CSI 503 – Data Structures and Algorithms

Proofs of Lemmas 1, 2 and 3 in Lecture 14

Handout 14.2

Lemma 1: Every pair of tasks in the set A returned by the algorithm is compatible.

Proof: To begin with, the algorithm adds a_1 to the set A. At that time, $A = \{a_1\}$. So, compatibility is trivial. Subsequently, every time the algorithm adds a new task a_i to A in Step 4, the starting time of a_i is at least as large as the largest finish time among all the tasks that are currently in A. Thus, the new task a_i is compatible with every task that is currently in A. Since this property holds every time a new task is added to A, at the end of the algorithm, every pair of tasks in A is compatible. ■

Lemma 2 (Greedy Choice Property): Let $\langle a_1, a_2, \ldots, a_n \rangle$ denote the task ordering after the sorting step. There is an optimal solution to the problem that includes task a_1.

Proof: Suppose $A' = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ is an optimal solution, also arranged according to the sorted order of finish times.

If $i_1 = 1$, then A' already includes a_1 and so the lemma follows trivially. So, assume that $i_1 \neq 1$. Consider the set $A'' = (A' - \{a_{i_1}\}) \cup \{a_1\}$. Since the finish time of a_1 is no more than that of a_{i_1}, any task with which a_{i_1} is compatible is also compatible with a_1. In other words, A'' also contains tasks that are pairwise compatible. Further, $|A''| = |A'|$. Therefore A'' is also an optimal solution to the problem. Since A'' contains a_1, the lemma is proven. ■

Lemma 3 (Optimal Substructure Property): If A is an optimal solution to the problem containing a_1, then $A - \{a_1\}$ is an optimal solution to the problem consisting of the set S_2 of tasks where $S_2 = \{a_i \ : \ s_i \geq f_1\}$. (That is, $A - \{a_1\}$ is an optimal solution to the subproblem specified by S_2 containing all the tasks that are compatible with a_1.)

Proof: The proof is by contradiction. Suppose there is a solution A' to the subproblem specified by S_2 such that $|A'| > |A - \{a_1\}|$. Since all the tasks in S_2 are compatible with A_1, the set $A'' = A' \cup \{a_1\}$ is a solution to the original problem containing a_1. However, $|A''|$ is larger than $|A|$, and this contradicts the assumption that A is an optimal solution to the original problem. Thus, $A - \{a_1\}$ is an optimal solution to the problem specified by S_2. ■