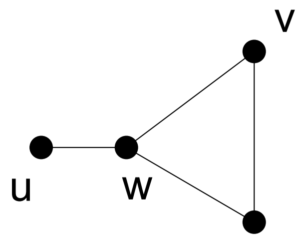


Figure 1: Counterexample for Problem 1a).

MATH 350: Graph Theory and Combinatorics. Fall 2012.

Assignment #1: Paths, Cycles and Trees. Solutions.

**1.** For each of the following statements decide if it is true or false, and either prove it or give a counterexample.

**a)** If $u, v, w$ are vertices of $G$, and there is an even length path from $u$ to $v$ and an even length path from $v$ to $w$ then there is an even length path from $u$ to $w$.

**Solution:** False. See Figure 1.

**b)** If $G$ is connected and has no path with length larger than $k$, then every two paths in $G$ of length $k$ have at least one vertex in common.

**Solution:** True. Suppose for a contradiction that $P_1$ and $P_2$ are two vertex disjoint paths of length $k$. Let vertices of $P_i$ be $v_1^i, v_2^i, \ldots, v_{k+1}^i$, in order. Let $Q$ be the a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be a short as possible. Let $v_n^1$ and $v_m^2$ be the ends of $Q$. We can suppose without loss of generality that $m, n \geq k/2 + 1$. Then a path obtained by taking the union of the subpath of $P_1$ from $v_1^1$ to $v_n^1$, the path $Q$ and the subpath of $P_2$ from $v_1^2$ to $v_m^2$ has at least $m + n \geq k + 2$ vertices, a contradiction.

**c)** If $u, v, w$ are vertices of $G$, and there is a cycle of $G$ containing $u$ and $v$, and a cycle containing $v$ and $w$, then there is a cycle containing $u$ and $w$.

**Solution:** False. Consider a graph $G$ with $V(G) = \{u, v, w\}$ and $E(G)$ consisting of a pair of edges joining $u$ to $v$ and a pair of edges joining $v$ to $w$.

**d)** If $e, f, g$ are edges of $G$, and there is a cycle containing $e$ and $f$, and a cycle containing $f$ and $g$, then there is a cycle containing $e$ and $g$.

**Solution:** True. Without loss of generality we may assume that $G$ is connected. The result follows immediately from the next claim.

**Claim:** If there exist does not exist a cycle containing edges $e$ and $g$ then there does not exist a vertex $u \in V(G)$ such that every path in $G$ sharing one end with $e$ and another with $g$ contains $u$.

**Proof:** The claim trivially holds if $e$ or $g$ is a loop, so we assume that neither is. Let $P$ with vertex set $v_1, v_2, \ldots, v_k$, in order, be a path with $e$ joining $v_1$ to $v_2$ and $g$ joining $v_{k-1}$ and $v_k$. Let $f_i \in E(P_i)$ be the edge with ends $v_i$ and $v_{i+1}$. Let $j$ be chosen minimum so that no cycle in $G$ contains $e$ and $f_j$. We will show that $u = v_j$ satisfies the claim.

Suppose not. Let $C$ be a cycle containing $e$ and $f_{j-1}$ and let $P'$ be a path from an end of $e$ to an end of $f$ avoiding $u$. Choose a subpath $Q$ of $P'$ with one end in $V(C)$ and another in $\{v_{j+1}, v_{j+2}, \ldots, v_k\}$ as short as possible. Then $C \cup Q \cup P$ contains a cycle containing both $e$ and $f_j$, a contradiction. (The last statement requires some case checking.)

**2.** Show that every non-null graph $G$ contains at least

$$|E(G)| - |V(G)| + \mathrm{comp}(G)$$

distinct cycles.

**Solution:** Proof by induction on $|E(G)| - |V(G)| + \mathrm{comp}(G) := \mathrm{rk}(G)$. If $\mathrm{rk}(G) = 0$ the statement trivially holds. For the induction step suppose $\mathrm{rk}(G) = k > 0$. Then by (3.1) $G$ is not a forest an so $G$ contains a cycle $C$. Let $e$ be a nd edge of $C$. Let $G' = G \setminus e$. Then $\mathrm{comp}(G') = \mathrm{comp}(G)$ by (2.7). Thus $\mathrm{rk}(G') = \mathrm{rk}(G) - 1$ and $G'$ contains at least $k-1$ distinct cycles by the induction hypothesis. The cycle $C$ is a subgraph of $G$, but not $G'$, and therefore $G$ contains at least $k$ distinct cycles.

**3.** Show that a loopless graph $G$ is a forest if and only if intersection of any two intersecting paths in $G$ is a path.

**Solution:** If $G$ contains a cycle $C$, consider two distinct vertices $u$ and $v$ of $C$. Let $P_1, P_2 \subseteq C$ be two paths with ends $u$ and $v$ such that $P_1 \cup P_2 = C$. Then $P_1 \cap P_2$ is an edgeless graph with two vertices $u$ and $v$ – not a path. Suppose now that $G$ is a forest. Let $P_1$ and $P_2$ be intersecting path in $G$. Then $P_1 \cap P_2$ is connected. Indeed, for all $u, v \in V(P_1 \cap P_2)$ the unique path $P_{uv}$ in $G$ with ends $u$ and $v$ is a subgraph of both $P_1$ and $P_2$, and thus of $P_1 \cap P_2$. Moreover, $P_1 \cap P_2$ contains no vertices of degree 3 or more. It follows from (1.1) and (3.1) that $P_1 \cap P_2$ has at most tow leaves and thus is a path by (3.3).

**4.** Let $T$ be a tree, and let $T_1, \ldots, T_n$ be connected subgraphs of $T$ so that $V(T_i \cap T_j) \neq \emptyset$ for all $i, j$ with $1 \leq i < j \leq n$. Show that $V(T_1 \cap T_2 \cap \ldots \cap T_n) \neq \emptyset$.

**Solution:** Proof by induction on $V(T)$. Base case $|V(T)| = 1$ is trivial. For the induction step, let $v$ be a leaf of $T$ and let $u$ be the unique vertex of $T$ adjacent to $v$. Let $T' = T \setminus v$ and let $T'_i = T' \setminus v$ for $i = 1, 2, \ldots, n$. If $V(T'_i \cap T'_j) \neq \emptyset$ for all $i, j$ with $1 \leq i < j \leq n$, then we can apply the induction hypothesis to $T'$ to complete the proof. Thus we may assume, without loss of generality, that $V(T'_1) \cap V(T'_2) = \emptyset$. It follows that $V(T_1) \cap V(T_2) = \{v\}$. Thus either $u \notin V(T_1)$ or $u \notin V(T_2)$. Without loss of generality, we have $V(T_1) = \{v\}$. Therefore $v \in V(T_i)$ for every $1 \leq i \leq n$ by the assumption and $v \in V(T_1 \cap T_2 \cap \ldots \cap T_n)$, as desired.

**5.** Let $v_1, v_2, v_3$ be distinct vertices of a graph $G$ such that $G \setminus v_1$, $G \setminus v_2$, $G \setminus v_3$ are all acyclic. Show that $G$ contains at most one cycle.

**Solution:** Suppose for a contradiction that $C_1$ and $C_2$ are two distinct cycles in $G$. We have $v_1, v_2, v_3 \in V(C_1 \cap C_2)$. Let $P$ be a path with ends in $V(C_1)$ so that $P \subseteq C_2$, $P \subsetneq C_1$, chosen to be as short as possible. (We can choose such a path as a subpath of $C_2$ with ends $v_1$ and $v_2$ satisfies the required conditions.) Then no internal vertex of $P$ belongs to $C_1$. Let $Q_1$ and $Q_2$ be the two paths in $C_1$ with the same ends as $P$. Then $C_1 = Q_1 \cup Q_2$ and $C_3 := Q_1 \cup P$ and $C_4 := Q_2 \cup P$ are also cycles. As $C_1, C_3$ and $C_4$ have only two vertices in common (the ends of $P$), one of $G \setminus v_1$, $G \setminus v_2$, $G \setminus v_3$ contains $C_1$, $C_3$ or $C_4$. A contradiction.