**MATH 350: Graph Theory and Combinatorics. Fall 2016.**
**Assignment #4: Ramsey theory, Matchings, Colorings**

Due Wednesday, November 16th, 2016, 14:30

---

**1.** Recall that $R(k, \ell)$ is the minimum integer $n$ such that every red/blue coloring of $E(K_n)$ contains a red $K_k$ or blue $K_\ell$.

**a)** Construct a red/blue coloring of $E(K_8)$ such that the coloring contains neither red $K_3$ nor blue $K_4$.

**Solution:** Consider the following red/blue coloring of $E(K_8)$ where only red edges are drawn (i.e., the non-edges in the figure are the blue edges in the coloring):

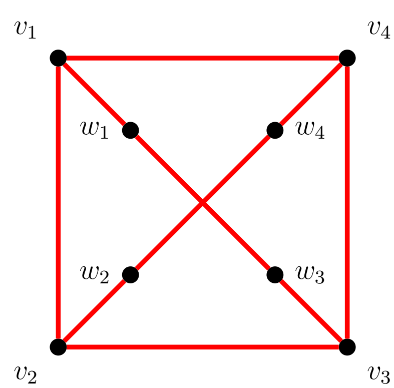


Figure 1: The red subgraph of a red/blue coloring of $E(K_8)$ which shows that $R(3, 4) > 8$.

Clearly, there is no red triangle. A blue $K_4$ corresponds to an independent set of size 4 in the depicted graph. Such an independent set would contain exactly two vertices from the outer 4-cycle. By symmetry, say the top-left $v_1$ and the right-bottom vertex $v_3$. However, it is impossible to add 2 inner vertices $w_i, w_j$ so that $\{v_1, v_3, w_i, w_j\}$ is independent.

**b)** Prove that $R(3, 4) = 9$.

**Solution:** By (a), it is enough to show that every red/blue coloring of $E(K_9)$ contains a red $K_3$ or blue $K_4$. First, recall that $R(3, 3) = 6$ and $R(2, 4) = 4$. Now suppose for contradiction there is a red/blue coloring of $E(K_9)$ containing neither red $K_3$ nor blue $K_4$. Fix a vertex $v$, and let $r_v$ and $b_v$ be the number of red and blue neighbors of $v$, respectively. First observe that $r_v \leq 3$ as otherwise by $R(2, 4) = 4$ we can find in the red neighborhood of $v$ a red edge (which together with $v$ forms a red triangle) or a blue $K_4$. Analogously, $b_v \leq R(3,3) - 1 = 5$. However, $r_v + b_v = 9 - 1 = 8$ since the graph is complete, so we conclude that $r_v = 3$ and $b_v = 5$. This applies to every vertex $v \in V(K_9)$ so the subgraph induced by red edges is 3-regular. But clearly, there is no 9-vertex 3-regular graph (the sum of the degrees must be even!).

**c)** Show that $R(4,4) \leq 18$.

**Solution:** We know from the lecture that $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$. Therefore,

$$R(4,4) \leq R(3,4) + R(4,3) = 2 \cdot R(3,4) = 18.$$

**2.** Recall that $R_k(3) := R_k(\overbrace{3,3,\ldots,3}^{k})$ is the minimum integer $n$ such that any $k$-coloring of $E(K_n)$ contains a monochromatic $K_3$.

Prove that $R_k(3) \leq 3k!$ for any integer $k \geq 1$.

**Solution:** Induction on $k$. Clearly, the formula holds for $k = 1$ and $k = 2$ as well ($R(3,3) = 6$). Suppose $k > 2$ and fix any $k$-coloring of $E(K_{3k!})$. Let $v$ be a vertex and $i \in \{1,\ldots,k\}$ be the most frequent color on the edges incident to $v$. Without loss of generality, $i = k$. Set $N$ to be the set of vertices $u$ such that $\{u,v\}$ has color $k$. We claim that $|N| \geq 3(k-1)!$ as otherwise the total number of vertices in $K_{3k!}$ would be at most

$$k \cdot ((3k-1)! - 1) + 1 = 3k! - (k-1) < 3k!.$$

Now either at least one edge with both endpoints in $N$ has color $k$, in which case we are done, or, we can apply induction on the $(k-1)$-coloring of $K_{|N|}$ that is induced by the coloring of the edges inside $N$. Since $|N| \geq 3(k-1)!$, the induction hypothesis yields a monochromatic triangle inside $N$.

**3.** Let $G$ be a 3-regular simple graph with no cut-edge, and let $e \in E(G)$ be an edge of $G$.

**a)** Show that $G$ contains a perfect matching $M_1$ such that $e \in M_1$.

**THIS IS A FIXED SOLUTION.** The earlier solution here was wrong.

Let $u$ and $w$ be the two endpoints of $e$, and let $H := G - u - w$. It is enough to show that $H$ has a perfect matching $M'$, since $M' + e$ will be a perfect matching of $G$ that contains $e$.

Let $V := V(G)$ and $W := V(H) = V \setminus \{u,w\}$. Suppose for contradiction $H$ does not have a perfect matching. By Tutte's theorem, there exists $S_0 \subseteq W$ such that $odd_H(\overline{S_0}) > |S_0|$, where $\overline{S_0} = W \setminus S_0$. First, we observe that the parity of $odd_H(\overline{S_0})$ and $|S_0|$ is the same. Indeed, recall that $|V|$ is even and that

$$|V| - 2 = |W| = \sum_{\substack{C \text{ even component} \\ \text{of } H[\overline{S_0}]}} |C| \; + \sum_{\substack{C \text{ odd component} \\ \text{of } H[\overline{S_0}]}} |C| \; + \; |S_0|.$$

Therefore, $odd_H(\overline{S_0}) \geq |S_0| + 2$, and for $S := S_0 \cup \{u, w\}$ we have

$$odd_G(V \setminus S) = odd_H(\overline{S_0}) \geq |S_0| + 2 = |S|.$$

Now we look closer to the situation in $G$ and the set of vertices $S$. The number of edges between $S$ and $V \setminus S$ is at most $3(|S| - 2) + 4 = 3|S| - 2$ because $u$ is adjacent to at most two vertices in $V \setminus S$ and the same holds also for $v$. On the other hand, there are at least $|S|$ odd components in $G[V \setminus S]$. As in the proof of Petersen's theorem in the lecture, each such odd connected component must receive at least 3 edges from the vertices in $S$ (only one edge would mean a cut-edge in $G$, only two edges violates the parity constraint). So the number of edges between $S$ and $V \setminus S$ must be at least $3|S|$; a contradiction.

**b)** Show that $G$ contains a perfect matching $M_2$ such that $e \notin M_2$.

**Solution:** This immediately follows from (a). Let $v$ be one of the endpoints of $e$ and let $f$ be one of the other two edges incident to $v$ (chosen aribtraririly). A perfect matching $M$ containing $f$ guaranteed by (a) clearly cannot contain $e$.

**4.** Recall that for a simple graph $G$, the chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of $G$ so that for every edge $e$ the endpoints of $e$ receive two different colors.

Let $G$ be a simple graph such that any two odd cycles $C_1$ and $C_2$ in $G$ it holds that $V(C_1) \cap V(C_2) \neq \emptyset$. Prove that $\chi(G) \leq 5$.

**Solution:** If $G$ contains no odd cycle, then $G$ is bipartite and $\chi(G) \leq 2$. Otherwise, let $C$ be an odd cycle of $G$ of the shortest length. The subgraph induced by the vertices of $C$ cannot contain any additional edges except the ones from the cycle, as otherwise we would have found a shorter odd cycle. On the other hand, let $W := V(G) \setminus V(C)$. If the induced subgraph $G[W]$ would contain an odd cycle, say $C'$, then we would have found two odd cycles in $G$ such that $V(C) \cap V(C') = \emptyset$ violating the assumption on $G$. So $G[W]$ is bipartite and can be colored with two colors, say $\{1, 2\}$. The vertices of $C$ can be colored with three new colors, say $\{3, 4, 5\}$. So together this forms a proper 5-coloring of $G$.

**5.** A simple graph $G = (V, E)$ is called *triangle-free* if no 3-vertex subgraph of $G$ is isomorphic to $K_3$.

Let $G$ be a triangle-free simple graph with $n$ vertices. Show that $G$ contains an independent set of size $\lfloor \sqrt{n} \rfloor$. Deduce that $R(3, \ell) \leq \ell^2$.

**Solution:** Let $\Delta$ be the maximum degree of $G$. The neighborhood of any vertex $v \in V(G)$ must form an independent set ($G$ is triangle-free!), so if $\Delta \geq \lfloor \sqrt{n} \rfloor$, then we are done. If $\Delta \leq \lfloor \sqrt{n} \rfloor - 1$, then by the greedy coloring algorithm $G$ can be colored with $\Delta + 1 = \lfloor \sqrt{n} \rfloor \leq \sqrt{n}$ colors. Therefore, the largest color class, which is indeed an independent set, have size at least

$$\frac{n}{\sqrt{n}} = \sqrt{n} \geq \lfloor \sqrt{n} \rfloor.$$

For the second part, let $n := \ell^2$. Consider any red/blue coloring of $E(K_n)$ and let $G$ be the $n$-vertex subgraph of $K_n$ induced by the red edges. Either $G$ contains a triangle, in which case we are done, or, by the previous $G$ contains an independent set of size at least $\lfloor \sqrt{n} \rfloor = \ell$. The independent set in $G$ corresponds to a blue $K_\ell$ in the coloring, so $R(3, \ell) \leq \ell^2$.