Math 417
Mathematical Programming

Homework IX

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21 novembre 2016

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1 Optimality Conditions

For the standard linear program defined by the data $(A, b, c) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n$ consider the optimality conditions:

$$\begin{aligned} A^{\mathsf{T}}y + s &= c \\ Ax &= b \\ x_i s_i &= 0 \\ x, s &\geq 0 \end{aligned}$$

We want to show that the solution set of the above system of equations is convex. First let that solution be denoted by $\mathcal{F}$ and let : $w = (x, y, x) \in \mathcal{F}$.

Lemma: Cartesian product preserves convexity (finite vector spaces). Let P_1 and P_2 be convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ and x and y elements of those sets respectively. Then quite clearly:

$$\lambda x + (1-\lambda)x \in P_1 \quad \mu y + (1-\mu)y \in P_2 \iff (x, y) \in P_1 \times P_2 \tag{1}$$

Now we convert the different conditions into different sets.

$$P_1 := \{y \mid A^{\mathsf{T}}y = c - s\} \quad \text{polyhedron so convex}$$

$$P_2 := \{x \mid Ax = b,\ x \geq 0\} \quad \text{polyhedron so convex}$$

$$P := P_1 \times \mathbb{R}^m \times P_2 \Rightarrow P \text{ is convex by the above lemma}$$

Now let $x, s \in \mathbb{R}^n$ and $x, s \geq 0$. Let $F \in \mathbb{R}^{n\times m\times n}$ denote the set such that $x_i s_i = 0$ for all $i = 1 \ldots n$ where x and s represent the first and last n elements of any vector $w \in F$. We prove directly that F is convex. Let $w \in F$:

$$\lambda w + (1-\lambda)w = (\lambda x + (1-\lambda)x, 0 \ldots 0, \lambda s + (1-\lambda)s)$$
$$(\lambda x_i + (1-\lambda)x_i)(\lambda s_i + (1-\lambda)s_i) = \lambda^2 x_i s_i + 2(1-\lambda)\lambda x_i s_i + (1-\lambda)^2 x_i s_i = 0$$

Now we have seen that the set F contains all its convex combinations so it is convex. Finally let $\mathcal{F} = P \cap F$. This set clearly convex by the preservation of intersection. Moreover it is the set of all possible solutions of the optimality conditions.

2 Linear equations in interior-points methods

We want to show that if

$$\begin{pmatrix} 0 & A^{\mathsf{T}} & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \Delta w = b \tag{2}$$

has a solution $\Delta w = (\Delta x, \Delta y, \Delta s)$ then the components Δx and Δs are uniquely determined. Since we don't consider the other component we look at a smaller system of equations:

$$\begin{pmatrix} 0 & I \\ A & 0 \\ S & X \end{pmatrix} w' = b' \tag{3}$$

Where b' is just the first and last n elements of b. Clearly if there is a solution of the bigger system 2 there is one for the smaller one. We simply removed the second column as we don't care about y, it is not necessarily uniquely determined since we can't assume that A^{T} has full rank. Now we immediately have that Δs is uniquely determined since $I\Delta s$ has to be equal to the first n elements of b. Moreover we have that $S\Delta x + X\Delta s$ is equal to the n last elements of b. Let r denote the last n elements of b:

$$S\Delta x + X\Delta s = r \tag{4}$$

$$\iff S\Delta x = r - X\Delta s \tag{5}$$

Now since Δs was shown to be uniquely determined the whole RHS of 5 is uniquely determined. Moreover S is a diagonal positive definite matrix. It is therefore invertible with its inverse being a diagonal matrix with $(s^{-1})_{ii} = (s_{ii})^{-1}$. We have no divsion by zero problem since positive definite means eigenvalues are strictly bigger than zero and the eigenvalues of a diagonal matrix are its diagonal entries. Therefore Δx is uniquely determined.

We want to emphasize the fact that removing the second column had no impact on the reasoning as by assumption the first matrix equation had a solution and therefore we can retrieve the y's determined by $A^{\mathsf{T}}\Delta y + I\Delta s = c$. In other words the y's are dependent on s which is uniquely determined and not the other way around.

3 Central path conditions

Now let $\tau > 0$

$$\begin{aligned} \varphi_\tau(a, b) = 0 & \\ & \iff a + b - \sqrt{(a-b)^2 + 4\tau} = 0 \\ & \iff (a+b)^2 = (a-b)^2 + 4\tau \\ & \iff ab = \tau \end{aligned}$$

And the equivalence when squaring is guaranteed by the sign restrictions on a and b. Let's define Φ

$$\Phi_\tau(x, y, s) := (Ax - b) + (A^\mathsf{T} y + s - c) + \sum_{i=1}^{n} \varphi_\tau(x_i, s_i) \tag{6}$$

The first two terms are the usual feasibility conditions. Moreover since central path conditions require $\tau > 0$ and $x, s \geq 0$, we immediately have $x, s > 0$. Each $\varphi_\tau(x_i, s_i)$ in the sum is therefore 0 if and only if $x_i s_i = 0$ as proven in a). Moreover each term in the sum is nonnegative so the sum is 0 if and only if every single term is 0. The equation $\Phi_\tau(x, y, s) = 0$ is therefore another way to write the central path conditions and fully equivalent.

4 Termination of inner-point algorithm

Let x be feasible for the primal linear program:

$$\min c^\top x \quad \text{s.t.} \quad Ax = b \tag{7}$$

And let (y, s) be feasible for the dual linear program

$$\max b^\top y \quad \text{s.t.} \quad A^\top y + s = c \tag{8}$$

Such that

$$x^\top s \leq \epsilon \quad \text{for some } \epsilon > 0 \tag{9}$$

We want to show two claims which are essentially symmmetrical.

$$c^\top \bar{x} \leq c^\top x \leq c^\top \bar{x} + \epsilon \tag{10}$$

$$b^\top \bar{y} - \epsilon \leq b^\top y \leq b^\top \bar{y} \tag{11}$$

Now the first inequality in 10 and the second 11 are direct from weak duality. In both cases the actual meat of what we are trying to prove is :

$$c^\top (x - \bar{x}) \leq x^\top s \leq \epsilon \tag{12}$$

$$b^\top (\bar{y} - y) \leq x^\top s \leq \epsilon \tag{13}$$

But:

$$x^\top s = x^\top (c - A^\top y) \tag{14}$$

$$= c^\top x - (Ax)^\top y \tag{15}$$

$$= c^\top x - b^\top y \tag{16}$$

Hence

$$c^\top (x - \bar{x}) \leq x^\top s \iff c^\top x - c^\top \bar{x} \leq c^\top x - b^\top y \iff c^\top \bar{x} \geq b^\top y \tag{17}$$

$$b^\top (\bar{y} - y) \leq x^\top s \iff b^\top \bar{y} - b^\top y \leq c^\top x - b^\top y \iff c^\top x \geq b^\top \bar{y} \tag{18}$$

And so the second part of the claims is direct from weak duality.