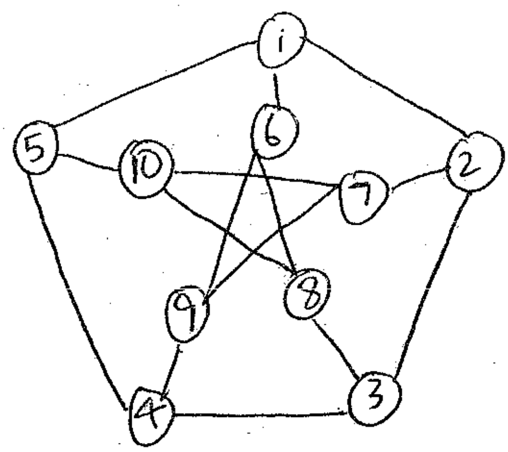
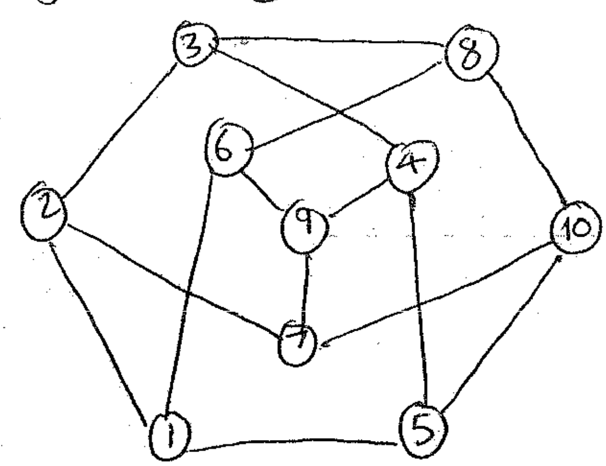
3032: Solutions – Jan. 2008

1. (a) Isomorphism is given by:

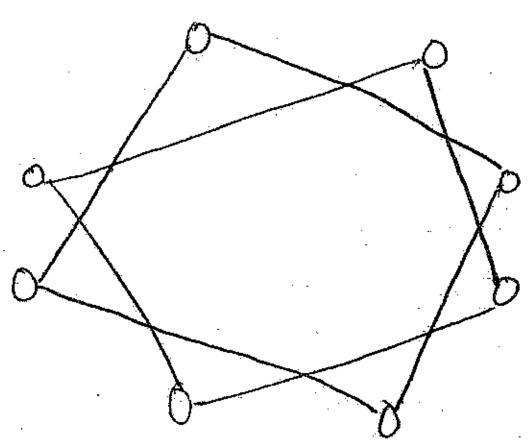


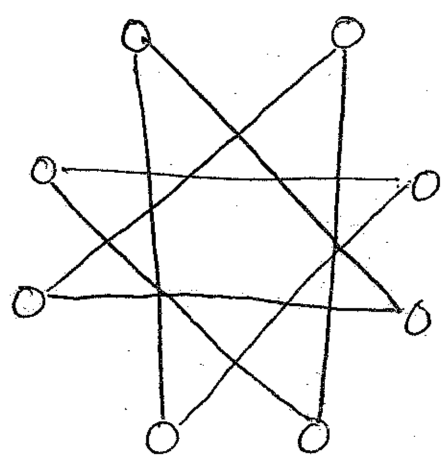


in notes

4

(b) Complements of the given graphs are:





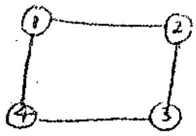
3

new

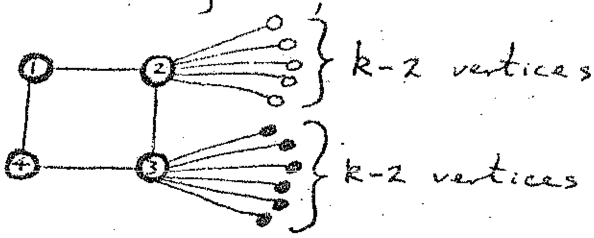
Since graphs are isomorphic iff their complements are — and the first is disconnected and the second is not — the original graphs are not isomorphic.

2

(c) Let G be k-regular of girth 4. G must contain a circuit of length 4:



Since G is k-regular, we can extend G:



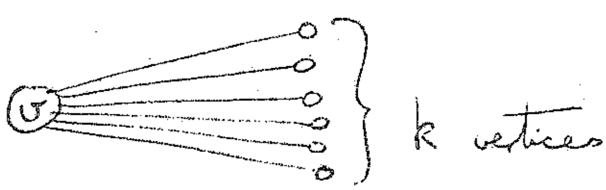
Since G has girth 4, all these vertices are distinct. So G has at least $4 + 2(k-2) = 2k$ vertices. 5

similar problem sheet

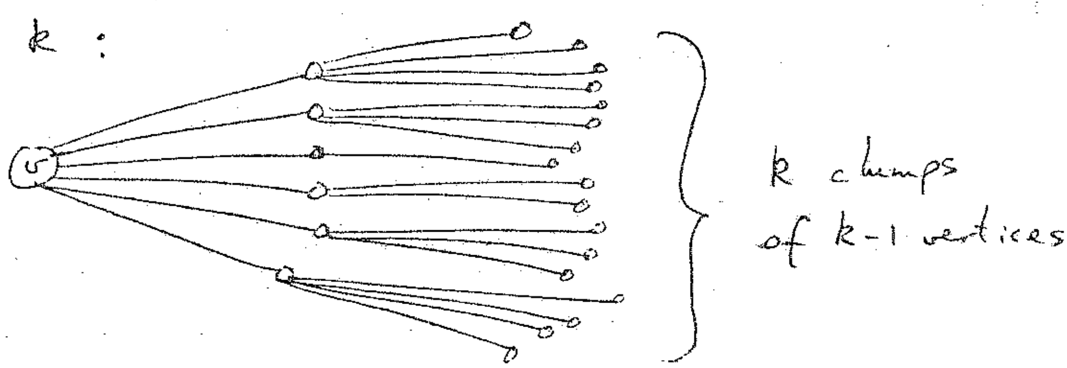
If G has exactly $2k$ vertices, can take $G = K_{k,k}$. 2

new

Let G be k-regular of girth 5. Let v be a vertex of G. Since G is k-regular, v must have degree k:



Again, each of the vertices adjacent to v must have degree k:



Since G has girth 5, all of these vertices must be distinct. So G has at least

$1 + k + k(k-1) = k^2 + 1$ vertices.

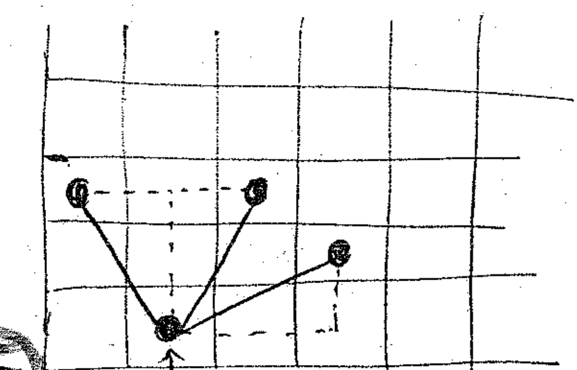
similar problem sheet

4

2) (a) An Euler tour for G is a closed chain which
1 includes all edges of G. G is Eulerian iff it has an Euler tour.

1 G is Eulerian iff G is connected and all vertices have even degree.

Take a graph G with vertices corresponding to the 64 squares of the chessboard, in which 2 vertices are adjacent iff a knight can move legitimately between the 2 corresponding squares. Then the knight can travel round the board performing each move just once iff G is semi-Eulerian, i.e., iff G has at most 2 vertices of odd degree.



But it can be seen from the diagram that there are at least 8 vertices of odd degree — so answer is <u>NO</u>.

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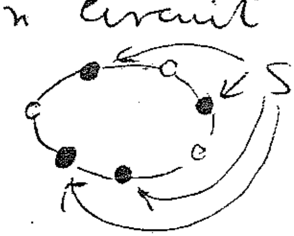
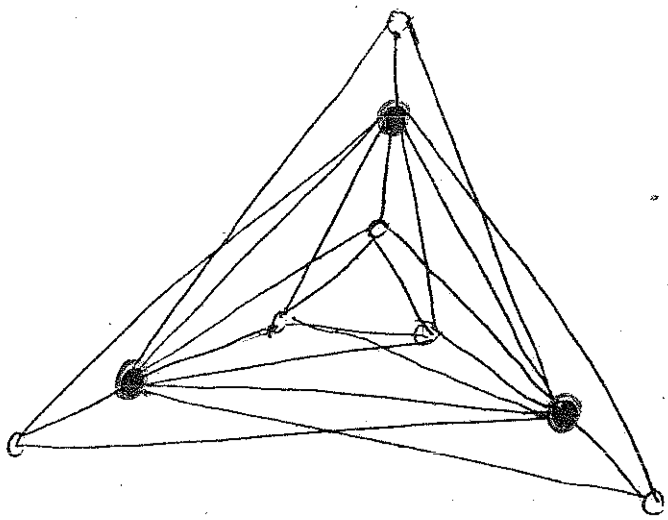
(b) Let C be a Hamilton circuit for G.

Then $c(C-S) \leq |S|$



But $c(G-S) \leq c(C-S)$, so result follows.

3



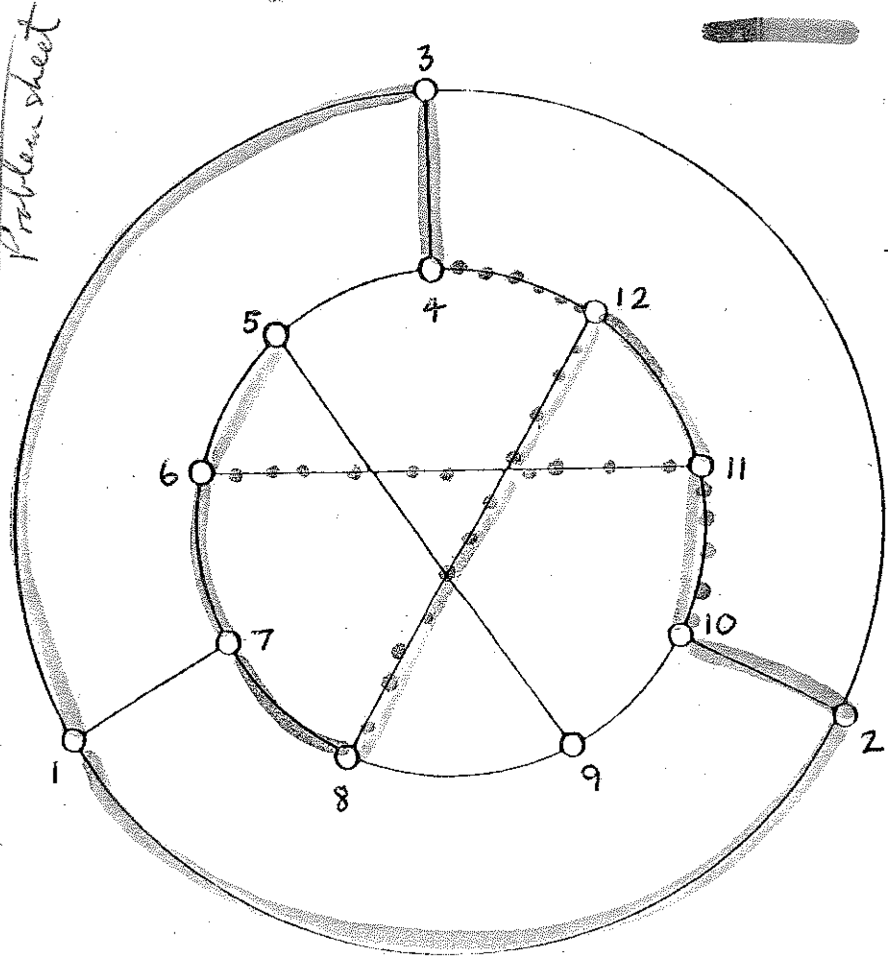
Take S to be the red vertices

Then $c(G-S) = 4 \not\leq |S| = 3$.

4 So G is not Hamiltonian.

(c)

new, but similar Petersen graph on Problem sheet



= initial form of Hamilton circuit

Case 1. Miss out edge $11 \to 12$ – then must have ••••• edges – impossible

Case 2. Include $11 \to 12$ (so $11 \to 10$ or $12 \to$

Get edges

Then both $5 \to 9$ and $5 \to 4$ are impossible

6

3) (a) Proof by induction on no. ε of edges of G.

<u>IH</u> : Fmla. holds for any connected plane graph with $< \varepsilon$ edges

Assume G a connected plane graph with ε edges.

<u>Case 1</u> . <u>G has no circuits</u> . Then G is a tree, and $\varepsilon = v - 1$ and $\varphi = 1$.

So $\varepsilon = v + \varphi - 2$, ie, $v - \varepsilon + \varphi = 2$.

<u>Case 2</u> . <u>G has at least one circuit</u> , and e is an edge in this circuit. Then $G - e$ is a connected plane graph with v vertices , $\varepsilon - 1$ edges, and $\varphi - 1$ faces.

So by the IH , $v - (\varepsilon - 1) + (\varphi - 1) = 2$,

so $v - \varepsilon + \varphi = 2$, and the result follows. 6

in notes

(b) $v(G^*) = \varphi(G)$, $\varepsilon(G^*) = \varepsilon(G)$, $d_{G^*}(f^*) = d_G(f)$ follow straight from the defn. of dual graph. 2

(i) Let G have dual G^*.

Then $\varepsilon = \varepsilon^*$, $\varphi = v^*$, $v = \varphi^*$.

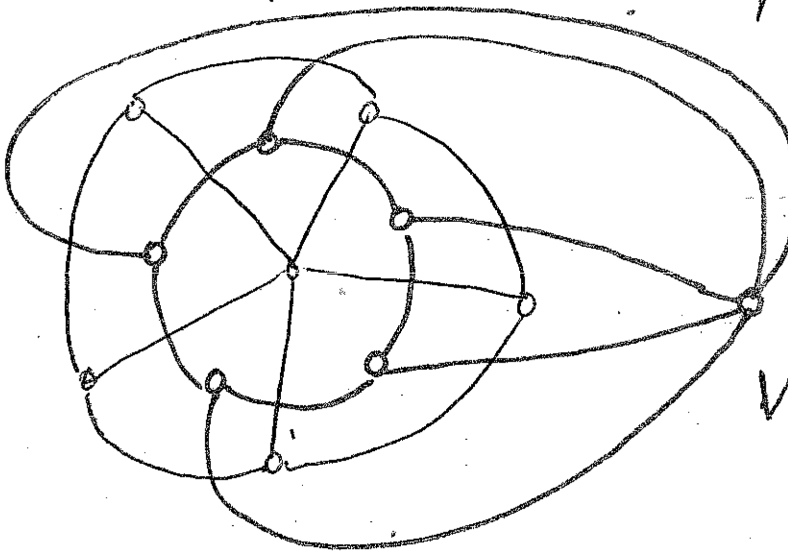
So if $G \cong G^*$, $v = v^* = \varphi$

From Euler's Fmla. , $v - \varepsilon + \varphi = v - \varepsilon + v = 2$

ie, $\varepsilon = 2v - 2$. 4

similar problem sheet

For each $v \geq 4$, the wheel W_{v-1} with $v-1$ spokes is self-dual. For example:

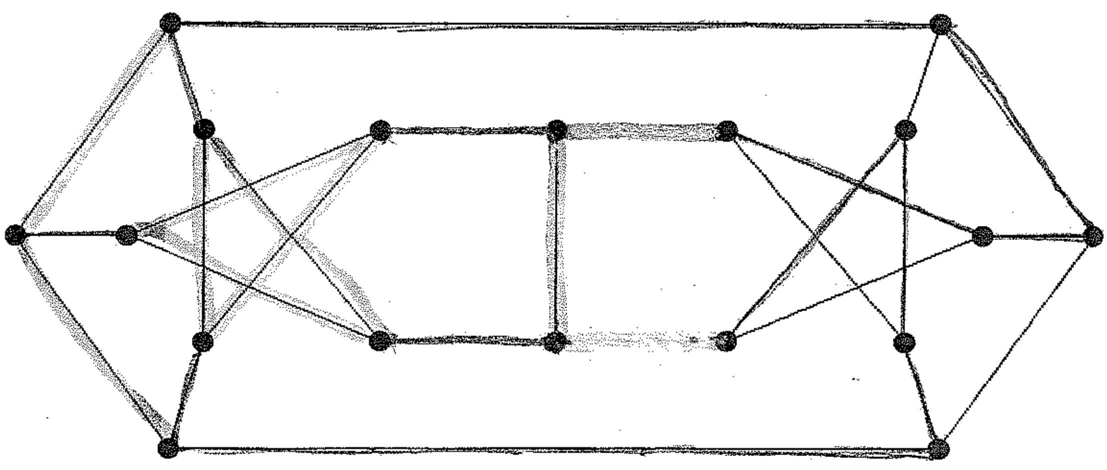


$W_5^* \cong W_5$

2

similar problem sheet

(c)



Take the red and green subgraph shown — and then contract all the red edges to get K_5.

new – diff Petersen graph in lectures

6

4) (a) D is a tournament if it is an orientation of a complete simple graph.

Clearly any vertex adjacent to a sink cannot be a sink. Similarly for sources.

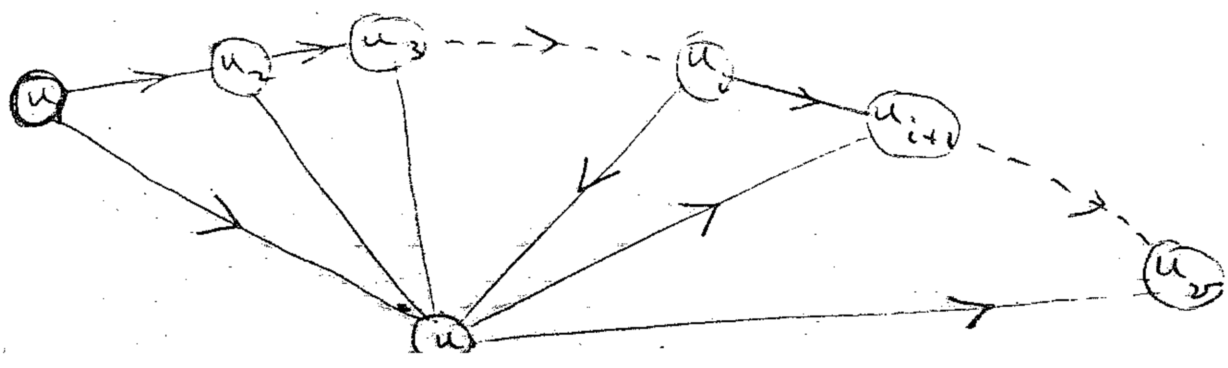
(b) D is semi-Hamiltonian iff $\exists$ a spanning dipath for D.

Prove every tournament semi-Hamiltonian by induction on ν.

① For $\nu = 2$, theorem is true ✓

② INDUCTIVE HYPOTHESIS. Every tournament on ν vertices is semi-Hamiltonian.

Let T be a tournament on $\nu+1$ vertices. Let u be a vertex of T. Then $T-u$ is a tournament on ν vertices, and by the inductive hypothesis, $T-u$ has a Hamilton dipath $u_1u_2, u_2u_3, \ldots, u_{\nu-1}u_\nu$.



CASE I. uu_1 is an arc of T.

Then $uu_1, u_1u_2, \ldots, u_{\nu-1}u_\nu$ is the required Hamilton dipath

CASE II. $u_\nu u$ is an arc of T.

Then $u_1u_2, \ldots, u_{\nu-1}u_\nu, u_\nu u$ is the Hamilton dipath.

CASE III. Otherwise.

Let i be the least no. for which u_iu and uu_{i+1} are in T.

Then $u_1u_2, \ldots, u_iu, uu_{i+1}, \ldots, u_{\nu-1}u_\nu$ is the required dipath ✓

This completes the induction. $\square$ 11

(c) Let D be a tournament and let v be a vertex of maximum outdegree in D.

Let u be any other vertex of D.

Then either ① u is an out-neighbour of v, or ② u is an out neighbour of an out neighbour w of v,

6 or ③ v and every out neighbour of v is an out neighbour of u — But ③ would give $d^{\rightarrow}(u) \geq d^{\rightarrow}(v) + 1$, contradicting the choice of v.

5) (a) Let G be an Eulerian map with dual graph G^*. So $d(v)$ is even each $v \in V$.
Let C^* be any circuit of G^*, and let V' be the set of vertices inside C^*.

$$\text{Then length } C^* = \underbrace{\sum_{v \in V'} d_G(v)}_{\text{even since each } d_G(v) \text{ even}} - \underbrace{\sum_{v \in V'} d_{G[V']}(v)}_{\text{even since } = 2\,\mathcal{E}_{G[V']}}$$

So length C^* is even, so G^* is bipartite.
Hence G is 2-face-colourable.

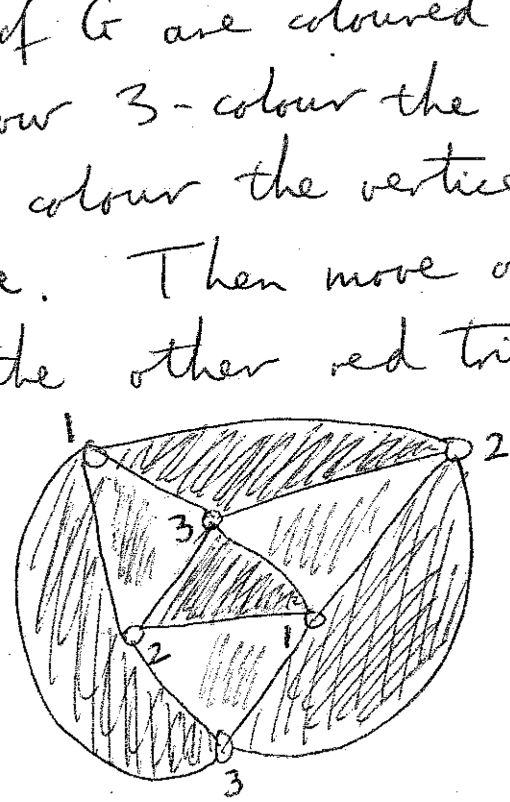
in notes

6

(b) Let G be plane, Eulerian, all faces bounded by triangles. So by part (a), G is 2 face colourable — say the faces of G are coloured red and green.

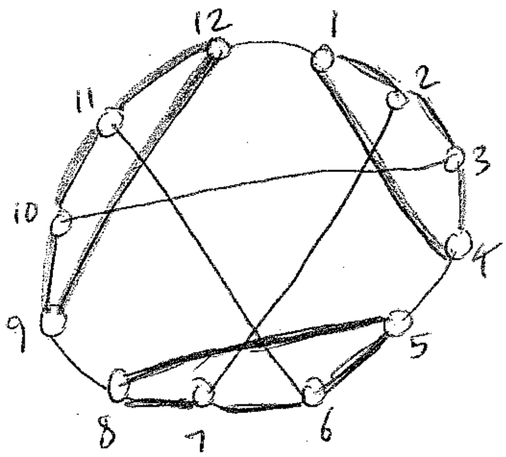
We can now 3-colour the vertices of G as follows: First colour the vertices of a red triangle 1, 2, 3 clockwise. Then move outwards, colouring the vertices of the other red triangles 1, 2, 3 clockwise:

similar homework problem

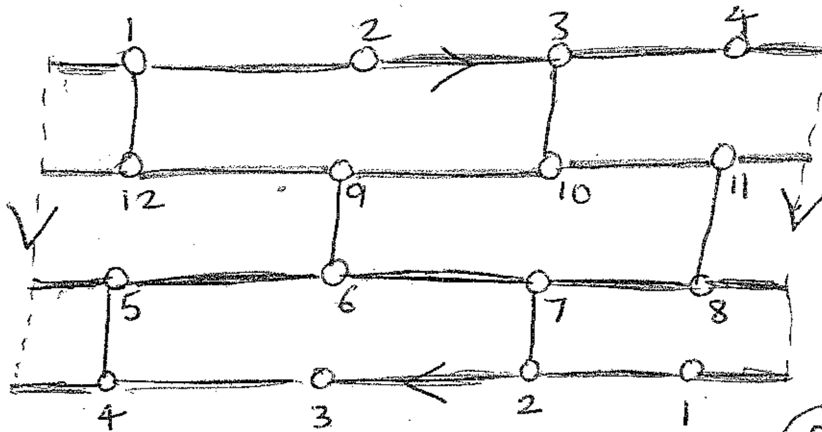


4

(c)



Franklin graph



Embedding on Klein bottle 5

Since all 6 faces of the embedding are mutually adjacent, need 6 colours, giving

$$\gamma(\text{Klein bottle}) \geq 6.$$ 2

Can use the above embedding to calculate the Euler characteristic: 2

$$n(\text{Klein bottle}) = v - \varepsilon + \varphi = 12 - 18 + 6 = 0$$

So Heawood's inequality gives

$$\gamma(\text{Klein bottle}) \leq \left[\tfrac{1}{2}\left(7 + \sqrt{49 - 24 \times 0}\right)\right]$$ 1

$$= 7.$$