PROJECT REPORT

PROJECT

Pricing lookback options with Monte-Carlo method

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Contents

1 Introduction

This report summarizes the work carried out within the framework of a project which aims at the implementation of lookback options pricing in its two possible forms call or put, and also the evaluation of the option's greeks with respect to the parameters on which it depends: level of the underlying, risk-free rate, volatility and maturity.
Numerical calculations relating to pricing were carried out in the C ++ programming language. A user interface to visualize the results is also made in Excel.

2 LookBack option

A lookback option is an exotic path dependant option whose payoff does not depend only on the underlying value at maturity (case of vanilla options) but also on the optimal level (maximum or minimum) reached by it during the option life period. There are several variants of the lookback options, the most commonly known are those with a floating strike or with a fixed strike. As part of this project, we will focus our study on the pricing of lookback options with a floating strike.

Our pricer should allow the valuation of lookback call or put options and to calculate their sensitivities with respect to the variation of its various parameters. For the call, the strike's value is set at the lowest price of the asset during the lifetime of the option and, for the put, it is set at the highest price of the asset. The payoff of the lookback call and put are given respectively by:

$$Call: \qquad S_T - \underset{t\in[T_0,T]}{min}(S_t)$$

$$Put: \qquad \underset{t\in[T_0,T]}{max}(S_t) - S_T$$

Where T_0 defines the issue date of the option, T the maturity of the option indexed on the underlying S.

3 LookBack's pricing

Knowing the payoff of the option, a risk-neutral pricing under a risk-free rate contant r will consist (in case of a call for instance) in the valuation of the following quantity:

$$\mathbb{E}^{\mathbb{Q}}(e^{-rT}(S_T - \underset{t\in[T_0,T]}{min}(S_t)))$$

This corresponds well to the mathematical expectation of the payoff of the option discounted at the risk-free rate r.
There are of course several pricing methodologies for this option. The project imposes a valuation method based on Monte Carlo simulations. For this, we obviously need a stock price diffusion model, which in the context of the project, corresponds to the Black & Scholes model given in the risk neutral approach by:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^Q$$

Where S_t defines the stock price at time t, r is the interest rate, σ the underlying's volatility and W^Q a brownian motion under the propability Q.

To simulate S, The discretized version of the Black&Scholes model that will be implemented corresponds to:

$$S_{t_{i+1}} = S_{t_i} e^{(r-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z_i}$$

Where h defines the descretization step and $(Z_i)_i$ is a collection of standardized centered i.i.d gaussian variables.

By arriving at this stage of analysis, we are able to define the inputs of our pricing function:

- The initial level of the underlying S_0.
- The risk-free rate r.
- The volatility of the underlying σ.
- The date of issue of the option T_0.
- The maturity of the contract T.
- The number of Monte Carlo simulations N_{sim}.
- The number of discretization points N.

Pricing by the Monte Carlo method will consist in simulating several paths of the underlying over the time interval $[T_0, T]$ according to the Black& Scholes dynamics given the parameters S_0, r and σ. This will make it possible to generate several possible scenarios of the payoff $(S_T - \underset{t\in[T0,T]}{min}(S_t))$ by evaluating the difference, for each path, of the value of the underlying at maturity and the minimum reached by it. The price is approximated by the empirical average of the updated payoff scenarios:

$$e^{-rT}\frac{1}{N_{sim}}\sum_{j=1}^{N_{sim}}(S_T - \underset{i\in[1,N]}{min}(S_{ti}))^{\{j\}}$$

We implemented a price calculation function in c ++ according to the principle stated above. To validate the price of our function, we have also implemented the closed formula for calculating the price of the lookback option at time t given by:

$$S_t\phi(a_1) - S_{min}e^{-rT}\phi(a_2) - \frac{S_t\sigma^2}{2r}(\phi(-a_1) - e^{-rT}(\frac{S_{min}}{S_t})^{\frac{2r}{\sigma^2}}\phi(-a_3))$$

Where:

$$S_{min} = \underset{u\in[0,t]}{min}(S_u)$$

,

$$a_1 = \frac{log(\frac{S_t}{S_{min}}) + (r + \frac{\sigma^2)}{2})(T-t)}{\sigma\sqrt{T-t}}, \qquad a_2 = a_1 - \sigma\sqrt{T-t}$$

$$a_3 = a_1 - \frac{2r}{\sigma}\sqrt{T-t}$$

We can use this closed formula to price the initial price of our option by setting t into 0, in this case $S_{min} = S_0$. The results of calculation and comparison of the two prices can be seen in the section 5 Computing results.

4 LookBack's greeks

In this section, we explain the calculation of the sensitivities of the lookback option. Our pricer should make it possible to assess the following sensitivities:

- The option's delta : $\Delta = \frac{\partial P}{\partial S}$
- The option's gamma : $\Gamma = \frac{\partial^2 P}{\partial S^2}$
- The option's vega: $\nu = \frac{\partial P}{\partial \sigma}$
- The option's theta : $\Theta = \frac{\partial P}{\partial T}$
- The option's rho : $\rho = \frac{\partial P}{\partial r}$

Where $P = P(S_0, r, \sigma, T_0, T)$ denotes The theoretical price of the option.

To calculate an approximation of these quantities, a classical method is to apply a centered finite difference scheme by adopting a step too small $\eta \longrightarrow 0$.

$$\Delta = \frac{\partial P}{\partial S} = \frac{P(S_0 + \eta, r, \sigma, T_0, T) - P(S_0 - \eta, r, \sigma, T_0, T)}{2\eta}$$

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{P(S_0 + \eta, r, \sigma, T_0, T) + P(S_0 - \eta, r, \sigma, T_0, T) - 2P(S_0, r, \sigma, T_0, T)}{\eta^2}$$

$$\nu = \frac{\partial P}{\partial \sigma} = \frac{P(S_0, r, \sigma + \eta, T_0, T) - P(S_0, r, \sigma - \eta, T_0, T)}{2\eta}$$

$$\Theta = \frac{\partial P}{\partial S} = \frac{P(S_0, r, \sigma, T_0, T + \eta) - P(S_0, r, \sigma, T_0, T - \eta)}{2\eta}$$

$$\rho = \frac{\partial P}{\partial S} = \frac{P(S_0, r + \eta, \sigma, T_0, T) - P(S_0, r - \eta, \sigma, T_0, T)}{2\eta}$$

For example, in the delta, prices $P(S_0 + \eta, r, \sigma, T_0, T)$ and $P(S_0 - \eta, r, \sigma, T_0, T)$ are calculated by Monte Carlo using the same randomness for both to reduce the variance.

For each sensitivity, a function on c ++ has been implemented to calculate it. We also wanted to validate our functions by comparing their results with the results of the finite differences applied to the closed price formula stated in the previous section.

The comparison results in the case of the call and the put can be seen in the next section.

5 Computing results

We set the parameters as follows: $S_0 = 100, r = 0.02, \sigma = 0.01, T = 1, N = 500, N_{sim} = 10000$.

Then we launch the calculation of the price and sensitivities of the lookback option by the c ++ functions. The following figure shows the outputs obtained for a call lookback and gives a comparison between the results from the Monte Carlo method and those from the closed price formula:

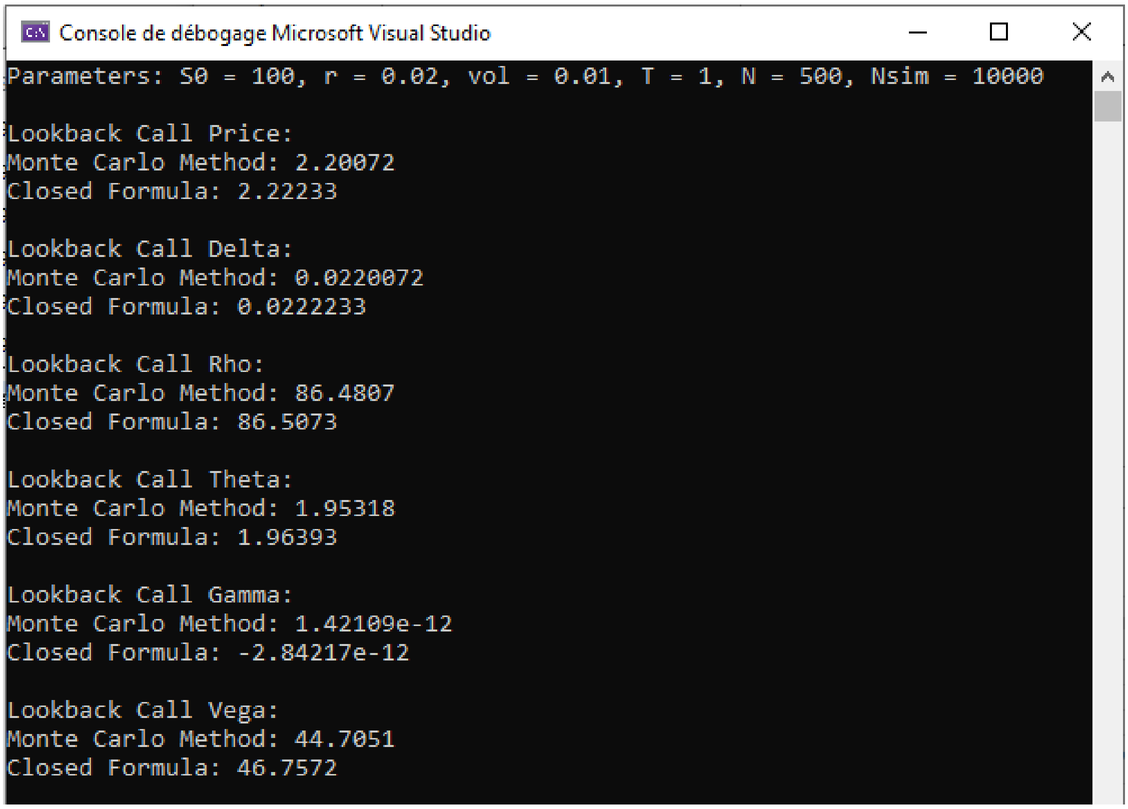

Figure 1: Computing results of call lookback's price and greeks

Overall, we notice that there is no significant difference between our calculation functions by the Monte Carlo method and the results of the closed formula, which validates our c ++ code.

It was also possible to validate the calculation for the put by comparing it with the calculation resulting from the closed formula. The following figure shows the results of execution of the functions associated with the calculation of the lookback put for same parameters:

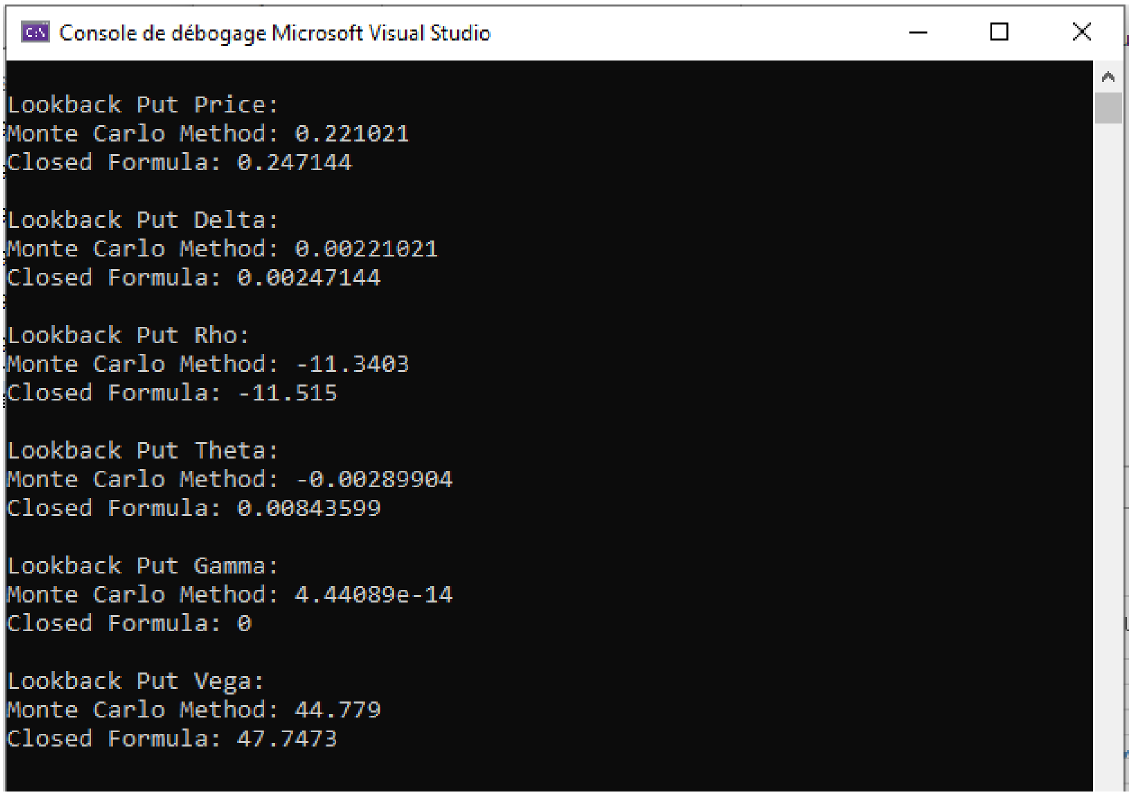

Figure 2: Computing results of put lookback's price and greeks

6 Graphic interface

We developed a graphical interface under VBA for the pricing of the lookback option. The interface is made up of several fields (see figure 3):

- a first field to enter the parameters of the option: S_0 the spot price, T_0 date of issue of the option which by default taken as the current date, and T the maturity of the option. The dates can be changed by manual entry or by pressing the change date button.
- a second field which concerns the parameters of the Black&Scholes model: the risk-free rate r and the volatility of the underlying σ.
- a last field to enter the parameters of the Monte Carlo method: N_{sim} number of simulations and N number of discretization points.

To the right of the interface, we can clearly see the possibility of choosing the type of the call or put option that we can check.

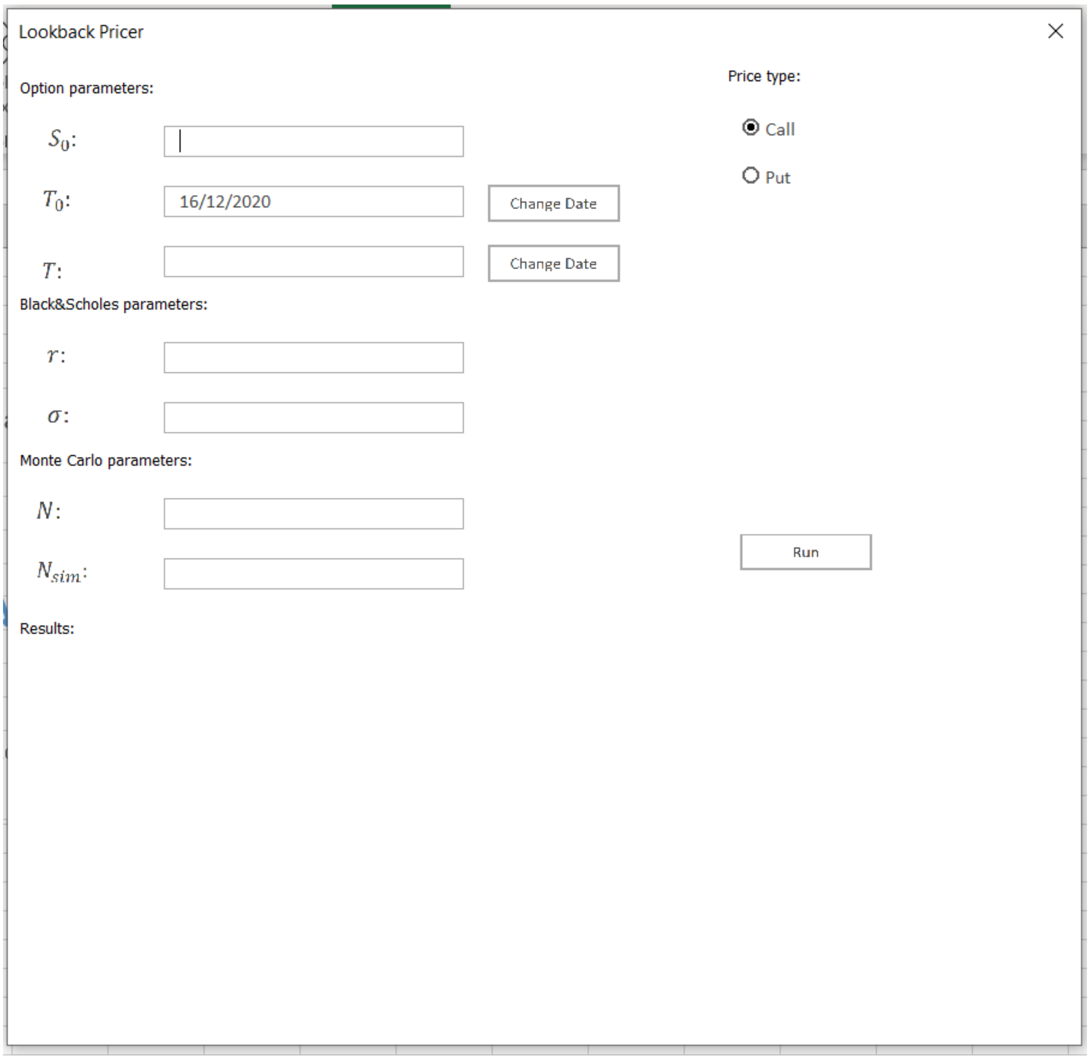

Figure 3: Lookback Pricer's graphical interface

For what comes, we fill in the parameters as follows $S_0 = 100, r = 0.02, \sigma = 0.01, T_0 = 16/12/2020, T = 16/12/2021, N = 500, N_{sim} = 10000$, and we choose to price a call lookback option (see figure 4).

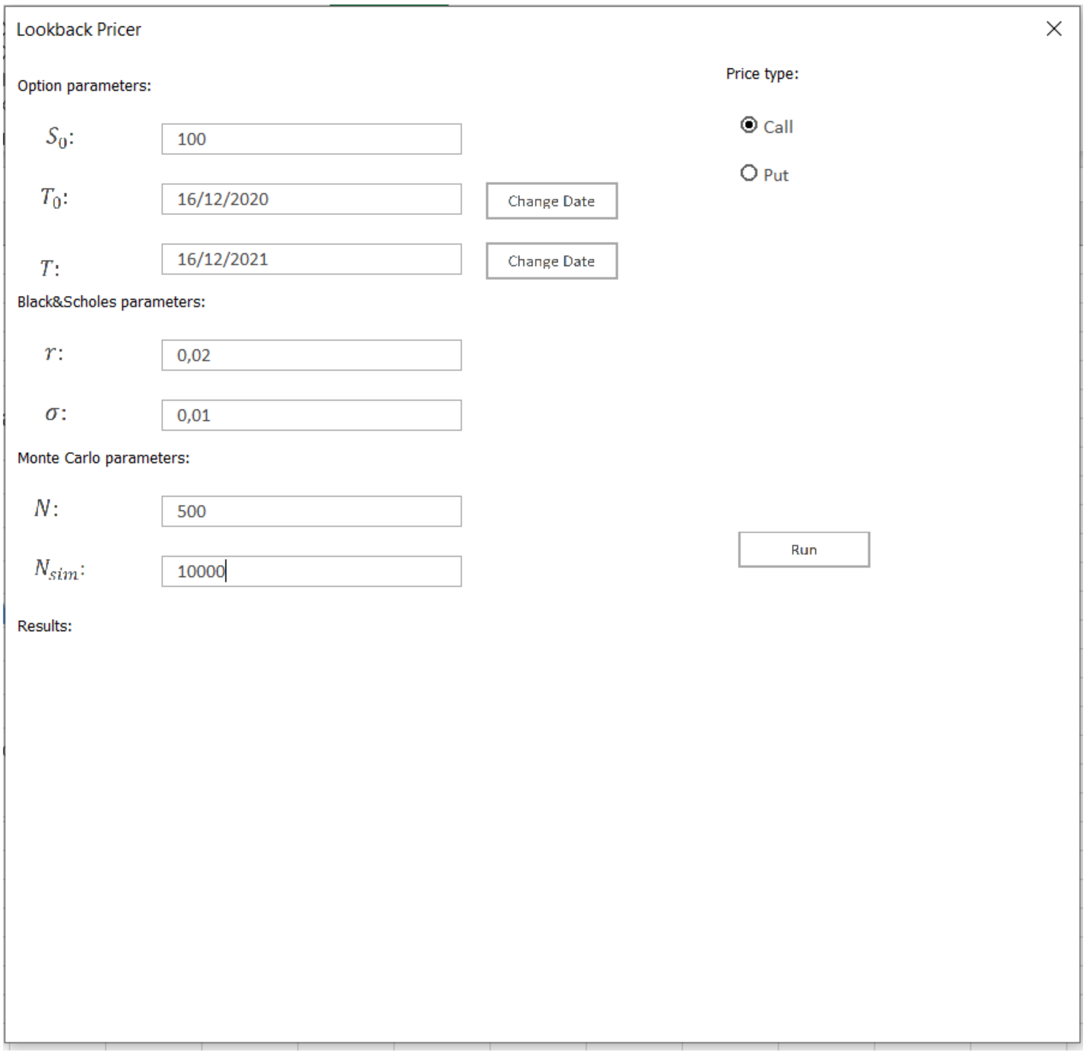

Figure 4: Lookback Pricer's graphical interface

If the parameters are not entered correctly, the color of the text box changes to red to indicate areas where the user has made a mistake. Once the value inside the text box is changed, it will return to its original color. This corrects subsequent errors due to a mistyped value. The screenshot of the user interface below shows this behavior:

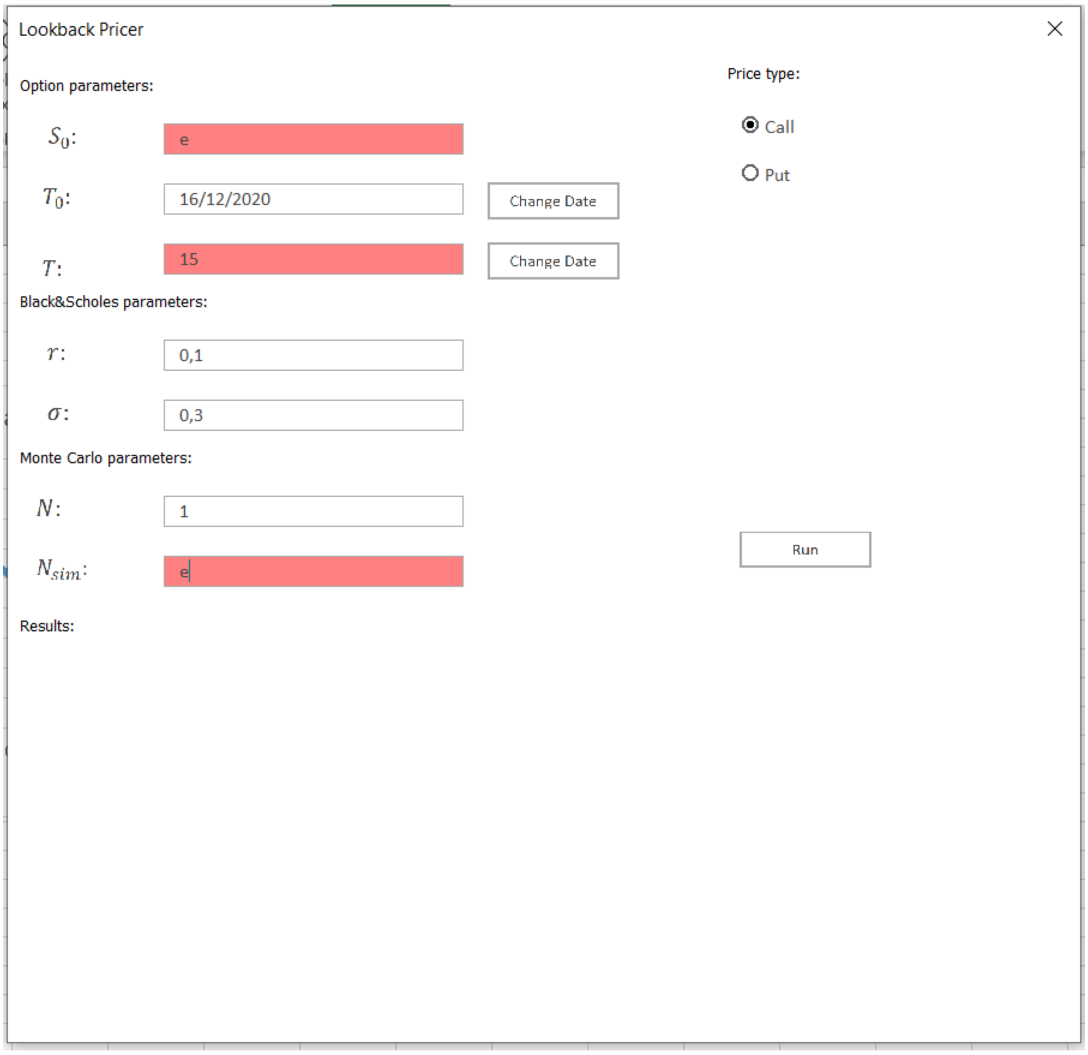

Figure 5: Error in when entering variables in Lookback Pricer's graphical interface

Once all the parameters have been entered and the type of option chosen, we can press the Run button to launch the calculations and retrieve the results (see figure 5).

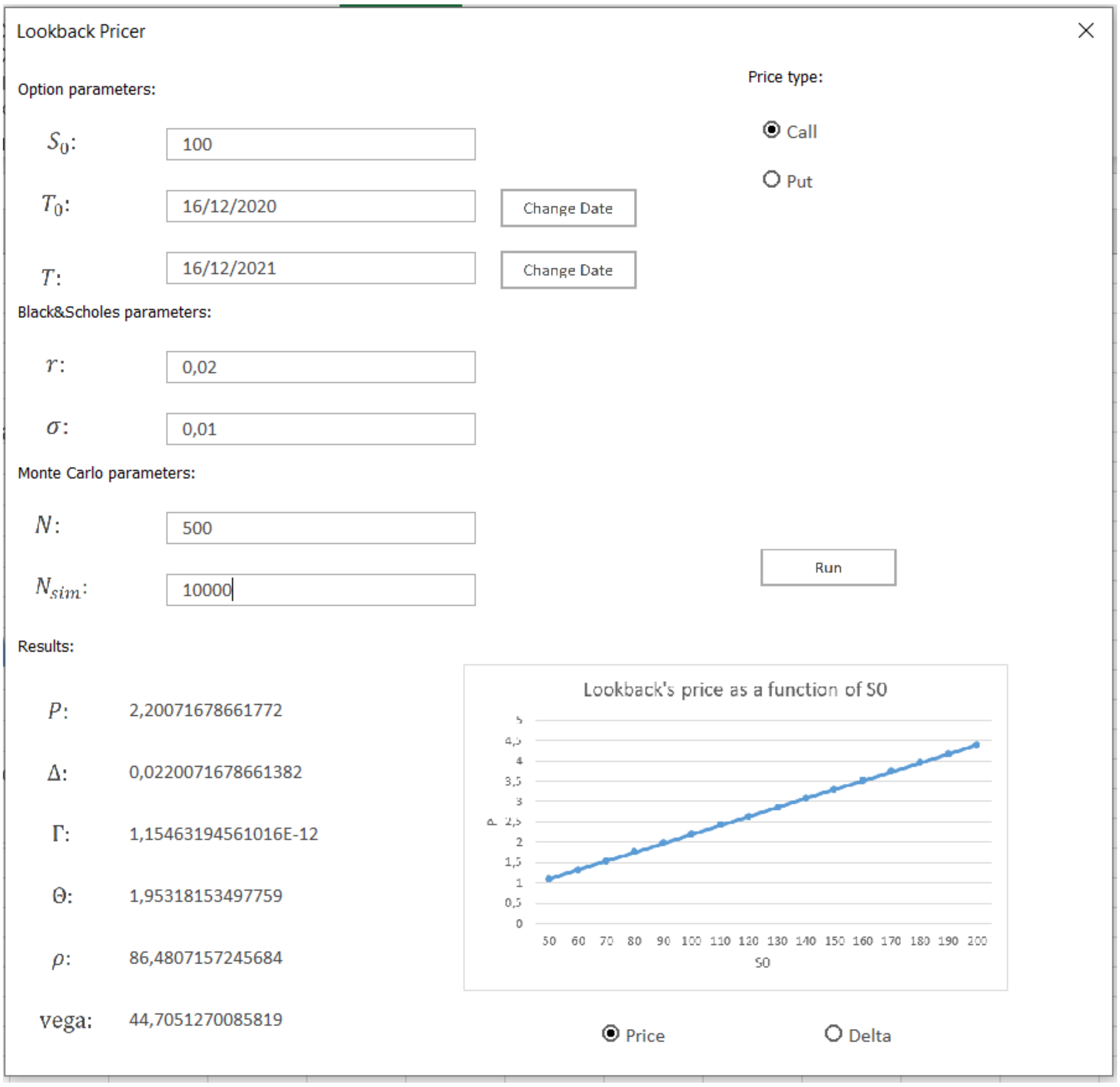

Figure 6: Lookback Pricer's graphical interface

On the results field, we get the price of the option noted P as well as all the relative sensitivities $\Delta, \Gamma, \Theta, \rho, \nu$. On the right side at the bottom, we can visualize the plot of the option price according to the different values of the underlying initials. We can also see the graph of the option's delta versus the price of the underlying stock on the calculation date T_0 by pressing the box associated with the delta (see figure 6).

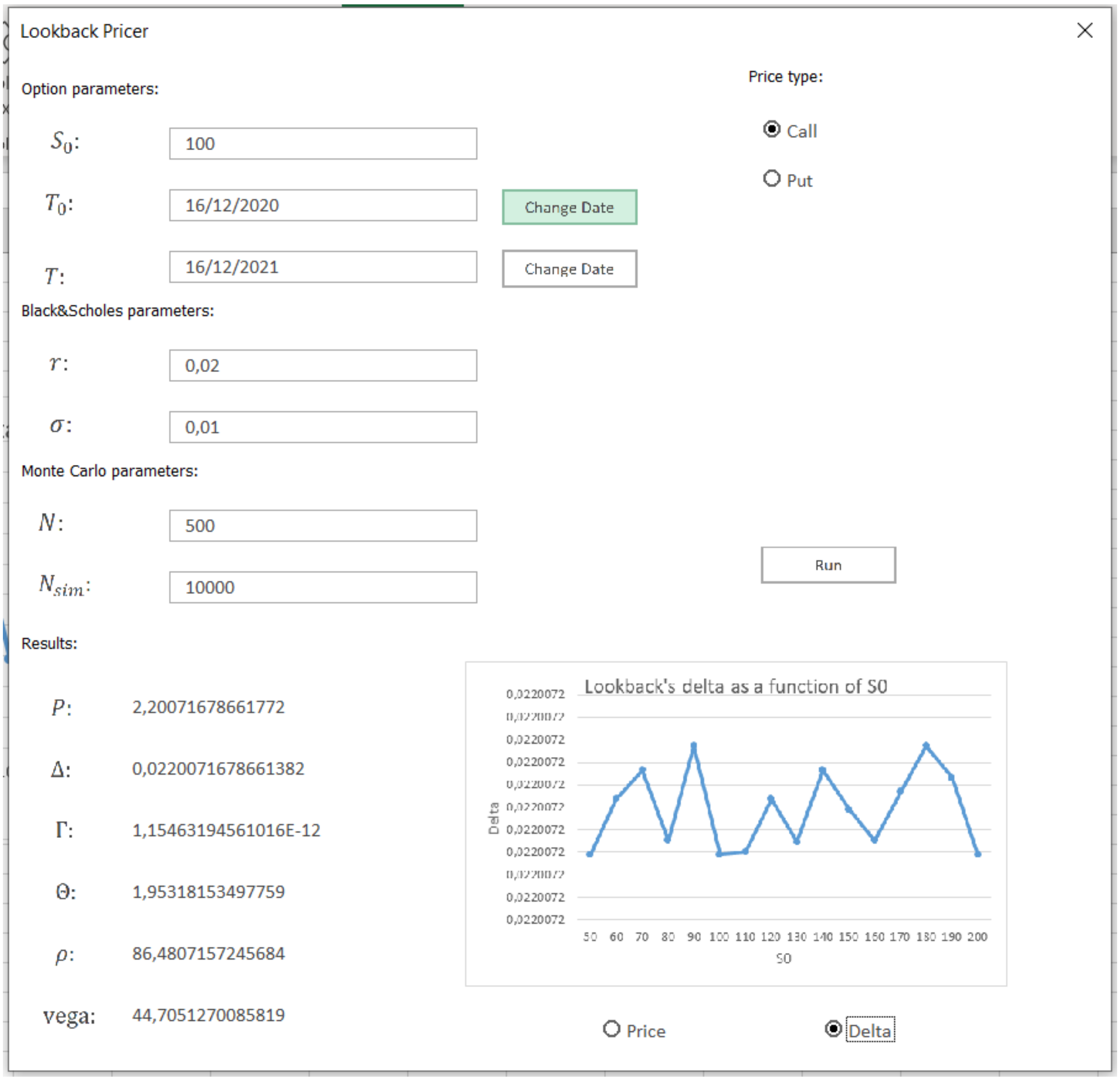

Figure 7: Lookback Pricer's graphical interface

For the delta plot, we can see that it is almost stable as a function of the values of the underlying which joins the fact that gamma is almost zero on the initial calculation date T_0, meaning that the value of the delta on the issue date of the option is insensitive to the value of the underlying.

In the following, we show the outputs of the interface in the case of a put lookback by keeping the same parameters as before:

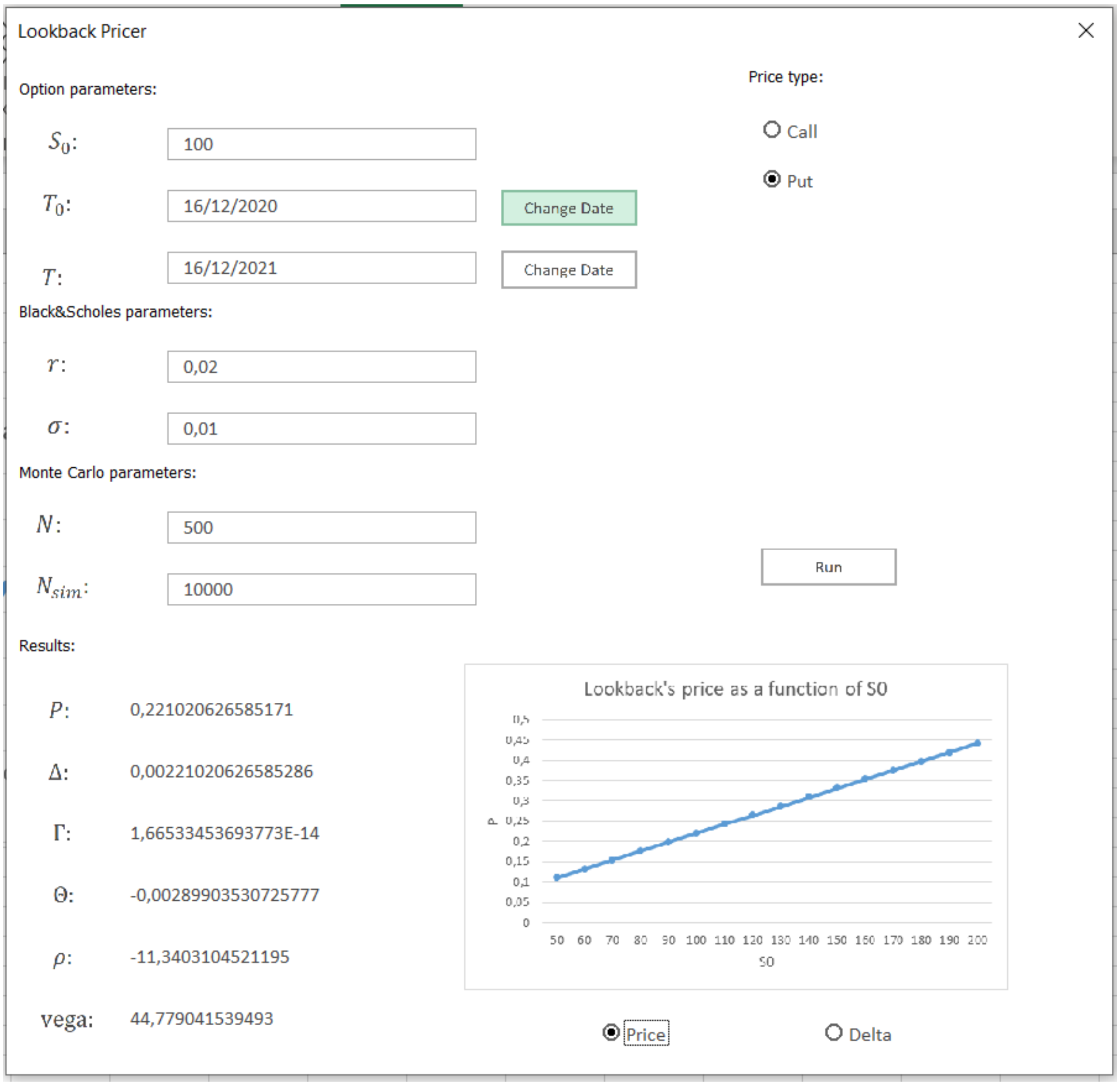

Figure 8: Lookback Pricer's graphical interface

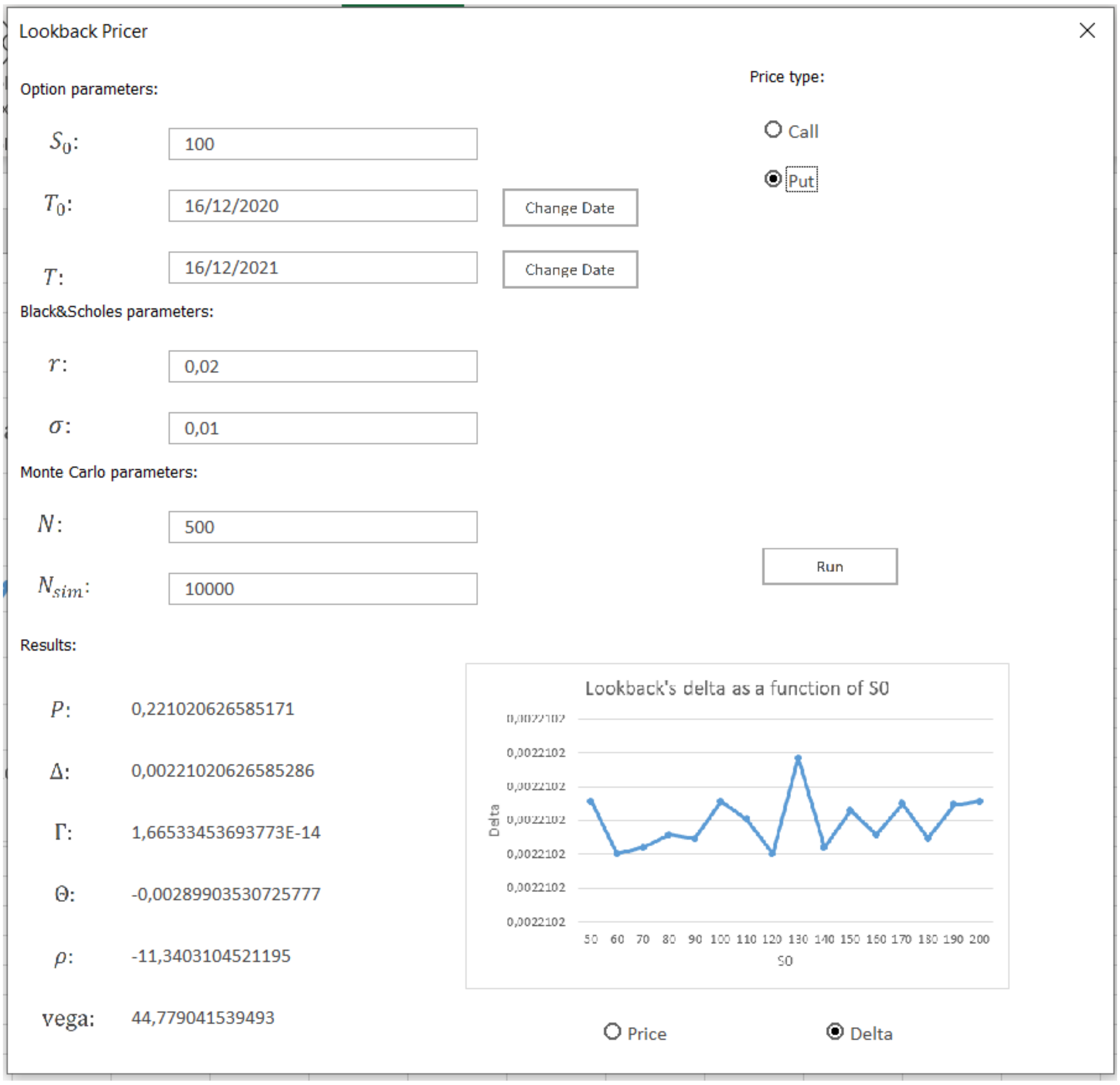

Figure 9: Lookback Pricer's graphical interface

7 Conclusion

In this project, we worked on the implementation of the pricing functions of a lookback option and the calculation of its greeks by the Monte Carlo method under c ++ with the Black&Scholes model. We also built a graphical interface under VBA to allow the user to retrieve the results in a fairly automated and interactive way.