Professor Kindred
Math 104, Graph Theory
Homework 9 Solutions
April 18, 2013

Introduction to Graph Theory, West
Section 6.1   33 (modified)
Grid graph problem
Section 6.3   6, 12, 32
Genus of complete graph problem

Problems you should be able to do: 6.2.1, 6.2.7, 6.3.3, 6.3.5, 6.3.34

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6.1.33 Let G be a triangulation, and let n_i be the number of vertices of degree i in G. Prove that
$$\sum_i (6-i)n_i = 12.$$
Use this fact to to show that any planar graph G must have a vertex of degree at most 5, i.e., $\delta(G) \leq 5$.

A triangulation of graph G with n vertices has $3n-6$ edges and hence degree-sum $\sum_v d(v) = 6n-12$. The degree-sum also equals $\sum_v d(v) = \sum_i in_i$, where n_i is the number of vertices of degree i in G, since the vertices of degree i contribute in_i to the total degree sum. Therefore,

$$6n - 12 = \sum_i in_i \quad \Longrightarrow \quad 12 = 6n - \sum_i in_i = 6\sum_i n_i - \sum_i in_i = \sum_i (6-i)n_i$$

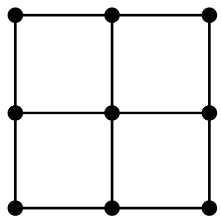
as desired.

Consider a planar graph H. We know there exists a maximal planar graph G such that H is a spanning subgraph of G. Consider a planar embedding of G; by our previous theorem, we know this is a triangulation. Thus, we have

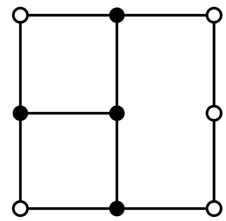
$$\sum_i (6-i)n_i = 12 \geq 0 \quad \text{and} \quad n_i \geq 0 \;\forall\, i.$$

There must exist at least one value of i for which $6-i > 0$ and $n_i > 0$; otherwise, all terms in the sum are 0 or negative, which would contradict the fact that the total sum is positive. This implies that there exists a value $0 \leq i < 6$ such that $n_i > 0$, i.e., there exists at least one vertex v with $d_G(v) < 6$. Thus, $\delta(G) \leq 5$. Because H is a subgraph of G, it follows that $\delta(H) \leq \delta(G) \leq 5$. ∎

Grid graph Consider the 3×3 grid graph G (as described at http://mathworld.wolfram.com/GridGraph.html). Use the characterization for outerplanar graphs given in problem 6.2.7 to prove that G is not outerplanar in two different ways: (a) using a subdivision of K_4, and (b) using a subdivision of $K_{2,3}$.

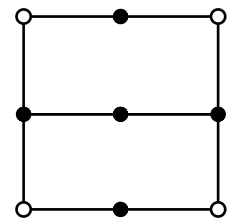


the 3×3 grid graph



subdivision of K_4

subdivided edge



subdivision of $K_{2,3}$

subdivided edge

■

6.3.6 Without using the Four Color Theorem, prove that every planar graph with at most 12 vertices is 4-colorable. Use this to prove that every planar graph with at most 32 edges is 4-colorable.

First consider the case where $G = (V, E)$ is a planar graph with 12 vertices and $\delta(G) \geq 5$. Then

$$30 = \frac{5 \cdot 12}{2} \leq \frac{1}{2} \sum_{v \in V} \delta(G) \leq \frac{1}{2} \sum_{v \in V} d(v) = |E| \leq 3|V| - 6 = 30$$

so it must be that equality holds throughout the above series of inequalities. Thus, G is a maximal planar graph that is 5-regular with 12 vertices and 30 edges. Since it is maximal planar, a planar embedding of it is a triangulation and so all faces have degree (length) 3. The only such graph is the icosahedral graph (see discussion of 6.1.28 on pages 242-243 of textbook), and an explicit 4-coloring of this graph is shown in Figure 1.

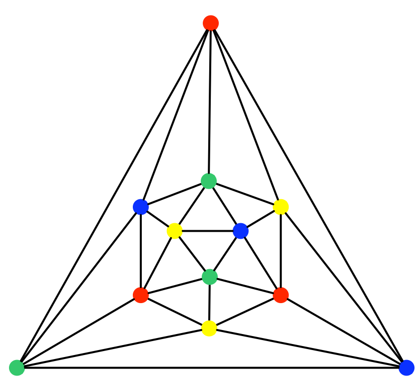


Figure 1: A 4-coloring of the icosahedral graph.

Hereafter, we assume G has less than 12 vertices or has $\delta(G) \leq 4$. We prove a bound on the minimum degree of a planar graph with less than 12 vertices.

Claim: A planar graph $G = (V, E)$ with $3 \leq |V| < 12$ has minimum degree $\delta(G) \leq 4$. Since G is planar and $|V| \geq 3$, we know

$$|E| \leq 3|V| - 6 \quad \Longrightarrow \quad 2|E| \leq 6|V| - 12.$$

Thus,

$$\delta(G) \leq \text{ avg degree of vertex in } G = \frac{2|E|}{|V|} \leq \frac{6|V| - 12}{|V|} = 6 - \frac{12}{|V|} < 6 - 1 = 5$$

Since the minimum degree of a graph is an integer value, it must be that $\delta(G) \leq 4$.

We prove by induction on $n = |V|$ that a planar graph G is 4-colorable when $\delta(G) \leq 4$.

Base case(s): For $n \leq 4$, it is clear that G is 4-colorable (by assigning a unique color to each vertex).

Induction hypothesis: Suppose that a planar graph G with n vertices and $\delta(G) \leq 4$ is 4-colorable.

Let G be a planar graph with $n + 1$ vertices and $\delta(G) \leq 4$.

Because $\delta(G) \leq 4$, we know there exists a vertex v in G such that $d(v) \leq 4$. Consider the graph $G - v$. By the induction hypothesis, we know that $G - v$ is 4-colorable since $G - v$ has n vertices, is still planar, and $\delta(G - v) \leq \delta(G) \leq 4$. Consider a 4-coloring of $G - v$, where the set of colors is $\{1, 2, 3, 4\}$.

Now, we replace the vertex v in the graph to obtain G. The worst case is that v has four neighbors, u_1, u_2, u_3, and u_4, in G and each neighbor has a distinct color, say colors 1, 2, 3, and 4, respectively. We use the notion of Kempe chains, as in the proof of the 5-Color Theorem, to complete the proof.

WLOG, suppose there exists an alternating 1-3 path (Kempe chain) from u_1 to u_3. Then, by the planarity of G, there cannot exist simultaneously an alternating 2-4 path from u_2 to u_4. Consider the graph induced by the vertices colored with colors 2 and 4. On the component of G[color 2,4 vtcs] that contains vertex u_2, swap colors of all color 2 and color 4 vertices. Note that the result is a proper 4-coloring of $G - v$, and observe also that v is then not adjacent to a vertex of color 2. Extend the 4-coloring to include v which is assigned the leftover color 2. The result follows by induction.

Finally, we consider any planar graph $G = (V, E)$ with $|E| \leq 32$. If $|V| \leq 12$, then by proof above, G is 4-colorable. Otherwise, $|V| \geq 13$, and we have

$$13\delta(G) \leq \sum_{v \in V} d(v) = 2|E| \leq 64 \quad \Longrightarrow \quad \delta(G) \leq \frac{64}{13} < 5.$$

Hence, in this case, too, we have that $\delta(G) \leq 4$, and so by the same inductive argument as that above, we have that G is 4-colorable. ■

> **6.3.12** Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem:
>
> If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor \frac{n}{3} \rfloor$ guards such that every point of the interior is visible to some guard.
>
> For $n \geq 3$, construct a polygon that requires $\lfloor \frac{n}{3} \rfloor$ guards.

Let G be an outerplanar graph with n vertices. We prove that G is 3-colorable via induction on n.

Base case: Any outerplanar graph with at most 3 vertices is clearly 3-colorable.

Induction hypothesis: Suppose the result is true for any outerplanar graph with n vertices.

Let G be an outerplanar graph with $n + 1$ vertices. Every simple outerplanar graph has a vertex of degree at most 2 (by Proposition 6.1.20). We can delete such a vertex v, 3-color $G - v$ by the induction hypothesis, and extend the coloring to v since there must be an unused color amongst the neighbors of v in G. Thus, the result holds by induction.

Next we claim that any art gallery laid out as a polygon with n segments can be guarded by $\lfloor \frac{n}{3} \rfloor$ guards so that every point of the interior is visible to some guard. The proof is comprised of the following steps:

1. Show that the polygon can be triangulated.
2. Show that for such a triangulation of the polygon, its vertices can be colored with three colors such that all three colors are present in every triangle of the triangulation.
3. Choose the vertices of the polygon assigned the least frequent color.

The art gallery is a drawing of an n-cycle in the plane. We add straight-line segments to obtain a maximal outerplane graph with n vertices. To do this, observe that polygons with three sides are already triangulated without adding segments. For $n > 3$, some corner can see some other corner across the interior of the polygon, i.e., there exists a diagonal of the polygon that lies entirely in the interior of the polygon.[1] We add this segment and proceed inductively on the two resulting polygons with fewer corners.

[1] To see that this is true, pick any corner vertex A and an adjacent vertex B. Consider a ray with the end point at A originally in the direction of B. Let it rotate from its initial position towards the interior of the

Since all vertices are on the boundary of the unbounded face and all bounded faces are triangles, it is a maximal outerplanar graph. Consider a proper 3-coloring of this graph, which exists by our previous proof. Since each bounded face is a triangle, its vertices are pairwise adjacent and receive distinct colors. Thus each color class contains a vertex of each triangle. Any vertex in a triangle sees all points in the triangle. Thus, guards placed at the vertices of a color class can see the entire gallery. Since the three color classes partition the set of vertices, the smallest class has at most $\lfloor \frac{n}{3} \rfloor$ vertices.

The bound of $\lfloor \frac{n}{3} \rfloor$ guards is best possible. The alcoves in the polygon below require their own guards; no guard can see into more than one of them. There are $\lfloor \frac{n}{3} \rfloor$ alcoves. When n is not divisible by 3, we can add the extra vertex (or two) anywhere. ∎

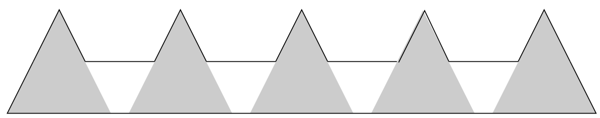


Figure 2: An art gallery that requires $\lfloor \frac{n}{3} \rfloor$ guards.

6.3.32 Construct an embedding of a 3-regular nonbipartite simple graph on the torus so that every face has even length.

It suffices to use K_4 as shown in Figure 3. It has two faces – one of degree/length 4 and one of degree/length 8. ∎

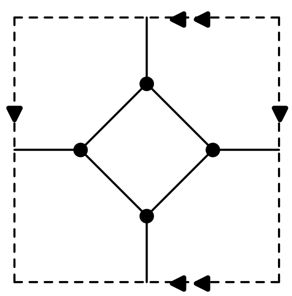


Figure 3: An embedding of K_4 on the torus so that every face has even length.

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polygon. Eventually, the ray will pass through another of the polygon's vertices. Let C be the first vertex when this happens. Then either AC or BC (or both) is the desired diagonal. By construction, there could not possibly be any vertices inside the triangle ABC. Had there been any, they would have been met earlier than C by the rotating ray.

Genus of complete graph Use the Euler characteristic for closed, orientable surfaces to prove that

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}$$

for $n \geq 3$.

Assume $n \geq 3$. Let $g = \gamma(K_n)$. Using the Euler characteristic, we know that

$$2 - 2g = n - e + f = n - \binom{n}{2} + f, \qquad (\star)$$

where $e = |E(K_n)| = \binom{n}{2}$. Furthermore, since $n \geq 3$, each face in an embedding of K_n has degree at least 2. Thus, we have

$$3f \leq \sum_{\text{faces}} d(\text{face}) = 2e \quad \Longrightarrow \quad f \leq \frac{2}{3}e = \frac{2}{3}\binom{n}{2}.$$

Substituting this expression for f into $(\star)$ gives us

$$\begin{aligned} 2 - 2g = n - \binom{n}{2} + f &\leq n - \binom{n}{2} + \frac{2}{3}\binom{n}{2} \\ &= n - \frac{1}{3}\binom{n}{2} \\ &= n - \frac{n(n-1)}{6} \\ &= \frac{7n - n^2}{6}. \end{aligned}$$

Solving for g, we have

$$g \geq \frac{n^2 - 7n + 12}{12} = \frac{(n-4)(n-3)}{12}.$$

Thus,

$$\gamma(K_n) = g \geq \frac{(n-3)(n-4)}{12}.$$

∎