# Lecture 7: Minimum Spanning Trees and Prim's Algorithm

CLRS Chapter 23

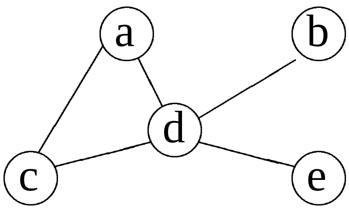
## Outline of this Lecture

- Spanning trees and minimum spanning trees.
- The minimum spanning tree (MST) problem.
- The generic algorithm for MST problem.
- Prim's algorithm for the MST problem.
  - The algorithm
  - Correctness
  - Implementation + Running Time

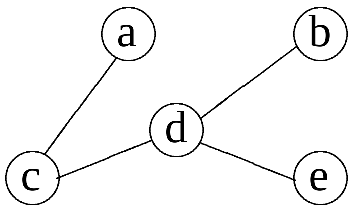
# Spanning Trees

**Spanning Trees:** A subgraph $T$ of a undirected graph $G = (V, E)$ is a spanning tree of $G$ if it is a tree and contains every vertex of $G$.

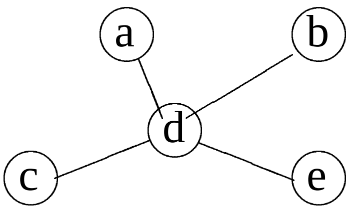
**Example:**



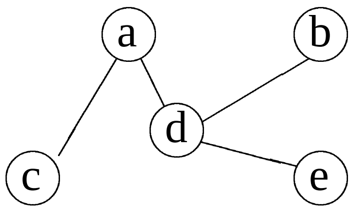

Graph




spanning tree 1




spanning tree 2



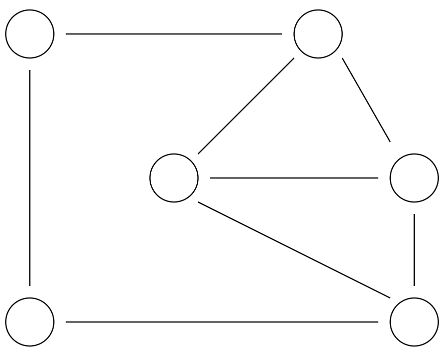

spanning tree 3

## Spanning Trees

**Theorem:** Every connected graph has a spanning tree.

**Question:** Why is this true?

**Question:** Given a connected graph $G$, how can you find a spanning tree of $G$?

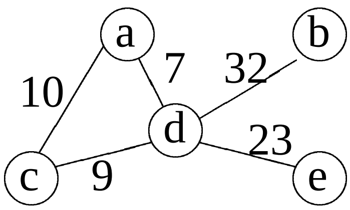


## Weighted Graphs

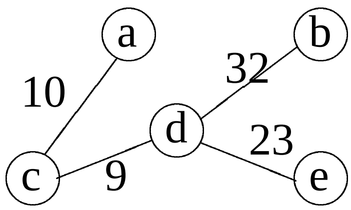
**Weighted Graphs:** A weighted graph is a graph, in which each edge has a weight (some real number).

**Weight of a Graph:** The sum of the weights of all edges.

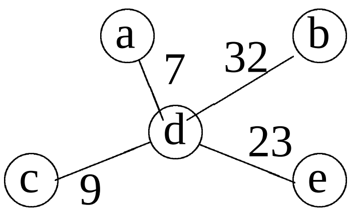
**Example:**



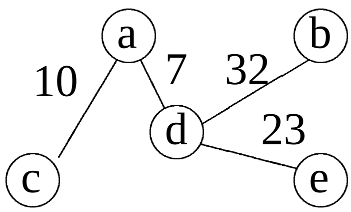

weighted graph




Tree 1. w=74




Tree 2, w=71
Minimum spanning tree

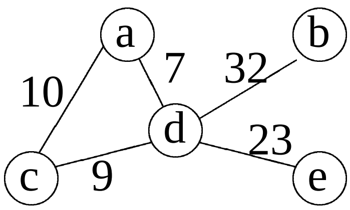



Tree 3, w=72

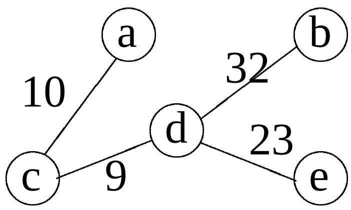
# Minimum Spanning Trees

A **Minimum Spanning Tree** in an undirected connected weighted graph is a spanning tree of minimum weight (among all spanning trees).

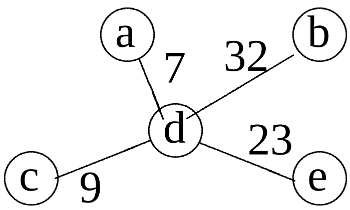
**Example:**



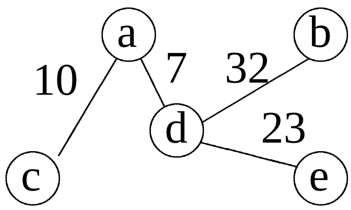

weighted graph




Tree 1. w=74




Tree 2, w=71
Minimum spanning tree

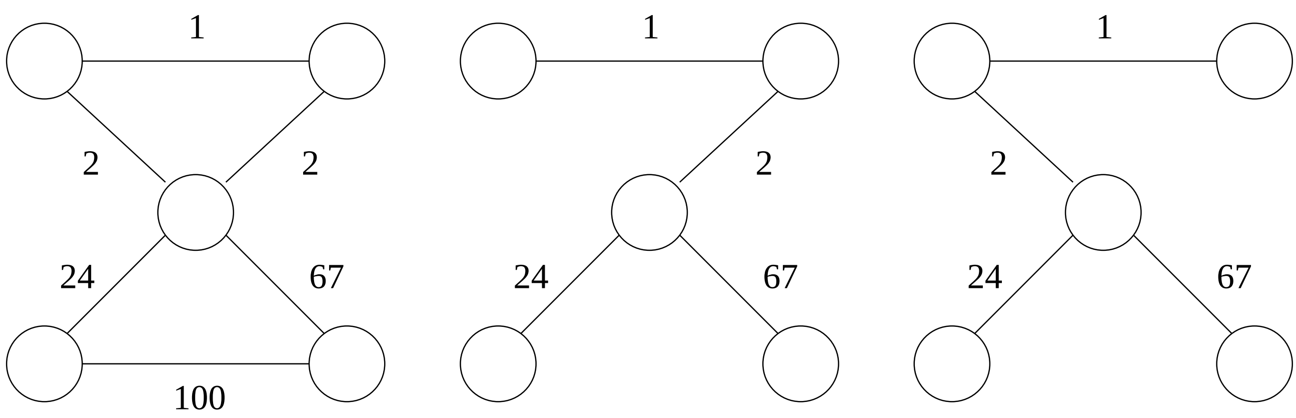



Tree 3, w=72

# Minimum Spanning Trees

**Remark:** The minimum spanning tree may not be unique. However, if the weights of all the edges are pairwise distinct, it is indeed unique (we won't prove this now).

**Example:**




weighted graph MST1 MST2

# Minimum Spanning Tree Problem

**MST Problem:** Given a connected weighted undirected graph $G$, design an algorithm that outputs a minimum spanning tree (MST) of $G$.

**Question:** What is most intuitive way to solve?

**Generic approach:** A tree is an acyclic graph.
The idea is to start with an empty graph and try to add edges one at a time, always making sure that what is built remains acyclic. And if we are sure every time the resulting graph always is a subset of some minimum spanning tree, we are done.

## Generic Algorithm for MST problem

Let $A$ be a set of edges such that $A \subseteq T$, where $T$ is a MST. An edge $(u, v)$ is a *safe edge* for $A$, if $A \cup \{(u, v)\}$ is also a subset of some MST.

If at each step, we can find a safe edge $(u, v)$, we can 'grow' a MST. This leads to the following generic approach:

```
Generic-MST(G, w)
 Let A=EMPTY;
 while A does not form a spanning tree
  find an edge (u, v) that is safe for A
  add (u, v) to A

 return A
```

How can we find a safe edge?

## How to find a safe edge

We first give some definitions. Let $G = (V, E)$ be a connected and undirected graph. We define:

**Cut** A **cut** $(S, V - S)$ of G is a partition of V.

**Cross** An edge $(u, v) \in E$ **crosses** the cut $(S, V - S)$ if one of its endpoints is in $S$, and the other is in $V - S$.

**Respect** A cut **respects** a set $A$ of edges if no edge in A crosses the cut.

**Light edge** An edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut.

## How to find a safe edge

**Lemma**

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function $w$ defined on $E$. Let $A$ be a subset of $E$ that is included in some minimum spanning tree for $G$, let $(S, V-S)$ be *any* cut of $G$ that respects $A$, and let $(u, v)$ be a light edge crossing the cut $(S, V - S)$. Then, edge $(u, v)$ is safe for $A$.

It means that we can find a safe edge by

1. first finding a cut that respects $A$,

2. then finding the light edge crossing that cut.

That light edge is a safe edge.

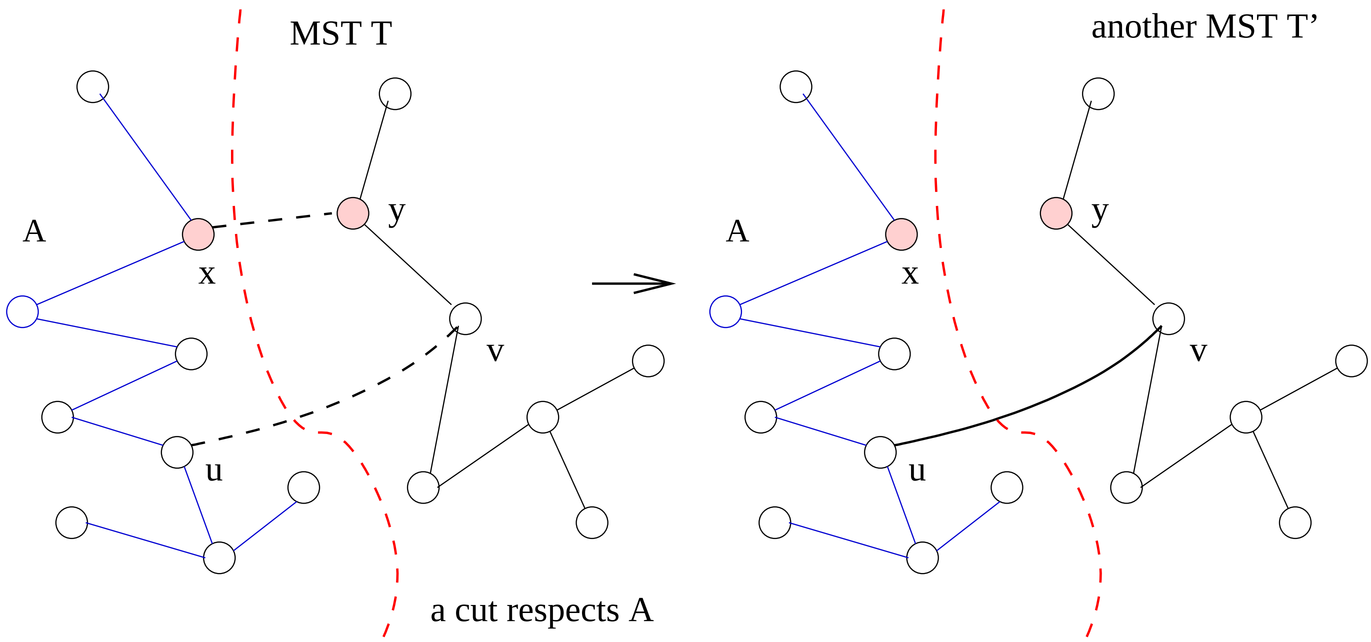
**Proof**

1. Let $A \subseteq T$, where $T$ is a MST. Suppose $(u, v) \notin T$.

2. The trick is to construct *another* MST $T^{'}$ that contains both $A$ and $(u, v)$, thereby showing $(u, v)$ is a safe edge for $A$.

3. Since $u$, and $v$ are on opposite sides of the cut $(S, V - S)$, there is at least one edge in $T$ on the path from $u$ to $v$ that *crosses* the cut. Let $(x, y)$ be such edge. Since the cut respects $A$, $(x, y) \notin A$.

   Since $(u, v)$ is a light edge crossing the cut, we have $w(x, y) \geq w(u, v)$.



4. Add $(u, v)$ to $T$, it creates a cycle. By removing an edge from the cycle, it becomes a tree again. In particular, we remove $(x, y)$ ($\notin A$) to make a new tree $T^{'}$.

5. The weight of $T^{'}$ is

$$\begin{aligned} w(T') &= w(T) - w(x, y) + w(u, v) \\ &\leq w(T) \end{aligned}$$

6. Since $T$ is a MST, we must have $w(T) = w(T^{'})$, hence $T^{'}$ is also a MST.

7. Since $A \cup \{(u, v)\}$ is also a subset of $T^{'}$ (a MST), $(u, v)$ is safe for $A$.

## Prim's Algorithm

The generic algorithm gives us an idea how to 'grow' a MST.

If you read the theorem and the proof carefully, you will notice that the choice of a cut (and hence the corresponding light edge) in each iteration is immaterial. We can select *any cut* (that respects the selected edges) and find the light edge crossing that cut to proceed.

The *Prim's* algorithm makes a nature choice of the cut in each iteration – it grows a single tree and adds a light edge in each iteration.

## Prim's Algorithm : How to grow a tree

Grow a Tree

- Start by picking any vertex $r$ to be the root of the tree.

- While the tree does not contain
  all vertices in the graph
  find shortest edge leaving the tree
  and add it to the tree .

Running time is $O((|V| + |E|) \log |V|)$.

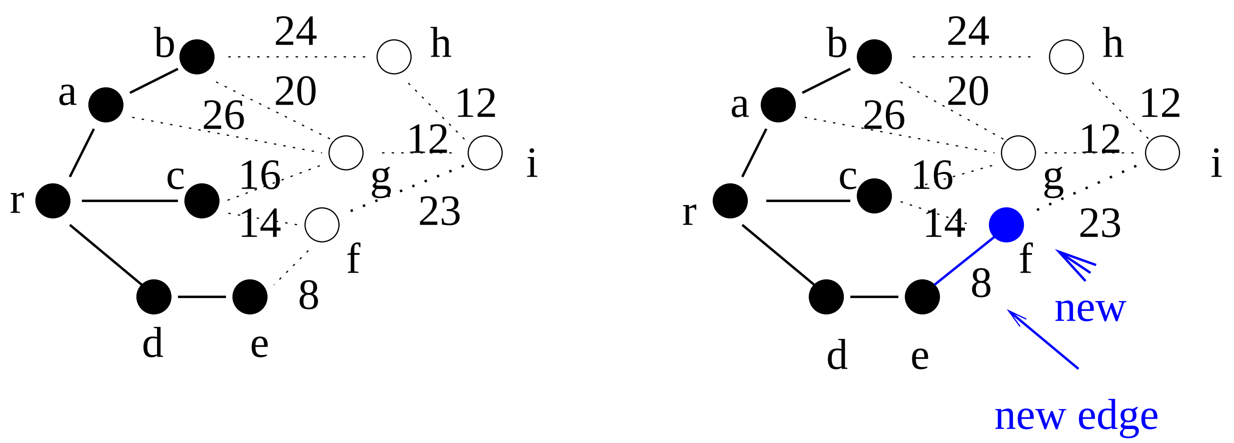
## More Details

**Step 0:** Choose any element $r$; set $S = \{r\}$ and $A = \emptyset$. (Take $r$ as the root of our spanning tree.)

**Step 1:** Find a lightest edge such that one endpoint is in $S$ and the other is in $V \setminus S$. Add this edge to $A$ and its (other) endpoint to $S$.

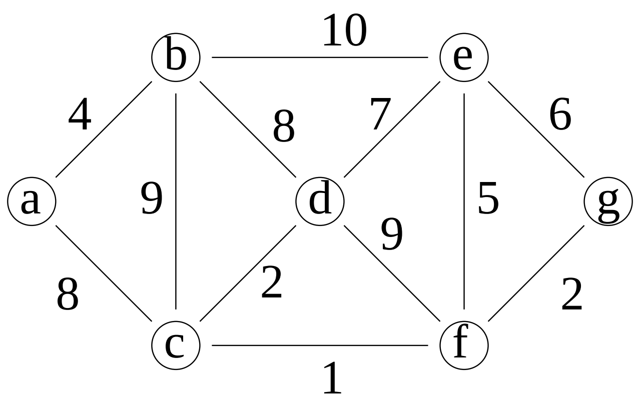
**Step 2:** If $V \setminus S = \emptyset$, then stop & output (minimum) spanning tree $(S, A)$. Otherwise go to Step 1.

The idea: expand the current tree by adding the lightest (shortest) edge leaving it and its endpoint.

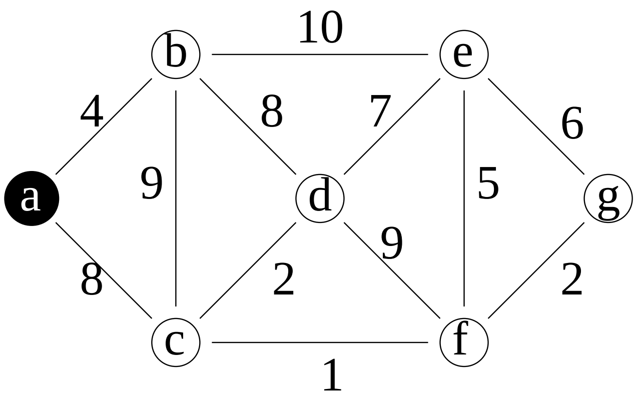


# Prim's Algorithm

## Worked Example




Connected graph




Step 0

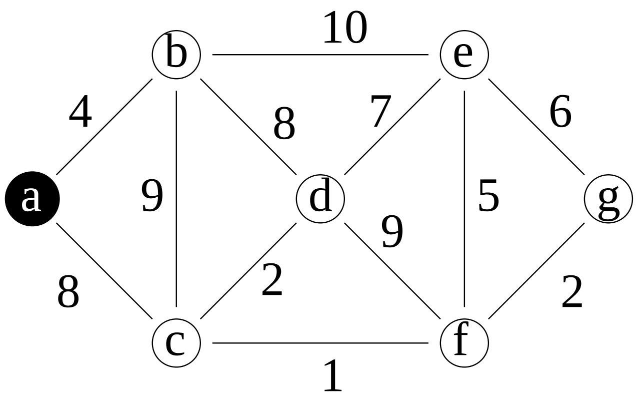
S={a}

V \ S = {b,c,d,e,f,g}

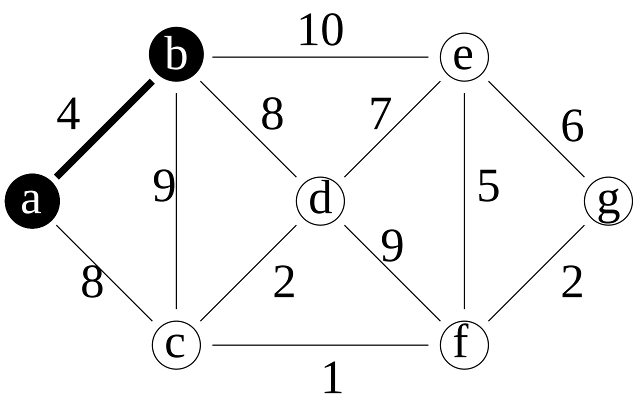
lightest edge = {a,b}

Prim’s Algorithm

## Prim’s Example – Continued

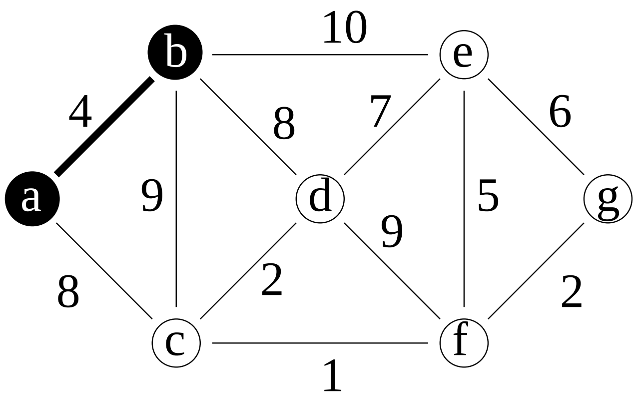



Step 1.1 before
S={a}
V \ S = {b,c,d,e,f,g}
A={}
lightest edge = {a,b}

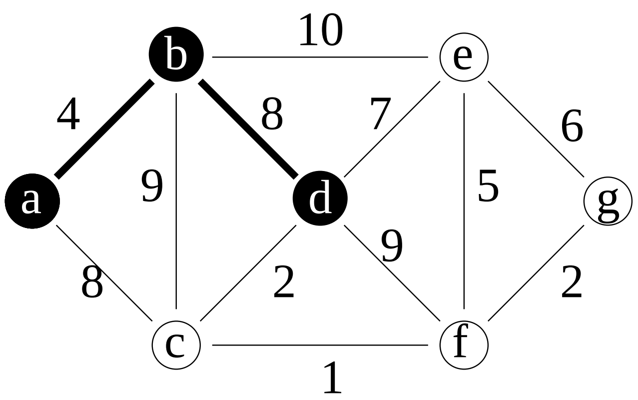



Step 1.1 after
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}

## Prim’s Example – Continued



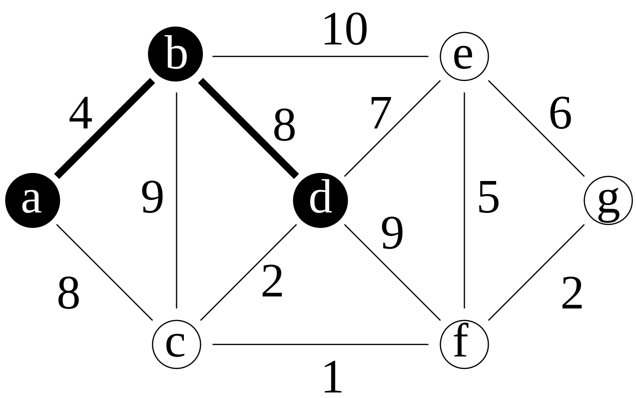

Step 1.2 before
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}




Step 1.2 after
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}

# Prim’s Algorithm

## Prim’s Example – Continued



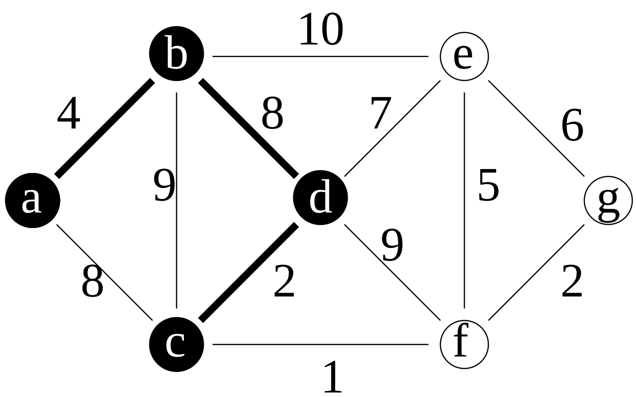

Step 1.3 before

S={a,b,d}

V \ S = {c,e,f,g}

A={{a,b},{b,d}}

lightest edge = {d,c}




Step 1.3 after

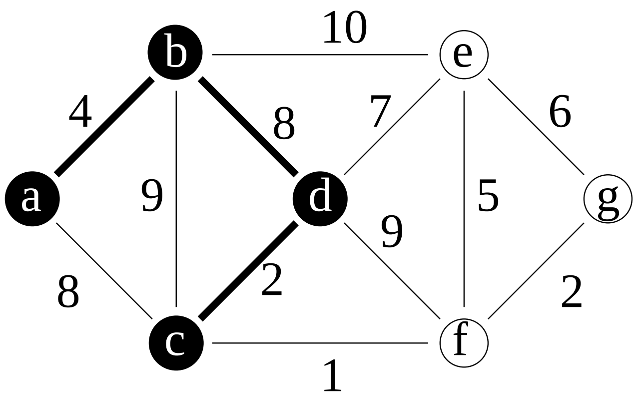
S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}

## Prim’s Example – Continued



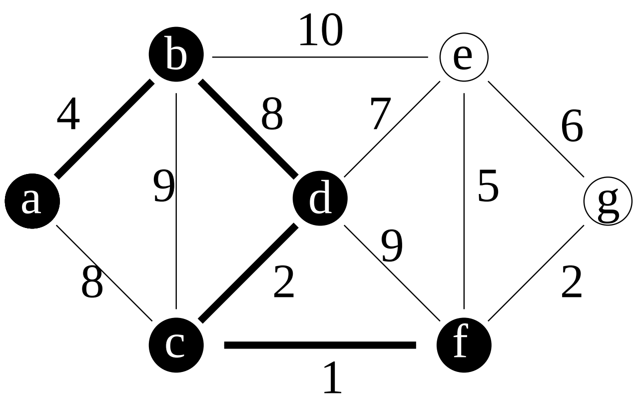

Step 1.4 before

S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}




Step 1.4 after

S={a,b,c,d,f}

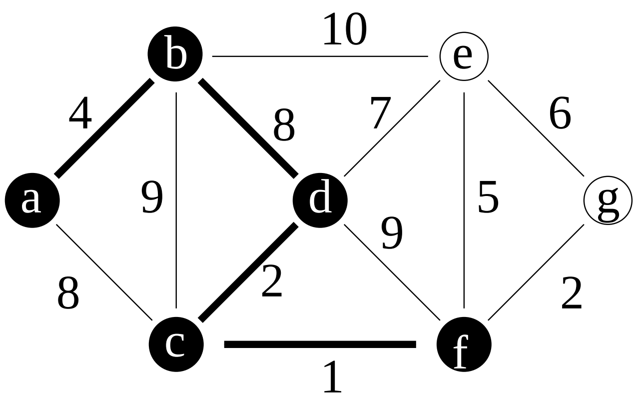
V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

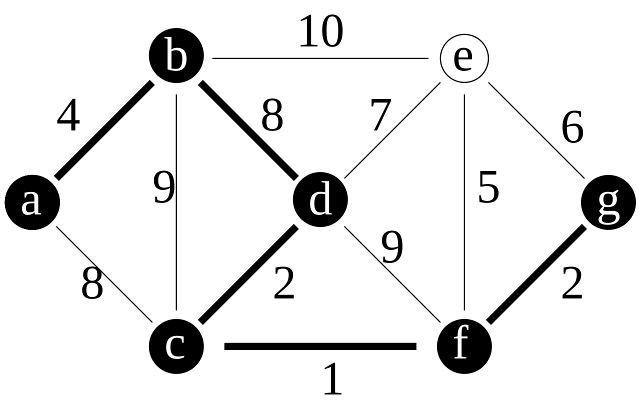
lightest edge = {f,g}

## Prim's Algorithm

## Prim's Example – Continued



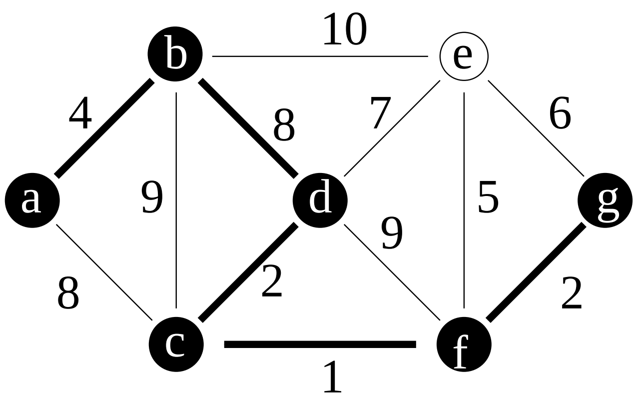

Step 1.5 before
S={a,b,c,d,f}
V \ S = {e,g}
A={{a,b},{b,d},{c,d},{c,f}}
lightest edge = {f,g}



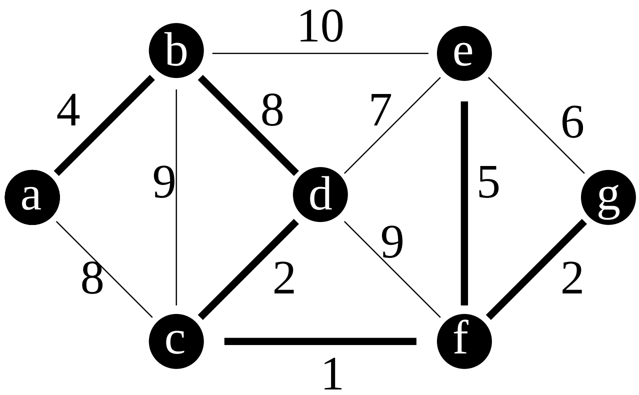

Step 1.5 after
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}

# Prim’s Algorithm

## Prim’s Example – Continued




Step 1.6 before
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}




Step 1.6 after
S={a,b,c,d,e,f,g}
V \ S = {}
A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}
MST completed

# Recall Idea of Prim's Algorithm

**Step 0:** Choose any element $r$ and set $S = \{r\}$ and $A = \emptyset$. (Take $r$ as the root of our spanning tree.)

**Step 1:** Find a lightest edge such that one endpoint is in $S$ and the other is in $V \setminus S$. Add this edge to $A$ and its (other) endpoint to $S$.

**Step 2:** If $V \setminus S = \emptyset$, then stop and output the minimum spanning tree $(S, A)$.
Otherwise go to Step 1.

<u>**Questions:**</u>

- Why does this produce a **Minimum** Spanning Tree?

- How does the algorithm find the lightest edge and update $A$ efficiently?

- How does the algorithm update $S$ efficiently?

## Prim's Algorithm

**Question:** How does the algorithm update $S$ efficiently?

**Answer:** Color the vertices. Initially all are white. Change the color to black when the vertex is moved to $S$. Use color[$v$] to store color.

**Question:** How does the algorithm find the lightest edge and update $A$ efficiently?

**Answer:**
(a) Use a priority queue to find the lightest edge.
(b) Use pred[$v$] to update $A$.

## Reviewing Priority Queues

**Priority Queue** is a data structure (can be implemented as a heap) which supports the following operations:

**insert($u, key$):**
Insert $u$ with the key value $key$ in $Q$.

**u = extractMin():**
Extract the item with the minimum key value in $Q$.

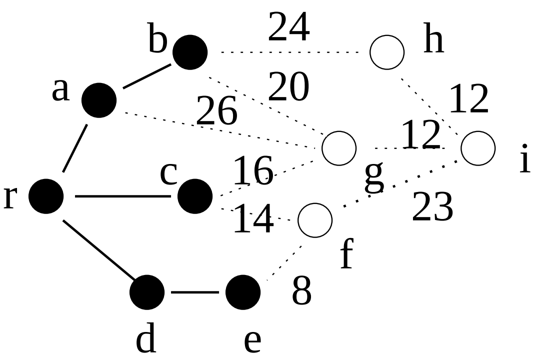
**decreaseKey($u, new\text{-}key$):**
Decrease $u$'s key value to $new\text{-}key$.

**Remark:** Priority Queues can be implemented so that each operation takes time $O(\log |Q|)$. See CLRS!

## Using a Priority Queue to Find the Lightest Edge

Each item of the queue is a triple $(u, pred[u], key[u])$, where

- $u$ is a vertex in $V \setminus S$,
- $key[u]$ is the weight of the lightest edge from $u$ to any vertex in $S$, and
- $pred[u]$ is the endpoint of this edge in $S$. The array is used to build the MST tree.



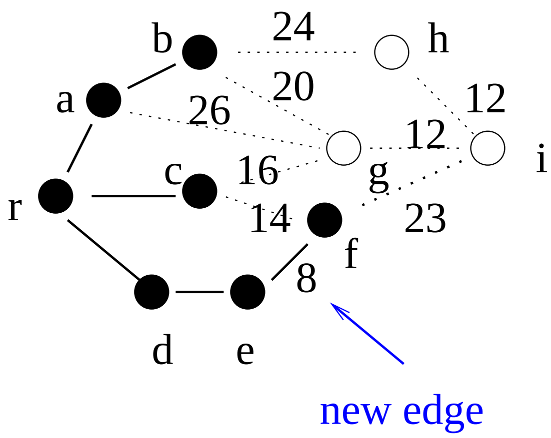

key[f] = 8, pred[f] = e

key[i] = infinity, pred[i] = nil

key[g] = 16, pred[g] = c

key[h] = 24, pred[h] = b

$\longrightarrow$ f has the minimum key




key[i] = 23, pred[i] = f

After adding the new edge and vertex f, update the key[v] and pred[v] for each vertex v adjacent to f

## Description of Prim's Algorithm

**Remark:** $G$ is given by adjacency lists. The vertices in $V \setminus S$ are stored in a priority queue with key=value of lightest edge to vertex in $S$.

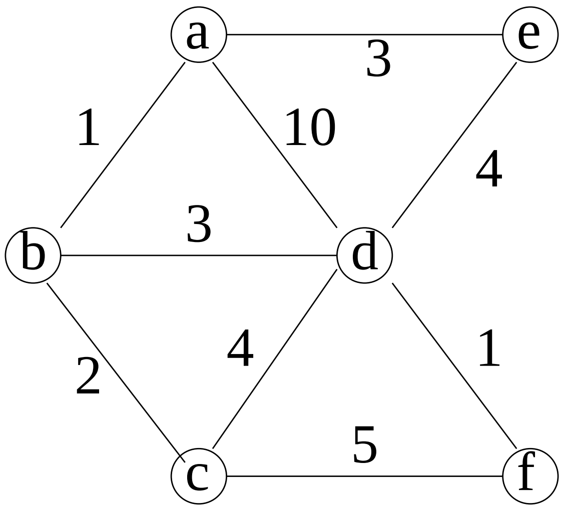
```
Prim(G, w, r)
{   for each u ∈ V                                   initialize
    {   key[u] = +∞;
        color[u] = W;
    }
    key[r] = 0;                                      start at root
    pred[r] = NIL;
    Q = new PriQueue(V);                             put vertices in Q
    while(Q is nonempty)                             until all vertices in MST
    {   u=Q.extraxtMin();                            lightest edge
        for each (v ∈ adj[u])
        {   if ((color[v] == W)&&(w[u,v] < key[v]))
                key[v] = w[u,v];                     new lightest edge
                Q.decreaseKey(v, key[v]);
                pred[v] = u;
        }
        color[u] = B;
    }
}
```

When the algorithm terminates, $Q = \emptyset$ and the MST is

$$T = \{\{v, pred[v]\} : v \in V \setminus \{r\}\}.$$

The pred pointers defi ne the MST as an inverted tree rooted at $r$.

# Example for Running Prim’s Algorithm




| u | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| key[u] | | | | | | |
| pred[u] | | | | | | |

## Analysis of Prim's Algorithm

Let $n = |V|$ and $e = |E|$. The data structure PriQueue supports the following two operations: (See CLRS)

- $O(\log n)$ to extract each vertex from the queue. Done once for each vertex $= O(n \log n)$.

- $O(\log n)$ time to decrease the key value of neighboring vertex. Done at most once for each edge $= O(e \log n)$.

Total cost is then

$$O((n + e) \log n)$$

# Analysis of Prim's Algorithm – Continued

```
Prim(G, w, r)  {
   for each (u in V)
   {
      key[u] = +infinity;                         2n
      color[u] = white;
   }

   key[r] = 0;                                    1
   pred[r] = nil;                                 1
   Q = new PriQueue(V);                           n

   while (Q. nonempty())                  1
   {
      u = Q.extractMin();                 O(log n)
      for each (v in adj[u])
      {
         if ((color[v] == white) &       1
             (w(u,v) < key[v])           1   O(deg(u) log n)
         {
            key[v] = w(u, v);            1
            Q.decreaseKey(v, key[v]);    O(log n)
            pred[v] = u;                 1
         }
      }
      color[u] = black;                   1
   }
}
```

$$\sum_{u \text{ in } V} [O(\log n) + O(deg(u) \log n)]$$

## Analysis of Prim's Algorithm – Continued

So the overall running time is

$$
\begin{aligned}
& T(n, e) \\
= \;& 3n + 2 + \sum_{u \in V} [O(\log n) + O(\deg(u) \log n)] \\
= \;& 3n + 2 + O\left[(\log n) \sum_{u \in V} (1 + \deg(u))\right] \\
= \;& 3n + 2 + O[(\log n)(n + 2e)] \\
= \;& O[(\log n)(n + 2e)] \\
= \;& O[(\log n)(n + e)] \\
= \;& O[(|V| + |E|) \log |V|].
\end{aligned}
$$