Lecture 11

Dynamic Programming

Dynamic programming is a method frequently applied to *optimization problems*, problems where we are looking for the best solution to a problem.

A famous example of an optimization problem is the "travelling salesman problem." A travelling salesman must visit n cities and return to the first city, minimizing the cost of his tour (in terms of gallons of gasoline used, miles travelled, or some other metric). An optimal solution is one that is no longer than any other solution (we can't say "shortest" because there may be more than one solution with the same length). We will see this problem again later in the semester when we study NP-complete problems.

Developing a dynamic programming algorithm consists of these steps:

1. Characterize the structure of an optimal solution.
2. Recursively define the value of an optimal solution.
3. Compute the value of an optimal solution in a bottom-up fashion.
4. Construct an optimal solution from computed information.

Not all problems easily lend themselves to this approach. For example, the problem of sorting (which can be thought of as optimizing the degree of disorder in an array according to some proper definition of disorder) is best done using a divide-and-conquer approach, even though on the surface each of the four steps above seems to apply.

Dynamic programming is best applied to problems having these two characteristics:

- Optimal substructure.
  Consider a problem P, broken down into two subproblems, P_1 and P_2. We must be able to efficiently combine optimal solutions to P_1 and P_2 into an optimal solution to P. The structure of an optimal solution must contain optimal solutions to the recursive subproblems.
- Overlapping subproblems.
  Having defined in the second step the value of an optimal solution, a first attempt at solving the problem may be to simply implement a solution as a recursive algorithm. Dynamic programming requires that the recursive sub-solutions be computed many times, i.e. that the straight recursive solution does a lot of unnecessary computation. Dynamic programming works by eliminating this redundancy.

Your book gives matrix-chain multiplication as the first example of dynamic programming. Before we dive into this rather involved problem, let's look at a very simple example, to get the feel for dynamic programming.

Suppose you want to compute the nth Fibonacci number, F_n. The optimal solution to the problem is simply F_n (this is a somewhat contrived use of the word "optimal" to illustrate dynamic programming :-).

Recall that the Fibonacci numbers are defined: $F(n)$ =

1, if n=1 or 2,

$F_{n-2} + F_{n-1}$ otherwise.

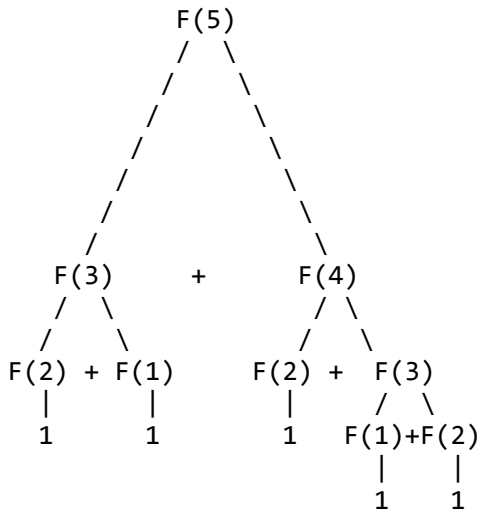
So the Fibonacci numbers are:

```
n  : 1 2 3 4 5 6 7  8  9  10 ...
F_n: 1 1 2 3 5 8 13 21 34 55 ...
```

Thus we have completed steps 1 and 2 for designing a dynamic programming algorithm to compute F_n. We are prepared to write a non-dynamic algorithm as simply:

```
F (n)
        if (n = 1) or (n = 2) then return 1
        return F(n-2) + F(n-1)
```

This is not an efficient algorithm for computing F_n. Let's look at the recursive calls made for F_4:



Note that F_2 is computed three times, and F_3 is computed twice. Each recomputation incurs extra recusive work that has already been done elsewhere. Note also the shape of this diagram; it looks like a binary tree of calls. We can see the height of the tree by going from the root to the rightmost leaf. The height is $\Theta(n)$.

The tree is not complete, but it is very full. What do we know about complete binary trees of height n? They have $\Theta(2^n)$ nodes. While this tree is not quite that big, it is very close. There are F_n 1's as leaves of the tree, so at least that many additions. From equation 2.15 in the book, we can tell that $F_n = \Theta(\phi^n)$, where ϕ is the Golden Ratio, about 1.618033988, so this algorithm does $\Omega(\phi^n)$ additions. This exponential behavior is very inefficient.

We can do much better with dynamic programming. Our problem satisfies the optimal substructure property: each solution is the sum of two other solutions. It also satisfies the overlapping subproblems property: each solution includes many sums of previous solutions. Thus we can use the 3rd and 4th steps to design an algorithm that uses the precomputed information.

Using a dynamic programming technique called *memoization*, we can make the recursive algorithm much faster. We assume there is an array A of integers whose first and second elements have been initialized to 1, and there is an integer called *unknown*, initially two, that keeps track of the index of the least Fibonacci number whose value is not known:

```
F (n)
        if n < unknown then return A[n]
        A[n] = F(n-2) + F(n-1)
        return A[n]
```

This algorithm is very similar to the previous one, but it uses the array A as a sort of "scratch" area to record previous results, rather than recompute compute them.

What is the running time of this recursive algorithm? If F_i is computed, it is stored in the array and never recomputed, so the algorithm basically traces a path from root to rightmost leaf of the tree, adding up all the results at each level in one addition. Thus the algorithm runs in time $\Theta(n)$. This is much better than the exponential-time algorithm.

What about the storage required? The first algorithm seems to have an advantage, since the second requires an array of size at least n to work. But even the first algorithm uses $O(n)$ storage, since a stack frame is consumed for each of the n levels of recursion.

This memoization technique can be applied to many recursive algorithms, and can really speed up your programs. Once you have computed em>F_n, finding the value of em>F_i where $i < n$ takes $\Theta(1)$ time, since it is just an array access! This technique can speed up programs even when the recursive algorithm to compute a function is no slower than the memoized algorithm; the first computation from both algorithms may take, say, $\Theta(n)$, but each subsequent memoized lookup is just $\Theta(1)$.

(It turns out when analyzing numerical algorithms like this, we need to be a little more careful and look at the asymptotic cost of multiplying and storing large numbers. In C, we quickly exhaust the 32 or 64 bits modern compilers give us for integers and must move to a large integer representation. Then the time needed to multiply m-bit numbers becomes important.)