Problem Set 3

MATH 776, Fall 2009, Mohr

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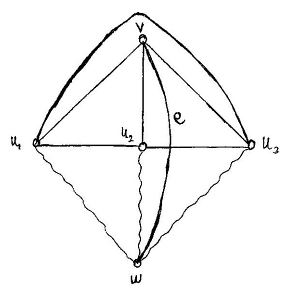
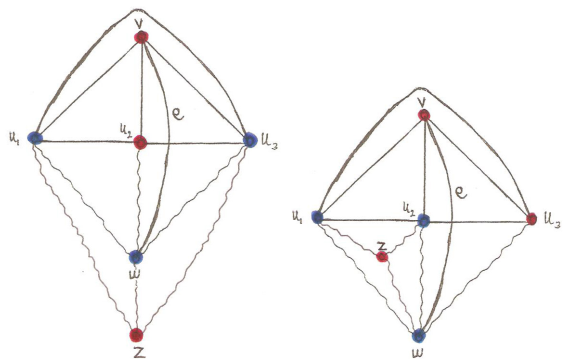
1 Problem 2

Proposition 1.1. *Adding a new edge to a maximal planar graph of order at least 6 always produces both a TK^5 and a $TK_{3,3}$ subgraph.*

Proof. Let G be a maximal planar graph of order at least 6. We have immediately that G is 3-connected. Before adding the new edge e to G, consider the vertices v and w which are to be its endpoints. As G is 3-connected, there are 3 disjoint vw-paths. Since v is not adjacent to w (else we could add no edge between them), we can find vertices u_1, u_2, and u_3 lying one on each path that are neighbors of v but not of w. By the edge-maximality of G, u_1, u_2, and u_3 induce a cycle. Finally, adding e gives the TK^5 (see figure - solid lines represent true edges, while wiggly lines represent topological edges).



To show the existence of a $TK_{3,3}$, repeat the same construction up to the point where we add e. Since G has order at least 6, there is another vertex z distinct from those previously mentioned. The construction allows for two cases: either z lies outside the region bounded by the topological cycle vu_1wu_2v or it lies inside one of the faces of this region (they are all equivalent). In either case, the edge-maximality of G gives the existence of new topological edges e_1, e_2, and e_3 that ensure a $TK_{3,3}$ (see figures).



□

2 Problem 3

Definition 2.1. *A graph is called* <u>*outerplanar*</u> *if it has a drawing in which every vertex lies on the boundary of the outer face.*

Proposition 2.2. *A graph is outerplanar if and only if it contains neither* K^4 *nor* $K_{2,3}$ *as a minor.*

Proof. ($\Rightarrow$) Let G be outerplanar. Define the graph G' by placing a new vertex v in the unbounded face of G and connecting v to every vertex of G. Since G is outerplanar, these edges can be added without inducing a crossing, and so G' is planar. By Kuratowski's Theorem, G' contains no K^5 nor $K_{3,3}$ minor. Hence, it must be that G contains no K^4 nor $K_{2,3}$ minor (else adding the vertex v and the corresponding edges would result in a K^5 or $K_{3,3}$ minor in G').
($\Leftarrow$) Let G contain no K^4 nor $K_{2,3}$ minor. Construct the graph G' as before. Evidently, G' contains no K^5 nor $K_{3,3}$ (else G would contain a K^4 or $K_{2,3}$ minor), and so G' is planar. A priori, we do not know where v is located in a planar drawing of G'. To address this, project G' to the sphere S^2. Using an appropriate homeomorphism from S^2 to S^2 and then projecting back to the plane, we can obtain a planar drawing of G' in which v lies in the unbounded face of G. The fact that v can reach every vertex of G without inducing a crossing implies that every vertex of G is on the boundary of the unbounded face. That is, G is outerplanar. □

3 Problem 4

Proposition 3.1. *A 2-connected plane graph is bipartite if and only if every face is bounded by an even cycle.*

Proof. ($\Leftarrow$) In a 2-connected plane graph, every face is bounded by a cycle. As the graph is bipartite, it contains no odd cycle, and so every face must be bounded by an even cycle.
($\Rightarrow$) In a 2-connected plane graph G, the facial cycles generate the entire cycle space. Hence, C can be written as the sum of even facial cycles. Now, any edge lies on

exactly two faces. Furthermore, an edge in the sum is canceled (i.e. does not appear in C) if and only if it appears in exactly two of the facial cycles generating G. Hence, the cancellation of edges does not change the parity of the number of edges in the sum, and so we conclude that C is even. Therefore, every cycle of G is even, and so G is bipartite. □

4 Problem 5

Proposition 4.1. *The set of critically 3-chromatic graphs is precisely the set of odd cycles.*

Proof. ($\supseteq$) Let G be an odd cycle on $2n+1$ vertices. We have immediately that G is non-bipartite, and so $\chi(G) \geq 3$. Observe that, for all $v \in V(G)$, $G - v$ is a path, and so is bipartite. In other words, $G - v$ is 2-chromatic. Hence, G is 3-chromatic (take any 2-coloring of $G - v$ and use a third color for v), and so critically 3-chromatic, as v was chosen arbitrarily.
($\subseteq$) Let G be critically 3-chromatic. Since G is 3-chromatic, it is non-bipartite, and so contains an odd cycle. Consider the cycle C in G of smallest size. Observe that there is no chord of C in G, as this would yeild a smaller odd cycle.
Suppose now, for the purpose of contradiction, that C is a proper subgraph of G. It follows that there is some vertex $v \in V(G)$ that does not lie on C. Hence, $C \subseteq G - v$. Since $G - v$ contains an odd cycle, it is non-bipartite, and so $\chi(G - v) = 3$, which is contrary to the fact that G is critically 3-chromatic. Therefore, it must be that $G = C$. That is, G is an odd cycle. □

5 Problem 6

Proposition 5.1. *Given $k \in \mathbb{N}$, there is a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leq k$ contains a cycle of length at least $c_k|G|$.*

Proof. We know that $|G| \leq \alpha(G)\chi(G)$. It follows that

$$\begin{aligned} \frac{|G|}{\alpha(G)} &\leq \chi(G) \\ &\leq \max\{\delta(H) \mid H \subseteq G\} + 1. \end{aligned}$$

Let H_0 be the subgraph witnessing $\max\{\delta(H) \mid H \subseteq G\}$ (that is, $\max\{\delta(H) \mid H \subseteq G\} = \delta(H_0)$).
Now, if $\delta(H_0) \geq 2$, we are guaranteed to find a cycle of length at least $\delta(H_0) + 1$ in H_0 (and so also in G). Hence, we can take c_k to be $\frac{1}{k}$ to obtain

$$\begin{aligned} \frac{|G|}{k} &\leq \frac{|G|}{\alpha(G)} && \text{(since } \alpha(G) \leq k\text{)} \\ &\leq \delta(H_0) + 1, \end{aligned}$$

as desired.
It remains to show that, given $k \geq 1$, every sufficiently large graph with $\alpha(G) \leq k$ contains a subgraph with minimum degree at least 2. Arguing by contrapositive, suppose that every subgraph of G has minimum degree at most 1. It must be that G is acyclic, and so G is a forest. As every forest is bipartite, we have that $\alpha(G) \geq \left\lceil \frac{|G|}{2} \right\rceil$ (one of the partite sets accounts for at least half of the vertices of G). By making G arbitrarily large, we can force $\alpha(G) > k$ for any k. Having established the contrapositive, we conclude that every sufficiently large graph with $\alpha(G) \leq k$ contains a subgraph with minimum degree at least 2, thus completing the proof. $\square$

6 Problem 7

Proposition 6.1. *For every $k \in \mathbb{N}$, there is a triangle-free k-chromatic graph.*

Proof. (by strong induction on k) For $|G| = 1$, the claim holds trivially. Assume inductively that there are triangle-free graphs $G_1, \ldots, G_{k-1}$ with $\chi(G_i) = i$ for $1 \leq i \leq k-1$. Let W denote the vertex set $V(G_1) \times \cdots \times V(G_{k-1})$ (the ordinary Cartesian product of the sets $V(G_i)$). Now, construct the graph $G = (V, E)$ with

$$\begin{aligned} V &= V(G_1) \cup \cdots \cup V(G_{k-1}) \cup W \\ E &= \{xy \mid xy \in G_i \text{ for some } i \text{ or } x \in W \text{ and } y = x_i \text{ for some } i\} \end{aligned}$$

where x_i denotes the i^{th} coordinate of x as a tuple. Observe that there are no edges between vertices of W nor between graphs G_i and G_j for $i \neq j$. We claim that G_k is triangle-free and k-chromatic, thus establishing the proposition.
Evidently, G_k is triangle-free, since, for $1 \leq i \leq k-1$, G_i is triangle-free and vertices of W have exactly one neighbor in each of the (disjoint) G_i.
It is clear that G_k is k-colorable, since we can inductively color the vertices of the G_i for $1 \leq i \leq k-1$ using at most $k-1$ colors. Since no edge lies between vertices of W, we can color all of W using a new color k. To see that fewer colors will not suffice, choose, for $1 \leq i \leq k-1$, a vertex $v_i \in V(G_i)$ such that v_i is colored differently than v_j for all $j < i$. Thus, $k-1$ distinct colors are represented in the collection of v_i. Now, the vertex $(v_1, \ldots, v_{k-1}) \in W$ is adjacent to each of the v_i, and so must be colored differently than each of them. Hence, G_k requires at least k colors, and so is k-chromatic. $\square$

7 Problem 8

Let $G = (V, E)$ and $H = (V', E')$ be graphs.

Definition 7.1. *The <u>Cartesian/direct product</u> $G \square H$ is the graph on $V \times V'$ with*

$$(a, b) \sim (c, d) \Leftrightarrow ((a = c) \wedge (b \sim d)) \vee ((b = d) \wedge (a \sim c)).$$

Definition 7.2. *The* *categorical/tensor product* $G \times H$ *is the graph on* $V \times V'$ *with*

$$(a, b) \sim (c, d) \Leftrightarrow (a \sim c) \wedge (b \sim d).$$

Definition 7.3. *The* *strong/normal product* $G \boxtimes H$ *is the graph on* $V \times V'$ *with*

$$(a, b) \sim (c, d) \Leftrightarrow ((a = c) \vee (a \sim c)) \wedge ((b = d) \vee (b \sim d)).$$

Definition 7.4. *The* *lexicographic/replacement product* $G[H]$ *is the graph on* $V \times V'$ *with*

$$(a, b) \sim (c, d) \Leftrightarrow (a \sim c) \vee ((a = c) \wedge (b \sim d)).$$

Proposition 7.5. $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$

Proof. Let $k = \max\{\chi(G), \chi(H)\}$. Observe first that $G \square H$ contains a copy of G (fix $b = d$ in $V(H)$ and let a and c run through $V(G)$). Similarly, $G \square H$ contains a copy of H. Hence, $\chi(G \square H) \geq k$. It remains to show that $G \square H$ is indeed k-colorable. To that end, let $g : V(G) \to \{0, \ldots, \chi(G) - 1\}$ define a proper coloring of G. Similarly, let $h : V(H) \to \{0, \ldots, \chi(H) - 1\}$ define a proper coloring of H. Define the coloring

$$f : V(G) \times V(H) \to \{0, \ldots, k - 1\}$$
$$f(u, v) = g(u) + h(v) \pmod{\mathrm{k}}.$$

Evidently, f makes use of k colors. To see that f indeed defines a proper k-coloring on $G \square H$, let $(u, v) \sim (u', v')$ in $G \square H$. The definition of adjacency in the Cartesian product allows for two cases. In one case, $u = u'$ and $v \sim v'$, which implies that $g(u) = g(u')$ and $h(v) \neq h(v')$. In the other case, $v = v'$ and $u \sim u'$, which implies that $h(v) = h(v')$ and $g(u) \neq g(u')$. In either case, we see that $g(u) + h(v) \neq g(u') + h(v')$. That is, $f(u, v) \neq f(u', v')$, and so f is indeed a proper k-coloring of $G \square H$. □

Proposition 7.6. $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$

Proof. The symmetry of the tensor product allows us to assume, without loss of generality, that $\chi(G) \leq \chi(H)$. Let g define a proper $\chi(G)$-coloring on G. Now, define the coloring f on $G \times H$ by $f(u, v) = g(u)$. Evidently, f uses $\chi(G)$ colors. To see that f indeed defines a proper coloring on $G \times H$, let $(u, v) \sim (u', v')$ in $G \times H$. It follows that

$$\begin{aligned}(u, v) \sim (u', v') &\Rightarrow u \sim u' \\ &\Rightarrow g(u) \neq g(u') \\ &\Rightarrow f(u, v) \neq f(u', v').\end{aligned}$$

Hence, $\chi(G \times H) \leq \chi(G)$. That is, $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. □

Proposition 7.7. $\max\{\chi(G), \chi(H)\} \leq \chi(G \boxtimes H) \leq \chi(G[H]) \leq \chi(G)\chi(H)$

Proof. For the first inequality, observe that $G \boxtimes H$ contains a copy of G (fix $b = d$ in $V(H)$ and let a and c run through $V(G)$). Similarly, $G \boxtimes H$ contains a copy of H. Hence, $\chi(G \boxtimes H) \geq \max\{\chi(G), \chi(H)\}$.
For the second inequality, observe that $G \boxtimes H \subseteq G[H]$ since

$$\begin{aligned}
& ((a = c) \vee (a \sim c)) \wedge ((b = d) \vee (b \sim d)) \\
\Leftrightarrow \quad & ((a = c) \wedge ((b = d) \vee (b \sim d))) \vee ((a \sim c) \wedge ((b = d) \vee (b \sim d))) \\
\Rightarrow \quad & ((a = c) \wedge ((b = d) \vee (b \sim d))) \vee (a \sim c)) \\
\Leftrightarrow \quad & ((a = c) \wedge (b = d)) \vee ((a = c) \wedge (b \sim d)) \vee (a \sim c) \\
\Leftrightarrow \quad & ((a = c) \wedge (b \sim d)) \vee (a \sim c) \qquad \text{(since loops are not allowed).}
\end{aligned}$$

Hence, $\chi(G \boxtimes H) \leq \chi(G[H])$.
For the third inequality, let g define a $\chi(G)$-coloring on G and h define a $\chi(H)$-coloring on H. Define the coloring f on $G[H]$ by $f(u, v) = (g(u), h(v))$. Evidently, f uses $\chi(G)\chi(H)$ colors. To see that f is indeed a proper coloring of $G[H]$, let $(u, v) \sim (u', v')$ in $G[H]$. It follows that

$$\begin{aligned}
(u, v) \sim (u', v') &\Leftrightarrow (u \sim u') \vee ((u = u') \wedge (v \sim v')) \\
&\Rightarrow (u \sim u') \vee (v \sim v') \\
&\Rightarrow (g(u) \neq g(u')) \vee (h(v) \neq h(v')) \\
&\Leftrightarrow (g(u), h(v)) \neq (g(u'), h(v')) \\
&\Leftrightarrow f(u, v) \neq f(u', v').
\end{aligned}$$

Hence, $\chi(G[H]) \leq \chi(G)\chi(H)$. □

8 Problem 9

Definition 8.1. *The* <u>*thickness*</u> $\Theta(G)$ *of a graph* G *is the minimum* k *so that* $G = \bigcup_{i=1}^{k} G_i$ *for some planar graphs* G_i, $1 \leq i \leq k$.

Proposition 8.2. *Any graph* G *with* $||G|| > 0$ *satisfies*

$$\left\lceil \frac{\delta(G)}{6} \right\rceil \leq \Theta(G) \leq \left\lceil \frac{\Delta G}{2} \right\rceil.$$

Proof. To establish the lower bound, observe that for some i, G_i accounts for a fraction of at least $\frac{1}{\Theta(G)}$ of the edges of G. It follows that,

$$\begin{aligned}
\Theta(G)||G_i|| &\geq ||G|| \\
&\geq \frac{1}{2}|G|\delta(G) \qquad \text{(by the Handshaking Lemma)} \\
&\geq \frac{1}{2}|G_i|\delta(G).
\end{aligned}$$

Hence, $\Theta(G) \geq \frac{|G_i|\delta(G)}{2||G_i||}$. In fact, $\Theta(G) \geq \left\lceil \frac{|G_i|\delta(G)}{2||G_i||} \right\rceil$, since $\Theta(G)$ must be an integer. Now, as any plane graph on n vertices has at most $3n - 6$ edges, we have that $||G_i|| \leq 3|G_i| - 6$, and so

$$\begin{aligned}\Theta(G) &\geq \left\lceil \frac{|G_i|\delta(G)}{2||G_i||} \right\rceil \\ &\geq \left\lceil \frac{|G_i|\delta(G)}{2(3|G_i| - 6)} \right\rceil \\ &\geq \left\lceil \frac{|G_i|\delta(G)}{6|G_i|} \right\rceil \\ &\geq \left\lceil \frac{\delta(G)}{6} \right\rceil,\end{aligned}$$

as desired.

To establish the upper bound, suppose that G is regular of even degree. A result of Petersen shows that G possesses a 2-factor. As a 2-factor is planar, we can take this 2-factor to be G_1 (i.e. the first graph contributing toward the thickness of G). Next, remove the edges of this 2-factor from G. By definition of 2-factor, this removal decreases the degree of every vertex of G by exactly 2. Hence, G minus these edges is still regular of even degree, and so possesses another 2-factor which we will take to be G_2. After $\frac{\Delta(G)}{2}$ iterations, the remaining graph will be regular of degree 0. Hence, we can represent G as the union of $\frac{\Delta(G)}{2}$ (planar) 2-factors, and so $\Theta(G) = \frac{\Delta(G)}{2}$ when G is regular of even degree.

It remains to show that, given any graph G, we can extend it to a regular graph of even degree without lowering the thickness. To begin, take a second copy of G (call it G') and add an edge between corresponding vertices of G and G' if the common degree of these vertices is less than $2\left\lceil \frac{\Delta(G)}{2} \right\rceil$. Note that the addition of these vertices and edges can only raise the thickness (if a set of planar graphs can cover this larger graph, then they can surely cover the original G). Call this newly constructed graph H. Now, if H is still not regular of degee $2\left\lceil \frac{\Delta(G)}{2} \right\rceil$, take another copy of H and add edges as before. Continuing in this way, we construct a regular graph of degree $2\left\lceil \frac{\Delta(G)}{2} \right\rceil$ (call it $\tilde{G}$). Combining our previous observations, we have that

$$\begin{aligned}\Theta(G) &\leq \Theta(\tilde{G}) \\ &= \frac{1}{2} \cdot 2\left\lceil \frac{\Delta(G)}{2} \right\rceil \\ &= \left\lceil \frac{\Delta(G)}{2} \right\rceil,\end{aligned}$$

as desired. $\square$

9 Problem 10

Definition 9.1. *A chordal graph is a graph with the property that any cycle longer than a C_3 induces a chord.*

Definition 9.2. *A near-triangulation is a connected graph with a planar embedding so that all inner faces are triangles.*

Proposition 9.3. *A finite graph is chordal and outerplanar if and only if it is a disjoint union of K^4-minor free near-triangulations.*

Proof. ($\Rightarrow$) Let G be chordal and outerplanar. Since G is outerplanar, we immediately have that G is K^4-minor free by 2.2. Since G is chordal, we see that G cannot contain a facial cycle larger than C_3 (anything larger induces a chord). Hence, every inner face of G is a triangle. That is, G is near-triangular.
($\Leftarrow$) Let G be a finite K^4-minor free near-triangulation. We first claim that G contains no $K_{2,3}$ minor. Suppose, for the purpose of contradiction, that this is not the case. A planar drawing of $K_{2,3}$ contains two facial cycles of size 4. Since G is near-triangular, it must be that these faces are subdivided into triangles in its $K_{2,3}$ minor. This subdivision yields a TK^4, which implies that G contains a K^4 minor - a contradiction. Now, as G contains no K^4 nor $K_{2,3}$ minor, we have that G is outerplanar by 2.2.
We show next that, if G is outerplanar and near-triangular, then it is chordal. To that end, let C be any cycle in G larger than C_3. Since G is near-triangular, the region bounded by C must be subdivided into triangles. Since G is outerplanar, there can be no vertex on the interior of this region, and so the subdivision is accomplished using chords. Hence, G is chordal. $\square$