Minimum Spanning Trees

Algorithms and Applications

Varun Ganesan

18.304 Presentation

Outline

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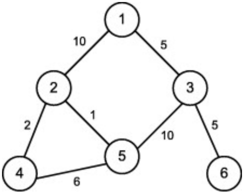
Graphs

- In graph theory, a **graph** is an ordered pair $G = (V, E)$ comprising a set of **vertices** or nodes together with a set of **edges**.



- **Edges** are 2-element subsets of V which represent a connection between two vertices.
  - Edges can either be directed or undirected.
  - Edges can also have a **weight** attribute.

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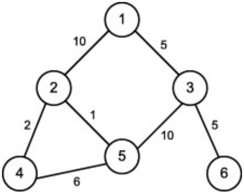

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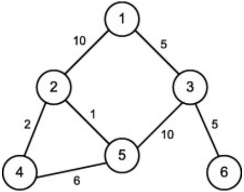

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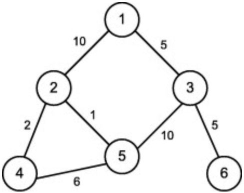

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Connectivity

- A graph is **connected** when there is a path between every pair of vertices.
- A **cycle** is a path that starts and ends with the same vertex.
- A **tree** is a connected, acyclic graph.

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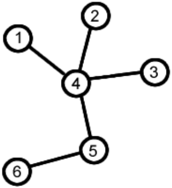

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1. Definitions
   - Graph Terminology
   - Minimum Spanning Trees
2. Common Algorithms
   - Kruskal's Algorithm
   - Prim's Algorithm
3. Applications

Spanning Trees

- Formally, for a graph $G = (V, E)$, the spanning tree is $E' \subseteq E$ such that:
  - $\exists u \in V : (u, v) \in E' \vee (v, u) \in E' \; \forall v \in V$
  - In other words: the subset of edges spans all vertices.
  - $|E'| = |V| - 1$
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Linear Graph

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What Makes A Spanning Tree The Minimum?

MST Criterion: When the *sum* of the edge weights in a spanning tree is the minimum over all spanning trees of a graph

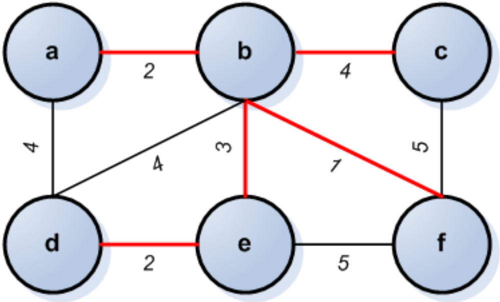

Figure: Suppose (b, f) is removed and (c, f) is added...

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Main Idea

- Start with $|V|$ disjoint components.
- Consider lesser weight edges first to incrementally connect components.
- Make certain to avoid cycles.
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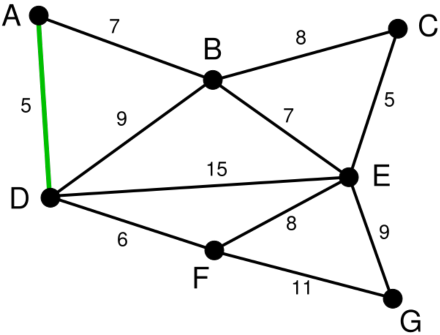
Pseudocode

Kruskal's Algorithm

$A = 0$
foreach $v \in G.V$:
MAKE-SET(v)
foreach (u, v) ordered by $weight(u, v)$, increasing:
if $FIND\text{-}SET(u) \neq FIND\text{-}SET(v)$:
$A = A \cup (u, v)$
$UNION(u, v)$
return A

Example



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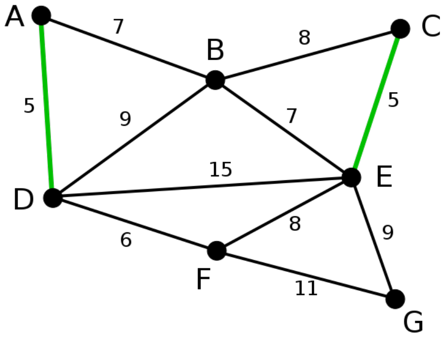

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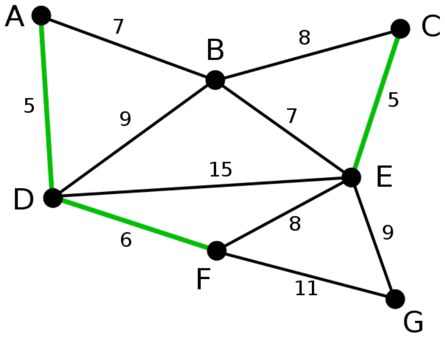

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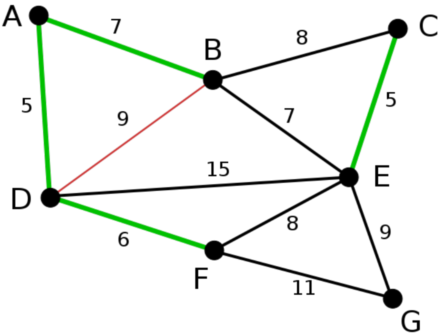

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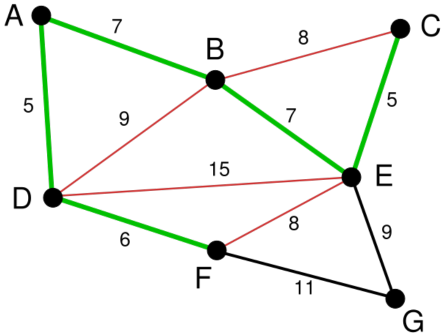

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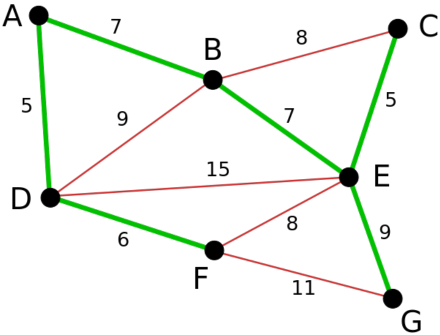

Proof Of Correctness

- Spanning Tree Validity
  - By avoiding connecting two already connected vertices, output has no cycles.
  - If G is connected, output must be connected.
- Minimality
  - Consider a lesser total weight spanning tree with at least one different edge $e = (u, v)$.
  - If e leads to less weight, then e would have been considered before some edge that connects u and v in our output.

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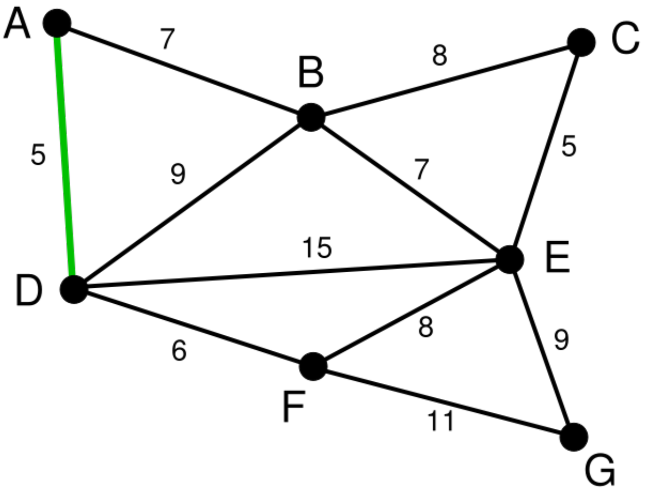

Proof of Correctness

Both the spanning tree and minimality argument are nearly identical for Prim's as they are for Kruskal's.

Some Applications

- ***Taxonomy***
- ***Clustering Analysis***
- ***Traveling Salesman Problem Approximation***