1 High-level Problem Description

The objective of the solution is to prevent and solve congestion in the MV networks by providing some type of signal to the aggregator. This signal can be price, firm/non-firm capacity contracts, etc. The problem can be divided into two layers, an upper-level layer (DSO operation) and a lower-level layer (set of aggregators). By approaching the problem with a bi-level modeling approach, both, DSO and aggregators can be modeled independently and can maximize their objectives while satisfying each other's constraints.

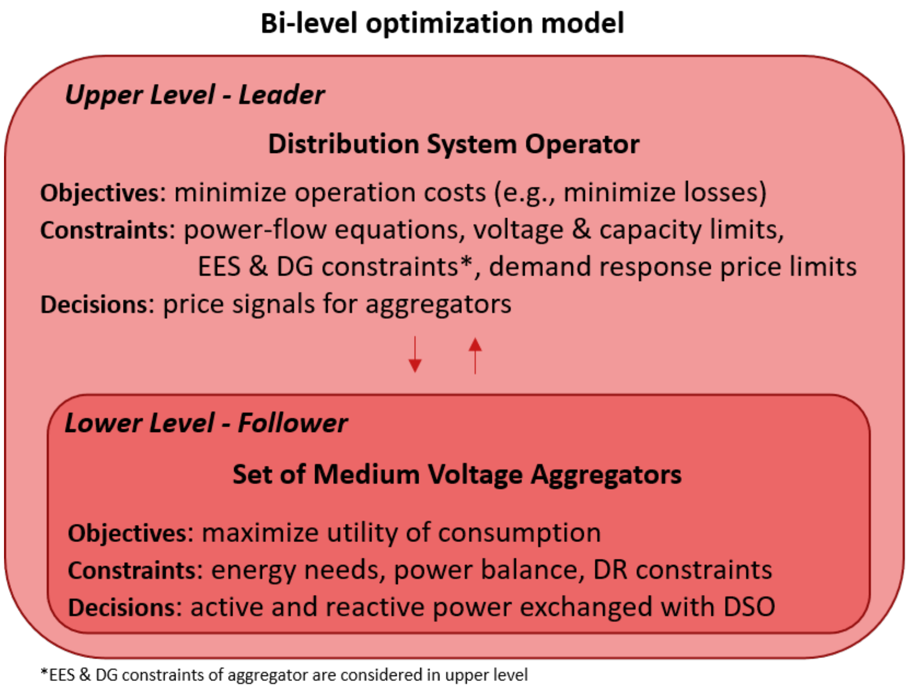

Figure 1: Graphical overview of the problem.

The objective of the DSO is to prevent congestion in the medium-voltage network. This can be indirectly achieved by minimizing losses or operating costs in the network. DSO is subjected to several network constraints. The network is modeled in steady-state as a radial distribution network with a pi-model of a line (including shunt components). Second-order cone approximation is implemented in order to obtain a convex representation of the non-linear system which can be solved with commercial solvers.

Same as DSO, each aggregator can have an objective such as to maximize profits, minimize discomfort or minimize costs, etc. The advantage of the bi-level approach is that in practice, the aggregator does not need to have information about the network. The aggregator decides on the optimal power dispatch (consumption/generation) based on the price signal obtained from DSO and communicates back the aggregated power to be exchanged with the medium voltage network. Aggregators base their decision based on the price of energy, their energy needs, and the utility to consume energy.

A graphical overview of the problem description is depicted in Fig.1. The following sections provide a derivation of the upper and lower-level problems. Derivation of equivalent MILP representation of the bi-level problem is derived as shown in Fig.2.

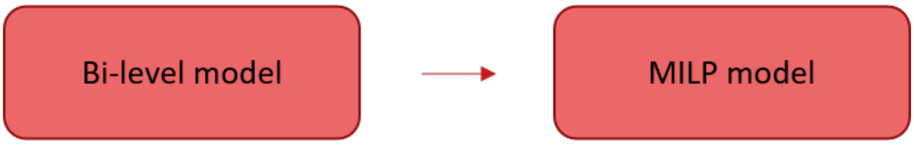

Figure 2: MILP problem derived from bi-level representation using KKT and Fortuny-Amat transformation.

Additional sections introduce modeling of uncertainty from renewable generation. The chance-constrained modeling approach is derived in the subsequent section. Finally, the Monte Carlo approach and implementation of random variations within the obtained aggregator profiles are derived to further consider uncertainties that DSOs experience in practice. This allows to evaluate the robustness of the solution by analyzing the impact of random variations on possible violations of network constraints.

2 Modeling - Upper Level Model

Decision vector $X_{DSO} = \{P_{ij,t}, P^S_{i,t}, P^{DG}_{g,t}, P^{EES,dc}_{s,t}, P^{EES,ch}_{s,t}, SOC_{s,t}, u^{EES}_{s,t}, F_{g,t}, u^{DG}_{s,t}, Q_{ij,t}, Q^S_{i,t}, Q^{DG}_{g,t}, I^{sqr}_{ij,t}, V^{sqr}_{ij,t}, c^{DR}_t\}$

Objective Function

$$\begin{aligned} \min_{X_{DSO}} \quad & \Delta T \sum_{t\in\Omega_T} \sum_{ij\in\Omega_l} c^R_t R_{ij,a(d_{ij})} I^{sqr}_{ij,t} \\ \text{s.t.} \quad & (2) - (22), (26) \end{aligned} \tag{1}$$

Nodal Power Balance

$$\begin{aligned} &\sum_{ij\in\Omega_l|_{j=k}} P_{ij,t} + \sum_{p\in\Omega_{PV}|_{p=k}} P^{PV}_{p,t} + \sum_{\omega\in\Omega_{WT}|_{\omega=k}} P^{WT}_{\omega,t} + \sum_{s\in\Omega_{EES}|_{s=k}} P^{EES,dc}_{s,t} + \sum_{g\in\Omega_{DG}|_{g=k}} P^{DG}_{g,t} \\ &+ P^S_{i,t} = P^D_{i,t} + \sum_{ij\in\Omega_l|_{i=k}} (P_{ij} + R_{ij,a(d_{ij})} I^{sqr}_{ij,t}) + \sum_{s\in\Omega_{EES}|_{s=k}} P^{EES,ch}_{s,t} \qquad \forall k \in \Omega_b, \forall t \in \Omega_T \end{aligned} \tag{2}$$

$$\begin{aligned} &\sum_{ij\in\Omega_l|_{j=k}} (Q_{ij,t} + \frac{B_{jj,a(d_{jj})}}{2} V^{sqr}_j) + \sum_{p\in\Omega_{PV}|_{p=k}} Q^{PV}_{p,t} + \sum_{\omega\in\Omega_{WT}|_{\omega=k}} Q^{WT}_{\omega,t} + \sum_{g\in\Omega_{DG}|_{g=k}} Q^{DG}_{g,t} \\ &+ Q^S_{i,t} = Q^D_{i,t} + \sum_{ij\in\Omega_l|_{i=k}} (Q_{ij} + X_{ij,a(d_{ij})} I^{sqr}_{ij,t} + \frac{B_{ii,a(d_{ii})}}{2} V^{sqr}_i) \qquad \forall k \in \Omega_b, \forall t \in \Omega_T \end{aligned} \tag{3}$$

Network Constraints

$$V^{sqr}_{i,t} - V^{sqr}_{j,t} = 2(R_{ij,a(d_{ij})} P_{ij,t} + X_{ij,a(d_{ij})} Q_{ij,t}) + Z^2_{ij,a(d_{ij})} I^{sqr}_{ij,t} \qquad \forall ij \in \Omega_l, \forall t \in \Omega_T \tag{4}$$

$$V^{sqr}_{j,t} I^{sqr}_{ij,t} \geq P^2_{ij,t} + Q^2_{ij,t} \qquad \forall ij \in \Omega_l, \forall t \in \Omega_T \tag{5}$$

$$\underline{V}^2 \leq V^{sqr}_{i,t} \leq \overline{V}^2 \qquad \forall i \in \Omega_b, \forall t \in \Omega_T \tag{6}$$

$$0 \leq I^{sqr}_{ij,t} \leq \overline{I}^2_{ij,t} \qquad \forall ij \in \Omega_l, \forall t \in \Omega_T \tag{7}$$

$$\begin{aligned} P^{Exch}_{i,t} = {} & P^S_{i,t} - P^D_{i,t} + \sum_{p \in \Omega_{PV}|_{p=i}} P^{PV}_{p,t} + \sum_{\omega \in \Omega_{WT}|_{\omega=i}} P^{WT}_{\omega,t} + \sum_{s \in \Omega_{EES}|_{s=i}} (P^{EES,dc}_{s,t} - P^{EES,ch}_{s,t}) \\ & + \sum_{g \in \Omega_{DG}|_{g=i}} P^{DG}_{g,t} \quad \forall i \in \Omega_b, \forall t \in \Omega_T \end{aligned} \tag{8}$$

$$Q^{Exch}_{i,t} = Q^S_{i,t} - Q^D_{i,t} + \sum_{p \in \Omega_{PV}|_{p=i}} Q^{PV}_{p,t} + \sum_{\omega \in \Omega_{WT}|_{\omega=i}} Q^{WT}_{\omega,t} + \sum_{g \in \Omega_{DG}|_{g=i}} Q^{DG}_{g,t} \tag{9}$$

$$\forall i \in \Omega_b, \forall t \in \Omega_T \tag{10}$$

$$\left(P^{Exch}_{i,t}\right)^2 + \left(Q^{Exch}_{i,t}\right)^2 \leq \left(\overline{S}^{Exch}_i\right)^2 \quad \forall i \in \Omega_b, \forall t \in \Omega_T \tag{11}$$

- **NOTE**: $V^{sqr}_{j,t} I^{sqr}_{ij,t} \geq P^2_{ij,t} + Q^2_{ij,t}$ is the second-order cone relaxation.
- **NOTE**: Considering voltage drops in HV/LV, MV/LV transformers and at the low voltage side gives Stedin ±4% room for voltage variations in the medium voltage network.
- **NOTE**: $P^S_{i,t}$ and $Q^S_{i,t}$ are set to 0 for all buses except the slack bus. Since there is no local consumption or generation at the slack bus, the equation (11) breaks down to $\left(P^S_{i,t}\right)^2 + \left(Q^S_{i,t}\right)^2 \leq \left(\overline{S}^{Exch}_i\right)^2 \quad \text{if } i{=}1, \forall t \in \Omega_T$
- **NOTE**: Bus i at time t is consuming active and reactive power when $P^{Exch}_{i,t} > 0$ and $Q^{Exch}_{i,t} > 0$ and is injecting active and reactive power into the MV network when $P^{Exch}_{i,t} < 0$ and $Q^{Exch}_{i,t} < 0$. Local demand and supply are in balance when $P^{Exch}_{i,t} = 0$ and $Q^{Exch}_{i,t} = 0$.

Energy Storage System

$$SOC_{s,t} = SOC_{s,t-1} + \frac{\Delta T}{\overline{SOE}_s}(\eta_s P^{ch}_{s,t} - \frac{P^{dc}_{s,t}}{\eta_s}) \qquad \forall s \in \Omega_{EES}, \forall t \in \Omega_T \tag{12}$$

$$\underline{SOC}_s \leq SOC_{s,t} \leq \overline{SOC}_s \qquad \forall s \in \Omega_{EES}, \forall t \in \Omega_T \tag{13}$$

$$P^{EES,ch}_{s,t} \leq \overline{P}^{EES}_s u^{EES}_{s,t} \qquad \forall s \in \Omega_{EES}, \forall t \in \Omega_T \tag{14}$$

$$P^{EES,dc}_{s,t} \leq \overline{P}^{EES}_s (1 - u^{EES}_{s,t}) \qquad \forall s \in \Omega_{EES}, \forall t \in \Omega_T \tag{15}$$

$$u^{EES}_{s,t} \in \{0,1\} \qquad \forall s \in \Omega_{EES}, \forall t \in \Omega_T \tag{16}$$

Distributed Generation

$$-P^{DG}_{g,t} \tan\left[\cos^{-1}(pf_g)\right] \leq Q^{DG}_{g,t} \leq P^{DG}_{g,t} \tan\left[\cos^{-1}(pf_g)\right] \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{17}$$

$$\left(P^{DG}_{g,t}\right)^2 + \left(Q^{DG}_{g,t}\right)^2 \leq \left(\overline{S}^{DG}_g\right)^2 \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{18}$$

$$R^d_g \leq P^{DG}_{g,t} - P^{DG}_{g,t-1} \leq R^u_g \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{19}$$

$$F_{g,t} = F_{g,t-1} - \frac{\Delta T P^{DG}_{g,t}}{\eta^f_g \mathrm{FC}_g \mathrm{H}_g} \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{20}$$

$$F_{g,t} \geq \underline{F}_g \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{21}$$

$$u^{DG}_{g,t} \in \{0,1\} \qquad \forall g \in \Omega_{DG}, \forall t \in \Omega_T \tag{22}$$

- **NOTE**: All distributed generation is modeled in the upper level. The reason for this is that it is not possible to derive KKT conditions for non-continuous variables. This is because the method depends on objective and constrains to be continuously differentiable.

Demand Response Price

$$(1 - k^{DSO})c^R_t \leq c^{DR}_t \leq c^R_t \tag{23}$$

- **NOTE**: k^{DSO} represents the portion of price which the DSO can influence.

3 Modeling - Lower Level Model ($\forall i \in \Omega_A$)

Decision vector $X_{AGG} = \{P^D_{i,t}, Q^D_{i,t}, P^D_{i,m,t}, P^{DR}_{i,t}\}$

Objective Function

$$\max_{X_{AGG}} \quad \sum_{t\in\Omega_T} \left[U_{i,t}(P^D_{i,t}) - \Delta T \left(c^{DR}_t P^{DR}_{i,t} + c^R_t P^{IL}_{i,t}\right)\right] \quad \forall i \in \Omega_A \tag{24}$$

$$U_{i,t}(P^D_{i,t}) = \Delta T \sum_{m\in\Omega_{M_i}} u^{DR}_{i,m,t} P^D_{i,m,t} \tag{25}$$

$$\begin{aligned} \min_{X_{AGG}} \quad & \Delta T \sum_{t\in\Omega_T} \left[c^{DR}_t P^{DR}_{i,t} + c^R_t P^{IL}_{i,t} - \sum_{m\in\Omega_{M_i}} u^{DR}_{i,m,t} P^D_{i,m,t}\right] \quad \forall i \in \Omega_A \\ \text{s.t.} \quad & (27) - (34) \end{aligned} \tag{26}$$

Demand Response

$$P^D_{i,t} = \sum_{m\in\Omega_{M_i}} P^D_{i,m,t} \qquad \forall t \in \Omega_T \quad (\lambda^1_{i,t}) \tag{27}$$

$$P^D_{i,m,t} \leq \overline{P}^D_{i,m,t} \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \quad (\mu^1_{i,m,t}) \tag{28}$$

$$P^D_{i,m,t} \geq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \quad (\mu^2_{i,m,t}) \tag{29}$$

$$E^D_i \leq \Delta T \sum_{t\in\Omega_T} P^D_{i,t} \leq k^D E^D_i \qquad (\mu^{3,7}_i) \tag{30}$$

$$P^D_{i,t} \geq P^{IL}_{i,t} \qquad \forall t \in \Omega_T \quad (\mu^4_{i,t}) \tag{31}$$

$$R^d_i \leq P^D_{i,t} - P^D_{i,t-1} \leq R^u_i \qquad \forall t \in \Omega_T \quad (\mu^{6,5}_{i,t}) \tag{32}$$

$$Q^D_{i,t} = P^D_{i,t} \tan\left(\arccos\left(pf_i\right)\right) \qquad \forall t \in \Omega_T \quad (\lambda^2_{i,t}) \tag{33}$$

$$P^D_{i,t} = P^{DR}_{i,t} + P^{IL}_{i,t} \qquad \forall t \in \Omega_T \quad (\lambda^3_{i,t}) \tag{34}$$

- **NOTE**: $\overline{P}^D_{i,m,t}$ is set to 1/4th of the maximal historical load of an aggregator from given simulation period. In the final implementation, this is independent of time (the value is constant for each aggregator). This could be tailored to the physical aggregated connection capacity.
- **NOTE**: $\underline{P}^D_{i,m,t}$ is set to zero. However, the first block can potentially be 0 only if there is no inflexible load.
- **NOTE**: Aggregated load operates at constant power factor ($\tan\left(\arccos\left(pf_i\right)\right) = \zeta_i$).
- **NOTE**: Implementation of utility functions: There is actually not a set Ω_M with all possible marginal utility/consumption blocks. In the script, set of utility functions is constructed from the input file. Each aggregator has dedicated 4 m blocks associated

with it. Eg.) $P^D_{1,1} = P^D_{1,1,1} + P^D_{1,2,1} + P^D_{1,3,1} + P^D_{1,4,1}$. Another aggregator has different set of m values such as $P^D_{2,1} = P^D_{2,5,1} + P^D_{2,6,1} + P^D_{2,7,1} + P^D_{2,8,1}$. Therefore, Ω_M is not a set in a conventional way since it is important to maintain decreasing nature of marginal utility blocks when constructing the utility functions. It could be said that each aggregator has its own ordered set Ω_{M_i}. Aggregator utility function is created by matching appropriate subscripts of $u^{DR}_{i,m,t}$ and $P^D_{i,m,t}$. Figure 3 shows how all possible combinations of $u^{DR}_{i,m,t}$ are first created and subsequently initialized.

- **NOTE**: The amount of energy consumed by an aggregator should not be simply constrained by the minimal requirement to prevent over-consumption. Therefore, multiplier $k^D \geq 1$ defines the upper limit (e.g. $k^D = 1.2 = 120\%$) on the total daily energy consumed by an aggregator.

```
# define set of marginal utility blocks
utils = {(aggs[a], dblocks[d + NMd*a], time[t]) for a in AggData.index
        for d in range(NMd) for t in UtilTimeMul.index}
# create marginal utility blocks from input
MargUtil = {(aggs[a], dblocks[d + NMd*a], time[t]):
            UtilTimeMul.loc[t,'UtilMultiplier']*DemandBlocks.loc[DemandBlocks['AGG']
            == AggData.loc[a,'AGG']]['MargUtil'].iloc[d]/1000*Sbase
            for a in AggData.index for d in range(NMd) for t in UtilTimeMul.index}
# assign set of marginal utility blocks to the model
model.UTILS = Set(initialize=utils)
# assign marginal utility blocks to the model
model.AGG_util = Param(model.UTILS, mutable=True, initialize=MargUtil,
                 within=NonNegativeReals)
```

Figure 3: Initialization of marginal utility blocks $u^{DR}_{i,m,t}$

4 KKT - Lower Level Model ($\forall i \in \Omega_A$)

Lagrangian of Aggregators

$$\begin{aligned}
L_i =&\Delta T \sum_{t=1} \left[c_t^{DR} P_{i,t}^{DR} + c_t^R P_{i,t}^{IL} - \sum_{m=1} u_{i,m,t}^{DR} P_{i,m,t}^D \right] + \sum_{t=1} \lambda_{i,t}^1 \left(P_{i,t}^D - \sum_{m=1} P_{i,m,t}^D \right) \\
&+ \sum_{t=1} \sum_{m=1} \mu_{i,m,t}^1 (P_{i,m,t}^D - \overline{P}_{i,m,t}^D) + \sum_{t=1} \sum_{m=1} \mu_{i,m,t}^2 (-P_{i,m,t}^D) \\
&+ \mu_i^3 \left(E_i^D - \Delta T \sum_{t=1} P_{i,t}^D \right) + \sum_{t=1} \mu_{i,t}^4 (P_{i,t}^{IL} - P_{i,t}^D) \\
&+ \sum_{t=1} \mu_{i,t}^5 (P_{i,t}^D - P_{i,t-1}^D - R_i^U) + \sum_{t=1} \mu_{i,t}^6 (R_i^D - P_{i,t}^D + P_{i,t-1}^D) \\
&+ \sum_{t=1} \lambda_{i,t}^2 \left[Q_{i,t}^D - P_{i,t}^D \tan\left(\arccos\left(pf_i\right)\right) \right] + \sum_{t=1} \lambda_{i,t}^3 \left(P_{i,t}^D - P_{i,t}^{DR} - P_{i,t}^{IL} \right) \\
&+ \mu_i^7 \left(\Delta T \sum_{t=1} P_{i,t}^D - k^D E_i^D \right)
\end{aligned} \tag{35}$$

Stationarity

$$\frac{\partial L_i}{\partial P_{i,t}^D} = \lambda_{i,t}^1 - \Delta T(\mu_i^3 - \mu_i^7) - \mu_{i,t}^4 + \mu_{i,t}^5 - \mu_{i,t}^6 - \mu_{i,t+1}^5 + \mu_{i,t+1}^6 - \lambda_{i,t}^2 \zeta_i + \lambda_{i,t}^3 = 0 \qquad \forall t < NT \tag{36}$$

$$\frac{\partial L_i}{\partial P_{i,t}^D} = \lambda_{i,t}^1 - \Delta T(\mu_i^3 - \mu_i^7) - \mu_{i,t}^4 + \mu_{i,t}^5 - \mu_{i,t}^6 - \lambda_{i,t}^2 \zeta_i + \lambda_{i,t}^3 = 0 \qquad t = NT \tag{37}$$

$$\frac{\partial L_i}{\partial P_{i,t}^{DR}} = c_t^{DR} - \lambda_{i,t}^3 = 0 \qquad \forall t \in \Omega_T \tag{38}$$

$$\frac{\partial L_i}{\partial Q_{i,t}^D} = \lambda_{i,t}^2 = 0 \qquad \forall t \in \Omega_T \tag{39}$$

$$\frac{\partial L_i}{\partial P_{i,m,t}^D} = -\Delta T u_{i,m,t}^{DR} - \lambda_{i,t}^1 + \mu_{i,m,t}^1 - \mu_{i,m,t}^2 = 0 \qquad \forall t \in \Omega_T \tag{40}$$

Where $\tan\left(\arccos\left(pf_i\right)\right) = \zeta_i$

Primal Feasibility ($h(x) = 0,\ g(x) \leq 0$)

$$P_{i,t}^{D} - \sum_{m=1} P_{i,m,t}^{D} = 0 \qquad \forall t \in \Omega_T \quad (\lambda_{i,t}^{1}) \tag{41}$$

$$P_{i,m,t}^{D} - \overline{P}_{i,m,t}^{D} \leq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \quad (\mu_{i,m,t}^{1}) \tag{42}$$

$$-P_{i,m,t}^{D} \leq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \quad (\mu_{i,m,t}^{2}) \tag{43}$$

$$E_{i}^{D} - \Delta T \sum_{t=1} P_{i,t}^{D} \leq 0 \qquad (\mu_{i}^{3}) \tag{44}$$

$$P_{i,t}^{IL} - P_{i,t}^{D} \leq 0 \qquad \forall t \in \Omega_T \quad (\mu_{i,t}^{4}) \tag{45}$$

$$P_{i,t}^{D} - P_{i,t-1}^{D} - R_{i}^{u} \leq 0 \qquad \forall t \in \Omega_T \quad (\mu_{i,t}^{5}) \tag{46}$$

$$R_{i}^{d} - P_{i,t}^{D} + P_{i,t-1}^{D} \leq 0 \qquad \forall t \in \Omega_T \quad (\mu_{i,t}^{6}) \tag{47}$$

$$Q_{i,t}^{D} - P_{i,t}^{D} \tan\left(\arccos\left(pf_i\right)\right) = 0 \qquad \forall t \in \Omega_T \quad (\lambda_{i,t}^{2}) \tag{48}$$

$$P_{i,t}^{D} - P_{i,t}^{DR} - P_{i,t}^{IL} = 0 \qquad \forall t \in \Omega_T \quad (\lambda_{i,t}^{3}) \tag{49}$$

$$\Delta T \sum_{t=1} P_{i,t}^{D} - k^{D} E_{i}^{D} \leq 0 \qquad (\mu_{i}^{7}) \tag{50}$$

Dual Feasibility

$$\mu_{i,m,t}^{1} \geq 0, \quad \mu_{i,m,t}^{2} \geq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \tag{51}$$

$$\mu_{i}^{3} \geq 0, \quad \mu_{i}^{7} \geq 0 \tag{52}$$

$$\mu_{i,t}^{4} \geq 0, \quad \mu_{i,t}^{5} \geq 0, \quad \mu_{i,t}^{6} \geq 0 \qquad \forall t \in \Omega_T \tag{53}$$

$$\lambda_{i,t}^{1}, \quad \lambda_{i,t}^{2}, \quad \lambda_{i,t}^{3} \quad \text{free} \qquad \forall t \in \Omega_T \tag{54}$$

Complementary Slackness ($(\mu \geq 0) + (g(x) \leq 0) + (\mu g(x) = 0) \rightarrow \mu(-g(x)) \geq 0$)

$$0 \leq \mu \perp -g(x) \geq 0$$

$$0 \leq \mu^1_{i,m,t} \perp (-P^D_{i,m,t} + \overline{P}^D_{i,m,t}) \geq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \tag{55}$$

$$0 \leq \mu^2_{i,m,t} \perp (P^D_{i,m,t}) \geq 0 \qquad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \tag{56}$$

$$0 \leq \mu^3_i \perp (-E^D_i + \Delta T \sum_{t=1} P^D_{i,t}) \geq 0 \tag{57}$$

$$0 \leq \mu^4_{i,t} \perp (-P^{IL}_{i,t} + P^D_{i,t}) \geq 0 \qquad \forall t \in \Omega_T \tag{58}$$

$$0 \leq \mu^5_{i,t} \perp (-P^D_{i,t} + P^D_{i,t-1} + R^u_i) \geq 0 \qquad \forall t \in \Omega_T \tag{59}$$

$$0 \leq \mu^6_{i,t} \perp (-R^d_i + P^D_{i,t} - P^D_{i,t-1}) \geq 0 \qquad \forall t \in \Omega_T \tag{60}$$

$$0 \leq \mu^7_i \perp (-\Delta T \sum_{t=1} P^D_{i,t} + k^D E^D_i) \geq 0 \tag{61}$$

Fortuny-Amat Transformation

$$0 \leq \mu \perp -g(x) \geq 0$$

$$0 \leq -g(x) \leq Mz$$

$$0 \leq \mu \leq M(1-z)$$

$$z \in \{0, 1\}$$

Equations in primal and dual feasibility capture lower limit of the Fortuny-Amat transformation. This leads to set of equations $\mathbf{g}(\mathbf{x}) \geq -\mathbf{Mz}$ and $\mu \leq \mathbf{M}(\mathbf{1}-\mathbf{z})$. [1]

$$P^D_{i,m,t} - \overline{P}^D_{i,m,t} \geq -M^1 z^1_{i,m,t} \qquad \mu^1_{i,m,t} \leq M^1(1 - z^1_{i,m,t}) \quad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \tag{62}$$

$$-P^D_{i,m,t} \geq -M^2 z^2_{i,m,t} \qquad \mu^2_{i,m,t} \leq M^2(1 - z^2_{i,m,t}) \quad \forall m \in \Omega_{M_i}, \forall t \in \Omega_T \tag{63}$$

$$E^D_i - \Delta T \sum_{t=1} P^D_{i,t} \geq -M^3 z^3_i \qquad \mu^3_i \leq M^3(1 - z^3_i) \tag{64}$$

$$P^{IL}_{i,t} - P^D_{i,t} \geq -M^4 z^4_{i,t} \qquad \mu^4_{i,t} \leq M^4(1 - z^4_{i,t}) \quad \forall t \in \Omega_T \tag{65}$$

$$P^D_{i,t} - P^D_{i,t-1} - R^u_i \geq -M^5 z^5_{i,t} \qquad \mu^5_{i,t} \leq M^5(1 - z^5_{i,t}) \quad \forall t \in \Omega_T \tag{66}$$

$$R^d_i - P^D_{i,t} + P^D_{i,t-1} \geq -M^6 z^6_{i,t} \qquad \mu^6_{i,t} \leq M^6(1 - z^6_{i,t}) \quad \forall t \in \Omega_T \tag{67}$$

$$\Delta T \sum_{t=1} P^D_{i,t} - k^D E^D_i \geq -M^7 z^7_i \qquad \mu^7_i \leq M^7(1 - z^7_i) \tag{68}$$

$$z^1_{i,m,t}, \quad z^2_{i,m,t} \qquad \in \{0,1\} \tag{69}$$

$$z^3_i, \quad z^7_i \qquad \in \{0,1\} \tag{70}$$

$$z^4_{i,t}, \quad z^5_{i,t}, \quad z^6_{i,t} \qquad \in \{0,1\} \tag{71}$$

- **NOTE**: All M values are set to $k\frac{S_{base}}{S_{base}}$ where $k = 100$. Alternatively $M = I_{base} = \frac{S_{base}}{V_{nom}}$.

5 Mixed-integer Linear Programming (MILP) Model

Decision vector $X = \{X_{DSO}, X_{AGG}, \lambda^{(1-3)}_{i,t}, \mu^{(1-2)}_{i,m,t}, \mu^3_i, \mu^{(4-6)}_{i,t}, z^{(1-2)}_{i,m,t}, z^3_i, z^{(4-6)}_{i,t}\}$

$$\begin{aligned} \min_X \quad & \Delta T \sum_{t\in\Omega_T} \sum_{ij\in\Omega_l} c^{DR}_t R_{ij,a(d_{ij})} I^{sqr}_{ij,t} \\ \text{s.t.} \quad & (2)-(22),(36)-(54),(62)-(71) \end{aligned} \tag{72}$$

6 Chance-Constraint Optimization

The probability of constraint violations should be limited to η according to Eq.73

$$P_r(P^{RES}_{i,t} \leq P^{fore}_{i,t}) \geq \eta \tag{73}$$

which means that if $\eta = 90\%$, we select a value $P^{RES}_{i,t}$ which is below a random realization $P^{fore}_{i,t}$ at least 90% of the time. Equation 73 can be expressed as

$$P_r(P^{fore}_{i,t} \leq P^{RES}_{i,t}) = \Phi_{i,t}(P^{RES}_{i,t}) \leq 1-\eta \tag{74}$$

where $\Phi_{i,t}(P^{RES}_{i,t})$ is the CDF of a random variable with given PDF, mean and variance. Finally, $P^{RES}_{i,t}$ which satisfies given conditions can be expressed as

$$P^{RES}_{i,t} \leq \Phi^{-1}_{i,t}(1-\eta) \tag{75}$$

Therefore, solar irradiance (s) and wind (w) can be obtained from

$$\begin{aligned} s^{PV}_{i,t} &\leq \Phi^{-1}_{B_{i,t}}(1-\eta) \qquad \forall t \in \Omega_T, \quad \forall t \in \Omega_T \\ w^{WT}_{i,t} &\leq \Phi^{-1}_{W_{i,t}}(1-\eta) \qquad \forall t \in \Omega_T, \quad \forall t \in \Omega_T \end{aligned} \tag{76}$$

where $\Phi^{-1}_{B_{i,t}}$ and $\Phi^{-1}_{W_{i,t}}$ represent Beta and Weibull CDF respectively. The irradiance and wind speed can be used as an input to determine the output power $P^{PV}_{i,t}$ and $P^{WT}_{i,t}$.

7 Monte Carlo & Variable Load

PV generation

Random variable representing the solar irradiation at the location of an aggregator i at time t follows Beta distribution and is completely described by shape parameters $\alpha_{i,t}$ and $\beta_{i,t}$. The Monte Carlo simulation draws random samples of the random variable at each iteration following described distribution as depicted in Fig:4.

Wind generation

The random variable representing the wind speed at the location of an aggregator i at time t follows Weibull distribution and is completely described by shape parameter $\kappa_{i,t}$ and scale parameter $\lambda_{i,t}$. The random realization of a random variable is sampled from the aforementioned distribution.

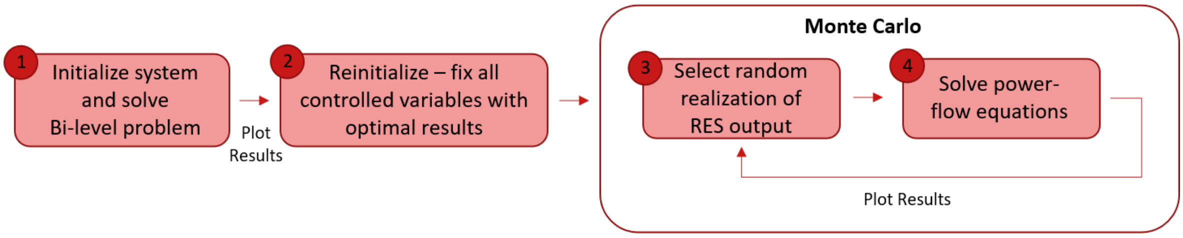

Figure 4: Flow diagram of the implementation with Monte Carlo.

Load

The output of the MILP model is a set of optimal load profiles. However, consumer demands are uncertain in nature and can not be guaranteed in practice. Generally, conventional demands follow bi-variate normal PDF which considers the positive correlation between active and reactive power. The active power follows a normal distribution with mean $P^D_{i,t}$ and variance $\sigma^2_{P^D_{i,t}}$ while the reactive power follows the same distribution with mean $Q^D_{i,t}$ and variance $\sigma^2_{Q^D_{i,t}}$. The correlation between active and reactive power is denoted as ρ. The random variables $P^{D,fore}_{i,t}$ and $Q^{D,fore}_{i,t}$ that are used in the Monte Carlo simulation are expressed as [2]

$$
\begin{aligned}
&P^{D,fore}_{i,t} \sim N(P^D_{i,t}, \sigma^2_{P^D_{i,t}}) \\
&Q^{D,fore}_{i,t} \sim N(A^D_{i,t}, D^D_{i,t})
\end{aligned}
\tag{77}
$$

where

$$
\begin{aligned}
&A^D_{i,t} = Q^D_{i,t} + \rho \frac{\sigma_{Q^D_{i,t}}}{\sigma_{P^D_{i,t}}}(P^{D,fore}_{i,t} - P^D_{i,t}) \\
&D^D_{i,t} = (1-\rho^2)\sigma^2_{Q^D_{i,t}}
\end{aligned}
\tag{78}
$$

8 Nomenclature

Sets

Ω_a	Set of conductor types.
Ω_b	Set of buses (nodes).
Ω_{DG}	Set of nodes with distributed generation.
Ω_{EES}	Set of nodes with energy storage systems.
Ω_l	Set of lines.
Ω_{PV}	Set of nodes with photo-voltaic (PV) systems.
Ω_T	Set of time periods.
Ω_{WT}	Set of nodes with wind parks.
Ω_A	Set of aggregators.
Ω_{M_i}	Set of marginal utility/consumption blocks.

Parameters

$B_{ij,a(d_{ij})}$	Shunt susceptance of node i of cable type a with length d [mS].
$R_{ij,a(d_{ij})}$	Resistance of line ij of cable type a with length d [mΩ].
$X_{ij,a(d_{ij})}$	Reactance of line ij of cable type a with length d [mΩ].
$Z_{ij,a(d_{ij})}$	Impedance of line ij of cable type a with length d [mΩ].
d_{ij}	Length of line ij [km].
R_g^d	Ramp-down rate of DG unit at node g [kW/h].
R_g^u	Ramp-up rate of DG unit at node g [kW/h].
FC_g	Fuel capacity of DG unit at node g [m^3].
H_g	Calorific value of DG unit at node g [kWh/m^3].
$\underline{F}_g$	Minimum fuel of DG unit at node g [m^3].
pf_g	Minimum power factor of DG unit at node g.
$P_{p,t}^{PV}$	Active power supplied by PV at node p in period t [kW].
$P_{i,t}^{IL}$	Inflexible active power demand at node i in period t [kW].
$P_{\omega,t}^{WT}$	Active power supplied by wind park at node ω in period t [kW].
$\overline{P}_s^{EES}$	Maximum charging/discharging power of the EES at node s [kW].
$Q_{p,t}^{PV}$	Reactive power supplied by PV at node p in period t [kvar].
$Q_{\omega,t}^{WT}$	Reactive power supplied by wind park at node ω in period t[kvar].
$\overline{S}_g^{DG}$	Maximum capacity of DG at node g [kVA]
$\overline{S}_i^{Exch}$	Maximum substation capacity at node i [kVA]
$\overline{V}$	Maximum voltage magnitude [V].
$\underline{V}$	Minimum voltage magnitude [V].
$\overline{SOE}_s$	Maximum state-of-energy of the EES at node s [kWh].
$\overline{SOC}_s$	Maximum state-of-charge of the EES at node s [%].
$\underline{SOC}_s$	Minimum state-of-charge of the EES at node s [%].
c_t^R	Retail price of energy [$/kWh].
k^{DSO}	Portion of retail price DSO can influence.
k^D	Upper limit multiplier on the total daily energy consumed by an aggregator.
$U_{i,t}$	Magnitude of utility of aggregator i at time t [$].
$u_{i,m,t}$	Magnitude of marginal utility block m of aggregator i at time t [$/kWh].
$\overline{P}_{i,m,t}^D$	Magnitude of marginal consumption block m of aggregator i at time t [kW].
E_i^D	Energy demand of aggregator i for considered time frame [kWh].
R_i^d	Ramp-down rate of aggregator at node i [kW/h].
R_i^u	Ramp-up rate of aggregator at node i [kW/h].
pf_i	Power factor of aggregator i.
M^{1-6}	Fortuny-Amat multiplier.
η_g^f	Fuel efficiency of DG at node g.
η_s	Charging/discharging efficiency of EES at node s.
ΔT	Time step [h].

Variables

$I_{ij,t}^{sqr}$	Squared current magnitude at line ij in period t $[A^2]$.
V_i^{sqr}	Squared voltage magnitude at node i in period t $[V^2]$.
$P_{ij,t}$	Active power flow at line ij in period t [kW].
$P_{i,t}^{S}$	Active power injection at node i in period t [kW].
$P_{g,t}^{DG}$	Active power supplied by DG unit at node g in period t [kW].
$P_{i,t}^{Exch}$	[kW].
$P_{i,t}^{DR}$	Active power demand associated with demand response at node i in period t [kW].
$P_{i,t}^{D}$	Active power demand at node i in period t [kW].
$P_{i,m,t}^{D}$	[kW].
$Q_{i,t}^{S}$	Reactive power injection at node i in period t [kW].
$Q_{ij,t}$	Reactive power flow at line ij in period t [kvar].
$Q_{g,t}^{DG}$	Reactive power supplied by DG unit at node g in period t [kvar].
$Q_{i,t}^{Exch}$	[kW].
$Q_{i,t}^{D}$	Reactive power demand a node i in period t [kvar].
$P_{s,t}^{EES,ch}$	Charging power of EES at node s in period t [kW].
$P_{s,t}^{EES,dc}$	Discharging power of EES at node s in period t [kW].
$SOC_{s,t}$	Sate-of-charge of EES at node s in period t [kWh].
$u_{s,t}^{EES}$	Binary variable associated to the charging (1) or discharging (0) operation of EES at node s in period t.
$u_{g,t}^{DG}$	Commitment status of DG unit at node g in period t.
c_t^{DR}	[$/kWh].
$\mu_{i,t}^{(1-3)}$	Lagrange multiplier of the KKT condition associated with inequality constraint.
$\lambda_{i,(m),(t)}^{(1-7)}$	Lagrange multiplier of the KKT condition associated with equality constraint..
$z_{i,(m),(t)}^{(1-7)}$	Binary variable associated with the Fortuny-Amat transformation.

References

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[2] D. P. Bertsekas and J. N. Tsitsiklis, *Introduction to Probability.* Athena Scientific; 1st edition, 2002.