Professor Kindred
Math 104 Graph Theory
Homework 7 Solutions
April 3, 2013

Introduction to Graph Theory, West
Section 5.1   20, variation of 25, 39
Section 5.2   9
Section 5.3   3, 8, 31
Section 7.1   22

Problems you should be able to do: 5.1.38, 5.1.47, 5.3.4(a)

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5.1.20 (!) Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

Proof 1:
Let $G = (V, E)$ be a graph whose odd cycles are pairwise intersecting. (If G has no odd cycle, then G is bipartite and hence 2-colorable, implying $\chi(G) \leq 2 \leq 5$; so we assume G has at least one odd cycle.) Let C be a *shortest* odd cycle in G. Let $G' = G - V(C)$.

Since all odd cycles are pairwise intersecting, when we remove $V(C)$ from G, we delete at least one vertex from every odd cycle of G. So it follows that G' has no odd cycles and hence is bipartite. Any bipartite graph is 2-colorable, so we have $\chi(G') \leq 2$.

In addition, since C is a shortest odd cycle, the graph induced by $V(C)$ does not contain any chords, i.e., does not contain any edges besides those in the cycle itself.[1] It follows then that $\chi(G[V(C)]) = \chi(C) = 3$. Thus, we can color G by coloring the vertices of G' with 2 colors and the vertices of C with 3 colors, distinct from the previous two used in G'. So

$$\chi(G) \leq \chi(G - V(C)) + \chi(G[V(C)]) = \chi(G') + \chi(C) \leq 2 + 3 = 5$$

Therefore, we have shown that $\chi(G) \leq 5$.

Proof 2:
We prove the contrapositive statement: If G is a graph with $\chi(G) \geq 6$, then there exist two odd cycles in G that are disjoint, i.e., that have no common vertex.

Suppose $\chi(G) \geq 6$, and consider an optimal coloring of G where the colors are labeled $1, 2, 3, \ldots, \chi(G)$. The subgraph induced by vertices colored 1, 2, and 3 must have an odd cycle C, else it would have been bipartite and we could replace these three colors by two. Similarly, the subgraph induced by vertices colored 4, 5, and 6 in the optimal coloring must also have an odd cycle C'. The two odd cycles C and C' are disjoint since no vertex of C has the same color as a vertex of C'. ∎

[1] In other words, the induced subgraph $G[V(C)]$ is isomorphic to an odd cycle.

Variation of problem 5.1.25 (+) Let G be the **unit-distance graph** in the plane[2]; $V(G) = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. (This is an infinite graph.)

(a) Prove that $\chi(G) \geq 3$.

(b) Prove that $\chi(G) \geq 4$. *(Hint: Find a finite graph H that is a subgraph of G and has chromatic number 4.)*

(c) Prove that $\chi(G) \leq 9$. *(Hint: Tile the plane with squares having side length 0.6.)*

(d) Prove that $\chi(G) \leq 7$. *(Hint: Use a different tiling of the plane, paying attention to the boundaries.)*

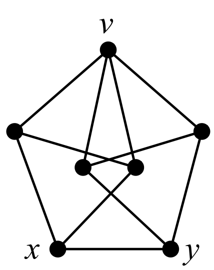
Remark: The *Hadwiger-Nelson problem* seeks the minimum number of colors needed to color the unit-distance graph in the plane. This problem is still open and currently all that can be said is that $4 \leq \chi(G) \leq 7$, which is what you will prove in parts (b) and (d) above.

Let $G = (V, E)$ be the unit-distance graph.

(a) Since an equilateral triangle may have side lengths all of 1, K_3 is a subgraph of G. Thus, $\chi(G) \geq \omega(G) \geq 3$.

(b) We seek a (finite) 4-chromatic subgraph of G to show that $\chi(G) \geq 4$.

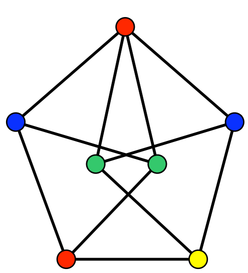
Consider the graph H known as the Moser spindle:



Notice that we can construct the graph at left as follows:

- adjoin 2 equilateral triangles along one side,
- duplicate this figure,
- join two identical figures at a single vertex v,
- fix the position of vertex v,
- rotate the figures until the bottom vtcs x and y are distance 1 apart.

Then all the edges in H are length 1, making H a subgraph of G.

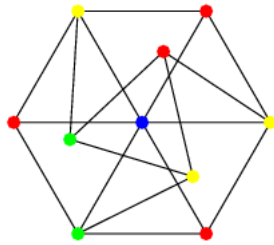


Given the coloring shown, we see that $\chi(H) \leq 4$. Also, since $\alpha(H) = 2$, we have that

$$\chi(H) \geq \frac{|V(H)|}{\alpha(H)} = \frac{7}{2} > 3,$$

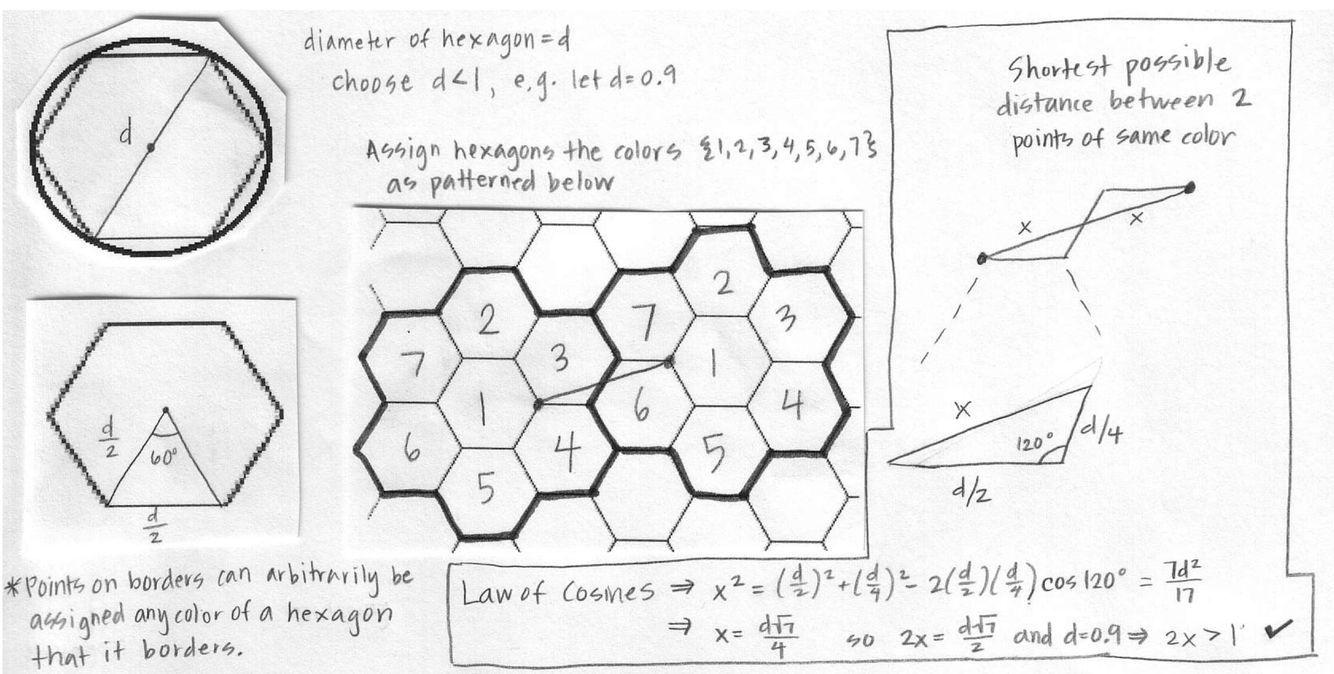
implying that $\chi(H) \geq 4$. Combining these two inequalities, we conclude that $\chi(H) = 4$. Since H is a subgraph of G, it follows that $\chi(G) \geq 4$.

Alternative: The Golomb graph is another 4-chromatic subgraph of G.



(c) To prove the upper bound, $\chi(G) \leq 9$, we construct a 9-coloring of G by tiling the plane using squares of side length 0.6. Note that each square of side length 0.6 has a diagonal length of $\sqrt{2(0.6)^2} < 1$, so no two points inside (or or on the border of) the square are distance 1 apart.

(d) We prove the upper bound, $\chi(G) \leq 7$, by exhibiting a proper 7-coloring of G. Since G is an infinite graph, we color the graph by regions, based on a tessellation of the plane using hexagons.



■

5.1.39 (!) Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \leq 1 + m\sqrt{m-1}$.

Suppose, for sake of contradiction, that $G = (V, E)$ is a k-chromatic graph with $|E| < \binom{k}{2}$. Let $S = \{1, 2, \ldots, k\}$ be the set of k colors in a minimum coloring of G. Then since $\binom{k}{2}$ is the number of pairs of distinct elements of S and $|E| < \binom{k}{2}$, there exists a pair of colors $i, j \in S$ such that no edge in E has endpoints with the colors i, j. In other words, no vertex of color i is adjacent to a vertex of color j. This suggests that we can replace all vertices of color j with color i to form a new proper coloring of G that uses only $(k-1)$ colors. $\Rightarrow\Leftarrow$. This contradicts the assumption that G is k-chromatic. Therefore, it is true that G has at least $\binom{k}{2}$ edges.

Assume $G = (V, E)$ is the union of m complete graphs of order m, where $m \geq 1$. (This means each of the complete graphs has $\binom{m}{2}$ edges.) Since the upper bound of edges in G occurs when the union of complete graphs is of disjoint sets of edges, we have $|E| \leq m \cdot \binom{m}{2}$. In addition, suppose $\chi(G) = k$. We know from our previous argument that $|E| \geq \binom{k}{2}$. We have

$$\frac{k(k-1)}{2} = \binom{k}{2} \leq |E| \qquad \text{and} \qquad |E| \leq m \cdot \binom{m}{2} = \frac{m^2(m-1)}{2}$$

$$\begin{aligned} k(k-1) \leq m^2(m-1) &\leq m^2(m-1) + m\sqrt{m-1} \\ &\leq (m\sqrt{m-1}+1)(m\sqrt{m-1}) \\ \Rightarrow \chi(G) = k &\leq m\sqrt{m-1}+1 \end{aligned}$$

Alternative Approach:

$$k(k-1) \leq m^2(m-1) \quad \Longrightarrow \quad k^2 - k - (m^2(m-1)) \leq 0$$

Using quadratic formula,

$$\begin{aligned} \frac{1-\sqrt{1+4m^2(m-1)}}{2} \leq k &\leq \frac{1+\sqrt{1+4m^2(m-1)}}{2} \\ k &\leq \frac{1}{2} + \sqrt{\frac{1}{4}(1+4m^2(m-1))} \\ &\leq \frac{1}{2} + \sqrt{\frac{1}{4} + m^2(m-1)} \\ &\leq \frac{1}{2} + \sqrt{\frac{1}{4}} + \sqrt{m^2(m-1))} \qquad (\star) \\ \Rightarrow \chi(G) = k &\leq 1 + m\sqrt{m-1} \end{aligned}$$

$(\star)$ Note: For nonnegative real numbers a, b, we have

$$\begin{aligned} a+b &\leq a + 2\sqrt{ab} + b \\ a+b &\leq (\sqrt{a}+\sqrt{b})^2 \\ \sqrt{a+b} &\leq \sqrt{a}+\sqrt{b} \end{aligned}$$

$\blacksquare$

5.2.9 (!) Prove that if G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.

Suppose G is k-critical, and let G' be the graph generated from G via Mycielski's construction. Since G is k-critical, we know that

$$\chi(G) = k, \quad \chi(G - e) \leq k - 1 \text{ for all } e \in E(G), \text{ and } \chi(G - v) \leq k - 1 \text{ for all } v \in V(G).$$

We claim that G' is $(k+1)$-critical. We know, from the proof of Mycielski's theorem, that $\chi(G') = k + 1$. Then, to show that G' is $(k+1)$-critical, it is sufficient to show that $\chi(G' - e) \leq k$ for all edges $e \in E(G')$. To do this, we consider three types of edges in G' (notation used below comes from description of Mycielski's construction in class).

Case 1: $e = wu_i$ for some i.

Since $\chi(G - v_i) \leq k - 1$, consider a coloring of $G - v_i$ using $k - 1$ colors $\{1, 2, \ldots, k-1\}$. Then extend this to a k-coloring of $G' - e$ in the following way:

- assign v_i color k,
- assign u_ℓ the color of v_ℓ for all ℓ,
- and assign w color k.

The only three vertices to have color k are v_i, u_i and w, but since no two of these are adjacent in $G' - e$, we have a proper k-coloring of $G' - e$.

Case 2: $e = v_iv_j$ for some $i \neq j$.

Since $\chi(G - e) \leq k - 1$, consider a coloring of $G - e$ using $k - 1$ colors $\{1, 2, \ldots, k-1\}$. Then extend this to a k-coloring of $G' - e$ in the following way:

- assign u_ℓ the color k for all ℓ,
- and assign w color 1.

Since the vertices in $\{u_\ell : 1 \leq \ell \leq |V(G)|\}$ form an independent set and are precisely the set of neighbors of w, we have a proper k-coloring of $G' - e$.

Case 3: $e = v_iu_j$ for some $i \neq j$.

If the edge $e = v_iu_j$ exists in G', then it must be that $v_i \sim v_j$ in G (based on definition of Mycielski construction). Since $\chi(G - v_iv_j) \leq k - 1$, consider a coloring of $G - v_iv_j$ using $k - 1$ colors $\{1, 2, \ldots, k-1\}$. Notice that v_i and v_j must have the same color in this coloring; otherwise, this coloring would be a proper coloring of G, implying that $\chi(G) \leq k - 1$, which is not the case.

Extend this coloring to a k-coloring of $G' - e$ in the following way:

- assign u_ℓ the color of v_ℓ for all ℓ,
- change color of v_j to color k,
- and assign w color k.

This is a proper k-coloring of $G' - e$.

In each case, we have that $\chi(G' - e) \leq k$ as desired. ∎

5.3.3 (-) Prove that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial of any graph.

Solution 1: If the polynomial $p(k) = k^4 - 4k^3 + 3k^2$ was a chromatic polynomial for some graph G, then when evaluated at a positive integer k, it would give the number of ways of coloring the graph G with k colors. Note that $p(2) = -4 < 0$, so it cannot count the proper 2-colorings in any graph, which implies that $p(k)$ cannot be a chromatic polynomial of any graph.

Solution 2: Alternatively, in $\chi(G;k)$, the degree is $|V(G)|$, and the second coefficient is $-|E(G)|$. Hence we seek a 4-vertex graph with four edges. The only such graphs are C_4 and the paw, seen in Figure 1, which have chromatic polynomials $k(k-1)(k^2-3k+3)$ and $k(k-1)^2(k-2)$, respectively. In particular, neither are equal to the given polynomial. ■

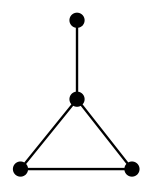


Figure 1: The paw graph.

5.3.8 (!) Prove that the number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k \geq 3$ and G is not a tree. What happens when $k = 2$?

If G is connected but not a tree, let T be a spanning tree of G, and choose $e \in E(G) - E(T)$ with $e = uv$. Every proper coloring of G must be a proper coloring of the subgraph T, and there are exactly $k(k-1)^{n-1}$ proper k-colorings of T by Proposition 5.3.3. It suffices to show that at least one of these is not a proper k-coloring of G. Since T is bipartite and $k \geq 3$, we can construct such a coloring by taking a 2-coloring of T and changing the endpoints of e, u and v, to a third color. While this is still a proper k-coloring of T (edge $e = uv$ does not exist in T), it is not a proper k-coloring of G since $u \sim v$ in G.

If $k = 2$, then T has exactly two proper 2-colorings, and these are both proper colorings of G if G is bipartite. (If G is not bipartite, then the given statement still holds when $k = 2$.) ■

5.3.31 The number $a(G)$ of acyclic orientations of G satisfies the recurrence

$$a(G) = a(G-e) + a(G \cdot e).$$

The number of spanning trees of G appears to satisfy the same recurrence; does the number of acyclic orientations of G always equal the number of spanning trees? Why or why not?

We know $\tau(G)$ satisfies the recurrence $\tau(G) = \tau(G-e) + \tau(G \cdot e)$. This is the recurrence satisfied by $a(G)$, but *the initial conditions are different*.

- An edgeless graph has one acyclic orientation (the empty one), but it has no spanning tree unless it has only one vertex.

- A connected graph containing a loop has spanning trees but no acyclic orientation.
- A tree of order n has one spanning tree and $2n-1$ acyclic orientations.
- The complete graph K_n has n^{n-2} spanning trees and $n!$ acyclic orientations; $n^{n-2} > n!$ if $n \geq 6$.

■

7.1.22 Use Brooks' Theorem on an appropriate graph to prove that if G is a simple graph with $\Delta(G) = 3$, then G is 4-edge-colorable. (Comment: The result is a special case of Vizing's Theorem; do not use Vizing's Theorem to prove this.)

Suppose G is a simple graph with $\Delta(G) = 3$. Assume G is connected. (If G is not connected, then repeat the subsequent argument for each component of G.)

Consider the line graph $L(G)$. Since G is connected, $L(G)$ is connected, and $\Delta(G) = 3$ implies that an edge of G is incident to at most 2 other edges at each of its endpoints, so $\Delta(L(G)) \leq 4$.

Note that if $L(G)$ is a 4-regular graph, then it must be that G is a 3-regular (simple) graph. This implies that G has at least 6 edges and so $L(G)$ has at least 6 vertices, so $L(G)$ is not K_5.

We consider three cases.
Case 1: $L(G)$ is an odd cycle. Then $\chi(L(G)) = 3$.

Case 2: $L(G)$ is a complete graph. Then $L(G)$ has at most 4 vertices, and so $\chi(L(G)) \leq 4$.

Case 3: $L(G)$ is neither an odd cycle nor a complete graph. By Brooks' Theorem, it follows that $\chi(L(G)) \leq \Delta(L(G)) \leq 4$.

Finally we have
$$\chi'(G) = \chi(L(G)) \leq 4$$
so G is 4-edge-colorable. ■