Lecture 8: Kruskal's MST Algorithm

CLRS Chapter 23

Main Topics of This Lecture

- Kruskal's algorithm
  Another, but different, greedy MST algorithm

- Introduction to UNION-FIND data structure.
  Used in Kruskal's algorithm
  Will see implementation in next lecture.

Idea of Kruskal’s Algorithm

The Kruskal’s Algorithm is based directly on the generic algorithm. Unlike Prim’s algorithm, we make a different choices of cuts.

Initially, trees of the forest are the vertices (no edges).

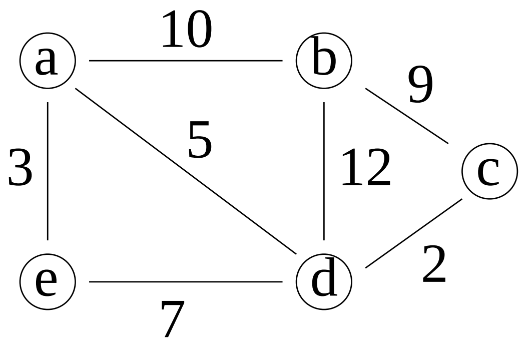
In each step add the cheapest edge that does not create a cycle.

Observe that unlike Prim’s algorithm, which only grows one tree, Kruskal’s algorithm grows a collection of trees (a forest).

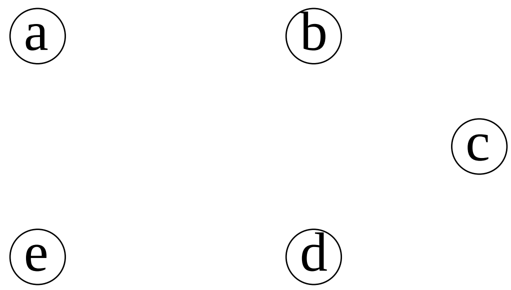
Continue until the forest ’merge to’ a single tree.
(Why is a single tree created?)

This is a *minimum* spanning tree
(we must prove this).

Outline by Example



original graph



forest → MST

E

edge	weight
{d, c}	2
{a, e}	3
{a, d}	5
{e, d}	7
{b, c}	9
{a, b}	10
{b, d}	12

Forest (V, A)

A={ }

$\boxed{\textbf{Outline of Kruskal's Algorithm}}$

Step 0: Set $A = \emptyset$ and $F = E$, the set of all edges.

Step 1: Choose an edge e in F of minimum weight, and check whether adding e to A creates a cycle.

- If "yes", remove e from F.
- If "no", move e from F to A.

Step 2: If $F = \emptyset$, stop and output the minimal spanning tree (V, A). Otherwise go to Step 1.

Remark: Will see later, after each step, (V, A) is a subgraph of a MST.

$\boxed{\textbf{Outline of Kruskal's Algorithm}}$

Implementation Questions:

- How does algorithm choose edge $e \in F$ with minimum weight?
- How does algorithm check whether adding e to A creates a cycle?

How to Choose the Edge of Least Weight

Question:
How does algorithm choose edge $e \in F$ with minimum weight?

Answer: Start by sorting edges in E in order of increasing weight.
Walk through the edges in this order.
(Once edge e causes a cycle it will always cause a cycle so it can be thrown away.)

How to Check for Cycles

Observation: At each step of the outlined algorithm, (V, A) is acyclic so it is a forest.

If u and v are in the same tree, then adding edge $\{u, v\}$ to A creates a cycle.

If u and v are not in the same tree, then adding edge $\{u, v\}$ to A does not create a cycle.

Question: How to test whether u and v are in the same tree?

High-Level Answer: Use a disjoint-set data structure
Vertices in a tree are considered to be in same set.
Test if Find-Set(u) = Find-Set(v)?

Low -Level Answer:
The UNION-FIND data structure implements this:

The UNION-FIND Data Structure

UNION-FIND supports three operations on collections of **disjoint sets**: Let n be the size of the universe.

Create-Set(u): $O(1)$

Create a set containing the single element u.

Find-Set(u): $O(\log n)$

Find the set containing the element u.

Union(u, v): $O(\log n)$

Merge the sets respectively containing u and v into a common set.

For now we treat UNION-FIND as a black box.
Will see implementation in next lecture.

Kruskal's Algorithm: the Details

Sort E in increasing order by weight w; $O(|E| \log |E|)$

/* After sorting $E = \langle \{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{|E|}, v_{|E|}\} \rangle$ */

$A = \{\,\}$;
for (each u in V) CREATE-SET(u); $O(|V|)$

for $e_i = (u_i, v_i)$ from 1 to $|E|$ do $O(|E| \log |V|)$
    if (FIND-SET(u_i) != FIND-SET(v_i))
    $\{$ add $\{u_i, v_i\}$ to A;
        UNION(u_i, v_i);
    $\}$
return(A);

Remark: With a proper implementation of UNION-FIND, Kruskal's algorithm has running time $O(|E| \log |E|)$.

Why Kruskal's Algorithm is correct?

Let A be the edge set which has been selected by Kruskal's Algorithm, and (u, v) be the edge to be added next. It suffices to show there is a cut which respects A, and (u, v) is the light edge crossing that cut.

1. Let $A^{'} = (V^{'}, E^{'})$ denote the tree of the forest A that contains u. Consider the cut $(V^{'}, V - V^{'})$.

2. Observe that there is no edge in A crosses this cut, so the cut respects A.

3. Since adding (u, v) to $A^{'}$ does not induce a cycle, (u, v) crosses the cut. Moreover, since (u, v) is currently the smallest edge, (u, v) is the light edge crossing the cut. This completes the correctness proof of Kruskal's Algorithm.

Why Kruskal's Algorithm is correct?