MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #4: Ramsey theory, Matchings, Colorings

Due Wednesday, November 16th, 2016, 14:30

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1. Recall that $R(k,\ell)$ is the minimum integer n such that every red/blue coloring of $E(K_n)$ contains a red K_k or blue K_ℓ.

a) Construct a red/blue coloring of $E(K_8)$ such that the coloring contains neither red K_3 nor blue K_4.

b) Prove that $R(3,4) = 9$.

c) Show that $R(4,4) \leq 18$.

[*Note that there exists a coloring of* $E(K_{17})$ *coming from number theory that has no monochromatic* K_4.]

2. Recall that $R_k(3) := R_k(\overbrace{3,3,\ldots,3}^{k})$ is the minimum integer n such that any k-coloring of $E(K_n)$ contains a monchromatic K_3.

Prove that $R_k(3) \leq 3k!$ for any integer $k \geq 1$.

3. Let G be a 3-regular simple graph with no cut-edge, and let $e \in E(G)$ be an edge of G.

a) Show that G contains a perfect matching M_1 such that $e \in M_1$.

b) Show that G contains a perfect matching M_2 such that $e \notin M_2$.

4. Recall that for a simple graph G, the chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of G so that for every edge e the endpoints of e receive two different colors.

Let G be a simple graph such that any two odd cycles C_1 and C_2 in G it holds that $V(C_1) \cap V(C_2) \neq \emptyset$. Prove that $\chi(G) \leq 5$.

5. A simple graph $G = (V,E)$ is called *triangle-free* if no 3-vertex subgraph of G is isomorphic to K_3.

Let G be a triangle-free simple graph with n vertices. Show that G contains an independent set of size $\lfloor \sqrt{n} \rfloor$. Deduce that $R(3,\ell) \leq \ell^2$.