Homework 3

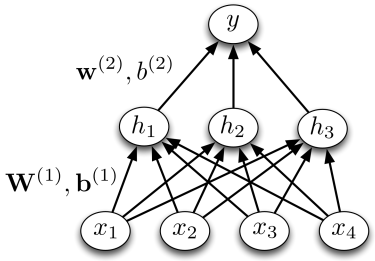
Deadline: Wednesday, Jan. 31, at 11:59pm.

Submission: You must submit your solutions as a PDF file through MarkUs[1]. You can produce the file however you like (e.g. LaTeX, Microsoft Word, scanner), as long as it is readable.

Late Submission: MarkUs will remain open until 2 days after the deadline; until that time, you should submit through MarkUs. If you want to submit the assignment more than 2 days late, please e-mail it to `csc321ta@cs.toronto.edu`. The reason for this is that MarkUs won't let us collect the homeworks until the late period has ended, and we want to be able to return them to you in a timely manner.

Weekly homeworks are individual work. See the Course Information handout[2] for detailed policies.

1. **Hard-Coding a Network. [2pts]** In this problem, you need to find a set of weights and biases for a multilayer perceptron which determines if a list of length 4 is in sorted order. More specifically, you receive four inputs $x_1, \ldots, x_4$, where $x_i \in \mathbb{R}$, and the network must output 1 if $x_1 < x_2 < x_3 < x_4$, and 0 otherwise. You will use the following architecture:



All of the hidden units and the output unit use a hard threshold activation function:

$$\phi(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

Please give a set of weights and biases for the network which correctly implements this function (including cases where some of the inputs are equal). Your answer should include:

- A 3×4 weight matrix $\mathbf{W}^{(1)}$ for the hidden layer
- A 3-dimensional vector of biases $\mathbf{b}^{(1)}$ for the hidden layer
- A 3-dimensional weight vector $\mathbf{w}^{(2)}$ for the output layer
- A scalar bias $b^{(2)}$ for the output layer

You do not need to show your work.

[1] `https://markus.teach.cs.toronto.edu/csc321-2018-01`

[2] `http://www.cs.toronto.edu/~rgrosse/courses/csc321_2018/syllabus.pdf`

2. **Backprop.** Consider a neural network with N input units, N output units, and K hidden units. The activations are computed as follows:

$$\begin{aligned}\mathbf{z} &= \mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)} \\ \mathbf{h} &= \sigma(\mathbf{z}) \\ \mathbf{y} &= \mathbf{x} + \mathbf{W}^{(2)}\mathbf{h} + \mathbf{b}^{(2)},\end{aligned}$$

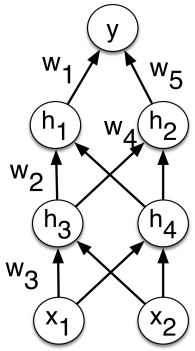
where σ denotes the logistic function, applied elementwise. The cost will involve both $\mathbf{h}$ and $\mathbf{y}$:

$$\begin{aligned}\mathcal{E} &= \mathcal{R} + \mathcal{S} \\ \mathcal{R} &= \mathbf{r}^\top \mathbf{h} \\ \mathcal{S} &= \frac{1}{2}\|\mathbf{y} - \mathbf{s}\|^2\end{aligned}$$

for given vectors $\mathbf{r}$ and $\mathbf{s}$.

- **[1pt]** Draw the computation graph relating $\mathbf{x}$, $\mathbf{z}$, $\mathbf{h}$, $\mathbf{y}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{E}$.
- **[3pts]** Derive the backprop equations for computing $\overline{\mathbf{x}} = \partial\mathcal{E}/\partial\mathbf{x}$. You may use σ' to denote the derivative of the logistic function (so you don't need to write it out explicitly).

3. **Sparsifying Activation Function.** **[4pts]** One of the interesting features of the ReLU activation function is that it sparsifies the activations and the derivatives, i.e. sets a large fraction of the values to zero for any given input vector. Consider the following network:



Note that each w_i refers to the weight on a *single* connection, not the whole layer. Suppose we are trying to minimize a loss function $\mathcal{L}$ which depends only on the activation of the output unit y. (For instance, $\mathcal{L}$ could be the squared error loss $\frac{1}{2}(y-t)^2$.) Suppose the unit h_1 receives an input of -1 on a particular training case, so the ReLU evaluates to 0. Based only on this information, which of the weight derivatives

$$\frac{\partial\mathcal{L}}{\partial w_1}, \quad \frac{\partial\mathcal{L}}{\partial w_2}, \quad \frac{\partial\mathcal{L}}{\partial w_3}$$

are **guaranteed** to be 0 for this training case? Write YES or NO for each. Justify your answers.