Introduction to Programming (in C++)

Multi-dimensional vectors

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Matrices

- A matrix can be considered a two-dimensional vector, i.e. a vector of vectors.

`my_matrix:`

3	8	1	0
5	0	6	3
7	2	9	4

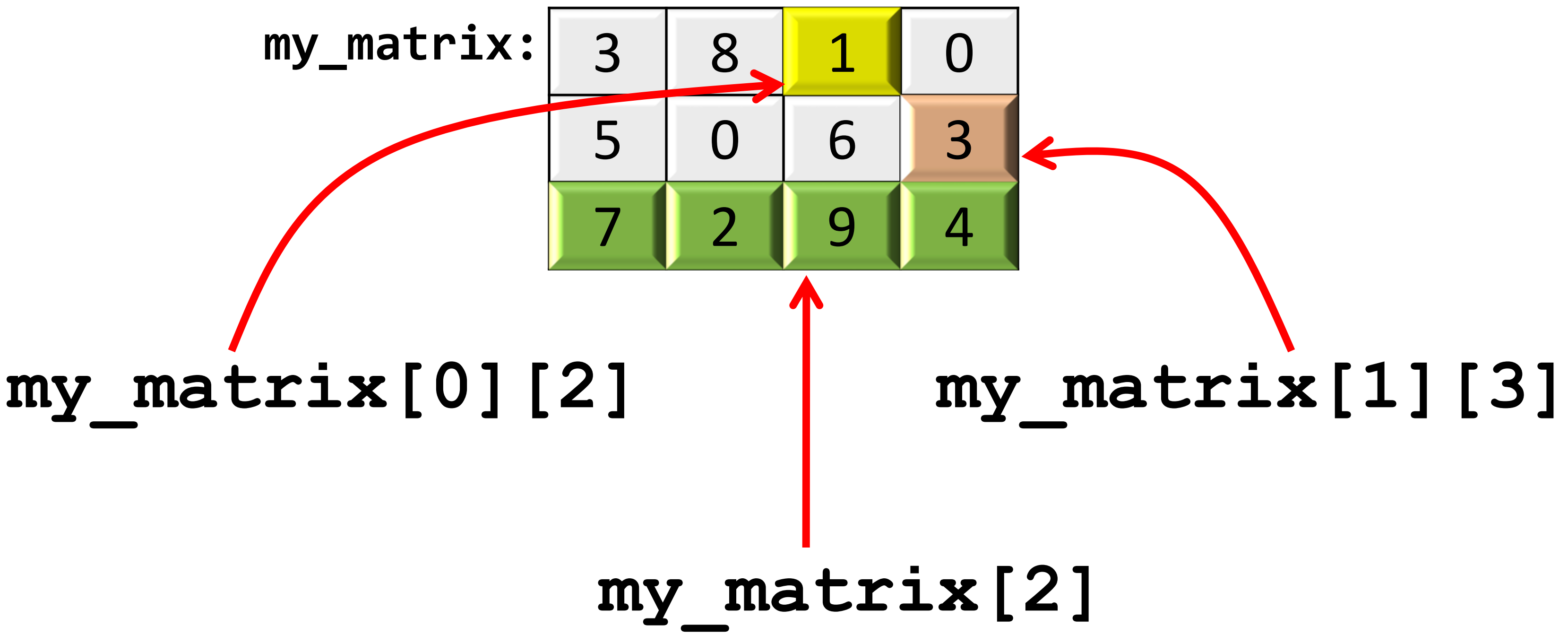
```
// Declaration of a matrix with 3 rows and 4 columns
vector< vector<int> > my_matrix(3,vector<int>(4));


// A more elegant declaration
typedef vector<int> Row;     // One row of the matrix
typedef vector<Row> Matrix; // Matrix: a vector of rows

Matrix my_matrix(3,Row(4));  // The same matrix as above
```

Matrices

- A matrix can be considered as a 2-dimensional vector, i.e., a vector of vectors.



n-dimensional vectors

- Vectors with any number of dimensions can be declared:

```
typedef vector<int> Dim1;
typedef vector<Dim1> Dim2;
typedef vector<Dim2> Dim3;
typedef vector<Dim3> Matrix4D;

Matrix4D my_matrix(5,Dim3(i+1,Dim2(n,Dim1(9))));
```

Sum of matrices

- Design a function that calculates the sum of two *n×m* matrices.

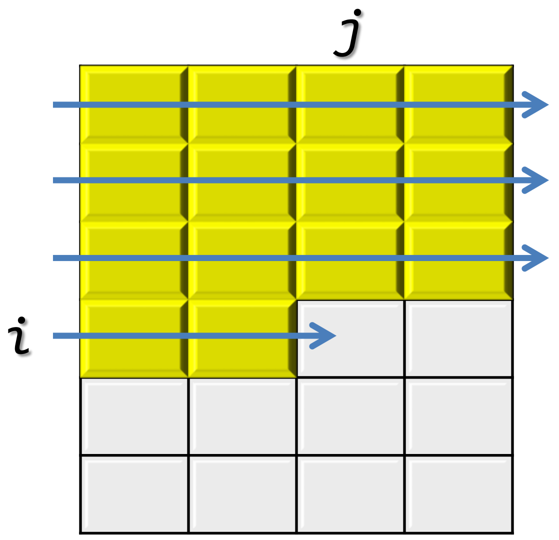
$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

```
typedef vector< vector<int> > Matrix;

Matrix matrix_sum(const Matrix& a,
                  const Matrix& b);
```

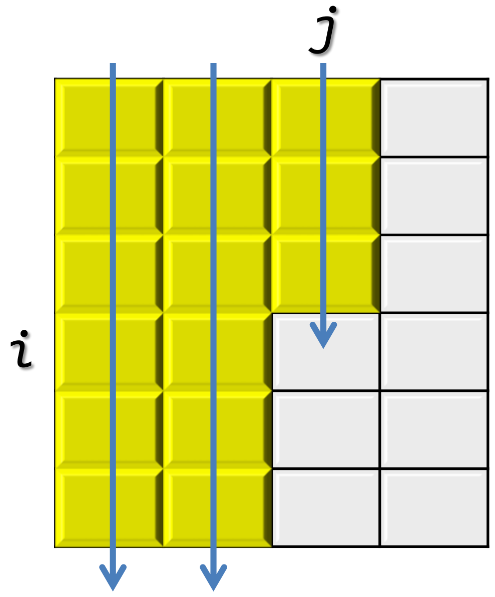
How are the elements of a matrix visited?

- By rows



```
For every row i
  For every column j
    Visit Matrix[i][j]
```

- By columns



```
For every column j
  For every row i
    Visit Matrix[i][j]
```

Sum of matrices (by rows)

```
typedef vector< vector<int> > Matrix;

Matrix matrix_sum(const Matrix& a, const Matrix& b) {
// Pre:   a and b are non-empty matrices
//        and have the same size
// Post: returns a+b (sum of matrices)

    int nrows = a.size();
    int ncols = a[0].size();
    Matrix c(nrows, vector<int>(ncols));

    for (int i = 0; i < nrows; ++i) {
        for (int j = 0; j < ncols; ++j) {
            c[i][j] = a[i][j] + b[i][j];
        }
    }
    return c;
}
```

Sum of matrices (by columns)

```
typedef vector< vector<int> > Matrix;

Matrix matrix_sum(const Matrix& a, const Matrix& b) {
// Pre:   a and b are non-empty matrices
//        and have the same size
// Post: returns a+b (sum of matrices)

    int nrows = a.size();
    int ncols = a[0].size();
    Matrix c(nrows, vector<int>(ncols));

    for (int j = 0; j < ncols; ++j) {
        for (int i = 0; i < nrows; ++i) {
            c[i][j] = a[i][j] + b[i][j];
        }
    }
    return c;
}
```

Transpose a matrix

- Design a procedure that transposes a square matrix in place:

```
void Transpose (Matrix& m);
```

$$\begin{array}{|c|c|c|}\hline 3 & 8 & 1 \\ \hline 0 & 6 & 2 \\ \hline 4 & 5 & 9 \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|}\hline 3 & 0 & 4 \\ \hline 8 & 6 & 5 \\ \hline 1 & 2 & 9 \\ \hline \end{array}$$

- Observation: we need to swap the upper with the lower triangular matrix. The diagonal remains intact.

Transpose a matrix

```
void swap(int& a, int& b) {
// Interchanges two values
    int c = a;
    a = b;
    b = c;
}

void Transpose(Matrix& m) {
// Pre:  m is a square matrix
// Post: m contains the transpose of the input matrix
    int n = m.size();
    for (int i = 0; i < n - 1; ++i) {
        for (int j = i + 1; j < n; ++j) {
            swap(m[i][j], m[j][i]);
        }
    }
}
```

Is a matrix symmetric?

- Design a procedure that indicates whether a matrix is symmetric:

```
bool is_symmetric(const Matrix& m);
```

3	0	4
0	6	5
4	5	9

symmetric

3	0	4
0	6	5
4	2	9

not symmetric

- Observation: we only need to compare the upper with the lower triangular matrix.

Is a matrix symmetric?

```
bool is_symmetric(const Matrix& m) {
// Pre:  m is a square matrix
// Post: returns true if m is symmetric, false otherwise

    int n = m.size();
    for (int i = 0; i < n - 1; ++i) {
        for (int j = i + 1; j < n; ++j) {
            if (m[i][j] != m[j][i]) return false;
        }
    }
    return true;
}
```

Search in a matrix

- Design a procedure that finds a value in a matrix. If the value belongs to the matrix, the procedure will return the location (i, j) at which the value has been found.

```
void search(const Matrix& m, int x,
            int& i, int& j);
// Pre:  m is a non-empty matrix
// Post: i and j define the location of a
//       cell that contains the value x in
//       m. In case x is not in m, then
//       i = j = -1
```

Search in a matrix

```
void search(const Matrix& m, int x, int& i, int& j) {
// Pre:  m is a non-empty matrix
// Post: i and j define the location of a cell that
//       contains the value x in M. In case x is not in m,
//       then i = j = -1

    int nrows = m.size();
    int ncols = m[0].size();

    for (i = 0; i < nrows; ++i) {
        for (j = 0; j < ncols; ++j) {
            if (m[i][j] == x) return;
        }
    }

    i = -1;
    j = -1;
}
```

Search in a sorted matrix

- A sorted matrix *m* is one in which

```
m[i][j] ≤ m[i][j+1]
m[i][j] ≤ m[i+1][j]
```

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

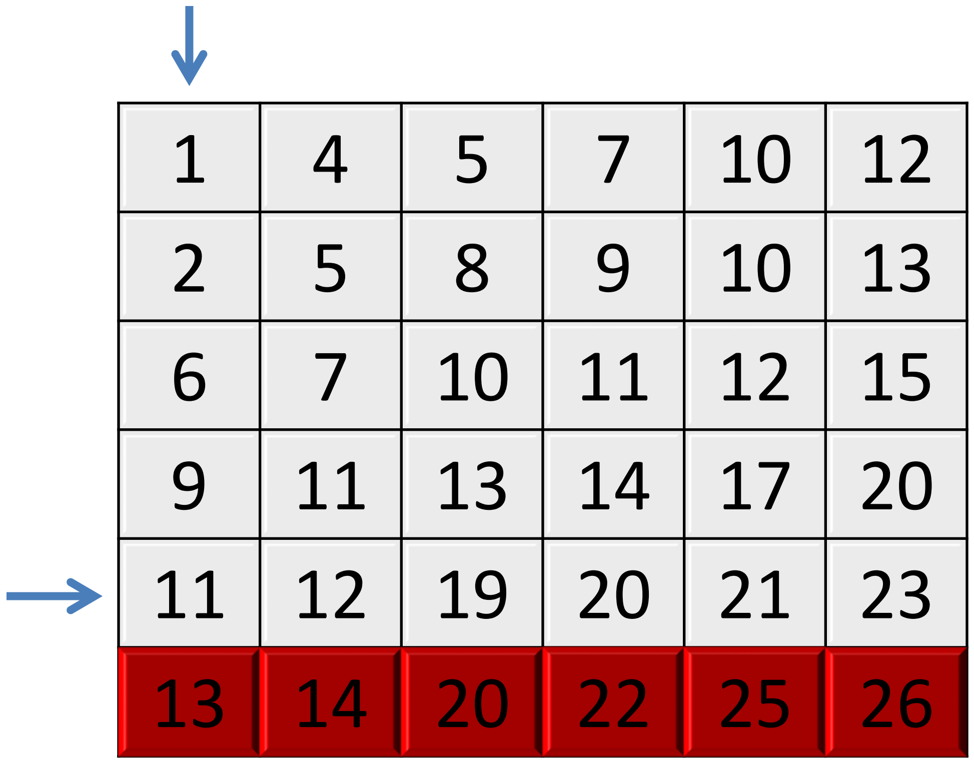
Search in a sorted matrix

- Example: let us find 10 in the matrix. We look at the lower left corner of the matrix.
- Since 13 > 10, the value cannot be found in the last row.

$$
\begin{array}{c|cccccc|}
 & \downarrow & & & & & \\
\hline
 & 1 & 4 & 5 & 7 & 10 & 12 \\
 & 2 & 5 & 8 & 9 & 10 & 13 \\
 & 6 & 7 & 10 & 11 & 12 & 15 \\
 & 9 & 11 & 13 & 14 & 17 & 20 \\
 & 11 & 12 & 19 & 20 & 21 & 23 \\
\rightarrow & 13 & 14 & 20 & 22 & 25 & 26 \\
\hline
\end{array}
$$

Search in a sorted matrix

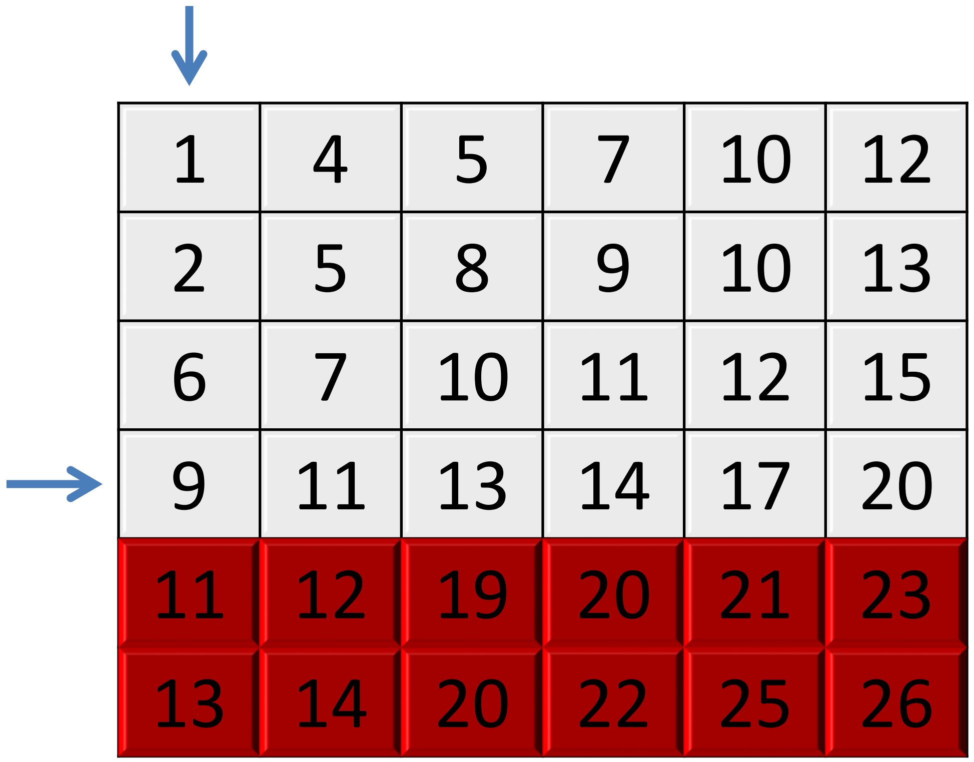
- We look again at the lower left corner of the remaining matrix.
- Since 11 > 10, the value cannot be found in the row.



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

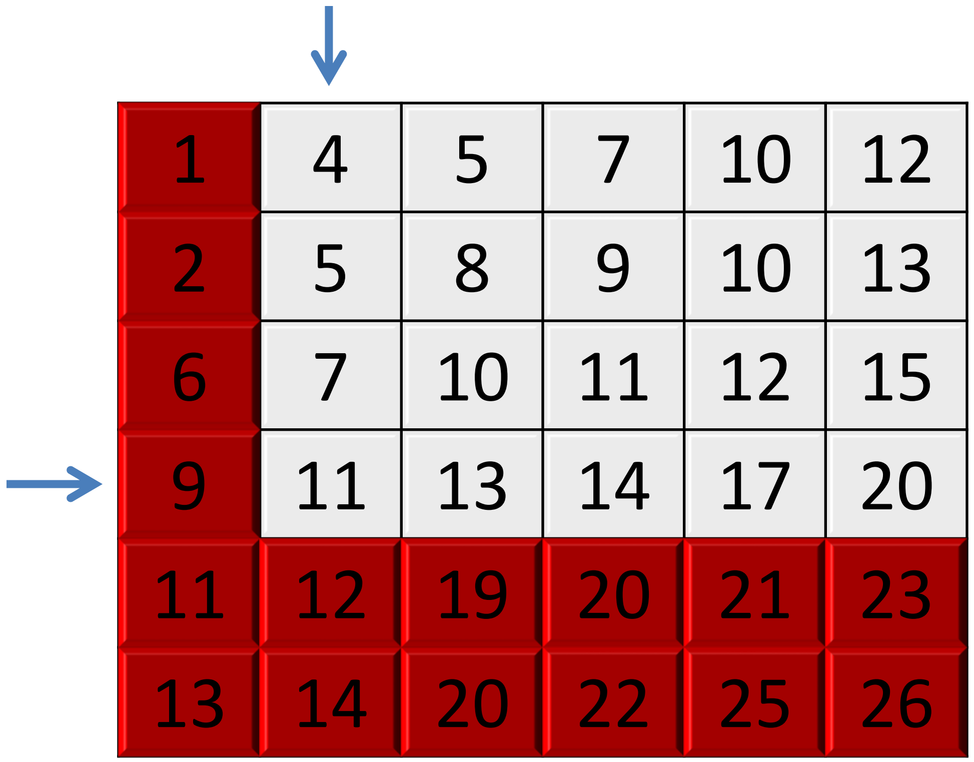
- Since 9 < 10, the value cannot be found in the column.



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

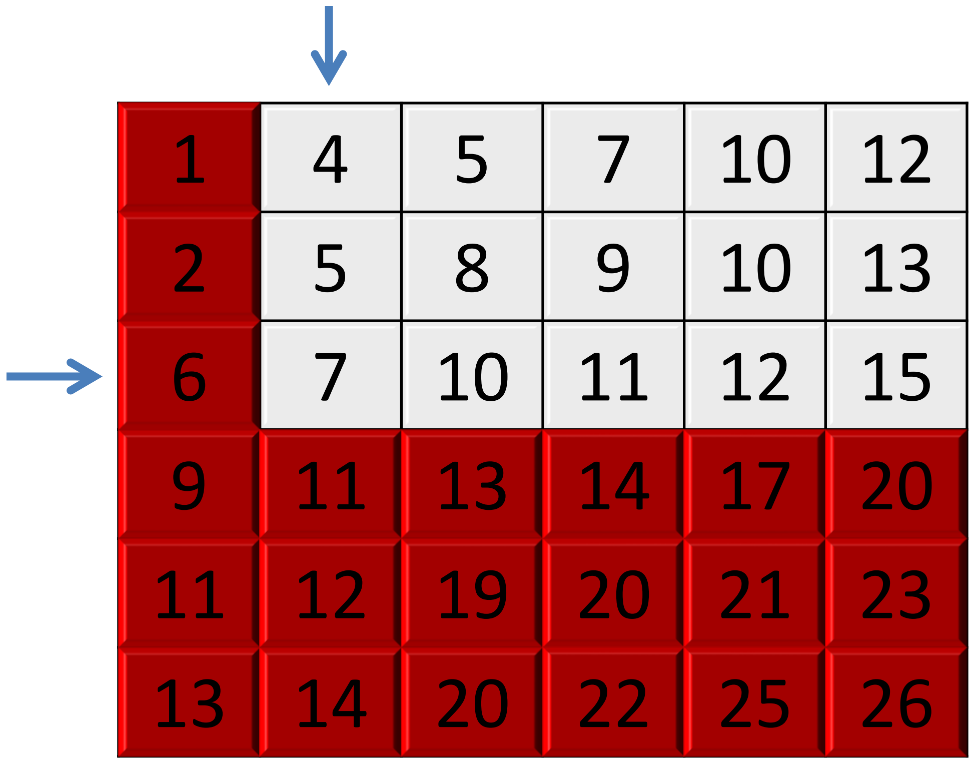
- Since 11 > 10, the value cannot be found in the row.



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

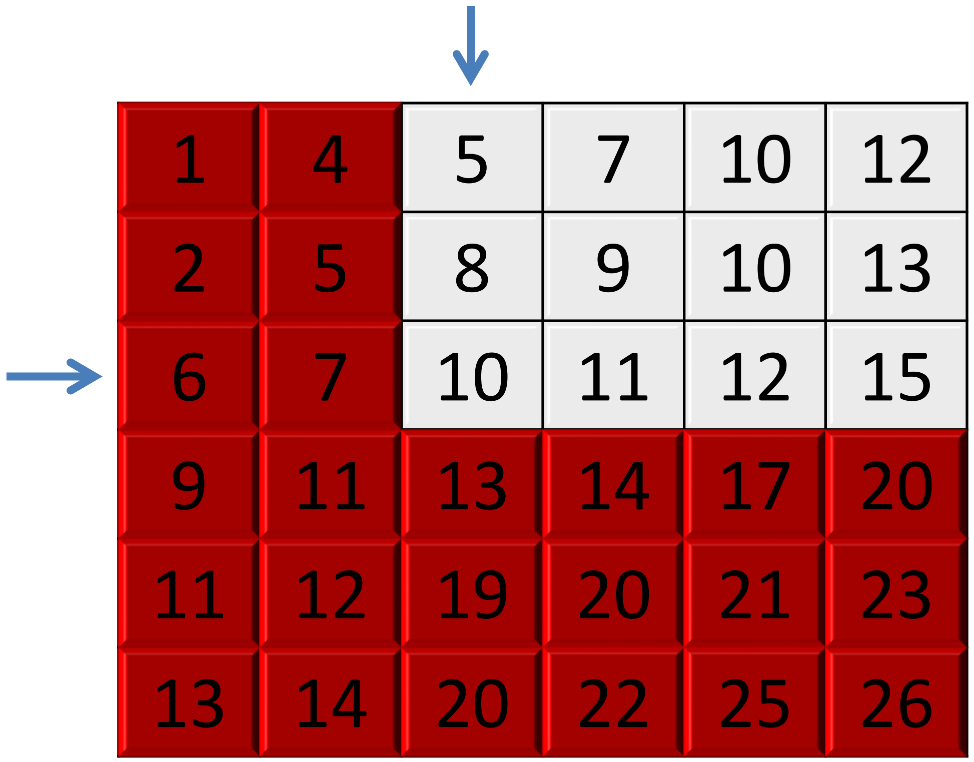
- Since 7 < 10, the value cannot be found in the column.



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

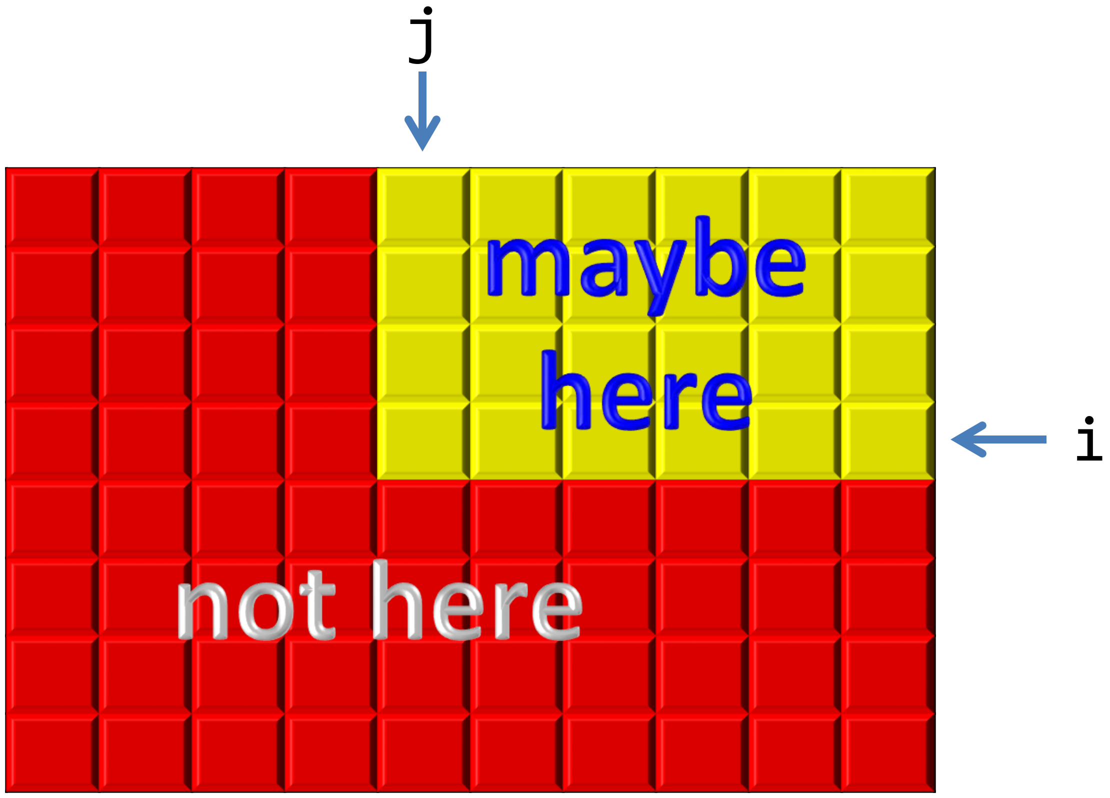
- The element has been found!



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- *Invariant*: if the element is in the matrix, then it is located in the sub-matrix ***[0…i, j…ncols-1]***



Search in a sorted matrix

```
void search(const Matrix& m, int x, int& i, int& j) {
// Pre:  m is non-empty and sorted by rows and columns in ascending order
// Post: i and j define the location of a cell that contains the value
//       x in m. In case x is not in m, then i=j=-1

    int nrows = m.size();
    int ncols = m[0].size();

    i = nrows - 1;
    j = 0;

    // Invariant: x can only be found in M[0..i,j..ncols-1]
    while (i >= 0 and j < ncols) {
        if (m[i][j] < x) j = j + 1;
        else if (m[i][j] > x) i = i - 1;
        else return;
    }

    i = -1;
    j = -1;
}
```

Search in a sorted matrix

- What is the largest number of iterations of a search algorithm in a matrix?

Unsorted matrix	nrows × ncols
Sorted matrix	nrows + ncols

- The search algorithm in a sorted matrix cannot start in all of the corners of the matrix. Which corners are suitable?

Matrix multiplication

- Design a function that returns the multiplication of two matrices.

$$\begin{array}{|c|c|c|c|}\hline 2 & -1 & 0 & 1 \\ \hline 1 & 3 & 2 & 0 \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline 1 & 2 & -1 \\ \hline 3 & 0 & 2 \\ \hline -1 & 1 & 3 \\ \hline 2 & -1 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 0 \\ \hline 8 & 4 & 11 \\ \hline \end{array}$$

```
Matrix multiply(const Matrix& a, const Matrix& b);

// Pre:  a is a non-empty n×m matrix,
//       b is a non-empty m×p matrix
// Post: returns a×b (an n×p matrix)
```

Matrix multiplication

```
Matrix multiply(const Matrix& a, const Matrix& b) {
// Pre:  a is a non-empty n×m matrix, b is a non-empty m×p matrix
// Post: returns a×b (an n×p matrix)

    int n = a.size();
    int m = a[0].size();
    int p = b[0].size();
    Matrix c(n, vector<int>(p));

    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < p; ++j) {
            int sum = 0;
            for (int k = 0; k < m; ++k) {
                sum = sum + a[i][k]*b[k][j];
            }
            c[i][j] = sum;
        }
    }
    return c;
}
```

Matrix multiplication

```
Matrix multiply(const Matrix& a, const Matrix& b) {
// Pre:  a is a non-empty n×m matrix, b is a non-empty m×p matrix
// Post: returns a×b (an n×p matrix)

    int n = a.size();
    int m = a[0].size();
    int p = b[0].size();
    Matrix c(n, vector<int>(p, 0));

    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < p; ++j) {
            for (int k = 0; k < m; ++k) {
                c[i][j] += a[i][k]*b[k][j];
            }
        }
    }
    return c;
}
```

Initialized to zero

The loops can be in any order

Accumulation

Matrix multiplication

```
Matrix multiply(const Matrix& a, const Matrix& b) {
// Pre:   a is a non-empty n×m matrix, b is a non-empty m×p matrix
// Post: returns a×b (an n×p matrix)

    int n = a.size();
    int m = a[0].size();
    int p = b[0].size();
    Matrix c(n, vector<int>(p, 0));

    for (int j = 0; j < p; ++j) {
        for (int k = 0; k < m; ++k) {
            for (int i = 0; i < n; ++i) {
                c[i][j] += a[i][k]*b[k][j];
            }
        }
    }
    return c;
}
```