Binary Heaps

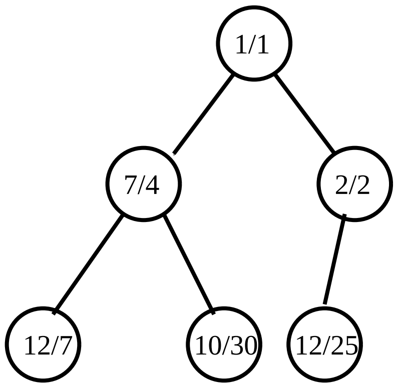
A *binary heap* is defined to be a binary tree with a key in each node such that:

1. All leaves are on, at most, two adjacent levels.

2. All leaves on the lowest level occur to the left, and all levels except the lowest one are completely filled.

3. The key in root is $\geq$ all its children, and the left and right subtrees are again binary heaps.

Conditions 1 and 2 specify shape of the tree, and condition 3 the labeling of the tree.



The ancestor relation in a heap defines a *partial order* on its elements, which means it is reflexive, anti-symmetric, and transitive.

1. *Reflexive:* x is an ancestor of itself.

2. *Anti-symmetric:* if x is an ancestor of y and y is an ancestor of x, then $x = y$.

3. *Transitive:* if x is an ancestor of y and y is an ancestor of z, x is an ancestor of z.

Partial orders can be used to model heirarchies with incomplete information or equal-valued elements. One of my favorite games with my parents is fleshing out the partial order of "big" old-time movie stars.

The partial order defined by the heap structure is weaker than that of the total order, which explains

1. Why it is easier to build.

2. Why it is less useful than sorting (but still very important).

Constructing Heaps

Heaps can be constructed incrementally, by inserting new elements into the left-most open spot in the array.

If the new element is greater than its parent, swap their positions and recur.

Since at each step, we replace the root of a subtree by a larger one, we preserve the heap order.

Since all but the last level is always filled, the height h of an n element heap is bounded because:

$$\sum_{i=1}^{h} 2^i = 2^{h+1} - 1 \geq n$$

so $h = \lfloor \lg n \rfloor$.

Doing n such insertions takes $\Theta(n \log n)$, since the last $n/2$ insertions require $O(\log N)$ time each.

Heapify

The bottom up insertion algorithm gives a good way to build a heap, but Robert Floyd found a better way, using a *merge* procedure called *heapify*.

Given two heaps and a fresh element, they can be merged into one by making the new one the root and trickling down.

```
Build-heap(A)
        $n = |A|$
        For $i = \lfloor n/2 \rfloor$ to 1 do
                Heapify(A,i)


Heapify(A,i)
        left = $2i$
        right = $2i + 1$
        if $(left \leq n)$ and $(A[left] > A[i])$ then
                max = left
                else max = i
        if $(right \leq n)$ and $(A(right] > A[max])$ then
                max = right
        if $(max \neq i)$ then
                swap(A[i],A[max])
                Heapify(A,max)
```

Rough Analysis of Heapify

Heapify on a subtree containing n nodes takes

$$T(n) \leq T(2n/3) + O(1)$$

The 2/3 comes from merging heaps whose levels differ by one. The last row could be exactly half filled. Besides, the asymptotic answer won't change so long the fraction is less than one.

Solve the recurrence using the Master Theorem.

Let $a = 1$, $b = 3/2$ and $f(n) = 1$.

Note that $\Theta(n^{\log_{3/2} 1}) = \Theta(1)$, since $\log_{3/2} 1 = 0$.

Thus Case 2 of the Master theorem applies.

The Master Theorem: Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows:

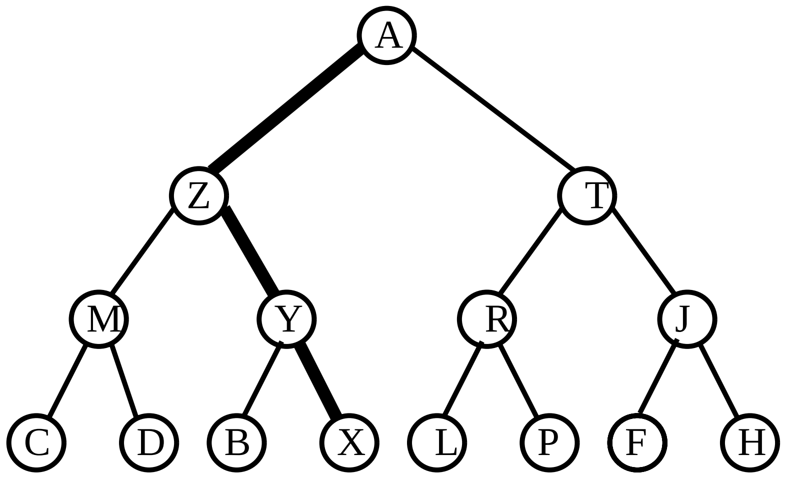
1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$, and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Exact Analysis of Heapify

In fact, Heapify performs better than $O(n \log n)$, because most of the heaps we merge are extremely small.



In a full binary tree on n nodes, there are $n/2$ nodes which are leaves (i.e. height 0), $n/4$ nodes which are height 1, $n/8$ nodes which are height 2, ...

In general, there are at most $\lceil n/2^{h+1} \rceil$ nodes of height h, so the cost of building a heap is:

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \lceil n/2^{h+1} \rceil O(h) = O(n \sum_{h=0}^{\lfloor \lg n \rfloor} h/2^h)$$

Since this sum is not quite a geometric series, we can't apply the usual identity to get the sum. But it should be clear that the series converges.

Proof of Convergence

Series convergence is the “free lunch” of algorithm analysis.

The identify for the sum of a geometric series is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

If we take the derivative of both sides, ...

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Multiplying both sides of the equation by x gives the identity we need:

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Substituting $x = 1/2$ gives a sum of 2, so Build-heap uses at most $2n$ comparisons and thus linear time.

The Lessons of Heapsort, I

"Are we doing a careful analysis? Might our algorithm be faster than it seems?"

Typically in our analysis, we will say that since we are doing at most x operations of at most y time each, the total time is $O(xy)$.

However, if we overestimate too much, our bound may not be as tight as it should be!

Heapsort

Heapify can be used to construct a heap, using the observation that an isolated element forms a heap of size 1.

```
Heapsort(A)
    Build-heap(A)
    for $i = n$ to 1 do
        swap(A[1],A[i])
        $n = n - 1$
        Heapify(A,1)
```

If we construct our heap from bottom to top using Heapify, we do not have to do anything with the last $n/2$ elements.

With the implicit tree defined by array positions, (i.e. the ith position is the parent of the $2i$th and $(2i+1)$st positions) the leaves start out as heaps.

Exchanging the maximum element with the last element and calling heapify repeatedly gives an $O(n \lg n)$ sorting algorithm, named *Heapsort*.

Heapsort Animations

The Lessons of Heapsort, II

Always ask yourself, "Can we use a different data structure?"

Selection sort scans throught the entire array, repeatedly finding the smallest remaining element.

```
For $i = 1$ to $n$
A:    Find the smallest of the first $n - i + 1$ items.
B:    Pull it out of the array and put it first.
```

Using arrays or unsorted linked lists as the data structure, operation A takes $O(n)$ time and operation B takes $O(1)$.

Using heaps, both of these operations can be done within $O(\lg n)$ time, balancing the work and achieving a better tradeoff.

Priority Queues

A *priority queue* is a data structure on sets of keys supporting the following operations:

- *Insert(S, x)* - insert x into set S
- *Maximum(S)* - return the largest key in S
- *ExtractMax(S)* - return and remove the largest key in S

These operations can be easily supported using a heap.

- *Insert* - use the trickle up insertion in $O(\log n)$.
- *Maximum* - read the first element in the array in $O(1)$.
- *Extract-Max* - delete first element, replace it with the last, decrement the element counter, then heapify in $O(\log n)$.

Applications of Priority Queues

Heaps as stacks or queues

- In a stack, *push* inserts a new item and *pop* removes the most recently pushed item.
- In a queue, *enqueue* inserts a new item and *dequeue* removes the least recently enqueued item.

Both stacks and queues can be simulated by using a heap, when we add a new *time* field to each item and order the heap according it this time field.

- To simulate the stack, increment the time with each insertion and put the maximum on top of the heap.
- To simulate the queue, decrement the time with each insertion and put the maximum on top of the heap (or increment times and keep the minimum on top)

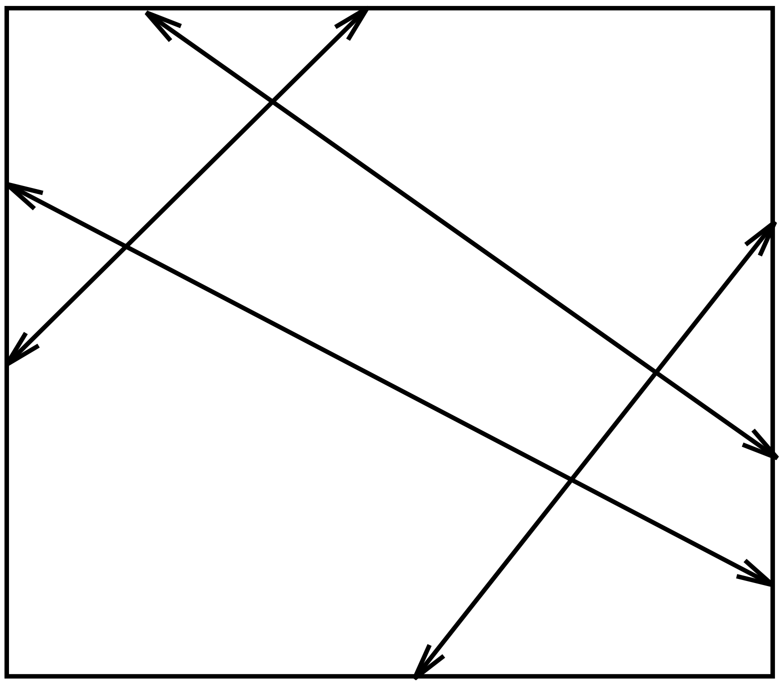
This simulation is not as efficient as a normal stack/queue implementation, but it is a cute demonstration of the flexibility of a priority queue.

Discrete Event Simulations

In simulations of airports, parking lots, and jai-alai – priority queues can be used to maintain who goes next.

The stack and queue orders are just special cases of orderings. In real life, certain people cut in line.

Sweepline Algorithms in Computational Geometry



In the priority queue, we will store the points we have not yet encountered, ordered by x coordinate. and push the line forward one stop at a time.