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Claw-free graphs — A survey

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Abstract

In this paper we summarize known results on claw-free graphs. The paper is subdivided into the following chapters and sections:

1. Introduction
2. Paths, cycles, hamiltonicity
   (a) Preliminaries
   (b) Degree and neighborhood conditions
   (c) Local connectivity conditions
   (d) Further forbidden subgraphs
   (e) Invariants
   (f) Squares
   (g) Regular graphs
   (h) Other hamiltonicity related results and generalizations
3. Matchings and factors
4. Independence, domination, other invariants and extremal problems
5. Algorithmic aspects
6. Miscellaneous
7. Appendix — List of all 2-connected nonhamiltonian claw-free graphs on $n \leqslant 12$ vertices.

1. Introduction

Claw-free graphs have been a subject of interest of many authors in the recent years. In this paper we try to summarize the known results concerning this family of graphs.

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The first motivation for studying properties of claw-free graphs apparently appeared from the Beineke's characterization of line graphs in [14,15]. However, the main impulse that turned the attention of the graph theory community to the class of claw-free graphs was given in late 70s and early 80s. During this period the matching properties of such graphs were observed in [183,120,184], and first results on hamiltonian properties were proved in [85,50,148,155]. However, probably more importantly, were the observations that the determination of the independence number is polynomial (see [151,172]) and that the Berge's Perfect Graph Conjecture holds (see [158]) in claw-free graphs.

In general, we follow the most common graph-theoretical terminology and notation, and for concepts not defined here we refer to [20]. Unless otherwise mentioned, throughout the paper by a *graph* we always mean a simple finite undirected graph G with vertex set $V(G)$ and edge set $E(G)$. We will always let $n = |V(G)|$ and, unlike in [20], we denote by $\langle M \rangle$ the subgraph induced by a set $M \subset V(G)$. By $\delta(G)$ and $\Delta(G)$ we will mean the minimum degree and maximum degree of G, respectively. For a subset A of $V(G)$, $N(A)$ is the set of all vertices of G that are adjacent to at least one vertex of A. If H is a graph, then we say that G is H-free if G does not contain a copy of H as an induced subgraph. By the *claw* (denoted C) we mean the complete bipartite graph $K_{1,3}$ (see Fig. 1). Thus, G is said to be *claw-free* if it does not contain an induced subgraph that is isomorphic to C.

Some of the results surveyed here are also mentioned in the two former surveys on claw-free graphs, one by Flandrin in [65] and the other by Li Mingchu and Liu Zhenhong in [135], and in the survey paper on hamiltonicity by Gould [87].

1.1. Families of claw-free graphs

There are several well-known and important families of graphs that are also claw-free, so we now recall some of these families.

(α) *Line graphs*: If G is a graph, then the *line graph* of G, usually denoted by $L(G)$, is obtained by associating one vertex to each edge of G, and two vertices of $L(G)$ being joined by an edge if and only if the corresponding edges in G are adjacent. In [14] and also [15], Beineke gives a characterization of line graphs in terms of forbidden induced subgraphs and the claw is one (among nine) of those subgraphs. Thus, every line graph is claw-free.

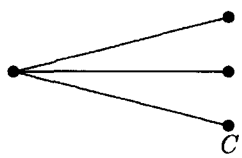

Fig. 1. The claw C.

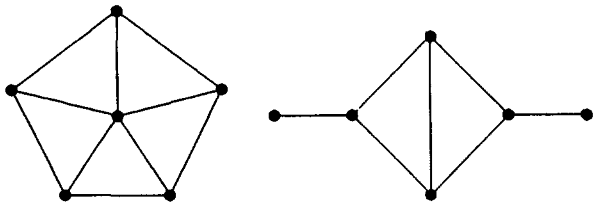

Fig. 2.

Šoltés [178] reduced the number of forbidden induced subgraphs characterizing connected line graphs to 7 for $n \geqslant 9$, with the claw remaining one of those seven forbidden subgraphs. In [179] line graphs are also characterized by 5 forbidden induced subgraphs and by describing the structure of neighborhoods of vertices.

(β) *Complements of triangle-free graphs.* It is easy to see G is a complement of a triangle-free graph if and only if its independence number satisfies $\alpha(G) \leqslant 2$ and that every such graph is claw-free.

(γ) *Inflation of a graph.* By the *inflation* of a graph H we mean a graph which is obtained by replacing each vertex x by a complete graph $K_{d(x)}$ and joining each edge to a different vertex of $K_{d(x)}$. Clearly, the inflation of every graph H is a claw-free graph and if H is regular then so is its inflation. Moreover, the inflation of an arbitrary planar cubic graph is a planar cubic claw-free graph which is hamiltonian if and only if the original graph is hamiltonian.

(δ) *Comparability graphs.* Consider a finite set of intervals on the real line such that no one completely contains another one (see [81]). A claw-free graph, called the *comparability graph* (or also the *interval graph* — cf. [48] and Theorems 6.6–6.8), is obtained by associating a vertex to each of these intervals and putting edges between vertices if and only if the corresponding intervals have nonempty intersection.

(ε) *Middle graphs.* The *middle graph* of H is obtained by inserting a vertex x_i in the 'middle' of each edge e_i, $1 \leqslant i \leqslant |E(H)|$, and adding the edge x_ix_j for $1 \leqslant i < j \leqslant |E(H)|$ if and only if e_i and e_j have a common vertex. The middle graph of every graph is also claw-free.

It is easy to see that all inflations and middle graphs are line graphs, but, on the other hand, the graphs H_1 and H_2 in Fig. 2 are examples of a complement of a triangle-free graph and of a comparability graph that are not line graphs.

(ζ) *Generalized line graphs.* Graphs such that the neighborhood of each vertex can be partitioned into at most two cliques are called *generalized line graphs*. They can also be characterized in the following way (see [148]): A generalized line graph is the complement of a locally bipartite graph (in [148] called 'almost bipartite'), where a graph is *locally bipartite* if and only if the neighborhood of each vertex induces a bipartite graph. It is easy to check that this family contains all line graphs (and therefore also all middle graphs and inflations). Nevertheless, some complements of

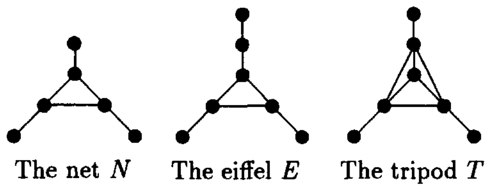

Fig. 3.

a triangle-free graphs are not generalized line graphs (see H_1 in Fig. 2) and some generalized line graphs are not line graphs (see H_2 in Fig. 2).

For additional properties of these graphs, see [16].

In the following we turn our attention to some properties of claw-free graphs that will be useful in the next sections. In many situations, it is useful to have local information about the structure of neighborhoods of vertices in a claw-free graph. It is an easy observation to realize that a graph G is claw-free if and only if the neighborhood of every vertex $x \in V(G)$ satisfies $\alpha(\langle N(x)\rangle) \leqslant 2$ (where $\alpha(H)$ is the independence number of H).

It is also easy to observe that, for every $x \in V(G)$, $\langle N(x)\rangle$ must be net-free (i.e., cannot contain as an induced subgraph a copy of the net N (see Fig. 3)). Since clearly $\langle N(x)\rangle$ is also claw-free, we see that neighborhoods of vertices can be of the following types:

— $\langle N(x)\rangle$ is disconnected and then, since $\alpha(\langle N(x)\rangle) \leqslant 2$, $\langle N(x)\rangle$ consists of two vertex-disjoint cliques,

— $\langle N(x)\rangle$ has connectivity 1 and then, by a result of Duffus et al. [50] (see Theorem 2d), $\langle N(x)\rangle$ is traceable; moreover, $\langle N(x)\rangle$ can be covered by two vertex-disjoint cliques such that all edges joining the cliques have a common vertex in one of the cliques,

— $\langle N(x)\rangle$ is k-connected with $k \geqslant 2$ and then, again by Theorem 2d.2, $\langle N(x)\rangle$ is hamiltonian.

Another related result was proved by Fouquet [76]. Recall that an odd hole or antihole in G is an induced subgraph of G which is isomorphic to a chordless cycle of odd length or to its complement, respectively.

Theorem 1.1 ([76]) (Generalized Ben Rebea's lemma). *Let G be a connected claw-free graph with $\alpha(G) \geqslant 4$. Then G does not contain any odd hole of size greater than 5.*

Corollary 1.2 ([76]). *Let G be a connected claw-free graph with $\alpha(G) \geqslant 3$. Then every vertex of G satisfies one (and only one) of the following conditions:*

(i) *$N(v)$ is covered by two complete graphs,*

(ii) *$N(v)$ contains an induced $\bar{C}_5$ (i.e. an antihole of size 5).*

Another characterization was obtained by Shepherd [174].

Theorem 1.3 ([174]). *A connected graph G is claw-free if and only if for every minimal cut set S and every $v \in S$, $\langle N(v) - S \rangle$ is the disjoint union of two complete graphs.*

Shepherd also introduced a subclass of the class of claw-free graphs by imposing a condition that, for every vertex $x \in V(G)$, its neighborhood $N_i(X)$ at any distance i has independence number at most two. Namely, G is said to be *distance claw-free* if $\alpha(\langle N_i(x) \rangle) \leqslant 2$ for every $v \in V(G)$ and for every i. It is surprising that this global property can be fully characterized in terms of local conditions (for the eiffel and tripod, see Fig. 3).

Theorem 1.4 ([174]). *A graph G is distance claw-free if and only if G is claw-free, eiffel-free and tripod-free.*

Two interesting and useful consequences of Theorem 1.4 are the following.

Corollary 1.5 ([174]). *For any graph G, claw-free and net-free $\Rightarrow$ distance claw-free $\Rightarrow$ claw-free.*

Corollary 1.6 ([174]). *A graph G is distance claw-free if and only if $\alpha(N_i(v)) = 2$ for $i = 1$ and $i = 2$.*

1.2. Extensions

Together with claw-free graphs, we will examine properties of the following superclasses:

(a) Families of graphs defined in terms of a set of forbidden subgraphs such that each of those forbidden subgraphs contains an induced claw, for example $K_{1,r}$-free graphs, $r \geqslant 4$.

(b) Graphs with some constraints on the claws including:

- (α) Locally claw-free graphs (see [165]). A graph G is said to be *locally claw-free* if and only if $\langle N(x) \rangle$ is claw-free for every $x \in V(G)$. It can be shown (see [165]) that a graph G is locally claw-free if and only if G is crown-free (see Fig. 9).
- (β) Almost claw-free graphs (see [165]). We say that G is *almost claw-free* if the centers of induced claws in G are independent and their neighborhoods are 2-dominated. More precisely, G is almost claw-free if there is a (possibly empty) independent set $A \subset V(G)$ such that $\alpha(N(x, G)) \leqslant 2$ for $x \notin A$ and $\gamma(N(x, G)) \leqslant 2 < \alpha(N(x, G))$ for $x \in A$, where γ denotes the domination number of a graph. Clearly, every claw-free graph is almost claw-free.

The almost claw-free graphs are contained in some of the other classes of extensions of claw-free graphs, as the next result indicates.

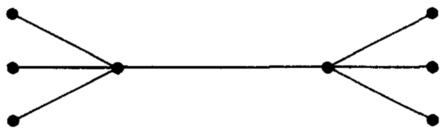

Fig. 4. The biclaw.

Proposition 1.7 ([165]). (i) *Every almost claw-free graph is locally claw-free.*
(ii) *Every almost claw-free graph is $K_{1,5}$-free.*

(γ) Graphs with independent centers of claws (see [123]).
(δ) Graphs without too many claws (see [74]).

(c) L_i-graphs (see [8]).

Asratian and Khachatrian [8] introduced the following classes of L_i-graphs (L for local). We say that G is a *L_i-graph* (i being an integer) if, for each triple of vertices u, v, w with $d(u,v)=2$ and $w \in N(u) \cap N(v)$, $d(u)+d(v) \geqslant |N(u) \cup N(v) \cup N(w)| - i$, or, equivalently, $|N(u) \cap N(v)| \geqslant |N(w) - (N(u) \cup N(v))| - i$. It can be shown (see [7]) that $L_0 \subset L_1 \subset L_2 \subset \cdots$ and that the class L_1 contains all claw-free graphs.

In the case of bipartite graphs, we also will be interested in the analogous concept of biclaw-free graphs. A *biclaw* is defined as the graph obtained from two vertex disjoint claws by adding an edge between the two vertices of degree 3 in each of the claws (see Fig. 4) and a bipartite graph is said to be *biclaw-free* if it does not contain any biclaw as an induced subgraph.

It is easy to get claw-free graphs using the above constructions (along with possibly choosing subgraphs of claw-free graphs, since the claw-free property remains when suppressing vertices). It is also easy to recognize if a given graph is claw-free (more precisely, it can be tested in a polynomial time with complexity at most $O(n^4)$ (see [5]). We can hope that some of these classes of graphs will have good behavior in comparison to some NP-complete problems for arbitrary graphs. Some results of this type have already been proved and are summarized later in Section 5.

2. Paths, cycles, hamiltonicity and related problems

(a) Preliminaries

Many of the results that are mentioned in this section are also included in the survey by Gould [87].

If $S \subset V(G)$, then by $c(G-S)$ we denote the number of components of $G-S$. We say that a graph G is *t-tough* if for every subset $S \subset V(G)$ with $c(G-S) > 1$ we have $|S| \geqslant tc(G-S)$. The *toughness* of G, denoted by $\tau(G)$, is the largest value of t such that G is t-tough.

Chvátal has shown in [40] that if G is not complete and has connectivity $\kappa(G)$, then $\tau(G) \leqslant \kappa(G)/2$. In the special case of claw-free graphs, Matthews and Sumner proved that equality holds.

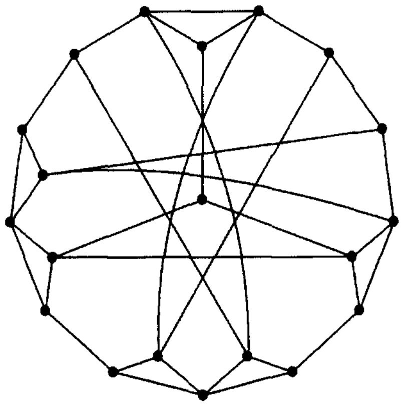

Fig. 5.

Theorem 2a.1 ([146]). *If G is a noncomplete claw-free graph, then*

$$\tau(G) = \kappa(G)/2.$$

Chvátal originally conjectured that if $\tau(G) > \frac{3}{2}$, then G is hamiltonian, but because of a family of counterexamples the conjecture has now been modified to the following.

Conjecture 2a.2 [40]. Every 2-tough graph is hamiltonian.

Based on this conjecture (which is still open) and an observation by Sumner (see [146]) that states that every claw-free with connectivity at least $n/4$ is hamiltonian, Matthews and Sumner [146] conjectured the following (which is also still open).

Conjecture 2a.3 [146]. If G is a 4-connected claw-free graph, then G is hamiltonian.

Matthews and Sumner in [146] also showed that the assumption G is a 3-connected claw-free graph does not imply hamiltonicity. In fact, an infinite family of 3-connected claw-free graphs that are not hamiltonian can be obtained from the cubic planar non-hamiltonian claw-free graph of Tutte by successive inflation. Matthews and Sumner also showed that every 3-connected claw-free graph on less than 20 vertices is hamiltonian and in [146] exhibited the following example (see Fig. 5) of a smallest 3-connected nonhamiltonian claw-free graph on 20 vertices (note that this graph is the line graph of a graph that is obtained from the Petersen graph by subdividing a perfect matching).

Since a 3-connected claw-free graph G is not in general hamiltonian, it is natural to ask how large a cycle G must have. This was done by Jackson and Wormald (see p. 63 in [19]).

Theorem 2a.4 ([19]). *Let G be a 3-connected claw-free graph on n vertices. Then G contains a cycle of length at least n^c for some positive constant c.*

In 2-connected claw-free graphs, Jackson [104] conjectured the following.

Conjecture 2a.5 [104]. If G is a 2-connected claw-free graph, then G has a Tutte cycle.

(A *Tutte cycle* in G is a cycle $C \subset G$ such that all bridges of C have at most three vertices of attachment on C. A *bridge* B of C is an equivalence class of the equivalence relation defined on $E(G) - E(C)$ by saying that e is equivalent to f if and only if there is a e, f-walk which is internally vertex disjoint from C, and the *vertices of attachment* of B are the elements of $V(B) \cap V(C)$).

The circumference of 2-connected claw-free graphs was investigated by Broersma et al. [24]. The following upper and lower bounds on $c(G)$ were proved using the relationship between the toughness and connectivity of a $K_{1,r}$-free graph.

Theorem 2a.6 ([24]). *If G is a 2-connected $K_{1,r}$-free graph on n vertices with circumference $c(G)$, then*

$$c(G) \geqslant 4\log_{r-1}(n) - A = \frac{4}{\log(r-1)}\log(n) - A,$$

where A is an appropriate absolute constant.

In [24] there is a construction that shows that for every r, $r \geqslant 3$, and sufficiently large n, there exists a 2-connected $K_{1,r}$-free graph $H_{r,n}$ on n vertices such that

$$c(H_{r,n}) < \begin{cases} \dfrac{4}{\log(r-2)}\log(n) + 4 & \text{if } r \geqslant 3, \\ 8\log(n+6) - 8\log(3) - 2 & \text{if } r = 3. \end{cases}$$

Hence, the order of magnitude of the lower bound in Theorem 2a.6 is best possible.

In some of the superclasses of the class of claw-free graphs that were defined in the introduction, the toughness of a graph can be smaller than the half of its connectivity. In $K_{1,r}$-free graphs, Broersma et al. [24] (and also later and independently Chen and Schelp [33]) proved the following.

Theorem 2a.7 ([24, 33]). *If G is a non-complete $K_{1,r}$-free graph of connectivity $\kappa(G)$, then*

$$\frac{\kappa(G)}{r-1} \leqslant \tau(G) \leqslant \frac{\kappa(G)}{2}.$$

In almost claw-free graphs a bound on the toughness as a function of the connectivity was proved by Broersma et al. [26].

Theorem 2a.8 ([26]). *If G is a noncomplete almost claw-free graph with connectivity κ, then*

$$\tau(G) \geqslant \min\{1, \tfrac{1}{2}\kappa\}.$$

Table 1
1-connected claw-free graphs

	$\delta \geqslant \frac{n-2}{3}$	$\sigma_3 \geqslant n-2$	$U_2 > \frac{2n-4}{3}$	$U_2 \geqslant \frac{2n-5}{3}$
Traceability	[147] (S)	[21,135] (S)	[54]	[11] (S)

Table 2
2-connected claw-free graphs

	$\delta \geqslant \frac{n-2}{3}$	$\sigma_2 \geqslant \frac{2n-5}{3}$	$\sigma_3 \geqslant n-2$
Hamiltonicity	[147] (S)	[69]	[21,135] (S)
Pancyclicity	[68]		

Table 3
2-connected claw-free graphs

	$U_2 > \frac{2n-2}{3}$	$U_2 \geqslant \frac{2n-3}{3}$	$U_2 \geqslant \frac{2n-5}{3}$	$U_2 \geqslant \frac{n-2}{2}$
Traceability				[54]
Hamiltonicity		[54]	[11] (S)	
Pancyclicity	[54]			

Also, an example is given in [26] showing that the equality $\tau(G) = \frac{1}{2}\kappa(G)$ fails in almost claw-free graphs with connectivity $\kappa > 2$.

Asratian et al. [7] proved an analogue for L_1-graphs.

Theorem 2a.9 ([7]). *If G is a 2-connected L_1-graph then G is 1-tough.*

An example is given in [7] of an infinite family of 1-tough L_1-graphs of arbitrary connectivity that are not $(1+\varepsilon)$-tough for any $\varepsilon > 0$.

(b) Degree and neighborhood conditions

For $1 \leqslant k \leqslant n$ we denote by $\sigma_k(G)$ the minimum of the degree sum $d(x_1) + \cdots + d(x_k)$ and by $U_k(G)$ the minimum of the neighborhood union $|N(x_1) \cup \cdots \cup N(x_k)|$, where the minimum is taken over all subsets $\{x_1, \ldots, x_k\}$ of k independent vertices of $V(G)$. We will keep the notation δ for the common value of σ_1 and U_1. The two parameters σ_k and U_k have been frequently used in sufficient conditions for hamiltonian properties in graphs.

For the sake of clarity and ease of reference the results concerning traceability, hamiltonicity and pancyclicity in claw-free graphs as a function of δ, σ_k and U_k have been placed in Tables 1–3 (depending on the connectivity of the graph). An 'S' (for

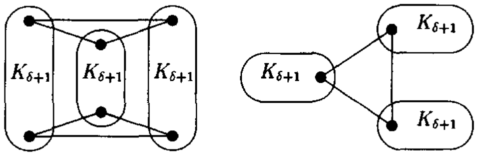

Fig. 6.

sharp) in Table 1 indicates that the bound cannot be improved. Even early results that have been improved are included in the table in order to exhibit the development of the area. Sharpness of some of these bounds can be checked using the graphs in Fig. 6.

When the connected graph G is not traceable, then Matthews and Sumner [147] have shown that the longest path has length at least $2\delta+2$, and this was improved to σ_2+2 by Liu et al. [135].

In the following two results, the bounds on degree in the previous results are improved in the presence of some additional assumptions. Liu and Wu in [136] assume regularity.

Theorem 2b.1 ([136]). *If G is claw-free, 2-connected and δ-regular such that $\delta \geqslant (n-1)/4$, then G is hamiltonian.*

Li Hao in [121] excludes the family $\mathscr{F}_1$ defined as follows: If G is in $\mathscr{F}_1$, then G can be decomposed into three disjoint subgraphs G_1, G_2, G_3 such that for any $i \neq j$, $1 \leqslant i,j \leqslant 3$, $E_G(G_i, G_j) = \{u_iu_j, v_iv_j\}$, where $u_i, v_i \in G_i$, (see also $\mathscr{F}_1$ in Fig. 8).

Theorem 2b.2 ([121]). *If G is a 2-connected claw-free graph with minimum degree $\delta \geqslant n/4$ which does not belong to $\mathscr{F}_1$, then G is hamiltonian. The bound $n/4$ is sharp.*

For non-hamiltonian 2-connected graphs, the circumference is proved to be at least $2\delta+4$ by Matthews and Sumner [147], and this was improved to σ_2+4 by Flandrin et al. [69] and also by Liu et al. [135]. Assuming only $n/2+k$ vertices of degree at least k, Dang [46] also gets the bound $2k+4$, except for a forbidden family of graphs. Denote by $\mathscr{F}'$ the family of 13 graphs which are obtained from G_1, G_2 or G_3 (see Fig. 7) by possibly adding some of the intermittent edges. Note that these graphs are all in the family $\mathscr{F}_1$ defined by Li [121].

Theorem 2b.3 ([46]). *Let G be a 2-connected claw-free graph having at least $n/2+k$ vertices of degree k for some integer k. Then either $G \in \mathscr{F}'$ and G has a cycle of length at least $2k+3$ or else G has a cycle of length at least* $\min\{n, 2k+4\}$.

Corollary 2b.4 ([46]). *Let G be a 2-connected claw-free graph with $n \geqslant 13$ having at least $n/2+k$ vertices of degree k for some integer k. Then G has a cycle of length at least* $\min\{n, 2k+4\}$.

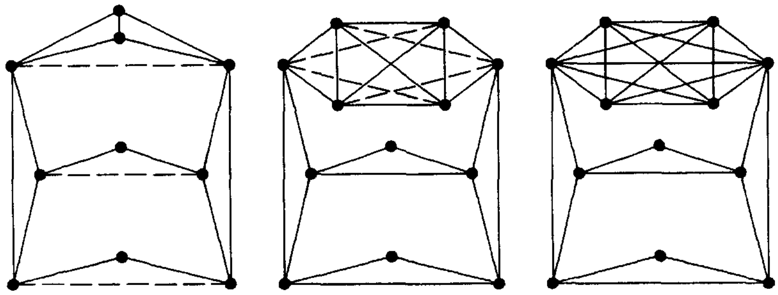

Fig. 7.

Table 4
3-connected claw-free graphs

	$\delta \geqslant \frac{n+10}{5}$	$\delta \geqslant \frac{n+5}{5}$	$\delta \geqslant \frac{n+7}{6}$	$\sigma_3 \geqslant n+1$	$U_2 \geqslant \frac{11(n-7)}{21}$
Traceability				[125]	
Hamiltonicity	[66]	[128]	[120]		[125]
Hamilton connectedness				[191,71]	

In another direction Tian in [185] also gets a lower bound on the circumference by placing some restrictions on the induced subgraphs isomorphic to P_4 (path with four vertices).

Theorem 2b.5 ([185]). *Let G be a 2-connected claw-free graph and S_c be the set of vertices with degree at least $c/2$, $3 \leqslant c \leqslant n$. If $|V(L) \cap S_c| \geqslant 1$ for each induced subgraph L isomorphic to P_4, then G has circumference at least c.*

For non-hamiltonian 3-connected claw-free graphs, Li Mingchu [126] verified 4δ as a lower bound for the circumference and Wang [189] showed that the value $\sigma_4 - 4$ is also a lower bound if $\delta \geqslant 8$ (see Table 4).

For nonhamiltonian k-connected claw-free graphs (Table 5), Flandrin et al. [68] proved that the circumference is at least $2\sigma_{k+1}/(k+1)+4$ if $k \geqslant 2$.

There are several additional degree and neighborhood conditions that imply hamiltonian type properties in k-connected claw-free graphs for $k \geqslant 2$. Fraisse [77] has shown that for any graph G (without the assumption of claw-freeness), if $U_t > t(n-1)/(t+1)$ for some integer t, $1 \leqslant t \leqslant k$, then G is hamiltonian.

Li and Virlouvet considered the same type of neighborhood union condition for claw-free graphs in [125], and proved the following result, which has several interesting corollaries, some of which have already been mentioned.

Table 5
k-connected claw-free graphs

	$\delta > \frac{2n}{2k+1} - 2$	$\sigma_{k+1} \geqslant n - k$	$\sigma_{k+2} \geqslant n - k - 1$	$U_t \geqslant t\frac{n-1}{t+1}$, $t \leqslant k$
Traceability			[1]	
Hamiltonicity	[125]	[192, 1]		Conjecture in [54] Proof in [125]

Theorem 2b.6 ([125]). *Let G be a k-connected ($k \geqslant 3$) claw-free graph. If there is some integer $t, t \leqslant 2k$, such that*

$$U_t(G) > \frac{t(4k - t + 1)}{2k(2k + 1)}(n - 2k - 1),$$

then G is hamiltonian.

Li Mingchu [127] gave a degree sum condition for claw-free graphs of sufficiently large order under which hamiltonicity implies pancyclicity.

Theorem 2b.7 ([127]). *If G is a hamiltonian claw-free graph of order $n > 100$ and $\sigma_3 \geqslant (n - 2)/3$, then G is pancyclic.*

Considering not only subsets of independent vertices but also arbitrary sets of vertices, Faudree et al. [55] define the *generalized r-degree*, $\delta_r(G)$, of a graph G by

$$\delta_r(G) = \min_{S \subseteq V(G), |S| = r} |\bigcup_{u \in S} N(u)|.$$

In [58] Faudree et al. used the generalized 2-degree δ_2 to give sufficient conditions for hamiltonian type properties.

Theorem 2b.8 ([58]). *Let G be a claw-free graph.*

(a) *If G is connected and $\delta_2(G) \geqslant (n + 1)/3$, then for n sufficiently large G is traceable.*

(b) *If G is 2-connected and $\delta_2(G) \geqslant (n + 1)/3$, then for n sufficiently large G is hamiltonian.*

(c) *If G is 3-connected and $\delta_2(G) \geqslant (n + 24)/3$, then for n sufficiently large G is hamiltonian-connected.*

(d) *If G is 3-connected, then there is a constant c such that if $\delta_2(G) \geqslant n/3 + c$, then G is pancyclic.*

In the presence of a graph with bounded independence number, the generalized degree δ_r is used in [55] to give additional sufficient conditions for hamiltonian type properties (see also Theorem 2b.16).

Theorem 2b.9 ([55]). *Let G be a graph of order n. Then for each pair of integers r and m ($1 \leqslant r \leqslant n$ and $3 \leqslant m < n$), and for each non-negative function $f(r,m)$ there exists a constant $C = C(r,m,f(r,m))$ such that if $\delta_r(G) \geqslant n/3 + C$ and $\alpha(G) \leqslant f(r,m)$, then*

(i) *G is traceable if $\delta(G) \geqslant r$ and G is connected,*
(ii) *G is hamiltonian if $\delta(G) \geqslant r+1$ and G is 2-connected, and*
(iii) *G is hamiltonian-connected if $\delta(G) \geqslant r+2$ and G is 3-connected.*

Generalizations to larger classes

(α) *$K_{1,4}$-free graphs.*

In 2-connected $K_{1,4}$-free graphs, Markus [142] proved the following minimum degree condition is sufficient for hamiltonicity.

Theorem 2b.10 ([142]). *Let G be a 2-connected $K_{1,4}$-free graph with $\delta \geqslant (n+2)/3$. Then G is hamiltonian.*

Chen and Schelp [33] extended the previous result for $K_{1,4}$-free graphs of connectivity at least 2 by considering a sum of degrees condition.

Theorem 2b.11 ([33]). *Let G be a k-connected $K_{1,4}$-free graph of order $n \geqslant 3$.*

(i) *If $\sigma_{k+1}(G) \geqslant n+k$, then G is hamiltonian.*
(ii) *If $\sigma_k(G) \geqslant n+k+1$, then G is hamiltonian-connected.*

From this result, we get as a corollary the following minimum degree conditions.

Corollary 2b.12 ([33]). *Let G be a k-connected $K_{1,4}$-free graph of order $n \geqslant 3$.*

(i) *If $\delta(G) \geqslant (n+k)/(k+1)$, then G is hamiltonian.*
(ii) *If $\delta(G) \geqslant (n+k+1)/k$, then G is hamiltonian-connected.*

(β) *$K_{1,r}$-free ($r \geqslant 5$) graphs.*

Markus [142] proved the following minimum degree condition for hamiltonicity in 2-connected $K_{1,r}$-free graphs.

Theorem 2b.13 ([142]). *Let G be a 2-connected $K_{1,r}$-free graph ($r \geqslant 5$) with $\delta \geqslant (n+r-3)/3$. Then G is hamiltonian unless $n = 2r-3$ and $G - E(G-T)$ is isomorphic to $K_{r-1,r-2}$, where T is any largest independent set in G.*

Chen and Schelp [33] proved the following strengthening of this result. We say that G is *q-edge-hamiltonian* if for every set F of q edges of G which induce a set of vertex disjoint paths of G, there is a hamiltonian cycle in G containing all edges of F.

Theorem 2b.14 ([33]). *Let k, q be nonnegative integers such that $k \geqslant q+1$ and let G be a $K_{1,r}$-free graph of order $n \geqslant (r-1)(k+1)(k+1-q)+q-1$. If G satisfies $\sigma_{k-q+1} \geqslant n+q+(k-q-1)k$, then G is q-edge-hamiltonian.*

The following degree conditions for hamiltonicity and hamilton-connectedness follow immediately from Theorem 2b.14 for $q=0$ and $q=1$.

Theorem 2b.15 ([33]). *Let G be a k-connected $K_{1,r}$-free graph of order n and let $k \geqslant 1$ and $r \geqslant 2$ be integers.*

(i) *If $n \geqslant (k+1)^2(r-1)-1$ and $\sigma_{k+1} \geqslant n+(k-1)k$, then G is hamiltonian.*

(ii) *If $n \geqslant k(k+1)(r-1)$ and $\sigma_k \geqslant n+(k-2)k+1$, then G is hamiltonian-connected.*

Using the fact that a generalized degree condition $\delta_t(G) \geqslant pn$ $(p>0)$ in a $K_{1,r}$-free graph G implies that the independence number $\alpha(G)$ is bounded, Faudree et al. [55] obtained the next result, which is really a corollary of Theorem 2b.9.

Theorem 2b.16 ([55]). *Let G be a $K_{1,r}$-free graph of order n where $3 \leqslant r < n$. For each integer t, $1 \leqslant t \leqslant n$, there exists a constant $C = C(t,r)$ such that if $\delta_t(G) \geqslant n/3 + C$, then*

(i) *G is traceable if $\delta(G) \geqslant t$ and G is connected,*

(ii) *G is hamiltonian if $\delta(G) \geqslant t+1$ and G is 2-connected, and*

(iii) *G is hamiltonian-connected if $\delta(G) \geqslant t+2$ and G is 3-connected.*

(γ) *Almost claw-free graphs.*

For 2-connected almost claw-free graphs, Broersma et al. [26] get the same minimum degree for hamiltonicity as in claw-free graphs.

Theorem 2b.17 ([26]). *If G is a 2-connected almost claw-free graph with $\delta(G) \geqslant (n-2)/3$, then G is hamiltonian.*

Considering triples of independent vertices, they proved in [26] that if G is a 2-connected almost claw-free graph with $\sigma_3(G) \geqslant n$, then G is hamiltonian, and they conjectured that $\sigma_3(G) \geqslant n-2$ implies hamiltonicity in 2-connected almost claw-free graphs. This conjecture was verified for $n \geqslant 79$ by Li and Tian [124] (see Theorem 2b.19).

(δ) *Graphs with independent claw centers.*

In graphs in which all centers of claws are independent, Hao Li et al. [123] recently proved a result in which they further decreased the lower bound on minimum degree under an additional assumption that G does not belong to a specified family of graphs. Namely, $G \in \mathcal{F}$ if G can be decomposed into three vertex disjoint subgraphs G_1, G_2, G_3 plus (2, 1 or 0) additional vertices u_i (i.e. G is a subgraph of one of the graphs $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in Fig 8).

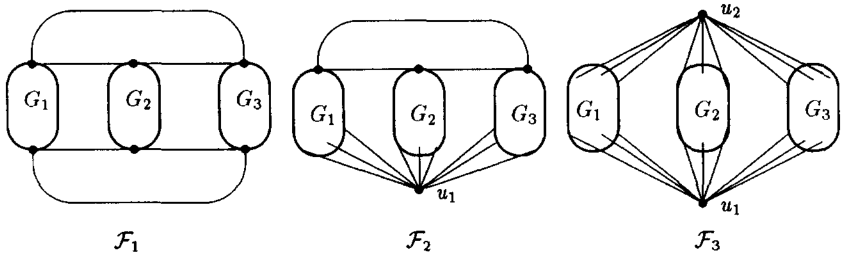

Fig. 8.

The crown $K_{1,1,3}$ The graph $K_{2,3}$

Fig. 9.

Theorem 2b.18 ([123]). *Let G be a 2-connected graph such that the set of vertices that are centers of induced claws in G is independent. If $\delta(G) \geqslant (n+3)/4$, then either $G \in \mathscr{F}$ or G is hamiltonian.*

Li Hao and Tian Feng proved the following result that generalizes several degree conditions for hamiltonicity in claw-free graphs and also verifies the conjecture of Broersma, Ryjáček and Schiermeyer on almost claw-free graphs for $n \geqslant 79$.

Theorem 2b.19 ([**124**]). *Let G be a 2-connected graph on $n \geqslant 79$ vertices such that the set of vertices that are centers of induced claws in G is independent. If $\sigma_3(G) \geqslant n-2$, then either $G \in \mathscr{F}_2 \cup \mathscr{F}_3$ or G is hamiltonian.*

For graphs of higher connectivities, Shen et al. [175] proved the following result.

Theorem 2b.20 ([175]). *Let G be a k-connected graph ($k \geqslant 2$) such that the set of vertices that are centers of induced claws in G is independent. If $\sigma_{k+1}(G) \geqslant n + k$, then G is hamiltonian.*

(ε) *$K_{1,1,3}$ and $K_{2,3}$-free graphs.*

These classes of graphs also contain the class of claw-free graphs. In [70], Flandrin et al. obtained (see Fig. 9).

Theorem 2b.21 ([70]). *Let G be a 2-connected crown-free and $K_{2,3}$-free graph such that $\sigma_3 \geqslant n + 1$. Then G is hamiltonian.*

(ζ) *Graphs with not too many claws.*

The δ-condition for hamiltonicity of Matthews and Sumner [147] can be extended if G does not have 'too many claws'. For any two independent vertices u, v of G, denote

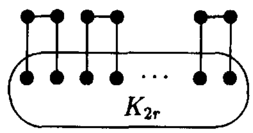

Fig. 10.

by $n_c(u,v)$ the number of claws containing both u and v and by $n_c(G)$ the maximum of $n_c(u,v)$ taken over all pairs of nonadjacent vertices of G. Flandrin and Li [74] proved:

Theorem 2b.22 ([74]). *Let G be a 2-connected graph of order $n \geqslant 29$ and minimum degree δ. Let l be an integer. If one of the following conditions is verified,*

1. $\delta = (n-2)/3$ *and* $n_c(G) < \delta/2$,
2. $\delta = (n+l)/3$, $-1 \leqslant l \leqslant 1$ *and* $n_c(G) < \delta - 1$, *or*
3. $\delta = (n+l)/3$, $2 \leqslant l \leqslant (n-3)/2$ *and* $n_c(G) < \delta(l+2) - 2$,

then G is hamiltonian.

(η) *Modified Fan condition*

Bedrossian et al. [13] define property $PC(k)$ as follows: G is said to satisfy $PC(k)$ if $\max\{d(x), d(y)\} \geqslant k/2$ for every pair (x,y) of nonadjacent vertices that belong to an induced claw or an induced $K_{1,3}+e$ in G. They prove the following three results.

Theorem 2b.23 ([13]). *If G is a 2-connected graph of order $n \geqslant 3$ satisfying $PC(k)$, $k \leqslant n$, then G has circumference at least k.*

Theorem 2b.24 ([13]). *If G is a 2-connected graph of order $n \geqslant 3$ satisfying $PC(n)$, then G is either a cycle, a pancyclic graph, $K_{n/2,n/2}$, $K_{n/2,n/2}-e$, or the graph in Fig. 10.*

Theorem 2b.25 ([13]). *If G is a 3-connected graph of order $n \geqslant 3$ satisfying $PC(n+1)$, then G is hamiltonian connected.*

(ϑ) *Conditions on independent triples with a common neighbor*

Flandrin et al. [75] combined a neighborhood intersection property along with a global degree condition to get the following two results.

Theorem 2b.26 ([75]). *Let G be a 2-connected graph of order $n \geqslant 27$ with minimum degree $\delta \geq (n-2)/3$. If for all triples of independent vertices u, v, w such that $N(u) \cap N(v) \cap N(w) \neq \emptyset$ we have $\max\{d(u), d(v), d(w)\} \geqslant n/2$, then G is hamiltonian.*

Theorem 2b.27 ([75]). *Let G be a 2-connected graph of order $n \geqslant 3$ with $\sigma_3 \geqslant n+2$. If for all triples of independent vertices u, v, w such that $N(u) \cap N(v) \cap N(w) \neq \emptyset$ we have $\max\{d(u), d(v), d(w)\} \geqslant n/2$, then G is hamiltonian.*

(ι) *Conditions on independent triples.*

Flandrin et al. [70] considered all triples of independent vertices and obtained the following general results.

Theorem 2b.28 ([70]). *If G is a 2-connected graph of order n such that $d(u) + d(v) + d(w) \geqslant n + |N(u) \cap N(v) \cap N(w)|$ for any independent set $\{u,v,w\}$, then G is hamiltonian.*

Theorem 2b.29 ([70]). *If G is a connected graph of order n such that $d(u) + d(v) + d(w) \geqslant n - 1 + |N(u) \cap N(v) \cap N(w)|$ for any independent set $\{u,v,w\}$, then G is traceable.*

(κ) *Biclaw-free bipartite graphs.*

An analogous result to those concerning the claw-free property and hamiltonicity is obtained for biclaw-free graphs by Flandrin et al. [67].

Theorem 2b.30 ([67]). *If G is connected, bipartite, balanced, biclaw-free with $\delta \geqslant \max(9, (n + 14)/6)$, then G is hamiltonian.*

The corresponding result for dominating cycles was proved by Barraez et al. [10].

Theorem 2b.31 ([10]). *If G is connected, bipartite, balanced, biclaw-free with $\delta \geqslant \max(24, (n + 69)/8)$, then every longest cycle in G is dominating.*

The following is conjectured by Li Hao [122].

Conjecture 2b.32 [122]. There exists a constant c such that every connected bipartite biclaw-free graph G with $\delta(G) \geqslant c$ is hamiltonian.

(c) Local connectivity conditions

It is easy to see that every graph satisfying a global degree condition must have limited diameter, i.e., every global lower bound on degrees implies at the same time an upper bound on the diameter of G (e.g., Ore's condition $\sigma_2 \geqslant n$ implies that G must have diameter at most 2). Local conditions, however, are applicable to graphs with arbitrarily large diameter.

If $\mathscr{P}$ is a property, then we say that G is *locally $\mathscr{P}$* if, for every $x \in V(G)$, $\langle N(x) \rangle$ has the property $\mathscr{P}$. As mentioned in the introduction, every locally connected claw-free graph is locally traceable and every locally 2-connected claw-free graph is locally hamiltonian. In this section we will be mainly interested in global consequences of local connectivity conditions.

Oberly and Sumner [153] proved the following result.

Theorem 2c.1 ([153]). *If G is a connected, locally connected claw-free graph on $n \geqslant 3$ vertices, then G is hamiltonian.*

Several authors observed that the assumptions of Theorem 2c.1 imply stronger cycle properties. Clark [44] (and also independently later on Shi Rong Hua [176] and Zhang [193]) proved the following.

Theorem 2c.2 ([44]). *Every connected, locally connected claw-free graph on at least three vertices is vertex pancyclic.*

Hendry [96] introduced the following concept. We say that G is *cycle extendable* if for every cycle C with $|V(C)| \geqslant n-1$ there is a cycle C' such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. G is *fully cycle extendable* if it is cycle extendable and each vertex is on a triangle.

Hendry [96] (and, later on, independently Ambartsumian et al. [6]) proved the following.

Theorem 2c.3 ([96]). *Every connected, locally connected claw-free graph on at least three vertices is fully cycle extendable.*

A graph G is a *chordal graph* if each cycle C_k in G of length $k \geqslant 4$ has a chord (i.e., an edge that joins two nonconsecutive vertices of C_k). Balakrishnan and Paulraja [9] showed that every 2-connected chordal graph is locally connected and from this and the result by Oberly and Sumner, they proved that every 2-connected claw-free chordal graph is hamiltonian. By the result of Hendry [96], we can, moreover, conclude the following.

Theorem 2c.4 ([9, 96]). *Every 2-connected claw-free chordal graph is fully cycle extendable.*

If we assume higher local connectivity, we can obtain stronger cycle and path properties. First result in this direction is by Chartrand et al. [31], who proved that if G is a connected, locally 3-connected claw-free graph then G is hamiltonian-connected. This result was improved by Clark [44] who proved that (i) any connected, locally 3-connected claw-free graph is panconnected and (ii) if G is a connected, locally 2-connected claw-free graph of diameter d then G is $(3,d)$-panconnected (we say that G is *(r,s)-panconnected* if for each pair u,v of vertices with $r \leqslant d(u,v) \leqslant s$ and for each m satisfying $d(u,v) \leqslant m \leqslant n-1$ there is a u,v-path of length m).

Finally, Kanetkar and Rao [111] proved the following.

Theorem 2c.5 ([111]). *If G is a connected, locally 2-connected claw-free graph, then G is panconnected.*

Broersma and Veldman [28] conjectured that in 3-connected claw-free graphs, the assumption can be further relaxed.

Conjecture 2c.6 [28]. Let G be a connected, locally connected claw-free graph of order at least 4. Then G is panconnected if and only if G is 3-connected.

We say that G is *k-hamiltonian* if $G - U$ is hamiltonian for every subset U of $V(G)$ with $0 \leqslant |U| \leqslant k$ $(k \geqslant 0)$. The following was observed by Chartrand et al. [31].

Theorem 2c.7 ([31]). *If G is a connected, locally $(k+1)$-connected claw-free graph, then G is k-hamiltonian.*

Broersma and Veldman conjectured in [28], and then Zhou [194] proved the following result.

Theorem 2c.8 ([194]). *Let G be a connected, locally k-connected claw-free graph $(k \geqslant 1)$. Then G is k-hamiltonian if and only if G is $(k+2)$-connected.*

On the other hand, there are several results in which the assumptions of the Oberly–Sumner theorem are replaced by weaker conditions which still imply hamiltonicity.

We say that G is *quasilocally connected* if every vertex cut set of G contains a vertex with a connected neighborhood. Zhang [193] proved that every quasilocally connected claw-free graph of order at least three is pancyclic. Ainouche et al. [2] strengthened this result showing that every quasilocally connected claw-free graph of order at least 3 is vertex pancyclic. They, moreover, showed that every vertex of a quasilocally connected claw-free graph of order $n \geq 3$ is on a special cycle of length i for each i with $3 \leqslant i \leqslant n$ (we say that a cycle C is *special* if C contains a vertex with connected neighborhood).

We say that G *has a pancyclic ordering* if $V(G)$ can be ordered such that the subgraph induced by the first k vertices is hamiltonian for any k, $3 \leqslant k \leqslant n$. The graph G is *vertex pancyclic orderable* if for every $x \in V(G)$, G has a pancyclic ordering such that x is the first vertex. Clearly, every vertex pancyclic orderable graph is vertex pancyclic. A graph, that is obtained by joining two cliques of the same order by a perfect matching, is an example of a vertex pancyclic claw-free graph that is not vertex pancyclic orderable.

A generalization of Hendry's concept of cycle extension was introduced in [56]. A nonhamiltonian cycle C is *k-chord extendable* if it can be extended to a cycle C' that has one additional vertex and uses at most k chords of C. A graph G is *k-chord extendable* if each nonhamiltonian cycle $C \subset G$ is *k-chord extendable*. The graph G is *fully k-chord extendable* if G is k-chord extendable and every vertex of G is on a triangle.

Let $M(G)$ be the set of all vertices of G that have connected neighborhood. It is easy to observe that G is quasilocally connected if and only if $M(G)$ is a dominating set and $\langle M(G) \rangle$ is connected. Faudree et al. [60] further strengthened the results of [193,2] proving the following.

Theorem 2c.9 ([60]). *Let G be a claw-free graph and put $M(G) = \{x \in V(G) | \langle N(x) \rangle$ is connected$\}$.*

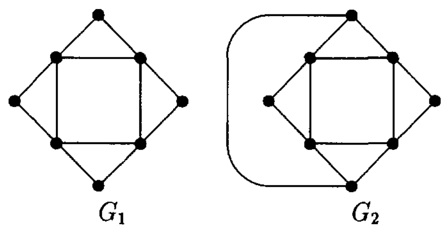

Fig. 11.

(i) *If $M(G)$ is a dominating set, $\langle M(G)\rangle$ has r components and $r \leqslant \kappa(G)$, then G is hamiltonian.*

(ii) *If $M(G)$ is a dominating set and $\langle M(G)\rangle$ is connected, then G is vertex pancyclic orderable.*

(iii) *If $M(G)$ is a dominating set, $\langle M(G)\rangle$ is connected and $G - M(G)$ is triangle-free, then G is fully 3-chord extendable.*

In [193], Zhang conjectured the following.

Conjecture 2c.10 [193]. If G is a 3-connected claw-free graph such that each vertex cut set is not an independent set, then G is hamiltonian.

Another approach is based on the idea of modifying the concept of vertex neighborhood. We say that an edge xy is a *neighboring edge* of a vertex v if $x \neq v \neq y$ and x or y is adjacent to v. The edge induced subgraph on the set of all neighboring edges of v will be called the *neighborhood of the second type* of v in G and denoted by $N_2(v, G)$ (this concept was first introduced by Sedláček [171]). The neighborhood (in the obvious sense) of G will be in this context referred to as the *neighborhood of the first type* of v and denoted by $N_1(v, G)$. A graph G is said to be *N_2-locally connected* if $N_2(v, G)$ is connected for every $v \in V(G)$. Clearly, every locally connected graph is N_2-locally connected. Ryjáček [163, 164] used this concept to prove the next theorem.

We say that G satisfies assumption (A) if for every induced subgraph H of G which is isomorphic to either G_1 or G_2 (see Fig. 11), there is at least one vertex $v \in V(G)$ of degree 4 in H such that $N_1(v, G)$ is connected.

Theorem 2c.11 ([163, 164]). *Let G be a connected, N_2-locally connected claw-free graph with minimum degree $\delta(G)$.*

(i) *If $\delta(G) \geqslant 2$, then G has a 2-factor.*

(ii) *If, moreover, G satisfies the assumption (A), then G is hamiltonian.*

(iii) *If, moreover, G is 3-connected, then G is pancyclic.*

It is conjectured in [164] that the assumption (A) is not needed in Theorem 2c.11 for 3-connected graphs.

Conjecture 2c.12 [164]. Every 3-connected N_2-locally connected claw-free graph is hamiltonian.

Chen and Xue [36] showed that if G is an N_2-locally connected claw-free graph satisfying (A), then every vertex of G is on a 3-cycle or a 4-cycle which can be extended to a hamiltonian cycle by a series of 1-extensions or 2-extensions (here, by a k-extension of a cycle C we mean a cycle C' such that $V(C) \subset V(C')$ and $|V(C') - V(C)| = k$).

Li Mingchu [131] used a similar idea to that of Zhang [193] to extend the main result of [164]. He proved that if $\delta(G) \geqslant 3$, G satisfies (A) and is N_2-quasilocally connected (i.e., if every vertex cut set of G contains a vertex with connected N_2-neighborhood), then every $x \in V(G)$ with connected $N_2(x)$ is contained in cycles of all possible lengths (and, specifically, G is pancyclic). In [130], Li Mingchu slightly further relaxed the assumption of local connectivity and proved a hamiltonicity result by replacing the condition (A) with an analogous one but with 6 induced subgraphs.

Chen Yufu et al. [34] introduced the following concept. For every $x \in V(G)$ with disconnected $\langle N(x) \rangle$ denote by $K_i(x)$ $(i = 1,2)$ the two components of $\langle N(x) \rangle$. G is said to be *strong 2-order neighbor connected* if for every $x \in V(G)$ with disconnected $\langle N(x) \rangle$ there are $y_i \in V(G) - \{x\}$ $(i=1,2)$ such that $|N(y_i) \cap K_i(x)| \geqslant 2$ and $|N(y_i) \cap N(K_{i+1}(x)) - \{x\}| \geqslant 2$ (i modulo 2).

Theorem 2c.13 ([34]). *Every connected, strong 2-order neighbor connected claw-free graph is hamiltonian.*

There are also several extensions to larger classes of graphs. In almost claw-free graphs, Ryjáček [165] showed the following.

Theorem 2c.14 ([165]). *Every connected, locally connected $K_{1,4}$-free almost claw-free graph is fully cycle extendable.*

The paper by Oberly and Sumner [153] contains the following conjecture on a generalization to $K_{1,r}$-free graphs.

Conjecture 2c.15 [153]. If G is a connected, locally r-connected $K_{1,r+2}$-free graph, then G is hamiltonian.

Shi Ronghua [177] observed that if G is a claw-free graph and if for every pair of vertices x, y with $d(x, y) = 2$, $|N(x) \cap N(y)| \geqslant 2$, then G is hamiltonian. Asratian et al. [7] extended this result to L_1-graphs in the following way.

Theorem 2c.16 ([7]). *Let G be a connected L_1-graph of order at least 3 such that $|N(u) \cap N(v)| \geqslant 2$ for every pair of vertices u, v with $d(u, v) = 2$. Then the following is true*:

(a) *G is hamiltonian unless $G \in \{G \mid K_{p,p+1} \subset G \subset K_p \vee \overline{K_{p+1}}$ for some $p \geqslant 2\}$, and*

(b) *each pair of vertices* x, y *with* $d(x, y) \geqslant 3$ *is connected by a hamiltonian path of* G.

(d) Further forbidden subgraphs

We begin this section by listing some additional connected claw-free graphs that have appeared as forbidden subgraphs in conditions that imply hamiltonian properties in graphs. Although the notation for such graphs is not standard, we will attempt to follow the most commonly used terms.

We denote by:

A, the antenna — the unique claw-free graph with degree sequence 333221,
B, the bull — the unique graph with degree sequence 33211,
D, the deer — the graph obtained from a bull by subdividing its bridges,
E, the eiffel — the graph obtained from a net by subdividing one bridge of the net,
H, the hourglass — the graph which is obtained by identifying a vertex in 2 distinct copies of K_3,
N, the net — the unique graph with degree sequence 333111,
S, the station — the unique claw-free graph with degree sequence 443331,
T, the tripod — the graph obtained from a K_4 by attaching 3 independent edges,
W, the wounded — the graph obtained from a bull by subdividing one of the bridges of the bull,
Z_i, $(i \geqslant 1)$ — the graph which is obtained by identifying a vertex of K_3 with an end-vertex of a path of length i.

These graphs appear in Fig. 12. (The graph Z_1 is also called the paw; see [154].)

Any connected graph G of order $n \geqslant 3$ that is P_3-free must be complete, so we will avoid P_3 as a forbidden subgraph. One of the earliest forbidden subgraph results for hamiltonicity in claw-free graphs is due to Goodman and Hedetniemi [85].

Theorem 2d.1 ([85]). *If* G *is a 2-connected graph that is* CZ_1*-free, then* G *is hamiltonian.*

The CZ_1-free property is also very strong, and it can be shown that any connected graph that is CZ_1-free is either a path, a cycle, or a complete graph with at most a matching missing. Thus, for a CZ_1-free, connectivity implies it is traceable, 2-connectivity implies it is either a cycle or is pancyclic, and 3-connectivity implies it is panconnected.

The larger class of CN-free graphs were studied by Duffus et al. [50], where they proved the following.

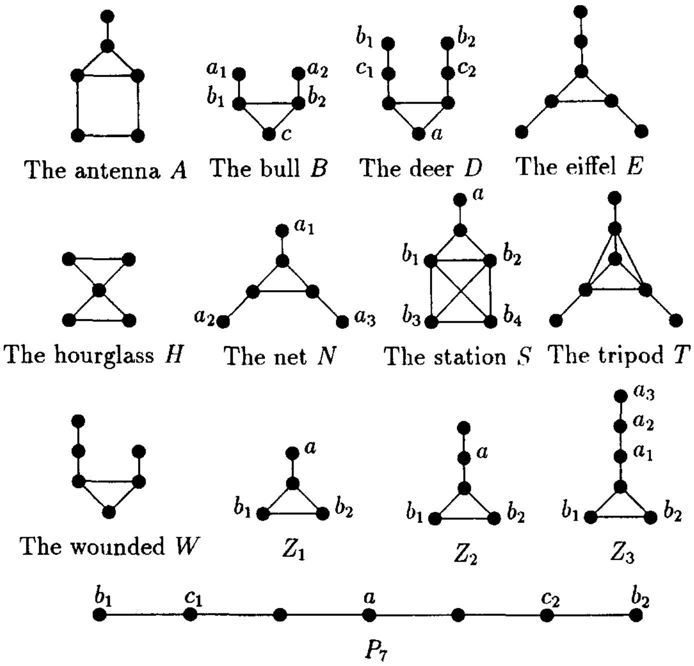

Fig. 12.

Theorem 2d.2 ([50]). *Let G be a CN-free graph.*
(i) *If G is connected, then G is traceable.*
(ii) *If G is 2-connected, then G is hamiltonian.*

Continuing in the same vein, Gould and Jacobson considered CZ_i-free graphs for $k = 2, 3$ in [88] and verified the following.

Theorem 2d.3 ([88]). *Let G be a 2-connected claw-free graph.*
(a) *If G is Z_2-free, then G is a cycle or is pancyclic.*
(b) *If G is Z_3-free and H-free, then G is hamiltonian.*
(c) *If G is H-free of diameter $\leqslant 3$, then G is homogeneously traceable.*

In the doctoral thesis of Gould [86] diameter 2 claw-free graphs were investigated and the following proved.

Theorem 2d.4 ([86]). *If G is a C-free graph of diameter at most 2, then G is hamiltonian.*

As was noted in subsection 2c, Hendry introduced the concepts of cycle extendable and fully cycle extendable in [96], and extended the result of Gould and Jacobson by proving the following.

Theorem 2d.5 ([96]). *If G is a 2-connected CZ_2-free graph of order $n \geqslant 10$, then G is cycle extendable.*

Broersma and Veldman generalized several of the forbidden subgraph conditions for hamiltonian and pancyclic results in [29]. In particular one consequence of these results is the following (cf. Theorem 2d.19(iii)).

Theorem 2d.6 ([29]). *If G is a 2-connected CP_6-free or CDP_7-free graph, then G is hamiltonian.*

A wide variety of authors have shown that various pairs of connected forbidden subgraphs imply that a graph G is hamiltonian or pancyclic. These pairs all included the claw as one of the forbidden subgraphs. This was not by accident as the following result of Bedrossian in [12], which characterizes all such pairs of graphs that avoid a P_3, shows.

Theorem 2d.7 ([12]). *Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to symmetry) $X = C$ and $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Note that each of the possible subgraphs for Y in the previous theorem are induced subgraphs of either P_6, N, or W, and that CP_6-free graphs were shown to be hamiltonian in [29], and CN-free graphs in [50]. Thus, the previous result gives one new graph W such that CW-free implies a 2-connected graph is hamiltonian, and characterizes all pairs of connected forbidden graphs in a 2-connected graph that imply hamiltonicity.

A pancyclic version of the previous theorem was also proved by Bedrossian [12].

Theorem 2d.8 ([12]). *Let X and Y be connected graphs with X, $Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being XY-free implies G is pancyclic if and only if (up to symmetry) $X = C$ and $Y = P_4, P_5, Z_1$, or Z_2.*

The maximal pairs in the pancyclic case are CP_5 and CZ_2, and it was proved in [88] that 2-connected CZ_2-free graphs are pancyclic or a cycle. The previous result adds the pair CP_5 to those forbidden graphs that imply pancyclicity and characterizes those pairs of forbidden graphs that imply pancyclicity.

Problem 2d.9. It would be interesting to characterize those pairs of connected graphs X and Y such that if G is a connected (3-connected) XY-free graph, then G is traceable (hamiltonian connected). Other hamiltonian properties such as homogeneously traceable, panconnected, pancyclic ordering, and fully cycle extendable could also be considered.

In general, 2-connected CP_6-free graphs are not pancyclic, but Faudree et al. [59] proved that the only exceptions are graphs of order at most 9. A graph G has a *k-pancyclic ordering* if the vertices of G can be ordered such that the graph induced by the first j vertices ($j \geqslant k$) is hamiltonian. Thus, a graph with a 3-pancyclic ordering has a pancyclic ordering. The following result in [59] gives some forbidden subgraph conditions for a k-pancyclic ordering.

Theorem 2d.10 ([59]). *Let G be a 2-connected C-free graph of order n.*
(i) *If G is P_5-free, then G is pancyclic and has a 5-pancyclic ordering.*
(ii) *If G is P_6-free and $n \geqslant 10$, then G is pancyclic and has a 6-pancyclic ordering.*
(iii) *If G is DP_7-free and $n \geqslant 13$, then G is pancyclic and has a 8-pancyclic ordering.*

The following theorem shows that together with P_7, the hourglass H can be also considered as a forbidden subgraph to obtain hamiltonian properties.

Theorem 2d.11 ([59]). *Let G be a 2-connected C-free graph of order n.*
(i) *If G is HP_7-free, then G is hamiltonian.*
(ii) *If G is HP_7-free and $n \geqslant 13$, then G is pancyclic or missing only one cycle.*

An example of a CHP_7-free graph that is missing exactly one cycle is the graph in Fig. 10.

Recall that a graph G is *k-chord extendable* if each nonhamiltonian cycle C in G can be extended to a cycle C' that has one additional vertex and uses at most k chords of C. The following result of Faudree et al. generalizes the result of Hendry [96].

Theorem 2d.12 ([59]). *If G is a 2-connected CZ_2-free graph of order at least* 10, *then G is 2-chord extendable.*

For any property that implies that a graph G is cycle extendable, it is natural to ask what is the smallest nonnegative integer k such that the same property implies that the graph G is k-chord extendable (or if such a k actually exists).

Shepherd in [173,174] investigated additional hamiltonian properties for CN-free graphs. In particular, the following was proved.

Theorem 2d.13 ([173]). *If G is a 3-connected CN-free graph, then G is pancyclic.*

A graph G is *k-leaf-connected*, $k \geqslant 2$, if for each subset S of k vertices, there is a spanning tree T of G whose leaves (i.e. the vertices of degree 1) are exactly the vertices in S. Thus, 2-leaf connected is just hamiltonian connected. The relationship between connectivity and leaf-connectivity was investigated by Shepherd [174].

Theorem 2d.14 ([174]). *Let G be a CN-free graph, $k \geqslant 2$. Then, G is $(k+1)$-connected if and only if G is k-leaf-connected.*

Corollary 2d.15. *Let G be a CN-free graph. Then G is hamiltonian-connected if and only if G is 3-connected.*

The results on CN-free graphs are further extended by Shepherd [174] to the class of distance claw-free graphs. Recall that the vertices at a distance i from v is denoted by $N_i(v)$ and called the *distance i neighborhood* of a vertex v. Clearly, G is claw-free if and only if $\alpha(N_1(v)) = 2$ for every $v \in V(G)$ (where $\alpha(H)$ denotes the independence number of H) and G is *distance claw-free* if $\alpha(N_i(v)) = 2$ for every $v \in V(G)$ and for every i. As mentioned in the introduction, G is distance claw-free if and only if it is claw-free, eiffel-free and tripod-free (see [174]). From this it follows immediately that

Proposition 2d.16 ([174]). *CN-free $\Rightarrow$ distance claw-free $\Rightarrow$ claw-free.*

With this characterization Shepherd proved the following in [174].

Theorem 2d.17 ([174]). *Let G be a distance claw-free graph.*
(i) *If G is 2-connected, then G is traceable.*
(ii) *If G is 3-connected, then G is hamiltonian.*

Also in [174] Shepherd proved the following characterization of CN-free graphs. A graph G is said to be distance 2-complete centered at v if $G - v$ has two components and in each component C and for each positive integer i, the vertices at distance i from v in $\langle C \cup v \rangle$ induce a complete graph.

Theorem 2d.18 ([174]). *A connected graph G is CN-free if and only if for every minimal cut set S and every v in S, $G - (S - \{v\})$ is distance 2-complete centered at v.*

Broersma and Veldman [29] generalized some of the previous results by admitting additional induced subgraphs but under special conditions on common neighbors of some of their vertices. If H is a subgraph of G and $x, y \in V(H)$, then we say that H has the property $\Phi(x,y)$ if x and y have a common neighbor outside H. See Fig. 12 for the induced graphs of the remaining theorems of this section.

Theorem 2d.19 ([29]). *Let G be a 2-connected claw-free graph.*

(i) *If every induced Z_1 of G satisfies $\Phi(a,b_1) \vee \Phi(a,b_2)$, then G is a cycle or G is pancyclic.*

(ii) *If every induced Z_2 of G satisfies $\Phi(a,b_1) \wedge \Phi(a,b_2)$, then G is a cycle or G is pancyclic.*

(iii) *If every induced subgraph of G isomorphic to P_7 or to D satisfies $\Phi(a,b_1) \vee \Phi(a,b_2) \vee (\Phi(a,c_1) \wedge \Phi(a,c_2))$, then G is hamiltonian.*

(iv) *If every induced B of G satisfies $\Phi(a_1,a_2) \vee \{[\Phi(a_1,b_2) \vee \Phi(a_1,c)] \wedge [\Phi(a_2,b_1) \vee \Phi(a_2,c)]\}$, and every induced F of G satisfies $\Phi(a,b_1) \vee \Phi(a,b_2) \vee \Phi(a,b_3) \vee \Phi(a,b_4)$, then G is hamiltonian.*

(v) *If G is H-free and every induced Z_3 of G satisfies $\Phi(a_1,b_1) \vee \Phi(a_1,b_2) \vee \Phi(a_2,b_1) \vee \Phi(a_2,b_2) \vee [\Phi(a_3,b_1) \wedge \Phi(a_3,b_2)]$, then G is hamiltonian.*

In the paper of Broersma and Veldman [29] the following was conjectured and then proved by Zhiquan Hu [102].

Theorem 2d.20 ([102]). *Let G be a 2-connected claw-free graph. If every induced N of G satisfies $(\Phi(a_1,a_2) \wedge \Phi(a_1,a_3)) \vee (\Phi(a_1,a_2) \wedge \Phi(a_2,a_3)) \vee (\Phi(a_1,a_3) \wedge \Phi(a_2,a_3))$, then G is hamiltonian.*

A weaker version of this statement, saying that 'a 2-connected claw-free graph G is hamiltonian provided every induced B of G satisfies $\Phi(a_1,a_2)$' was also conjectured in [29], and then proved independently in [166].

Broersma et al. proved in [27] the following common generalization of the minimum degree condition for hamiltonicity and of Theorems 2c.1, 2d.6 and 2d.11(i). For any $u, v \in V(G)$ we say that $\{u,v\}$ is a

- *2-pair* if u, v are at distance 2 in G,
- *B-pair* if u, v are vertices of degree 1 in an induced subgraph of G isomorphic to B,
- *H-pair* if u, v are nonadjacent vertices of degree 2 in an induced subgraph of G isomorphic to H,
- *D-pair* if $\{u,v\} = \{a,c_1\}$ or $\{u,v\} = \{a,c_2\}$ in an induced subgraph of G isomorphic to D,
- *P_7-pair* if $\{u,v\} = \{a,c_1\}$ or $\{u,v\} = \{a,c_2\}$ in an induced subgraph of G isomorphic to P_7.

Theorem 2d.21 ([27]). *Let G be a 2-connected claw-free graph on n vertices. If each 2-pair $\{u,v\} \subset V(G)$ satisfies at least one of the following conditions, then G is hamiltonian.*

(1) $\min\{d(u), d(v)\} \geqslant \frac{1}{3}(n-2)$;

(2) $|N(u) \cap N(v)| \geqslant 2$;

(3) *$\{u,v\}$ is a 3-pair in $G-w$, and there exists a path ux_1x_2v of length 3 in $G-w$ such that $wx_2 \in E(G)$, where $w \in N(u) \cap N(v)$;*

(4) *$\{u,v\}$ is neither a D-pair nor a P_7-pair in G;*

(5) *$\{u,v\}$ is neither an H-pair nor a P_7-pair in G;*

(6) *$\{u,v\}$ is a 3-pair in $G-w$, and not a B-pair in $G-w$, where $w \in N(u) \cap N(v)$;*

(7) *$\{u,v\}$ is a 3-pair in $G-w$, and there are at least two internally disjoint (u,v)-paths of length 3 in $G-w$, where $w \in N(u) \cap N(v)$.*

(e) Invariants

The well-known theorem by Chvátal and Erdős [42] states that if G satisfies $\alpha(G) \leq \kappa(G)$, then G is hamiltonian (or, equivalently, every nonhamiltonian graph of connectivity κ contains an independent set with $\kappa + 1$ vertices). Flandrin and Li [72] proved that in a 3-connected claw-free graph the assumptions can be weakened.

Theorem 2e.1 ([72]). *If G is a claw-free graph with connectivity $\kappa \geqslant 3$ and $\alpha(G) \leqslant 2\kappa$, then G is hamiltonian.*

Ainouche et al. [2] showed that every nonhamiltonian claw-free graph contains an independent set I of $\kappa + 1$ vertices such that no two vertices of I have a common neighbor. Thus, they proved the following.

Theorem 2e.2 ([2]). *If G is a k-connected claw-free graph ($k \geqslant 2$) with $\alpha(G^2) \leqslant k$, then G is hamiltonian.*

Examples are given in [2] showing that Theorem 2e.2 and the result of Flandrin and Li [72] are independent. This theorem also extends the result of Zhang [192] (saying that, in k-connected claw-free graphs, $\sigma_{\kappa+1} \geqslant n - \kappa$ implies hamiltonicity) and the theorem that every 2-connected claw-free graph of diameter at most 2 is hamiltonian (which is originally by Gould [86]). It also implies the following observation (where $\gamma(G)$ denotes the domination number of G).

Corollary 2e.3 ([165]). *If G is a k-connected claw-free graph ($k \geqslant 2$) with $\gamma(G) \leqslant k$, then G is hamiltonian.*

The result of Gould [86] as extended by Flandrin and Li [73] in the following way. Denote by $D(G)$ the diameter of G and put $D'(G) = \max\{D(\langle V(G) - \{x\}\rangle) \mid x \in V(G)\}$. Let $\mathscr{L}$ be the class of graphs that are obtained by taking $2k + 1$ ($k \geqslant 1$) vertex disjoint cliques of size at least three, choosing a pair of vertices x_i, y_i ($i = 1, \ldots, 2k + 1$) in each of them and making $\langle x_1, \ldots, x_{2k+1}\rangle$ and $\langle y_1, \ldots, y_{2k+1}\rangle$ complete graphs.

Theorem 2e.4 ([73]). *Every 2-connected claw-free graph G with $D'(G) \leqslant 3$ is hamiltonian except when $G \in \mathscr{L}$.*

The following analogue of the Chvátal–Erdős theorem for traceability was also proved in [2].

Theorem 2e.5 ([2]). *If G is a k-connected claw-free graph with $\alpha(G^2) \leqslant k + 1$, then G is traceable.*

The special case $k = 1$ gives the following corollary.

Corollary 2e.6 ([2]). *Every connected claw-free graph of diameter at most 2 is traceable.*

For any subgraph $H \subset G$ denote by $\alpha_3(H)$ the maximum number of vertices of H that are pairwise at distance at least three in G. Broersma and Lu [25] further extended Theorem 2e.2.

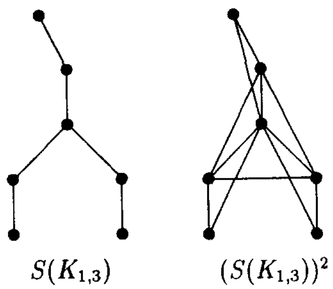

Fig. 13.

Theorem 2e.7 ([25]). *Let G be a 2-connected claw-free graph and let H be a subgraph of G. If $\alpha_3(H) \leqslant \kappa(G)$, the G has a cycle which contains all vertices of H.*

Since $\sigma_{k+1}(S) \geqslant n-k$ implies $\alpha_3(S) \leqslant k$, a consequence of Theorem 2c.7 is the following.

Corollary 2e.8 ([25]). *Let G be a k-connected claw-free graph ($k \geqslant 2$) of order n and let $S \subset V(G)$. If $\sigma_{k+1}(S) \geqslant n-k$, then G has a cycle which contains all vertices of S.*

(f) Squares

Harary and Schwenk in [93] characterized those trees whose squares are hamiltonian in terms of the forbidden subgraph $S(K_{1,3})$, the graph obtained from a claw $K_{1,3}$ by subdividing each edge (see Fig. 13). It is easily checked that $(S(K_{1,3}))^2$ is not hamiltonian, and consequently that the square of any tree with a $S(K_{1,3})$ is not hamiltonian.

Theorem 2f.1 ([93]). *If T is a tree of order at least 3, then the square T^2 is hamiltonian if and only if T is $S(K_{1,3})$-free.*

Although the square of a connected graph of even order is not, in general, hamiltonian, it does have a perfect matching (see [32,180,151]). However, much more can be said about the square of a connected claw-free graph. Matthews and Sumner [146] investigated the squares of claw-free graphs and proved the following.

Theorem 2f.2 ([146]). *If G is a connected claw-free graph, then*

(i) *G^2 is vertex pancyclic,*

(ii) *the total graph of G is hamiltonian.*

The results of Matthews and Sumner were improved by Gould and Jacobson [89] by showing that G^2 is vertex pancyclic if G is a connected Y-free graph, where Y

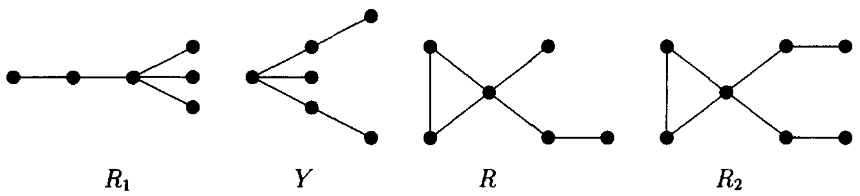

Fig. 14.

is the graph obtained from $S(K_{1,3})$ by deleting an endvertex (see Fig. 14), or if G is connected, $S(K_{1,3})$-free, and in addition, $K_{1,4}NR$-free, where N is the net and R is the graph obtained by identifying a vertex of a K_3 with one of the central vertices of a P_4 (see Fig. 14).

Theorem 2f.3 ([89]). (i) *If G is a connected $K_{1,4}$-free and Y-free graph, then G^2 is vertex pancyclic.*

(ii) *If G is a connected $K_{1,4}$-free, $S(K_{1,3})$-free, N-free and R-free graph with $n \geqslant 3$, then G^2 is vertex pancyclic.*

Similar results were obtained by Flandrin [64], where, in particular, (ii) of the previous result was improved by replacing $K_{1,4}$ by R_1, where R_1 is obtained from a $K_{1,4}$ by subdividing 1 edge, and replacing R by R_2, where R_2 is obtained by identifying a vertex of a K_3 with the central vertex of a P_5, (see Fig. 14).

Theorem 2f.4 ([64]). *If G is a connected $K_{1,4}NR_1R_2$-free graph, then G^2 is vertex pancyclic.*

The following, which improves the results of Gould and Jacobson and also of Flandrin, was conjectured by Gould and Jacobson [89] and proved by Hendry and Vogler [95].

Theorem 2f.5 ([95]). *If G is a connected $S(K_{1,3})$-free graph with at least three vertices, then G^2 is vertex pancyclic.*

Using the concept of a fully cycle extendable graph, the previous result was also strengthened by Hendry in a later paper [96], and he also conjectured that if G is a 2-connected graph (without being claw-free), then G^2 is fully cycle extendable.

Theorem 2f.6 ([96]). *If G is a connected $S(K_{1,3})$-free graph with at least three vertices, then G^2 is fully cycle extendable.*

Sekanina [172] and independently later on Karaganis [112] proved that for any tree T, the cube T^3 is hamiltonian. Since the endvertices of the tree can be pruned, an immediate consequence of this is the following result.

Theorem 2f.7 ([172]). *If G is a connected graph, then G^3 has a pancyclic ordering.*

However, more can be said about the cube of a connected graph. Šoltés (personal communication) proved that such a graph is fully cycle extendable.

Theorem 2f.8. *If G is a connected graph, then G^3 is fully cycle extendable.*

It would be interesting to know if full cycle extendibility can be replaced by k-chord cycle extendable for some small integer k, say, $k \leqslant 3$, in all or even some of the previous results on the square and cube of a connected graph. At least $k = 3$ is necessary for the cube of a graph.

(g) Regular graphs

A strong motivation for considering claw-free regular graphs is the result of Jackson on regular graphs in [103].

Theorem 2g.1 ([103]). *Every 2-connected, k-regular graph of order $n \leqslant 3k$ is hamiltonian.*

The Petersen graph, which is non-hamiltonian and 3-regular of order 10, implies that for $k = 3$ the degree of regularity in the previous result cannot be reduced. Also, for $k \geqslant 4$ there are non-hamiltonian 2-connected k-regular graphs with $3k + 5$ vertices, so the degree of regularity cannot be reduced significantly in the result of Jackson. A minor improvement was made by Zhu et al. [137], where it was shown for $k \geqslant 5$ that a 2-connected k-regular graph of order at most $3k + 3$ is hamiltonian.

For claw-free graphs a smaller degree of regularity is sufficient to imply that a graph is hamiltonian, as the following result of Liu and Wu [136] proves.

Theorem 2g.2 ([136]). *Every 2-connected, k-regular claw-free graph of order $n \leqslant 4k+1$ is hamiltonian.*

There is an example of a nonhamiltonian graph in [136] to show that for $k = 4$ the previous result of Liu and Wu is sharp, but it is not known if this is true for large k. Li Mingchu has also proved a sharp result on the circumference of a regular 2-connected claw-free graph.

Theorem 2g.3 ([129]). *Every 2-connected, k-regular claw-free graph of order $n \geqslant 4k-2$ has a cycle of length at least $4k - 2$.*

If the connectivity is increased, then a smaller degree of regularity is sufficient to give hamiltonicity. This was shown by Li and Liu [132].

Theorem 2g.4 ([132]). *Every 3-connected, k-regular claw-free graph of order $n \leqslant 5k-5$ is hamiltonian.*

It is not known if the degree of regularity in the previous result is sharp. Of course, it was conjectured by Matthews and Sumner [146] that any 4-connected claw-free graph is hamiltonian. The next result of Plummer [161] is a special case of the conjecture of Matthews and Sumner.

Theorem 2g.5 ([161]). *If G is a 4-connected, 4-regular, claw-free graph containing a K_4, then G is hamiltonian.*

The previous results come out of a stronger result about the structure of 4-connected, 4-regular claw-free graphs that contain a K_4. These results also lead to the next conjecture.

Conjecture 2g.6 [161]. Every 4-connected 4-regular claw-free graph in which each vertex lies in exactly two triangles is hamiltonian.

It should be noted that the previous conjecture is equivalent to another well-known conjecture by Thomassen [184] that any 4-connected line graph is hamiltonian (see also [174, p. 174]) and that both these conjectures (if true) follow from Conjecture 2a.3 by Matthews and Sumner.

The toughness of cubic graphs was investigated by Jackson and Katerinis in [105], and they proved the following result.

Theorem 2g.7 ([105]). *Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if $G = K_4$, $G = K_2 \times K_3$, or G is the inflation of a 3-connected graph.*

Corollary 2g.8 ([105]). *Every 3/2-tough cubic graph is claw-free.*

Corollary 2g.8 and Theorem 2a.1 immediately give the following.

Corollary 2g.9. *Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if G is 3-connected and claw-free.*

(h) Other hamiltonicity related results and generalizations

There are numerous generalizations of paths and cycles and hamiltonian properties in graphs, and one of these generalizations is the k-walk. A *k-walk* in G is a closed spanning walk that visits each vertex of G at most k-times, and if each vertex is visited exactly k-times, then it is called an *exact k-walk*. Thus, if G has at least three vertices, then a 1-walk in G is a hamiltonian cycle of G.

Jackson and Wormald in [106] investigated k-walks in claw-free graphs. In particular, they proved the following, which implies the existence of k-walks in $K_{1,k+1}$-free graphs.

Theorem 2h.1 ([106]). *Let G be a connected $K_{1,k+1}$-free graph. Then*
(i) *G has a k-walk, and*
(ii) *if $\delta(G) \geqslant k$, then G has an exact k-walk.*

In the special case $k = 1$, we obtain the following corollary.

Corollary 2h.2 ([106]). *Every connected claw-free graph G has a 2-walk. If, moreover, $\delta(G) \geqslant 2$, then G has an exact 2-walk.*

As the connectivity of G increases, the restriction on the claw-free property can be decreased, as the following result indicates.

Theorem 2h.3 ([106]). *If $j \geqslant 1$, $k \geqslant 3$ and G is j-connected and $K_{1,j(k-2)+1}$-free, then G has a k-walk.*

The previous general result does not imply the result for connected graphs that preceded it, but it is conjectured that the result can be sharpened so that this is true.

Conjecture 2h.4 [106]. If $j \geqslant 1$, $k \geqslant 2$ and G is j-connected and $K_{1,jk+1}$-free, then G has a k-walk.

The previous conjecture and the conjecture of Matthews and Sumner [146] leads to the following question concerning r-connected $K_{1,r}$-free graphs.

Question 2h.5 [106]. If $r \geqslant 4$ and G is r-connected and $K_{1,r}$-free, then does G have a 1-walk?

Oberly and Sumner [153] proved that every connected, locally connected claw-free graph is hamiltonian (Theorem 2c.1), so the following theorem is another extension of that well-known result.

Theorem 2h.6 ([106]). *For $k \geqslant 1$, every connected, locally connected $K_{1,k+2}$-free graph with at least three vertices has a k-walk.*

The next conjecture is the locally connected analogue of one of the previous conjectures.

Conjecture 2h.7 [106]. If $j \geqslant 1$, $k \geqslant 1$ and G is connected, locally j-connected and $K_{1,(j+1)k+1}$-free, then G has a k-walk.

In [62] Favaron et al. investigated k-walks in almost claw-free graphs, and generalized some of the results for claw-free graphs of Jackson and Wormald [106]. Also, additional information on the nature of k-walks are given. Denote by $v(x, C)$ the number of visits by a k-walk C of a vertex $x \in V(G)$.

Theorem 2h.8 ([62]). *If G is a connected almost claw-free graph of order $n \geq 3$, then*

(i) *G has a 2-walk,*

(ii) *for any $x \in V(G)$ there is a 2-walk C such that $v(x, C) = 1$ if and only if x is not a cutvertex of G,*

(iii) *if $\langle N(x) \rangle$ is connected then there is a shortest 2-walk C such that $v(x, C) = 1$, and*

(iv) *if $B = \{x \in V(G) | \; \langle N(x) \rangle$ is connected and x not the center of a claw$\}$, then there is a shortest 2-walk C such that $v(x, C) = 1$ for every $x \in B$.*

Part (iv) is a common generalization of results of Oberly and Sumner [153] and of Jackson and Wormald [106].

In the special case of claw-free graphs, there is the following corollary.

Corollary 2h.9 ([62]). *Let G be a connected claw-free graph.*

(i) *If $x \in V(G)$, then there is a 2-walk that visits x exactly once if and only if x is not a cutvertex of G.*

(ii) *There is a shortest 2-walk that visits once all vertices with a connected neighborhood.*

Some of the degree conditions that implied a graph was hamiltonian, such as those of Ore and Dirac, were generalized by Bondy and Chvátal by considering closure conditions (adding an edge without changing the hamiltonian property of the graph). A closure-type condition was considered by Broersma [30] and, in claw-free graphs, this lead to the following result. Two vertices u, v of G are said to be a K_4-pair if u and v are the vertices of degree two of an induced subgraph of G which is isomorphic to $K_4 - e$.

Theorem 2h.10 ([30]). *Let $\{u, v\}$ be a K_4-pair of a claw-free graph G. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

3. Matchings and factors

The basic result concerning perfect matchings or 1-factors in claw-free graphs was proved independently by Sumner [180] and Las Vergnas [119]

Theorem 3.1 ([180, 119]). *If G is a connected claw-free graph of even order, then G has a 1-factor.*

Since the claw does not have a perfect matching, the previous result implies that each connected subgraph of even order in a graph G has a 1-factor if and only if G is claw-free. Another consequence is that if G is a connected graph of even order n

without a 1-factor, then for each $4 \leqslant k \leqslant n$ there is a connected subgraph of order k without a 1-factor.

The classical result of Tutte says that a graph G fails to have a 1-factor if and only if there is some separating set S such that the number of odd components $c_0(G-S)$ in $G-S$ exceeds $|S|$. A set S with the property that $c_0(G-S) > |S|$ is called an antifactor set. Sumner showed in [181] that in a graph G without a 1-factor, an antifactor set S can be chosen such that each vertex in S is the center of a claw.

Theorem 3.2 ([181]). *If G is an even connected graph that does not have a perfect matching then there is a set $S \subset V(G)$ such that $c_0(G-S) > |S|$ and every vertex of S is adjacent to vertices in at least three odd components of $G-S$.*

In fact, if the antifactor set S is minimal, then each vertex of S is the center of a claw.

Corollary 3.3 ([181]). *If G is a connected graph of even order that does not have a perfect matching and S is a minimal antifactor set for G, then every element of S is a claw center.*

Corollary 3.4 ([181]). *If a k-connected graph G of even order has fewer than k claw centers, then G has a 1-factor.*

The previous results imply that the theorem of Tutte can be restated in the following form.

Theorem 3.5. *A graph G has a 1-factor if and only if there does not exist a set S of claw centers of G such that $c_0(G-S) > |S|$.*

Additional properties of antifactor sets in graphs without 1-factors and the relationship of antifactor sets and the block structure of a graph can be found in [181].

The result of Sumner and Las Vergnas can be improved significantly as the connectivity of the graph increases.

Theorem 3.6 ([181]). *If $r \geqslant 2$, then every $(r-1)$-connected $K_{1,r}$-free graph of even order has a 1-factor.*

The result of Sumner and Las Vergnas for claw-free graphs has been generalized in several ways, including the following two results due to Nebeský [150] and Ryjáček [165]. Each have the Sumner and Las Vergnas result as a corollary.

Theorem 3.7 ([150]). *Let G be a graph of even order. Assume that there exists a connected spanning subgraph F of G such that for every set U of four vertices in G, if $\langle U\rangle_F$ is isomorphic to the claw, then $\langle U\rangle_G$ is isomorphic to K_4. Then G has a 1-factor.*

Theorem 3.8 ([165]). *Every connected almost claw-free graph of even order has a 1-factor.*

Recall that a graph is an L_1-graph is for each triple of vertices u, v, w with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$, then $d(u) + d(v) \geqslant |N(u) \cup N(v) \cup N(w)| - 1$. The class of L_1-graphs includes the claw-free graphs, and Asratian et al. [7] extended the 1-factor result for claw-free graphs to L_1-graphs.

Theorem 3.9 ([7]). *If G is a connected L_1-graph of even order, then G has a 1-factor.*

If a graph G has odd order, then it cannot have a 1-factor (or perfect matching). However, other factors that approximate 1-factors have been investigated for such graphs. More generally, if H_1, H_2 are graphs, then by a $H_1, \{H_2\}$-*factor* of G we mean a factor H such that exactly one component of H is isomorphic to H_1 and all the other components of H are isomorphic to H_2. Also, a factor is said to be *strong* if its components are induced subgraphs. Natural factors to consider for odd order graphs are $K_3, \{P_2\}$-factors, $P_3, \{P_2\}$-factors, and $C_{2k+1}, \{P_2\}$-factors, which are called *perfect 2-matchings*. This type of problem was investigated by Lonc and Ryjáček [138] and Ryjáček [163].

In [138], a complete characterization was given for classes of claw-free graphs of odd order that fail to have

(i) a $K_3, \{P_2\}$-factor,
(ii) a strong $P_3, \{P_2\}$-factor,
(iii) a perfect 2-matching.

From these characterizations one can easily obtain the following assertion, which was originally proved in [163].

Corollary 3.10 ([163]). *Let G be a connected claw-free graph with odd number $n \geqslant 3$ vertices. If G has at most one vertex of degree 1 then G has a perfect 2-matching.*

If N_2-locally connected is added to the claw-free property, then the existence of a 2-factor was also shown in [163], and mentioned in Section 2c.

Theorem 3.11 ([163]). *If G is a connected, N_2-locally connected claw-free graph with minimum degree $\delta(G) \geqslant 2$, then G has a 2-factor.*

Less is known about the existence of k-factors in claw-free graphs for $k \geqslant 2$. However, one of the first results was due to Choudum and Paulraj [38].

Theorem 3.12 ([38]). *Let $k \geqslant 1$ be an integer. If G is a connected claw-free graph with $k|V(G)|$ even and with minimum degree $\delta(G) \geqslant 2k$, then G has a k-factor.*

The minimum degree condition sufficient to insure a k-factor in a claw-free graph was weakened by Egawa and Ota [52].

Theorem 3.13 ([52]). *Let $k \geqslant 2$ be an integer. If G is a connected claw-free graph with $k|V(G)|$ even and with minimum degree $\delta(G) \geqslant \lceil (9k+12)/8 \rceil$, then G has a k-factor.*

This previous theorem of Egawa and Ota is an extension of a result of Nishimura in [152], in which the same lower bound on δ was shown to imply the existence of a k-factor in line graphs. Egawa and Ota also have an analogous result for $K_{1,r}$-free graphs in the same article.

Theorem 3.14 ([52]). *Let r for $(r \geqslant 3)$ and k be positive integers. If k is odd, we assume that $k \geqslant r-1$. Let G be a connected $K_{1,r}$-free graph with $k|V(G)|$ even, and suppose that the minimum degree of G is at least $(r^2/4(r-1))k+(3r-6)/2+(r-1)/4k$. Then G has a k-factor.*

Examples are given which verify that the condition $k \geqslant r-1$ is necessary. Although the minimum degree condition in the previous results might not be sharp, examples are given to verify that they are of the correct order of magnitude.

If G is a graph with a perfect matching and $1 \leqslant k \leq (n-2)/2$, then we say that G is *k-extendable* if every matching of size k is a subset of a perfect matching in G. Also, a graph G is *bicritical* if $G-u-v$ has a perfect matching for every pair of vertices u, v, so a bicritical graph is 1-extendable. The extendibility of matchings was studied by Plummer [160], and the following two complementary results were proved.

Theorem 3.15 ([160]). *If G is a k-extendable claw-free graph, then $\delta(G) \geqslant 2k$.*

Theorem 3.16 ([160]). *If G is a $(2k+1)$-connected claw-free graph, then G is k-extendable.*

There are some interesting special cases of the previous results for planar graphs. For example, it follows that every 3-connected claw-free graph is 1-extendable, but it is known that no planar graph is 3-extendable. The following was proved in [160].

Theorem 3.17 ([160]). (i) *The only* 2*-extendable* 3*-connected claw-free planar graph is the icosahedron.*

(ii) *No toroidal claw-free graph is* 3*-extendable.*

In [168], Ryjáček extended the minimum connectivity condition result for claw-free graphs of Plummer [160] to $K_{1,r}$-free graphs with independent claw centers (and hence, as a corollary, also to almost claw-free graphs).

Theorem 3.18 ([168]). *Let G be an even $(2k+1)$-connected $K_{1,k+3}$-free graph such that the set of vertices that are centers of induced claws is independent. Then G is k-extendable.*

The extendibility of matchings for 3-connected and 4-regular 4-connected claw-free graphs was studied by Plummer [161]. This led to a series of results which give some improvement on his more general previous results for 2-extendable graphs. First two results for 3-connected graphs.

Theorem 3.19 ([161]). *Let G be a 3-connected even order claw-free graph. Then G is 2-extendable if and only if G does not contain two independent edges $e_i = a_ib_i$ $(i = 1,2)$ such that $G - a_1 - a_2 - b_1 - b_2$ consists of precisely two components which are both odd.*

Theorem 3.20 ([161]). *Let G be a 3-connected cubic even order claw-free graph and $\{e_1, e_2\}$ $(e_i = a_ib_i, i = 1,2)$ a set of two independent edges in G. Then $\{e_1, e_2\}$ extends to a perfect matching of G if and only if $G - a_1 - a_2 - b_1 - b_2$ is not disconnected with one of its components a single vertex.*

Additional characterizations of cubic 3-connected claw-free graphs and 2-extendibility of matchings can also be found in [161]. The conditions of the next result are also sufficient for a graph to be hamiltonian (see Theorem 2g.2).

Theorem 3.21 ([161]). *Let G be a 4-connected, 4-regular claw-free graph containing a K_4. Then either $G = K_5$ or G is 2-extendable.*

Further general information on matchings and factors in graphs can be found in the survey paper by Akiyama and Kano [3] and in the excellent book *Matching Theory* by Lovász and Plummer [140].

4. Independence, domination, other invariants and extremal problems

We begin with a discussion of independence, domination and irredundance in a graph. Recall that a set S of vertices of a graph G is *independent* if no pair of vertices of S are adjacent, is *dominating* if the closed neighborhood of S is $V(G)$, and is *redundant* if the closed neighborhood of some proper subset of S is the same as the closed neighborhood of S. A set S is *irredundant* if it is not redundant. For any graph G we define the following parameters.

$\alpha(G)$ = the maximum cardinality of a maximal independent set in G (i.e. the independence number of G),
$\gamma(G)$ = the minimum cardinality of a minimal dominating set in G (i.e. the domination number of G),
$i(G)$ = the minimum cardinality of a maximal independent set in G (i.e., the independent domination number of G),
$\Gamma(G)$ = the maximum cardinality of a minimal dominating set in G,
$ir(G)$ = the minimum cardinality of a maximal irredundant set in G, and
$IR(G)$ = the maximum cardinality of a maximal irredundant set in G.

Using the definitions, it is straightforward to verify the following relationships between these parameters.

Theorem 4.1. *For any graph G,*

$$ir(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \alpha(G) \leqslant \Gamma(G) \leqslant IR(G).$$

In one of the first papers concerning independence and domination for claw-free graphs Allan and Laskar [4] proved (and, as mentioned in [61], it was also proved independently by Jaeger and Payan) that $\gamma(G) = i(G)$.

Theorem 4.2 ([4]). *If G is claw-free, then* $\gamma(G) = i(G)$.

Since every induced subgraph of a claw-free graph is claw-free, it follows that claw-free graphs are *domination perfect* (i.e., if G is claw-free, then $\gamma(H) = i(H)$ for every induced subgraph H of G). The concept of domination perfect graphs was introduced by Sumner and Moore in [183], and they showed that a graph G is domination perfect if and only if each induced subgraph H of G with $\gamma(G) = 2$ has $i(G) = 2$.

Bollobás and Cockayne [18] extended the result of Allan and Laskar [4] from claw-free graphs to graphs that are $K_{1,r}$-free.

Theorem 4.3 ([18]). *If G is* $K_{1,k+1}$*-free, then*

$$i(G) \leqslant \gamma(G)(k-1) - (k-2).$$

Zverovich and Zverovich [195] attacked the question of characterizing domination perfect graphs. However, Fulman [78] showed that the statement in [195] fails and he gave counterexamples to it. In addition, as a generalization of the result of Allan and Laskar [4], he proved that a graph G is domination perfect if it does not contain an induced subgraph that is isomorphic to one of the eight graphs in Fig. 15 (Note that both graphs G_6 and G_7 are not domination perfect graphs, so these 8 graphs do not give a forbidden subgraph characterization of domination perfect graphs).

Topp and Volkmann [186] generalized the result of Fulman by proving that $\gamma(G) = i(G)$ for a class of graphs that are characterized in terms of 16 forbidden subgraphs, each of which has order at most 8 and contains as an induced subgraph some of the graphs in Fig. 1. (Note that, in the notation of [186], the graphs H_2 and H_3 are redundant since H_5 contains an induced H_2 and H_6 contains an induced H_3.)

A forbidden subgraph characterization of graphs G such that $\gamma(G) = i(G)$ is impossible, since adding a new vertex v adjacent to each vertex of any graph H gives a graph G with $\gamma(G) = 1 = i(G)$.

The result of Bollobás and Cockayne [18] was extended by Zverovich and Zverovich [195], where it was shown that $i(G) \leqslant \gamma(G)(k-1) - (k-2)$ provided G does not contain two induced subgraphs $K_{1,k+1}$ having different centers and an edge in common.

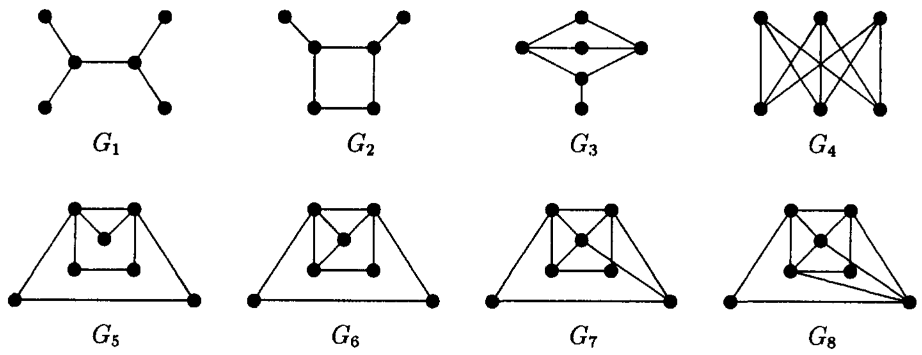

Fig. 15.

In [63] Fink and Jacobson also generalized the result of Allan and Laskar by considering the more general concepts of k-domination and k-dependence. A set $S \subset V(G)$ is *k-dependent* if $\Delta(\langle S \rangle) \leqslant k$. A set $D \subset V(G)$ is *k-dominating* if every $x \in V(G) - D$ has at least k neighbors in D. The *k-domination number* of G, $\gamma_k(G)$, is the minimum cardinality of a k-dominating set in G, and the *j-dependent-k-domination* number $i(j,k;G)$ of G is the minimum cardinality of a j-dependent k-dominating set in G.

Note that $i(j,k;G)$ does not exist for all pairs j,k; but, if it does exists, then $\gamma_k(G) \leqslant i(j,k;G)$. Furthermore, $\gamma(G) = \gamma_1(G)$ and $i(G) = i(0,1;G)$. The following was proved for claw-free graphs. Recall that a paw is a triangle and an edge sharing a vertex, and an hourglass H is two triangles sharing a vertex (see Fig. 12).

Theorem 4.4 ([63]). *If G is a claw-free graph, then*
(i) *if $\Delta(G) \geqslant k$, then $\gamma_k(G) < \gamma_{2k}(G)$,*
(ii) *$i(2k-2,k;G) = \gamma_k(G)$, and*
(iii) *if G is either paw-free or H-free and $(H+e)$-free, then $i(k-1,k;G) = \gamma_k(G)$.*

Favaron [61] proved sufficient conditions for equality of some of the previous parameters in terms of forbidden subgraphs. For example, the following was proved.

Theorem 4.5 ([61]). *If G is claw-free and D-free (deer-free), then $ir(G) = i(G)$.*

Denote by L the graph which is obtained by taking two distinct copies of K_4 and by adding three independent edges between these complete graphs. This leads to the following sufficient condition for $\Gamma(G) = IR(G)$.

Theorem 4.6 ([61]). *If G is claw-free, D-free, and L-free, then $\Gamma(G) = IR(G)$.*

Gernert [80] determined an upper bound on the domination number $\gamma(G)$ of a 2-connected claw-free graph G.

Theorem 4.7 ([80]). *If G is 2-connected and claw-free of order n, then $\gamma(G) \leqslant \lceil n/3 \rceil$. Also, this bound is sharp, since $\gamma(C_n) = \lceil n/3 \rceil$.*

We next consider the relationship between the chromatic number $\chi(G)$ and the clique number $\omega(G)$ of a claw-free graph G. Sumner [182] considered the relationship between the chromatic number and the clique number for graphs with certain trees as forbidden subgraphs. In particular, the tree $K_{1,3}$ was considered, giving the next result.

Theorem 4.8 ([182]). *If G is claw-free and is not an odd cycle, then $\chi(G) \leqslant [((\omega(G))^2 + 1)/2]$.*

Clearly, $\chi(G) \geqslant \omega(G)$ in general, but it is of interest to determine conditions that imply $\chi(G) = \omega(G)$, or at least $\chi(G)$ is close to $\omega(G)$. The classical result of Vizing [188] implies that $\chi(G) \leqslant \omega(G)+1$ for any line graph G (recall that line graphs have been characterized in terms of forbidden subgraphs, one being the claw, by Beineke [14] and more recently improved by Šoltés [178]). In [37], Choudum extended these results for line graphs by determining a class of graphs defined in terms of forbidden subgraphs (one also being a claw) for which $\chi(G) \leqslant \omega(G) + 1$, and this result was improved by Javdekar [107]. Also, in [107] it was conjectured that if G is a claw-free and $(K_5 - e)$-free graph, then $\chi(G) \leqslant \omega(G) + 1$.

Kierstead and Schmerl [115] showed that if G is a graph that is claw-free and $(K_5 - e)$-free, then $\chi(G) \leqslant \omega(G)+2$. They, in fact, showed that if three additional graphs were forbidden as induced subgraphs, then the stronger conclusion that $\chi(G) \leqslant \omega(G)+1$ is true. However, Kierstead improved on this result in [113] by proving the following.

Theorem 4.9 ([113]). *If G is claw-free and $(K_5 - e)$-free, then $\chi(G) \leqslant \omega(G) + 1$.*

If an appropriate restriction is placed on the maximum degree $\Delta(G)$ of a claw-free and $(K_5 - e)$-free graph, then $\chi(G) = \omega(G)$. This was proved by Kierstead and Schmerl in [116].

Theorem 4.10 ([116]). *If G is claw-free and $(K_5 - e)$-free with maximum degree at most $2\omega(G) - 5$, then $\chi(G) = \omega(G)$.*

It should be noted that the previous result is sharp in the sense that for each $k \geqslant 4$ there is a graph G which is claw-free, $(K_5 - e)$-free and such that $\omega(G) = k$, $\Delta(G) = 2k - 3$ and $\chi(G) = k + 1$.

If the forbidden induced subgraph $K_5 - e$ is replaced by the graph $K_{2s+3} - e$ in a claw-free graph, then there is still a restriction on the chromatic number as a function of the clique number. This was proved by Kierstead [114]. In the following result, $R(n,m)$ denotes the *Ramsey number* for the pair of complete graphs K_n and K_m.

Theorem 4.11 ([114]). *If G is claw-free and $(K_{2s+3}-e)$-free, then*

$$\chi(G) \leqslant \max\{\omega(G)+s, R(3,4s-1)\}.$$

Corollary 4.12 ([114]). *If G is claw-free, $(K_{2s+3}-e)$-free and $\omega(G)$ is sufficiently large, then*

$$\chi(G) \leqslant \omega(G)+s.$$

It would be nice to be able to drop the condition that $\omega(G)$ is sufficiently large in the previous corollary, but that problem still remains open.

Problem 4.13 [114]. *Show that if G is claw-free and $(K_{2s+3}-e)$-free, then*

$$\chi(G) \leqslant \omega(G)+s$$

(*i.e. even when* $\omega(G)+s < R(3,4s-1)$).

The result, similar to that of Kierstead in [113], was proved by Dhurandhar [49] by replacing the graph K_5-e by the graph $K_5-K_{1,2}$.

Theorem 4.14 ([49]). *If G is a claw-free and $K_5-K_{1,2}$-free graph, then $\chi(G) \leqslant \omega(G)+1$.*

The independence number $\alpha(G)$ of claw-free graphs has also been investigated. While considering neighborhood union properties in claw-free graphs that imply a graph is hamiltonian, Li and Virlouvet [125] proved the following result involving the independence number of a graph.

Theorem 4.15 ([125]). *Let G be a claw-free graph of order n. Then for any integer $t \leqslant \alpha$, there exists some independent set H of t vertices in G such that*

$$\left|\bigcup_{v\in H} N(v)\right| \leqslant \frac{t(\alpha(G)-t-1)}{\alpha(G)(\alpha(G)-1)}(n-\alpha(G)).$$

For the special case $t=1$, this gives the following corollary.

Corollary 4.16 ([125]). *For any claw-free graph G of order n we have $\delta(G) \leqslant 2(n-\alpha(G))/\alpha(G)$.*

Also while investigating hamiltonian properties of $K_{1,r}$-free graphs satisfying degree or neighborhood type conditions, similar results to those of Li and Virlouvet on the independence number of $K_{1,r}$-free graphs were proved by Faudree et al. [57]. For the neighborhood union of pairs of vertices the following was proved.

Theorem 4.17 ([57]). *If G is a $K_{1,r+1}$-free graph with $(r \geqslant 2)$ such that the cardinality of the neighborhood union of each pair of non-adjacent vertices is at least k, then $\alpha(G) \leqslant s$, where s is the largest solution to*

$$ks(s-1) = r(n-s)(2s-r-1).$$

The corresponding result for the sum of degrees of independent sets of vertices in $K_{1,r}$-free graphs is the following condition.

Theorem 4.18 ([57]). *If G is a $K_{1,r+1}$-free graph such that $\sigma_p = px$, for some p with $1 \leqslant p \leqslant \alpha(G)$, then*

$$\alpha(G) \leqslant nr/(x+r).$$

In the special case when $p = 1$ (i.e. when $\sigma_p(G) = \delta(G)$), the previous result gives the following corollary.

Corollary 4.19 ([57]). *If G is a $K_{1,r+1}$-free graph with minimum degree δ, then*

$$\alpha(G) \leqslant nr/(\delta+r).$$

In the special case of claw-free graphs (i.e., when $r=2$ in the three previous results), we have the following three consequences.

Theorem 4.20 ([57]). *If G is a claw-free graph such that the cardinality of the neighborhood union of pairs of non-adjacent vertices is at least k, then $\alpha(G) \leqslant s$, where s is the larger solution to $ks(s-1) = 2(n-s)(2s-3)$.*

For example, for $k = n/3 + c$ (where c is some small constant) this implies $\alpha(G) \leqslant 11$.

Theorem 4.21 ([57]). *If G is a claw-free graph such that $\sigma_p = px$, for some p with $1 \leqslant p \leqslant \alpha(G)$, then $\alpha(G) \leqslant 2n/(x+2)$.*

The next result was also proved independently by Li and Virlouvet [125].

Corollary 4.22 ([57, 125]). *If G is a claw-free graph with minimum degree δ, then $\alpha(G) \leqslant 2n/(\delta+2)$.*

In [167] Ryjáček and Schiermeyer determined upper bounds on the independence number $\alpha(G)$ for $K_{1,r}$-free graphs in terms of the number of edges of G, the degree sequence of G, the number of vertices that are the centers of various size claws of G, and the connectivity $\kappa(G)$ of G. In particular, for the number of edges, the following was proved.

Theorem 4.23 ([167]). *Let G be a $K_{1,r+1}$-free graph ($r \geqslant 2$) having n vertices and m edges. Then*

$$\alpha(G) = 1 \quad \text{if } m = \tbinom{n}{2},$$
$$\alpha(G) \leqslant \tfrac{1}{2}\left(2n + 2r - 1 - \sqrt{8m + (2r-1)^2}\right) \quad \text{if } 0 \leqslant m < \tbinom{n}{2}.$$

If the degree sequence of the graph is known, then more can be said about the independence number of the graph.

Theorem 4.24 ([167]). *Let G be a $K_{1,r+1}$-free graph ($r \geqslant 2$) with degree-sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. Then*

$$\alpha(G) \leqslant \max\left\{ k \mid k + \frac{1}{r}\sum_{i=1}^{k} d_i \leqslant n \right\}.$$

For the connectivity of G the following was proved.

Theorem 4.25 ([167]). *Let G be a $K_{1,r+1}$-free graph ($r \geqslant 2$) with connectivity κ. Then*

$$\alpha(G) \leqslant \frac{(r-1)n - \kappa + 2}{r}.$$

In the case of claw-free graph (i.e. when $r = 2$ in the previous results), we have the following summary result.

Theorem 4.26 ([167]). *Let G be a claw-free graph of order n, size m, connectivity κ and degree-sequence $d_1 \leqslant d_2 \cdots \leqslant d_n$. Then*

(i) $\alpha(G) = 1$ *if* $m = \binom{n}{2}$, $\alpha(G) \leqslant \frac{1}{2}(2n + 3 - \sqrt{8m+9})$ *if* $0 \leqslant m < \binom{n}{2}$;

(ii) $\alpha(G) \leqslant \max\{k \mid k + \frac{1}{2}\sum_{i=1}^{k} d_i \leqslant n\}$;

(iii) $\alpha(G) \leqslant \frac{1}{2}(n - \kappa + 2)$.

Any almost claw-free graph is $K_{1,5}$-free, so the previous results give some bounds on the independence number of an almost claw-free graph. However, it was also proved in [167] that the independence number of an almost claw-free graph with minimum degree δ can be at most $\frac{2}{3}n$ if $\delta = 1$, $\frac{4}{7}n$ if $\delta = 2$ and $2n/(\delta + 1)$ if $\delta \geqslant 3$.

The number of independent sets of vertices in claw-free graphs was investigated by Hamidoune in [90]. Let $s_k(G)$ be the number of independent k-subsets of $V(G)$ in the graph G. Hamidoune proved that the sequence $\{s_k\}$ is log concave, and thus is unimodal, and he verified the following inequality.

Theorem 4.27 ([90]). *If G is a claw-free graph, then*

$$s_k^2(G) \geqslant (1 + 1/k)s_{k+1}(G)s_{k-1}(G) + s_k(G).$$

The following conjecture also appears in [90].

Conjecture 4.28 [90]. If G is a claw-free graph, then

$$s_k^2(G) \geqslant \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\alpha(G) - k}\right) s_{k+1}(G) s_{k-1}(G).$$

Woodall [190] introduced the concept of *binding number*, which is defined as follows: $\text{bind}(G) = \min\{|N(S)|/|S|\}$, where the minimum is taken over all nonempty subsets $S \subset V(G)$ such that $N(S) \neq V(G)$, and where $N(S)$ is the neighborhood of the set S. It was proved by Woodall [190] that for every graph G of order n, $\text{bind}(G) \leqslant (n-1)/(n - \delta(G))$.

Goddard showed in [84] that the binding number $\text{bind}(G)$ for claw-free graphs G with some conditions on the connectivity and minimum degree could be determined precisely. In particular, the maximum possible value of the binding number determined by Woodall is attained for a large class of claw-free graphs, as the following result indicates.

Theorem 4.29 ([84]). *Let G be a claw-free graph of order n. If the connectivity of G is at least $\delta - 1$ and $n \neq \delta + 2$, then $\text{bind}(G) = (n-1)/(n-\delta)$.*

Classical extremal problems such as the determination of Ramsey numbers and Turán numbers can be considered for the family of claw-free graph. This was done by Matthews [144], where he introduced the concepts of claw-free Ramsey and Turán numbers by adding the restriction that G is claw-free to the obvious definition. That is, the *claw-free Ramsey number* $\text{rc}(s,t)$ is the minimum integer n such that every claw-free graph on n vertices contains K_s or $\overline{K_t}$; similarly, the *claw-free Turán number* $\text{tc}(n,s)$ is the minimum integer n such that every claw-free graph having n vertices and $\text{tc}(n,s)$ edges contains K_s. The next two theorems summarize the results obtained by Matthews on the extremal numbers $\text{rc}(s,t)$ and $\text{tc}(n,s)$.

Theorem 4.30 ([144]). (i) $\text{rc}(s,t) \leqslant r(s,t)$,

(ii) $\text{rc}(s,t) \leqslant \text{rc}(s-1,3) + \text{rc}(s,t-1)$,

(iii) $\text{rc}(3,t) = (5t-3)/2$,

(iv) $\text{rc}(s,3) = r(s,3)$,

(v) $6t - 9 \geqslant rc(4,t) \geq \begin{cases} 4t-3 & \textit{if } n \equiv 1 \,(\text{mod} 3), \\ 4t-4 & \textit{otherwise.} \end{cases}$

Theorem 4.31 ([144]). (i) $\text{tc}(n,3) = n+1$ *for* $n > 3$,

(ii)

$$\text{tc}(12k+r,4) = \begin{cases} 30k+1 & \textit{for } r = 0,1, \\ 30k+2 & \textit{for } r = 2, \\ 30k+3r-5 & \textit{for } r = 3,\ldots,10, \\ 30k+26 & \textit{for } r = 11, \end{cases}$$

except $\text{tc}(4,4) = 6$ *and* $\text{tc}(5,4) = 9$,

(iii) $\text{tc}(n,s) \leqslant (n/2)[r(s-1,3) - 1]$.

5. Algorithmic aspects

As noted in the introduction, it is obvious that claw-freeness in a given graph can be tested in a polynomial time with complexity at most $O(n^4)$, since it is sufficient to consider only all subgraphs induced by 4 vertices. However, Alon and Boppana [5], in the context of boolean functions, point out that the recognition of a claw-free graph can be done in time $O(n^{3.5})$.

Theorem 5.1 ([5]). *There is a algorithm for the recognition of a claw-free graph that can be done in time* $O(n^{3.5})$.

If some additional assumptions are made, this bound can be lowered. For example, Fouquet [76] gets an easy algorithm to recognize a connected claw-free, W_5-free graph with independence number at least 3, where W_5 denotes the wheel with 5 spokes.

Theorem 5.2 ([76]). *There is a* $O(n^3)$*-algorithm to recognize a connected claw-free,* W_5*-free graph* G *with* $\alpha(G) \geqslant 3$.

We are now interested in determining for which problems the complexity of some algorithms is improved (that is lowered) when dealing with a claw-free graph. Some results of this type can found in a survey article by Johnson [108].

5.1. NP-complete

Some problems that are *NP-complete* in general graphs still remain *NP-complete* for the family of claw-free graphs. This is the case of the determination of domination number $\gamma(G)$, which was proved by Hedetniemi and Laskar [94].

Theorem 5.3 ([94]). *The determination of the domination number* $\gamma(G)$ *of a claw-free graph is NP-complete.*

The computation of the chromatic number $\chi(G)$ of a claw-free graph G is NP-complete, since it is even true for line graphs. This follows since a vertex coloring in $L(G)$ corresponds to an edge coloring in G, and determination of the edge chromatic number is NP-complete.

The maximum clique problem, although polynomial for line graphs (see the survey article by Johnson [108], remains NP-complete for claw-free graphs since the maximum independent set problem is NP-complete in triangle-free graphs (see [158]).

Theorem 5.4 ([158]). *The determination of the maximum independent set is NP-complete for triangle-free graphs, and so the maximal clique problem is NP-complete for claw-free graphs.*

Bertossi [17] proved that the complexity of hamiltonian problems is not improved for the class of claw-free, and in fact not even for line graphs.

Theorem 5.5 ([17]). *Recognizing hamiltonian line graphs (and hence also recognizing hamiltonian claw-free graphs) is an NP-complete problem.*

However, there are some polynomial algorithms concerning cycle and path problems in some subclasses of claw-free graphs defined by additional forbidden subgraphs. Zhang considered extending cycles in connected, locally connected, and claw-free graphs in [193] and proved the following.

Theorem 5.6 ([193]). *There is a polynomial algorithm for constructing a cycle C' of length $|V(C)|+1$ for any cycle C of length$< n$ in a connected, locally connected claw-free graph.*

In [174] Shepherd considered algorithms for finding paths and cycles in CN-free graphs, and proved the following.

Theorem 5.7 ([174]). *There is an $O(n^6)$-algorithm for finding a hamiltonian path (respectively hamiltonian cycle) in a connected (respectively 2-connected) CN-free graph.*

5.2. Polynomial

Some problems which were *NP-complete* in general graphs become *polynomial* in claw-free graphs. The most well-known of them is the determination of the independence number and the recognition of perfect graphs. Results of this type are detailed below.

5.2.1. Independence number

Three different approaches have been taken in producing polynomial algorithms for determining the independence number $\alpha(G)$ of a graph G. An important result underlying two of the approaches is due to Edmonds [51], where he exhibited a polynomial algorithm for finding a maximal matching in a graph. A key idea that was used in [51] was the notion of an *augmenting path*, which is, given a matching M, a path whose edges are alternately in M and $E(G)-M$, and whose end vertices are not incident to any edge of M.

When going to the line graph of a graph, a maximum matching is transformed into a maximum independent set. Using this idea, Sbihi [169] gives a polynomial algorithm for finding a maximum independent set in a claw-free graph, and Minty, [149] describes a different but also polynomial algorithm to find an independent set with maximum weight in a weighted claw-free graph.

Theorem 5.8 ([169, 149]). *There is a polynomial algorithm to find an independent set of maximum weight in a weighted claw-free graph, and so there is a polynomial algorithm to determine $\alpha(G)$.*

Both of these algorithms use the idea of *augmenting path* defined as follows. For a given independent set S in G, a path P is generated whose vertices are alternately in S and $G-S$, $V(P)\cap V(G-S)$ is an independent set, and the endvertices of P are in $G-S$ and have no neighbors in S that are not in P.

The second approach is to use directly the result of Edmonds [51], which clearly implies that finding a maximum independent set in the line graph of a graph is polynomial. Lovász and Plummer [140, pp. 479–480] define a reduction procedure which is a sequence of transformations such that at each step the independence number is decreased by either one or two, and the final graph is a line graph. They also remark that the resulting algorithm can be implemented with complexity $O(n^4)$.

Using techniques analogous to those of Lovász and Plummer, the existence of a polynomial algorithm for the determination of the independence number was verified for some superclasses of claw-free graphs. De Simone and Sassano [47] considered the class of bull-free and chair-free graphs (where the chair is a graph which is obtained from the claw by subdividing one of the edges), and proved the following.

Theorem 5.9 ([47]). *There exists a polynomial algorithm that determines the independence number $\alpha(G)$ of a bull-free and chair-free graph G.*

Using analogous techniques that were developed independently, Hammer et al. also gave sharper polynomial algorithms for the determination of the independence number for special subclasses of claw-free graphs determined by additional forbidden subgraphs. They considered CAN-free graphs in [91] and later CN-free graphs in [92], and proved the next two results.

Theorem 5.10 ([91]). *There is an $O(n^3)$-algorithm for determining the independence number in CAN-free graphs.*

Theorem 5.11 ([91]). *There is an $O(n^3)$-algorithm for determining the independence number in CN-free graphs.*

More recently, in [97], Hertz and de Werra exhibited an infinite set of forbidden graphs that characterize a class of graphs for which there exists a polynomial algorithm to determine the independence number of the graph.

Theorem 5.12 ([97]). *There is polynomial algorithm that determines the independence number α for the class of graphs characterized by the infinite set of forbidden subgraphs ($k \geqslant 0$) in Fig.* 16.

Finally, Giles and Trotter [82] study the independence number problem as a maximization problem of a linear function on independent set polyhedra associated with claw-free graphs.

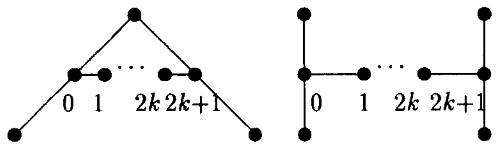

Fig. 16.

5.2.2. Perfect graphs

Perfect graphs are another class of graphs for which an NP-complete problem becomes polynomial in the presence of claw-freeness. First let us recall some definitions, results, and conjectures. Let $\theta(G)$ be the minimum number of cliques to cover the vertices of G. Recall that a graph G is said to be *perfect* if it satisfies one of the following three equivalent properties:

(i) for every induced subgraph H of G, $\chi(H) = \omega(H)$,

(ii) for every induced subgraph H of G, $\theta(H) = \alpha(H)$.

(iii) For every induced subgraph H of G, $\omega(H)\alpha(H) \geqslant |V(H)|$.

Also, recall from the introduction that an odd hole (respectively antihole) in a graph G is an induced subgraph of G which is isomorphic to a chordless cycle of odd length (respectively its complement). Berge's strong perfect graph conjecture states the following.

Conjecture 5.13. (Strong perfect graph) A graph is perfect if and only if it does not contain odd holes nor odd antiholes with at least 5 vertices.

Parthasarathy and Ravindra [156] proved that claw-free graphs satisfy the strong perfect graph conjecture, and subsequently an alternative proof was given by Giles and Trotter [82] and Giles et al. [83].

Theorem 5.14 ([156, 82, 83]). *Claw-free graphs satisfy the strong perfect graph conjecture.*

A graph G is *critically imperfect* if G is not perfect but each proper induced sugbraph of G is perfect. In [156] the strong perfect graph conjecture was restated using the concept of critically imperfect and Theorem 5.14.

Conjecture 5.15 [156]. No critically imperfect graph contains an induced claw.

A related result is one of Olariu concerning the superclass of the *k-pan-free* graphs. The *k-pan* is obtained from a chordless cycle C_k, $(k \geqslant 4)$, by adding a new vertex x and precisely one edge between x and the cycle C_k. Clearly, every claw-free graph is pan-free but not conversely. In [155], Olariu proved the following for pan-free graphs

Theorem 5.16 ([155]). *Pan-free graphs satisfy the strong perfect graph conjecture.*

The problem of the recognition of perfect graphs is known to be NP-complete (see [108]). However, if making the additional assumption of claw-freeness, Chvátal and Sbihi [43] get a polynomial algorithm for the recognition of claw-free perfect graphs. Using the Ben Rebea Lemma (if G is a connected claw-free graph with $\alpha(G) \geqslant 3$, and if G contains an odd antihole, then it contains a hole of length five), they proved the following.

Theorem 5.17 ([43]). *There is a polynomial algorithm for recognizing claw-free perfect graphs.*

When one restricts consideration to just the subclass of the claw-free perfect graphs, several problems which were NP-complete even for claw-free graphs become polynomial. Hsu [98] and jointly Hsu and Nemhauser [99–101] observed that for any vertex u in a claw-free perfect graph G the complement of the subgraph of G induced on $N(u)$ is bipartite, and used this to prove the existence of polynomial algorithms. In particular, the following was proved.

Theorem 5.18 ([98]). *There is an* $O(n^4)$*-algorithm for determining the chromatic number* $\chi(G)$ *(minimum vertex coloring) of a claw-free perfect graph* G.

Theorem 5.19 ([99]). (i) *There is an* $O(n^{5.5})$*-algorithm for determining a minimum clique cover in a claw-free perfect graph.*

(ii) *There is an* $O(n^{3.5})$*-algorithm for determining a maximum clique in a claw-free perfect graph.*

Theorem 5.20 ([100]). *There is a polynomial algorithm for determining a minimum weighted clique cover in a weighted claw-free perfect graph.*

Using combinatorial arguments different algorithms were described by Hsu and Nemhauser [101] to verify the following.

Theorem 5.21 ([101]). *There is a polynomial algorithm* ($O(n^4)$) *for determining a maximum weighted clique and a minimum weighted clique cover in a weighted claw-free perfect graph. Also, there is a polynomial algorithm for determining a minimum cardinality vertex coloring in a claw-free perfect graph.*

5.2.3. Breadth first search tree

In [145], Matthews made the following observation about the *breadth first search* tree (BFS-tree) in a claw-free graph.

Lemma 5.22 ([145]). *If* u *is an interior node in a BFS-tree in a claw-free graph with children* $c_1, c_2, \ldots, c_k$, *then the subgraph induced in* G *by* $c_1, c_2, \ldots, c_k$ *is a complete graph.*

Using this observation Sumner gave efficient algorithms for finding perfect matchings in connected claw-free graphs and hamiltonian cycles in the square of a claw-free graph.

Theorem 5.23 ([145]). *Let G be a claw-free graph with m edges: Then*
(i) *there is an* $O(m)$*-algorithm for finding a perfect matching in G, and*
(ii) *an* $O(m)$*-algorithm for finding a hamiltonian cycle in the square G^2 of G.*

This algorithm in Theorem 5.23 can be modified to provide an algorithm for finding a cycle of any length i, $3 \leqslant i \leqslant n$, containing any specified vertex.

6. Miscellaneous

Ellingham et al. [53] proved that the edge reconstruction conjecture is true in the class of claw-free graphs (i.e., every claw-free graph with at least four edges is uniquely determined by the collection of its edge-deleted subgraphs).

Krasikov [118] extended the result of Ellingham et al. [53] to $K_{1,r}$-free graphs.

Theorem 6.1 ([118]). *Let G be a $K_{1,r}$-free graph which is not edge reconstructible. Then* $\Delta(G) = O(r(\log r)^{1/2})$.

Corollary 6.2 ([118]). *If c is a sufficiently large constant, then every $K_{1,r}$-free graph with average degree* $> 2\log r + \log\log r + c$ *is edge reconstructible.*

Posa proved that, for $n \geqslant 6$, every graph with $m \geqslant 3n - 6$ contains two (vertex) disjoint cycles and every graph with $m \geqslant n+4$ contains two edge disjoint cycles; these results are sharp. Matthews [144] improved the first result and showed that the second one cannot be improved in the case of claw-free graphs.

Theorem 6.3 ([144]). *If G is a claw-free graph with $m \geqslant n + 6$, then G contains at least two disjoint cycles and this bound is sharp.*

Markus [141] extended this result as follows.

Theorem 6.4 ([141]). *Let G be a claw-free graph and let $k \geqslant 1$. If* $m \geqslant n + (3k-1)(3k-4)/2 + 1$, *then G contains k disjoint cycles and this bound is sharp.*

For $K_{1,r}$-free graphs, Markus and Snevily [143] obtained the following extension.

Theorem 6.5 ([143]). *If G is a $K_{1,r}$-free graph with $r \geqslant 4$ and $m \geqslant n + 2r - 1$, then G contains at least 2 disjoint cycles and this bound is sharp.*

Paulraja [157] proved that if G is a connected claw-free graph such that every edge of G lies on a cycle C_k of length $k \leqslant 5$, then G has a spanning eulerian subgraph.

In [134] Liu Yiping proved that if G is a 2-connected claw-free graph with $\delta \leqslant (n-5)/2$, then G contains a cycle with at least $\delta(\delta - 1)$ diagonals.

Galeana-Sánchez [79] showed that if G is a digraph which is obtained as an orientation of a claw-free graph such that each of its oriented cycles of length at least 5 has 2 diagonals, then G has a kernel.

Knor et al. [117] studied centers and peripheries in line graphs. They proved, among others, that every line graph is a center of some line graph. The paper is concluded with the following conjecture.

Conjecture 6.6 [117]. Every claw-free graph is a center of some claw-free graph.

A graph G is said to be *perfectly orderable* if $V(G)$ admits a linear order $<$ such that no induced path with vertices a, b, c, d and edges ab, bc, cd has $a < b$ and $d < c$. In connection with characterization of totally balanced (0,1)-matrices, Chvátal [41] characterized perfectly orderable claw-free graphs in terms of nine well-described infinite families of forbidden induced subgraphs.

Recall that G is said to be *k-extendable* if every matching of size k in G is a subset of a perfect matching, and G is *bicritical* if $\langle V(G) - \{u, v\}\rangle$ has a perfect matching for every $u, v \in V(G)$. Plummer [159] characterized all maximal planar claw-free graphs and, from this, he obtained the following.

Theorem 6.7 ([159]). *If G is a maximal planar claw-free graph with $|V(G)| \geqslant 4$ then G is panconnected and, if G is even, then G is bicritical. Moreover, G is not 2-extendable unless G is the icosahedron.*

In [161], Plummer studied the structure of claw-free regular graphs with higher connectivities and he proved the following.

Theorem 6.8 ([161]). *A graph G is cubic, 3-connected and claw-free if and only if $G = K_4, G = C_3 \times K_2$, or G is the 3-inflation of a cubic 3-connected graph H.*

Theorem 6.9 ([161]). *If G is a 4-connected 4-regular claw-free graph and G contains a K_4, then either $G = K_5$ or $V(G)$ can be partitioned into disjoint sets of four vertices such that each four-vertex set induces a K_4 in G.*

Recently, Nebeský (personal communication) obtained the following result.

Theorem 6.10. *Let G be a 2-connected claw-free graph such that every edge of G is on a cycle of length at least 5, and let $\mathscr{P}$ be a partition of $V(G)$ such that, for every $X \in \mathscr{P}$, $|X| \geqslant 2$ and $\langle X\rangle$ is connected. Denote by $E_{\mathscr{P}}$ the set of all edges of G which have vertices in different sets from $\mathscr{P}$. Then*

$$|E_{\mathscr{P}}| \geqslant 2(|\mathscr{P}| - 1).$$

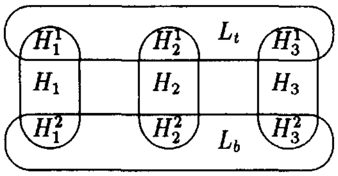

Fig. 17. List of all possible graphs H_j ($j = 1,2,3$).

From this, Nebeský obtains the following corollary.

Corollary 6.11. *Let G be a 2-connected claw-free graph such that every edge of G is on a cycle of length at most 5. Then G is upper embeddable.*

Appendix: List of all 2-connected nonhamiltonian claw-free graphs on $n \leqslant 12$ vertices [4]

n: number of vertices
(i) number of nonisomorphic connected claw-free graphs
(ii) number of nonisomorphic 2-connected claw-free graphs
(iii) number of nonisomorphic 2-connected nonhamiltonian claw-free graphs

n	(i)	(ii)	(iii)
1	1	0	0
2	1	0	0
3	2	1	0
4	5	3	0
5	14	8	0
6	50	32	0
7	191	126	0
8	881	619	0
9	4494	3332	4
10	26 389	20 910	16
11	184 749	157 721	84
12	1 728 403	1 590 329	408

All 2-connected nonhamiltonian claw-free graphs on $n \leqslant 12$ vertices can be obtained (up to isomorphism) by the following construction.

1. Take three (not necessarily distinct) vertex-disjoint copies H_1, H_2, H_3 of some of the graphs in Fig. 18 and denote by H_j^1, H_j^2 the two cliques induced by the double-circled vertices of H_j, $j = 1,2,3$.

2. Take two vertex disjoint graphs L_t, L_b on at least 3 vertices such that either

(a) both are complete graphs or

(b) one of them is complete and the other is isomorphic to L_1 or to L_2 in Fig. 19.

[4] Authors' thanks for this chapter are to O. Zýka, Charles University, Prague, for providing computer search, and to J. Brousek, University of West Bohemia, Pilsen, for checking all the possible cases.

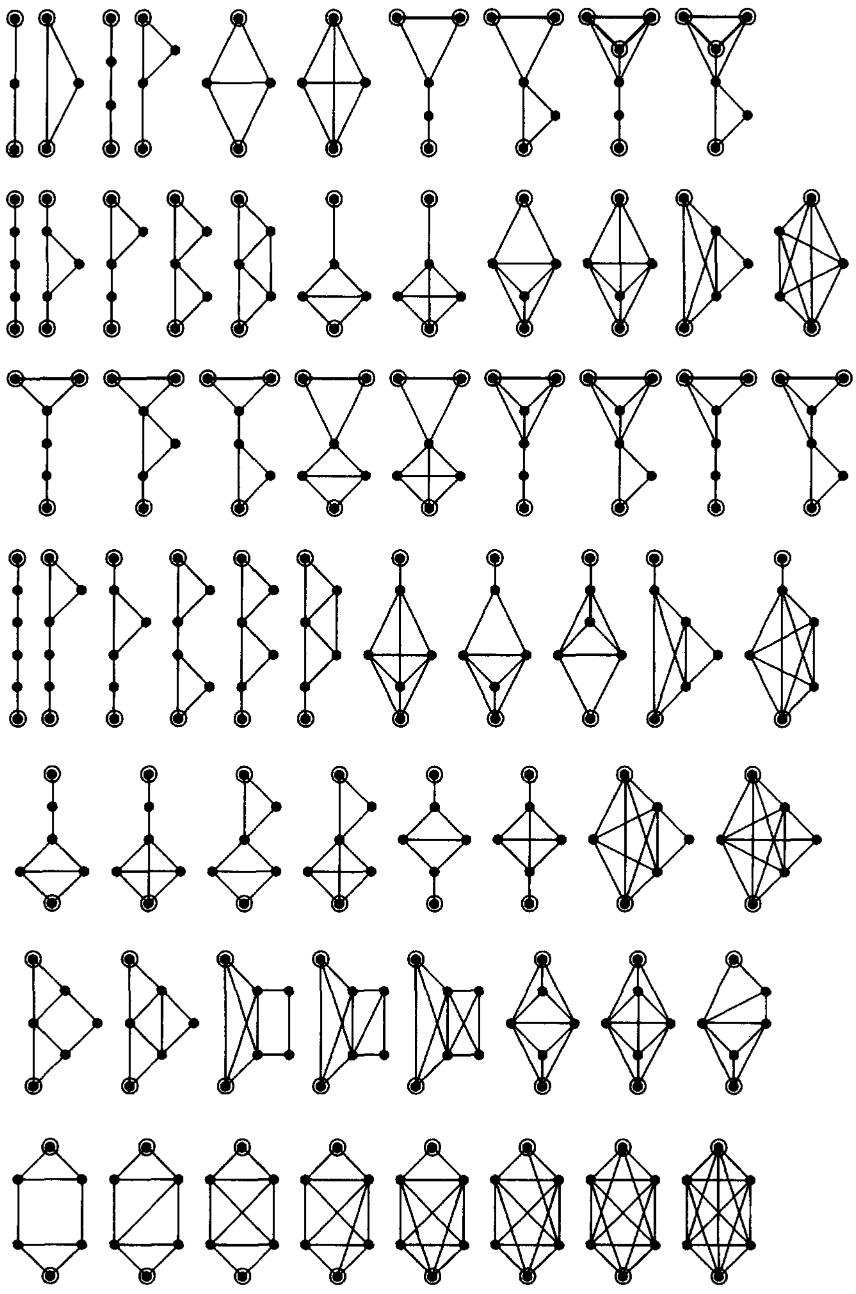

Fig. 18. The graphs L_i ($i = 1, 2$).

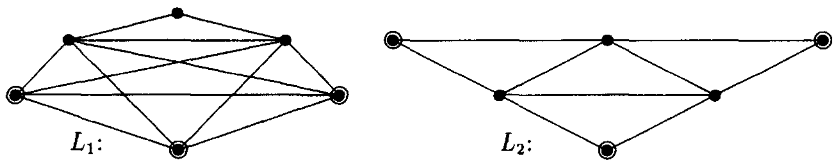

Fig. 19.

3. For each H_j, identify $H_j^1$and H_j^2 with some subclique in L_t and L_b or a one-element H_j^k with a double-circled vertex in L_i in the case $2b$, respectively.

Orders of H_j and L_i must be chosen and the identification must be done such that H_1, H_2, H_2 remain vertex disjoint and the resulting graph has at most 12 vertices (see Fig. 17).

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[5] Recently, the following result was proved by Ryjáček [198]:
If G is a claw-free graph, then there is a graph $cl(G)$ such that
(i) G is a spanning subgraph of $cl(G)$,
(ii) $cl(G)$ is the line graph of a triangle-free graph, and
(iii) the length of a longest cycle in G and in $cl(G)$ is the same.
This result implies that Conjecture 2a.3 is equivalent with the Thomassen's conjecture (every 4-connected line graph is hamiltonian). Using a result of B. Jackson [196] and S. Zhan [199], this also implies that every 7-connected claw-free graph is hamiltonian. Li Hao [197] improved this result to 6-connected graphs with at most 33 vertices of degree 6.