

Introduction to Graph Theory, West
Section 3.3 10, 16, 22
Section 4.1 9, 25
Section 4.3 2

Problems you should be able to do: 3.3.6, 4.1.1, 4.1.14

3.3.10 For every graph G , prove that $\beta(G) \leq 2\alpha'(G)$. For each $k \in \mathbb{N}$, construct a simple graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$. (Note that this shows that the aforementioned bound is best possible for general graphs.)

If G has a maximum matching of size k , then the $2k$ endpoints of these edges form a set of vertices covering all edges, as any uncovered edge could be added to the matching to obtain a larger one. Hence, we have a vertex cover of size $2k$, so $\beta(G) \leq 2k = 2\alpha'(G)$.

A graph G that is the disjoint union of k copies of K_3 has $\alpha'(G) = k\alpha'(K_3) = k$ and $\beta(G) = k\beta(K_3) = 2k$, so the inequality is best possible.

Note: These values also hold for the complete graph K_{2k+1} , since, in general, for any positive integer n , $\alpha'(K_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta(K_n) = n - 1$. More generally, every disjoint union of complete graphs of odd order satisfies $\beta(G) = 2\alpha'(G)$. ■

3.3.16 Let G be a k -regular graph of even order that remains connected when any $k - 2$ edges are deleted. Prove that G has a perfect matching.

Suppose G is a k -regular graph with an even number of vertices that remains connected when any $k - 2$ edges are deleted. By Tutte's theorem, it suffices to show that $o(G - S) \leq |S|$ for any $S \subseteq V(G)$. Since $|V(G)|$ is even, this condition holds when $S = \emptyset$, so assume $S \neq \emptyset$. Let H be an odd component of $G - S$, and let m be the number of edges joining H to S . We know that, by the degree-sum formula,

$$\underbrace{2|E(H)|}_{\text{even}} = \sum_{v \in V(H)} d_H(v) = k \underbrace{|V(H)|}_{\text{odd}} - m.$$

Thus, either k and m are both even or they are both odd (i.e., same parity).

Since removing $k - 2$ edges does not disconnect the graph, we also know that $m \geq k - 1$. But since m and k have the same parity, we, in fact, know that $m \geq k$. Summing over all

odd components of $G - S$ yields at least $ko(G - S)$ edges between S and $V(G) - S$. So we have

$$k|S| = \sum_{v \in S} d_G(v) \geq \# \text{ of edges between } S \text{ and } V(G) - S \geq ko(G - S).$$

Dividing by k , we have $|S| \geq o(G - S)$, so we have satisfied Tutte's condition, and therefore, by Tutte's theorem, G has a perfect matching. ■

3.3.22 Let G be an X, Y -bipartite graph. Let H be the graph obtained from G by adding one vertex to Y if $|V(G)|$ is odd and then adding edges to make Y a clique.

- (a) Prove that G has a matching of size $|X|$ if and only if H has a perfect matching.
- (b) Prove that if G satisfies Hall's Condition ($|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's condition ($o(H - T) \leq |T|$ for all $T \subseteq V(H)$).
- (c) Use parts (a) and (b) to obtain Hall's Theorem from Tutte's Theorem.

Given an X, Y -bipartite graph G on n vertices, let H be the graph obtained from G by adding one vertex to Y if n is odd and then adding edges to make Y into a clique.

- (a) Suppose G has a matching of size $|X|$. Each edge of this matching has as endpoints one vertex of X and one vertex of Y . Since the vertices of $H - X$ are a clique, we can pair the remaining unmatched vertices of $H - X$ arbitrarily to obtain a perfect matching in H . Note that the number of unmatched vertices in $H - X$ is even because
 - (i) if X, Y have same parity, then n is even and there are $|Y| - |X|$ unmatched vertices in $H - X$, or if
 - (ii) if X, Y have different parity, then n is odd and a vertex was added to Y to form H , so there are $|Y| + 1 - |X|$ unmatched vertices.

Conversely, if H has a perfect matching, it must use $|X|$ edges to saturate X , since X is an independent set. These edges form the desired matching of G .

- (b) Assume G satisfies Hall's Condition (namely $|N(S)| \geq |S|$ for all $S \subseteq X$). Let $T \subseteq V(H)$. Then, in $H - T$,
 - vertices of $X - T$ who had all their neighbors in $Y \cap T$ are isolated vertices,
 - and vertices of $X - T$ with neighbors in $Y - T$ together with vertices of $Y - T$ form a component.

Let $S = \{x \in X : N(x) \subseteq Y \cap T\}$. It follows that

$$\begin{array}{ccccccc} \text{by above} & & \text{by Hall's} & & \text{by def'n of } S & & \\ \text{observations} & & \text{condition} & & & & \\ \downarrow & & \downarrow & & \downarrow & & \\ o(H - T) & \leq & 1 + |S| & \leq & 1 + |N(S)| & \leq & 1 + |Y \cap T| \leq 1 + |T|. \end{array}$$

Now, as explained in Remark 3.3.4 of text, we know that $o(H - T) + |T|$ has the same parity as n , which is even. Thus, $o(H - T)$ and $|T|$ have the same parity. Combining this fact with the above conclusion that $o(H - T) \leq 1 + |T|$, it follows that $o(H - T) \leq |T|$. Therefore, H satisfies Tutte's Condition.

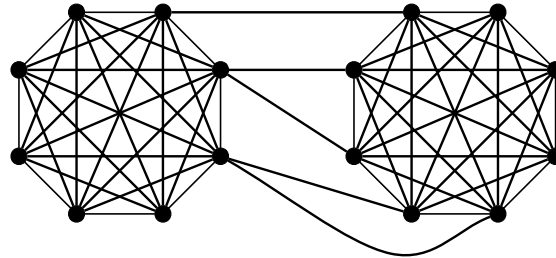
- (c) Let G be a bipartite graph with bipartition X, Y . If G has a matching saturating X , then certainly any subset of X must have as many neighbors as elements in order to be saturated by the matching. So G must satisfy Hall's condition.

Conversely, assume G satisfies Hall's condition. Then form H from G as described previously. By part (b), H must satisfy Tutte's condition. So, by Tutte's Theorem, H has a perfect matching M . Now, by part (a), it follows that G has a matching of size $|X|$, i.e., a matching that saturates X .

■

4.1.9 For each choice of integers k, l, m with $0 < k \leq l \leq m$, construct a simple graph with $\kappa(G) = k$, $\kappa'(G) = l$, and $\delta(G) = m$.

Begin with two disjoint copies of K_{m+1} . Pick k vertices from one clique and l vertices from the other. Add l edges between them in such a way that each of the selected vertices is the endpoint of at least one of these l added edges. (So each of the l selected vertices from one clique has exactly one of the l edges incident to it.) Call the resulting connected graph G . An example of this construction is shown below for $k = 3, l = 5$, and $m = 7$.



$$k = 3, l = 5, m = 7$$

- *Claim:* $\delta(G) = m$. Since there are $m + 1$ vertices in each clique and at most m of them have edges added to them, there is a vertex whose degree is m after edges are added. All vertices had degree m before additional edges were added, so their degrees are bounded below by m .
- *Claim:* $\kappa'(G) = l$. Removing the l edges we added to the two cliques disconnects the graph, so $\kappa'(G) \leq l$. No smaller set of edges can be deleted to disconnect the graph since the edge-connectivity of each of the cliques is $m \geq l$.
- *Claim:* $\kappa(G) = k$. Removing the k vertices we originally selected from one of the two cliques disconnects the graph, so $\kappa(G) \leq k$. No smaller set of vertices can be deleted to disconnect the graph since

- the connectivity of each of the cliques is $m \geq k$,
- $l \geq k$ (removing the l vertices selected from one of the cliques disconnects the graph),
- and every one of the k designated vertices has at least one edge incident to it that connects the two cliques.

■

4.1.25 Let G be a simple graph with diameter 2, and let $[S, \bar{S}]$ be a minimum edge cut with $|S| \leq |\bar{S}|$.

- Prove that every vertex of S has a neighbor in \bar{S} .
- Use part (a) and Corollary 4.1.13 to prove that $\kappa'(G) = \delta(G)$.

Suppose throughout that G is a graph on n vertices with diameter 2, and assume $[S, \bar{S}]$ is a minimum edge cut with $|S| \leq |\bar{S}|$.

- (a) By way of contradiction, suppose there exists a vertex $u \in S$ such that $u \not\sim v$ for all $v \in \bar{S}$. Then, since $\text{diam}(G) = 2$, for any $v \in \bar{S}$, there must exist $w \in S$ such that $u \sim w \sim v$. So every vertex of \bar{S} has a neighbor in S . This means

$$|[S, \bar{S}]| \geq |\bar{S}| \geq \frac{n}{2}.$$

However, since u has no neighbors in \bar{S} , we have $d(u) \leq |S| - 1 < \frac{n}{2}$, and now

$$\delta(G) \leq d(u) < \frac{n}{2} \leq |[S, \bar{S}]| = \kappa'(G).$$

$\Rightarrow \Leftarrow$ This contradicts the fact that $\kappa'(G) \leq \delta(G)$ (Whitney's inequality).

- (b) The contrapositive of Corollary 4.1.13 states the following:

If G is a simple graph and $|S| \leq \delta(G)$ for some nonempty proper subset S of $V(G)$, then $|[S, \bar{S}]| \geq \delta(G)$.

By part (a), we have

$$|S| \leq |[S, \bar{S}]| = \kappa'(G) \leq \delta(G).$$

Therefore, applying the contrapositive of Corollary 4.1.13, we have $|[S, \bar{S}]| \geq \delta(G)$. Hence, we have

$$\delta(G) \leq |[S, \bar{S}]| = \kappa'(G) \leq \delta(G).$$

It follows that $\kappa'(G) = \delta(G)$.

■

4.3.2 In the network below, find a maximum flow from s to t . Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.

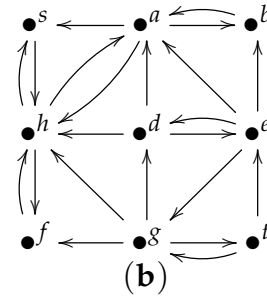
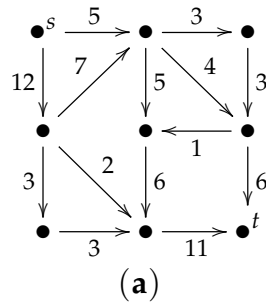
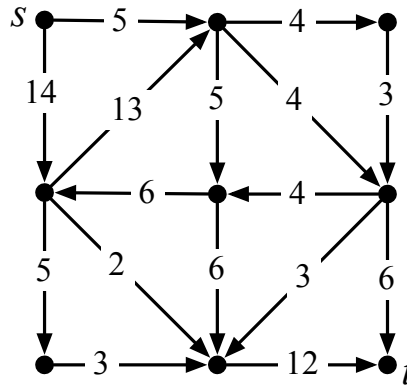


Figure 1: (a) Optimal flow X^* w/flow value = 17 (b) Residual network of optimal flow

Using the Ford-Fulkerson algorithm, we obtain a maximum flow that has value 17. One such resulting max flow is given in Figure 1(a), although there are other possible solutions with slight variations in the flow. However, all have flow value 17.

Examining the residual network of the max flow (disregarding capacities on the arcs) as in Figure 1(b), we determine which vertices are reachable by a walk from the source vertex s :

$$S = \text{vertices reachable by walk from } s = \{s, h, a, b, f\}.$$

We now calculate the capacity of the $s - t$ cut $[S, \bar{S}]$:

$$\text{cap}[S, \bar{S}] = c(a, e) + c(a, d) + c(b, e) + c(h, g) + c(f, g) = 4 + 5 + 3 + 2 + 3 = 17.$$

Since we have exhibited an $s - t$ cut with capacity equal to the flow value, we know that both are optimal by strong duality. This means the $s - t$ cut is a minimum $s - t$ cut and the flow is a maximum flow. ■