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Scalar one-loop integrals for QCD

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Abstract: We construct a basis set of infra-red and/or collinearly divergent scalar one-loop integrals and give analytic formulas, for tadpole, bubble, triangle and box integrals, regulating the divergences (ultra-violet, infra-red or collinear) by regularization in $D = 4 - 2\epsilon$ dimensions. For scalar triangle integrals we give results for our basis set containing 6 divergent integrals. For scalar box integrals we give results for our basis set containing 16 divergent integrals. We provide analytic results for the 5 divergent box integrals in the basis set which are missing in the literature. Building on the work of van Oldenborgh, a general, publicly available code has been constructed, which calculates both finite and divergent one-loop integrals. The code returns the coefficients of $1/\epsilon^2, 1/\epsilon^1$ and $1/\epsilon^0$ as complex numbers for an arbitrary tadpole, bubble, triangle or box integral.

Keywords: Scalar, One-loop, Feynman integral, QCD.

Contents

1. Introduction

The advent of the LHC requires a concerted effort to evaluate hard scattering processes at next-to-leading order in QCD. This requires both the calculation of tree graphs, for the leading order and real parton emission contributions, and the calculation of one-loop diagrams, for the virtual contributions. For most approaches to the calculation of one-loop amplitudes, the knowledge of *scalar* one-loop integrals is sufficient. For integrals with all massive internal lines these integrals are all known, both analytically [1, 2, 3] and numerically [4, 5]. This paper therefore concentrates on integrals with some vanishing internal masses; these integrals can contain infra-red and collinear singularities.

Many results have been presented in the literature for integrals with infra-red divergences regulated by introducing a small mass λ for the divergent lines, especially in the important paper of Beenakker and Denner [6]. Regulation with a small mass has been the method of choice in the calculation of electroweak processes in which massless lines are relatively rare. In QCD processes massless gluons are ubiquitous; in addition, light-quark lines are often treated as massless in the high energy limit. Consequently in QCD the method of choice for the regulation of collinear and infra-red divergent integrals is dimensional regularization. Therefore the results given in this paper are regulated dimensionally, although we shall exploit the relationship between the two methods of regulation where appropriate.

Although some of the integral results presented in this paper are (to the best of our knowledge) new, the majority are not. However the analysis of the basis set required for a complete treatment of divergent box integrals at one loop is new. The results for the known integrals are dispersed through the literature and we believe it will be of use to collect the results in one place. Note that the general box integral in D-dimensions has been calculated in ref. [7]. However the results of that reference require considerable further manipulation before they can be used in practical calculations.

Given the complete set of tadpole, bubble, triangle and box integrals one can construct the scalar integrals for diagrams with greater than four legs [8, 9, 10]. Thus a scalar pentagon in D dimensions, I_5^D, can be written as a sum of the five box diagrams obtainable by removing one propagator if we neglect terms of order ϵ

$$I_5^D = \sum_{i=1}^{5} c_i I_4^{D\ (i)} + \mathcal{O}(\epsilon)\,. \tag{1.1}$$

The general one loop N-point integral in $D = 4-2\epsilon$ dimensions for $N \geq 6$ can be recursively obtained as a linear combination of pentagon integrals [8, 9] provided that the external momenta are restricted to four dimensions[1].

$$I_N^D = \sum_{i=1}^{N} d_i I_{N-1}^{D\ (i)}\,. \tag{1.2}$$

Thus for the purposes of next-to-leading order calculations, higher point functions $N > 4$ can be always reduced to sums of boxes. Note that for the case $N \geq 7$ the coefficients d_i in Eq. (1.2) are not unique.

2. Definitions and notation

2.1 Definition of integrals

We work in the Bjorken-Drell metric so that $l^2 = l_0^2 - l_1^2 - l_2^2 - l_3^2$. As illustrated in Fig. 1 the definition of the integrals is as follows

$$\begin{aligned}
I_1^D(m_1^2) &= \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \int d^D l \; \frac{1}{(l^2 - m_1^2 + i\varepsilon)}\,,\\
I_2^D(p_1^2; m_1^2, m_2^2) &= \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \int d^D l \; \frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)}\,,\\
I_3^D(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) &= \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma}\\
\times \int d^D l \; &\frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)((l+q_2)^2 - m_3^2 + i\varepsilon)}\,,\\
I_4^D(p_1^2, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; m_1^2, m_2^2, m_3^2, m_4^2) &= \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma}\\
\times \int d^D l \; &\frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)((l+q_2)^2 - m_3^2 + i\varepsilon)((l+q_3)^2 - m_4^2 + i\varepsilon)}\,,
\end{aligned} \tag{2.1}$$

where $q_n \equiv \sum_{i=1}^{n} p_i$ and $q_0 = 0$ and $s_{ij} = (p_i + p_j)^2$. For the purposes of this paper we take the masses in the propagators to be real. Near four dimensions we use $D = 4 - 2\epsilon$.

[1] Relations which have the same structure as Eq. (1.2) can also be derived, without the restriction that the external momenta lie in four dimensions. For the details of these relations we refer the reader to refs. [11, 12, 13, 14].

(For clarity the small imaginary part which fixes the analytic continuations is specified by $+i\varepsilon$). μ is a scale introduced so that the integrals preserve their natural dimensions, despite excursions away from $D = 4$. We have removed the overall constant which occurs in D-dimensional integrals

$$r_\Gamma \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\Gamma(1-\epsilon)} + \mathcal{O}(\epsilon^3) = 1 - \epsilon\gamma + \epsilon^2\left[\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right] + \mathcal{O}(\epsilon^3)\,. \tag{2.2}$$

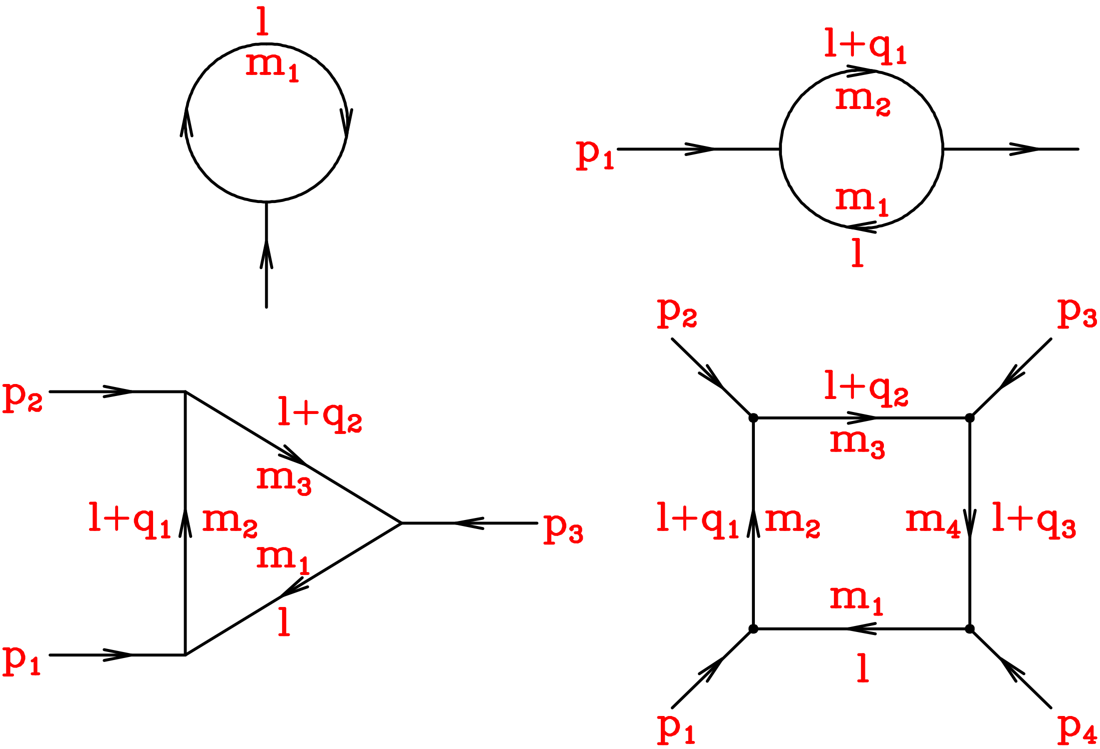

Figure 1: The notation for the one-loop tadpole, bubble, triangle and box integrals.

The final results are given in terms of logarithms and dilogarithms. The logarithm is defined to have a cut along the negative real axis. The rule for the logarithm of a product is

$$\begin{aligned}
\ln(ab) &= \ln a + \ln b + \eta(a,b)\,,\\
\eta(a,b) &= 2\pi i\left[\theta(-\mathrm{Im}(a))\theta(-\mathrm{Im}(b))\theta(\mathrm{Im}(ab)) - \theta(\mathrm{Im}(a))\theta(\mathrm{Im}(b))\theta(-\mathrm{Im}(ab))\right]\,.
\end{aligned} \tag{2.3}$$

So that

$$\begin{aligned}
\ln(ab) &= \ln a + \ln b, \;\; \text{if } \mathrm{Im}(a) \text{ and } \mathrm{Im}(b) \text{ have different signs}\,,\\
\ln\left(\frac{a}{b}\right) &= \ln a - \ln b, \;\; \text{if } \mathrm{Im}(a) \text{ and } \mathrm{Im}(b) \text{ have the same sign}\,.
\end{aligned} \tag{2.4}$$

The dilogarithm is defined as

$$\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dz}{z}\ln(1-z) \tag{2.5}$$

$$= \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots \;\; \text{for } |x| \le 1\,. \tag{2.6}$$

A number of the most commonly useful dilogarithm identities are given in ref. [15]. In addition to the common identities, we also used an identity from ref. [16] in the calculation of the new box integrals

$$
\begin{aligned}
&\mathrm{Li}_2(1-aP^2)+\mathrm{Li}_2(1-aQ^2)-\mathrm{Li}_2(1-as)-\mathrm{Li}_2(1-at)= \\
&\mathrm{Li}_2\Big(1-\frac{P^2}{s}\Big)+\mathrm{Li}_2\Big(1-\frac{P^2}{t}\Big)+\mathrm{Li}_2\Big(1-\frac{Q^2}{s}\Big)+\mathrm{Li}_2\Big(1-\frac{Q^2}{t}\Big) \\
-&\mathrm{Li}_2\Big(1-\frac{P^2Q^2}{st}\Big)+\frac{1}{2}\ln^2\Big(\frac{s}{t}\Big)\,,
\end{aligned}
\tag{2.7}
$$

which holds for $a=(P^2+Q^2-s-t)/(P^2Q^2-st)$.

2.2 Analytic continuation

The integrals given in sec. 4 are calculated in the spacelike region, $s_{ij}<0, p_i^2<0$. In this region, the denominator of the Feynman parameter integrals is positive definite and the $i\,\varepsilon$ prescription can be dropped. The analytic continuation is performed by restoring the $i\,\varepsilon$

$$
\begin{aligned}
p_i^2 &\to p_i^2+i\,\varepsilon\,, \\
s_{ij} &\to s_{ij}+i\,\varepsilon\,, \\
m_i &\to m_i-i\,\varepsilon\,.
\end{aligned}
\tag{2.8}
$$

The one-loop integrals which we present are all expressed in terms of two types of transcendental functions,

$$
\ln\left(\prod_{i=1}^n x_i\right)\,, \quad \text{and} \quad \mathrm{Li}_2\left(1-\prod_{i=1}^n x_i\right)\,. \tag{2.9}
$$

The individual x_i can be chosen such they that vary only on the first Riemann sheet of the logarithm, $-\pi<\arg(x_i)<\pi$. A complete specification of the continuation of terms of the form eq. (2.9) has been given in ref. [6]. The continuation prescription for the logarithm is

$$
\ln\left(\prod_{i=1}^n x_i\right)\to\sum_{i=1}^n\ln\left(x_i\right)\,. \tag{2.10}
$$

The continuation procedure for the dilogarithm is similar. For $\left|\prod_{i=1}^n x_i\right|<1$ we use [6, 12]

$$
\begin{aligned}
\mathrm{Li}_2\left(1-\prod_{i=1}^n x_i\right)&\to\mathrm{Li}_2\left(1-\prod_{i=1}^n x_i\right)+\ln\left(1-\prod_{i=1}^n x_i\right)\left[\ln\left(\prod_{i=1}^n x_i\right)-\sum_{i=1}^n\ln(x_i)\right] \\
&=\frac{\pi^2}{6}-\mathrm{Li}_2\left(\prod_{i=1}^n x_i\right)-\ln\left(1-\prod_{i=1}^n x_i\right)\sum_{i=1}^n\ln(x_i)\,.
\end{aligned}
\tag{2.11}
$$

If $\left|\prod_{i=1}^n x_i\right|>1$ it is expedient to make the transformation

$$
\mathrm{Li}_2\left(1-\prod_{i=1}^n x_i\right)=-\mathrm{Li}_2\left(1-\frac{1}{\prod_{i=1}^n x_i}\right)-\frac{1}{2}\ln^2\left(\prod_{i=1}^n x_i\right)\,. \tag{2.12}
$$

We can then continue the resulting expression using eqs. (2.10, 2.11) as before. The continuation procedure given in ref. [17] can be shown to be equivalent to the above.

3. Basis set of soft and collinear divergent integrals

After Feynman parametrization and integration over $d^D l$, we have for the triangle and box integrals

$$I_3^D(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) = -\frac{\mu^{2\epsilon}\Gamma(1+\epsilon)}{r_\Gamma} \prod_{i=1}^{3} \int_0^1 da_k \, \frac{\delta(1-\sum_k a_k)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{1+\epsilon}}, \quad (3.1)$$

$$I_4^D(p_1^2, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; m_1^2, m_2^2, m_3^2, m_4^2) = \frac{\mu^{2\epsilon}\Gamma(2+\epsilon)}{r_\Gamma} \prod_{i=1}^{4} \int_0^1 da_k \, \frac{\delta(1-\sum_k a_k)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{2+\epsilon}}, \quad (3.2)$$

where Y is the so-called modified Cayley matrix

$$Y_{ij} \equiv \frac{1}{2}\left[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\right]. \quad (3.3)$$

3.1 Landau conditions

The necessary conditions for eqs. (3.1,3.2) to contain a singularity are due to Landau [18, 19]. If we introduce the bilinear form D derived from the modified Cayley matrix,

$$D = \sum_{i,j} a_i a_j Y_{ij}, \quad (3.4)$$

eqs. (3.1, 3.2) contain singularities if $D = 0$ and one of the following conditions is satisfied for all values of j

$$\textbf{either } a_j = 0 \textbf{ or } \frac{\partial D}{\partial a_j} = 0. \quad (3.5)$$

In general we have two classes of solutions of eq. (3.5). In the first class we have solutions for the a_i which are implicit functions of the masses and external momenta. Both physical and anomalous thresholds fall into this class. The leading Landau singularity occurs when $\frac{\partial D}{\partial a_j} = 0$ is satisfied for all j. Such solutions of the Landau equations will lead to divergences at the kinematic points where there are anomalous and physical thresholds.

The second class of solution, which is of interest here, is the case where the external virtualities and internal masses have fixed values and the Landau conditions have solutions for arbitrary values of the other external invariants, s_{ij}. Only these solutions will lead to soft and collinear divergences which are relevant for next-to-leading order calculations.

As an example we consider the triangle shown in Fig. (2a) which contains a soft singularity. In this case the denominator is given by

$$D = (m_2^2 + m_3^2 - p_2^2) a_2 a_3 + m_2^2 a_2^2 + m_3^2 a_3^2. \quad (3.6)$$

This expression satisfies the Landau conditions for $a_2 = a_3 = 0$ and a_1 arbitrary. A second example is the triangle shown in Fig. (2b) which contains a collinear singularity. In this case the denominator reads

$$D = (m_3^2 - p_2^2) a_2 a_3 + (m_3^2 - p_3^2) a_1 a_3 + m_3^2 a_3^2, \quad (3.7)$$

which satisfies the Landau conditions for $a_3 = 0$ and a_1, a_2 arbitrary.

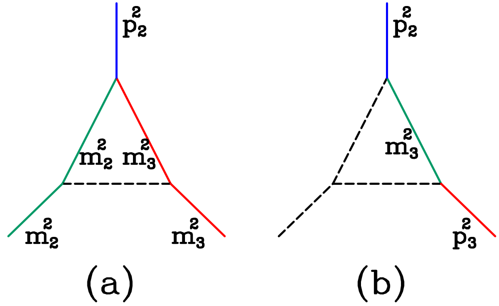

Figure 2: Examples of triangle diagrams with divergences.

3.2 Soft and collinear divergences

From the Landau conditions it follows that a necessary condition for a soft or collinear singularity is that for at least one value of the index i [20]

$$Y_{i+1\,i+1} = Y_{i+1\,i+2} = Y_{i+1\,i} = 0\,, \qquad \text{soft singularity}\,, \tag{3.8}$$

$$Y_{i\,i} = Y_{i+1\,i+1} = Y_{i\,i+1} = 0\,, \qquad \text{collinear singularity}\,. \tag{3.9}$$

The indices in eqs. (3.8, 3.9) should be interpreted mod N, where N is the number of external legs. Thus the structure of the Cayley matrices for integrals having a soft or collinear divergence is as follows

$$Y_{\text{soft}} = \begin{pmatrix} \dots & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots \\ \dots & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad Y_{\text{collinear}} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.10}$$

In order to have a divergence, we must have at least one internal mass equal to zero, i.e. at least one vanishing diagonal element of Y.

3.3 Basis set for triangle integrals

We list here the basis set of six divergent triangle integrals as shown in Fig. 3. All other divergent triangle integrals can be derived from this set. First we have two integrals with no internal masses,

1. $I_3^D(0, 0, p_3^2; 0, 0, 0)$
2. $I_3^D(0, p_2^2, p_3^2, 0; 0, 0, 0)$.

Second, we have three integrals with one massive internal line,

3. $I_3^D(0, p_2^2, p_3^2; 0, 0, m^2)$
4. $I_3^D(0, p_2^2, m^2; 0, 0, m^2)$
5. $I_3^D(0, m^2, m^2; 0, 0, m^2)$.

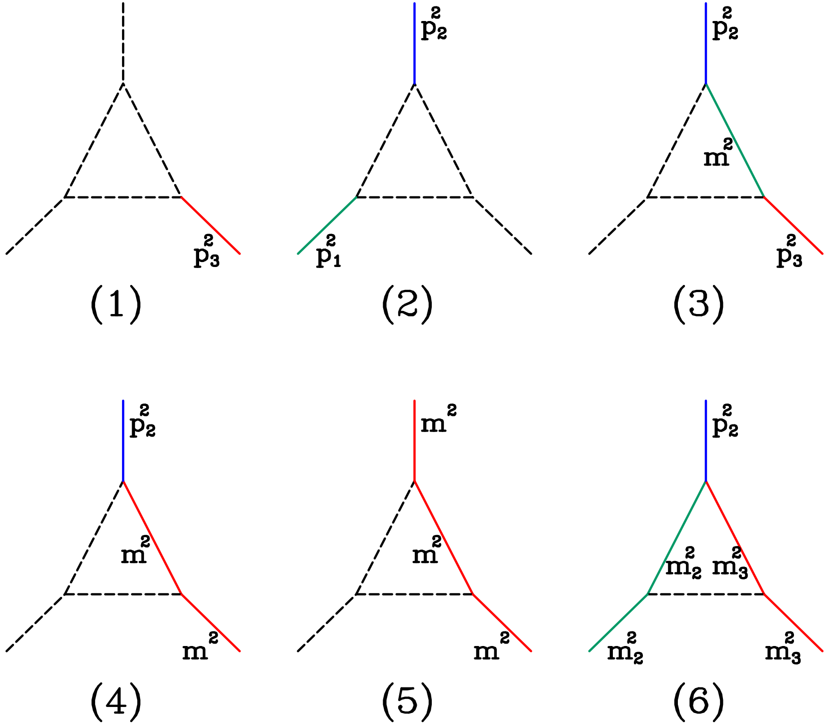

Figure 3: The six divergent triangle integrals. Lines with a zero internal mass $m_i = 0$ (for internal lines) or a zero virtuality, $p_i^2 = 0$, (for external lines) are shown dashed and are unlabelled. Solid lines have a non-zero internal mass, or a non-zero virtuality. Lines with the same color have the same internal mass and/or virtuality.

Last, we have one integral with two massive internal lines,

6. $I_3^D(m_2^2, p_2^2, m_3^2; 0, m_2^2, m_3^2)$.

The set is in fact overcomplete since the second integral can be obtained from the $m^2 \to 0$ limit of the third integral. However from a numerical point of view it is expedient to categorize the integrals by the number of vanishing internal masses and to treat the two cases separately.

3.4 Basis set for box integrals

In this section we demonstrate that a basis set sufficient to describe all box integrals with collinear or soft singularities can be constructed from sixteen integrals, illustrated in Fig. 4. All other divergent box integrals can be derived from this set. [2] The integrals are characterized by the number of internal masses which are equal to zero. Each one of these divergent integrals has a characteristic modified Cayley determinant as shown in Fig. 5 satisfying the conditions of eqs. (3.8, 3.9).

[2]As discussed in sec. 3.1, at specific kinematic points there are threshold singularities derivable from the Landau conditions which can lead to singular integrals not derivable from our basis set.

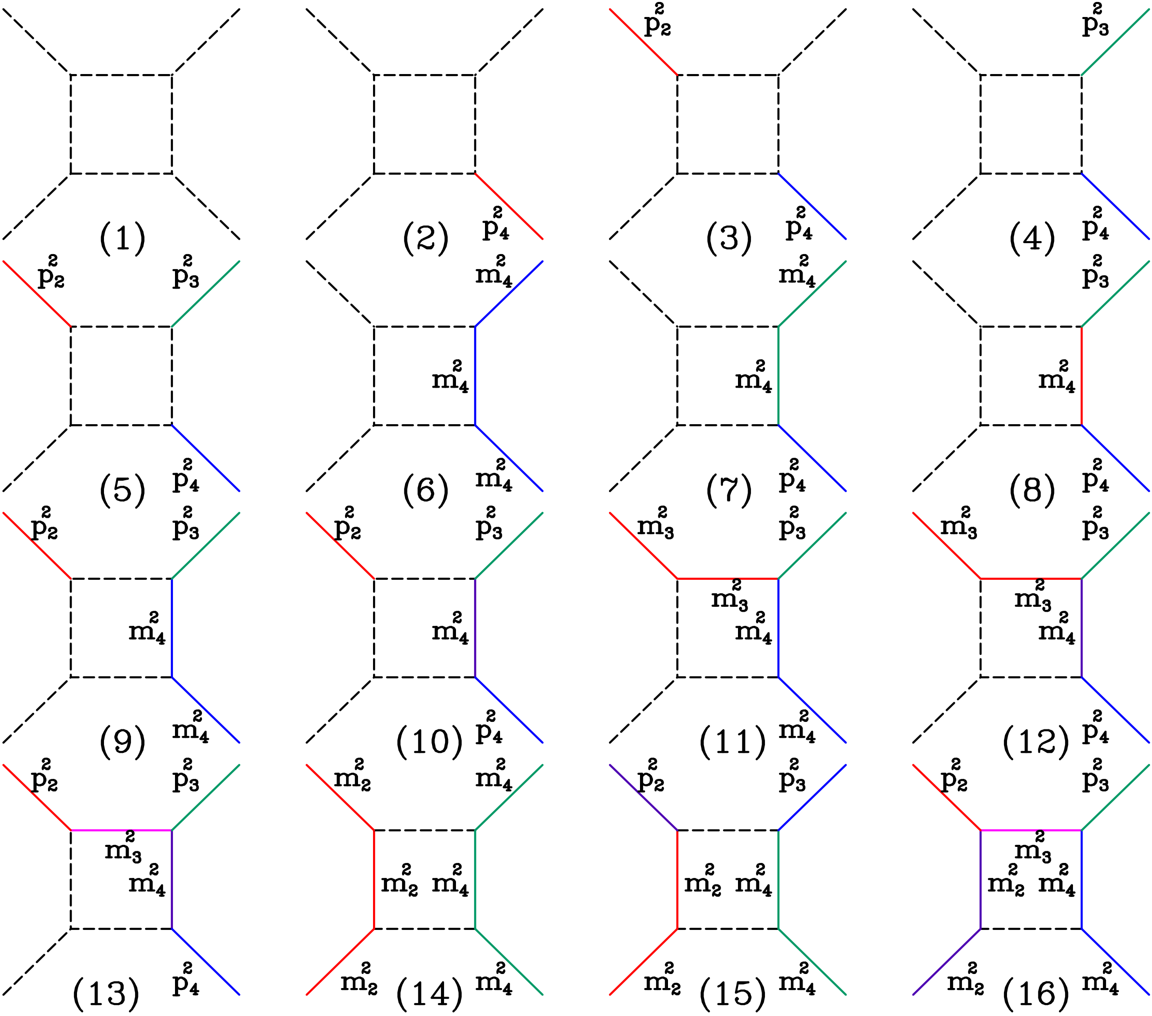

Figure 4: The sixteen divergent box integrals. Lines with a zero internal mass $m_i = 0$ (for internal lines) or a zero virtuality, $p_i^2 = 0$, (for external lines) are shown dashed and are unlabelled. Solid lines have a non-zero internal mass, or a non-zero virtuality. Lines with the same color have the same internal mass or virtuality.

3.4.1 Integrals with no internal masses

There are five integrals with no internal masses,

1. $I_4^D(0,0,0,0; s_{12}, s_{23}; 0,0,0,0)$
2. $I_4^D(0,0,0,p_4^2; s_{12}, s_{23}; 0,0,0,0)$
3. $I_4^D(0,p_2^2,0,p_4^2; s_{12}, s_{23}; 0,0,0,0)$
4. $I_4^D(0,0,p_3^2,p_4^2; s_{12}, s_{23}; 0,0,0,0)$
5. $I_4^D(0,p_2^2,p_3^2,p_4^2; s_{12}, s_{23}; 0,0,0,0)$.

$$\begin{array}{cccc}
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix} \\
(1) & (2) & (3) & (4) \\[2ex]
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & \times \end{pmatrix} \\
(5) & (6) & (7) & (8) \\[2ex]
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ 0 & \times & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \\
(9) & (10) & (11) & (12) \\[2ex]
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix} \\
(13) & (14) & (15) & (16)
\end{array}$$

Figure 5: The structure of the modified Cayley determinant for the sixteen divergent box integrals.

This set of integrals is sufficient to describe all the integrals with massless internal lines, but it is overcomplete since in ref. [21] an expression is given for box integral 3, which can also give results for integrals 1 and 2 by taking the appropriate limit. However, as before, for reasons of numerical expediency it is convenient to retain the overcomplete basis.

3.4.2 Integrals with one non-zero internal mass

If we have one non-zero internal mass we can take this without loss of generality to be the last one (m_4). In this case the modified Cayley matrix, eq. (3.3) is

$$Y = \begin{pmatrix} 0 & -\frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & -\frac{1}{2}p_2^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & -\frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix}. \tag{3.11}$$

With s_{12}, s_{23} fixed we can apply four conditions to potentially create a soft or collinear divergence, namely

$$p_1^2 = 0,\ p_2^2 = 0,\ p_3^2 = m_4^2,\ p_4^2 = m_4^2. \tag{3.12}$$

However performing the interchange $p_1^2 \leftrightarrow p_2^2, p_3^2 \leftrightarrow p_4^2$, (with m_4 fixed) corresponds to a relabelling of the diagram. In addition setting either $p_3^2 = m_4^2$ without setting $p_2^2 = 0$, or

$p_4^2 = m_4^2$ without setting $p_1^2 = 0$ does not lead to a divergence. If we denote the application of the four conditions, eq. (3.12) on $p_1^2, p_2^2, p_3^2, p_4^2$ by (i, j, k, l) we have following 15 cases:

$$\begin{array}{ll} 6.) & (1,2,3,4) \\ 7.) & (1,2,3) \equiv (1,2,4) \\ 8.) & (1,2) \\ 9.) & (1,4) \equiv (2,3) \equiv (1,3,4) \equiv (2,3,4) \\ 10.) & (1) \equiv (2) \equiv (1,3) \equiv (2,4) \\ & (3) \equiv (4) \equiv (3,4) \equiv \text{finite}\,. \end{array} \tag{3.13}$$

To be quite explicit, the notation $(1, 2, 3)$ corresponds to setting $p_1^2 = 0, p_2^2 = 0$ and $p_3^2 = m_4^2$, etc. The first five lines in eq. (3.13) correspond to the integrals 6-10 given below,

6. $I_4^{\{D=4-2\epsilon\}}(0, 0, m^2, m^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

7. $I_4^{\{D=4-2\epsilon\}}(0, 0, m^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

8. $I_4^{\{D=4-2\epsilon\}}(0, 0, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

9. $I_4^{\{D=4-2\epsilon\}}(0, p_2^2, p_3^2, m^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

10. $I_4^{\{D=4-2\epsilon\}}(0, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2)$.

The last case in eq. (3.13) does not lead to a divergent integral.

3.4.3 Integrals with two adjacent internal masses

Without loss of generality we can take the two non-zero adjacent internal masses to be m_3 and m_4. In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & -\frac{1}{2}p_1^2 & \frac{1}{2}m_3^2 - \frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & \frac{1}{2}m_3^2 - \frac{1}{2}p_2^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ \frac{1}{2}m_3^2 - \frac{1}{2}s_{12} & \frac{1}{2}m_3^2 - \frac{1}{2}p_2^2 & m_3^2 & \frac{1}{2}m_3^2 + \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_3^2 + \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix}. \tag{3.14}$$

A necessary condition to have any divergence is $p_1^2 = 0$. This gives the integral 13. Applying either $p_2^2 = m_3^2$ or $p_4^2 = m_4^2$ gives a pair of integrals related by relabelling, integral 12. Applying both $p_2^2 = m_3^2$ and $p_4^2 = m_4^2$ gives integral 11,

11. $I_4^{\{D=4-2\epsilon\}}(0, m_3^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

12. $I_4^{\{D=4-2\epsilon\}}(0, m_3^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

13. $I_4^{\{D=4-2\epsilon\}}(0, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$.

3.4.4 Integrals with two opposite internal masses

Without loss of generality we can take the two non-zero opposite internal masses to be m_2 and m_4. In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & m_2^2 & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix}. \tag{3.15}$$

Here we can only have a soft divergence since there is no pair of adjacent zero internal masses. Setting $p_1^2 = m_2^2, p_4^2 = m_4^2$ or $p_2^2 = m_2^2, p_3^2 = m_4^2$ gives two integrals related by relabelling (15). Setting both conditions gives integral 14,

14. $I_4^{\{D=4-2\epsilon\}}(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

15. $I_4^{\{D=4-2\epsilon\}}(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)\,.$

3.4.5 Integral with three internal masses

Without loss of generality we can take $m_1 = 0$. With only one zero mass, there can only be a soft singularity. This requires the two adjacent external lines to satisfy the conditions, $p_1^2 = m_2^2, p_4^2 = m_4^2$

16. $I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2)\,.$

The concludes the listing of our basis set of the divergent box integrals.

4. Results for integrals

4.1 Tadpole integrals

The result for the tadpole integral is given by

$$\begin{aligned} I_1^D(m^2) &= -\mu^{2\epsilon}\,\Gamma(-1+\epsilon)\,[m^2 - i\varepsilon]^{1-\epsilon} \\ &= m^2 \left(\frac{\mu^2}{m^2 - i\varepsilon}\right)^{\epsilon} \left\{\frac{1}{\epsilon} + 1\right\} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.1}$$

The $1/\epsilon$ pole corresponds to an ultraviolet divergence and the analytic continuation has been made explicit in this case.

4.2 Bubble integrals

Following 't Hooft and Veltman [1], the result for this bubble integral in our notation is,

$$\begin{aligned} I_2^D(s; m_1^2, m_2^2) &= \frac{\mu^{2\epsilon}\Gamma(\epsilon)}{r_\Gamma} \int_0^1 d\gamma\; [-\gamma(1-\gamma)s + \gamma m_2^2 + (1-\gamma)m_1^2 - i\varepsilon]^{-\epsilon} \\ &= \mu^{2\epsilon} \left\{\frac{1}{\epsilon} - \int_0^1 d\gamma\; \ln\big(-\gamma(1-\gamma)s + \gamma m_2^2 + (1-\gamma)m_1^2 - i\varepsilon\big)\right\} + \mathcal{O}(\epsilon) \end{aligned}$$

$$= \mu^{2\epsilon} \left\{ \frac{1}{\epsilon} - \ln(s - i\varepsilon) - \int_0^1 d\gamma \ \ln\left(\gamma^2 - \gamma(1 - \frac{m_2^2}{s} + \frac{m_1^2}{s}) + \frac{m_1^2}{s} - \frac{i\varepsilon}{s}\right) \right\} + \mathcal{O}(\epsilon)$$

$$= \mu^{2\epsilon} \left\{ \frac{1}{\epsilon} + 2 - \ln(s - i\varepsilon) + \sum_{i=1}^{2} \left[\gamma_i \ln\left(\frac{\gamma_i - 1}{\gamma_i}\right) - \ln(\gamma_i - 1) \right] \right\} + \mathcal{O}(\epsilon)\,, \tag{4.2}$$

where $\gamma_{1,2}$ are the two roots of the quadratic equation,

$$\gamma_{1,2} = \frac{s - m_2^2 + m_1^2 \pm \sqrt{(s - m_2^2 + m_1^2)^2 - 4s(m_1^2 - i\varepsilon)}}{2s}\,. \tag{4.3}$$

The special limits for this integral are

$$\begin{aligned} I_2^D(s; 0, m^2) &= \left(\frac{\mu^2}{m^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} + 2 + \frac{m^2 - s}{s} \ln\left(\frac{m^2 - s - i\varepsilon}{m^2}\right) \right\} + \mathcal{O}(\epsilon)\,, \\ I_2^D(s; 0, 0) &= \left(\frac{\mu^2}{-s - i\varepsilon}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} + 2 \right\} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.4}$$

As before, the $1/\epsilon$ pole corresponds to an ultraviolet divergence and the analytic continuation has been made explicit in these cases.

4.3 Divergent triangle integrals

In this section we give the explicit results for the six divergent triangles. The results for the triangles have been presented already by many authors and are given here only for completeness. The results are reported in the spacelike region below all thresholds. The analytic continuation is performed using the prescription given in sec. 2.2. Each expression for a triangle integral stands for the 6 different labellings of the triangle obtained by repeated application of the following identities

$$\begin{aligned} I_3^D(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) &= I_3^D(p_2^2, p_3^2, p_1^2; m_2^2, m_3^2, m_1^2)\,, \\ I_3^D(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) &= I_3^D(p_1^2, p_3^2, p_2^2; m_2^2, m_1^2, m_3^2)\,. \end{aligned}$$

4.3.1 Triangle 1: $I_3^D(0, 0, p_3^2; 0, 0, 0)$

$$\begin{aligned} I_3^D(0, 0, p_3^2; 0, 0, 0) &= \frac{\mu^{2\epsilon}}{\epsilon^2} \left\{ \frac{(-p_3^2)^{-\epsilon}}{p_3^2} \right\} \\ &= \frac{1}{p_3^2} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln\left(\frac{-p_3^2}{\mu^2}\right) + \frac{1}{2} \ln^2\left(\frac{-p_3^2}{\mu^2}\right) \right\} + \mathcal{O}(\epsilon). \end{aligned} \tag{4.5}$$

4.3.2 Triangle 2: $I_3^D(0, p_2^2, p_3^2; 0, 0, 0)$

$$I_3^D(0, p_2^2, p_3^2, 0; 0, 0, 0) = \frac{\mu^{2\epsilon}}{\epsilon^2} \left\{ \frac{(-p_2^2)^{-\epsilon} - (-p_3^2)^{-\epsilon}}{p_2^2 - p_3^2} \right\}$$

$$= \frac{1}{p_2^2 - p_3^2} \left\{ \frac{1}{\epsilon} \ln\left(\frac{-p_3^2}{-p_2^2}\right) + \frac{1}{2} \left[\ln^2\left(\frac{-p_2^2}{\mu^2}\right) - \ln^2\left(\frac{-p_3^2}{\mu^2}\right) \right] \right\} + \mathcal{O}(\epsilon)\,. \tag{4.6}$$

In the limit $p_2^2 \to p_3^2$ we define $r = (p_3^2 - p_2^2)/p_2^2$ and use following expansion

$$I_3^D(0, p_2^2, p_3^2, 0; 0, 0, 0) = \frac{1}{p_2^2}\left\{ -\frac{1}{\epsilon}\left(1 - \frac{r}{2}\right) + \ln\left(\frac{-p_2^2}{\mu^2}\right) + \frac{r}{2}\left(1 + \ln\left(\frac{-p_2^2}{\mu^2}\right)\right) \right\} + \mathcal{O}(\epsilon, r^2)\,. \tag{4.7}$$

4.3.3 Triangle 3: $I_3^D(0, p_2^2, p_3^2; 0, 0, m^2)$

$$I_3^D(0, p_2^2, p_3^2; 0, 0, m^2) = \frac{1}{p_2^2 - p_3^2}\left(\frac{\mu^2}{m^2}\right)^\epsilon \left\{ \frac{1}{\epsilon}\ln\left(\frac{m^2 - p_3^2}{m^2 - p_2^2}\right) + \mathrm{Li}_2\left(\frac{p_2^2}{m^2}\right) - \mathrm{Li}_2\left(\frac{p_3^2}{m^2}\right) \right.$$
$$\left. + \ln^2\left(\frac{m^2 - p_2^2}{m^2}\right) - \ln^2\left(\frac{m^2 - p_3^2}{m^2}\right) \right\} + O(\epsilon)\,. \tag{4.8}$$

In the limit $p_2^2 \to p_3^2$ we define $r = (p_3^2 - p_2^2)/(m^2 - p_2^2)$ and use following expansion

$$I_3^D(0, p_2^2, p_3^2, 0; 0, 0, 0) = \frac{1}{m^2 - p_2^2}\left\{ \left(1 - \frac{r}{2}\right)\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2}\right) - \frac{m^2 + p_2^2}{p_2^2}\ln\left(\frac{m^2 - p_2^2}{m^2}\right) \right.$$
$$\left. - \frac{r}{2p_2^2}\left[\frac{m^4 - 2p_2^2 m^2 - p_2^4}{p_2^2}\ln\left(\frac{m^2 - p_2^2}{m^2}\right) + m^2 + p_2^2\right] \right\} + \mathcal{O}(\epsilon, r^2)\,. \tag{4.9}$$

Rewriting eq. (4.8) in the following form makes the $m \to 0$ limit and the agreement with eq. (4.6) manifest

$$I_3^D(0, p_2^2, p_3^2; 0, 0, m^2) = \frac{1}{p_2^2 - p_3^2}\left\{ \frac{1}{\epsilon}\ln\left(\frac{m^2 - p_3^2}{m^2 - p_2^2}\right) + \frac{1}{2}\left[\ln^2\left(\frac{-p_2^2}{\mu^2}\right) - \ln^2\left(\frac{-p_3^2}{\mu^2}\right)\right] \right.$$
$$+ \ln\left(\frac{(m^2 - p_2^2)}{-p_2^2}\right)\ln\left(\frac{-p_2^2(m^2 - p_2^2)}{\mu^2 m^2}\right) - \ln\left(\frac{(m^2 - p_3^2)}{-p_3^2}\right)\ln\left(\frac{-p_3^2(m^2 - p_3^2)}{\mu^2 m^2}\right)$$
$$\left. - \mathrm{Li}_2\left(\frac{m^2}{p_2^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{p_3^2}\right) \right\} + O(\epsilon)\,. \tag{4.10}$$

4.3.4 Triangle 4: $I_3^D(0, p_2^2, m^2; 0, 0, m^2)$

$$I_3^D(0, p_2^2, m^2; 0, 0, m^2) = \left(\frac{\mu^2}{m^2}\right)^\epsilon \frac{1}{p_2^2 - m^2}$$
$$\times \left\{ \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\ln\left(\frac{m^2}{m^2 - p_2^2}\right) + \frac{\pi^2}{12} + \frac{1}{2}\ln^2\left(\frac{m^2}{m^2 - p_2^2}\right) - \mathrm{Li}_2\left(\frac{-p_2^2}{m^2 - p_2^2}\right) \right\} + \mathcal{O}(\epsilon)\,. \tag{4.11}$$

4.3.5 Triangle 5: $I_3^D(0, m^2, m^2; 0, 0, m^2)$

$$I_3^D(0, m^2, m^2; 0, 0, m^2) = \left(\frac{\mu^2}{m^2}\right)^\epsilon \frac{1}{m^2}\left(-\frac{1}{2\epsilon} + 1 \right) + \mathcal{O}(\epsilon)\,. \tag{4.12}$$

4.3.6 Triangle 6: $I_3^D(m_2^2, s, m_3^2; 0, m_2^2, m_3^2)$

The result for this triangle integral can be obtained from ref. [6], eq. (C3), by the normal replacement rule [22] which is true in the case of a soft singularity

$$\ln \lambda^2 \to \frac{r_\Gamma}{\epsilon} + \ln \mu^2 + \mathcal{O}(\epsilon)\,. \tag{4.13}$$

For $s - (m_2 - m_3)^2 \neq 0$ we have

$$\begin{aligned} I_3^{\{D=4\}}(m_2^2, s, m_3^2; 0, m_2^2, m_3^2) &= \frac{x_s}{m_2 m_3 (1-x_s^2)} \\ \times \Bigg\{ \ln(x_s)\Big[-\frac{1}{\epsilon} - \frac{1}{2}\ln(x_s) + 2\ln(1-x_s^2) + \ln(\frac{m_1 m_3}{\mu^2})\Big] \\ -\frac{\pi^2}{6} + \mathrm{Li}_2(x_s^2) + \frac{1}{2}\ln^2\frac{m_2}{m_3} + \mathrm{Li}_2(1 - x_s\frac{m_2}{m_3}) + \mathrm{Li}_2(1 - x_s\frac{m_3}{m_2})\Bigg\} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{4.14}$$

where $x_s = -K(s + i\varepsilon, m_2, m_3)$ and K is given by

$$\begin{aligned} K(z, m, m') &= \frac{1-\sqrt{1-4mm'/[z-(m-m')^2]}}{1+\sqrt{1-4mm'/[z-(m-m')^2]}} && z \neq (m-m')^2 \\ K(z, m, m') &= -1 && z = (m-m')^2\,. \end{aligned} \tag{4.15}$$

For $s - (m_2 - m_3)^2 = 0$ this becomes

$$\begin{aligned} I_3^{\{D=4\}}(m_2^2, s, m_3^2; 0, m_2^2, m_3^2) &= \frac{1}{2m_2 m_3} \\ \times \left\{ \frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m_1 m_3}\right) - 2 - \frac{m_3 + m_2}{m_3 - m_2}\ln\left(\frac{m_2}{m_3}\right) \right\} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.16}$$

4.4 Divergent box integrals

In this section we give the explicit results for the sixteen divergent boxes. The results are reported in the spacelike region below all thresholds. The analytic continuation is performed using the prescription given in sec. 2.2. Each expression for a box integral stands for the 8 different labellings of the box obtained by repeated application of the following identities

$$\begin{aligned} I_4^D(p_1^2, p_2^2, p_3^2, p_4^2, s_{12}, s_{23}, m_1^2, m_2^2, m_3^2, m_4^2) &= I_4^D(p_2^2, p_3^2, p_4^2, p_1^2, s_{23}, s_{12}, m_2^2, m_3^2, m_4^2, m_1^2)\,, \\ I_4^D(p_1^2, p_2^2, p_3^2, p_4^2, s_{12}, s_{23}, m_1^2, m_2^2, m_3^2, m_4^2) &= I_4^D(p_4^2, p_3^2, p_2^2, p_1^2, s_{12}, s_{23}, m_1^2, m_4^2, m_3^2, m_2^2)\,. \end{aligned} \tag{4.17}$$

Where we have found the integrals in the literature we give references. To the best of our knowledge the results for boxes 9, 10, 11, 12, 13 are new.

4.4.1 Box 1: $I_4^D(0, 0, 0, 0; s_{12}, s_{23}; 0, 0, 0, 0)$

$$\begin{aligned} I_4^D(0, 0, 0, 0; s_{12}, s_{23}; 0, 0, 0, 0) &= \frac{\mu^{2\epsilon}}{s_{12}s_{23}} \\ \times \left\{ \frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon}\Big) - \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) - \pi^2 \right\} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.18}$$

This result is taken from [11].

4.4.2 Box 2: $I_4^D(0,0,0,p_4^2;s_{12},s_{23};0,0,0,0)$

$$I_4^D(0,0,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{12}s_{23}}$$
$$\times \left\{ \frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} - (-p_4^2)^{-\epsilon}\Big) - 2\,\mathrm{Li}_2(1-\frac{p_4^2}{s_{12}}) - 2\,\mathrm{Li}_2(1-\frac{p_4^2}{s_{23}})\right.$$
$$\left. -\ \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) - \frac{\pi^2}{3}\right\} + \mathcal{O}(\epsilon)\,. \tag{4.19}$$

This integral is given in [23, 11]. An alternative formulation with three dilogarithms is given in [21].

4.4.3 Box 3: $I_4^D(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0)$

$$I_4^D(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{23}s_{12}-p_2^2p_4^2}$$
$$\times \left\{ \frac{2}{\epsilon^2}\left((-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} - (-p_2^2)^{-\epsilon} - (-p_4^2)^{-\epsilon}\right)\right.$$
$$-\ 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{23}}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{12}}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{23}}\right)$$
$$\left. +\ 2\,\mathrm{Li}_2\left(1-\frac{p_2^2p_4^2}{s_{12}s_{23}}\right) - \ln^2\left(\frac{-s_{12}}{-s_{23}}\right)\right\} + \mathcal{O}(\epsilon)\,. \tag{4.20}$$

This result is taken from [11]. As for all the integrals, the analytic continuation of this result follows the procedure detailed in sec. 2.2. An alternative form in which the analytic continuation is manifest is given in ref. [21].

When the denominator in the overall factor in eq. (4.20) becomes small and the entries in the two pairs $(s_{12},\, s_{23})$ and (p_2^2, p_4^2) have the opposite sign we can expand in $r = 1-\frac{p_2^2p_4^2}{s_{12}s_{23}}$

$$I_4^D(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{1}{s_{23}s_{12}}$$
$$\times \left\{ -\frac{1}{\epsilon}(2+r) + (2-\frac{1}{2}r) + (2+r)\left(\ln\left(\frac{-s_{12}}{\mu^2}\right) + \ln\left(\frac{-s_{23}}{-p_4^2}\right)\right)\right.$$
$$\left. +\ 2\left[\mathrm{L}_0\left(\frac{p_4^2}{s_{23}}\right) + \mathrm{L}_0\left(\frac{p_4^2}{s_{12}}\right)\right] + r\left[\mathrm{L}_1\left(\frac{p_4^2}{s_{23}}\right) + \mathrm{L}_1\left(\frac{p_4^2}{s_{12}}\right)\right]\right\} + \mathcal{O}(\epsilon, r^2)\,, \tag{4.21}$$

where

$$\mathrm{L}_0(z) = \frac{\ln(z)}{1-z}, \quad \mathrm{L}_1(z) = \frac{\mathrm{L}_0(z)+1}{1-z} \tag{4.22}$$

Thus in this region the residue of the overall pole at $s_{23}s_{12} = p_2^2p_4^2$ vanishes and we obtain a numerically stable expression. The other region $s_{12}, s_{23} > 0$ and $p_2^2, p_4^2 < 0$ or vice versa ($s_{12}, s_{23} < 0$ and $p_2^2, p_4^2 > 0$) is the region of the Landau pole which gives a large contribution to the imaginary part.

4.4.4 Box 4: $I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$

$$
\begin{aligned}
&I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{12}s_{23}} \\
&\times \left\{\frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon}+(-s_{23})^{-\epsilon}-(-p_3^2)^{-\epsilon}-(-p_4^2)^{-\epsilon}\Big)+\frac{1}{\epsilon^2}\Big((-p_3^2)^{-\epsilon}(-p_4^2)^{-\epsilon}\Big)/(-s_{12})^{-\epsilon}\right. \\
&\left. -2\,\mathrm{Li}_2\left(1-\frac{p_3^2}{s_{23}}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{23}}\right)-\ln^2\left(\frac{-s_{12}}{-s_{23}}\right)\right\}+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{4.23}
$$

This result is taken from ref. [11]. (See also ref. [21]).

4.4.5 Box 5: $I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$

$$
\begin{aligned}
&I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{(s_{23}s_{12}-p_2^2p_4^2)} \\
&\times \left\{\frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon}+(-s_{23})^{-\epsilon}-(-p_2^2)^{-\epsilon}-(-p_3^2)^{-\epsilon}-(-p_4^2)^{-\epsilon}\Big)\right. \\
&+\frac{1}{\epsilon^2}\Big((-p_2^2)^{-\epsilon}(-p_3^2)^{-\epsilon}\Big)/(-s_{23})^{-\epsilon}+\frac{1}{\epsilon^2}\Big((-p_3^2)^{-\epsilon}(-p_4^2)^{-\epsilon}\Big)/(-s_{12})^{-\epsilon} \\
&\left. -2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{23}}\right)+2\,\mathrm{Li}_2\left(1-\frac{p_2^2p_4^2}{s_{12}s_{23}}\right)-\ln^2\left(\frac{-s_{12}}{-s_{23}}\right)\right\}+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{4.24}
$$

This result is taken from [11]. As for all the integrals, the analytic continuation of this result follows the procedure detailed in sec. 2.2. An alternative form in which the analytic continuation is manifest is given in ref. [21].

When the denominator in the overall factor in eq. (4.24) becomes small and the entries in the two pairs $(s_{12},\, s_{23})$ and (p_2^2, p_4^2) have the opposite sign we can expand in $r = 1-\frac{p_2^2p_4^2}{s_{12}s_{23}}$

$$
\begin{aligned}
&I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{1}{s_{23}s_{12}} \\
&\times \left\{-\frac{1}{\epsilon}\left(1+\frac{1}{2}r\right)-(1+\frac{1}{2}r)\left[\ln\left(\frac{\mu^2}{-s_{12}}\right)+\ln\left(\frac{-p_3^2}{-s_{23}}\right)-2-(1+\frac{p_4^2}{s_{23}})\mathrm{L}_0\left(\frac{p_4^2}{s_{23}}\right)\right]\right. \\
&\left. +\ r\left[\mathrm{L}_1\left(\frac{p_4^2}{s_{23}}\right)-\mathrm{L}_0\left(\frac{p_4^2}{s_{23}}\right)-1\right]\right\}+\mathcal{O}(\epsilon,r^2)\,,
\end{aligned}
\tag{4.25}
$$

with $\mathrm{L}_0, \mathrm{L}_1$ as in eq. (4.22).

4.4.6 Box 6: $I_4^D(0,0,m^2,m^2;s_{12},s_{23};0,0,0,m^2)$

$$
\begin{aligned}
&I_4^{\{D=4-2\epsilon\}}(0,0,m^2,m^2;s_{12},s_{23};0,0,0,m^2) = -\frac{1}{s_{12}(m^2-s_{23})}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} \\
&\times\left\{\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\Big(2\ln(\frac{m^2-s_{23}}{m^2})+\ln(\frac{-s_{12}}{m^2})\Big)+2\ln(\frac{m^2-s_{23}}{m^2})\ln(\frac{-s_{12}}{m^2})-\frac{\pi^2}{2}\right\}+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{4.26}
$$

The result for the real part of this integral in the region $s_{12}>0, s_{23}<0$ is given in ref. [24] eq. (A4) (note differing definition of ϵ).

4.4.7 Box 7: $I_4^D(0,0,m^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

$$I_4^D(0,0,m^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \left(\frac{\mu^2}{m^2}\right)^\epsilon \frac{1}{s_{12}(s_{23}-m^2)}$$
$$\times \left\{ \frac{3}{2}\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\left[2\ln\left(1-\frac{s_{23}}{m^2}\right) + \ln\left(\frac{-s_{12}}{m^2}\right) - \ln\left(1-\frac{p_4^2}{m^2}\right)\right]\right.$$
$$\left. -2\,\mathrm{Li}\left(\frac{s_{23}-p_4^2}{s_{23}-m^2}\right) + 2\ln\left(\frac{-s_{12}}{m^2}\right)\ln\left(1-\frac{s_{23}}{m^2}\right) - \ln^2\left(1-\frac{p_4^2}{m^2}\right) - \frac{5\pi^2}{12}\right\} + \mathcal{O}(\epsilon) \tag{4.27}$$

This integral is obtained from eq. (A4) (first equation) of ref. [25]. (The real part was given earlier in eq. (6.75) of [26].)

4.4.8 Box 8: $I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

$$I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{s_{12}(s_{23}-m^2)}$$
$$\times \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\left[\ln\frac{-s_{12}}{\mu^2} + \ln\frac{(m^2-s_{23})^2}{(m^2-p_3^2)(m^2-p_4^2)}\right]\right.$$
$$-2\,\mathrm{Li}_2\left(1-\frac{m^2-p_3^2}{m^2-s_{23}}\right) - 2\,\mathrm{Li}_2\left(1-\frac{m^2-p_4^2}{m^2-s_{23}}\right) - \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-p_4^2)}{s_{12}m^2}\right)$$
$$-\frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{-s_{12}}{\mu^2}\right) - \frac{1}{2}\ln^2\left(\frac{-s_{12}}{m^2}\right) + 2\ln\left(\frac{-s_{12}}{\mu^2}\right)\ln\left(\frac{m^2-s_{23}}{m^2}\right)$$
$$\left. -\ln\left(\frac{m^2-p_3^2}{\mu^2}\right)\ln\left(\frac{m^2-p_3^2}{m^2}\right) - \ln\left(\frac{m^2-p_4^2}{\mu^2}\right)\ln\left(\frac{m^2-p_4^2}{m^2}\right)\right\} + \mathcal{O}(\epsilon). \tag{4.28}$$

This integral was constructed from the expression eq. (B6) of ref. [27], which is valid for the real part in the region $s_{12} > 0, s_{23} < 0$.

4.4.9 Box 9: $I_4^D(0,p_2^2,p_3^2,m^2;s_{12},s_{23};0,0,0,m^2)$

$$I_4^D(0,p_2^2,p_3^2,m^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{s_{12}(s_{23}-m^2)}\left\{\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{s_{12}}{p_2^2}\frac{(m^2-s_{23})}{\mu m}\right)\right.$$
$$\left. + \ \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-s_{23})}{m^2p_2^2}\right) + 2\,\mathrm{Li}_2\left(1-\frac{s_{12}}{p_2^2}\right) + \frac{\pi^2}{12} + \ln^2\left(\frac{s_{12}}{p_2^2}\frac{(m^2-s_{23})}{\mu m}\right)\right\} + \mathcal{O}(\epsilon)\,. \tag{4.29}$$

4.4.10 Box 10: $I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

$$I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{(s_{12}s_{23}-m^2s_{12}-p_2^2p_4^2+m^2p_2^2)}$$
$$\times \left\{\frac{1}{\epsilon}\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right) + \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-s_{23})}{p_2^2m^2}\right) - \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-p_4^2)}{s_{12}m^2}\right)\right.$$
$$+ \ 2\,\mathrm{Li}_2\left(1-\frac{m^2-s_{23}}{m^2-p_4^2}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right) + 2\,\mathrm{Li}_2\left(1-\frac{p_2^2(m^2-p_4^2)}{s_{12}(m^2-s_{23})}\right)$$
$$\left. + \ 2\ln\left(\frac{\mu m}{m^2-s_{23}}\right)\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right)\right\} + \mathcal{O}(\epsilon)\,. \tag{4.30}$$

4.4.11 Box 11: $I_4^D(0, m_3^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

$$
\begin{aligned}
I_4^D(0, m_3^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2) &= \frac{1}{(m_3^2 - s_{12})(m_4^2 - s_{23})} \\
\times \Bigg\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln\left(\frac{(m_4^2 - s_{23})(m_3^2 - s_{12})}{m_3 m_4 \mu^2}\right) &+ 2\ln\left(\frac{m_3^2 - s_{12}}{m_3\mu}\right)\ln\left(\frac{m_4^2 - s_{23}}{m_4\mu}\right) \\
- \ \frac{\pi^2}{2} + \ln^2\left(\frac{m_3}{m_4}\right) - \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^+}{\gamma_{34}^+ - 1}\right) - \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^-}{\gamma_{34}^- - 1}\right)\Bigg\} &+ \mathcal{O}(\epsilon)\,, \qquad (4.31)
\end{aligned}
$$

where

$$
\gamma_{ij}^{\pm} = \frac{1}{2}\left[1 - \frac{m_i^2 - m_j^2}{p_3^2} \pm \sqrt{(1 - \frac{m_i^2 - m_j^2}{p_3^2})^2 - \frac{4m_j^2}{p_3^2}}\right]. \qquad (4.32)
$$

and $\gamma_{ij}^+ + \gamma_{ji}^- = 1$. Assuming $m_4^2 > m_3^2$, in the limit $p_3^2 \to 0$ we obtain,

$$
\frac{\gamma_{34}^+}{\gamma_{34}^+ - 1} \to 1 + \mathcal{O}(p_3^2), \quad \frac{\gamma_{34}^-}{\gamma_{34}^- - 1} \to \frac{m_4^2}{m_3^2} + \mathcal{O}(p_3^2)\,, \qquad (4.33)
$$

and this expression reduces to the form given in eq. (6.77) of Höpker [26]

$$
\begin{aligned}
I_4^D(0, m_3^2, 0, m_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2) &= \frac{1}{(m_3^2 - s_{12})(m_4^2 - s_{23})} \\
\times \Bigg\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{(m_4^2 - s_{23})(m_3^2 - s_{12})}{m_3 m_4 \mu^2}\right) \\
+ \ 2\ln\left(\frac{m_3^2 - s_{12}}{m_3\mu}\right)\ln\left(\frac{m_4^2 - s_{23}}{m_4\mu}\right) - \frac{\pi^2}{2} - \ln^2\left(\frac{m_3}{m_4}\right)\Bigg\} &+ \mathcal{O}(\epsilon)\,. \qquad (4.34)
\end{aligned}
$$

4.4.12 Box 12: $I_4^D(0, m_3^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

$$
\begin{aligned}
I_4^D(0, m_3^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2) &= \frac{1}{(s_{12} - m_3^2)(s_{23} - m_4^2)} \\
\times \Bigg\{ \frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{(m_4^2 - s_{23})(m_3^2 - s_{12})}{(m_4^2 - p_4^2) m_3\mu}\right) &+ 2\ln\left(\frac{m_4^2 - s_{23}}{m_3\mu}\right)\ln\left(\frac{m_3^2 - s_{12}}{m_3\mu}\right) \\
- \ln^2\left(\frac{m_4^2 - p_4^2}{m_3\mu}\right) - \frac{\pi^2}{12} + \ln\left(\frac{m_4^2 - p_4^2}{m_3^2 - s_{12}}\right)\ln\left(\frac{m_4^2}{m_3^2}\right) &- \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^+}{\gamma_{34}^+ - 1}\right) - \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^-}{\gamma_{34}^- - 1}\right) \\
- \ 2\,\mathrm{Li}_2\Big(1 - \frac{(m_4^2 - p_4^2)}{(m_4^2 - s_{23})}\Big) - \mathrm{Li}_2\Big(1 - \frac{(m_4^2 - p_4^2)}{(m_3^2 - s_{12})}\frac{\gamma_{43}^+}{\gamma_{43}^+ - 1}\Big) &- \mathrm{Li}_2\Big(1 - \frac{(m_4^2 - p_4^2)}{(m_3^2 - s_{12})}\frac{\gamma_{43}^-}{\gamma_{43}^- - 1}\Big)\Bigg\} \\
+ \ \mathcal{O}(\epsilon)\,, \qquad (4.35)
\end{aligned}
$$

where $\gamma_{ij}^{\pm}$ is given in eq. (4.32). In the limit $p_3^2 = 0$, we get

$$
\begin{aligned}
I_4^D(0, m_3^2, 0, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2) &= \frac{1}{(s_{12} - m_3^2)(s_{23} - m_4^2)} \\
\times \Bigg\{ \frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{(m_4^2 - s_{23})(m_3^2 - s_{12})}{(m_4^2 - p_4^2) m_3\mu}\right) &+ 2\ln\left(\frac{m_4^2 - s_{23}}{m_3\mu}\right)\ln\left(\frac{m_3^2 - s_{12}}{m_3\mu}\right)
\end{aligned}
$$

$$
- \ln^2\left(\frac{m_4^2-p_4^2}{m_3\mu}\right) - \frac{\pi^2}{12} + \ln\left(\frac{m_4^2-p_4^2}{m_3^2-s_{12}}\right)\ln\left(\frac{m_4^2}{m_3^2}\right) - \frac{1}{2}\ln^2\left(\frac{m_4^2}{m_3^2}\right)
$$

$$
- 2\ \mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_4^2-s_{23})}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\Big) - \mathrm{Li}_2\Big(1-\frac{m_3^2}{m_4^2}\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\Big)\Bigg\} + \mathcal{O}(\epsilon)\,. \tag{4.36}
$$

4.4.13 Box 13: $I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$

$$
I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2) = \frac{1}{\Delta}\left\{\frac{1}{\epsilon}\ln\left(\frac{(m_3^2-p_2^2)(m_4^2-p_4^2)}{(m_3^2-s_{12})(m_4^2-s_{23})}\right)\right.
$$

$$
- 2\,\mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)}{(m_3^2-s_{12})}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)}{(m_4^2-s_{23})}\frac{\gamma_{34}^+}{\gamma_{34}^+-1}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)}{(m_4^2-s_{23})}\frac{\gamma_{34}^-}{\gamma_{34}^--1}\Big)
$$

$$
- 2\,\mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_4^2-s_{23})}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\frac{\gamma_{43}^+}{\gamma_{43}^+-1}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\frac{\gamma_{43}^-}{\gamma_{43}^--1}\Big)
$$

$$
+ 2\,\mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)(m_4^2-p_4^2)}{(m_3^2-s_{12})(m_4^2-s_{23})}\Big) + 2\ln\left(\frac{m_3^2-s_{12}}{\mu^2}\right)\ln\left(\frac{m_4^2-s_{23}}{\mu^2}\right)
$$

$$
- \ln^2\left(\frac{m_3^2-p_2^2}{\mu^2}\right) - \ln^2\left(\frac{m_4^2-p_4^2}{\mu^2}\right) + \ln\left(\frac{m_3^2-p_2^2}{m_4^2-s_{23}}\right)\ln\left(\frac{m_3^2}{\mu^2}\right) + \ln\left(\frac{m_4^2-p_4^2}{m_3^2-s_{12}}\right)\ln\left(\frac{m_4^2}{\mu^2}\right)
$$

$$
\left. - \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^+}{\gamma_{34}^+-1}\right) - \frac{1}{2}\ln^2\left(\frac{\gamma_{34}^-}{\gamma_{34}^--1}\right)\right\} + \mathcal{O}(\epsilon)\,, \tag{4.37}
$$

where $\gamma_{ij}^{\pm}$ is given in eq. (4.32) and

$$
\begin{aligned}
\Delta &= (s_{12}s_{23} - m_3^2 s_{23} - m_4^2 s_{12} - p_2^2 p_4^2 + m_3^2 p_4^2 + m_4^2 p_2^2) \\
&= (m_3^2-s_{12})(m_4^2-s_{23}) - (m_3^2-p_2^2)(m_4^2-p_4^2)\,.
\end{aligned} \tag{4.38}
$$

In the limit $p_3^2 \to 0$ this simplifies to

$$
I_4^D(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2) = \frac{1}{\Delta}\left\{\frac{1}{\epsilon}\ln\left(\frac{(m_3^2-p_2^2)(m_4^2-p_4^2)}{(m_3^2-s_{12})(m_4^2-s_{23})}\right)\right.
$$

$$
- 2\,\mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)}{(m_3^2-s_{12})}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)}{(m_4^2-s_{23})}\Big) - \mathrm{Li}_2\Big(1-\frac{m_4^2}{m_3^2}\frac{(m_3^2-p_2^2)}{(m_4^2-s_{23})}\Big)
$$

$$
- 2\,\mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_4^2-s_{23})}\Big) - \mathrm{Li}_2\Big(1-\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\Big) - \mathrm{Li}_2\Big(1-\frac{m_3^2}{m_4^2}\frac{(m_4^2-p_4^2)}{(m_3^2-s_{12})}\Big)
$$

$$
+ 2\,\mathrm{Li}_2\Big(1-\frac{(m_3^2-p_2^2)(m_4^2-p_4^2)}{(m_3^2-s_{12})(m_4^2-s_{23})}\Big) + 2\ln\left(\frac{m_3^2-s_{12}}{\mu^2}\right)\ln\left(\frac{m_4^2-s_{23}}{\mu^2}\right)
$$

$$
- \ln^2\left(\frac{m_3^2-p_2^2}{\mu^2}\right) - \ln^2\left(\frac{m_4^2-p_4^2}{\mu^2}\right) + \ln\left(\frac{m_3^2-p_2^2}{m_4^2-s_{23}}\right)\ln\left(\frac{m_3^2}{\mu^2}\right) + \ln\left(\frac{m_4^2-p_4^2}{m_3^2-s_{12}}\right)\ln\left(\frac{m_4^2}{\mu^2}\right)
$$

$$
\left. - \frac{1}{2}\ln^2\left(\frac{m_4^2}{m_3^2}\right)\right\} + \mathcal{O}(\epsilon)\,. \tag{4.39}
$$

4.4.14 Box 14: $I_4^D(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

We can obtain this doubly IR divergent box integral from eq. (2.13) of ref. [6], using the simple replacement rule in eq. (4.13). We obtain

$$I_4^D(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2) = \frac{-2}{m_2 m_4 s_{12}} \frac{x_{23} \ln(x_{23})}{1 - x_{23}^2} \left\{ \frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{-s_{12}}\right) \right\} + \mathcal{O}(\epsilon), \tag{4.40}$$

The variable x_{23} is defined in terms of the function K, eq. (4.15), such that

$$x_{23} = -K(s_{23} + i\varepsilon, m_2, m_4)\,. \tag{4.41}$$

In the limit $s_{23} - (m_2 - m_4)^2 \to 0$ we have

$$I_4^D(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2) = \frac{1}{m_2 m_4 s_{12}} \left\{ \frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{-s_{12}}\right) \right\} + \mathcal{O}(\epsilon, (1 - x_{23})^2)\,. \tag{4.42}$$

4.4.15 Box 15: $I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

We can obtain this IR divergent box integral from eq. (2.11) of ref. [6], using the simple replacement rule eq. (4.13). We obtain

$$\begin{aligned}
& I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2) = \frac{x_{23}}{m_2 m_4 s_{12}(1 - x_{23}^2)} \\
& \times \Bigg\{ \ln x_{23} \left[-\frac{1}{\epsilon} - \frac{1}{2}\ln x_{23} - \ln\left(\frac{\mu^2}{m_2 m_4}\right) - \ln\left(\frac{m_2^2 - p_2^2}{-s_{12}}\right) - \ln\left(\frac{m_4^2 - p_3^2}{-s_{12}}\right) \right] \\
& - \operatorname{Li}_2(1 - x_{23}^2) + \frac{1}{2}\ln^2 y + \sum_{\rho=\pm 1} \operatorname{Li}_2(1 - x_{23} y^\rho) \Bigg\} + \mathcal{O}(\epsilon),
\end{aligned} \tag{4.43}$$

where

$$y = \frac{m_2}{m_4} \frac{(m_4^2 - p_3^2)}{(m_2^2 - p_2^2)}\,, \tag{4.44}$$

and the variable x_{23} is defined in eq. (4.41).

For $m_4^2 - p_3^2$ small it is useful for numerical purposes to rewrite eq. (4.43) in the form

$$\begin{aligned}
& I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2) = \frac{x_{23}}{m_2 m_4 s_{12}(1 - x_{23}^2)} \\
& \times \Bigg\{ \ln x_{23} \left[-\frac{1}{\epsilon} - \ln x_{23} - \ln\left(\frac{\mu^2}{m_2^2}\right) - 2\ln\left(\frac{m_2^2 - p_2^2}{-s_{12}}\right) \right] \\
& - \operatorname{Li}_2(1 - x_{23}^2) + \operatorname{Li}_2(1 - x_{23} y) - \operatorname{Li}_2(1 - \frac{y}{x_{23}}) \Bigg\} + \mathcal{O}(\epsilon),
\end{aligned} \tag{4.45}$$

and similarly for $m_2^2 - p_2^2$ small.

In the limit $x_{23} \to 1$ (i.e $s_{23} = (m_2 - m_4)^2$) we obtain

$$I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2) == \frac{1}{2m_2 m_4 s_{12}}$$
$$\times \left\{ \frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m_2 m_4}\right) + \ln\left(\frac{m_2^2 - p_2^2}{-s_{12}}\right) + \ln\left(\frac{m_4^2 - p_3^2}{-s_{12}}\right) - 2 - \frac{1+y}{(1-y)} \ln y \right\}$$
$$+ \mathcal{O}(\epsilon, (1 - x_{23})^2) \,. \tag{4.46}$$

4.4.16 Box 16: $I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2)$

We can calculate this IR divergent box integral from eq. (2.9) of ref. [6], using the simple replacement rule eq. (4.13). We obtain

$$I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2) = \frac{x_{23}}{m_2 m_4 (s_{12} - m_3^2)(1 - x_{23}^2)}$$
$$\times \Bigg\{ - \frac{\ln(x_{23})}{\epsilon} - 2\ln(x_{23}) \ln\left(\frac{m_3 \mu}{m_3^2 - s_{12}}\right) + \ln^2(x_2) + \ln^2(x_3) - \mathrm{Li}_2(1 - x_{23}^2)$$
$$+ \mathrm{Li}_2(1 - x_{23} x_2 x_3) + \mathrm{Li}_2(1 - \frac{x_{23}}{x_2 x_3}) + \mathrm{Li}_2(1 - \frac{x_{23} x_2}{x_3}) + \mathrm{Li}_2(1 - \frac{x_{23} x_3}{x_2}) \Bigg\} + \mathcal{O}(\epsilon), \tag{4.47}$$

where $x_{23} \equiv -K(s_{23}, m_2, m_4)$, $x_2 \equiv -K(p_2^2, m_2, m_3)$ and $x_3 \equiv -K(p_3^2, m_3, m_4)$.

In the limit $x_{23} \to 1$ (i.e $s_{23} = (m_2 - m_4)^2$) we obtain

$$I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2) = \frac{1}{2m_2 m_4 (s_{12} - m_3^2)}$$
$$\times \left\{ \frac{1}{\epsilon} + 2\ln\left(\frac{m_3 \mu}{m_3^2 - s_{12}}\right) - \frac{1 + x_2 x_3}{1 - x_2 x_3}\Big[\ln(x_2) + \ln(x_3)\Big] - \frac{x_3 + x_2}{x_3 - x_2}\Big[\ln(x_2) - \ln(x_3)\Big] - 2 \right\}$$
$$+ \mathcal{O}(\epsilon, (1 - x_{23})^2) \,. \tag{4.48}$$

Special choices of p_2^2, p_3^2 and (non-zero) values of the masses m_2^2, m_3^2, m_4^2 will not lead to further divergences.

4.5 Special cases for box integrals

In this section we give some examples of non-singular limits of the basis set of box integrals. The first example illustrates that the basis is overcomplete. If we look at the $m^2 \to 0$ limit of box 8, eq. (4.28), we find after little work that it reproduces the result for box integral 4, eq. (4.23). That this should be the case is clear from the form of the modified Cayley matrix Y, as shown in fig. 5. Taking the limit $m^2 \to 0$ does not introduce any new singularities of the form given in eqs. (3.8, 3.9). In a similar way one can show, for instance, that box 13 reduces to box 10 in the limit $m_3^2 \to 0$ and box 10 goes to box 5 in the limit $m_4^2 \to 0$. Since these limits, which are analytically simple, can sometimes be numerically delicate, we choose to treat these integrals as separate cases and to categorize the integrals by the number of the internal masses which are non-vanishing.

Divergent box	Special Case	Reference
Box 8	$I_4^D(0,0,p_3^2,p_3^2;s_{12},s_{23};0,0,0,m^2)$	[26], eq. (6.71)
	$I_4^D(0,0,0,p_4^2;s_{12},s_{23};0,0,0,m^2)$	[28], eq. (A17)
Box 11	$I_4^D(0,m_3^2,0,m_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$	[26], eq. (6.77)
	$I_4^D(0,m^2,0,m^2;s_{12},s_{23};0,0,m^2,m^2)$	[26], eq. (6.70)
	$I_4^D(0,m^2,p_3^2,m^2;s_{12},s_{23},0,0,m^2,m^2)$	[29], eq. (30)
		[25], eq. (A4), third eqn.
Box 12	$I_4^D(0,m_3^2,0,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$	[27], eq. (B7)
	$I_4^D(0,m_3^2,0,m_3^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$	[26], eq. (6.74)
	$I_4^D(0,m_3^2,0,p_4^2;s_{12},s_{23};0,0,m_3^2,m_3^2)$	[26], eq. (6.78)
	$I_4^D(0,m^2,0,p_4^2;s_{12},s_{23};0,0,m^2,m^2)$	[25], eq. (A4), second eqn.
	$I_4^D(0,m^2,p_3^2,p_4^2;s_{12},s_{23},0,0,m^2,m^2)$	[25], eq. (A4), fourth eqn.
Box 13	$I_4^D(0,p^2,0,p^2;s_{12},s_{23};0,0,m^2,m^2)$	[26], eq. (6.72)
	$I_4^D(0,m_4^2,0,m_3^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$	[26] eq. (6.79)
	$I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,m^2,m^2)$	[28] (v2), eq. (A19)
Box 16	$I_4^D(m^2,p_2^2,p_3^2,m^2;s_{12},s_{23};0,m^2,m^2,m^2)$	[25], eq. (A4), sixth eqn.
	$I_4^D(m^2,0,p_3^2,m^2;s_{12},s_{23};0,m^2,m^2,m^2)$	[25], eq. (A4), fifth eqn.
	$I_4^D(m^2,0,0,m^2;s_{12},s_{23};0,m^2,m^2,m^2)$	[24], eq. (A3)
	$I_4^D(m_1^2,0,0,m_1^2;s_{12},s_{23};0,m_1^2,m_2^2,m_1^2)$	[26], eq. (6.73)
	$I_4^D(m_1^2,0,0,m_2^2;s_{12},s_{23};0,m_1^2,m_1^2,m_2^2)$	[26], eq. (6.76)

Table 1: Special cases of the 16 basis integrals available in the literature.

The second illustration gives an example of an integral which is obtainable from one of our basis set integrals by taking a non-singular limit. If we look at the first entry of tab. 1 and take the limit $p_3^2 = p_4^2$ we reproduce the result of ref. [26], eq. (6.71). A little care is required since ref. [26] only gives the result for the real part of the integral in a physical region and has a different ϵ-dependent overall factor. Table 1 details the examples which we have found in the literature which are non-singular limits of our basis integrals. We have checked that we are in agreement with all these special cases.

5. Numerical procedure and checks

We have constructed a numerical code which for any N-point integral returns the three complex coefficients in the Laurent series

$$I_N^D = \frac{a_{-2}}{\epsilon^2} + \frac{a_{-1}}{\epsilon^1} + a_0\,, \qquad N \leq 4\,. \tag{5.1}$$

For the IR divergent triangles and boxes we use the analytic results of sec. 4. For the UV divergent tadpoles and two-point functions we use the FF library of van Oldenborgh [4]. For the finite integrals the coefficients a_{-2}, a_{-1} are equal to zero and we use the FF library for the coefficient a_0. The code for the box integrals is unable to handle the phase space point where $\det Y = 0$ and one is sitting exactly at the threshold given by the leading Landau singularity. The code assumes that all internal masses are real. An extension of the code which also handles complex masses, appropriate for unstable particles, is a matter of analytic continuation and programming, rather than additional calculation.

The code classifies the integrals in terms of the number of non-zero internal masses and then reduces the integrals to standard forms by relabelling the integral where necessary. Subsequently on the basis of the elements of the modified Cayley matrix the code identifies the appropriate divergent integral and evaluates it, or uses the FF library.

To perform a numerical check for the divergent box integrals we make use the identity [10]

$$I_4^D = \frac{1}{2}\left(-\sum_{i=1}^{4} c_i I_3^D[i] + (3-D)c_0 I_4^{D+2}\right), \tag{5.2}$$

where $I_3^D[i]$ denotes the D dimensional triangle integral obtained from the box integral I_4^D by removing the i-th propagator and the coefficients c_i are given by

$$c_i = \sum_{j=1}^{4}(Y^{-1})_{ij}\,, \qquad c_0 = \sum_{i=1}^{4} c_i\,. \tag{5.3}$$

The six-dimensional box, because it is finite, can be computed numerically using standard Feynman parameters. We introduce an $i\varepsilon$ prescription if needed and we set ε equal to a small number. Assuming that the simpler, potentially divergent triangle integrals have been calculated correctly, using eq. (5.2) we obtain a rather powerful check of the numerical implementation of the box integrals, for both space-like and time-like values of the external invariants p_i^2 and s_{ij}.

The principal results presented in this paper, as well as the code can be downloaded from the website `http://qcdloop.fnal.gov`.

6. Conclusions and outlook

The essential new results of this paper are the classification of the infrared and collinear divergent triangle and box integrals, the calculation of the box integrals which were missing from the literature, and the provision of a code which returns a numerical answer for any one-loop scalar integral, divergent or finite, for four or less external legs. We believe that the problem of one-loop scalar integrals is now completely solved as far as next-to-leading order calculations are concerned.

In conjunction with a procedure for determining the coefficients with which scalar integrals with four or less external legs appear in physical amplitudes we are in principle able to calculate the one-loop amplitude for any process. Amplitudes with massive internal lines, or massless internal lines, or both can now be treated in a seamless and uniform way.

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A. Useful auxiliary integrals

In this appendix we report on two integrals which were useful for the calculation of boxes 11, 12 and 13. The first one is defined as

$$V(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) = \int_0^1 d\gamma \, \frac{(m_3^2 - m_4^2 - p_2^2 + s_{23})}{\gamma(m_3^2 - p_2^2) + (1-\gamma)(m_4^2 - s_{23})} \times \ln\left(\frac{-\gamma(1-\gamma)p_3^2 + \gamma m_3^2 + (1-\gamma)m_4^2}{m_3^2}\right). \tag{A.1}$$

The result for the integral V is given in terms of the roots $\gamma_\pm$ of the quadratic equation

$$-\gamma(1-\gamma)p_3^2 + \gamma m_3^2 + (1-\gamma)m_4^2 = 0\,. \tag{A.2}$$

Hence $\gamma_+\gamma_- = \frac{m_4^2}{p_3^2}$ and $(1-\gamma_+)(1-\gamma_-) = \frac{m_3^2}{p_3^2}$ and

$$\gamma_\pm = \frac{(p_3^2 + m_4^2 - m_3^2) \pm \sqrt{(p_3^2 + m_4^2 - m_3^2)^2 - 4m_4^2 p_3^2}}{2p_3^2}\,. \tag{A.3}$$

Let us define the position of the pole in the integrand as γ_0,

$$\gamma_0 = \frac{m_4^2 - s_{23}}{m_4^2 - s_{23} - m_3^2 + p_2^2}\,. \tag{A.4}$$

In terms of these variables V is given by

$$V(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) = \int_0^1 d\gamma \, \frac{1}{\gamma - \gamma_0} \ln\left(\frac{(\gamma-\gamma^+)(\gamma-\gamma^-)}{(1-\gamma^+)(1-\gamma^-)}\right). \tag{A.5}$$

With this notation the result for V is

$$V(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) = -\mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\frac{\gamma^+}{\gamma^+ - 1}\Big) - \mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\frac{\gamma^-}{\gamma^- - 1}\Big) + \mathrm{Li}_2\Big(\frac{1}{1-\gamma^+}\Big) + \mathrm{Li}_2\Big(\frac{1}{1-\gamma^-}\Big) + 2\,\mathrm{Li}_2\Big(\frac{1}{\gamma_0}\Big)\,. \tag{A.6}$$

If we set $p_2^2 = m_3^2$ the pole is at $\gamma_0 = 1$ and we get

$$V(m_3^2, p_3^2; s_{23}; m_3^2, m_4^2) = \mathrm{Li}_2\Big(\frac{1}{1-\gamma^+}\Big) + \mathrm{Li}_2\Big(\frac{1}{1-\gamma^-}\Big)\,. \tag{A.7}$$

If we further set $m_3^2 = m_4^2 = m^2$ we have that $\gamma^+ + \gamma^- = 1$ and we obtain

$$V(m^2, p_3^2; s_{23}; m^2, m^2) = \mathrm{Li}_2\Big(\frac{1}{1-\gamma^+}\Big) + \mathrm{Li}_2\Big(\frac{1}{\gamma^+}\Big) = -\frac{1}{2}\ln^2\left(\frac{\gamma^+ - 1}{\gamma^+}\right). \tag{A.8}$$

In the limit $p_3^2 \to 0$ we have

$$\frac{\gamma_{34}^+}{\gamma_{34}^+ - 1} \to 1 + \mathcal{O}(p_3^2), \quad \frac{\gamma_{34}^-}{\gamma_{34}^- - 1} \to \frac{m_4^2}{m_3^2} + \mathcal{O}(p_3^2)\,, \tag{A.9}$$

and eq. (A.6) simplifies to

$$\begin{aligned} V(p_2^2, 0; s_{23}; m_3^2, m_4^2) &= \mathrm{Li}_2\Big(1 - \frac{(m_3^2 - p_2^2)}{(m_4^2 - s_{23})}\Big) + \mathrm{Li}_2\Big(1 - \frac{m_4^2}{m_3^2}\Big) \\ &\quad - \mathrm{Li}_2\Big(1 - \frac{m_4^2(m_3^2 - p_2^2)}{m_3^2(m_4^2 - s_{23})}\Big). \end{aligned} \tag{A.10}$$

If we further set $p_2^2 = m_3^2$ we get

$$V(m_3^2, 0; s_{23}; m_3^2, m_4^2) = \mathrm{Li}_2\Big(1 - \frac{m_4^2}{m_3^2}\Big). \tag{A.11}$$

Finally, if we also set $m_4^2 = m_3^2 = m^2$ we obtain

$$V(m^2, 0; s_{23}; m^2, m^2) = 0\,. \tag{A.12}$$

Now consider a related integral

$$\begin{aligned} W(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) &= \int_0^1 d\gamma \, \frac{(m_3^2 - m_4^2 - p_2^2 + s_{23})}{\gamma(m_3^2 - p_2^2) + (1-\gamma)(m_4^2 - s_{23})} \\ &\quad \times \ln\Big(\frac{-\gamma(1-\gamma)p_3^2 + \gamma m_3^2 + (1-\gamma)m_4^2}{\gamma m_3^2 + (1-\gamma)m_4^2}\Big). \end{aligned} \tag{A.13}$$

Since

$$W(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) = V(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) - V(p_2^2, 0; s_{23}; m_3^2, m_4^2)\,, \tag{A.14}$$

we find

$$\begin{aligned} W(p_2^2, p_3^2; s_{23}; m_3^2, m_4^2) &= \mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\Big) + \mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\frac{m_4^2}{m_3^2}\Big) \\ &\quad - \mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\frac{\gamma^+}{\gamma^+ - 1}\Big) - \mathrm{Li}_2\Big(1 - \frac{\gamma_0 - 1}{\gamma_0}\frac{\gamma^-}{\gamma^- - 1}\Big) \\ &\quad + \mathrm{Li}_2\Big(\frac{1}{1 - \gamma^+}\Big) + \mathrm{Li}_2\Big(\frac{1}{1 - \gamma^-}\Big) - \mathrm{Li}_2\Big(1 - \frac{m_4^2}{m_3^2}\Big). \end{aligned} \tag{A.15}$$

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