad (18)$$

which can be reduced to the isothermal version

$$\left[ \Gamma_2^2 - \left( x_1^{-2} \Gamma_1^{-1} + \frac{h\alpha_1}{2\Gamma_1} \right) \Gamma_2 - \frac{h\alpha_1}{2} \right] (\Gamma_2 - \Gamma_1) = 0. \quad (19)$$

By equations (15), we can write this equation (19) as

$$\left[ (\Gamma_1 x_1)^2 \left( \frac{\Gamma_2}{\Gamma_1} \right)^2 - \left( 1 + \frac{h\alpha_1 x_1^2}{2} \right) \frac{\Gamma_2}{\Gamma_1} - \frac{h\alpha_1 x_1^2}{2} \right] \times \left( 1 - \frac{\Gamma_1}{\Gamma_2} \right) \Gamma_2 x_1^{-2} = 0, \quad i.e. \quad \left[ M_1^2 \frac{1}{X^2} - \left( 1 + \frac{1}{\beta_1} \right) \frac{1}{X} - \frac{1}{\beta_1} \right] (1 - X) = 0, \quad (20)$$

which is exactly equation (13) multiplied by a factor  $(X - 1)/(X^2 \beta_1)$ , and we can throw away the factor  $(1 - X)$ , since  $X = 1$  gives trivial results. This outcome indicates that **if we take into account (dynamical) MHD energy conservation by combining equations (10) and (11) to attain equation (18), the shock condition in the polytropic model will reduce to the one-temperature shock condition in the isothermal model under  $\gamma \rightarrow 1$ .** We may further conclude that equation (5) for (dynamical) MHD energy conservation in the polytropic model is consistent with the very restriction that  $a_1 = a_2$  (i.e.  $(k_B T_u/\mu)^{1/2} = a_u = a_d = (k_B T_d/\mu)^{1/2}$ ,  $T_u = T_d$ : both sides of the shock have the same temperature) in the  $\gamma \rightarrow 1$  isothermal model.

## 2 WHEN THE EFFECTIVE ENTROPY IS NOT CONSERVED: A ‘NEW’ ISOTHERMAL MHD MODEL

The problem of non-conservation of (dynamical)<sup>4</sup> MHD energy comes from the condition that  $\gamma \rightarrow 1$  for an isothermal gas. So why shall  $\gamma \rightarrow 1$  for an isothermal gas, if we regard it as the ratio of the isobaric heat capacity and the isovolumetric heat capacity which can be larger than 1 (e.g. 4/3 for the relativistic perfect gas and 5/3 for the non-relativistic perfect gas)? The answer is to conserve the effective entropy.

In the polytropic model mentioned above, the equation of effective entropy conservation (equation (6) in Wang & Lou (2008)) is

$$\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) \left( \ln \frac{p}{\rho^\gamma} \right) = 0, \quad (21)$$

<sup>4</sup> The qualifier ‘dynamical’ here denotes that there is no heat flux.

<sup>3</sup> To be consistent, we also have  $x_{s1} = x_1$  and  $x_{s2} = x_2$ .

from which and other MHD ODEs, we can obtain the relation  $\beta = \alpha^\gamma m^q$  (see equations (2)). If meanwhile the gas is isothermal, i.e.  $p = a^2 \rho$ , the self-similar transformation (1) reduces to that (expressions (6) in my work)<sup>5</sup>

$$\begin{aligned} \rho(r, t) &= \frac{\alpha(x)}{4\pi G t^2}, & M(r, t) &= \frac{a^3 t}{G} m(x), \\ u(r, t) &= a v(x), & B_{||}(r, t) &= \frac{ab(x)}{\sqrt{G} t}. \end{aligned} \quad (22)$$

Therefore, we must have  $n = 1$ ,  $k = A^2 = \gamma a^2$  (, where  $A$  is the adiabatic sound speed),  $h = \lambda$ ,  $\omega = b$ , and  $\alpha^\gamma m^q = \alpha/\gamma$ , which means that  $\gamma \rightarrow 1$  and  $q = 0$ , then we arrive at the case of the last section that the effective entropy is conserved.

Actually, in my work as well as Lou & Shi (2014), Yu & Lou (2005) and Yu et al. (2006), equation (21) does not show up, but the effective entropy is still conserved when  $\gamma \rightarrow 1$  **for the model itself**, since  $s = \ln(p/\rho^\gamma) = 2\ln a = \text{const.}$ . But for two-temperature shocks, because  $a_u \neq a_d$ , the effective entropy varies across the shock front.

If we discard the conservation of the effective entropy  $\ln(p/\rho^\gamma)$  **for the model itself** and regard  $\gamma > 1$  as an adjustable parameter of the model, equation (5) is no longer badly formulated, and we can include it into our model to derive the isothermal MHD shock condition that conserves MHD energy (dynamically), but this leads to a completely different type of isothermmal shocks, which is beyond the range of our present paper. By the self-similar transformation (22), we can write equation (5) for (dynamical) MHD energy conservation in self-similar variables

$$\begin{aligned} a_i^2 \left[ \frac{(v_i - x_{si})^2}{2} + \frac{\gamma}{\gamma - 1} + \lambda \alpha_i x_{si}^2 \right] \\ = a_j^2 \left[ \frac{(v_j - x_{sj})^2}{2} + \frac{\gamma}{\gamma - 1} + \lambda \alpha_j x_{sj}^2 \right], \end{aligned} \quad (23)$$

where all the variables involved follow the same definitions of my draft and we have used equation (3) to simplify the formula. This equation is shown by formulae (14) to be equivalent to

$$X^3 + \left[ \frac{(\tau^2 - 1)\gamma}{2(\gamma - 1)} \beta_1 - 1 - \frac{M_1^2 \beta_1}{4} \right] X^2 + \frac{M_1^2 \beta_1}{4} = 0, \quad (24)$$

which can be combined with equation (12) for momentum conservation to eliminate  $\tau$  and give an equation for the complete isothermal shock condition with (dynamical) MHD energy conservation, that is independent of the strength of the shock  $\tau$ , which is similar to what has been done in Wang & Lou (2008) for the polytropic model. Finally we obtain a quadratic equation

$$\begin{aligned} f(X) &= \frac{2 - \gamma}{(\gamma - 1)\beta_1} X^2 + \left[ \frac{M_1^2}{2} + \frac{\gamma}{\gamma - 1} \left( 1 + \frac{1}{\beta_1} \right) \right] X \\ &\quad - \frac{(\gamma + 1)M_1^2}{2(\gamma - 1)} = 0, \end{aligned} \quad (25)$$

for which we have thrown away the trivial factor  $X - 1$ . When  $1 < \gamma < 2$ , the coefficient of the  $X^2$  term  $(2 - \gamma)/[(\gamma - 1)\beta_1]$

is positive, and we have

$$f(X = 0) = -\frac{(\gamma + 1)M_1^2}{2(\gamma - 1)} < 0, \quad (26)$$

$$f(X = 1) = \frac{1}{\gamma - 1} \left( \gamma + \frac{2}{\beta_1} - M_1^2 \right). \quad (27)$$

Therefore, quadratic equation (25) always has one positive root  $X_+$  and one negative root  $X_-$ , and only the positive one is physical. When the upstream is given, i.e.  $(i, j) = (u, d)$ , we require that  $X_+ > 1$ , which means that  $|u_u - u_s| > |u_d - u_s|$ , where  $|u_u - u_s|$  and  $|u_d - u_s|$  are the upstream flow speed and the downstream flow speed relative to the shock front (Landau & Lifshitz 1987), so we must have

$$f(X = 1) > 0, \text{ i.e. } \gamma + \frac{2}{\beta_1} - M_1^2 > 0, \quad (28)$$

which exactly corresponds to the physical requirement that the upstream flow speed relative to the shock speed must exceed the upstream fast magneto-sonic speed, i.e.  $a_u^2 (v_u - x_{su})^2 > A_u^2 + v_{Au}^2$ , where  $v_{Au} = \sqrt{\lambda \alpha_u} \cdot x_{su} a_u$ , and  $A_u = a_u \gamma^{1/2}$  is the upstream adiabatic sound speed. Once attain the positive root  $X = X_+ > 1$ , we can calculate  $\tau$  (which is no longer an adjustable parameter now) by

$$\tau^2 = \frac{X - 1}{X} \left( \frac{M_1^2}{X} - \frac{1 + X}{\beta_1} \right) + \frac{1}{X}, \quad (29)$$

the right side of which must be larger than 1 due to the second law of thermodynamics, given the upstream.

When the properties of the downstream are given, i.e.  $(i, j) = (d, u)$ , we require that  $X = X_+ < 1$ , and  $\tau < 1$ , thus,  $f(X = 1) > 0$ , and the right hand side of formula (29) should be positive and smaller than 1. We can further prove that  $a_u^2 (v_u - x_{su})^2 > A_u^2 + v_{Au}^2$  is still valid, which is now equivalent to  $\gamma + 2/\beta_2 < M_2^2$ , where the definitions of  $M_2$  and  $\beta_2$  are similar to those of  $M_1$  and  $\beta_1$ , i.e.  $M_2 \equiv u_2/a_2 = M_1/(X\tau)$  and  $\beta_2 \equiv 8\pi p_2/(B_2^2) = \beta_1 \tau^2/X$ . By these definitions and the expression of  $\tau^2$  (29), we have

$$\begin{aligned} \left( \gamma + \frac{2}{\beta_2} - M_2^2 \right) \tau^2 &= \tau^2 \gamma + \frac{2X}{\beta_1} - \frac{M_1^2}{X^2} \\ &= \left[ \frac{\gamma(X - 1) - 1}{X^2} \right] M_1^2 + \frac{1}{\beta_1} \left[ (2 - \gamma)X + \frac{\gamma}{X} \right] + \frac{\gamma}{X}. \end{aligned} \quad (30)$$

Since  $X < 1$ , and  $1 < \gamma < 2$ , from equation (25), we note down an inequality

$$\begin{aligned} \frac{(\gamma + 1)M_1^2}{2X^2} - \frac{(\gamma - 1)M_1^2}{2X} - \frac{\gamma}{X} \\ = \left[ (2 - \gamma) + \frac{\gamma}{X} \right] \frac{1}{\beta_1} > \left[ (2 - \gamma)X + \frac{\gamma}{X} \right] \frac{1}{\beta_1}, \end{aligned} \quad (31)$$

which is substituted to formula (30) to eliminate the second term of the right side and gives

$$\begin{aligned} \left( \gamma + \frac{2}{\beta_2} - M_2^2 \right) \tau^2 &< \frac{(\gamma + 1)(X - 1)M_1^2}{2X^2} < 0, \text{ i.e.} \\ \gamma + \frac{2}{\beta_2} - M_2^2 &< 0. \end{aligned} \quad (32)$$

The above analysis implies that equation (25) is the correct equation for the isothermal MHD shock condition with (dynamical) MHD energy conservation, from which we obtained the preferred ‘one to one’ relation between the upstream and the downstream.

<sup>5</sup>  $r = atx$ .

Besides, if we have  $\gamma \rightarrow 1$ , equation (25) can be put into the form

$$\begin{aligned}
 & (\gamma - 1)f(X)\beta_1 \\
 &= (2 - \gamma)X^2 + \left[ \frac{(\gamma - 1)M_1^2\beta_1}{2} + \gamma(1 + \beta_1) \right] X \\
 & - \frac{(\gamma + 1)\beta_1 M_1^2}{2} = 0, \text{ i.e.} \\
 & X^2 + (1 + \beta_1)X - \beta_1 M_1^2 = 0, \tag{33}
 \end{aligned}$$

which is exactly the quadratic equation (13) for the one-temperature MHD shock condition. By this equation and equation (12), we can further show that

$$\begin{aligned}
 & \frac{X^2 - 1}{\beta_1} + X\tau^2 - 1 + \frac{1 - X}{X}M_1^2 \\
 &= \frac{X^2 - 1}{\beta_1} + X\tau^2 - 1 + (1 - X) + \frac{1 - X^2}{\beta_1} \\
 &= X(\tau^2 - 1) = 0, \text{ i.e.} \\
 & \tau = 1, \tag{34}
 \end{aligned}$$

where the first line is just the left hand side of equation (13) divided by  $\beta_1 X$ , and for the second line we have used equation (33) multiplied by  $(1/X - 1)/\beta_1$ . This outcome supports our argument that (dynamical) MHD energy conservation for isotherm MHD shocks under  $\gamma \rightarrow 1$  is consistent with the very restriction that  $a_1 = a_2$  (i.e.  $(k_B T_u/\mu)^{1/2} = a_u = a_d = (k_B T_d/\mu)^{1/2}$ ,  $T_u = T_d$ : both sides of the shock have the same temperature).

I have already written and tested the codes for this new type of isothermal MHD shocks under (dynamical) MHD energy conservation. By now, I have verified that the shock condition (25) is indeed reversible, but it would be more difficult to construct relevant shock solutions, in that  $\tau$  is not adjustable here.

### 3 THE (DYNAMICAL) ISOTHERMAL MHD ENERGY CONSERVATION FOR ISOTHERMAL MHD SHOCKS

Since equation (23) for MHD energy conservation is based on the presupposition that gas around the shock layer experience an ‘isentropic’ process (of no ‘entropy source’), i.e.  $\partial S/\partial t + \mathbf{u} \cdot \nabla S = \partial S/\partial t + u \partial S/\partial r = 0$ , the MHD energy conservation considered in the last section can be called (dynamical) isentropic MHD energy conservation.

In this section, I investigate another version of MHD energy conservation, i.e. (dynamical) isothermal MHD energy conservation, which assumes that gas near the shock layer shares the same temperature (though the temperatures of different sides of the shock can be different). According to the energy flux formula (i.e. equations (23) of Lou & Shen (2004)), the isothermal counterpart of equation (5) is

$$\left[ \frac{(u_s - u)^2}{2} + a^2 \ln \left( \frac{\rho}{\rho_c} \right) + \frac{\langle B_t^2 \rangle}{4\pi\rho} \right]_u^d = 0, \tag{35}$$

in which we have used the expression of entropy per unit mass  $S$ , i.e.

$$S = S_0 + \frac{R}{\gamma + 1} \ln \left( \frac{a^2 \rho^{1-\gamma}}{\gamma - 1} \right) = \frac{R\gamma}{\gamma - 1} - R \ln \left( \frac{\rho}{\rho_c} \right), \tag{36}$$

where  $RT = a^2$ . The problem here is that we have to do self-similar transformation of  $\rho_c$  to express equation (35) in self-similar variables, while choice of such transformation is sort of arbitrary. And I always doubt the significance of studying such MHD energy conservation in the isothermal model. Actually, we have already assumed that different parts (which are not separated by shocks) of the system share the same invariant temperature, which implies that heat conduction may be efficient in our model. Thus, considering other possibly effective mechanisms involved around the shock (e.g. radiation transfer, cosmic rays, and chemical reaction), it should be allowed that non-zero heat flux exists across the shock front, so that MHD energies (per unit mass) of the two sides of the shock can be different.

Anyway, I just try two possible types of self-similar transformation: (1)  $\rho_c^{-1} = 4\pi G t^{4-2\gamma} r^{2\gamma-2} a^{2\gamma-2}$ , (2)  $\rho_c^{-1} = 4\pi G t^2$ , where the first ensures that  $S_0$  is independent of  $a$ , while the second means that  $\rho_c$  itself is independent of  $a$ . Then I can put equation (35) into the conventional form of our paper by formulae (14):

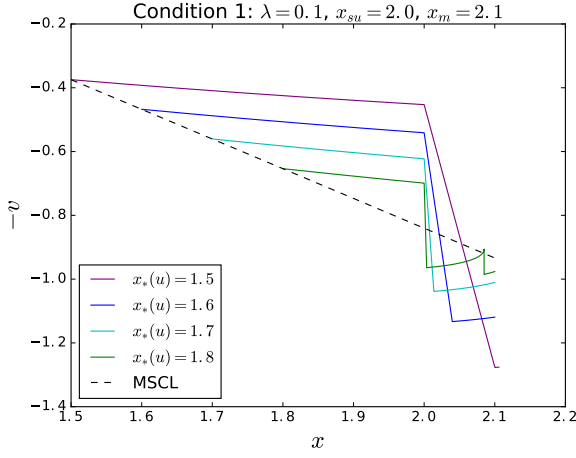
$$\begin{aligned}
 & \frac{2(X - 1)}{\beta_1} + \frac{M_1^2}{2} \left( \frac{1}{X^2} - 1 \right) + \tau^2 \ln X + \tau^2 (1 - \gamma) \tau^2 \\
 & + (\tau^2 - 1) [\ln \alpha_1 + (2\gamma - 2) \ln x_1] = 0, \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{2(X - 1)}{\beta_1} + \frac{M_1^2}{2} \left( \frac{1}{X^2} - 1 \right) + \tau^2 \ln X + (\tau^2 - 1) \alpha_1 = 0, \tag{38}
 \end{aligned}$$

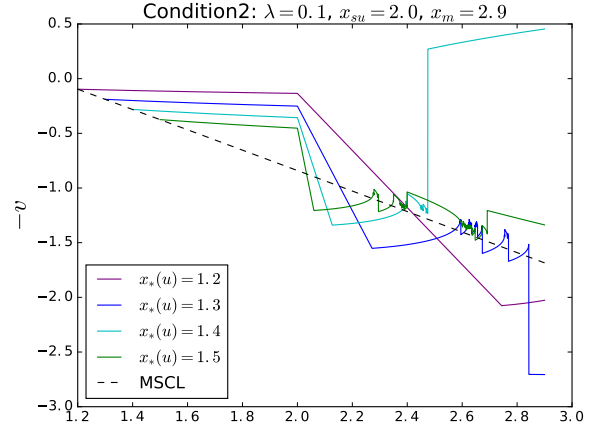
of which equation (37) corresponds to  $\rho_c^{-1} = 4\pi G t^{4-2\gamma} r^{2\gamma-2} a^{2\gamma-2}$  (Condition 1), while equation (38)  $\rho_c^{-1} = 4\pi G t^2$  (Condition 2). These equations are transcendental and cannot be evaluated analytically. I have to combine them with equation (12) for momentum conservation to solve  $X$  and  $\tau$  numerically.

Unfortunately, by limited numerical trials (for  $\gamma = 5/3$ , and  $\lambda = 0.1$ ), for both Condition 1 and Condition 2, I cannot find any isothermal MHD shock under such (dynamical) isothermal MHD energy conservation. The problem is i) in many regions of the solution space the transcendental equation set have no solution at all, and ii) when we have  $X > 1$ , we always meanwhile have  $\tau < 1$  (and vice via:  $X < 1$  with  $\tau > 1$ ), which violates the second law of thermodynamics<sup>6</sup>. Figure 1-4 show some **FALSE** isothermal MHD shock solutions under such (dynamical) isothermal MHD energy conservation. (Even though these **FALSE** solutions are unphysical, they are indeed **reversible**.) I also plot some solutions under (dynamical) isentropic MHD energy conservation prescribed by the last section in Figure 5-6, which show that both Type 1 and Type 2 two temperature shock condition can produce some results that do not break (dynamical) isentropic MHD energy conservation. In these figures, some solutions hit the MSCL and show instability, which is not our main concern here, since we only want to know features of the shock and are not at that stage of constructing complete shock solutions. So please ignore those meaningless curves after hitting the MSCL with instability.

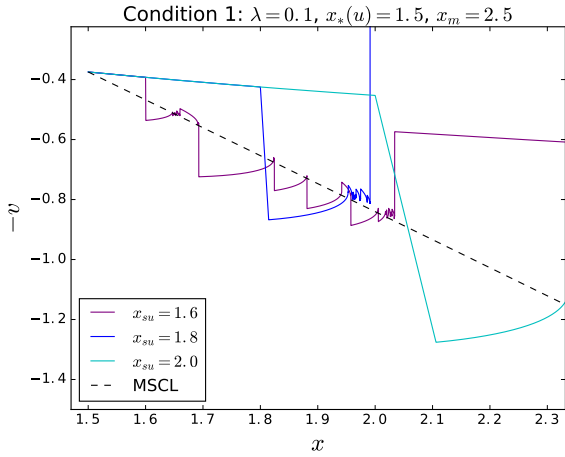
<sup>6</sup> Later on I realized that Condition 2 is itself contradictory, since here the requirement of  $S_d - S_u > 0$  becomes  $\ln(\alpha_u/\alpha_d) > 0$ . And this contradicts the natural requirement for shock propagation that  $|u_s - u_u| > |u_s - u_d|$  (which is equivalent to  $\alpha_u/\alpha_d < 1$ ).



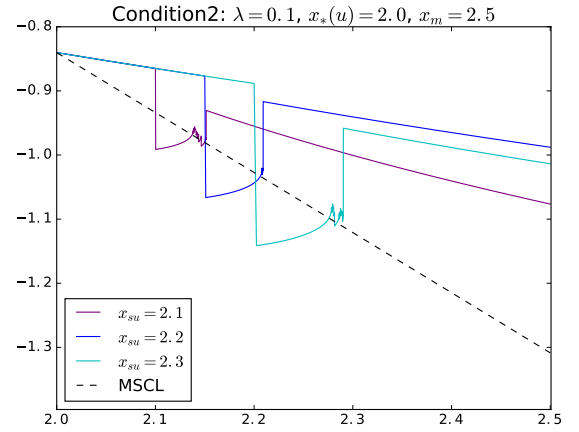
**Figure 1.**  $-v$  versus  $x$  plots of some **FALSE** Shocks under Condition 1 (dynamical) isothermal MHD energy conservation for  $\gamma = 5/3$ , and  $\lambda = 0.1$ . Here we have  $\tau < 1$  but  $X > 1$ . It is found that only when  $x_{su}$  is not too far away from  $x_*(u)$ , the transcendental equation set of Condition 1 has solutions.



**Figure 3.**  $-v$  versus  $x$  plots of some **FALSE** Shocks under Condition 2 (dynamical) isothermal MHD energy conservation for  $\gamma = 5/3$ , and  $\lambda = 0.1$ . Here we have  $\tau < 1$  but  $X > 1$ . It is found that only when  $x_*(u)$  is large enough, the transcendental equation set of Condition 2 has solutions.



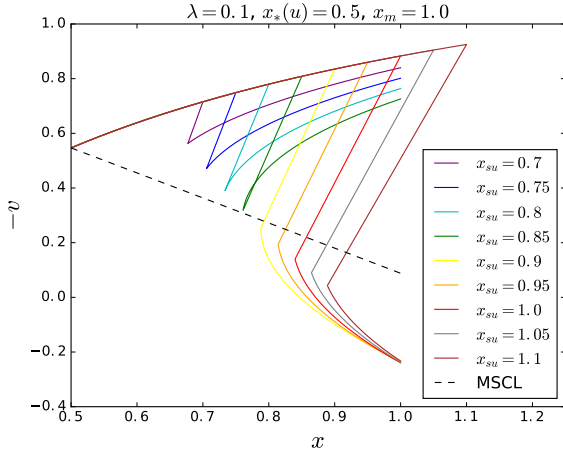
**Figure 2.**  $-v$  versus  $x$  plots of some **FALSE** Shocks under Condition 1 (dynamical) isothermal MHD energy conservation for  $\gamma = 5/3$ , and  $\lambda = 0.1$ . Here we have  $\tau < 1$  but  $X > 1$ .



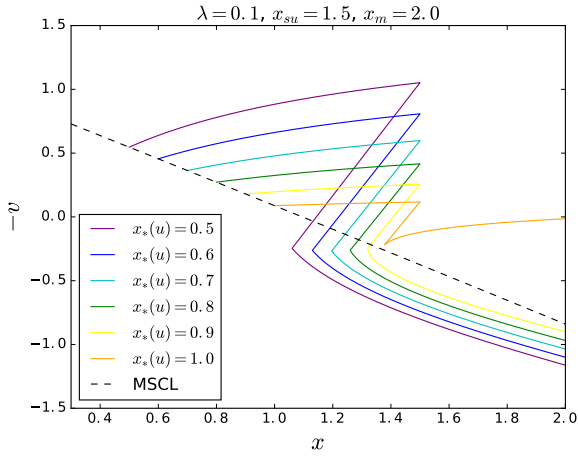
**Figure 4.**  $-v$  versus  $x$  plots of some **FALSE** Shocks under Condition 2 (dynamical) isothermal MHD energy conservation for  $\gamma = 5/3$ , and  $\lambda = 0.1$ . Here we have  $\tau < 1$  but  $X > 1$ .

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**Figure 5.**  $-v$  versus  $x$  plots of some Shocks under (dynamical) isentropic MHD energy conservation in Section 2 for  $\gamma = 5/3$ , and  $\lambda = 0.1$ . I found that when  $\gamma$  is not too close to 1, Type 1 isothermal two-temperature MHD shock condition can give results which do not break (dynamical) isentropic energy conservation when  $x_{su} - x_*(u)$  is small enough.



**Figure 6.**  $-v$  versus  $x$  plots of some Shocks under (dynamical) isentropic MHD energy conservation in Section 2 for  $\gamma = 5/3$ , and  $\lambda = 0.1$ .