Map Coloring Theorems

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Outline

1. Motivation

2. Duality

3. Coloring graphs

4. Coloring algorithm

5. Euler's Formula

6. Art Galleries

Motivation

Definition.

A *k-coloring* of a map is an assignment of k colors, one to each country, in such a way that no two countries sharing a border have the same color.

A coloring of a map of Europe in thirteen colors

Motivation

Definition.

> A map is k-colorable if it has a coloring with at most k colors.

A coloring of a map of Europe in five colors

Motivation

De Morgan (1982)

> *A student of mine asked me today to give him a reason for a fact which I did not know was a fact - and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured - four colours may be wanted, but not more - the following is the case in which four colours are wanted. Query cannot a necessity for five or more be invented.*

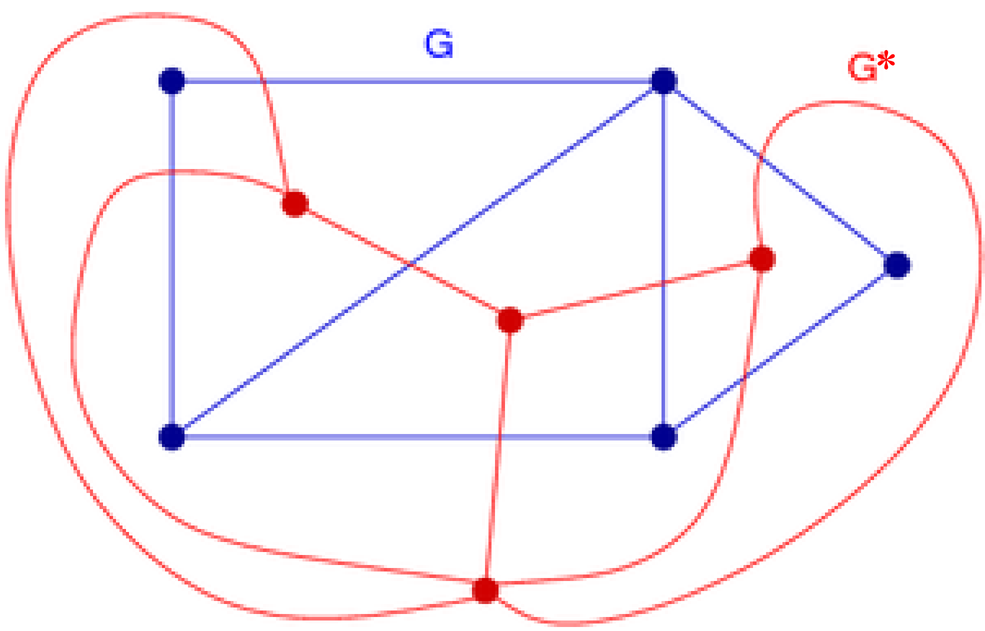
The 4-Color Conjecture (1852)

> Is every map 4-colorable?

Duality

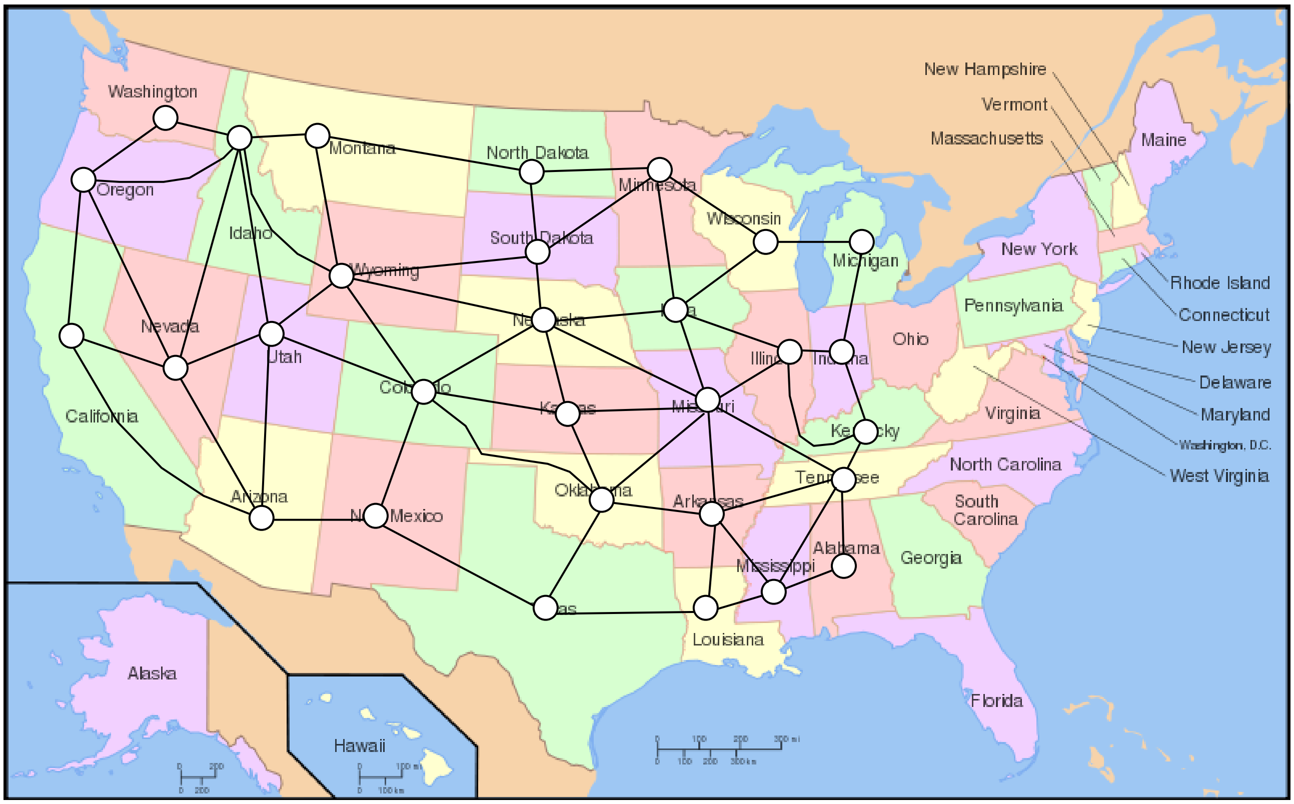
Definition.

The dual G^* of a map G is obtained by replacing each state with its capital and then putting an edge between two capitals if the states share a border.



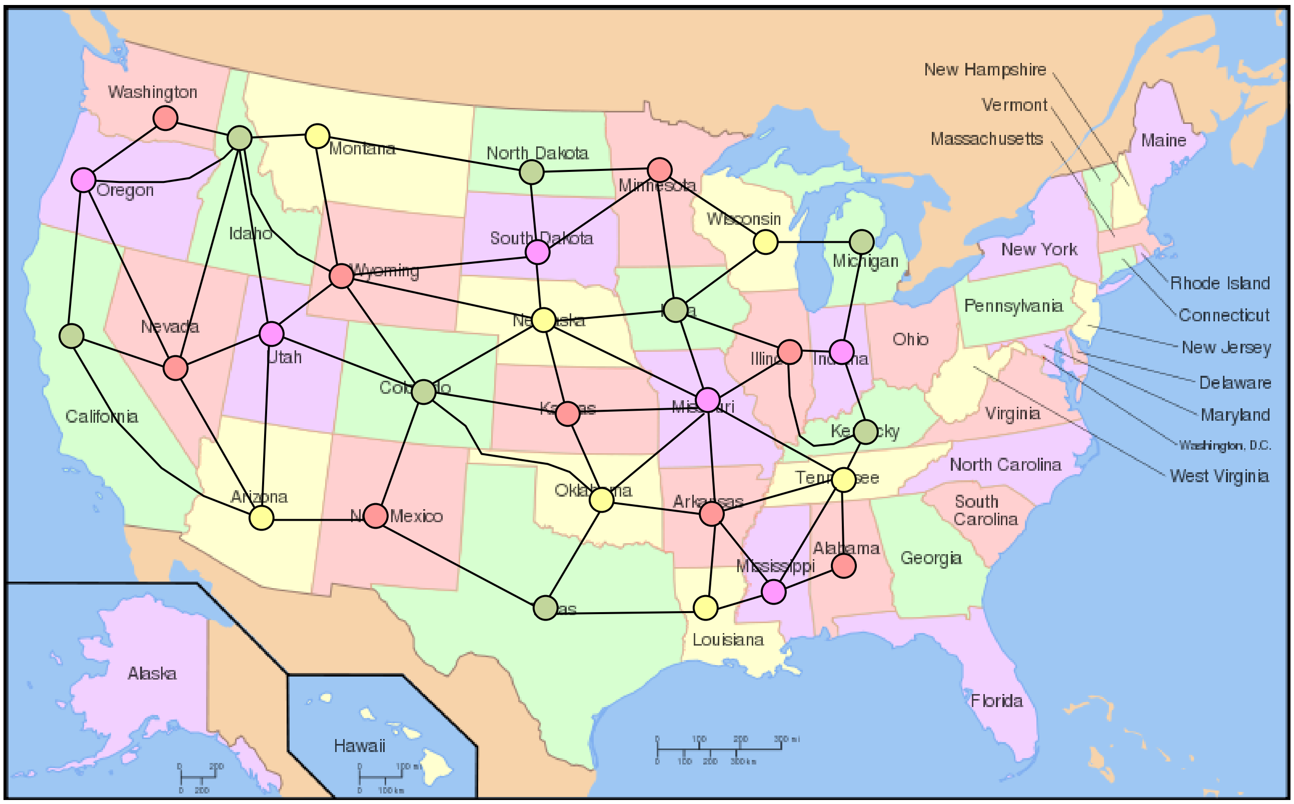
A simple example of duality

Duality



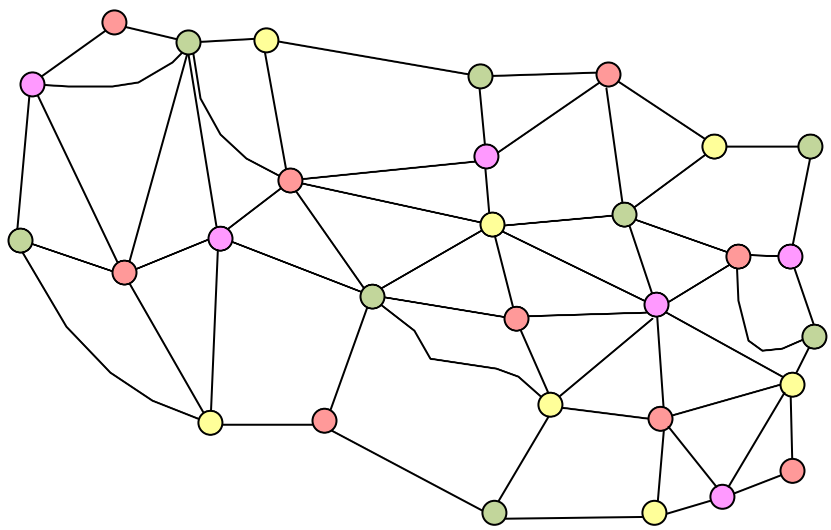
The dual of the USA

Duality



The dual of the USA

Duality

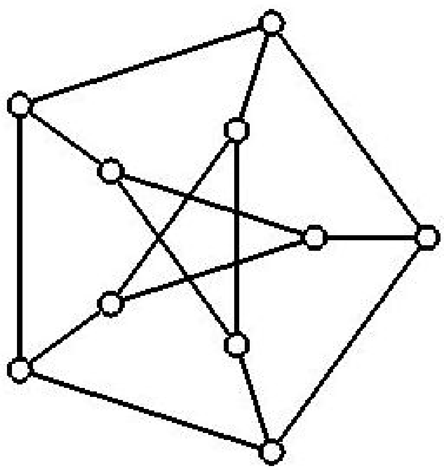


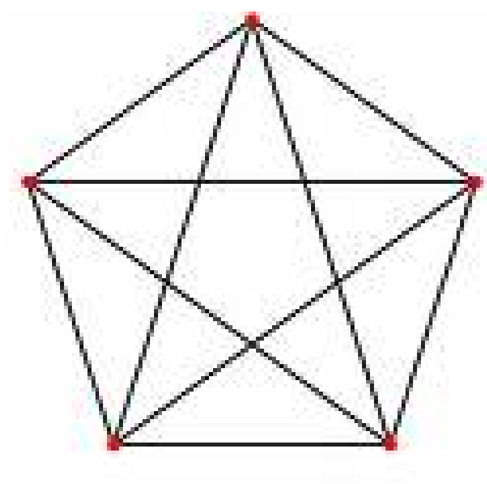
Coloring the dual of the USA

Coloring graphs

Definition.

A graph consists of vertices together with edges joining pairs of vertices.





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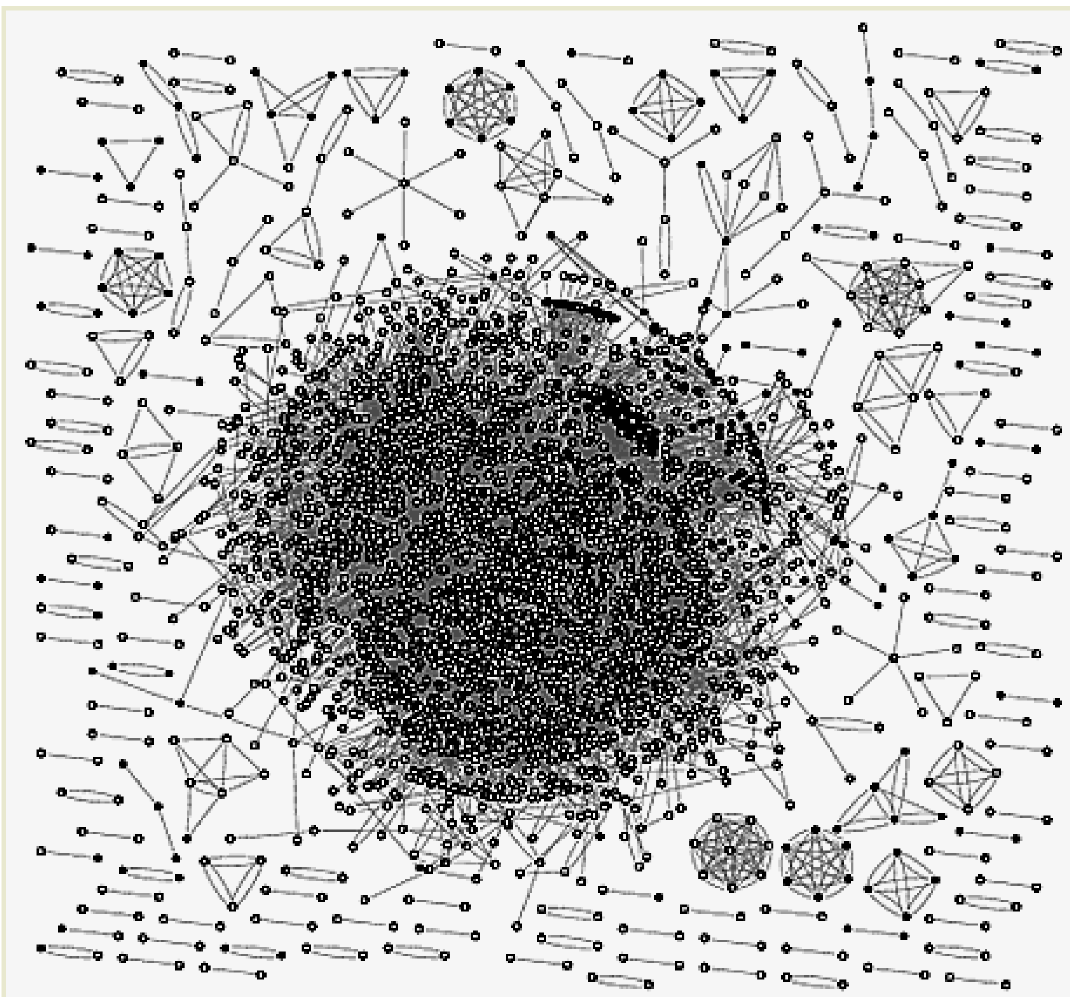
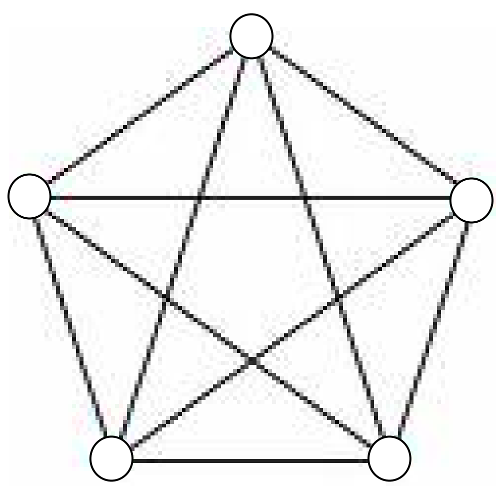


Coloring graphs

Definition.

A k-coloring of a graph is an assignment of k colors, one to each vertex, in such a way that no two adjacent vertices share the same color.

The chromatic number of a graph G is the minimum k for which a k-coloring exists. It is written $\chi(G)$.



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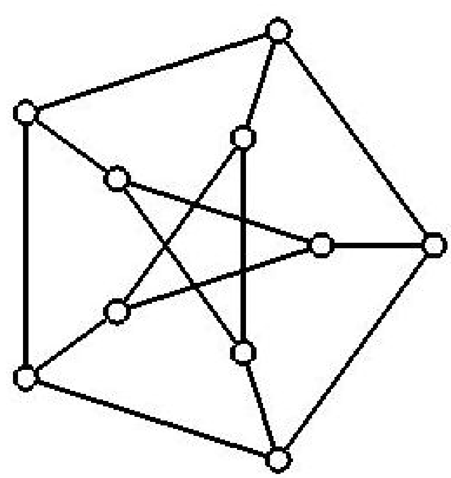
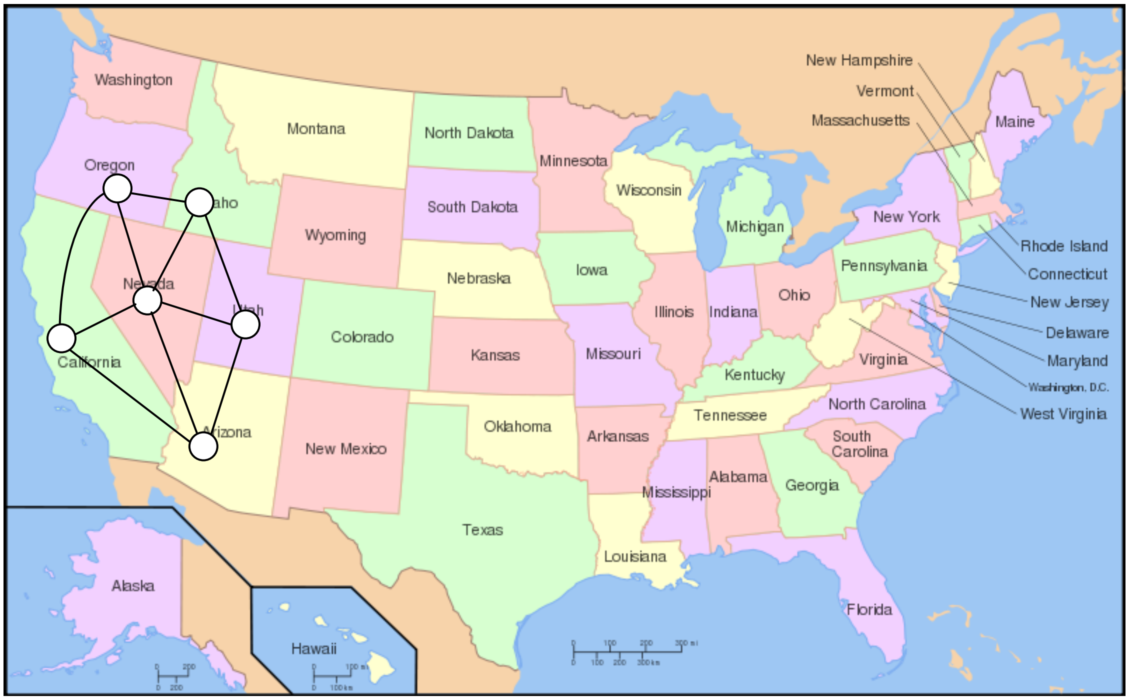
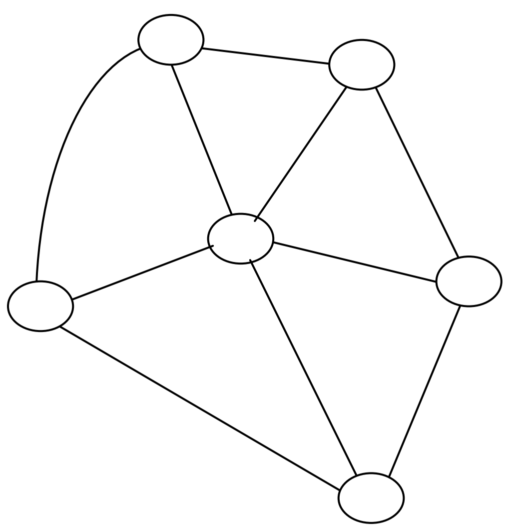


Coloring graphs



A 4-coloring of the USA



Coloring graphs

Observation 1.

If a graph G contains a graph H, then

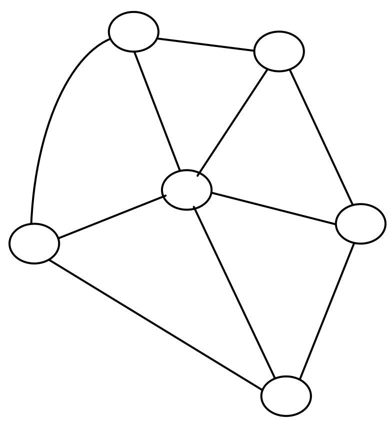
$$\chi(G) \quad \geq \quad \chi(H)$$

Therefore the chromatic number of the USA is 4.

Coloring algorithm

Definition.

The degree of a vertex in a graph is the number of edges out of that vertex.

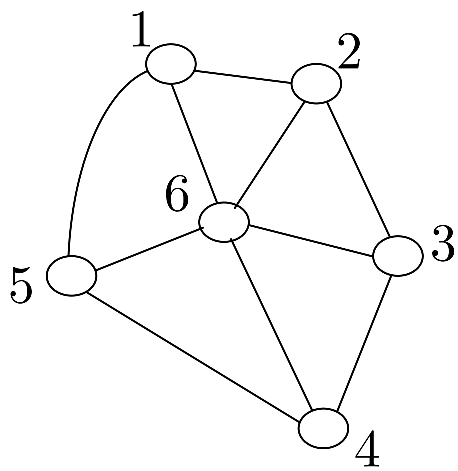


The minimum degree of a graph is the minimum of all the degrees of the vertices in the graph.

Coloring algorithm

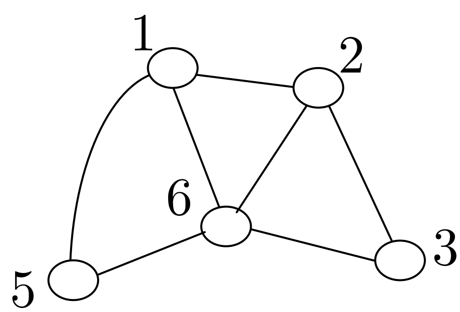
```
"
Uvctv"ykvj"c"itcrj"I3"ykvj"p"xgtvkegu0"
"
"   Hqt"k"?"3"vq"p"fq"
"
        30  Hkpf"c"xgtvgz"xk"qh"uocnnguv"
            fgitgg"kp"Ik0"
        40  Tgrnceg"Ik"ykvj"Ik"/"xk0"
"
Eqnqt"xp"ykvj"eqnqt"30""
"
"   Hqt"k"?"p/3"vq"3"fq"
"
        Eqnqt"xk"ykvj"vjg"hktuv"cxckncdng"
        eqnqt"pqv"wugf"qp"vjg"pgkijdqtu"
        qh"xk"coqpiuv"xl"hqt"l"@"k0"
```

Coloring algorithm



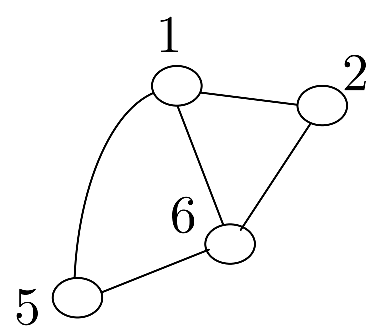
Coloring algorithm

Pick a vertex of smallest degree, say 4.
Delete it



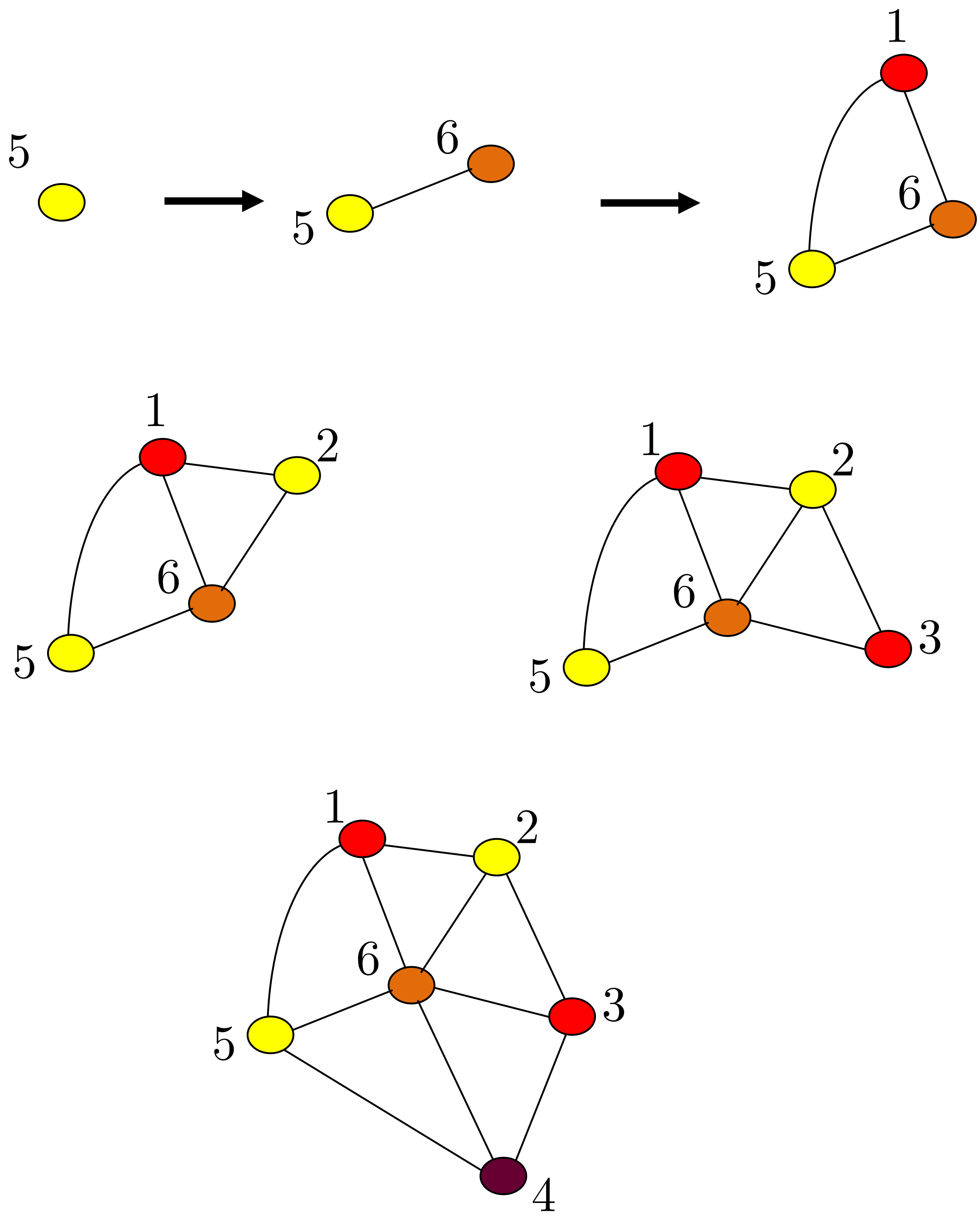
Coloring algorithm

Pick vertex of smallest degree, say 3.
Delete it.



Pick vertex of smallest degree, say 2.
Delete it.

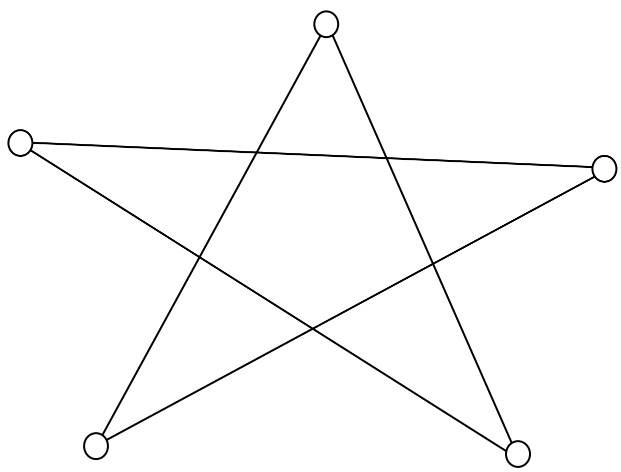
Coloring algorithm

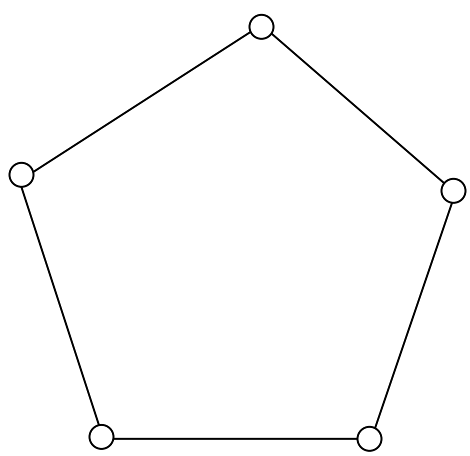


Euler's Formula

Definition.

A planar graph is a graph which can be drawn in the plane without any two edges crossing.





Euler's Formula

Definition.

> A face of a graph drawn without crossings is one of the lakes that you get when you pour water on the picture and treat the edges as walls.

Euler's Formula

If G is a graph drawn without crossings in the plane, let $f(G), n(G), e(G)$ be the number of faces, vertices and edges in the graph.

Euler's Formula (1752)

For any connected plane graph G

$$n(G) - f(G) + e(G) = 2$$

Euler's Formula

How to prove Euler's Formula:

Basic idea

If you delete one edge from a cycle in the graph, then the number of faces drops by 1.

So continuing deleting edges in cycles,

$$n(G) - f(G) + e(G)$$

never changes until there are no cycles left.

But if there are no cycles left, then the graph is a tree. For a tree we have

$$e(G) = n(G) - 1 \qquad f(G) = 1$$

and so the value of the formula is 2.

Euler's Formula

Definition.

The degree of a face surrounded by a cycle is the number of edges in that cycle.

Observation.

If $d(F)$ is the degree of face F in a graph G, then

$$\sum_F d(F) = 2e(G)$$

This follows from the fact that if we add up the number of edges on the boundary of each face, then each edge of the graph is counted twice.

Euler's Formula

Theorem (Plato, <360bc)

There is only one connected planar graph where all faces are quadrilaterals and all vertices have degree three.

How to prove Plato's Theorem:

Euler's Formula says

$$n(G) - e(G) + f(G) = 2$$

Adding up degrees of faces says

$$4f(G) = 2e(G)$$

Adding up degrees of vertices says

$$3n(G) = 2e(G)$$

Euler's Formula

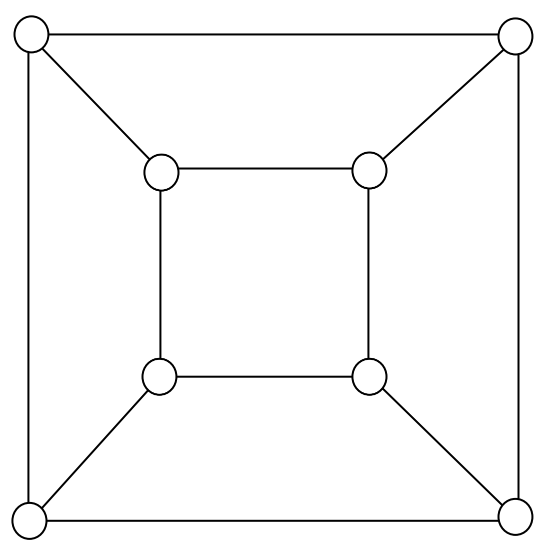
The only solution to the equations

$$n - e + f = 2$$

$$4f = 2e$$

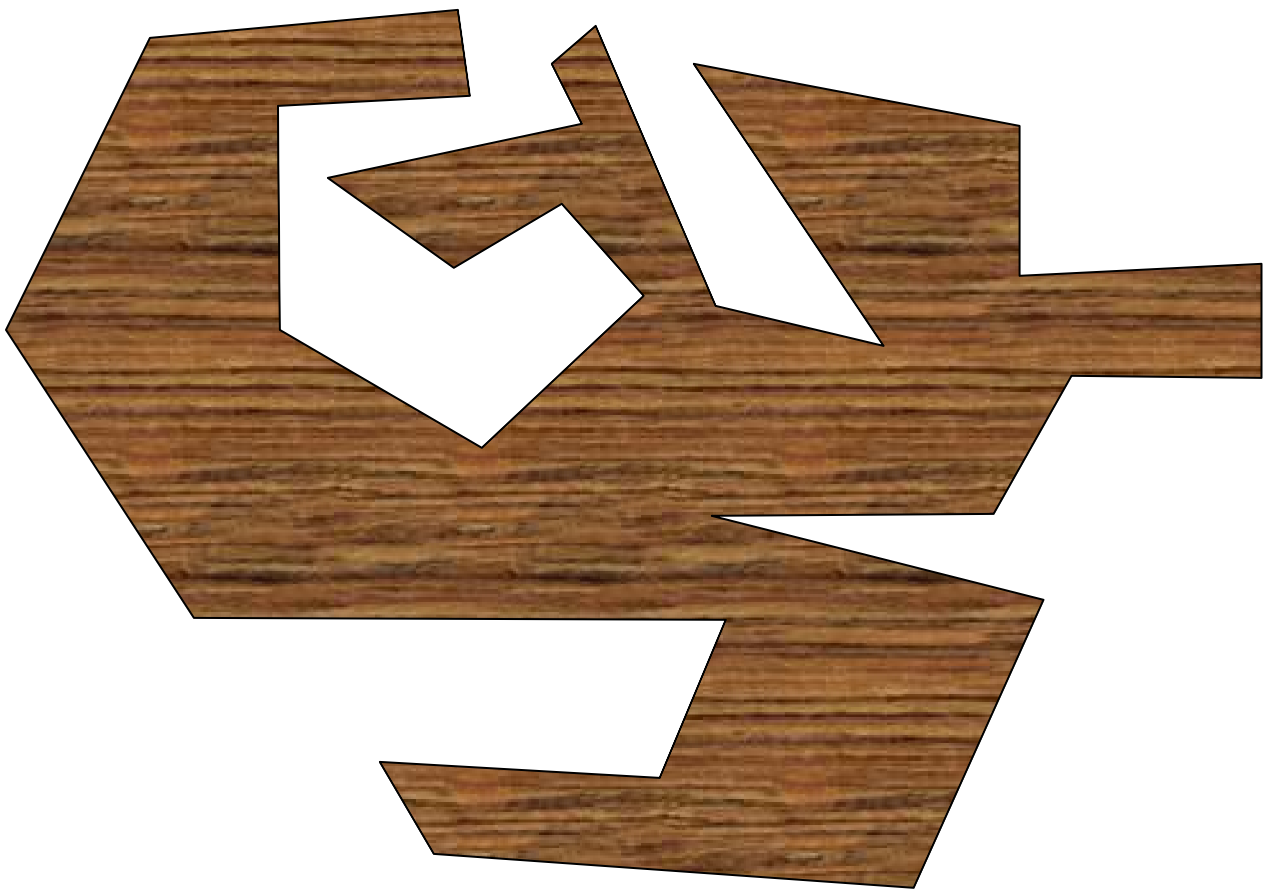
$$3n = 2e$$

is $n = 8, e = 12, f = 6$. We recognize the cube:



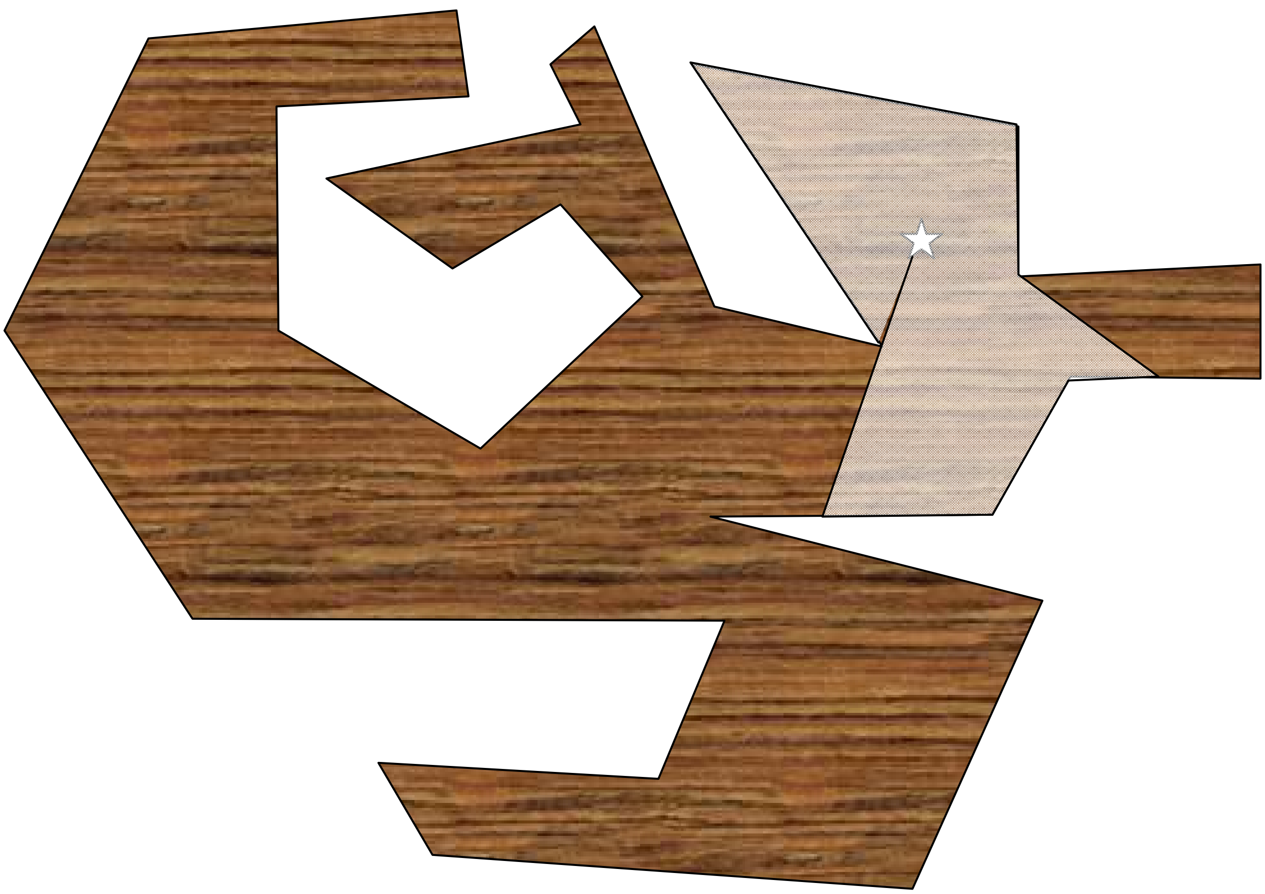
Art Galleries

An art gallery is a region in the plane bounded by a simple polygon.



Art Galleries

A guard is a point and the visibility of a guard is the set of points which can be joined to the guard by a straight line that does not cross through a wall of the art gallery.



Art Galleries

For each art gallery, what is the smallest number of guards required so that every point of the art gallery is visible to at least one guard?

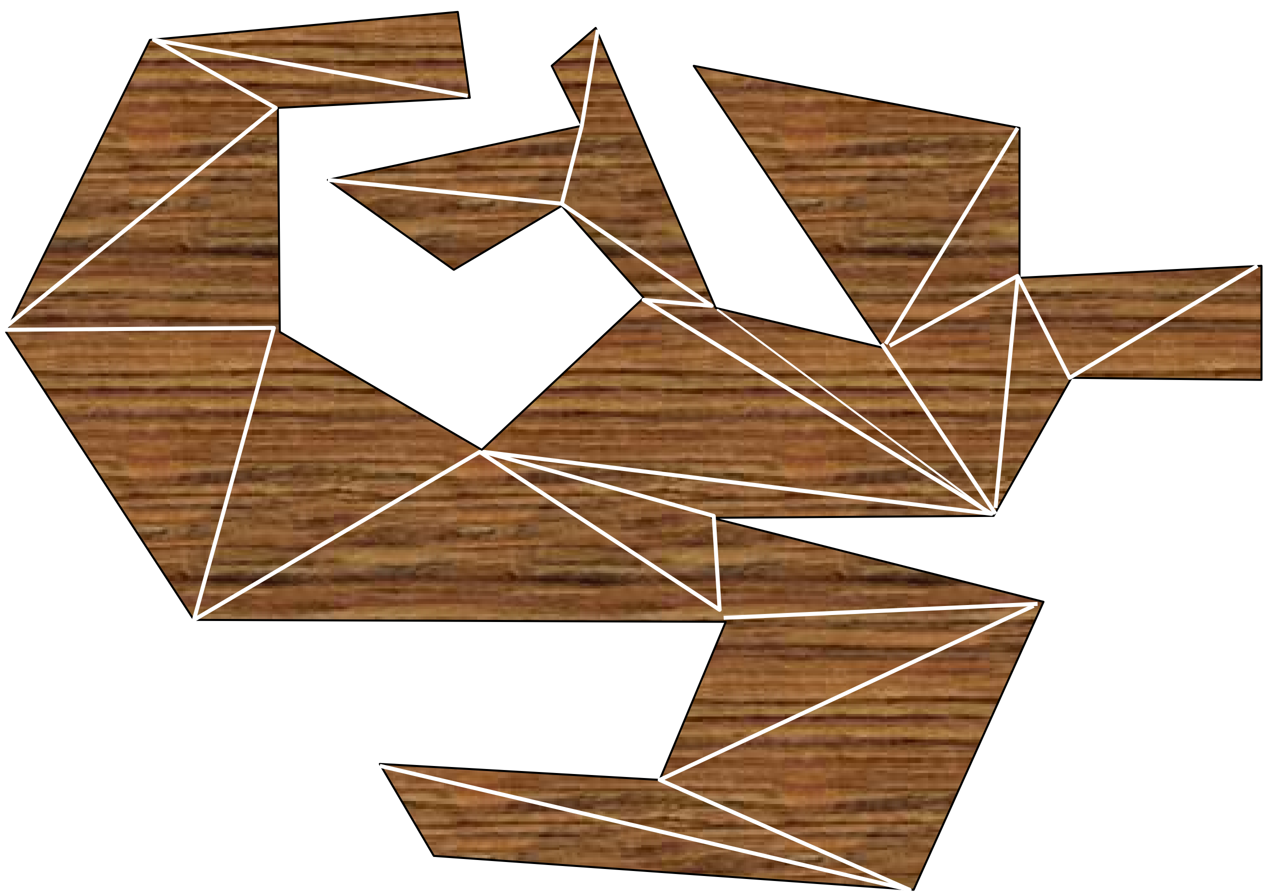
Art Gallery Theorem (Chvátal, 1975)

For any art gallery with n corners, $n/3$ guards are enough to guard the whole gallery.

Art Galleries

How to prove the art gallery theorem

Triangulate the polygon



Art Galleries

Then we get a planar graph $\mathcal{A}$ with n vertices. $\mathcal{A}$ has a vertex of degree two. Remove the vertex of degree two from $\mathcal{A}$. We get another triangulation of an art gallery. Again find a vertex of degree two and remove it.

So we conclude that $\mathcal{A}$ is built up from a single vertex by repeatedly adding a vertex of degree at most two.

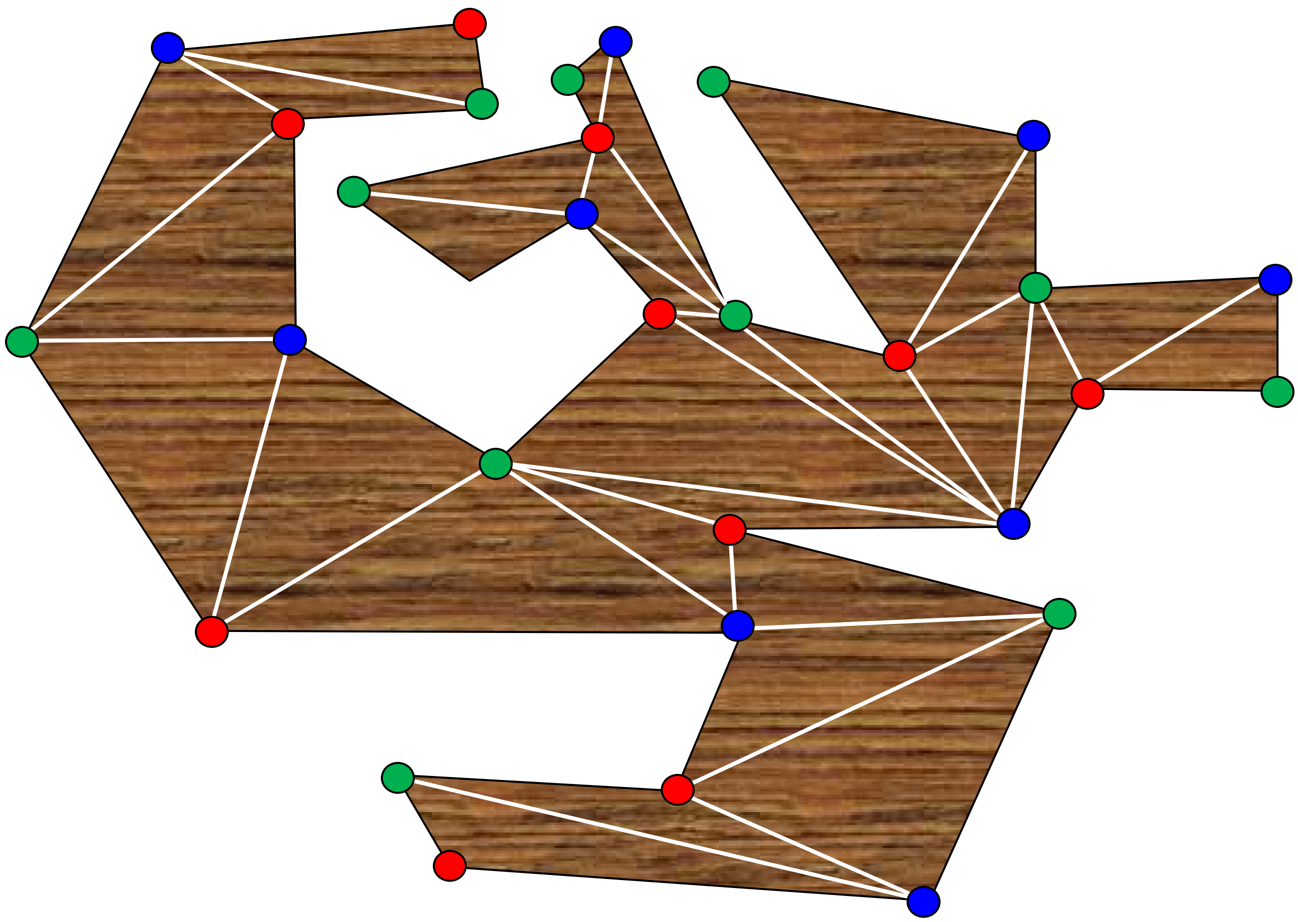
By the coloring algorithm it is 3-colorable.

Pick color 1. Then every triangle has color 1 on it since three colors are needed to color a triangle.

Then place guards at vertices of color 1 and they see the whole art gallery.

Art Galleries

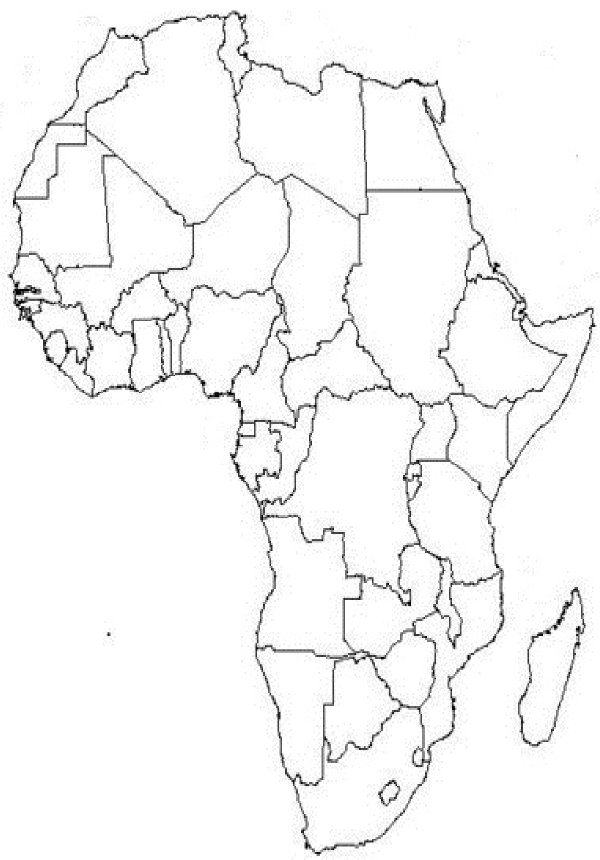
How to prove the art gallery theorem



(1) Can you draw an art gallery with
   (a) 6 corners and cannot be guarded by 1 guard?
   (b) 7 corners and cannot be guarded by 2 guards?
   (c) $3n$ corners that cannot be guarded by $n-1$ guards?

(3)(a) Color the map below with four colors. Can it be colored with three colors?

(b) Find the chromatic number of each continent shown below.



Is it possible to hang a picture on a wall using a string and two nails in such a way that the picture does not fall but if either nail is removed then the picture falls down.