CSC 321

1. (a) $\theta_i^{(t+1)} = \theta_i^t - \alpha a_i(\theta_i^t - r_i)$

(b) $e_i^{(t+1)} = \theta_i^{(t+1)} - r_i = \theta_i^t - \alpha a_i e_i^t - r_i$

$= e_i^t - \alpha a_i e_i^t$

$= (1-\alpha a_i)e_i^t$

(c) $e_i^{(t+1)} = (1-\alpha a_i)e_i^t$

$\Rightarrow e_i^t = (1-\alpha a_i)\, e_i^{t-1}$

$= (1-\alpha a_i)^2 e_i^{t-2}$

$\vdots$

$= (1-\alpha a_i)^k e_i^{t-k}$

set $k=t$

$e_i^t = (1-\alpha a_i)^t e_i^0$

① when $-1<1-\alpha a_i<1$, the procedure is stable.

$\begin{cases} \alpha a_i < 2 \\ \alpha a_i > 0 \end{cases} \Rightarrow \{ 0 < \alpha < \frac{2}{a_i}$

② when $1-\alpha a_i>1$ or $1-\alpha a_i<-1$, the procedure is unstable

$\begin{cases} 1-\alpha a_i > 1 \\ \text{or} \\ 1-\alpha a_i < -1 \end{cases} \Rightarrow \begin{cases} \alpha < 0 \\ \text{or} \\ \alpha > \frac{2}{a_i} \end{cases}$

(d) $C(\theta^{(t)}) = \sum_{i=1}^{N} \frac{a_i}{2}(e_i^{(t)})^2 = \sum_{i=1}^{N} \frac{a_i}{2}((1-\alpha a_i)^t e_i^0)^2$

$= \sum_{i=1}^{N} \frac{a_i}{2}((1-\alpha a_i)^t(\theta_i^{(0)} - r_i))^2$

$= \sum_{i=1}^{N} \frac{a_i}{2}(\underbrace{(1-\alpha a_i)^{2t}}(\theta_i^{(0)2} + r_i^2 - 2\theta_i^{(0)} r_i))$

dominant term because as $t \to \infty$, $(1-\alpha a_i)^{2t} \to 0$

2. (a) $E(y) = E(\sum_j m_j w_j x_j) = \sum_j w_j x_j E(m_j) = \sum_j \frac{1}{2} w_j x_j$
$= \frac{1}{2} \mathbf{w}^T \mathbf{x}$

$Var(y) = Var(\sum_j m_j w_j x_j) = \sum_j w_j^2 x_j^2 Var(m_j)$
$= \sum_j w_j^2 x_j^2 \frac{1}{4}$

(b) $E(y) = \tilde{y} = \sum_j \tilde{w}_j x_j = \frac{1}{2} \sum_j w_j x_j$
$\Rightarrow \tilde{w}_j = \frac{1}{2} w_j$

(c) $\mathcal{E} = \frac{1}{2N} \sum_{i=1}^{N} E[(y^{(i)} - t^{(i)})^2]$
$= \frac{1}{2N} \sum_{i=1}^{N} E[(y^{(i)})^2 + (t^{(i)})^2 - 2y^{(i)} t^{(i)}]$
$= \frac{1}{2N} \sum_{i=1}^{N} Var(y^{(i)}) + (E[y^{(i)}])^2 + (t^{(i)})^2 - 2t^{(i)} E[y^{(i)}]$
$= \frac{1}{2N} \sum_{i=1}^{N} Var(y^{(i)}) + \tilde{y}^2 + (t^{(i)})^2 - 2t^{(i)} \tilde{y}$
$= \frac{1}{2N} \sum_{i=1}^{N} (\tilde{y} - t^{(i)})^2 + Var(y^{(i)})$
$= \frac{1}{2N} \sum_{i=1}^{N} (\tilde{y} - t^{(i)})^2 + \sum_j w_j^{(i)^2} x_j^{(i)^2} \cdot \frac{1}{4}$
$= \frac{1}{2N} \sum_{i=1}^{N} (\tilde{y} - t^{(i)})^2 + \sum_j (\frac{1}{2} w_j^{(i)})^2 x_j^{(i)^2}$
$= \frac{1}{2N} \sum_{i=1}^{N} (\tilde{y} - t^{(i)})^2 + \sum_j \tilde{w}_j^{(i)^2} x_j^{(i)^2}$

Thus, $R = \sum_j \tilde{w}_j^{(i)^2} x_j^{(i)^2}$