Math417
Mathematical Programming

Homework V

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Contents

1

$$\min c^{\mathsf{T}} x \quad s.t. \quad Ax = b \tag{1}$$

Let $x = x^{+} - x^{-}$ and $x^{+}x^{-} \geq 0$
Then we have

$$\min c^{\mathsf{T}}(x^{+} - x^{-}) \quad : \quad A(x^{+} - x^{-}) = b \tag{2}$$

We put this problem in standard form by substitution:

$$\hat{x} = (x^{+}, x^{-}) \quad \hat{c} = (c, -c) \quad \hat{A} = [A| - A]$$

And the problem defined by the hat versions of c x and a and the same b vector is clearly equivalent. However now our vector $\hat{x}$ is restrained to be non negative so we can take the dual.

$$\max b^{\mathsf{T}} y \quad : \quad \hat{A}^{\mathsf{T}} y \leq \hat{c} \tag{3}$$

The constraints of the dual can be rewritten as follows:

$$A^{\mathsf{T}} y \leq c \qquad \text{and} \qquad - A^{\mathsf{T}} y \leq -c \quad \Rightarrow A^{\mathsf{T}} y = c$$

Thus

$$\max b^{\mathsf{T}} y \quad \hat{A}^{\mathsf{T}} y = c \tag{4}$$

is the simplified dual problem.

2

	$\inf(P) = -\infty$	$\inf(P) \in \mathbb{R}$	$\inf(P) = +\infty$
$\sup(D) = -\infty$	✓	×	✓
$\sup(D) \in \mathbb{R}$	×	✓	×
$\sup(D) = +\infty$	×	×	✓

There are only two theorems we need to invoke to show that the situations mentioned above are impossible. Strong Duality: (1,2)(2,3) because for a LP a solution exists for the primal iff it exist for the dual. Weak Duality: (2,1)(3,1)(3,2) because for a LP the supremum of the dual is always less or equal to the infinimum of the primal.
(1,1) Unbounded Primal which means infeasible dual:
Let c = (-1,-1) Ax = b, A = I, b = (0,1)
(1,3) For a problem infeasible for both the primal and the dual simply take a 2x2 matrix will a entries equal to 1 for A (so $A = A^\mathsf{T}$) and b and c with different entries. (e.g. b = (1,2) and c= (3,4))
(2,2) A solution exists for the primal ($\iff$ exists for the dual)
Look for example (lp from third homework set)
(3,3) Unbounded dual so infeasible primal

$$b = (1,1) \quad A^\mathsf{T} = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \end{Bmatrix} \quad c = (1,1)$$

Clearly we can let the second component of y go to infinity and so the problem is unbounded.
(2,2) From a past assignment where we solved the problem graphically:

$$\begin{aligned} \min \quad 5x_1 + 7x_2 + 4x_3 + 8x_4 + 9x_5 + 10x_6 \quad \text{s.t.} \quad x_1 + x_2 &= 11 \\ x_3 + x_4 &= 10 \\ x_5 + x_6 &= 9 \\ x_1 + x_3 + x_5 &= 18 \\ x_i &\geq 0 \end{aligned}$$

3

We have that 3.10 holds we know that $\exists r \, s.t. \, u_r < 0$ Hence we have that :

$$c^\intercal z = c^\intercal x + tu_r \leq c^\intercal x \tag{5}$$

Since $u_r < 0$ and $t \geq 0$ by construction. We need to have that

$$Az(t) = b \tag{6}$$

Let J be the set of indices with $|J| = m$ and K its complement. We start the process by choosing a BFP x with $x_K = 0$. So we construct z_J the following way:

$$z_J = B^{-1}(b - ta_r) = x_J - td \tag{7}$$

Where we have defined d as :

$$d := B^{-1}a_r \tag{8}$$

With B being the square matrix with the linearly independent columns a_J and N the matrix with the column vectors a_K. Now we have

$$Az(t) = Bz_J + Nz_K = b \tag{9}$$

Now assume $d_i \leq 0 \quad \forall i \in J$ Well then it is easy to see that the feasible vector z(t) is also non-negative for any positive t:

$$x_j - td_j \geq 0 \tag{10}$$

Therefore we can make t as large as we want. Looking at 5 we can see that therefore the objective function can be made as small as possible, i.e. the problem is unbounded. ■

4 Simplex algorithm iteration

Our aim is to find for which index $r \in K$ is the vector

$$u_k := c_k - a_k^{\mathsf{T}} y \tag{11}$$

Smaller than zero. The first step is to compute y which is defined as

$$y := (B^{\mathsf{T}})^{-1} c_J \tag{12}$$

So we need to compute $(B^{\mathsf{T}})^{-1}$ which is pretty straightfoward:

$$(B^{\mathsf{T}}) = \begin{Bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \quad \Rightarrow \quad (B^{\mathsf{T}})^{-1} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \tag{13}$$

$$\therefore y = (0, 0, -2, 0) \tag{14}$$

We the compute the different possible u_r:

$$u_2 = -3 - (1, 3, 0, 0)^{\mathsf{T}} (0, 0, -2, 0) = -3 \tag{15}$$

$$u_3 = -4 - (1, 1, 0, 1)^{\mathsf{T}} (0, 0, -2, 0) = -4 \tag{16}$$

$$u_6 = 0 - (0, 0, 1, 0)^{\mathsf{T}} (0, 0, -2, 0) \quad = 1 \tag{17}$$

From this we can see that u_6 is not a contender. We can choose between $r = 2$ or $r = 3$ and choose the later. Now let's compute d_J

$$d_J = B^{-1} a_3 = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{Bmatrix} = (0, 1, 1, 1) \tag{18}$$

Now we take choose the value of $\hat{t}$:

$$s = \operatorname{argmin} \left\{ \frac{x_j}{d_j} \mid d_j > 0 \right\} = 4 \Rightarrow \hat{t} = \frac{x_4}{d_4} = 2 \tag{19}$$

Remember

$$x = (2, 0, 0, 2, 6, 0, 3)$$

$$
\begin{aligned}
\hat{x}_1 &= x_1 - 2 \times 0 = 2\\
\hat{x}_2 &= 0 \qquad\qquad\; = 0\\
\hat{x}_3 &= \hat{t} \qquad\qquad\; = 2\\
\hat{x}_4 &= x_4 - 2 \times 1 = 0\\
\hat{x}_5 &= x_5 - 2 \times 1 = 1\\
\hat{x}_6 &= 0 \qquad\qquad\; = 0\\
\hat{x}_7 &= x_7 - 2 \times 1 = 1
\end{aligned}
$$

We can now write down $\hat{x}$

$$\hat{x} = (2, 0, 2, 0, 1, 0, 1)$$

Both $\hat{x}$ and x are not degenerate as all the basic components are non-zero for their respective index sets:

$$J = \{1, 4, 5, 7\} = \{1, 3, 5, 7\}$$

Dump

Let's derive the dual

$$\begin{aligned}
\min \quad & c^\top x + \lambda^\top (b - Ax) \\
g \quad : \quad & \lambda \mapsto \inf\{c^\top x + \lambda^\top (b - Ax)\} \\
g(\lambda) \leq\ & c^\top \bar{x} + \lambda^\top (b - A\bar{x}) = c^\top \bar{x} \\
g(\lambda) =\ & \lambda^\top b + \inf\{(c - A^\top \lambda)^\top x\}
\end{aligned}$$

$$\inf\{(c - A^\top \lambda)^\top x\} = \begin{cases} 0, & x \geq 0 \quad c - A^\top \lambda > 0 \\ 0, & x \leq 0 \quad c - A^\top \lambda < 0 \\ 0, & c - A^\top \lambda = 0 \\ -\infty & \text{elsewhere} \end{cases}$$

So we can formulate the dual problem is as following:

$$\max_{\lambda \in \mathbb{R}^m} g(\lambda) = b^\top \lambda \quad s.t. \quad \begin{cases} x \neq 0 & A^\top \lambda \neq c \\ x = 0 & A^\top \lambda = c \end{cases} \tag{20}$$