Fourier Series and Physics

Chapter 13

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MAT337

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1. Steady State Heat Equation on the Unit Disc

- **Problem Statement:** Calculate the steady state temperature of a unit disc provided the temperature on the boundary of the disc is a given continuous function $f(\theta), \theta \in [-\pi, \pi]$, $f(\pi) = f(-\pi)$.

1. Steady State Heat Equation on the Unit Disc

- **Problem Statement:** Calculate the steady state temperature of a unit disc provided the temperature on the boundary of the disc is a given continuous function $f(\theta), \theta \in [-\pi, \pi]$, $f(\pi) = f(-\pi)$.
- **Question:** What do we know about temperature from physical observations?

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1. Temperature is a continuous function.
2. *Heat* is transfered over the boundaries of a region if the temperatures across the boundary are different.
3. In the steady state solution no net heat should be transferred to any arbitrary region of the unit disc. So for any region this equation must be valid.

$$0 = \int_{\text{Boundary}} \frac{\partial u}{\partial n} \cdot ds$$

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Since this is true for <u>any</u> region, the steady state heat equation is the following.

$$\boxed{\begin{aligned} &\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ &u(1, \theta) = f(\theta) \end{aligned}}$$

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- A separable solution u is one for which exist C^2 functions R and Θ such that:

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- Separable solutions are easy to find, as the relatively complicated heat equation reduces to solving two ODEs.

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$$\boxed{\begin{aligned} r^2R'' + rR' - cR &= 0 \\ \Theta'' + c\Theta &= 0 \end{aligned}}$$

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- Depending on the sign of c we can determine Θ; ODEs of this format are discussed in details extensively in chapter 12.

$$\boxed{\Theta(\theta) = \begin{cases} A\cos\left(\sqrt{c}\theta\right) + B\sin\left(\sqrt{c}\theta\right) & c > 0 \\ A + B\theta & c = 0 \\ Ae^{\sqrt{-c}\theta} + Be^{-\sqrt{-c}\theta} & c < 0 \end{cases}}$$

So the final form of the solutions for Θ is the following after imposing the conditions on Θ; note that we are not interested in the trivial solution $\Theta = 0$.

$$c \in \mathbb{N} \cup \{0\}$$

$$\Theta(\theta) = \begin{cases} A_0 & c = 0 \\ A_n \cos(n\theta) + B_n \sin(n\theta) & n \equiv c \in \mathbb{N} \end{cases}$$

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$$\frac{dR}{dt} = \frac{dR}{dr}\frac{dr}{dt} = R'r$$

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So a typical separable solution given the value of $c = n$ would be the following.

$$\boxed{\begin{array}{l} u_n(r,\theta) = A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \\ n \in \mathbb{N} \cup \{0\} \end{array}}$$

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Note that $u_n(1,\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$

Idea: What if the Fourier series of f was convergent on the disk, was continuous, and converged to f itself? If so, would the following equation satisfy $\Delta u = 0$?

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

$$u(1,\theta) = f(\theta) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta)$$

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3. Convergence of the Temperature Equation

There are a number of problems with these equations.

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1. How do we know u converges?
2. Finite linear combinations of separable solutions are harmonic ($\Delta u = 0$); what about this infinite sum?
3. Uniqueness and existence of a solution are not verified yet.

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- We can find an upper bound for A_0, A_n, and B_n for $n \in \mathbb{N}$.

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cdot d\theta$$
$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) \cdot d\theta$$
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- Recall the $L1$ norm on $C([-\pi, \pi])$.

$$||f||_{L1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| \cdot d\theta$$

- This allows us to find an upper bound on the fourier coefficients of f. Since f is continuous over its domain, which is compact, it must be bounded by the extreme value theorem; so it has a finite $L1$ norm.

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- Consider the temperature function.

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} u_n(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n r^n \sin(n\theta)$$

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- We can find an opper bound on each u_n.

$$||u_n||_\infty \leq |A_n|r^n + |B_n|r^n \leq 4r^n||f||_{L1} \equiv M_n$$

- We note that for any $r \in [0, R]$ where $R \in [0, 1)$, the sum of the upper bounds converges.

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} 4r^n ||f||_{L1} = \frac{4||f||_{L1}}{1-r}$$

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- This allows us to utilize the Weierstrass-M test to conclude that $u(r, \theta)$ converges uniformly over $\overline{\mathbb{D}_R}$, where $R < 1$.
- This means $u(r, \theta)$ exists and is a continuous function over any disc $\overline{\mathbb{D}_R}$, where $R < 1$.

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1. Suppose u does exist on $\overline{\mathbb{D}}$; how do we know it satisfies $\Delta u = 0$?

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1. Suppose u does exist on $\overline{\mathbb{D}}$; how do we know it satisfies $\Delta u = 0$?
2. How do we know if $u(1, \theta)$ exists? If so, how do we know it converges to $f(\theta)$?

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- Recall Corollary 8.3.2; Suppose a sequence of functions f_n in $C^1([-\pi, \pi])$ exist such that for some c and some function $g \in C([-\pi, \pi])$:

$$\lim_{n\to\infty} f_n(c) \text{ exists}$$
$$\lim_{n\to\infty} f_n'(x) = g(x) \text{ uniform convergence}$$

Then f_n converges uniformly to some function f where:

$$f(x) = \lim_{n\to\infty} f_n(c) + \int_c^x g(t) \cdot dt$$

$$\implies f'(x) = g(x)$$

We will repurpose the theorem above through the following lemma to be utilized for investigating $u(r,\theta)$.

Lemma

Statement: *Suppose $u_n(x,y)$ are C^1 functions on an open set R. Suppose $\sum_{n=0}^{\infty} u_n$ converges uniformly to u and $\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial x}(x,y)$ converges uniformly to v. Then:*

$$\frac{\partial u}{\partial x}(x,y) = v(x,y)$$

Lemma

Proof: *Consider an arbitrary point* (x_0, y_0) *in* R. *We will restrict ourselves to a ball containing this point. Define* w_n *and* w *as follows over this ball.*

$$w_n(x,y) \equiv \sum_{k=0}^{n} u_k(x,y) = \sum_{k=0}^{n} u_k(x_0,y) + \int_{x_0}^{x} \frac{\partial}{\partial x} \sum_{k=0}^{n} u_k(t,y) \cdot dt$$

$$w(x,y) \equiv u(x_0,y) + \int_{x_0}^{x} v(t,y) \cdot dt$$

Use 8.3.2 to infer that w_n *converges uniformly to* w. *But we know* w_n *converges to* u, *so* $w = u$.

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} = v$$

Theorem

Statement: *The function $u(r,\theta)$ satisfies the heat equation $\Delta u = 0$ in the open disc $\mathbb{D}$.*

Theorem

Proof: *Each u_n is a solution to the heat equation $\Delta u_n = 0$. Therefore, it suffices to prove the following.*

$$\Delta u = \sum_{n=0}^{\infty} \Delta u_n$$

Therefore, it is sufficient to prove that the sum of the partials in Δu are convergent; this fact in conjunction with the previous lemma proves this theorem.

Theorem

Proof Continued: *Recall* $u_n(r,\theta) = A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$ *and* $|A_n|, |B_n| \leq 2||f||_{L1}$.

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$$\left|\frac{\partial u_n}{\partial r}\right| = |nA_n r^{n-1}\cos(n\theta) + nB_n r^{n-1}\sin(n\theta)| \leq 4nr^{n-1}||f||_{L1}$$

$$\left|\frac{\partial^2 u_n}{\partial r^2}\right| = |n(n-1)r^{n-2}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)| \leq 4n^2 r^{n-2}||f||_{L1}$$

$$\left|\frac{\partial u_n}{\partial \theta}\right| = |-nA_n r^n \sin(n\theta) + nB_n r^n\cos(n\theta)| \leq 4nr^n||f||_{L1}$$

$$\left|\frac{\partial^2 u_n}{\partial \theta^2}\right| = |-n^2 A_n r^n\cos(n\theta) - n^2 B_n r^n \sin(n\theta)| \leq 4n^2 r^n||f||_{L1}$$

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Applying the ratio test shows the sum of the bounds of these partials converge with a radius of convergence of 1.

Theorem

Proof Continued: *So the Weierstrass-M test applies and the partials all converge uniformly. This means the order of the partial derivative operator and the sum can exchange. Therefore:*

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So u satisfies the heat equation in the open disc $\mathbb{D}$.

- So far we have proven that $u(r,\theta)$ converges uniformly for any closed disc $\overline{\mathbb{D}}_R$ where $R \in [0,1)$, and that u satisfies the heat equation.

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- There are two unanswered questions:
  1. Does u converge to $f(\theta)$ as $r \to 1^-$?
  2. Is the solution to the heat equation unique?

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Let us expand the fourier coefficients in the expansion of $u(r,\theta)$ inside a disc $\overline{\mathbb{D}}_R$ for $R \in [0,1)$.

$$
\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(t) r^n \cos(nt)\cos(n\theta)\cdot dt \\
&+ \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) r^n \sin(nt)\sin(n\theta)\cdot dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(t) r^n \cos(n(\theta - t))\cdot dt
\end{aligned}
$$

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But the sum of the bounds converges as $R < 1$. This means the sum of the integrand must converge uniformly by the Weierstrass-M test; so the order of the sum and the integration can exchange. Therefore:

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$$\boxed{\begin{aligned} u(r,\theta) &= \int_{-\pi}^{\pi} P(r,\theta - t) f(t) \cdot dt \\ P(r,\theta) &\equiv \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta)\right) \end{aligned}}$$

Theorem

Statement: *The Poisson kernanl can be calculated using the function below.*

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(t)+r^2}$$

Theorem

Proof: *We will use complex analysis to prove this theorem. Review 13.9 for a brief review on this topic.*

$$
\begin{aligned}
2\pi P(r,t) &= 1 + 2\sum_{n=1}^{\infty} r^n \cos(nt) = 1 + \sum_{n=1}^{\infty} r^n \left(e^{int} + e^{-int}\right) \\
&= 1 + \sum_{n=1}^{\infty} \left(re^{it}\right)^n + \sum_{n=1}^{\infty} \left(re^{-it}\right)^n = 1 + \frac{re^{it}}{1 - re^{it}} + \frac{re^{-it}}{1 - re^{-it}} \\
&= \frac{\left(1 - re^{it}\right)\left(1 - re^{-it}\right) + re^{it}\left(1 - re^{-it}\right) + re^{-it}\left(1 - re^{it}\right)}{\left(1 - re^{it}\right)\left(1 - re^{-it}\right)} \\
&= \frac{1 - r^2}{1 - r\left(e^{it} + r^{-it}\right) + r^2} = \frac{1 - r^2}{1 - 2r\cos(t) + r^2}
\end{aligned}
$$

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5. For any $\delta \leq 0$, $\lim_{r \to 1^-} \max_{\delta < |t| \leq \pi} P(r, t) = 0$

To prove property (4): $||P||_{L1} = 1$, one can integrate the closed form of the Poisson's kernal. But an easier solution would be to use the reseult we have proven thus far. If f is a 2π-periodic continuous function over $\mathbb{R}$ and its Fourier coeffieicnts are A_n, B_n, then:

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Select $f(\theta) \equiv 1$. But we know the Fourier series of f:

$$f(\theta) \sim 1 \implies A_0 = 1, A_n = 0, B_n = 0$$

$$\boxed{\int_{-\pi}^{\pi} P(r,t)\cdot dt = 1}$$

To prove property (5) we use the mononocity of f over $[0, \pi]$ and $[-\pi, 0]$ and its evenness.

$$\lim_{r\to 1^-} \max_{\delta\leq|t|\pi} P(r, t) = \lim_{r\to -1} \frac{1-r^2}{1-2r\cos(\delta)+r^2} = \frac{0}{2(1-\cos(\delta))} = 0$$

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$$\boxed{\forall\delta > 0,\ \lim_{r\to 1^-} \max_{\delta\leq|t|\leq\pi} P(r,t) = 0}$$

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- Note that Poisson's kernal resembles a *spike function* centered around the origin. This means that the temperature of the boundary points near $t = \theta$ contribute the most to the temperature at the point (r, θ).
- The closer a point gets to the boundary($r \to 1^-$), by property (5), the *sharper* the kernal becomes. This means the closer we get to the boundary the more significant the contribution of the boundary temperatures around $t = \theta$ becomes and the less significant the temperature of the points away from it.

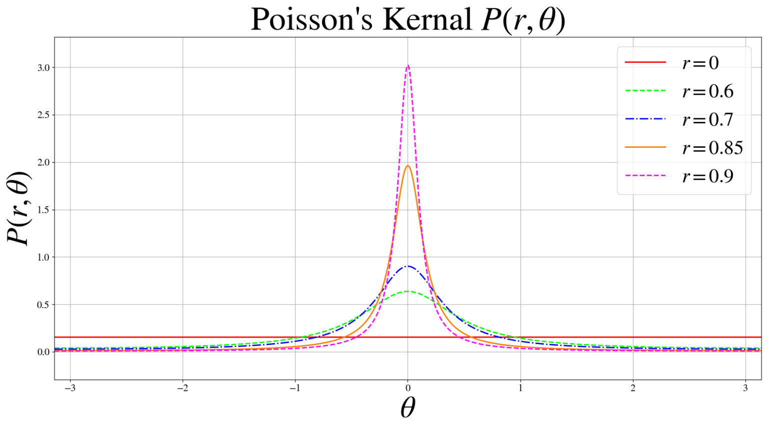

Figure: The Poisson kernel effectively tells us which parts of $f(\theta)$ are more important in determining the temperature at (r,θ).

4. Poisson's Theorem

We finally have attained the tools to establish the convergence of $u(r, \theta)$ on the boundary of $\mathbb{D}$ and prove **Poisson's Theorem**.

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We finally have attained the tools to establish the convergence of $u(r,\theta)$ on the boundary of $\mathbb{D}$ and prove **Poisson's Theorem**.

Theorem

Statement: *$u(r,\theta)$ satisfies the following condition and converges uniformly to the boundary temperature.*

$$\lim_{r\to 1^-} u(r,\theta) = f(\theta)$$

Moreover, by defining the value of the function $u(r=1,\theta) = f(\theta)$ we obtain a continuous and harmonic function $u(r,\theta)$ over the unit disc $\mathbb{D}$ which agrees with the boundary conditions $f(\theta)$.
*In other words, $u(r,\theta)$ is **a solution** to the steady state heat equation.*

Theorem

Proof: *We will prove for any $\epsilon > 0$ there exists a $R < 1$ such that for all $r \in (R, 1)$, we have $||f - u(r, \cdot)||_\infty < \epsilon$.*

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Recall that $||P(r, \cdot)||_{L1} = 1$. *So:*

$$f(\theta) - u(r, \theta) = f(\theta) \int_{-\pi}^{\pi} P(r, t) \cdot dt - \int_{-\pi}^{\pi} f(\theta - t) P(r, t) \cdot dt$$

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Since f *is continuous over* $[-\pi, \pi]$, $|f|$ *attains its maximum* M. *Now define* $\epsilon_0 \equiv (4\pi M + 1)^{-1} \epsilon$.

Theorem

Proof Continued: *Since f is continuous over $[-\pi, \pi]$, it must be uniformly continuous. So there is some δ such that:*

$$|a - b| < 2\delta \implies |f(a) - f(b)| < \epsilon_0$$

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$$\left| \int_{-\delta}^{\delta} P(r,t)(f(\theta) - f(\theta - t)) \cdot dt \right| \leq \int_{-\pi}^{\pi} \epsilon_0 P(r,t) \cdot dt = \epsilon_0$$

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Recall property (5) of Poisson's kernal. We can select R close enough to 1 so that for any $r \in (R, 1)$, $P(r,t)$ will be bounded by ϵ_0 for $|t| > \delta$.

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$$\left| \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} (f(\theta) - f(\theta - t))P(r,t) \cdot dt \right| \leq \int_{-\pi}^{\pi} 2M\epsilon_0 \cdot dt = 4M\pi\epsilon_0$$

Theorem

Proof Continued: *Now we can combine the preceeding results to get the following for the selected* δ *and* r.

$$||f - u(r, \cdot)||_\infty \leq \int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi} P(r, t)(f(\theta) - f(\theta - t)) \cdot dt$$
$$\leq (4\pi M + 1)\epsilon_0 = \epsilon$$

- We have finally proven that the function u constructed at the beginning of this presentation is continuous over $\mathbb{D}$, agrees with the boundary conditions on $r = 1$, and satisfies the steady state heat equation.

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- In other words, we have proven u is **a** solution.
- But this does not mean we have found the general solution to the heat equation; there could be other solutions to the problem which are not in the form of a (in)finite some of separable solutions.
- In the next section we will prove that u is **the** solution, implying that our choice of separable solutions has not restricted us to solving for a restricted class of solutions, but allows us to solve for *all* the solutions.

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So far we have only restricted ourselves to the span of separable solutions to the steady state solutions. In this section, we prove the maximum principle. This theorem applies to all solutions of the heat equation.

5. The Maximum Principle

So far we have only restricted ourselves to the span of separable solutions to the steady state solutions. In this section, we prove the maximum principle. This theorem applies to all solutions of the heat equation.

Theorem

Statement: *Suppose u is continuous on $\overline{\mathbb{D}}$ and $\Delta u = 0$ on the open disk $\mathbb{D}$. Then:*

$$\max_{(r,\theta)\in\overline{\mathbb{D}}} u(r,\theta) = \max_{-\pi\leq\theta\leq\pi} u(1,\theta)$$

Theorem

Proof: *First, we will prove that a function* v *where* $\Delta v \geq \epsilon > 0$ *cannot obtain its maximum in the interior of the disc.*

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$$v_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) = 0$$

And the second order derivatives have to be non-positive.

$$v_{rr}(r_0, \theta_0) \leq 0 \text{ and } v_{\theta\theta}(r_0, \theta_0) \leq 0$$

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And the second order derivatives have to be non-positive.

$$v_{rr}(r_0, \theta_0) \leq 0 \text{ and } v_{\theta\theta}(r_0, \theta_0) \leq 0$$

This results in a contradiction.

$$\epsilon \leq \Delta v = v_r r + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} \leq 0 \implies \impliedby$$

5. The Maximum Principle

Theorem

Proof Continued: *We now define a sequence of functions v_n defined as follows.*

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Note that $\Delta v_n = \frac{4}{n}$, meaning that it can only obtain its maximum on the boundary.
Note that v_n converges uniformly to u.

$$\begin{aligned}\max_{(r,\theta)\in\overline{\mathbb{D}}} u(r,\theta) &= \lim_{n\to\infty} \max_{(r,\theta)\in\overline{\mathbb{D}}} v_n(r,\theta) \\ &= \lim_{n\to\infty} \max_{-\pi\leq\theta\leq\pi} v_n(1,\theta) = \max_{-\pi\leq\theta\leq\pi} u(1,\theta)\end{aligned}$$

- Note that using merely creating a sequence of functions that have a negative laplacian, we can establish the minimum principle. In conjunction with the maximum principle we conclude that the maximum and minimum temperature for any harmonic equation on the unit disc occur on the boundary.

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- This means that if the boundary of a disc is imposed to be uniformly 0 for all θ, the only possible solution is $u(r, \theta) = 0$, as both the maximum and the minimum have to be 0.

Theorem

Solution: *The solution to any given steady state equation exists and is unique.*

Theorem

Proof: *The fact that a solution exists is already established. The following is readily a solution.*

$$u(r,\theta) \equiv \int_{-\pi}^{\pi} P(r,\theta - t)f(t) \cdot dt$$

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Proof: *The fact that a solution exists is already established. The following is readily a solution.*

$$u(r,\theta) \equiv \int_{-\pi}^{\pi} P(r,\theta - t)f(t) \cdot dt$$

Suppose there is another solution $v(r,\theta)$ to the steady state solution. Then the function $w \equiv u - v$ satisfies the following set of equations.

$$\Delta w = \Delta(u - v) = \Delta u - \Delta v = 0$$
$$w(1,\theta) = u(1,\theta) - v(1,\theta) = f(\theta) - f(\theta) = 0$$

Theorem

Proof Continued: *But we proved that if the boundary condition is uniformly* 0*, then the only solution is the trivial solution* $w(r,\theta) = 0$. *This implies:*

$$u = v$$

So the solution to the steady state heat equation with any given 2π*-periodic boundary condition on the unit disc is unique.*

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In solving the heat equation on the boundary disc, we have gathered a number of important conclusions about the Fourier series which we now enumerate.

Theorem

Statement: *If two continuous 2π-periodic functions f and g have equal Fourier series, then they are equal functions.*

6. Conclusion about Fourier Series

Theorem

Proof: *The solution to the unit disc heat equation is entirely determined by its Fourier series. So the heat equations with the boundary equations equal to f and g have the same solution $u(r,\theta)$.*

$$f(\theta) = \lim_{r \to 1^-} u(r,\theta) = g(\theta)$$

$$f = g$$

6. Conclusion about Fourier Series

Theorem

Statement: *If the Fourier series of a function f converges uniformly, then the limit is f.*

Theorem

Proof: *Suppose the limit of the Fourier series is* g. *Since the convergence is uniform and each finite Fourier sum is continuous,* g *has to be continuous.* g *is also* 2π*-periodic as it is the limit of* 2π*-periodic functions.*

Theorem

Proof: *Suppose the limit of the Fourier series is g. Since the convergence is uniform and each finite Fourier sum is continuous, g has to be continuous. g is also 2π-periodic as it is the limit of 2π-periodic functions.*

We can calculate the Fourier coefficients of g as follows. Suppose we want to calculate B_n^g where the superscript denotes the fact that the coefficient belogns to the function g.

$$B_n^g = \int_{-\pi}^{\pi} g(t)\sin(nt) \cdot dt$$

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$$B_n^g = \int_{-\pi}^{\pi} g(t)\sin(nt)\cdot dt$$

$$B_n^g = \frac{1}{\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty}\left(A_0^f + \sum_{k=1}^{N} A_k^f\cos(kt) + B_k^f\sin(kt)\right)\sin(nt)\cdot dt$$

Theorem

Proof Continued: *Since the convergence of the Fourier series of f is uniform, the limit and the integral can exchange orders.*

$$B_n^g = \lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(A_0^f + \sum_{k=1}^{N} A_k^f \cos(kt) + B_k^f \sin(kt) \right) \sin(nt) \cdot dt$$

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Likewise we can prove that the Fourier coefficients of f and g are equivalent. From the previous theorem we now that if the Fourier coefficients are equal, then so are the functions.

$$f = g$$

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7. The Vibrating String Problem

- Suppose we fix two ends of a string and know that waves travel at a speed of v along the string. We define the function $y(x, t)$ to be the displacement of the string from its equilibrium position at x and at time t.

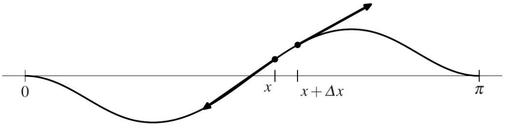

Figure: Taken from Real Analysis and Applications, Theory in Practice

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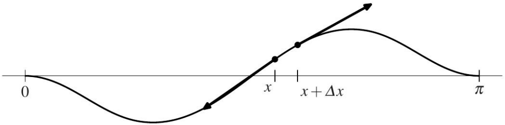

Figure: Taken from Real Analysis and Applications, Theory in Practice

- The equation describing the propogation of the wave along the string is as follows.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

- We again assume the solution $y(x,t)$ is seperable, namely:

$$y(x,t) = X(x)T(t)$$

7. The Vibrating String Problem

- We again assume the solution $y(x,t)$ is seperable, namely:

$$y(x,t) = X(x)T(t)$$

- This allows us to rewrite the said partial differential equation as two separate ODEs, as done before for the heat equation.

$$\boxed{\begin{aligned} X''(x) - \frac{c}{v^2} &= 0 \\ T''(t) - cT(t) &= 0 \end{aligned}}$$

For some constant $c \in \mathbb{R}$.

This differential equation is typically solved with the following boundary and initial conditions.

$$y(0) = y(\pi) = 0$$

$$y(x,0) = f(x)$$

$$\frac{\partial y(x,0)}{\partial t} = g(x)$$

Then we postulate that the infinite linear combination of seperable solutions are also solutions. This in conjunction with our knowledge of Fourier series gives us the following solution:

$$
\boxed{
\begin{aligned}
y(x,t) &= \sum_{n=1}^{\infty} A_n \sin(nx)\cos(nvt) + B_n \sin(nx)\sin(nvt) \\
A_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\cdot dt \\
B_n &= \frac{2}{nv\pi}\int_0^{\pi} g(t)\sin(nt)\cdot dt
\end{aligned}
}
$$

Note: For the purposes of this presentation we have not examined closely the conditions we need to impose on f, g. For example we tacitly assumed that f and g are continuous functions.

Thanks for Your Attention!