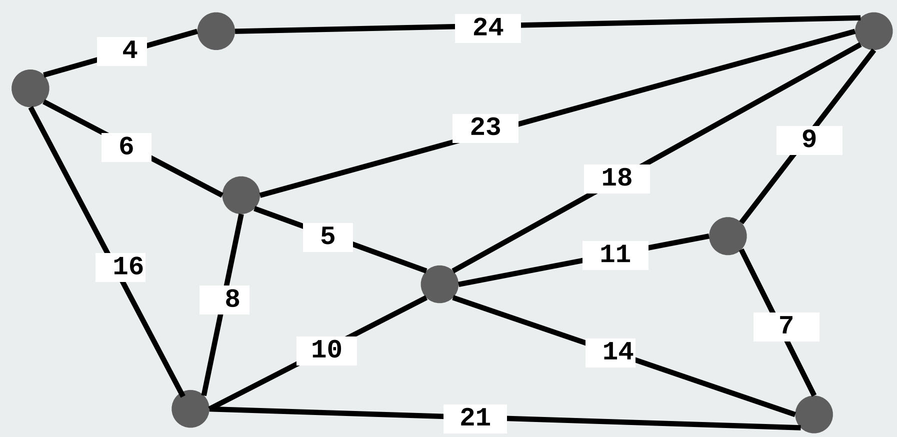
Minimum Spanning Trees

- weighted graph API
- cycles and cuts
- Kruskal’s algorithm
- Prim’s algorithm
- advanced topics

References:
Algorithms in Java, Chapter 20
http://www.cs.princeton.edu/introalgsds/54mst

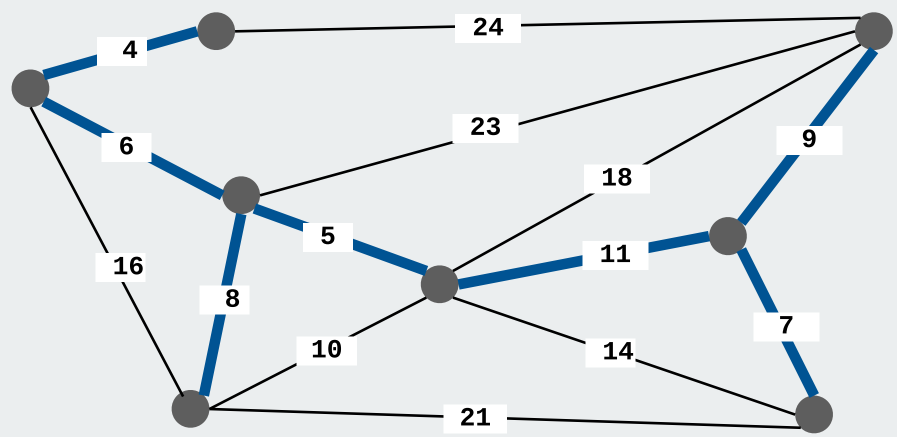
Minimum Spanning Tree

Given. Undirected graph G with positive edge weights (connected).
Goal. Find a min weight set of edges that connects all of the vertices.



G

Given. Undirected graph G with positive edge weights (connected).
Goal. Find a min weight set of edges that connects all of the vertices.



weight(T) = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

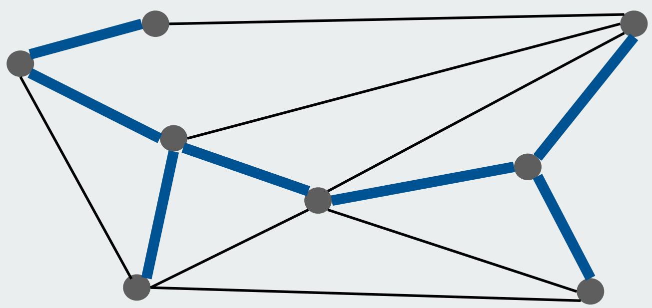
Brute force: Try all possible spanning trees
- problem 1: not so easy to implement
- problem 2: far too many of them

Ex: [Cayley, 1889]: V^{V-2} spanning trees on the complete graph on V vertices.

MST Origin

Otakar Boruvka (1926).

- Electrical Power Company of Western Moravia in Brno.
- Most economical construction of electrical power network.
- Concrete engineering problem is now a cornerstone problem-solving model in combinatorial optimization.



Otakar Boruvka

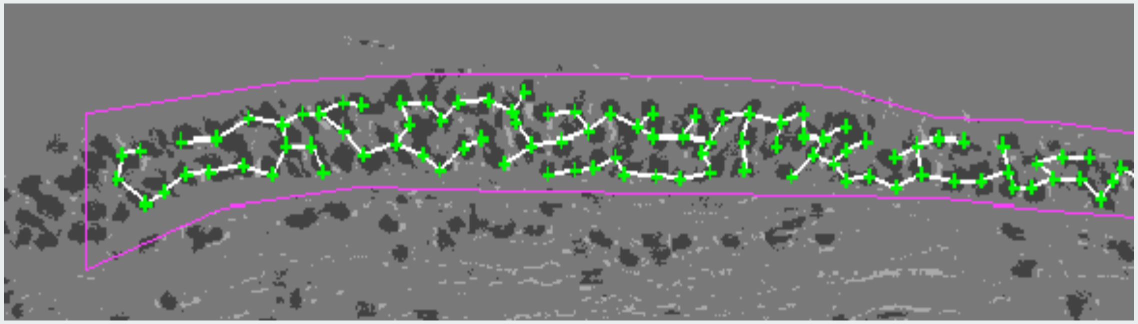
Applications

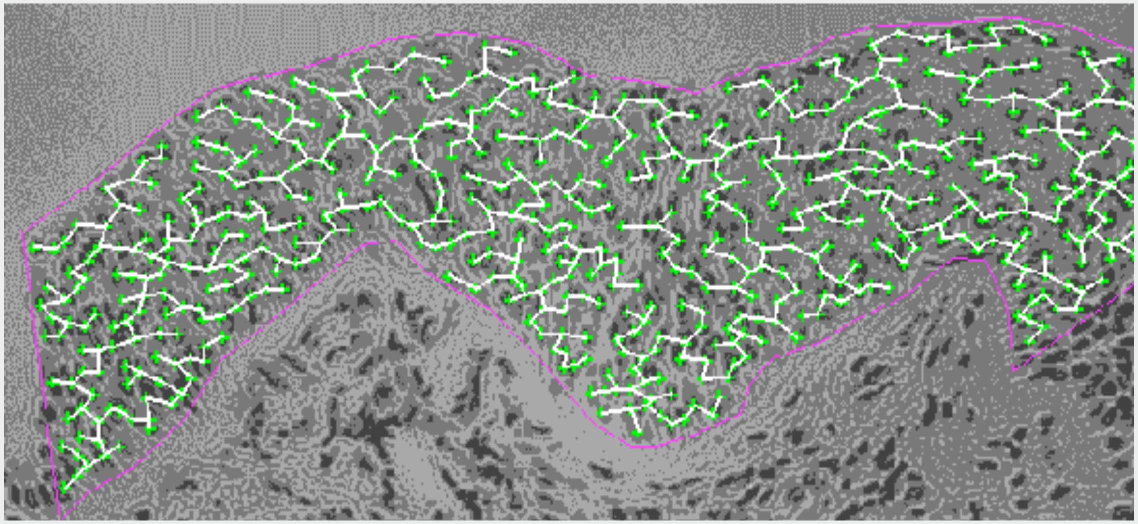
MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

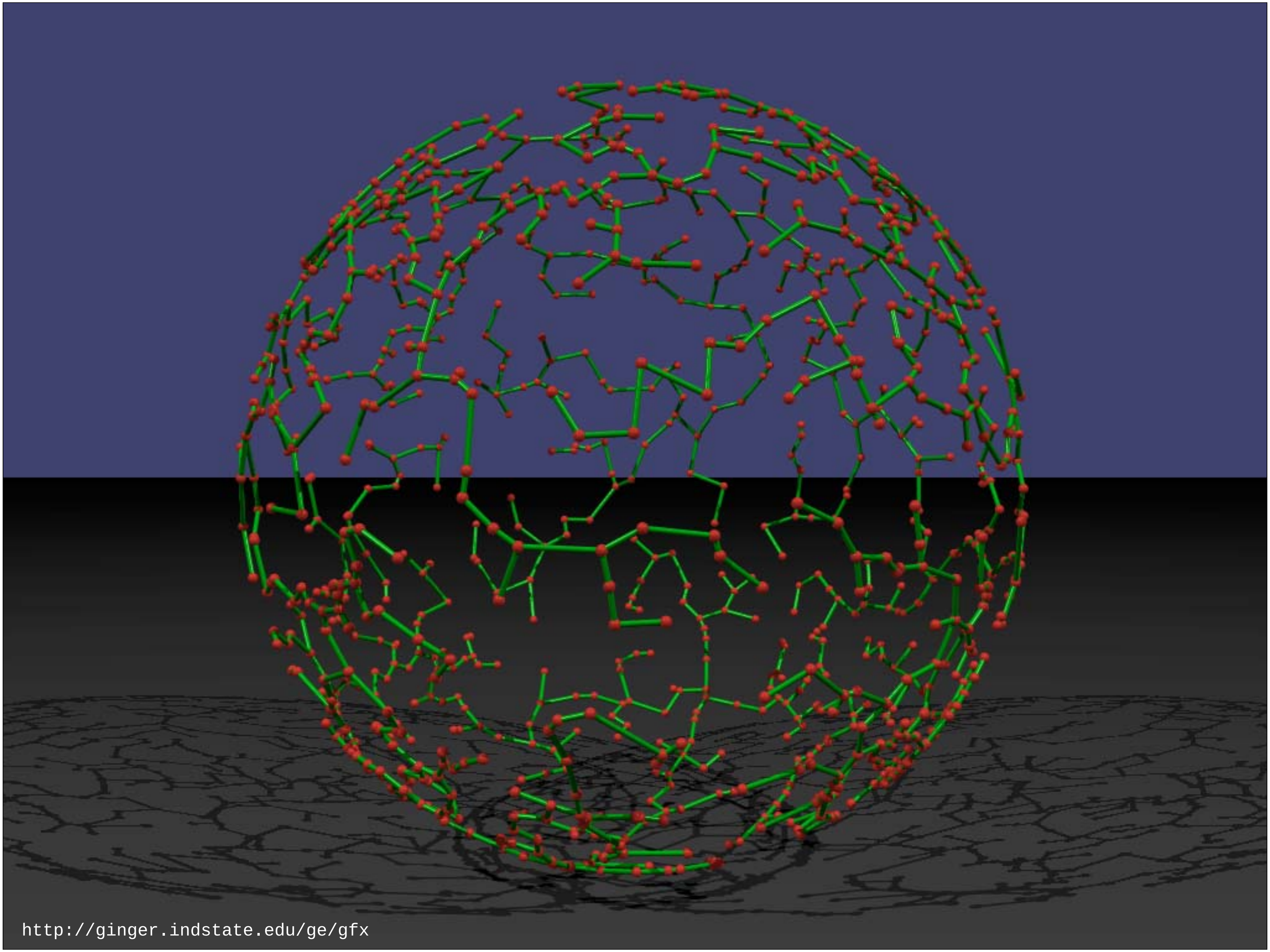
Medical Image Processing

MST describes arrangement of nuclei in the epithelium for cancer research





http://www.bccrc.ca/ci/ta01_archlevel.html


http://ginger.indstate.edu/ge/gfx

Two Greedy Algorithms

Kruskal's algorithm. Consider edges in ascending order of cost. Add the next edge to T unless doing so would create a cycle.

Prim's algorithm. Start with any vertex s and greedily grow a tree T from s. At each step, add the cheapest edge to T that has exactly one endpoint in T.

Proposition. Both greedy algorithms compute an MST.

> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit." - Gordon Gecko

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Weighted Graph API

`public class WeightedGraph`		
	`WeightedGraph(int V)`	create an empty graph with V vertices
`void`	`insert(Edge e)`	insert edge e
`Iterable<Edge>`	`adj(int v)`	return an iterator over edges incident to v
`int`	`V()`	return the number of vertices
`String`	`toString()`	return a string representation

iterate through all edges (once in each direction)

Weighted graph data type

Identical to **Graph.java** but use **Edge** adjacency sets instead of **int**.

```
public class WeightedGraph
{
   private int V;
   private SET<Edge>[] adj;

   public Graph(int V)
   {
      this.V = V;
      adj = (SET<Edge>[]) new SET[V];
      for (int v = 0; v < V; v++)
         adj[v] = new SET<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.v, w = e.w;
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }

}
```

Weighted edge data type

```
public class Edge implements Comparable<Edge>
{
   private final int v, int w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v; }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int weight()
   {  return weight; }

   // See next slide for edge compare methods.

}
```

Edge abstraction needed for weights

slightly tricky accessor methods
(enables client code like this)

```
for (int v = 0; v < G.V(); v++)
{
   for (Edge e : G.adj(v))
   {
      int w = e.other(v);

      // edge v-w
   }
}
```

Weighted edge data type: compare methods

Two different compare methods for edges

- **`compareTo()`** so that edges are **`Comparable`** (for use in **`SET`**)
- **`compare()`** so that clients can compare edges by weight.

```
public final static Comparator<Edge> BY_WEIGHT = new ByWeightComparator();

private static class ByWeightComparator implements Comparator<Edge>
{
   public int compare(Edge e, Edge f)
   {
      if (e.weight < f.weight) return -1;
      if (e.weight > f.weight) return +1;
      return 0;
   }
}

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

- weighted graph API
- **cycles and cuts**
- Kruskal's algorithm
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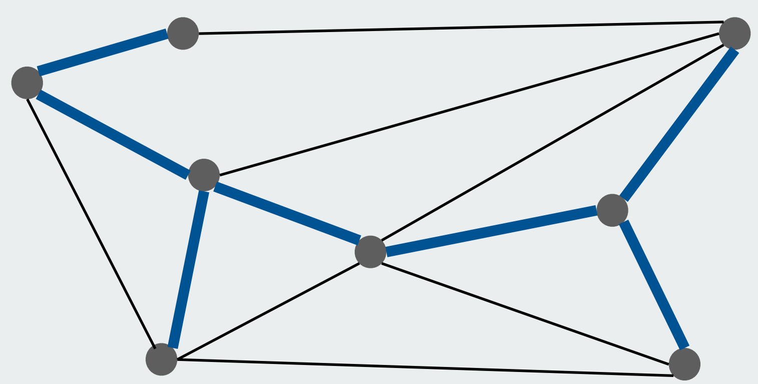
Spanning Tree

MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.

Def. A spanning tree of a graph G is a subgraph T that is connected and acyclic.

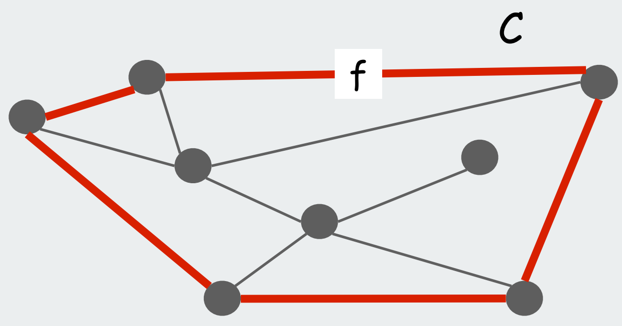


Property. MST of G is always a spanning tree.

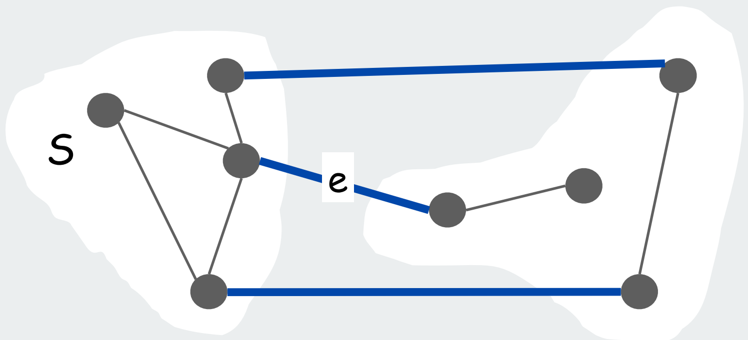
Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.

Cut property. Let S be any subset of vertices, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.



f is not in the MST



e is in the MST

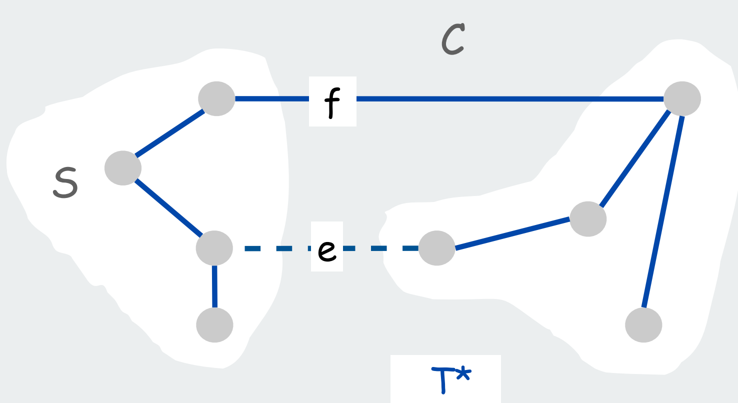
Cycle Property

Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. [by contradiction]

- Suppose f belongs to T*. Let's see what happens.
- Deleting f from T* disconnects T*. Let S be one side of the cut.
- Some other edge in C, say e, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, $\text{cost}(T) < \text{cost}(T^*)$.
- Contradicts minimality of T*. ▪



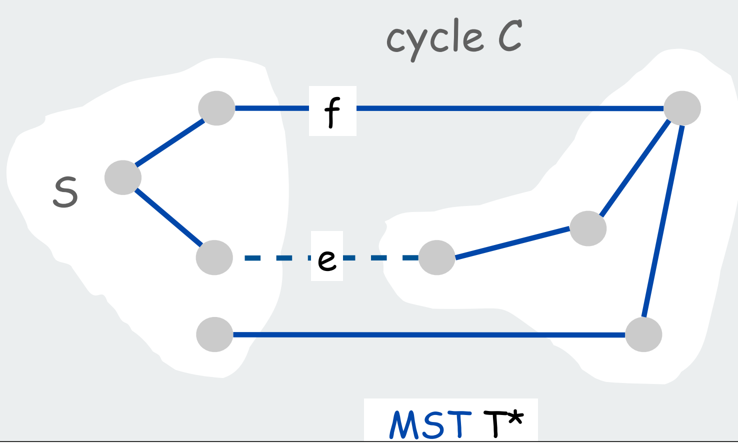
Cut Property

Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of vertices, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. [by contradiction]

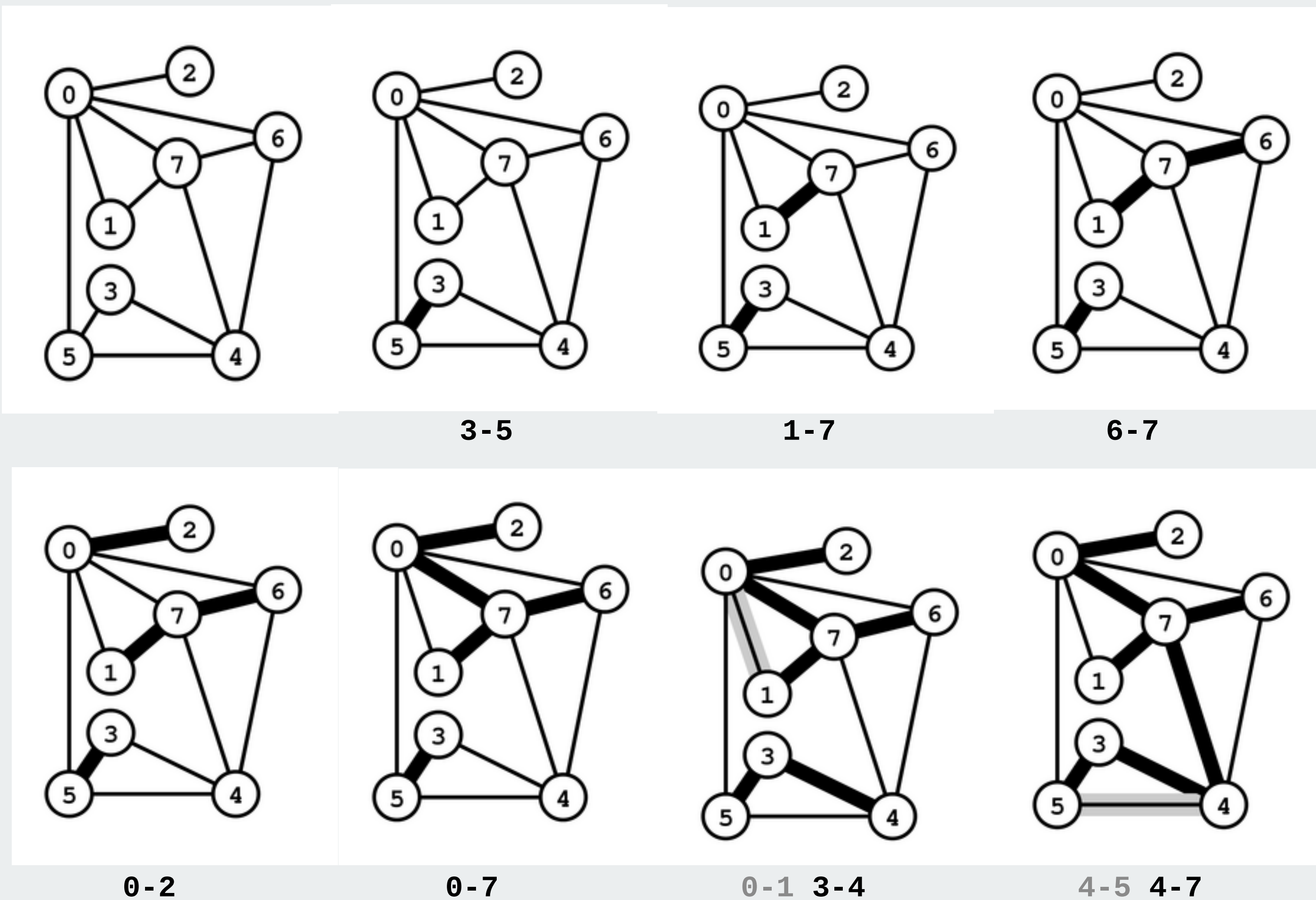
- Suppose e does not belong to T*. Let's see what happens.
- Adding e to T* creates a (unique) cycle C in T*.
- Some other edge in C, say f, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, cost(T) < cost(T*).
- Contradicts minimality of T*. ▪



- weighted graph API
- cycles and cuts
- **Kruskal's algorithm**
- Prim's algorithm
- advanced algorithms
- clustering

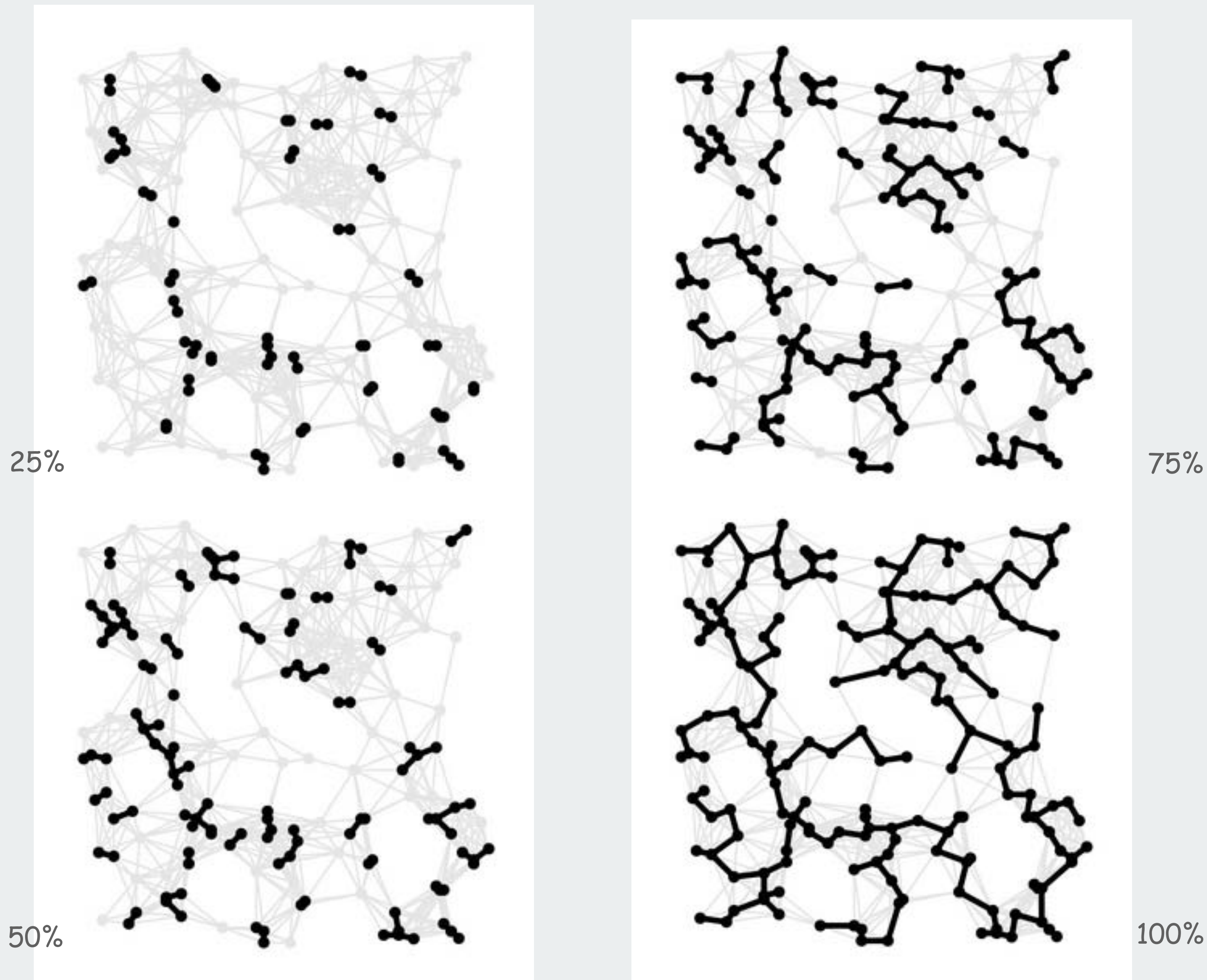
Kruskal's Algorithm: Example

Kruskal's algorithm. [Kruskal, 1956] Consider edges in ascending order of cost. Add the next edge to T unless doing so would create a cycle.



3-5	0.18
1-7	0.21
6-7	0.25
0-2	0.29
0-7	0.31
0-1	0.32
3-4	0.34
4-5	0.40
4-7	0.46
0-6	0.51
4-6	0.51
0-5	0.60

Kruskal's algorithm example

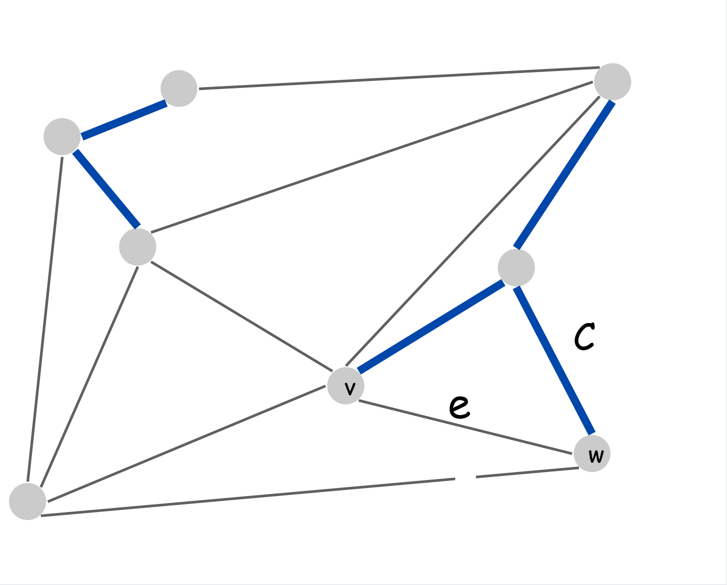


Kruskal's algorithm correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. [case 1] Suppose that adding e to T creates a cycle C

- e is the max weight edge in C (weights come in increasing order)
- e is not in the MST (cycle property)

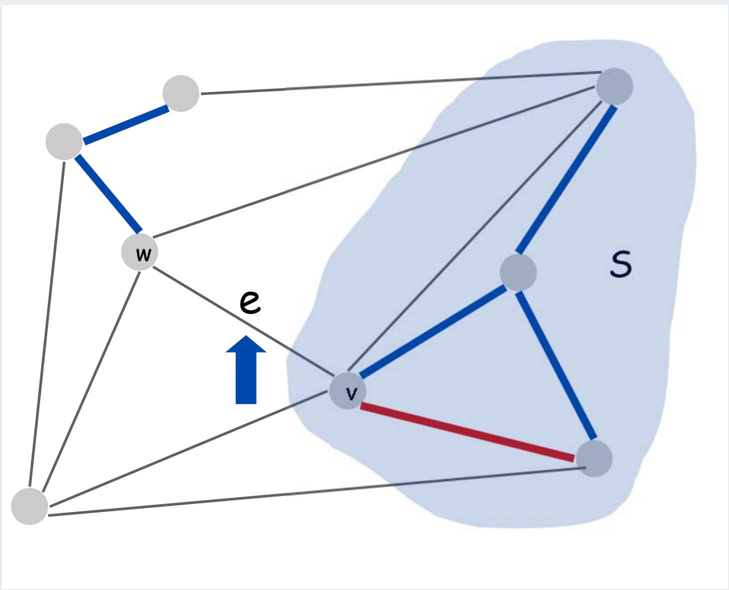


Kruskal's algorithm correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. [case 2] Suppose that adding e = (v, w) to T does not create a cycle

- let S be the vertices in v's connected component
- w is not in S
- e is the min weight edge with exactly one endpoint in S
- e is in the MST (cut property) ■



Kruskal's algorithm implementation

Q. How to check if adding an edge to T would create a cycle?

A1. Naïve solution: use DFS.

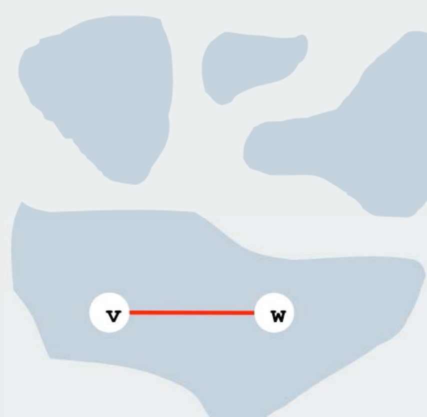
- O(V) time per cycle check.
- O(E V) time overall.

Kruskal's algorithm implementation

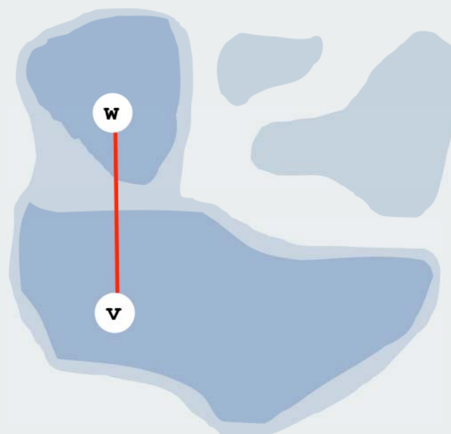
Q. How to check if adding an edge to T would create a cycle?

A2. Use the union-find data structure from lecture 1 (!).

- Maintain a set for each connected component.
- If v and w are in same component, then adding v-w creates a cycle.
- To add v-w to T, merge sets containing v and w.



Case 1: adding v-w creates a cycle



Case 2: add v-w to T and merge sets

Kruskal's algorithm: Java implementation

```
public class Kruskal
{
   private SET<Edge> mst = new SET<Edge>();

   public Kruskal(WeightedGraph G)
   {
      Edge[] edges = G.edges();
      Arrays.sort(edges, Edge.BY_WEIGHT);

      UnionFind uf = new UnionFind(G.V());
      for (Edge e: edges)
         if (!uf.find(e.either(), e.other()))
         {
            uf.unite(e.either(), e.other());
            mst.add(edge);
         }

   }

   public Iterable<Edge> mst()
   {  return mst;  }
}
```

sort edges by weight

greedily add edges to MST

return to client iterable sequence of edges

Easy speedup: Stop as soon as there are V-1 edges in MST.

Kruskal's algorithm running time

Kruskal running time: Dominated by the cost of the sort.

Operation	Frequency	Time per op
sort	1	E log E
union	V	log* V †
find	E	log* V †

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

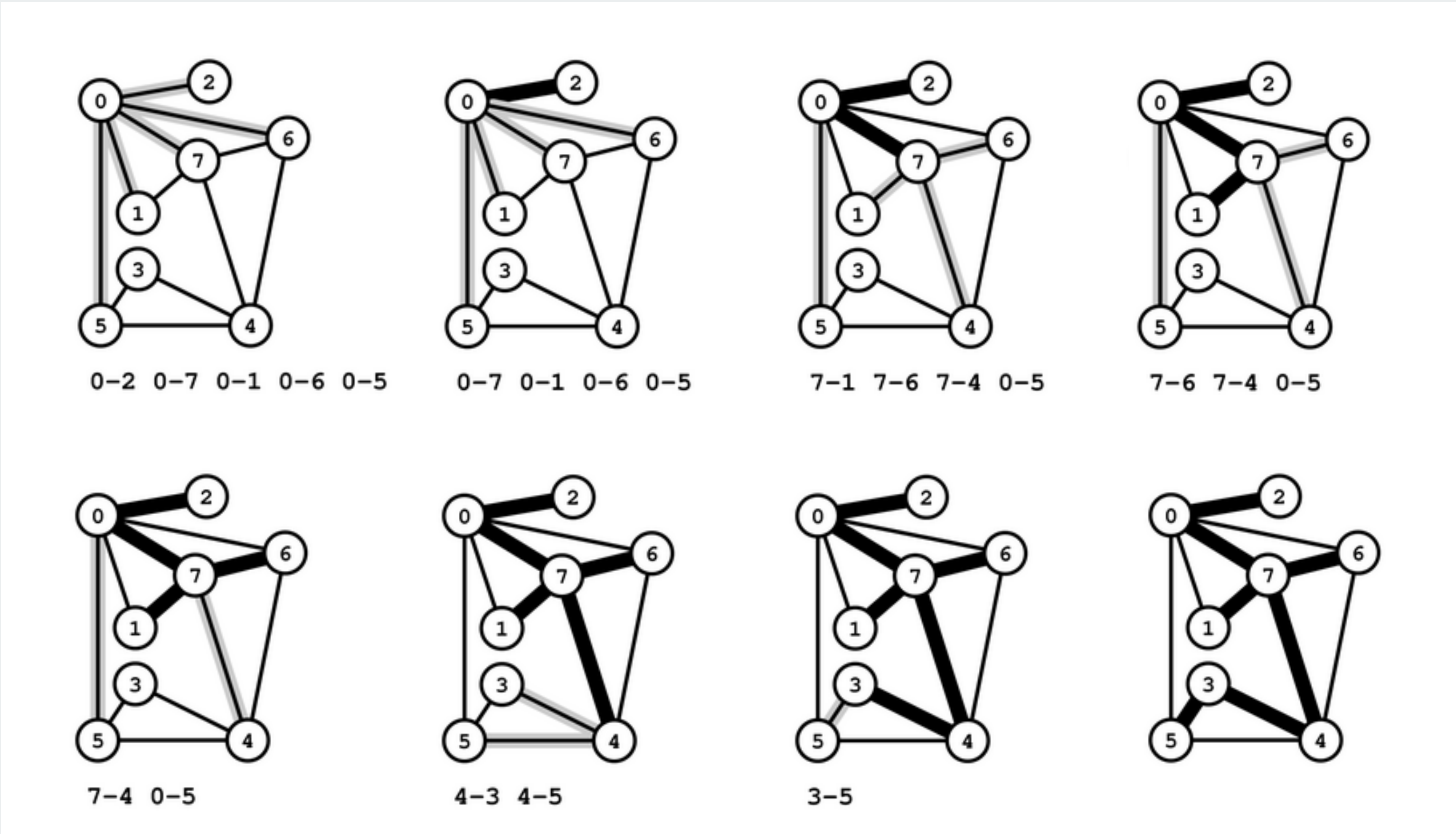
Remark 1. If edges are already sorted, time is proportional to E log* V

Remark 2. Linear in practice with PQ or quicksort partitioning
(see book: don't need full sort)

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- **Prim's algorithm**
- advanced topics

Prim's algorithm example

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]
Start with vertex 0 and greedily grow tree T. At each step, add cheapest edge that has exactly one endpoint in T.



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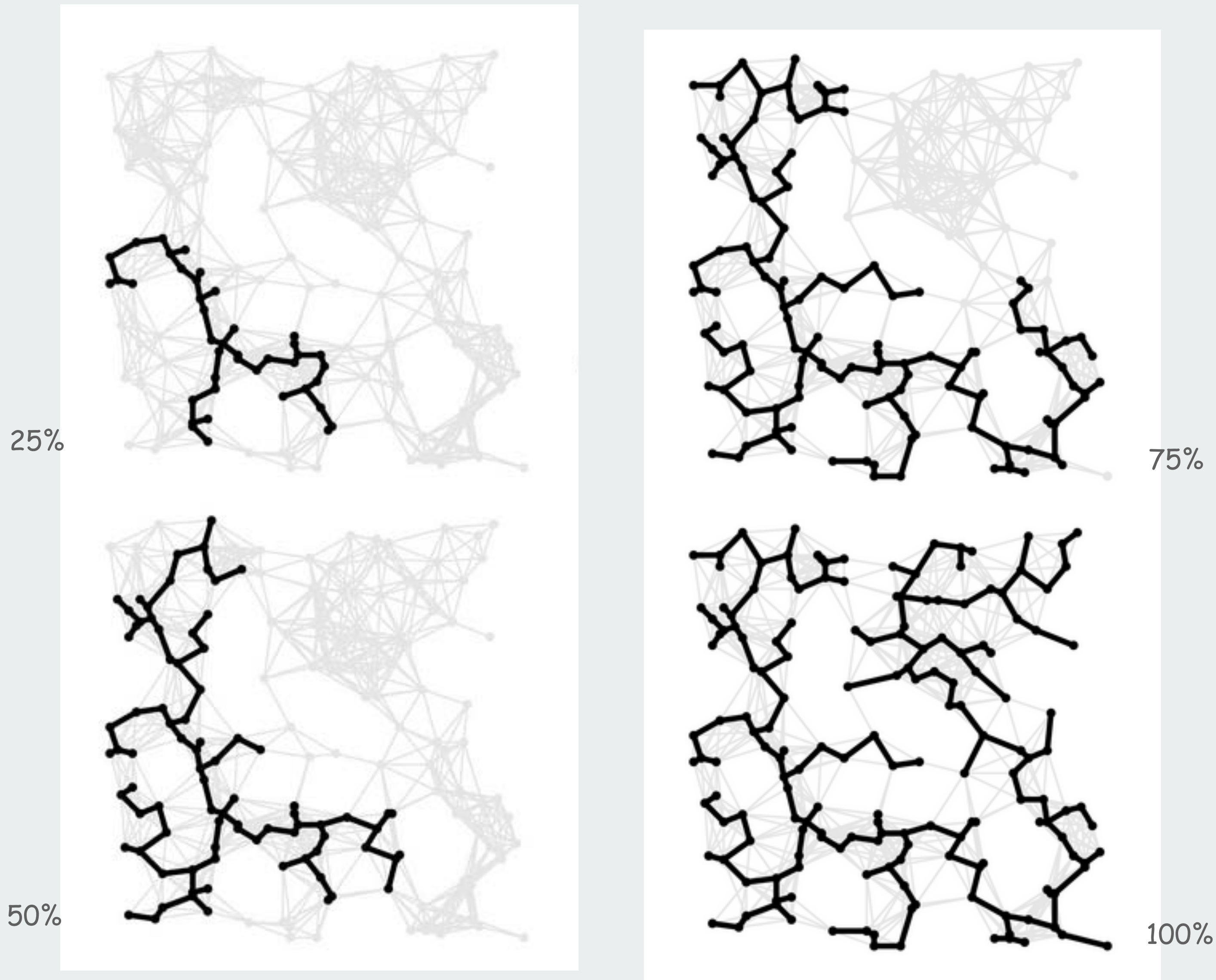
Prim's Algorithm example

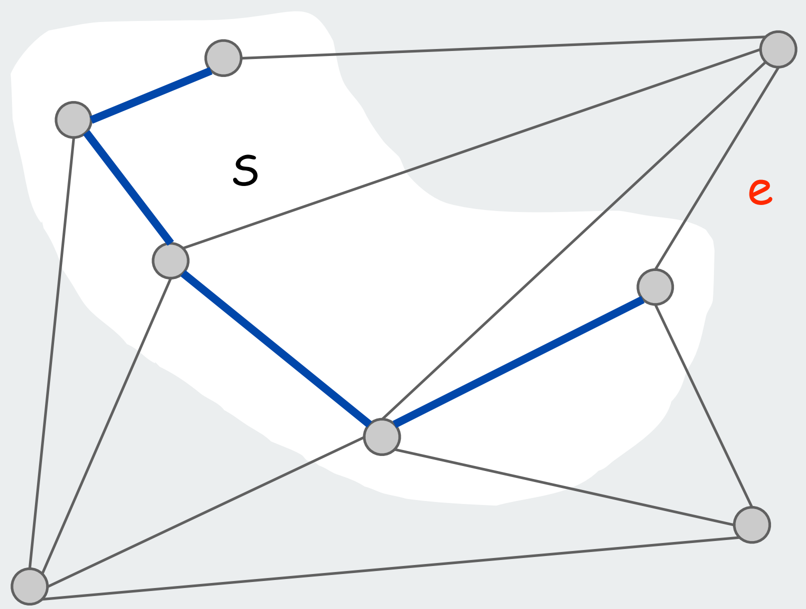


Prim's algorithm correctness proof

Proposition. Prim's algorithm computes the MST.

Pf.

- Let S be the subset of vertices in current tree T.
- Prim adds the cheapest edge e with exactly one endpoint in S.
- e is in the MST (cut property) ▪



Prim's algorithm implementation

Q. How to find cheapest edge with exactly one endpoint in S?

A1. Brute force: try all edges.

- O(E) time per spanning tree edge.
- O(E V) time overall.

Prim's algorithm implementation

Q. How to find cheapest edge with exactly one endpoint in S?

A2. Maintain a priority queue of vertices connected by an edge to S

- Delete min to determine next vertex v to add to S.
- Disregard v if already in S.
- Add to PQ any vertex brought closer to S by v.

Running time.

- log V steps per edge (using a binary heap).
- E log V steps overall.

Note: This is a lazy version of implementation in Algs in Java

lazy: put all adjacent vertices (that are not already in MST) on PQ
eager: first check whether vertex is already on PQ and decrease its key

Key-value priority queue

Associate a value with each key in a priority queue.

API:

`public class MinPQplus<Key extends Comparable<Key>, Value>`

	`MinPQplus()`	create a key-value priority queue
`void`	`put(Key key, Value val)`	put key-value pair into the priority queue
`Value`	`delMin()`	return value paired with minimal key
`Key`	`min()`	return minimal key

Implementation:

- start with same code as standard heap-based priority queue
- use a parallel array `vals[]` (value associated with `keys[i]` is `vals[i]`)
- modify `exch()` to maintain parallel arrays (do exch in `vals[]`)
- modify `delMin()` to return `Value`
- add `min()` (just returns `keys[1]`)

Lazy implementation of Prim's algorithm

```
public class LazyPrim
{
   Edge[] pred = new Edge[G.V()];
   public LazyPrim(WeightedGraph G)
   {
      boolean[] marked = new boolean[G.V()];
      double[] dist = new double[G.V()];
      MinPQplus<Double, Integer> pq;
      pq = new MinPQplus<Double, Integer>();
      dist[s] = 0.0;
      marked[s] = true;
      pq.put(dist[s], s);
      while (!pq.isEmpty())
      {
         int v = pq.delMin();
         if (marked[v]) continue;
         marked(v) = true;
         for (Edge e : G.adj(v))
         {
            int w = e.other(v);
            if (!done[w] && (dist[w] > e.weight()))
            {
               dist[w] = e.weight(); pred[w] = e;
               pq.insert(dist[w], w);
            }
         }
      }
   }
}
```

`pred[v]` is edge attaching v to MST

marks vertices in MST

distance to MST

key-value PQ

get next vertex

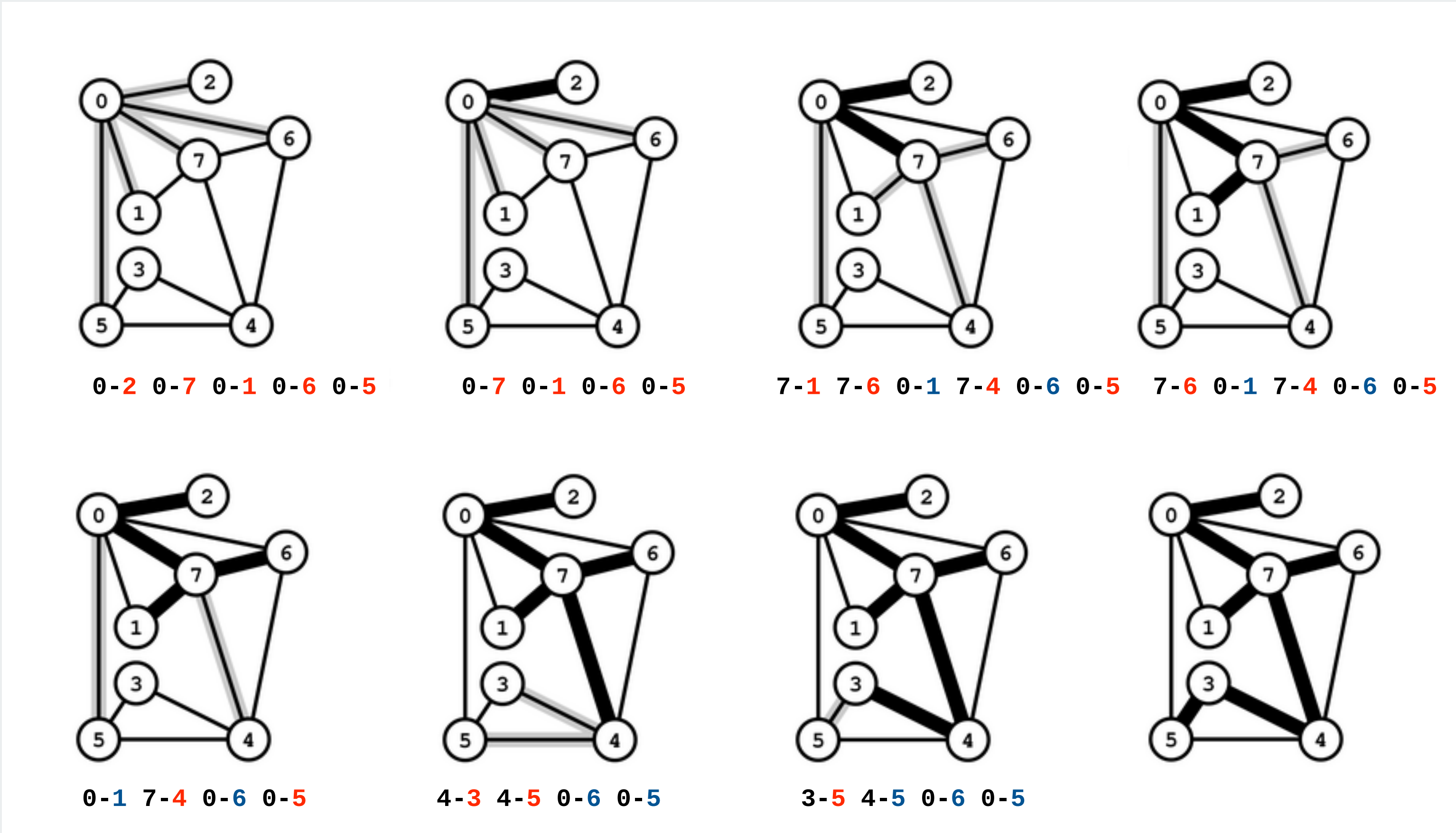
ignore if already in MST

add to PQ any vertices brought closer to S by v

Prim's algorithm (lazy) example

Priority queue key is distance (edge weight); value is vertex

Lazy version leaves obsolete entries in the PQ
therefore may have multiple entries with same value



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3-5	0.18
4-5	0.40
4-6	0.51
4-7	0.46
6-7	0.25

red: pq value (vertex)
blue: obsolete value

Eager implementation of Prim's algorithm

Use indexed priority queue that supports

- contains: is there a key associated with value v in the priority queue?
- decrease key: decrease the key associated with value v

[more complicated data structure, see text]

Putative "benefit": reduces PQ size guarantee from E to V

- not important for the huge sparse graphs found in practice
- PQ size is far smaller in practice
- widely used, but practical utility is debatable

Removing the distinct edge costs assumption

Simplifying assumption. All edge weights w_e are distinct.

Fact. Prim and Kruskal don't actually rely on the assumption
(our proof of correctness does)

Suffices to introduce tie-breaking rule for `compare()`.

Approach 1:

```
public int compare(Edge e, Edge f)
{
   if (e.weight < f.weight) return -1;
   if (e.weight > f.weight) return +1;
   if (e.v < f.v) return -1;
   if (e.v > f.v) return +1;
   if (e.w < f.w) return -1;
   if (e.w > f.w) return +1;
   return  0;
}
```

Approach 2: add tiny random perturbation.

- weighted graph API
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- **advanced topics**

Advanced MST theorems: does an algorithm with a linear-time guarantee exist?

Year	Worst Case	Discovered By
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

deterministic comparison based MST algorithms

Year	Problem	Time	Discovered By
1976	Planar MST	E	Cheriton-Tarjan
1992	MST Verification	E	Dixon-Rauch-Tarjan
1995	Randomized MST	E	Karger-Klein-Tarjan

related problems

Euclidean MST

Euclidean MST. Given N points in the plane, find MST connecting them.

- Distances between point pairs are Euclidean distances.

Brute force. Compute $N^2 / 2$ distances and run Prim's algorithm.

Ingenuity. Exploit geometry and do it in O(N log N)

[stay tuned for geometric algorithms]

Scientific application: clustering

k-clustering. Divide a set of objects classify into k coherent groups.
distance function. numeric value specifying "closeness" of two objects.

Fundamental problem.
Divide into clusters so that points in different clusters are far apart.

Applications.
- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

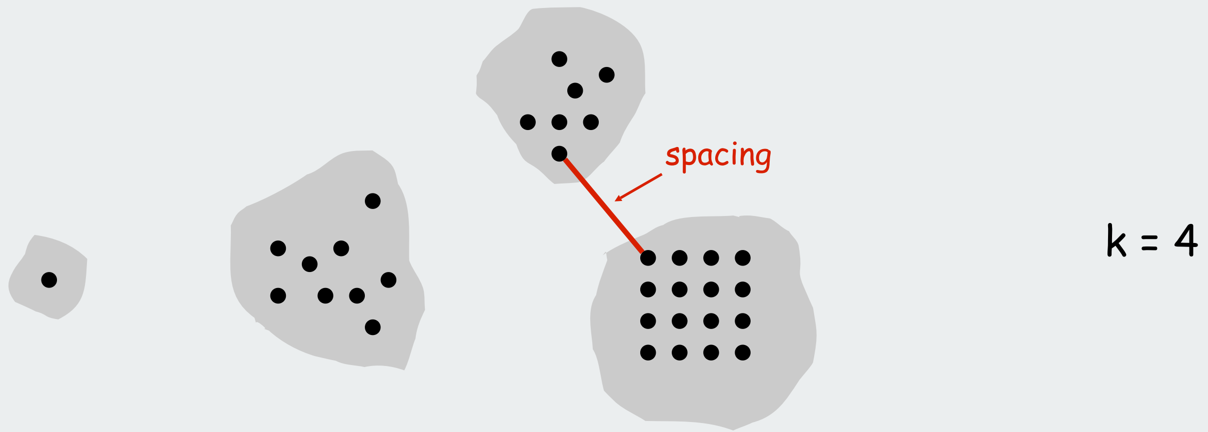
Outbreak of cholera deaths in London in 1850s.
Reference: Nina Mishra, HP Labs

k-clustering of maximum spacing

k-clustering. Divide a set of objects classify into k coherent groups.
distance function. Numeric value specifying "closeness" of two objects.

Spacing. Min distance between any pair of points in different clusters.

k-clustering of maximum spacing.
Given an integer k, find a k-clustering such that spacing is maximized.



Single-link clustering algorithm

"Well-known" algorithm for single-link clustering:

- Form V clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat until there are exactly k clusters.

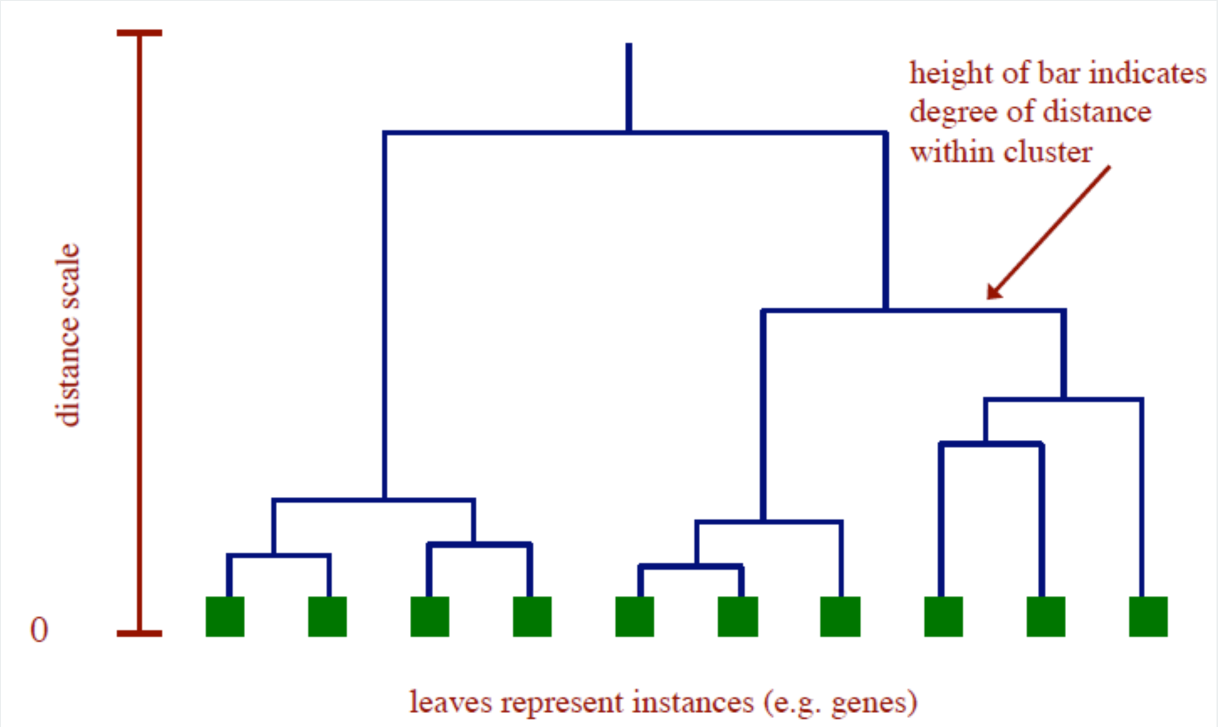
Observation. This procedure is precisely Kruskal's algorithm (stop when there are k connected components).

Property. Kruskal's algorithm finds a k-clustering of maximum spacing.

Clustering application: dendrograms

Dendrogram.
Scientific visualization of hypothetical sequence of evolutionary events.

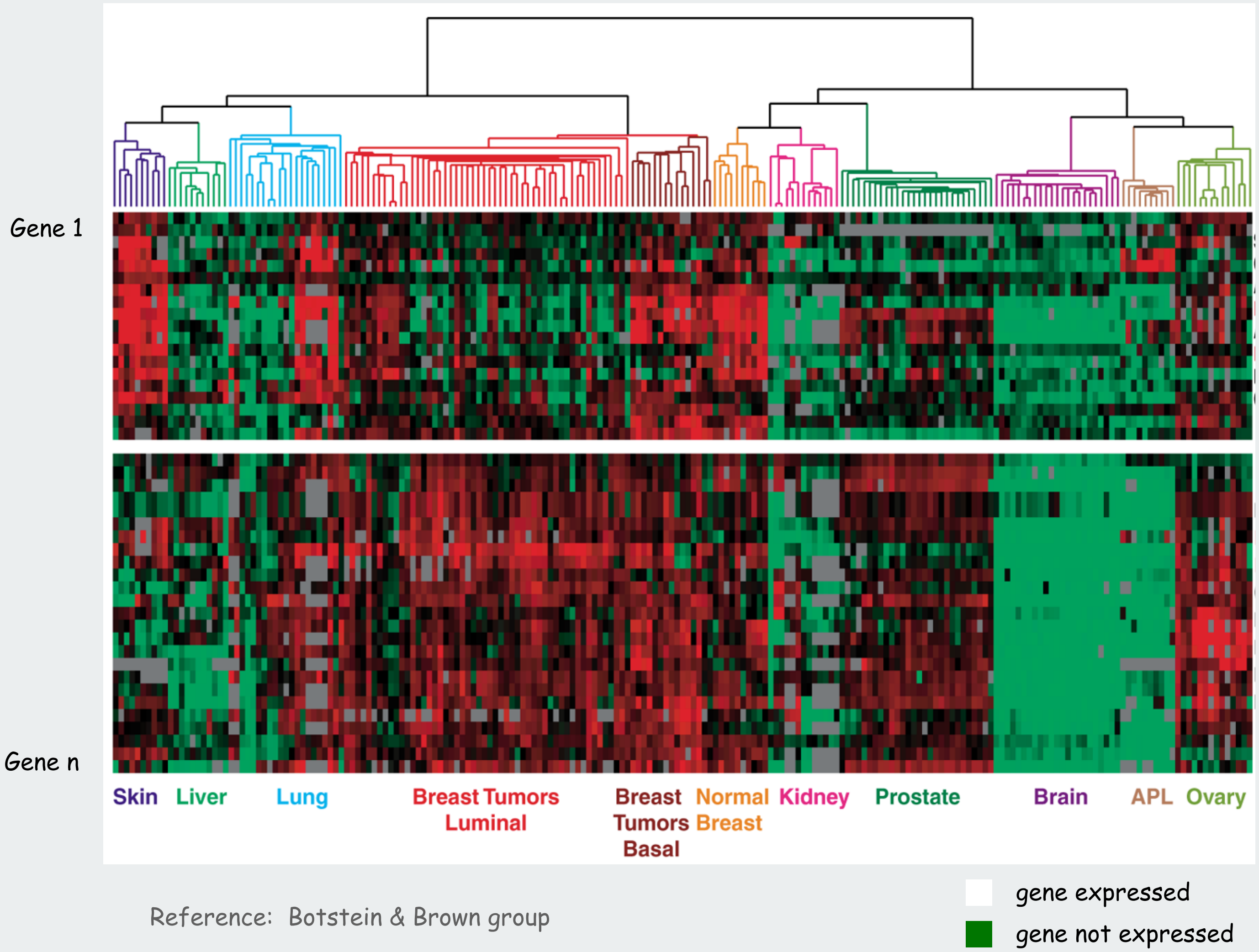
- Leaves = genes.
- Internal nodes = hypothetical ancestors.



Reference: http://www.biostat.wisc.edu/bmi576/fall-2003/lecture13.pdf

Dendrogram of cancers in human

Tumors in similar tissues cluster together.



Reference: Botstein & Brown group