McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 316

FUNCTIONS OF A COMPLEX VARIABLE

Examiner: Professor P. Rusell
Associate Examiner: Professor V. Jaskic

Date: Friday December 17, 2004
Time: 2:00 P.M – 5:00 P.M

INSTRUCTIONS

1. Please answer all 5 questions.
2. Please answer in exams booklets provided.
3. This is a closed book exam.
4. No calculators are allowed.
5. No Dictionaries are allowed
6. This exam consists of the cover page and 2 pages of questions.

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1. (i) Determine a branch of the logarithm analytic in

$$D = \{z \mid \mathrm{Re}(z) = 0 \;\Rightarrow\; Im(z) < 0\}$$

such that $0 < if(-i) < \pi$. Find $f(1)$ and $f(-1)$.

(ii) What is the domain of analyticity of $f(iz - 1)$?

(iii) Show that $g(z) = \exp\left(\frac{1}{2} f\left(\frac{z+i}{z-i}\right)\right)$ defines a branch $\left(\frac{z+i}{z-i}\right)^{\frac{1}{2}}$ that is analytic at 0. Find $g(0)$ and $g'(0)$. Is $g(1)$ defined?

(iv) Find the Taylor expansion of $f(z)$ about $z_0 = -1$. Determine the radius of convergence.

2. Find all possible Laurent expansions of

$$f(z) = \frac{z}{z^2 - 1}$$

about (a) $z_0 = 0$, (b) $z_0 = 1$.

3. Determine whether the following functions have an isolated singularity at the point z_0 indicated. If so classify the singularity as removable, pole or essential, and find the residue at z_0

(i) $(1 - z^2)^{\frac{1}{2}}$, principal branch, $z_0 = 1$

(ii) $z^n e^{\frac{1}{z}}, n \in \mathbf{Z}$, $z_0 = 0$

(iii) $\dfrac{z}{f(z)}$, $z_0 = 0$, where $f(z)$ is a branch of $\sin^{-1} z$ such that $f(0) = 0$

(iv) $z^{-7}(2 - z^2 - 2\cos z)$, $z_0 = 0$

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4. In this problem all functions are supposed to have an isolated singularity at z_0.

(i) Define: f has finite order d at z_0.

(ii) The logarithmic derivative of f is defined to be $\delta(f) = \dfrac{f'}{f}$. Show that $\delta(fg) = \delta(f) + \delta(g)$.

(iii) Suppose that f has finite order d at z_0. Show that Res $(\delta(f), z_0) = d$. Hint:Write $f(z) = (z - z_0)^d g(z)$

(iv) Let $P(z)$ be a polynomial of degree N. Show that for r sufficiently large

$$\oint_{|z|=r} \frac{P'(z)}{P(z)} dz = N.$$

5. Evaluate the following integrals. Use residues and contour integration where appropriate.

(a) $\displaystyle\oint_{|z|=4} \frac{e^z}{z^2 + \pi^2} dz$

(b) $\displaystyle\int_{-\pi}^{\pi} \frac{\cos\theta}{2 + \cos\theta} d\theta$