MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #1: Paths, Cycles and Trees

Due Wednesday, October 5th, 2016, 14:30

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1. For each of the following statements decide whether it is true or false, and either prove it, or give a counterexample.

a) Let G be a graph on $n \geq 2$ vertices with the vertex set $V = \{v_1, \ldots, v_n\}$. There exists two distinct vertices v_i and v_j such that $\deg(v_i) = \deg(v_j)$.

Solution: *The question was stated ambigously, since the answer depends on whether we consider only simple graphs or not. Both answers will be accepted, if they were correctly argued.*

If we assume that G is simple, then the statement is True. Every vertex in G has the degree between 0 and $n-1$, and there are n vertices in total. If all the degrees would be different, then G must contain a vertex u with $\deg(u) = 0$, and a vertex v with $\deg(v) = n-1$. However, that means that u is an isolated vertex (in particular, u is not adjacent to v), and v is a vertex adjacent to all the $n-1$ vertices different from v (in particular, v is adjacent to u); a contradiction.

If G does not have to be simple, so in particular, multiple edges are allowed, the statement is False, as can be seen in Figure 1.

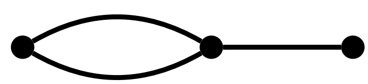

Figure 1: A counterexample for Problem 1a) in the case G does not have to be a simple graph.

b) Let G be a graph and u, v, w be three vertices of G. If there is a cycle in G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

Solution: False. See Figure 2.

c) Let G be a graph and e, f, g be three edges of G. If there is a cycle in G containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

Solution: True. Let $e = \{u_1, v_1\}$ and $f = \{u_2, v_2\}$. Fix an arbitrary cycle C_1 containing e and f. Without loss of generality,

$$C_1 = \underbrace{u_2, \ldots, u_1}_{P}, e, \underbrace{v_1, \ldots, v_2}_{Q}, f, u_2$$

where P and Q are the paths on C_1 between u_1 and u_2, and v_1 and v_2, respectively, that both avoid the edges e and f. Clearly, P and Q are vertex-disjoint.

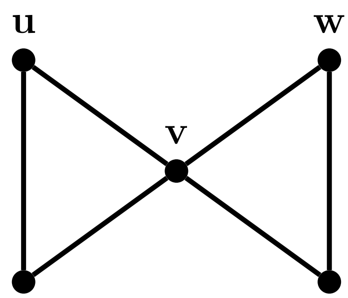

Figure 2: A counterexample for Problem 1b).

Now, let C_2 be a cycle containing f and g. We may assume C_2 does not contain the edge e, otherwise there is nothing to prove. Define p to be the first vertex on P starting from u_1 that is contained in C_2. Such a vertex must exist, since u_2 is a vertex of C_2 (note it might be that $p = u_1$). Analogously, let q be the first vertex on Q starting from v_1 that is contained in C_2 (again, it might be that $q = v_1$). Let R_1 be the path on C_1 between p and q that contains the edge e. It follows from the construction that $V(R_1) \cap V(C_2) = \{p, q\}$. Now set R_2 to be the path on C_2 between p and q that contains the edge g. The union of the edges of R_1 and R_2 forms a cycle that contains both e and g.

d) Let T be a tree on n vertices and let $v \in V(T)$ be a vertex of degree k. Then T contains at least k leaves, i.e., vertices of degree 1.

Solution: True. Let L be the set of leaves in T. Since T is a tree, $|V(T)| = |E(T)| + 1$. On the other hand,

$$2|V(T)| - 2 = 2|E(T)| = \sum_{u \in V(T)} \deg(u) = |L| + k + \sum_{\substack{u \in V(T) \setminus L \\ u \neq v}} \deg(u).$$

Since every vertex $u \in V \setminus L$ has degree at least 2, it follows that

$$\sum_{\substack{u \in V(T) \setminus L \\ u \neq v}} \deg(u) \geq 2(|V(T)| - |L| - 1) = 2|V(T)| - 2|L| - 2.$$

Combining the two derivations together, we conclude that

$$2|V(T)| - 2 \geq |L| + k + 2|V(T)| - 2|L| - 2 = k - |L| + 2|V(T)| - 2,$$

which after rearranging the terms yields $|L| \geq k$.

2. Let $G = (V, E)$ be a graph, and let $\overline{G}$ be the complement of G, i.e., the graph $(V, \overline{E})$, where $\overline{E} := \binom{V}{2} \setminus E$. Show that if G is not connected, then $\overline{G}$ is connected.

Solution: Let C be an arbitrary connected component of G, and let $D := V(G) \setminus C$. Since G is not connected, $D \neq \emptyset$. Fix two vertices $u \in C$ and $v \in D$. It follows that in the graph $\overline{G}$, any vertex in C is connected to any vertex in D by an edge. Moreover, for any two vertices $c_1, c_2 \in C$, there is a path of length two in $\overline{G}$ between c_1 and c_2 via v. Analogously, for any two vertices $d_1, d_2 \in D$, there is a path of length two between d_1 and d_2 via u. So $\overline{G}$ is connected.

3. Let G be a graph with $|V(G)| \geq 1$ where for every pair of vertices $u, v \in V(G)$, there is a path in G from u to v of length at most k. Show that if G is not a tree, then it contains a cycle of length at most $2k+1$.

Solution: Clearly, G is connected and contains a cycle. Let C be a cycle in G of the smallest length and let $v_1, v_2, \dots, v_\ell$ be the vertices of C in order. Suppose for a contradiction that $\ell \geq 2k+2$. Let P be the shortest path from v_1 to v_{k+2} in G. Then P has length at most k and it follows that $P \subsetneq C$. Thus there exists a subpath Q of P with distinct ends $v_i, v_j \in V(P)$ and otherwise disjoint from C. The union of Q with each of the two paths in C with ends v_i and v_j is a cycle, and so each of these cycles must have length at least ℓ. The sum of their lengths, however, is equal to $\ell + 2|E(Q)| \leq \ell + 2|E(P)| \leq \ell + 2k < 2\ell$, a contradiction.

4. Let G be a connected graph which contains no path with length larger than k. Show that every two paths in G of length k have at least one vertex in common.

Solution: Suppose for a contradiction that P_1 and P_2 are two vertex-disjoint paths of length k. Let vertices of P_i, where $1 \leq i \leq 2$, be $v_1^i, v_2^i, \dots, v_{k+1}^i$, in order. Let Q be a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be as short as possible. Let v_n^1 and v_m^2 be the ends of Q, where $1 \leq n, m \leq k+1$. We can suppose without loss of generality that $m, n \geq \lceil k/2 + 1 \rceil$. Then a path obtained by taking the union of the subpath of P_1 from v_1^1 to v_n^1, the path Q and the subpath of P_2 from v_1^2 to v_m^2 has at least $m + n \geq k+2$ vertices, a contradiction.

5. Let T be a tree, and let $T_1, \dots, T_k$ be connected subgraphs of T so that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq k$. Show that

$$\bigcap_{i=1}^{k} V(T_i) \neq \emptyset.$$

Solution: Proof by induction on $|V(T)|$. The base case $|V(T)| = 1$ is trivial. For the induction step, let v be a leaf of T and let u be the unique vertex of T adjacent to v. Let $T' = T \setminus v$ and let $T_i' = T_i \setminus v$ for $i = 1, 2, \dots, k$. If $V(T_i' \cap T_j') \neq \emptyset$ for all i, j with $1 \leq i < j \leq k$, then we can apply the induction hypothesis to T' to complete the proof. Thus we may assume, without loss of generality, that $V(T_1') \cap V(T_2') = \emptyset$. It follows that $V(T_1) \cap V(T_2) = \{v\}$. Thus either $u \notin V(T_1)$ or $u \notin V(T_2)$. Without loss of generality, we have $V(T_1) = \{v\}$. Therefore $v \in V(T_i)$ for every $1 \leq i \leq k$ by the assumption and $v \in V(T_1 \cap T_2 \cap \dots \cap T_k)$, as desired.