1.2.1 Secondary goals

- Learn proof techniques
- Learn some mathematics
- Have fun: Algorithms can be beautiful and ever poetic.

1.3 Typical Functions

- **polynomials** – $n, n^{\frac{1}{2}}, n^2, n^{10}, n^k$, etc (grows fast)
- **exponentials** – $2^n, 3^n, e^n, 11^{2n}$, etc (grows very fast)
- **logarithms** – $\log n = \log_2 n, \ln n, \log_{10} n$, etc (grows slowly)
- **poly logarithms** – $(\log n)^2$ (grows slowly)
- **log-logarithmic** – $\boldsymbol{\log\log n}$ (grows slowly)
- $\boldsymbol{\log^* n}$ – the number of times you have to take the log of a number before you get something less than **1**. (grows very slowly)

1.3.1 Illustrating the Growth of $\log\log n$ and $\log^* n$

- **Given:** Let N be the number of particles in the Universe. It is estimated that there are over 10^{80} particles in the universe.
  1. **Question:** What is the $\log\log$ of that number?
     - **Answer:** We know that $10^3 \approx 2^{10}$. This means that 10^{80} can be represented as $10^{3(27)}$, which is about 2^{270}. The $\log 2^{270} \approx 270$ and the $\log\log 2^{270}$ is about 8.1.
  2. **Question:** What is the $\log^*$ of that number?
     - **Answer:** $5 \leq \log^*(10^{80}) \leq 6$

Moral: $\log\log n$ **and** $\log^* n$ **are very slowly growing functions.**

1.4 Review of logs

Here are some basic properties of logs:

- $\log_a x = y \Longleftrightarrow x = a^y$
- $\log_a x = y$
- $\log_a(x \cdot y) = \log_a x + \log_a y$
- $\log_a x^y = y \cdot \log_a x$
- $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
- **Note that 'a' is the base of the log. If omitted, assume base of 2.**

1.4.1 Examples of logs

1. **Claim:** $2^{\log n} = n$.
   - **Proof:** Take the log of both sides:
     $\Rightarrow \log 2^{\log n} = \log n$
2. **Claim:** $n^{\frac{1}{\log n}} = 2$.
   - **Proof:** Let $n = 2^{\log n}$.
     $\Rightarrow (2^{\log n})^{\frac{1}{\log n}} = 2^1 = 2$.
3. **Claim:** $3^{\log n} = n^{\log 3}$.
   - **Proof:** Let $3 = 2^{log 3}$.
     $\Rightarrow (2^{\log 3})^{\log n} = 2^{\log 3 \cdot \log n} = (2^{\log n})^{\log 3} = n^{\log 3}$
4. **Question:** How to convert $\log_a n$ to $\log_b n$?
   - **Answer:** $\frac{\log_b n}{\log_a n} = \log_a b$.

1.5 Definitions of: $O(f(n)), \Omega(f(n)), \Theta(f(n)), o(f(n)), \omega(f(n))$

1.5.1 Definitions of $O(f(n))$ ("Big OH") ("Order")

We use O-notation to give an upper bound on a function, to within a constant factor. Figure 1.1 shows the intuition behind O-notation.

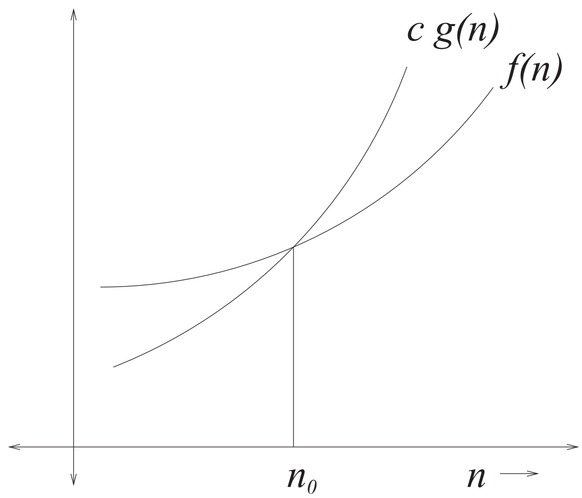

Figure 1.1: $f(n) = O(g(n))$

The following are three equivalent definitions of $O(f(n))$:

Definition 1. *$f(n) \in O(g(n))$ means $\exists$ constant $c > 0$ and some number n_0 such that for $n > n_0$, $f(n) \leq c \cdot g(n)$*

Definition 2. *$f(n) \in O(g(n))$ if $\exists c > 0$ such that for sufficiently large n, $c \cdot g(n) \geq f(n)$*

Definition 3. *$f(n) \in O(g(n))$ if $\exists c > 0$ such that,*

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) \leq c.$$

Examples:

- $n = O(n)$
- $n = O(n^2)$
- $\log n = O(\log n^2)$

We are interested in positive monotonically increasing functions and positive or non negative values of n.

1.5.2 Definitions of $\Omega(f(n))$ ("Omega")

Just as O-notation provides an asymptotic *upper* bound on a function, Ω-notation provides an **asymptotic lower bound**.

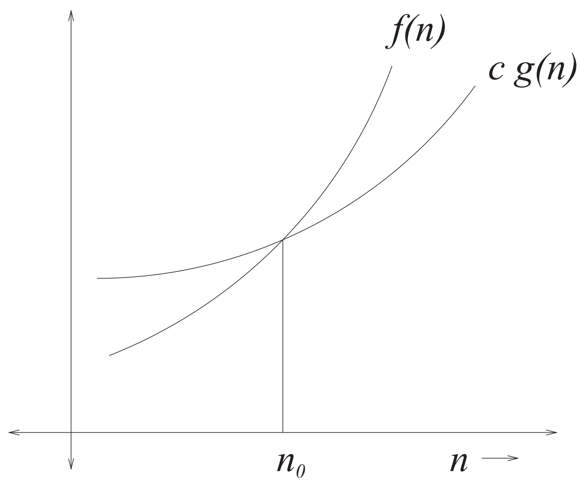

Figure 1.2: $f(n) = \Omega(g(n))$

The following are three equivalent definitions of $\Omega(f(n))$:

Definition 4. *$f(n) \in \Omega(g(n))$ if $\exists c > 0$ and $n_0 > 0$ such that $\forall_n > n_0$, $f(n) \geq c \cdot g(n)$.*

Definition 5. *$f(n) \in \Omega(g(n))$ if $\exists c > 0$ such that for sufficiently large n, $f(n) \geq c \cdot g(n)$.*

Definition 6. *$f(n) \in \Omega(g(n))$ if $\exists c > 0$ such that,*

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) > c.$$

Examples:

- $n^2 = \Omega(n)$.
- $n^2 = \Omega(n^2)$.
- $n^c = \Omega(\log n)$ for any constant c.

1.5.3 Definitions of $\Theta(f(n))$ ("Theta")

Θ-notation is used to provide **asymptotic tight bound** on a given function.

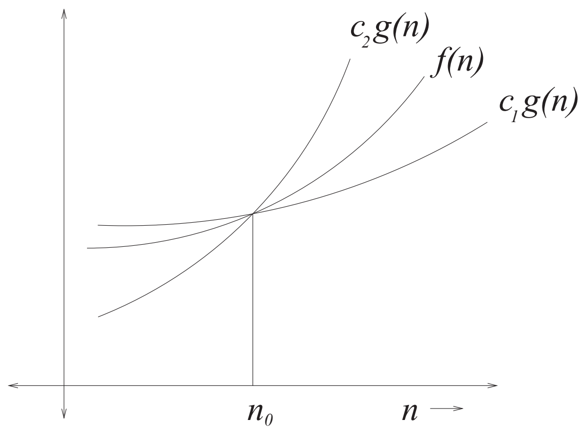

Figure 1.3: $f(n) = \Theta(g(n))$

The following are three definitions of $\Theta(f(n))$:

Definition 7. *$f(n) \in \Theta(g(n))$ means that $\exists c_1, c_2 > 0$ and $n_0 > 0$ such that $\forall_n > n_0$ $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ where $c_1 \leq c \leq c_2$*

Definition 8. *$f(n) \in \Theta(g(n))$ if $\exists c$ such that*

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = c.$$

Definition 9. *$f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$*

Counter Example

Shows that the second definition and third definition of $\boldsymbol{\Theta(g(n))}$ are not equivalent.

- Let $\boldsymbol{f(n) = 3^n}$,if $\boldsymbol{n}$ is even, $\boldsymbol{f(n) = 2 * 3^n}$, if $\boldsymbol{n}$ is odd, and $\boldsymbol{g(n) = 3^n}$.
- Then $\boldsymbol{f(n)}$ is still strictly increasing, but $\boldsymbol{\frac{f(n)}{g(n)} = 1}$, if $\boldsymbol{n}$ is even, and $\boldsymbol{\frac{f(n)}{g(n)} = 2}$, if $\boldsymbol{n}$ is odd.
- So the limit of $\boldsymbol{\frac{f(n)}{g(n)}}$ does not exit.
- Therefore, the third definition is correct.

1.5.4 Definition of $o(f(n))$ ("little oh")

Definition 10. $f(n) \in o(g(n))$ *means*

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0.$$

Also read "$g(n)$ dominates $f(n)$"

Examples:

- **Given:** Suppose $a > b$.
  1. **Question:** Does n^a dominate n^b ?
     - **Answer:** Yes. Because,

     $$\lim_{n\to\infty}\left(\frac{n^b}{n^a}\right) = 0.$$

  2. **Question:** What about $\log_a n$ and $\log_b n$. Does $log_a n$ dominate $\log_b n$?
     - **Answer:** No. Because,

     $$\lim_{n\to\infty}\left(\frac{\log_b n}{\log_a n}\right) = \log_a b \neq 0.$$

1.5.5 Definition of $\omega(f(n))$ ("little omega")

Definition 11. *If $f(n)$ is $\Omega(g(n))$, but not $\Theta(g(n))$, then $f(n)$ is said to be $\omega(g(n))$.*

Definition 12. $f(n) \in \omega(g(n)) \iff g(n) \in o(f(n))$

1.6 Examples:

1.6.1 True & False Questions:

1. **Claim:** $\log_2 n \in o(\log_{10} n)$
   - **Answer:** FALSE.
   - **Counter Example:** Let $n = 1000$,
     $\Rightarrow \log_2 1000 \approx 10$ and $\log_{10} 1000 = 3$.
     $10 > 3$ therefore, $\log_2 n$ is not $\in o(\log_{10} n)$.
2. **Claim:** $\log_2 n \in \Theta(\log_2 n)$.
   - **Answer:** TRUE.
   - **Proof:**

$$\lim_{n\to\infty}\left(\frac{\log_2 n}{c\cdot\log_2 n}\right) = c \Rightarrow \log_2 n \in \Theta(\log_2 n).$$

3. **Claim:** $\log_{10} n \in \Theta(\log_2 n)$.
   - **Answer:** TRUE.
   - **Proof:** Let $\log_{10} n = \frac{\log_2 n}{\log_2 10}$,

$$\lim_{n\to\infty}\left(\frac{\frac{\log_2 n}{\log_2 10}}{c\cdot\log_2 n}\right) = \frac{1}{c\cdot\log_2 10} \Rightarrow \log_{10} n \in \Theta(\log_2 n).$$

4. **Claim:** $3^n \in O(2^n)$.
   - **Answer:** FALSE.
   - **Counter Example:**

$$\lim_{n\to\infty}\left(\frac{2^n}{3^n}\right) = 0 \Rightarrow 2^n \in o(3^n).$$