Solution for Final Review Problems[1]

Final time and location: Dec. 15, 2010, Wednesday, 9-12am, Main Gymnasium, Rows 23, 25

(1) Let $f(z)$ be the principal branch of z^i.

(a) Find $f(1+i)$.

(b) Show that

$$f(z_1)f(z_2) = \lambda f(z_1 z_2)$$

for all $z_1, z_2 \neq 0$, where $\lambda = 1, e^{2\pi}$ or $e^{-2\pi}$.

Solution. (a)

$$\begin{aligned} f(1+i) &= (1+i)^i = \exp(i \operatorname{Log}(1+i)) = \exp(i(\ln\sqrt{2} + \frac{\pi i}{4})) \\ &= e^{-\pi/4}(\cos(\frac{\ln 2}{2}) + i\sin(\frac{\ln 2}{2})) \end{aligned}$$

(b) We have

$$\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \operatorname{Arg}(z_1 z_2) + 2n\pi$$

for some integer n. Since $-\pi < \operatorname{Arg}(z_1) \le \pi$, $-\pi < \operatorname{Arg}(z_2) \le \pi$ and $-\pi < \operatorname{Arg}(z_1 z_2) \le \pi$,

$$-3\pi < 2n\pi < 3\pi \Rightarrow -1 \le n \le 1$$

Therefore,

$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = \operatorname{Log}(z_1 z_2) + 2n\pi i$$

with $n \in \{-1, 0, 1\}$ and

$$\begin{aligned} \frac{f(z_1)f(z_2)}{f(z_1 z_2)} &= \frac{\exp(i \operatorname{Log} z_1)\exp(i \operatorname{Log} z_2)}{\exp(i \operatorname{Log}(z_1 z_2))} \\ &= \exp(i \operatorname{Log} z_1 + i \operatorname{Log} z_2 - i \operatorname{Log}(z_1 z_2)) \\ &= \exp(-2n\pi) \in \{e^{-2\pi}, 1, e^{2\pi}\} \end{aligned}$$

(2) Do the following:

(a) Find $\sin(\frac{\pi}{3} + i)$.

(b) Find the Taylor series of $(\sin z)^2$ at $z = 0$.

(c) Show that

$$|\sin(z)| \ge \sinh(|y|)$$

for all $z \in \mathbb{C}$, where $y = \operatorname{Im}(z)$.

[1] http://www.math.ualberta.ca/~xichen/math31110f/fpsol.pdf

(d) Let C_R denote the semicircle $\{|z-i| = R, \mathrm{Im}(z) \geq 1\}$. Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2 \sin z} = 0$$

Solution. (a)

$$\begin{aligned}
\sin\left(\frac{\pi}{3} + i\right) &= \frac{1}{2i}(e^{i(\pi/3+i)} - e^{-i(\pi/3+i)}) \\
&= \frac{1}{2i}(e^{-1}e^{\pi i/3} - ee^{-\pi i/3}) \\
&= \frac{1}{2i}\left(e^{-1}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) - e(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})\right) \\
&= \frac{\sqrt{3}}{4}\left(e + \frac{1}{e}\right) + \frac{i}{4}\left(e - \frac{1}{e}\right)
\end{aligned}$$

(b) We have

$$\begin{aligned}
(\sin z)^2 &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = -\frac{1}{4}(e^{2iz} + e^{-2iz} - 2) \\
&= -\frac{1}{4}\sum_{n=0}^{\infty} \frac{(2i)^n z^n}{n!} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-2i)^n z^n}{n!} + \frac{1}{2} \\
&= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2i)^{2n} z^{2n}}{(2n)!} = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} z^{2n}}{(2n)!} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} z^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} z^{2n}}{(2n)!}
\end{aligned}$$

(c) By triangle inequality,

$$\begin{aligned}
|\sin z| &= \left|\frac{e^{iz} - e^{-2iz}}{2i}\right| \geq \frac{1}{2}\left||e^{iz}| - |e^{-iz}|\right| \\
&= \frac{1}{2}\left||e^{ix-y}| - |e^{-ix+y}|\right| = \frac{1}{2}|e^{-y} - e^{y}| \\
&= |\sinh y| = \sinh(|y|)
\end{aligned}$$

(d) When $z \in C_R$, $\mathrm{Im}(z) \geq 1$. Therefore, $|\sin z| \geq \sinh(1)$. And since

$$|z| = |(z-i) + i| \geq |z-i| - 1 = R - 1$$

for $z \in C_R$,

$$\left|\frac{1}{z^2 \sin z}\right| \leq \frac{1}{(R-1)^2 \sinh(1)}$$

for $z \in C_R$. Therefore,

$$\left| \int_{C_R} \frac{dz}{z^2 \sin z} \right| \le \frac{1}{(R-1)^2 \sinh(1)} \int_{C_R} |dz| = \frac{\pi R}{(R-1)^2 \sinh(1)}$$

Since

$$\lim_{R \to \infty} \frac{\pi R}{(R-1)^2 \sinh(1)} = 0$$

we conclude

$$\lim_{R \to \infty} \int_{C_R} \frac{dz}{z^2 \sin z} = 0$$

(3) For each of the following functions, do the following:
- find all its singularities in $\mathbb{C}$;
- write the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a) $f(z) = (1-z)\exp\left(\frac{1}{z^2}\right)$

(b) $f(z) = \frac{1}{z^2+1}$

(c) $f(z) = \tan z$

(d) $f(z) = \frac{e^z}{z^2(z-1)}$

Solution. (a) $f(z)$ has a singularity at 0. At $z = 0$,

$$\begin{aligned}(1-z)\exp\left(\frac{1}{z^2}\right) &= (1-z)e^{1/z^2} = (1-z)\sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n-1}} \\ &= 1 - z + \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}\end{aligned}$$

So the principal part is

$$\sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}$$

Consequently, $f(z)$ has an essential singularity at 0 and

$$\operatorname*{Res}_{z=0} f(z) = -\frac{1}{1!} = -1$$

(b) $f(z)$ has two singularities at $\pm i$. We write

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{i}{2}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$$

At i, the principal part of $f(z)$ is

$$-\frac{i}{2}\frac{1}{z-i}$$

it has a pole of order 1 and

$$\operatorname*{Res}_{z=i} f(z) = -\frac{i}{2}$$

At $-i$, the principal part of $f(z)$ is

$$\frac{i}{2}\frac{1}{z+i}$$

it has a pole of order 1 and

$$\operatorname*{Res}_{z=-i} f(z) = \frac{i}{2}$$

(c) $f(z)$ has singularities at $\{\cos z = 0\} = \{z = k\pi + \pi/2 : k \in \mathbb{Z}\}$. At $z = k\pi + \pi/2$, we let $w = z - k\pi - \pi/2$. Then

$$\begin{aligned}
\tan(z) &= \tan\left(w + k\pi + \frac{\pi}{2}\right) = -\cot(w) \\
&= -\frac{\cos w}{\sin w} = -\left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}\right)^{-1} \\
&= -\frac{1}{w}\left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!}\right)^{-1} \\
&= -\frac{1}{w}\left(1 - \frac{w^2}{2!} + \frac{w^4}{4!} - ...\right)\left(1 - \frac{w^2}{3!} + \frac{w^4}{5!} - ...\right)^{-1} \\
&= -\frac{1}{w} + \sum_{n=0}^{\infty} a_n w^n \\
&= -\frac{1}{z - k\pi - \pi/2} + \sum_{n=0}^{\infty} a_n (z - k\pi - \pi/2)^n
\end{aligned}$$

So the principal part of $f(z)$ at $k\pi + \pi/2$ is

$$-\frac{1}{z - k\pi - \pi/2}$$

$f(z)$ has a pole of order 1 at $k\pi + \pi/2$ and

$$\operatorname*{Res}_{z=k\pi+\pi/2} f(z) = -1$$

(d) $f(z)$ has two singularities at 0 and 1. At $z = 0$,

$$\begin{aligned}
\frac{e^z}{z^2(z-1)} &= -\frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}z^n\right) \\
&= -\frac{1}{z^2}\left(1+\frac{z}{1!}+...\right)(1+z+...) \\
&= -\frac{1}{z^2}(1+2z)+\sum_{n=0}^{\infty}a_nz^n = -\frac{1}{z^2}-\frac{2}{z}+\sum_{n=0}^{\infty}a_nz^n
\end{aligned}$$

So the principal part of $f(z)$ at 0 is

$$-\frac{1}{z^2}-\frac{2}{z}$$

it has a pole order 2 at 0 and

$$\operatorname*{Res}_{z=0} f(z) = -2$$

At $z = 1$, we let $w = z - 1$ and then

$$\begin{aligned}
\frac{e^z}{z^2(z-1)} &= \frac{e^{w+1}}{(1+w)^2w} = \frac{e}{w}(e^w(1+w)^{-2}) \\
&= \frac{e}{w}\left(\sum_{n=0}^{\infty}\frac{w^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n(n+1)w^n\right) \\
&= \frac{e}{w}\left(1+\frac{w}{1!}+...\right)(1-2w+...) \\
&= \frac{e}{w}+\sum_{n=0}^{\infty}a_nw^n = \frac{e}{z-1}+\sum_{n=0}^{\infty}a_n(z-1)^n
\end{aligned}$$

So the principal part of $f(z)$ at 1 is

$$\frac{e}{z-1}$$

it has a pole of order 1 and then

$$\operatorname*{Res}_{z=1} f(z) = e$$

(4) Let $f(z)$ be an entire function. If $|f(z)| \le |z|^2$ for all z, then $f(z) = az^2$ for some constant $a \in \mathbb{C}$ satisfying $|a| \le 1$.

Proof. Since $f(z)$ is entire,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all z.

By Cauchy Integral Formula,

$$f^{(n)}(0) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)n!}{z^{n+1}} dz$$

for all n and $R > 0$. Since $|f(z)| \leq |z|^2$,

$$\left| \frac{f(z)n!}{z^{n+1}} \right| \leq \frac{n!}{|z|^{n-1}} = \frac{n!}{R^{n-1}}$$

for $|z| = R$. Therefore,

$$\left| \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)n!}{z^{n+1}} dz \right| \leq \left(\frac{1}{2\pi} \right) \left(\frac{n!}{R^{n-1}} \right) (2\pi R) = \frac{n!}{R^{n-2}}.$$

And since $\lim_{R\to\infty} n!/R^{n-2} = 0$ for $n > 2$,

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)n!}{z^{n+1}} dz = 0$$

and hence $f^{(n)}(0) = 0$ for all $n > 2$. And since $\lim_{R\to 0} n!/R^{n-2} = 0$ for all $n < 2$,

$$\lim_{R\to 0} \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)n!}{z^{n+1}} dz = 0$$

and hence $f^{(n)}(0) = 0$ for all $n < 2$. In conclusion,

$$f(z) = \frac{f''(0)}{2} z^2 = az^2$$

for some constants a.

Finally, by $|az^2| \leq |z|^2$, we obtain that $|a| \leq 1$. $\square$

(5) Let

$$f(z) = \frac{z^3}{z^2 - 3z + 2}$$

Find the Laurent series of $f(z)$ in each of the following domains:
(a) $1 < |z| < 2$;
(b) $2 < |z| < \infty$;
(c) $0 < |z - 1| < 1$.

Solution. We write $f(z)$ as a sum of partial fractions:

$$f(z) = z + 3 + \frac{8}{z-2} - \frac{1}{z-1}$$

Then

(a) For $1 < |z| < 2$,

$$\begin{aligned}
f(z) &= z + 3 + \frac{8}{z-2} - \frac{1}{z-1} \\
&= z + 3 - \frac{4}{1-(z/2)} - \frac{1}{z}\left(\frac{1}{1-(1/z)}\right) \\
&= z + 3 - 4\sum_{n=0}^{\infty} 2^{-n} z^{n} - z^{-1}\sum_{n=0}^{\infty} z^{-n} \\
&= z + 3 - \sum_{n=0}^{\infty} 2^{2-n} z^{n} - \sum_{n=0}^{\infty} z^{-n-1} \\
&= -z - 1 - \sum_{n=2}^{\infty} 2^{2-n} z^{n} - \sum_{n=1}^{\infty} z^{-n}
\end{aligned}$$

(b) For $2 < |z| < \infty$,

$$\begin{aligned}
f(z) &= z + 3 + \frac{8}{z-2} - \frac{1}{z-1} \\
&= z + 3 + \frac{8}{z}\left(\frac{1}{1-(2/z)}\right) - \frac{1}{z}\left(\frac{1}{1-(1/z)}\right) \\
&= z + 3 + 8z^{-1}\sum_{n=0}^{\infty} 2^{n} z^{-n} - z^{-1}\sum_{n=0}^{\infty} z^{-n} \\
&= z + 3 + \sum_{n=0}^{\infty} 2^{n+3} z^{-n-1} - \sum_{n=0}^{\infty} z^{-n-1} \\
&= z + 3 + \sum_{n=0}^{\infty} (2^{n+3} - 1) z^{-n-1} \\
&= z + 3 + \sum_{n=1}^{\infty} (2^{n+2} - 1) z^{-n}
\end{aligned}$$

(c) For $0 < |z-1| < 1$,

$$\begin{aligned} f(z) &= z+3+\frac{8}{z-2}-\frac{1}{z-1} \\ &= z+3+\frac{8}{(z-1)-1}-\frac{1}{z-1} \\ &= z+3-8\sum_{n=0}^{\infty}(z-1)^n-\frac{1}{z-1} \\ &= -4-7(z-1)-8\sum_{n=2}^{\infty}(z-1)^n-\frac{1}{z-1} \end{aligned}$$

(6) Compute the integrals:

(a) $\displaystyle\int_0^{\pi}\frac{1}{(2-\cos\theta)^2}d\theta$

(b) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(2x)}{1+x+x^2}dx$

Solution. (a) We parameterize the circle $|z|=1$ with $z=e^{i\theta}$ for $-\pi \le \theta \le \pi$. Then

$$\begin{aligned} \int_0^{\pi}\frac{1}{(2-\cos\theta)^2}d\theta &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{(2-\cos\theta)^2}d\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{(2-(e^{i\theta}+e^{-i\theta})/2)^2}d\theta \\ &= 2\int_{-\pi}^{\pi}\frac{1}{(4-e^{i\theta}-e^{-i\theta})^2}d\theta \\ &= 2\int_{-\pi}^{\pi}\frac{e^{2i\theta}}{(4e^{i\theta}-e^{2i\theta}-1)^2}d\theta \\ &= -2i\int_{-\pi}^{\pi}\frac{e^{i\theta}}{(4e^{i\theta}-e^{2i\theta}-1)^2}d(e^{i\theta}) \\ &= -2i\int_C\frac{z}{(4z-z^2-1)^2}dz \end{aligned}$$

and

$$\begin{aligned}
\int_C \frac{z}{(4z-z^2-1)^2}dz &= 2\pi i \operatorname*{Res}_{z=2-\sqrt{3}} \frac{z}{(4z-z^2-1)^2} \\
&= 2\pi i \operatorname*{Res}_{z=2-\sqrt{3}} \frac{z}{(z-(2-\sqrt{3}))^2(z-(2+\sqrt{3}))^2} \\
&= 2\pi i \left.\left(\frac{z}{(z-(2+\sqrt{3}))^2}\right)'\right|_{z=2-\sqrt{3}} \\
&= \frac{\sqrt{3}\pi}{9} i
\end{aligned}$$

Therefore,

$$\int_0^\pi \frac{1}{(2-\cos\theta)^2} d\theta = \frac{2\sqrt{3}\pi}{9}$$

(b) Obviously,

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\cos(2x)}{1+x+x^2}dx &= \operatorname{Re}\left(\int_{-\infty}^\infty \frac{e^{2iz}}{1+z+z^2}dz\right) \\
&= \operatorname{Re}\left(\lim_{R\to\infty}\int_{-R}^R \frac{e^{2iz}}{1+z+z^2}dz\right)
\end{aligned}$$

We integrate along the closed contour going from $-R$ to R and then the semicircle $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$ counterclockwise. Then

$$\begin{aligned}
&\int_{-R}^R \frac{e^{2iz}}{1+z+z^2}dz + \int_{C_R} \frac{e^{2iz}}{1+z+z^2}dz \\
&= 2\pi i \operatorname*{Res}_{z=(-1+\sqrt{3}i)/2} \frac{e^{2iz}}{1+z+z^2} \\
&= 2\pi i \left.\frac{e^{2iz}}{z-(-1-\sqrt{3}i)/2}\right|_{z=(-1+\sqrt{3}i)/2} \\
&= \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}}(\cos 1 - i\sin 1)
\end{aligned}$$

by Cauchy Integral Theorem.

For z on C_R, $|e^{2iz}| = |e^{-2y}| \leq 1$, $|z^2+z+1| \geq |z|^2 - |z| - 1 = R^2 - R - 1$ and hence

$$\left|\frac{e^{2iz}}{1+z+z^2}\right| \leq \frac{1}{R^2-R-1}.$$

Therefore,

$$\left|\int_{C_R} \frac{e^{2iz}}{1+z+z^2} dz\right| \leq \frac{\pi R}{R^2 - R - 1}.$$

And since

$$\lim_{R\to\infty} \frac{\pi R}{R^2 - R - 1} = 0,$$

we conclude that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{2iz}}{1+z+z^2} dz = 0.$$

Thus,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{2iz}}{1+z+z^2} dz = \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}} (\cos 1 - i \sin 1)$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x+x^2} dx = \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}} \cos 1.$$

(7) Compute the following contour integrals. You may apply Cauchy integral theorem and its corollaries wherever possible.

(a)

$$\int_L \overline{z} dz,$$

where L is the polygonal path ABC with $A = 0$, $B = 1 + i$ and $C = 1 - i$.

(b)

$$\int_L z^2 dz$$

where L is the curve in part (a).

(c)

$$\int_C \frac{dz}{\sin^2 z}$$

where C is the circle $|z| = 10$ oriented counter-clockwise.

(d)

$$\int_C \frac{z^{2009}}{z^{2010} + z + 1} dz$$

where C is the circle $|z| = 2$ oriented counter-clockwise.

Solution. (a)

$$\begin{aligned}\int_L \overline{z}dz &= \int_{AB} \overline{z}dz + \int_{BC} \overline{z}dz\\
&= \int_0^1 \overline{(1+i)t}d((1+i)t)\\
&\quad + \int_0^1 \overline{(1-t)(1+i)+t(1-i)}d((1-t)(1+i)+t(1-i))\\
&= (1+i)\int_0^1 (1-i)tdt - 2i\int_0^1 (1-(1-2t)i)dt\\
&= t^2\Big|_0^1 - 2i\left(t + \frac{i}{4}(1-2t)^2\right)\Big|_0^1 = 1-2i\end{aligned}$$

(b) Since z^2 is entire, z^2 has a complex anti-derivative $z^3/3$ in $\mathbb{C}$ and

$$\int_L z^2 dz = \frac{z^3}{3}\Big|_0^{1-i} = -\frac{2}{3} - \frac{2}{3}i$$

(c) $1/(\sin z)^2$ has singularities at $k\pi$ for $k \in \mathbb{Z}$. Hence

$$\int_{|z|=10} \frac{dz}{(\sin z)^2} = 2\pi i \sum_{k=-3}^{3} \operatorname*{Res}_{z=k\pi} \frac{1}{(\sin z)^2}$$

At $z = k\pi$, we let $w = z - k\pi$ and then

$$\begin{aligned}\frac{1}{(\sin z)^2} &= \frac{1}{(\sin(w+k\pi))^2} = \frac{1}{(\sin w)^2}\\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}\right)^{-2}\\
&= \frac{1}{w^2}\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!}\right)^{-2}\\
&= \frac{1}{w^2}\sum_{m=0}^{\infty}(m+1)\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!}\right)^{m}\end{aligned}$$

So the Laurent series of $1/(\sin w)^2$ at $w=0$ only has terms w^n with n even. Therefore,

$$\operatorname*{Res}_{z=k\pi} \frac{1}{(\sin z)^2} = \operatorname*{Res}_{w=0} \frac{1}{(\sin w)^2} = 0$$

Consequently,

$$\int_{|z|=10} \frac{dz}{(\sin z)^2} = 0$$

(d) We first show that all zeroes of $z^{2010}+z+1$ lie in $|z|<2$. Otherwise, suppose that $z^{2010}+z+1=0$ for some $|z|\geq 2$. Then $1+z^{-2009}+z^{-2010}=0$. But

$$\begin{aligned}|1+z^{-2009}+z^{-2010}| &\geq 1-\frac{1}{|z|^{2009}}-\frac{1}{|z|^{2010}}\\ &\geq 1-\frac{1}{2^{2009}}-\frac{1}{2^{2010}}>0\end{aligned}$$

for $|z|\geq 2$. Contradiction. So all zeroes of $z^{2010}+z+1$ lie in $|z|<2$. Therefore, $z/(z^{2010}+z+1)$ is analytic in $|z|>2$. It follows that

$$\begin{aligned}\int_C \frac{z^{2009}}{z^{2010}+z+1}dz &= -2\pi i \operatorname*{Res}_{z=\infty} \frac{z^{2009}}{z^{2010}+z+1}\\ &= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2}\left(\frac{z^{-2009}}{z^{-2010}+z^{-1}+1}\right)\\ &= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z(1+z^{2009}+z^{2010})} = 2\pi i\end{aligned}$$