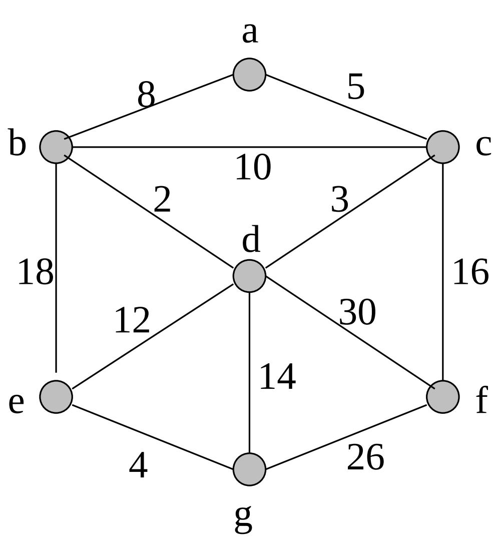
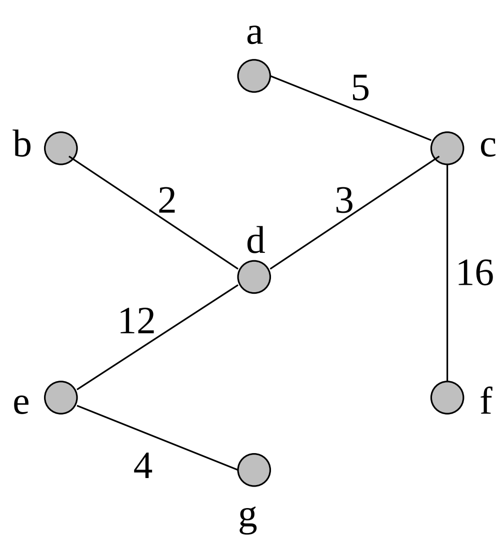
Minimum Spanning Trees

- **Given an *undirected* graph $G = (V, E)$, with edge costs c_{ij}.**
- **A spanning tree T of G is a cycle-free subgraph that spans all the nodes.**
- **The *cost* of T is the *sum* of the costs of the edges in T.**
- **MST is the smallest cost spanning tree.**

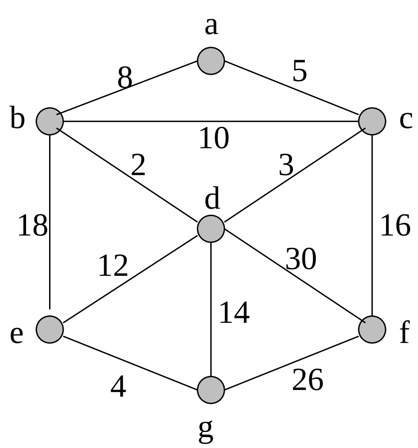


Graph G

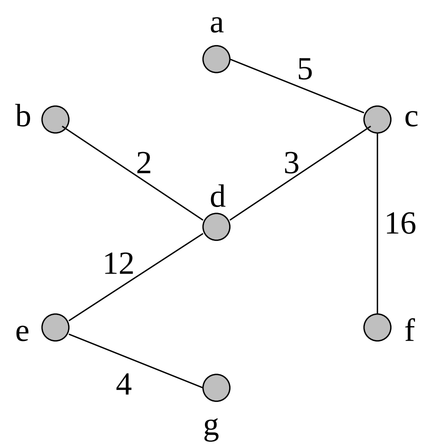


MST

Applications of MST



Graph G



MST

- **Direct applications: interconnection of entities.**

  1. **electrical devices (circuit boards)**
  2. **utilities (gas, oil)**
  3. **computers or communication devices by high speed lines.**
  4. **cable service customers**

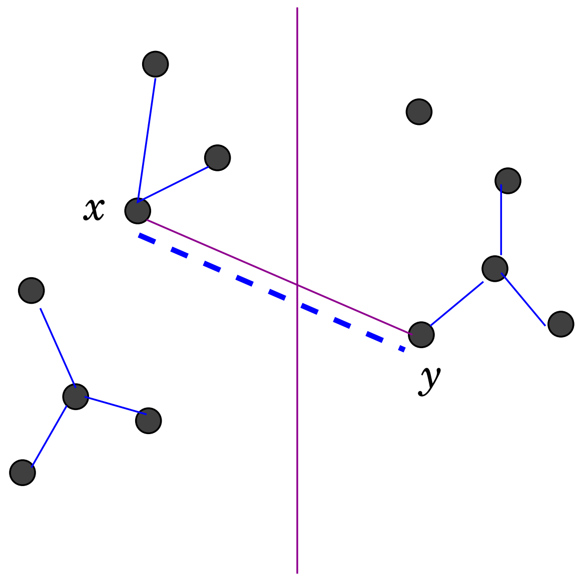
- **Indirect applications.**

  1. **Optimal message passing.**
  2. **Data storage.**
  3. **Cluster analysis**

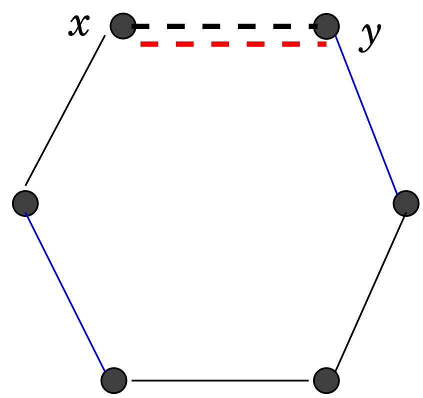
Optimality Conditions

- **Greedy incremental flavor: add one edge at a time.**
- **Each step colors an edge of G blue (accept) or red (reject).**
- **Color Invariant (CI): $\exists$ MST containing all blue edges, and no red edges.**
- **Recall that a cut in $G = (V, E)$ is a partition of it vertices $(X, V - X)$.**

Blue and Red Rules



Cut Rule (Blue)



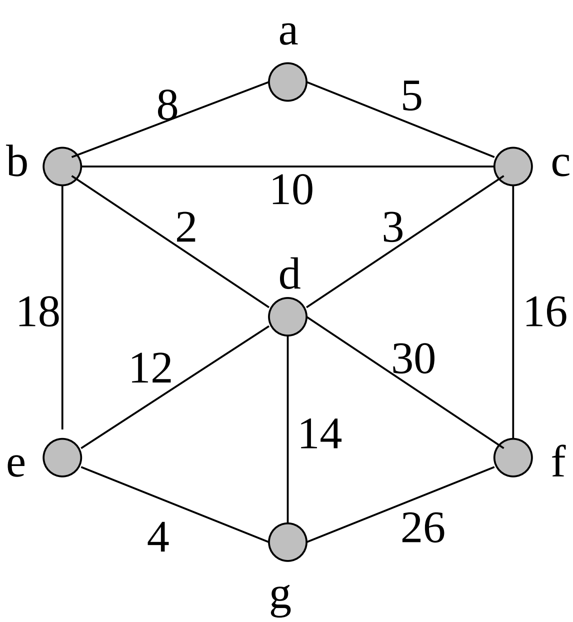
Red Rule (Cycle)

Blue (Cut) Rule: ***Select a cut not crossed by any blue edge. Among the uncolored edges crossing the cut, make the minimum cost edge blue.***

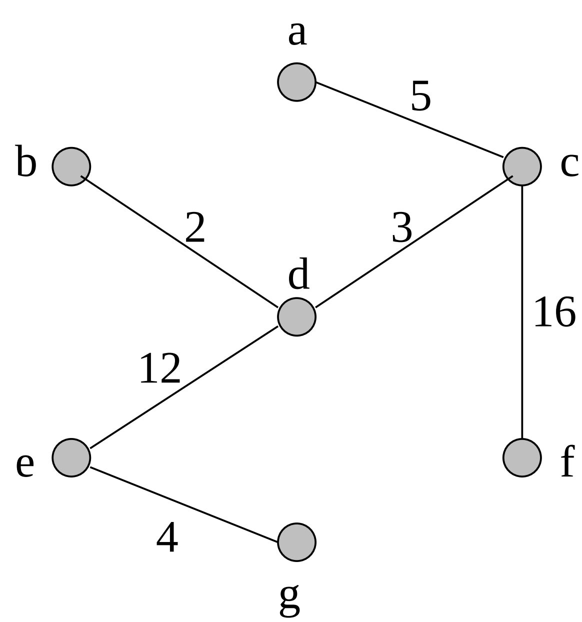
Red (Cycle) Rule: ***Select a simple cycle with no red edges. Among all uncolored edges of the cycle, make the maximum cost one red.***

Generic MST Algorithm

Theorem: ***Apply red and blue rules in arbitrary order until neither rule applies. The resulting set of blue edges forms a MST.***



Graph G

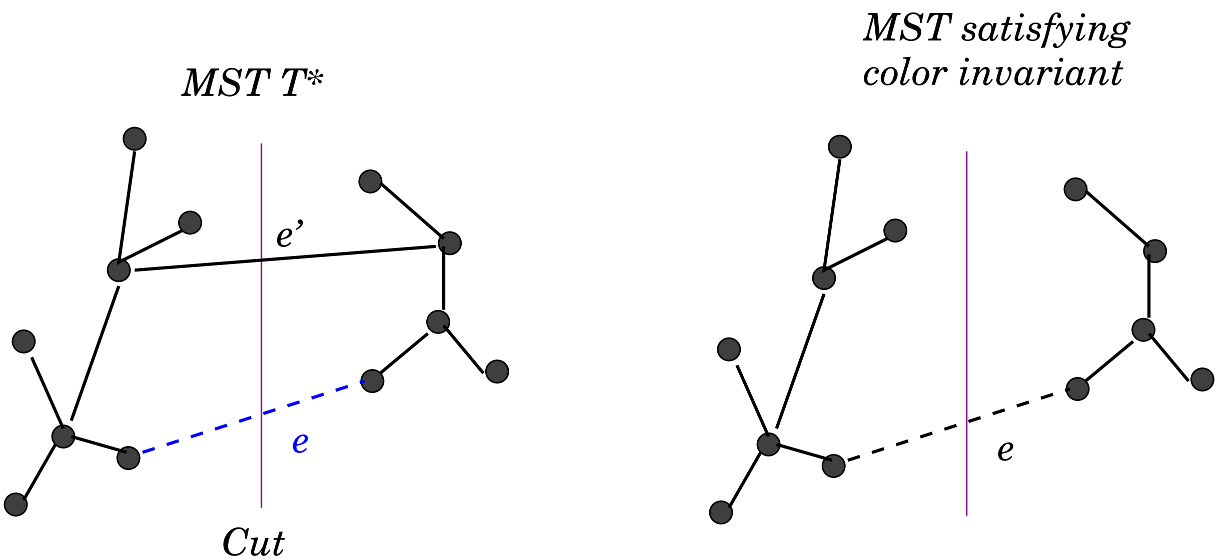


MST

- **Proof has two parts:**
  1. **The Color Invariant (CI) holds.**
  2. **All edges are ultimately colored.**

Correctness of Blue Rule

- **Let T^* be the MST guaranteed by the CI before the last coloring step.**

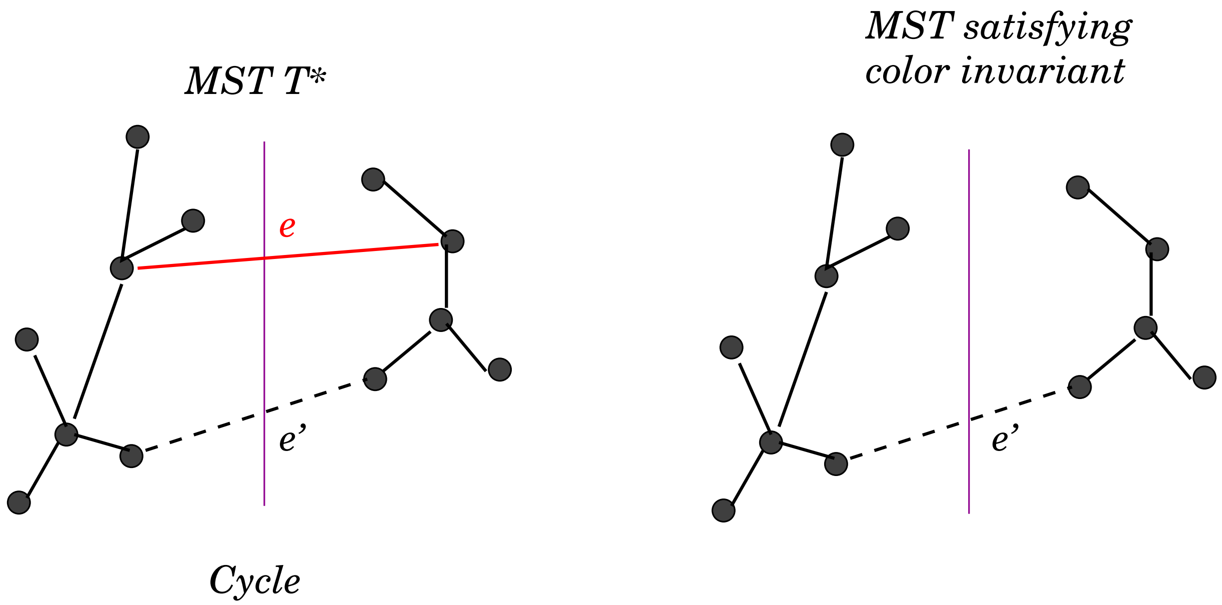


- **Suppose the last step was to color e blue. Consider the cut $(X, V - X)$ to which blue rule applied. Some edge e' of T^* must cross this cut.**

- **The graph $T^* + e$ contains a cycle containing both e and e', and $cost(e) \leq cost(e')$. (Why?)**

- **So $T^* + e - e'$ is also a MST.**

Correctness of Red Rule



- **Suppose edge e colored red. If $e \notin T^*$, then T^* still satisfies CI. Otherwise, consider $T^* - e$. It has two components.**

- **The cycle used in coloring e has some edge e' with one end in each of these components.**

- **By choice, $cost(e') \leq cost(e)$. Thus, the tree $T^* - e + e'$ is also an MST.**

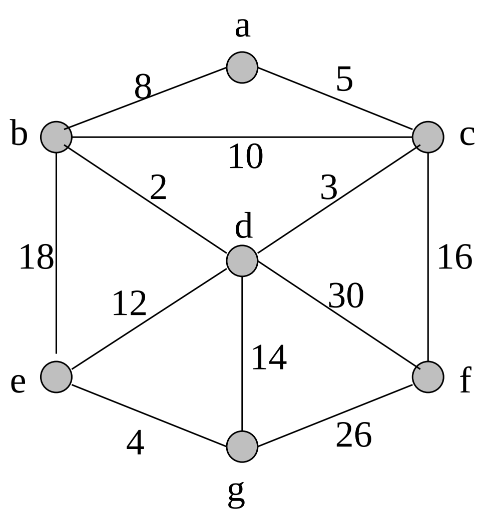
Proof of Completion

- **Blue edges form a forest.**
- **Suppose edge e left uncolored, and neither rule applies.**
- **If both endpoints of e in same blue tree B, then red rule applies to the cycle $B + e$.**
- **If endpoints in different blue trees B_1, B_2, then blue rule applies across the cut separating B_1 and B_2.**
- **Thus, the generic MST algorithm is correct.**

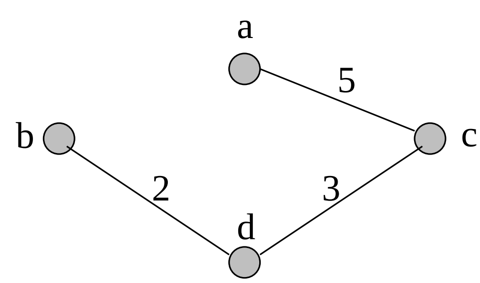
Prim's Algorithm

- **Start at any node s, and set $T = \{s\}$.**
- **repeat $n-1$ times**

  Let T be the current tree.
  Choose a minimum cost *uncolored*
  (u, v) with $u \in T$ and $v \notin T$.
  Color (u, v) blue, and add v to T.



Graph G



After 3 steps.

Prim's Algorithm

1. **Input: Graph $G = (V, E)$. For each vertex v, $A(v)$ is list of its neighbors.**

2. $key(w)$ **is the cost of the cheapest edge** (v, w) **with** $v \in T$**.** $blue(v, w)$ **is the identity of this edge.**

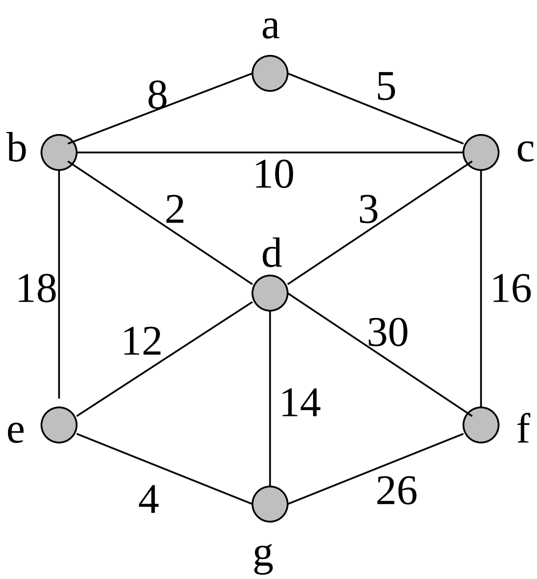
3. **Initialize:**

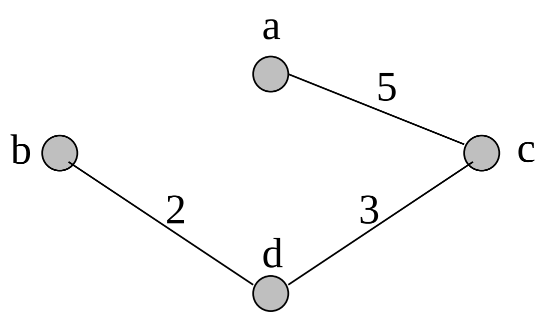
   - $key(v) = \infty$, **for all** $v \in V$.
   - $H = makeHeap()$.
   - $v = s$.

4. **while** $v \neq NIL$ **do**

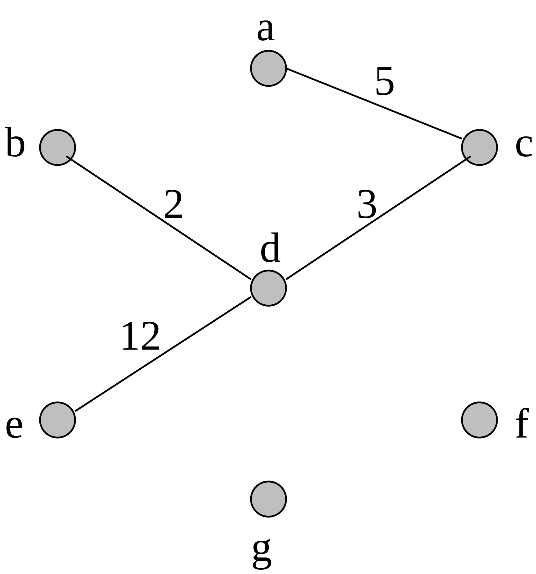
   - **for** $w \in A(v)$
   -     **if** $cost(v, w) < key(w)$
   -       $key(w) = cost(v, w)$
   -       $blue(w) = (v, w)$
   -     **if** $w \notin H$ $insert(w, H, key(w))$
   -     **else** $DecreaseKey(w, H, key(w))$
   - **end for**
   - $v = ExtractMin(H)$.

Illustration

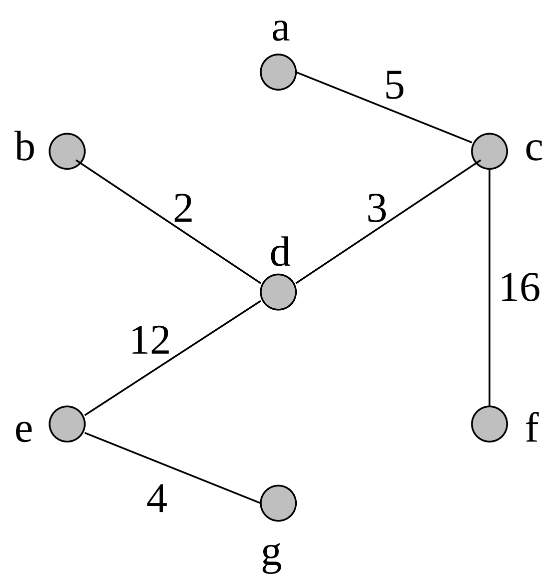




After 3 steps

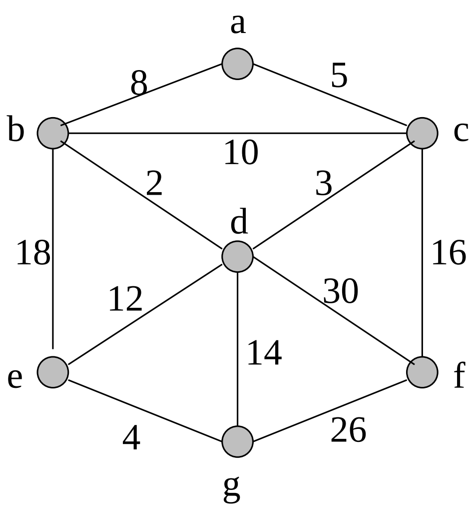


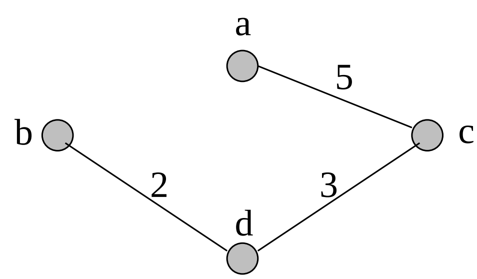
After 4 steps



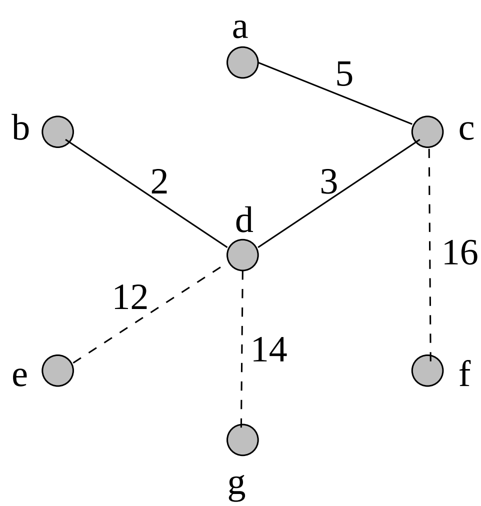
Final MST

Illustration





After 3 steps



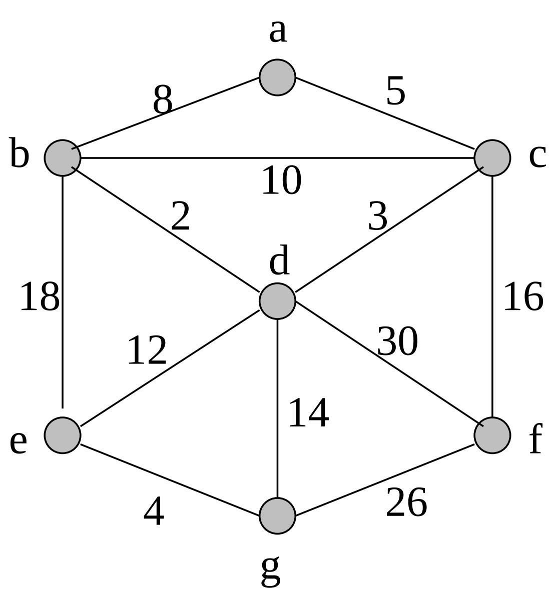
Blue edges and keys after 3 steps

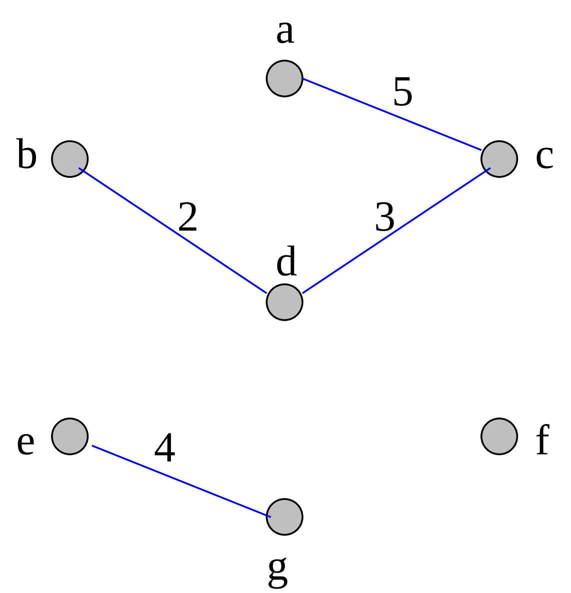
Kruskal's Algorithm

- **Initially make each node of V a singleton tree.**
- **Scan edges of E in non-decreasing order of cost.**
- **scan edge e:**

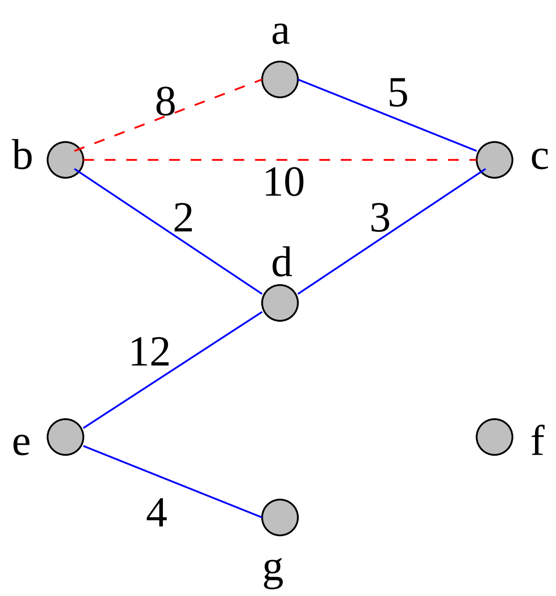
  If both endpoints of e in the same tree, color it red. Otherwise color e blue, and merge the two trees.

Illustration

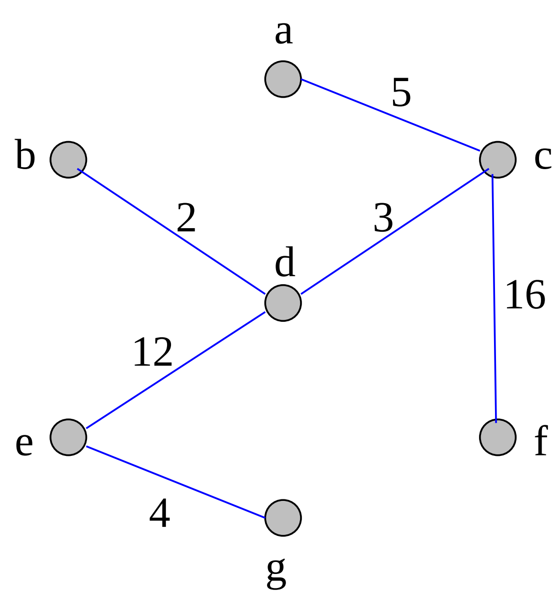




After 4 edges scanned



8, 10 colored red. 12 colored blue

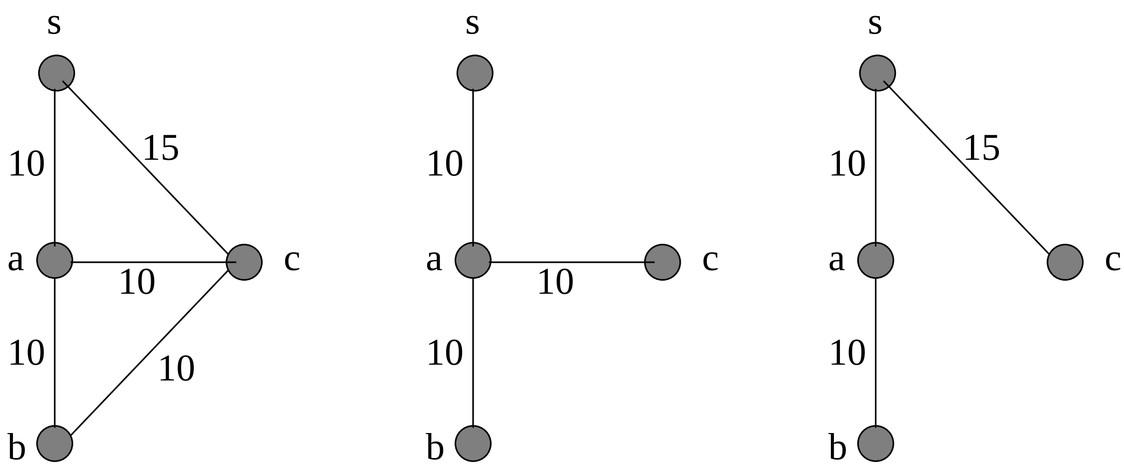


Final MST

Complexity Results

- **Heap implementation of Prim:** $O(m \log_2 n)$.
- d**-Heap:** $O(nd \log_d n \ + \ m \log_d n)$.
- **Fibonacci Heap:** $O(m + n \log n)$.
- **Kruskal with Union-Find:** $O(m \log n \ + \ m\alpha(n))$.
- **Round-Robin:** $O(m \log \log n)$.
- **Latest theoretical bound:** $O(m\alpha(n))$.

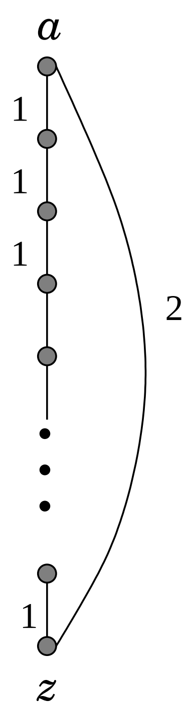
Balancing MST and SP



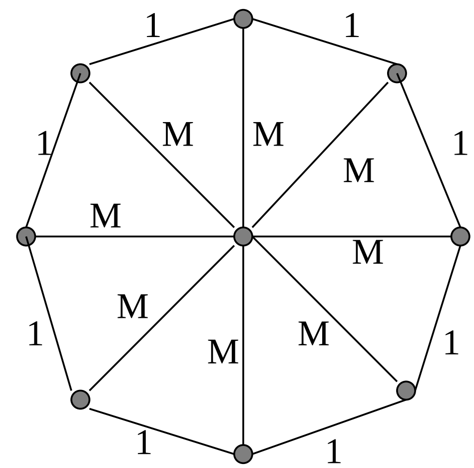
Graph G *MST* *SPT*

- **MST minimizes *total* interconnection cost.**
- **SPT minimizes *individual* path lengths, from a root source.**
- **How does each do with the other cost metric?**
- **In Fig. 2,** $d_{mst}(a,c) = 20$**, but** $d_{spt}(a,c) = 15$.
- **In Fig. 3,** $cost(spt) = 35$**, but** $cost(mst) = 30$**.**

A Pathological Example



(i)



(ii)

- **Fig. (i) shows an example where $d_{mst}(a, z) = n$, while $d_{spt} = 2$.**

- **As $n \to \infty$, this ratio is unbounded.**

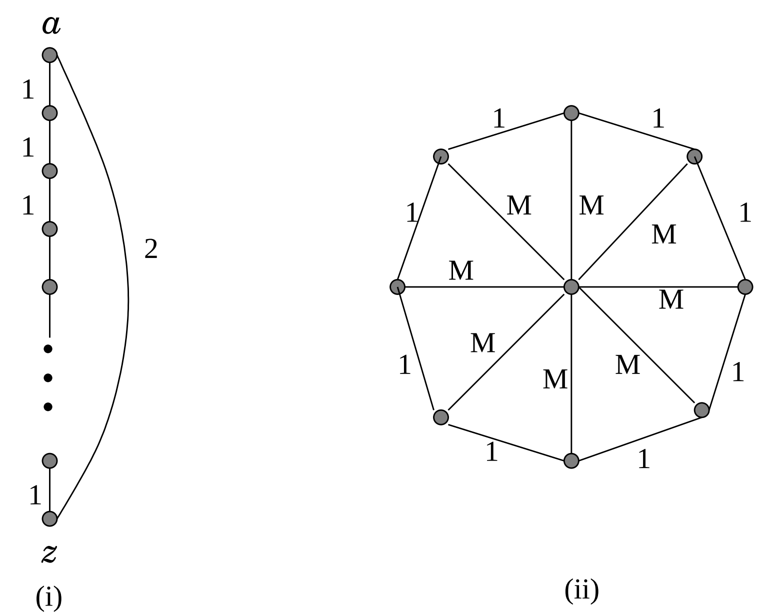
- **Fig. (ii) shows an example where $cost(spt) = nM$, while $cost(mst) = n - 1 + M$.**

- **As $M \to \infty$, this ratio is unbounded.**

Balanced Tree Theorem

Theorem: ***Pick any constant $\alpha > 1$, and let $\beta = 1 + \frac{2}{\alpha - 1}$. Given $G = (V, E)$ and a node s, there always exists a spanning tree T rooted at s such that***

- $d_T(s, v) \leq \alpha \cdot d_{spt}(s, v)$, ***for any*** $v \in V$.
- $cost(T) \leq \beta \cdot cost(MST)$.

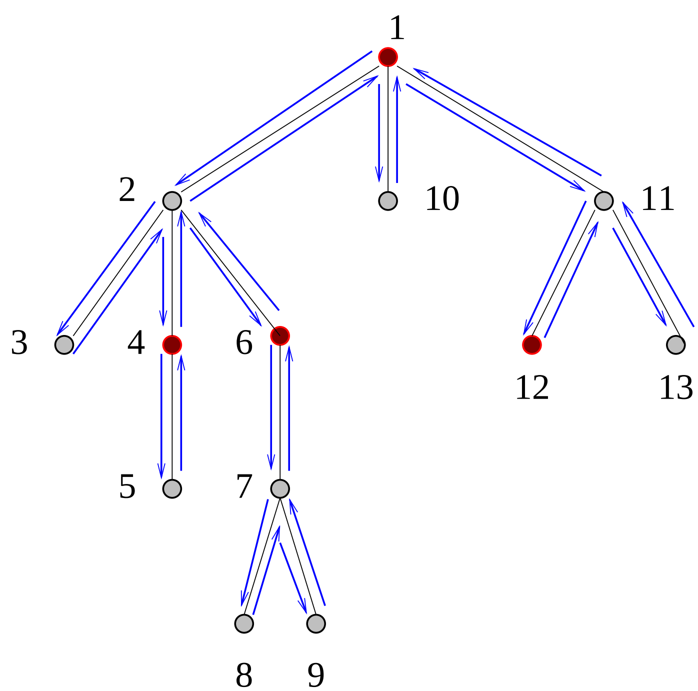


- **Which T for these pathological cases?**

Pre-Order Numbering

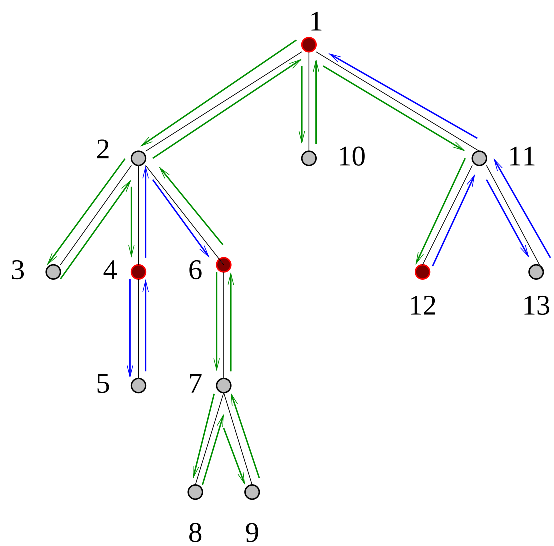
Pre-Order Lemma: ***Let T be a spanning tree with root s. Let $z_0, z_1, \ldots, z_k$ be any $k+1$ nodes listed in their pre-order sequence. Then,***

$$\sum_{i=1}^{k} d_T(z_{i-1}, z_i) \;\leq\; 2cost(T).$$



- **Draw T in the plane. Let W be the doubling "walk" around T (pre-order).**
- **Each edge visted twice, $cost(W) = 2cost(T)$.**

Proof

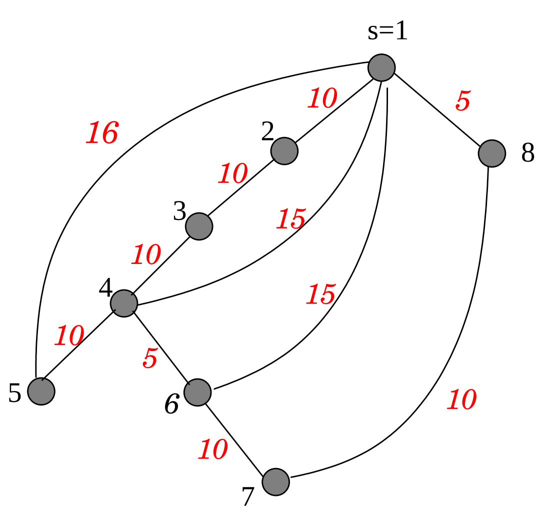


- $d_T(u, v)$ **is tree path length.**
- **Due to pre-ordering, first occurrence of** z_{i-1} **is before that of** z_i**.**
- **Mark off portions of** W **that join first occurence of** z_{i-1} **to that of** z_i**.**
- **No edge of** W **marked more than once. Thus,**

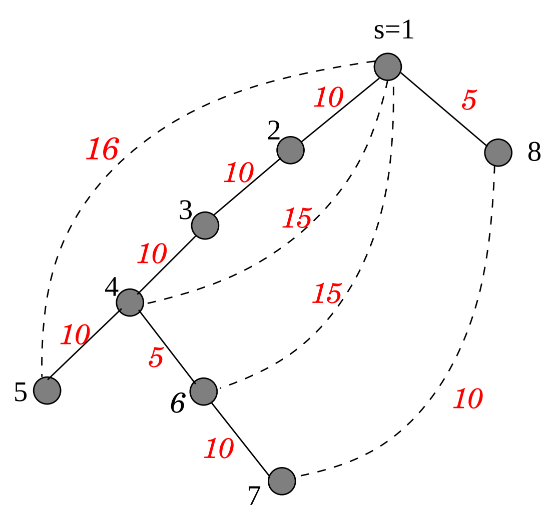
$$\sum_{i=1}^{k} d_T(z_{i-1}, z_i) \leq cost(W) \leq 2cost(T).$$

Algorithm

- $\beta = 1 + \frac{2}{\alpha - 1}$. **E.g.** $\alpha = 2$, $\beta = 3$.
- **Compute MST and its pre-order numbers, starting with** s.
- **Compute** $d_{spt}(s, v)$, **for all** v.
- **Initialize** $H = MST$.
- **for each node** v **in pre-order do**
  if $d_H(s, v) > \alpha \cdot d_{spt}(s, v)$ **then**
  add to H **all edges of path** $P_{spt}(s, v)$.
- **Output SPT of** H, **rooted at** s.

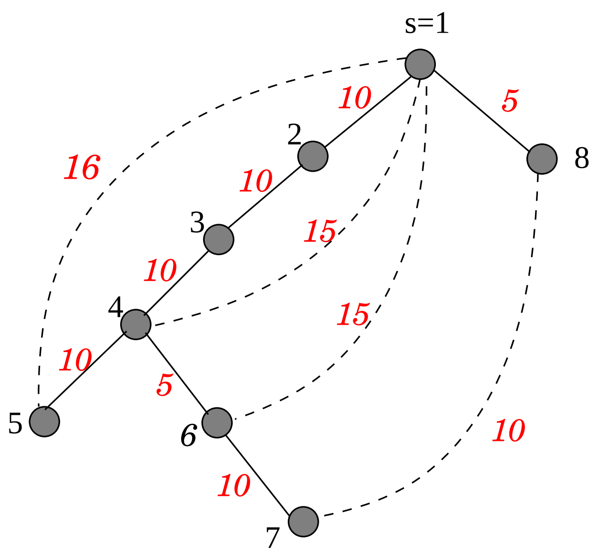


Graph G

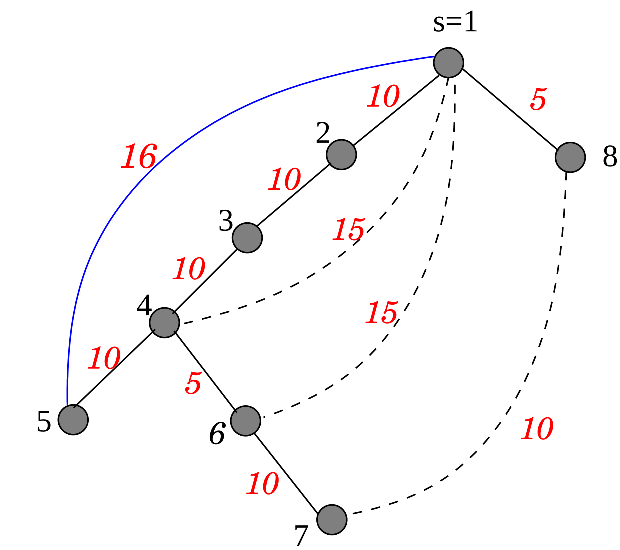


MST

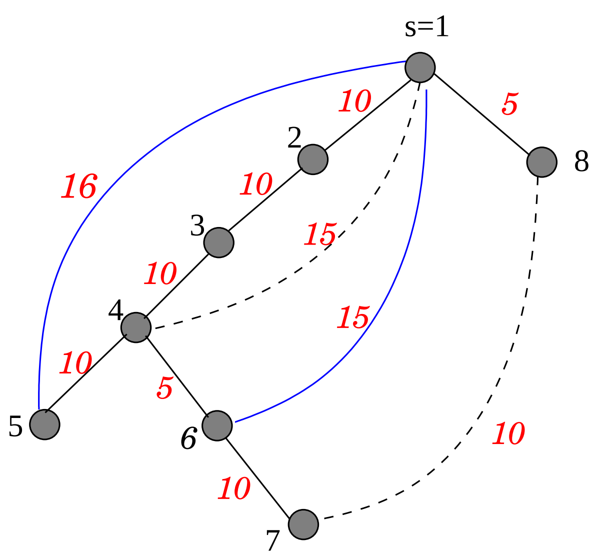
Illustration



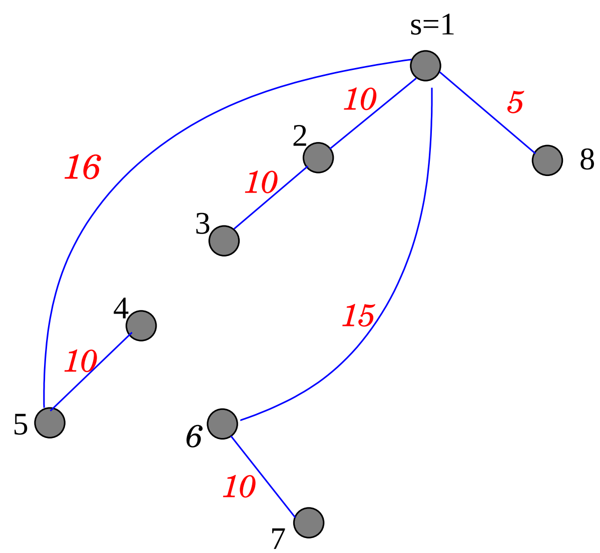
MST



(1,5) added



(1,6) added



Final tree T output

Analysis

- T **satisfies** $d_T(s,v) \le \alpha \cdot d_{spt}(s,v)$**; algorithm adds SP whenever needed.**
- **Let** $z_0, z_1, \ldots, z_k$ **be the pre-order sequence of vertices that caused SP edges to be added to** H**.**
- **When** z_i **is examined,** H **contains the shortest path to** z_{i-1}**. So, we must have**

$$d_{spt}(s, z_{i-1}) + d_{mst}(z_{i-1}, z_i) \; > \; \alpha \cdot d_{spt}(s, z_i).$$

- **Sum these over all** i**:**

$$\begin{aligned}
\alpha d_{spt}(s, z_1) - d_{spt}(s, z_0) &< d_{mst}(z_0, z_1) \\
\alpha d_{spt}(s, z_2) - d_{spt}(s, z_1) &< d_{mst}(z_1, z_2) \\
&\vdots \\
\alpha d_{spt}(s, z_k) - d_{spt}(s, z_{k-1}) &< d_{mst}(z_{k-1}, z_k)
\end{aligned}$$

- $\sum_{i=1}^{k} (\alpha - 1) d_{spt}(s, z_i) < \sum_{i=1}^{k} d_{mst}(z_{i-1}, z_i)$

Analysis

- **Thus, we have**

$$\begin{aligned}\sum_{i=1}^{k}(\alpha-1)d_{spt}(s,z_i) &< \sum_{i=1}^{k} d_{mst}(z_{i-1},z_i)\\ &\leq 2cost(MST)\end{aligned}$$

- **Thus,**

$$\sum_{i=1}^{k} d_{spt}(s,z_i) < \frac{2}{\alpha-1}cost(MST)$$

- **Since** $cost(H) = cost(MST) + \sum_{i=1}^{k} d_{spt}(s,z_i)$**, we get**

$$cost(H) < \left(1+\frac{2}{\alpha-1}\right)cost(MST)$$

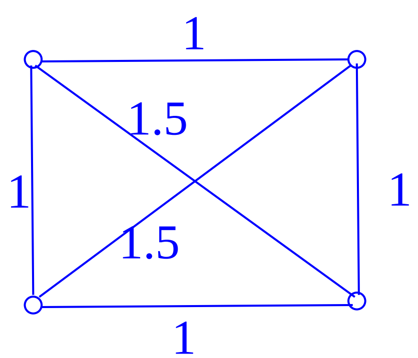
- **Theorem proved.**

Graph Spanners

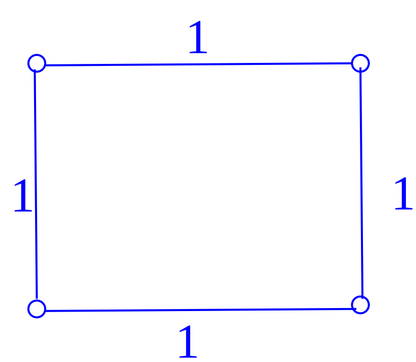
- **A subgraph $H = (V, E')$ is a t-spanner of graph $G = (V, E)$, where $E' \subseteq E$, if for all pairs $u, v \in V$,**

$$dist_H(u, v) \leq t \cdot dist_G(u, v)$$

- **The parameter t is called the stretch factor.**
- **Total cost of H is the sum of costs of edges in H.**
- **Determine H with minimum total cost such that stretch factor is at most t.**



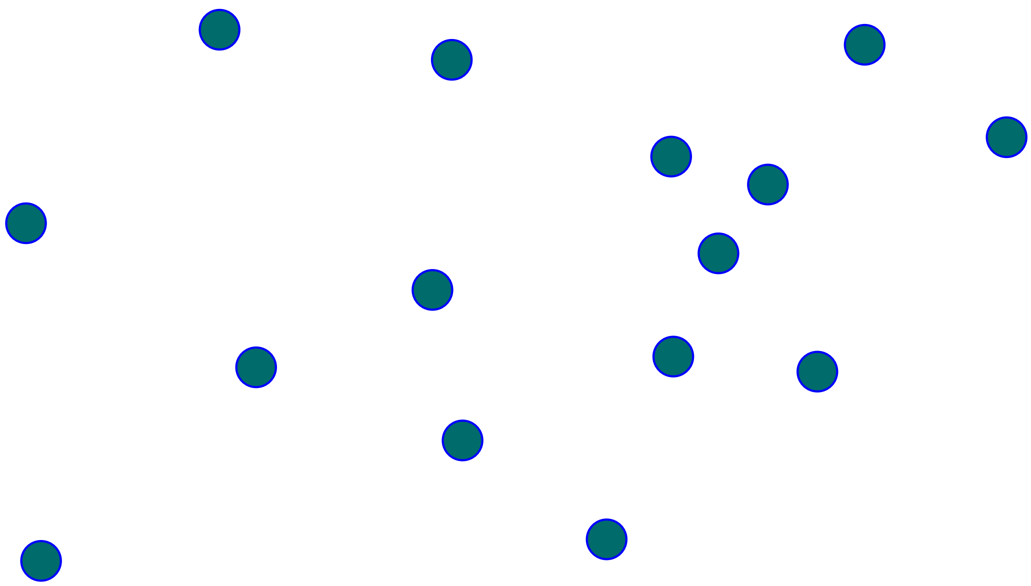
Graph G



Spanner H

Graph Spanners

- We want a small-size subgraph H that approximates well the shortest paths between all pairs of vertices in G.



- For instance, in building transportation networks, tradeoff between stretch factor and total network cost.
- What's the best H for this example, using Euclidean distances?

Spanner Algorithm

- **The following simple, elegant algorithm works.**

Spanner (G, t)
    $H \leftarrow (V, \emptyset)$;
    Sort edges of G **in increasing cost order;**
    Let $e_1, e_2, \ldots, e_m$ **be this order;**
    for $i = 1$ **to** m **do**
        Let $e_i = (u, v)$;
        if $dist_H(u, v) > t \cdot w(e_i)$
            $H \leftarrow H \cup \{e_i\}$;
        end
    end
Output H;

Correctness

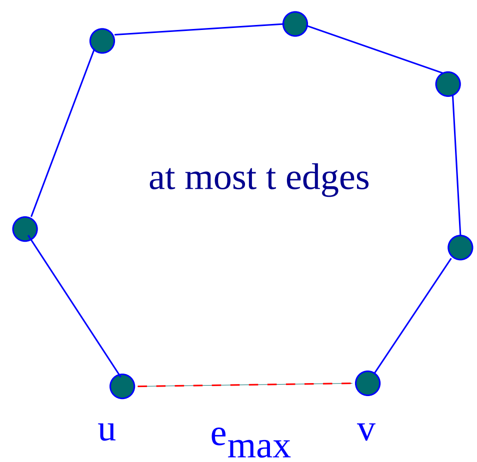
Lemma: The subgraph H output by Spanner is a t-spanner of G.

- **Let $P(u, v)$ be the shortest path in G.**
- **Consider an edge e on this path.**
- **e was either added to H by spanner algorithm, or there is a path of length $t \cdot w(e)$ in H.**
- **Concatenating these approximate paths in H, we get a path in H whole length is at most $t \cdot P(u, v)$.**

Correctness

Lemma: If C is a cycle in H, $size(C) > t + 1$.

- **Suppose $\exists$ a cycle C with $\leq t + 1$ edges.**
- **Let $e_{\max} = (u, v)$ be its most costly edge.**
- **When algorithm scans $e_{\max}$, all other edges of C have been added.**

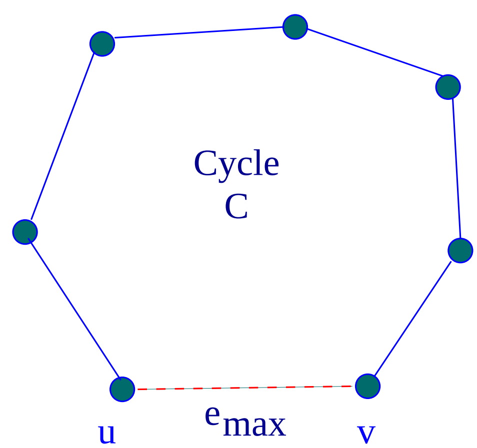


- **Thus, there is already a path from u to v of cost $\leq t \cdot e_{\max}$ (each of the t edges is $\leq w(e_{\max})$).**
- **So, edge $e_{\max}$ will not be added.**

Correctness

Lemma: $w(C-e) > t \cdot w(e)$, **for any** $e \in C$.

- **Suppose** $\exists$ **a cycle** C **and edge** e **with** $w(C-e) \leq t \cdot w(e)$.



- **So,** $w(C) \leq (t+1)w(e) \leq (t+1)w(e_{\max})$.
- **When considering** $e_{\max}$**, it won't be added!**

Correctness

Lemma: MST **contained in** H**.**

- **Straightforward. Prove it yourself.**
- **The key is to show**

$$w(E_v) \;\leq\; \frac{2w(MST)}{t-1}$$

where E_v **is edges of** $H - MST$ **incident to vertex** v**.**

- **This is sufficient to prove**

$$\begin{aligned} w(H) &\leq w(MST) + \frac{1}{2}\sum_v w(E_v) \\ &\leq w(MST)\left(1 + \frac{n}{t-1}\right) \end{aligned}$$

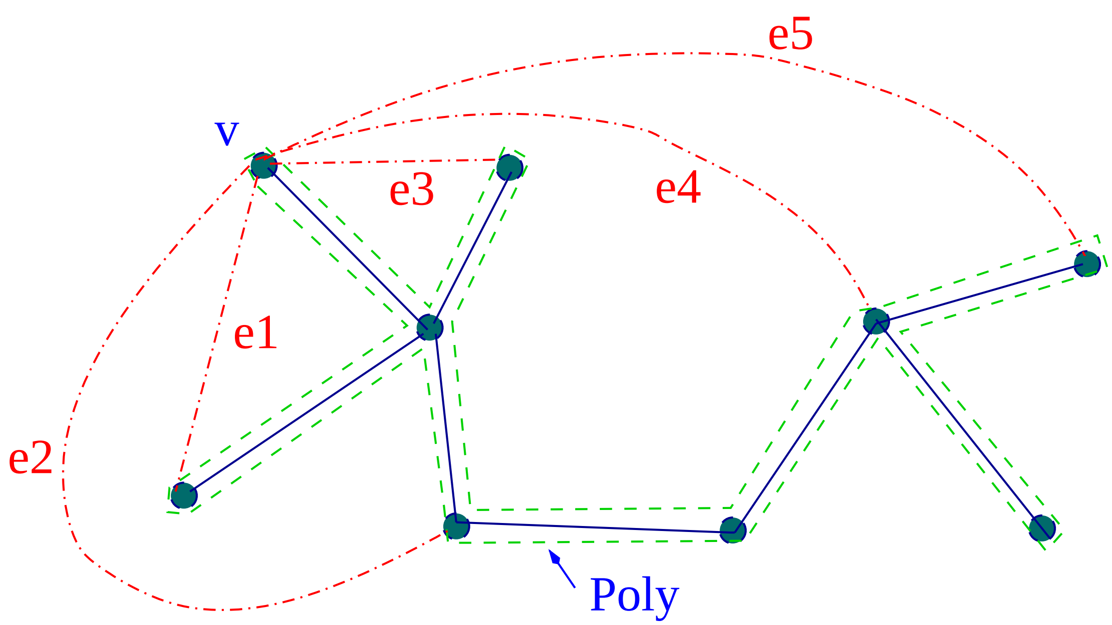
Proof

Theorem: $w(E_v) \leq \frac{2w(MST)}{t-1}$.

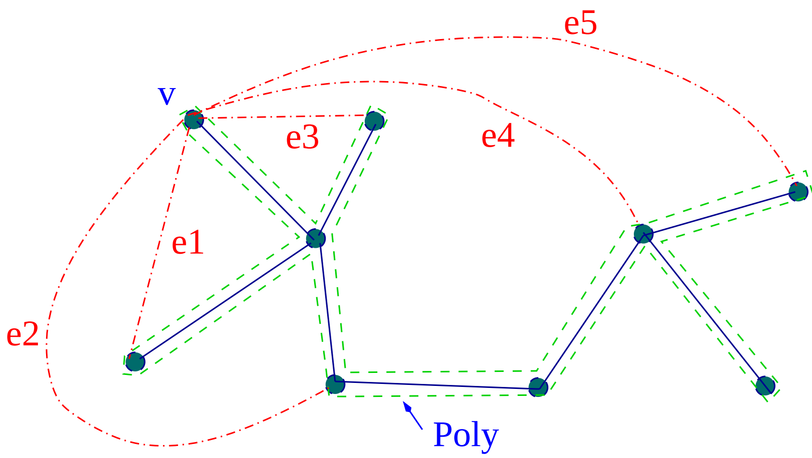
- **Let** $e_1, e_2, \ldots, e_k$ **be edges in** E_v.
- **For any edge** $e \in H$**, and any path** P **in** H **joining the endpoints of** e**, we have**

$$t \cdot w(e) < w(P)$$

- **Let Poly be the polygon defined by doubling MST.**



Proof



- W_i **is perimeter of Poly after** $e_1, \ldots, e_i$ **added. Initially,** $W_0 = MST$**.**
- **What happens when** $e_i = (u, v)$ **is added?**

$$\begin{aligned} W_i &= W_{i-1} + w(e_i) - w(P(u,v)) \\ &< W_{i-1} - w(e_i) \cdot (t-1) \\ (t-1)w(e_i) &< W_{i-1} - W_i \end{aligned}$$

- **Thus,** $(t-1)\sum_{i=1}^{k} w(e_i) \leq W_0 - W_k$**.**
- **Since** $W_0 = 2w(MST)$ **and** $W_k > 0$**,**

$$E_v = \sum_{i=1}^{k} w(e_i) < \frac{2w(MST)}{t-1}.$$

Spanner Results

- **Althofer et al. result gives spanners with $O(n^{1+\frac{2}{t-1}})$ edges.**
- **Peleg-Schaffer shows that there are n-vertex graphs, whose all t-spanners contain $\Omega(n^{1+\frac{1}{t}})$ edges.**
- **Much better results know if distances are Euclidean.**
- **In Euclidean k-space (constant k) spanners with $O(n)$ edges exist for any constant stretch factors t.**
- **Such spanners possible with $O(MST)$ weight, $O(1)$ degree, $O(\log n)$ diameter.**