Complex Variables: A Physical Approach

With Applications and `MatLab` Tutorials

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To my father, Henry Alfred Krantz: my only true hero.

Table of Contents

Preface

Complex variables is one of the grand old ladies of mathematics. Originally conceived in the pursuit of solutions of polynomial equations, complex variables blossomed in the hands of Euler, Argand, and others into the free-standing subject of complex analysis.

Like the negative numbers and zero, complex numbers were at first viewed with some suspicion. To be sure, they were useful tools for solving certain types of problems. But what were they precisely and where did they come from? What did they correspond to in the real world?

Today we have a much more concrete, and more catholic, view of the matter. First, we now know how to construct the complex numbers using rigorous mathematical techniques. Second, we understand how complex eigenvalues arise in the study of mechanical vibrations, how complex functions model incompressible fluid flow, and how complex variables enable the Fourier transform and the solution of a variety of differential equations that arise from physics and engineering.

It is essential for the modern undergraduate engineering student, as well as the math major and the physics major, to understand the basics of complex variable theory. The need then is for a textbook that presents the elements of the subject while requiring only a solid background in the calculus of one and several variables. This is such a text. There are, of course, other solid books for such a course. The book of Brown and Churchill has stood for many editions. The book of Saff and Snider, a more recent offering, is well-written and incisive. The book of Derrick features stimulating applications. What makes this text distinctive are the following features:

(1) We work in ideas from physics and engineering beginning in Chapter 1, and continuing throughout the book. Applications are an integral part of the presentation at every stage.

(2) Every chapter contains exercises that illustrate the applications.

(3) There are both exercises and text examples that illustrate the use of computer algebra systems in complex analysis.

(4) A very important attribute (and one not well represented in any other book) is that this text presents the subject of complex analysis as a natural continuation of the calculus. Most complex analysis texts exhibit the subject as a freestanding collection of ideas, independent of other parts of mathematical analysis and having its own body of techniques and tricks. This is in fact a misrepresentation of the discipline and leads to copious misunderstanding and misuse of the ideas. We are able to present complex analysis as part and parcel of the world view that the student has developed in his or her earlier course work. The result is that students can master the material more effectively and use it with good result in other courses in engineering and physics.

(5) The book has stimulating exercises at the three levels of drill, exploration, and theory. There is a comfortable balance between theory and applications.

(6) Most sections have examples that illustrate both the theory and the practice of complex variables.

(7) The book has many illustrations which clarify key concepts from complex variable theory.

(8) We use differential equations to illustrate important concepts throughout the book.

(9) We integrate `MatLab` exercises and examples throughout.

The subject of complex variables has many aspects—from the algebraic features of a complete number field, to the analytic properties imposed by the Cauchy integral formula, to the geometric qualities coming from the idea of conformality. The student must be acquainted with all components of the field. This text speaks all the languages, and shows the student how to deal with all the different approaches to complex analysis. The examples illustrate all the key concepts, while the exercises reinforce the basic skills, and provide practice in all the fundamental ideas.

As noted, we shall integrate `MatLab` activities throughout. Computer algebra systems have become an important and central tool in modern mathematical science, and `MatLab` has proved to be of particular utility in the engineering world. `MatLab` is particularly well adapted to use in complex variable theory. Here we show the student, in a natural context, how `MatLab` calculations can play a role in complex variables.

There is too much material in this book for a one-semester course. Some thought must be given as to how to design a course from this book. Any course should cover Chapters 1 through 5. Finishing off with Sections 7.1 through 7.3 and Chapter 8 will give a very basic grounding in the subject. Chapters 10 and 11 are great for applications and instructors can dip into them as time permits.

A more thoroughgoing course would want to cover the remainder of Chapter 7 and at least some of Chapter 6. As noted, Chapters 10 and 11 give the student a detailed glimpse of how complex variables are used in the real world. Chapter 9, on harmonic functions, is more advanced material and should perhaps be saved for a two-term course. Chapter 12 is dessert, for those who want to explore computer tools that can be used in the study of complex variables.

Complex variables is a vibrant area of mathematical research, and it interacts fruitfully with many other parts of mathematics. It is an essential tool in applications. This text will illustrate and teach all facets of the subject in a lively manner that will speak to the needs of modern students. It will give them a powerful toolkit for future work in the mathematical sciences, and will also point to new directions for additional learning.

I conclude by thanking my editor Bob Stern for encouraging me to write this book and providing all needed assistance. He engaged some exceptionally careful and proactive reviewers who provided valuable advice and encouragement. Working with Taylor & Francis is always a pleasure.

— SGK

Chapter 1

Basic Ideas

1.1 Complex Arithmetic

1.1.1 The Real Numbers

The real number system consists of both the rational numbers (numbers with terminating or repeating decimal expansions) and the irrational numbers (numbers with infinite, nonrepeating decimal expansions). The real numbers are denoted by the symbol $\mathbb{R}$. We let $\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}\ ,\ y \in \mathbb{R}\}$ (Figure 1.1).

1.1.2 The Complex Numbers

The complex numbers $\mathbb{C}$ consist of $\mathbb{R}^2$ equipped with some special algebraic operations. One defines

$$\begin{aligned} (x, y) + (x', y') &= (x + x', y + y')\,, \\ (x, y) \cdot (x', y') &= (xx' - yy', xy' + yx'). \end{aligned}$$

These operations of $+$ and $\cdot$ are commutative and associative.

EXAMPLE 1 We may calculate that

$$(3, 7) + (2, -4) = (3 + 2, 7 + (-4)) = (5, 3)\,.$$

Also

$$(3, 7) \cdot (2, -4) = (3 \cdot 2 - 7 \cdot (-4), 3 \cdot (-4) + 7 \cdot 2) = (34, 2)\,.$$

□

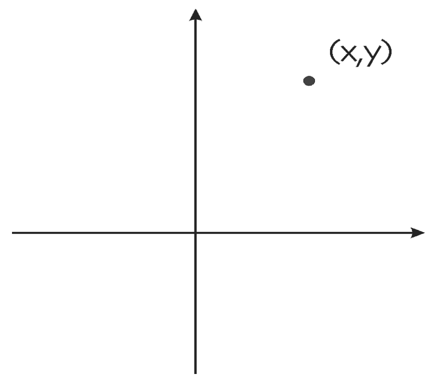

Figure 1.1: A point in the plane.

Of course we sometimes wish to subtract complex numbers. We define

$$z - w = z + (-w)\,.$$

Thus if $z = (11, -6)$ and $w = (1, 4)$ then

$$z - w = z + (-w) = (11, -6) + (-1, -4) = (10, -10)\,.$$

We denote $(1, 0)$ by 1 and (0,1) by i. We also denote $(0, 0)$ by 0. If $\alpha \in \mathbb{R}$, then we identify α with the complex number $(\alpha, 0)$. Using this notation, we see that

$$\alpha \cdot (x, y) = (\alpha, 0) \cdot (x, y) = (\alpha x, \alpha y)\,. \tag{1.1}$$

In particular,

$$1 \cdot (x, y) = (1, 0) \cdot (x, y) = (x, y)\,.$$

We may calculate that

$$x \cdot 1 + y \cdot i = (x, 0) \cdot (1, 0) + (y, 0) \cdot (0, 1) = (x, 0) + (0, y) = (x, y)\,.$$

Thus every complex number (x, y) can be written in one and only one fashion in the form $x \cdot 1 + y \cdot i$ with $x, y \in \mathbb{R}$. We usually write the number even more succinctly as $x + iy$.

Example 2 The complex number $(-2, 5)$ is usually written as

$$(-2, 5) = -2 + 5i\,.$$

The complex number $(4,9)$ is usually written as

$$(4,9) = 4 + 9i\,.$$

The complex number $(-3,0)$ is usually written as

$$(-3,0) = -3 + 0i = -3\,.$$

The complex number $(0,6)$ is usually written as

$$(0,6) = 0 + 6i = 6i\,.$$

$\square$

In this more commonly used notation, laws of addition and multiplication become

$$\begin{aligned}(x+iy)+(x'+iy') &= (x+x')+i(y+y'),\\ (x+iy)\cdot(x'+iy') &= (xx'-yy')+i(xy'+yx').\end{aligned}$$

Observe that $i \cdot i = -1$. Indeed,

$$i \cdot i = (0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1) + i(0\cdot 1 + 1\cdot 0) = -1 + 0i = -1\,.$$

This is historically the single most important fact about the complex numbers—that they provide negative numbers with square roots. More generally, the complex numbers provide *any polynomial equation* with roots. We shall develop these ideas in detail below.

Certainly our multiplication law is consistent with the scalar multiplication introduced in line (1.1).

Insight: The multiplicative law presented at the beginning of Section 1.1.2 may at first seem strange and counter-intuitive. Why not take the simplest possible route and define

$$(x,y)\cdot(x',y') = (xx',yy')\,? \tag{1.2}$$

This would certainly be easier to remember, and is consistent with what one might guess. The trouble is that definition (1.2), while simple, has a number of liabilities. First of all, it would lead to

$$(1,0)\cdot(0,1) = (0,0) = 0\,.$$

Thus we would have the product of two nonzero numbers equaling zero—an eventuality that we want to always avoid in any arithmetic. Second, the main point of the complex numbers is that we want a negative number to have a square root. That would not happen if (1.2) were our definition of multiplication.

The definition at the start of Section 1.1.2 is in fact a very clever idea that creates a *new number system* with many marvelous new properties. The purpose of this text is to acquaint you with this new world. □

EXAMPLE 3 The fact that $i \cdot i = -1$ means that the number -1 has a square root. This fact is at first counterintuitive. If we stick to the real number system, then only nonnegative numbers have square roots. In the complex number system, *any* number has a square root—in fact any nonzero number has two of them.[1] For example,

$$(1+i)^2 = 2i$$

and

$$(-1-i)^2 = 2i\,.$$

Later in this chapter we will learn how to find both the square roots, and in fact all the nth roots, of any complex number. □

EXAMPLE 4 The syntax in `MatLab` for complex number arithmetic is simple and straightforward. Refer to the basic manual [PRA] for key ideas. A complex number in `MatLab` may be written as `a + bi` or `a + b*i`.

In order to calculate $(3-2i)\cdot(1+4i)$ using `MatLab`, one enters the code

```
>>(3 - 2i)*(1 + 4i)
```

Here `>>` is the standard `MatLab` prompt. `MatLab` instantly gives the answer `11 + 10i`. □

[1]The number 0 has just one square root. It is the only root of the polynomial equation $z^2 = 0$. All other complex numbers α have two distinct square roots. They are the roots of the polynomial equation $z^2 = \alpha$ or $z^2 - \alpha = 0$. The matter will be treated in greater detail below. In particular, we shall be able to put these ideas in the context of the Fundamental Theorem of Algebra.

The symbols z, w, ζ are frequently used to denote complex numbers. We usually take $z = x + iy$, $w = u + iv$, $\zeta = \xi + i\eta$. The real number x is called the *real part* of z and is written $x = \mathrm{Re}\, z$. The real number y is called the *imaginary part* of z and is written $y = \mathrm{Im}\, z$.

EXAMPLE 5 The real part of the complex number $z = 4 - 8i$ is 4. We write

$$\mathrm{Re}\, z = 4\,.$$

The imaginary part of z is -8. We write

$$\mathrm{Im}\, z = -8\,.$$

□

EXAMPLE 6 Addition of complex numbers corresponds exactly to addition of vectors in the plane. Specifically, if $z = x + iy$ and $w = u + iv$ then

$$z + w = (x + u) + i(y + v)\,.$$

If we make the correspondence

$$z = x + iy \leftrightarrow \mathbf{z} = \langle x, y\rangle$$

and

$$w = u + iv \leftrightarrow \mathbf{w} = \langle u, v\rangle$$

then we have

$$\mathbf{z} + \mathbf{w} = \langle x, y\rangle + \langle u, v\rangle = \langle x + u, y + v\rangle\,.$$

Clearly

$$(x + u) + i(y + v) \leftrightarrow \langle x + u, y + v\rangle\,.$$

But complex multiplication *does not* correspond to any standard vector operation. Indeed it cannot. For the standard vector dot product has no concept of multiplicative inverse; and the standard vector cross product has no concept of multiplicative inverse. But one of the main points of the complex number operations is that they turn this number system into a *field*: every nonzero number does indeed have a multiplicative inverse. This is a very special property of two-dimensional space. There is no other Euclean space (except of course the real line) that can be equipped with commutative operations of addition and multiplication so that **(i)** every number has an additive inverse and **(ii)** every nonzero number has a multiplicative inverse. We shall learn more about these ideas below. □

The complex number $x - iy$ is by definition the complex *conjugate* of the complex number $x + iy$. If $z = x + iy$, then we denote the conjugate[2] of z with the symbol $\overline{z}$; thus $\overline{z} = x - iy$.

1.1.3 Complex Conjugate

Note that $z + \overline{z} = 2x$, $z - \overline{z} = 2iy$. Also

$$\overline{z + w} = \overline{z} + \overline{w} \,,$$
$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \,.$$

A complex number is real (has no imaginary part) if and only if $z = \overline{z}$. It is imaginary (has no real part) if and only if $z = -\overline{z}$.

Example 7 Let $z = -7 + 6i$ and $w = 4 - 9i$. Then

$$\overline{z} = -7 - 6i$$

and

$$\overline{w} = 4 + 9i \,.$$

Notice that

$$\overline{z} + \overline{w} = (-7 - 6i) + (4 + 9i) = -3 + 3i \,,$$

and that number is exactly the conjugate of

$$z + w = -3 - 3i \,.$$

Notice also that

$$\overline{z} \cdot \overline{w} = (-7 - 6i) \cdot (4 + 9i) = 26 - 87i \,,$$

and that number is exactly the conjugate of

$$z \cdot w = 26 + 87i \,.$$

□

[2]Rewriting history a bit, we may account for the concept of "conjugate" as follows. If $p(z) = az^2 + bz + c$ is a polynomial with real coefficients, and if $z = x + iy$ is a root of this polynomial, then $\overline{z} = x - iy$ will also be a root of that same polynomial. This assertion is immediate from the quadratic formula, or by direct calculation. Thus $x + iy$ and $x - iy$ are *conjugate roots* of the polynomial p.

EXAMPLE 8 Conjugation of a complex number is a straightforward operation. But `MatLab` can do it for you. The `MatLab` code

```
>>conj(8 - 7i)
```

yields the output

```
8 + 7i.
```

□

Exercises

1. Let $z = 13 + 5i$, $w = 2 - 6i$, and $\zeta = 1 + 9i$. Calculate $z + w$, $w - \zeta$, $z \cdot \zeta$, $w \cdot \zeta$, and $\zeta - z$.

2. Let $z = 4 - 7i$, $w = 1 + 3i$, and $\zeta = 2 + 2i$. Calculate $\overline{z}$, $\overline{\zeta}$, $\overline{z - w}$, $\overline{\zeta + z}$, $\overline{\zeta \cdot w}$.

3. If $z = 6 - 2i$, $w = 4 + 3i$, and $\zeta = -5 + i$, then calculate $z + \overline{z}$, $z + 2\overline{z}$, $z - \overline{w}$, $z \cdot \overline{\zeta}$, and $w \cdot \overline{\zeta}^2$.

4. If z is a complex number then $\overline{z}$ has the same distance from the origin as z. Explain why.

5. If z is a complex number then $\overline{z}$ and z are situated symmetrically with respect to the x-axis. Explain why.

6. If z is a complex number then $-\overline{z}$ and z are situated symmetrically with respect to the y-axis. Explain why.

7. Explain why addition in the real numbers is a special case of addition in the complex numbers. Explain why the two operations are logically consistent.

8. Explain why multiplication in the real numbers is a special case of multiplication in the complex numbers. Explain why the two operations are logically consistent.

9. Use `MatLab` to calculate the conjugates of $9 + 4i$, $6 - 3i$, and $2 + i$.

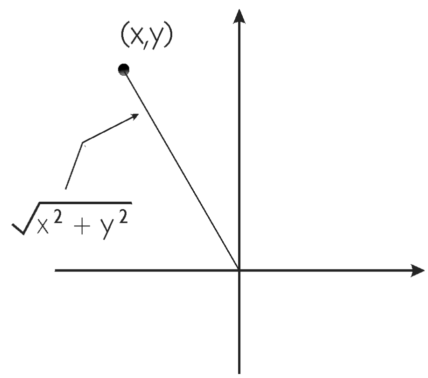

Figure 1.2: Distance to the origin or modulus.

10. Let $z = 10 + 2i$, $w = 4 - 6i$. Use `MatLab` to calculate $z \cdot w$, $\overline{z} \cdot w$, $z + w$, and $z - \overline{w}$.

11. Let $z = a + ib$ and $w = c + id$ be complex numbers. These correspond, in an obvious way, to points (a, b) and (c, d) in the plane, and these in turn correspond to vectors $Z = \langle a, b \rangle$ and $W = \langle c, d \rangle$.

Verify that addition of z and w as complex numbers corresponds in a natural way to addition of the vectors Z and W. What does multiplication of the complex numbers z and w correspond to vis a vis the vectors?

1.2 Algebraic and Geometric Properties

1.2.1 Modulus of a Complex Number

The ordinary Euclidean distance of (x, y) to $(0, 0)$ is $\sqrt{x^2 + y^2}$ (Figure 1.2). We also call this number the *modulus* of the complex number $z = x + iy$ and we write $|z| = \sqrt{x^2 + y^2}$. Note that

$$z \cdot \overline{z} = x^2 + y^2 = |z|^2 . \tag{1.3}$$

The distance from z to w is $|z - w|$. We also have the easily verified formulas $|zw| = |z||w|$ and $|\mathrm{Re}\, z| \le |z|$ and $|\mathrm{Im}\, z| \le |z|$.

The very important *triangle inequality* says that

$$|z + w| \le |z| + |w| \,.$$

We shall discuss this relation in greater detail below. For now, the interested reader may wish to square both sides, cancel terms, and see what the inequality reduces to.

EXAMPLE 9 The complex number $z = 7 - 4i$ has modulus given by

$$|z| = \sqrt{7^2 + (-4)^2} = \sqrt{65} \,.$$

The complex number $w = 2 + i$ has modulus given by

$$|w| = \sqrt{2^2 + 1^2} = \sqrt{5} \,.$$

Finally, the complex number $z + w = 9 - 3i$ has modulus given by

$$|z + w| = \sqrt{9^2 + (-3)^2} = \sqrt{90} \,.$$

According to the triangle inequality,

$$|z + w| \le |z| + |w| \,,$$

and we may now confirm this arithmetically as

$$\sqrt{90} \le \sqrt{65} + \sqrt{5} \,.$$

□

EXAMPLE 10 `MatLab` can perform modulus calculations quickly and easily. The `MatLab` code

```
>>abs(6 - 8i)
```

yields the output

```
10.
```

The input

```
>>abs(2 + 7i)
```

yields the output

```
7.2801.
```

□

1.2.2 The Topology of the Complex Plane

If P is a complex number and $r > 0$, then we set

$$D(P,r) = \{z \in \mathbb{C} : |z - P| < r\}$$

and

$$\overline{D}(P,r) = \{z \in \mathbb{C} : |z - P| \leq r\}.$$

The first of these is the *open disc with center P and radius r*; the second is the *closed disc with center P and radius r* (Figure 1.3). Notice that the closed disc includes its boundary (indicated in the figure with a solid line for the boundary) while the open disc does not (indicated in the figure with a dashed line for the boundary). We often use the simpler symbols D and $\overline{D}$ to denote, respectively, the discs $D(0,1)$ and $\overline{D}(0,1)$.

We say that a set $U \subseteq \mathbb{C}$ is *open* if, for each $P \in U$, there is an $r > 0$ such that $D(P,r) \subseteq U$. Thus an open set is one with the property that each point P of the set is surrounded by neighboring points (that is, the points of distance less than r from P) that are still in the set—see Figure 1.4. Of course the number r will depend on P. As examples, $U = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$ is open, but $F = \{z \in \mathbb{C} : \operatorname{Re} z \leq 1\}$ is not (Figure 1.5). Observe that, in these figures, we use a *solid line* to indicate that the boundary is included in the set; we use a *dotted line* to indicate that the boundary is not included in the set.

A set $E \subseteq \mathbb{C}$ is said to be *closed* if $\mathbb{C} \setminus E \equiv \{z \in \mathbb{C} : z \notin E\}$ (the complement of E in $\mathbb{C}$) is open. [Note that when the universal set is understood—in this case $\mathbb{C}$—we sometimes use the notation ${}^cE$ to denote the complement.] The set F in the last paragraph is closed.

It is *not* the case that any given set is either open or closed. For example, the set $W = \{z \in \mathbb{C} : 1 < \operatorname{Re} z \leq 2\}$ is *neither* open *nor* closed (Figure 1.6).

We say that a set $E \subset \mathbb{C}$ is *connected* if there do not exist nonempty disjoint open sets U and V such that $U \cap E \neq \emptyset$, $V \cap E \neq \emptyset$, and $E = (U \cap E) \cup (V \cap E)$. Refer to Figure 1.7 for these ideas. We say that U and V *separate* E. It is a useful fact that if E is an open set, then E is connected if and only if it is path-connected; this means that any two points of E can be connected by a continuous path or curve that lies entirely in the set. See Figure 1.8.

In practice we recognize a connected set as follows. If $E \subseteq \mathbb{C}$ is a set and there is a proper subset $S \subseteq E$ (proper means that S is not all of E) such

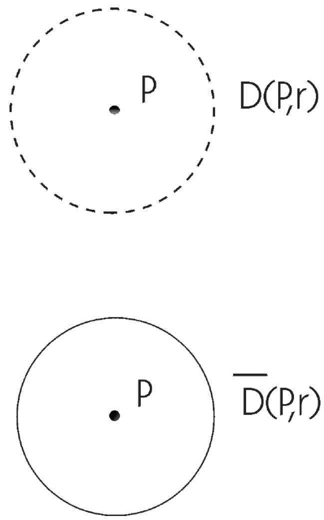

Figure 1.3: An open disc and a closed disc.

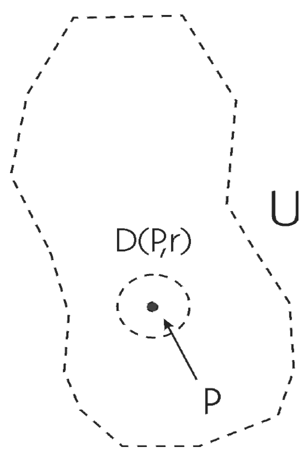

Figure 1.4: An open set.

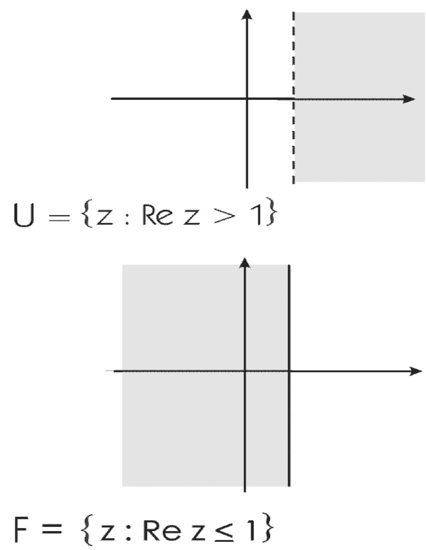

Figure 1.5: An open set and a nonopen set.

that S is both open and closed, then $U = S$ and $V = {}^cS$ are both open and separate E so that E is disconnected. Thus connectedness of E means that *there is no proper subset of E that is both open and closed.*

Much of our analysis in this book will be on domains in the plane. A *domain* is a connected open set. We also use the word *region* alternatively with "domain."

1.2.3 The Complex Numbers as a Field

Let 0 denote the complex number $0 + i0$. If $z \in \mathbb{C}$, then $z + 0 = z$. Also, letting $-z = -x - iy$, we have $z + (-z) = 0$. So every complex number has an additive inverse, and that inverse is unique. One may also readily verify that $0 \cdot z = z \cdot 0 = 0$ for any complex number z.

Since $1 = 1 + i0$, it follows that $1 \cdot z = z \cdot 1 = z$ for every complex number z. If $z \neq 0$, then $|z|^2 \neq 0$ and

$$z \cdot \left(\frac{\overline{z}}{|z|^2} \right) = \frac{|z|^2}{|z|^2} = 1\,. \tag{1.4}$$

So every nonzero complex number has a multiplicative inverse, and that

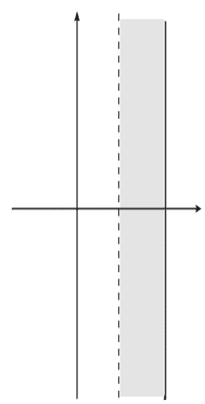

Figure 1.6: A set that is neither open nor closed.

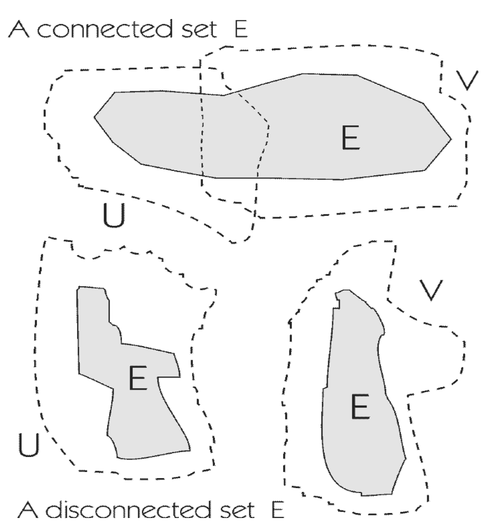

Figure 1.7: A connected set and a disconnected set.

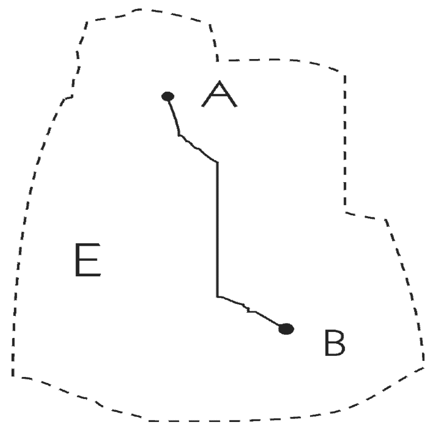

Figure 1.8: An open set is connected if and only if it is path-connected.

inverse is unique. It is natural to define $1/z$ to be the multiplicative inverse $\overline{z}/|z|^2$ of z and, more generally, to define

$$\frac{z}{w} = z \cdot \frac{1}{w} = \frac{z\overline{w}}{|w|^2} \qquad \text{for } w \neq 0\,. \tag{1.5}$$

We also have $\overline{z/w} = \overline{z}/\overline{w}$.

It must be stressed that $1/z$ makes good sense as an intuitive object but *not as a complex number*. A complex number is, by definition, one that is written in the form $x + iy$—which $1/z$ most definitely is not. But we have declared

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{|z|^2} = \frac{x}{|z|^2} - i \cdot \frac{y}{|z|^2}\,,$$

and this is definitely in the form of a complex number.

EXAMPLE 11 The idea of multiplicative inverse in the complex numbers is at first counterintuitive. So let us look at a specific instance.

Let $z = 2 + 3i$. It is all too easy to say that the multiplicative inverse of z is

$$\frac{1}{z} = \frac{1}{2 + 3i}\,.$$

The trouble is that, as written, $1/(2 + 3i)$ is *not* a complex number. Recall that a complex number is a number of the form $x + iy$. But our discussion preceding this example enables us to clarify the matter.

Because in fact the multiplicative inverse of $2+3i$ is

$$\frac{\overline{z}}{|z|^2} = \frac{2-3i}{13}\,.$$

The advantage of looking at things this way is that the multiplicative inverse is in fact now a complex number; it is

$$\frac{2}{13} - i\frac{3}{13}\,.$$

And we may check directly that this number does the job:

$$(2+3i)\cdot\left(\frac{2}{13} - i\frac{3}{13}\right) = \left(2\cdot\frac{2}{13} + 3\cdot\frac{3}{13}\right) + i\left(2\cdot\left(-\frac{3}{13}\right) + 3\cdot\frac{2}{13}\right) = 1+0i = 1\,.$$

□

Multiplication and addition satisfy the usual distributive, associative, and commutative laws. Therefore $\mathbb{C}$ is a *field* (see [HER]). The field $\mathbb{C}$ contains a copy of the real numbers in an obvious way:

$$\mathbb{R} \ni x \mapsto x + i0 \in \mathbb{C}\,. \tag{1.6}$$

This identification respects addition and multiplication. So we can think of $\mathbb{C}$ as a field extension of $\mathbb{R}$: it is a larger field which contains the field $\mathbb{R}$.

1.2.4 The Fundamental Theorem of Algebra

It is not true that every nonconstant polynomial with real coefficients has a real root. For instance, $p(x) = x^2 + 1$ has no real roots. The Fundamental Theorem of Algebra states that every polynomial with complex coefficients has a complex root (see the treatment in Sections 4.1.4, 6.3.3). The complex field $\mathbb{C}$ is the *smallest* field that contains $\mathbb{R}$ and has this so-called algebraic closure property.

Exercises

1. Let $z = 6 - 9i$, $w = 4 + 2i$, $\zeta = 1 + 10i$. Calculate $|z|$, $|w|$, $|z+w|$, $|\zeta - w|$, $|z\cdot w|$, $|z+w|$, $|\zeta\cdot z|$. Confirm directly that

$$|z+w| \le |z| + |w|\,,$$

$$|z \cdot w| = |z||w|,$$
$$|\zeta \cdot z| = |\zeta||z|.$$

2. Find complex numbers z, w such that $|z| = 5$, $|w| = 7$, $|z + w| = 9$.

3. Find complex numbers z, w such that $|z| = 1$, $|w| = 1$, and $z/w = i^3$.

4. Let $z = 4 - 6i$, $w = 2 + 7i$. Calculate z/w, w/z, and $1/w$.

5. Sketch these discs on the same set of axes: $D(2 + 3i, 4)$, $D(1 - 2i, 2)$, $\overline{D}(i, 5)$, $\overline{D}(6 - 2i, 5)$.

6. Which of these sets is open? Which is closed? Why or why not?

(a) $\{x + iy \in \mathbb{C} : x^2 + 4y^2 \leq 4\}$
(b) $\{x + iy \in \mathbb{C} : x < y\}$
(c) $\{x + iy \in \mathbb{C} : 2 \leq x + y < 5\}$
(d) $\{x + iy \in \mathbb{C} : 4 < \sqrt{x^2 + 3y^2}\}$
(e) $\{x + iy \in \mathbb{C} : 5 \leq \sqrt{x^4 + 2y^6}\}$

7. Consider the polynomial $p(z) = z^3 - z^2 + 2z - 2$. How many real roots does p have? How many complex roots? Explain.

8. The polynomial $q(z) = z^3 - 3z + 2$ is of degree three, yet it does *not* have three distinct roots. Explain.

9. Use `MatLab` to calculate $|3 + 6i|$, $|4 - 2i|$, and $|8 + 7i|$.

10. Let $z = 2 - 6i$ and $w = 9 + 3i$. Use `MatLab` to calculate z/w, $w/\overline{z}^2$, and $z \cdot (w + \overline{z})/\overline{w}$.

11. Use `MatLab` to test whether any of $-i$, i, or $1 + i$ is a root of the polynomial $p(z) = z^3 - 3z + 4i$.

12. Use `MatLab` to find all the complex roots of the polynomial $p(z) = z^4 - 3z^3 + 2z - 1$. Call the roots α_1, α_2, α_3, α_4. Calculate expicitly the product

$$Q(z) = (z - \alpha_1) \cdot (z - \alpha_2) \cdot (z - \alpha_3) \cdot (z - \alpha_4).$$

Observe that $Q(z) = p(z)$. Is this a coincidence?

13. Use `MatLab` if convenient to produce a fourth-degree polynomial that has roots $2-3i$, $4+7i$, $8-2i$, and $6+6i$. This polynomial is unique up to a constant multiple. Explain why.

14. Write a fourth degree polynomial $q(z)$ whose roots are 1, -1, i, and $-i$. These four numbers are all the fourth roots of 1. Explain therefore why q has such a simple form.

15. If z is a nonzero complex number, then it has a reciprocal $1/z$ that is also a complex number. Now if Z is the planar vector corresponding to z, then what vector does $1/z$ correspond to? [**Hint:** Think in terms of reflection in a circle.]

1.3 The Exponential and Applications

1.3.1 The Exponential Function

We define the complex exponential as follows:

(1.7) If $z = x$ is real, then

$$e^z = e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as in calculus. Here ! denotes the usual "factorial" operation:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1\,.$$

(1.8) If $z = iy$ is pure imaginary, then

$$e^z = e^{iy} \equiv \cos y + i \sin y.$$

[This identity, due to Euler, is discussed below.]

(1.9) If $z = x + iy$, then

$$e^z = e^{x+iy} \equiv e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y).$$

This tri-part definition may seem a bit mysterious. But we may justify it formally as follows (a detailed discussion of complex power series will come later). Consider the definition

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,. \tag{1.10}$$

This is a natural generalization of the familiar definition of the exponential function from calculus.

We may write this out as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \,. \tag{1.11}$$

In case $z = x$ is real, this gives the familiar

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \,.$$

In case $z = iy$ is pure imaginary, then (1.11) gives

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \frac{y^6}{6!} - i\frac{y^7}{7!} + - \cdots \,. \tag{1.12}$$

Grouping the real terms and the imaginary terms we find that

$$e^{iy} = \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + - \cdots\right] + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + - \cdots\right] = \cos y + i \sin y \,. \tag{1.13}$$

This is the same as the definition that we gave above in (1.8).

Part (1.9) of the definition is of course justified by the usual rules of exponentiation.

An immediate consequence of this new definition of the complex exponential is the following complex-analytic definition of the sine and cosine functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \,, \tag{1.14}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \,. \tag{1.15}$$

Note that when $z = x + i0$ is real this new definition is consistent[3] with the familiar Euler formula from calculus:

$$e^{ix} = \cos x + i \sin x. \tag{1.16}$$

[3] The key fact here is that, since $e^{ix} = \cos x + i \sin x$ then $e^{-ix} = \cos x - i \sin x$. Thus also $e^{iz} = \cos z + i \sin z$ and $e^{-iz} = \cos z - i \sin z$.

It is sometimes useful to rewrite equation (1.14) as

$$\begin{aligned}
\cos z &= \frac{e^{iz}+e^{-iz}}{2} \\
&= \frac{e^{ix-y}+e^{-ix+y}}{2} \\
&= \frac{(\cos x + i\sin x)e^{-y} + (\cos x - i\sin x)e^{y}}{2} \\
&= \cos x \cdot \frac{e^{y}+e^{-y}}{2} - i\sin x \cdot \frac{e^{y}-e^{-y}}{2} \\
&= \cos x \cosh y - i \sin x \sinh y\,.
\end{aligned}$$

Similarly, one can show that

$$\sin z = \sin x \cosh y + i \cos x \sinh y\,.$$

1.3.2 Laws of Exponentiation

The complex exponential satisfies familiar rules of exponentiation:[4]

$$e^{z+w} = e^{z}\cdot e^{w} \quad \text{and} \quad \left(e^{z}\right)^{w} = e^{zw} \ \text{ for } w \text{ an integer}\,. \tag{1.17}$$

Note that we may rewrite the second of these formulas as

$$\left(e^{z}\right)^{n} = \underbrace{e^{z}\cdots e^{z}}_{n\ \text{times}} = e^{nz}. \tag{1.18}$$

1.3.3 The Polar Form of a Complex Number

A consequence of our first definition of the complex exponential—see (1.8)—is that if $\zeta \in \mathbb{C}$, $|\zeta| = 1$, then there is a unique number θ, $0 \le \theta < 2\pi$, such that $\zeta = e^{i\theta}$ (see Figure 1.9). Here θ is the (signed) angle between the positive x axis and the ray $\overrightarrow{0\zeta}$.

Now if z is any nonzero complex number, then

$$z = |z|\cdot\left(\frac{z}{|z|}\right) \equiv |z|\cdot\zeta \tag{1.19}$$

[4]The formular $(e^{z})^{w}$ requires further elucidation. The expression *does* makes sense for w not an integer, but the complex logarithm function must be used in the process. See the development below.

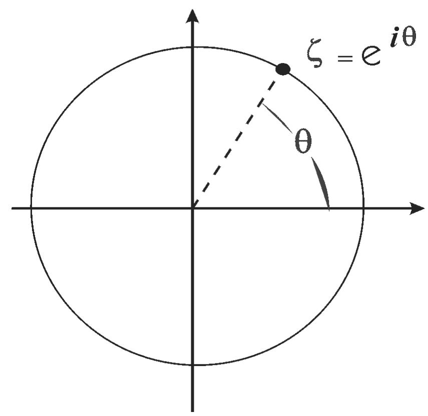

Figure 1.9: Polar coordinates of a point in the plane.

where $\zeta \equiv z/|z|$ has modulus 1. Again, letting θ be the angle between the positive real axis and $\overrightarrow{0\zeta}$, we see that

$$\begin{aligned} z &= |z| \cdot \zeta \\ &= |z| e^{i\theta} \\ &= r e^{i\theta} \ , \end{aligned} \tag{1.20}$$

where $r = |z|$. This form is called the *polar* representation for the complex number z. (Note that some classical books write the expression $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ as $z = r\mathrm{cis}\,\theta$. The reader should be aware of this notation, though we shall not use it in the present book.)

EXAMPLE 12 Let $z = 1 + \sqrt{3}i$. Then $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$. Hence

$$z = 2 \cdot \left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2} \right) . \tag{1.21}$$

The number in parentheses is of unit modulus and subtends an angle of $\pi/3$ with the positive x-axis. Therefore

$$1 + \sqrt{3}i = z = 2 \cdot e^{i\pi/3}. \tag{1.22}$$

□

It is often convenient to allow angles that are greater than or equal to 2π in the polar representation; when we do so, the polar representation is no longer unique. For if k is an integer, then

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ &= \cos(\theta+2k\pi) + i\sin(\theta+2k\pi) \\ &= e^{i(\theta+2k\pi)}\,. \end{aligned} \tag{1.23}$$

Remark: Of course the inverse of the exponential function is the (complex) logarithm. This is a rather subtle idea, and will be investigated in Section 2.5.

Exercises

1. Calculate (with your answer in the form $a+ib$) the values of $e^{\pi i}$, $e^{(\pi/3)i}$, $5e^{-i(\pi/4)}$, $2e^{i}$, $7e^{-3i}$.

2. Write these complex numbers in polar form: $2+2i$, $1+\sqrt{3}i$, $\sqrt{3}-i$, $\sqrt{2}-i\sqrt{2}$, i, $-1-i$.

3. If $e^z = 2-2i$ then what can you say about z? [**Hint:** There is more than one answer.]

4. If $w^5 = z$ and $|z| = 3$ then what can you say about $|w|$?

5. If $w^5 = z$ and z subtends an angle of $\pi/4$ with the positive x-axis, then what can you say about the angle that w subtends with the positive x-axis? [**Hint:** There is more than one answer to this question.]

6. Calculate that $|e^z| = e^x$. Also $|\cos z|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$ and $|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$.

7. If $w^2 = z^3$ then how are the polar forms of z and w related?

8. Write all the polar forms of the complex number $-\sqrt{2}+i\sqrt{6}$.

9. If $z = re^{\imath\theta}$ and $w = se^{i\psi}$ then what can you say about the polar form of $z+w$? What about $z\cdot w$?

10. Use `MatLab` to calculate $e^{i\pi/3}$, e^{1-i}, and $e^{-3\pi i/4}$. [**Hint:** The `MatLab` symbol for π is `pi`. The symbol for exponentiation is `^`. Be sure to use `*` for multiplication when appropriate.]

11. Use `MatLab` functions to calculate the polar form of the complex numbers $2-5i$, $3+7i$, $6+4i$. [**Hint:** The trignometric functions in `MatLab` are given by `sin( )`, `cos( )`, `tan( )` and the inverse trigonometric functions by `asin( )`, `acos( )`, and `atan( )`.]

12. Use `MatLab` to convert these complex numbers in polar form to standard rectilinear form: $4e^{5i}$, $-6e^{-3i}$, $2e^{\pi^2 i}$.

13. Use `MatLab` to calculate the rectangular form of the complex numbers $\sqrt{3}e^{i\pi/3}$, $\sqrt{8}e^{-2\pi/3}$, $\sqrt{5}e^{i\pi/6}$, and $\sqrt{2}e^{-\pi/3}$.

14. Let $w = 3e^{i\pi/3}$. Calculate w^2, w^3, $1/w$ and $w+1$. Use `MatLab` if you wish.

15. Explain why there is no complex number z such that $e^z = 0$.

16. Suppose that z and w are complex numbers that are related by the formula $z = e^w$. Each of z and w corresponds to a vector in the plane. How are these vectors related?

1.3.4 Roots of Complex Numbers

The properties of the exponential operation can be used, together with the polar representation, to find the n^{th} roots of a complex number.

EXAMPLE 13 To find all sixth roots of 2, we let $re^{i\theta}$ be an arbitrary sixth root of 2 and solve for r and θ. If

$$\left(re^{i\theta}\right)^6 = 2 = 2 \cdot e^{i0} \tag{1.24}$$

or

$$r^6 e^{i6\theta} = 2 \cdot e^{i0}\,, \tag{1.25}$$

then it follows that $r = 2^{1/6} \in \mathbb{R}$ and $\theta = 0$ solve this equation. So the real number $2^{1/6} \cdot e^{i0} = 2^{1/6}$ is a sixth root of two. This is not terribly surprising, but we are not finished.

We may also solve

$$r^6 e^{i6\theta} = 2 = 2 \cdot e^{2\pi i}. \tag{1.26}$$

Notice that we are taking advantage of the ambiguity built into the polar representation: The number 2 may be written as $2 \cdot e^{i0}$, but it may also be written as $2 \cdot e^{2\pi i}$ or as $2 \cdot e^{4\pi i}$, and so forth.

Hence

$$r = 2^{1/6}\ ,\ \ \theta = 2\pi/6 = \pi/3. \tag{1.27}$$

This gives us the number

$$2^{1/6} e^{i\pi/3} = 2^{1/6}\big(\cos\pi/3 + i\sin\pi/3\big) = 2^{1/6}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{1.28}$$

as a sixth root of two. Similarly, we can solve

$$\begin{aligned} r^6 e^{i6\theta} &= 2 \cdot e^{4\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{6\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{8\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{10\pi i} \end{aligned}$$

to obtain the other four sixth roots of 2:

$$2^{1/6}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{1.29}$$

$$-2^{1/6} \tag{1.30}$$

$$2^{1/6}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \tag{1.31}$$

$$2^{1/6}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right). \tag{1.32}$$

These are in fact all the sixth roots of 2. □

Remark: Notice that, in the last example, the process must stop after six roots. For if we solve

$$r^6 e^{i6\theta} = 2 \cdot e^{12\pi i}\,,$$

then we find that $r = 2^{1/6}$ as usual and $\theta = 2\pi$. This yields the complex root

$$z = 2^{1/6} \cdot e^{2\pi i} = 1^{1/6},$$

and that simply repeats the first root that we found. If we were to continue with $14\pi i$, $16\pi i$, and so forth, we would just repeat the other roots.

Example 14 Let us find all third roots of i. We begin by writing i as

$$i = e^{i\pi/2}. \tag{1.33}$$

Solving the equation

$$(re^{i\theta})^3 = i = e^{i\pi/2} \tag{1.34}$$

then yields $r = 1$ and $\theta = \pi/6$.

Next, we write $i = e^{i5\pi/2}$ and solve

$$(re^{i\theta})^3 = e^{i5\pi/2} \tag{1.35}$$

to obtain that $r = 1$ and $\theta = 5\pi/6$.

Finally we write $i = e^{i9\pi/2}$ and solve

$$(re^{i\theta})^3 = e^{i9\pi/2} \tag{1.36}$$

to obtain that $r = 1$ and $\theta = 9\pi/6 = 3\pi/2$.

In summary, the three cube roots of i are

$$\begin{aligned} e^{i\pi/6} &= \frac{\sqrt{3}}{2} + i\frac{1}{2}, \\ e^{i5\pi/6} &= -\frac{\sqrt{3}}{2} + i\frac{1}{2}, \\ e^{i3\pi/2} &= -i. \end{aligned}$$

$\square$

It is worth taking the time to sketch the six sixth roots of 2 (from Example 13) on a single set of axes. Also sketch all the third roots of i on a single set of axes. Observe that the six sixth roots of 2 are equally spaced about a circle that is centered at the origin and has radius $2^{1/6}$. Likewise, the three cube roots of i are equally spaced about a circle that is centered at the origin and has radius 1.

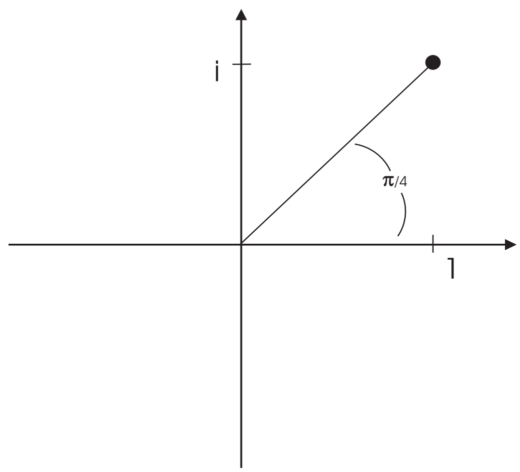

Figure 1.10: The argument of $1 + i$.

1.3.5 The Argument of a Complex Number

The (nonunique) angle θ associated to a complex number $z \neq 0$ is called its *argument*, and is written $\arg z$. For instance, $\arg(1 + i) = \pi/4$. See Figure 1.10. But it is also correct to write $\arg(1+i) = 9\pi/4, 17\pi/4, -7\pi/4$, etc. We generally choose the argument θ to satisfy $0 \leq \theta < 2\pi$. This is the *principal branch* of the argument—see Sections 2.5, 5.5 where the idea is applied to good effect.

Under multiplication of complex numbers (in polar form), arguments are additive and moduli multiply. That is, if $z = re^{i\theta}$ and $w = se^{i\psi}$, then

$$z \cdot w = re^{i\theta} \cdot se^{i\psi} = (rs) \cdot e^{i(\theta+\psi)}. \tag{1.37}$$

1.3.6 Fundamental Inequalities

We next record a few inequalities.

The Triangle Inequality: If $z, w \in \mathbb{C}$, then

$$|z + w| \leq |z| + |w|. \tag{1.38}$$

More generally,

$$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|. \tag{1.39}$$

For the verification of (1.38), square both sides. We obtain

$$|z+w|^2 \le (|z|+|w|)^2$$

or

$$(z+w)\cdot(\overline{z+w}) \le (|z|+|w|)^2\,.$$

Multiplying this out yields

$$|z|^2 + z\overline{w} + w\overline{z} + |w|^2 \le |z|^2 + 2|z||w| + |w|^2\,.$$

Cancelling like terms yields

$$2\mathrm{Re}\,(z\overline{w}) \le 2|z||w|$$

or

$$\mathrm{Re}\,(z\overline{w}) \le |z||w|\,.$$

It is convenient to rewrite this as

$$\mathrm{Re}\,(z\overline{w}) \le |z\overline{w}|\,. \tag{1.40}$$

But it is true, for any complex number ζ, that $|\mathrm{Re}\,\zeta| \le |\zeta|$. Our argument runs both forward and backward. So (1.40) implies (1.38). This establishes the basic triangle inequality.

To give an idea of why the more general triangle inequality is true, consider just three terms. We have

$$\begin{aligned} |z_1+z_2+z_3| &= |z_1+(z_2+z_3)| \\ &\le |z_1| + |z_2+z_3| \\ &\le |z_1| + (|z_2|+|z_3|)\,, \end{aligned}$$

thus establishing the general result for three terms. The full inequality for n terms is proved similarly.

The Cauchy-Schwarz Inequality: If $z_1, \dots, z_n$ and $w_1, \dots, w_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} z_j w_j \right|^2 \leq \left[\sum_{j=1}^{n} |z_j|^2 \right] \cdot \left[\sum_{j=1}^{n} |w_j|^2 \right]. \tag{1.41}$$

To understand why this inequality is true, let us begin with some special cases. For just one summand, the inequality says that

$$|z_1 w_1|^2 \leq |z_1|^2 |w_1|^2 ,$$

which is clearly true. For two summands, the inequality asserts that

$$|z_1 w_1 + z_2 w_2|^2 \leq (|z_1|^2 + |z_2|^2) \cdot (|w_1|^2 + |w_2|^2) .$$

Multiplying this out yields

$$|z_1 w_1|^2 + 2\mathrm{Re}\,(z_1 w_1 \overline{z_2 w_2}) + |z_2 w_2|^2 \leq |z_1|^2|w_1|^2 + |z_1|^2|w_2|^2 + |z_2|^2|w_1|^2 + |z_2|^2|w_2|^2 .$$

Cancelling like terms, we have

$$2\mathrm{Re}\,(z_1 w_1 \overline{z_2 w_2}) \leq |z_1|^2|w_2|^2 + |z_2|^2|w_1|^2 .$$

But it is always true, for $a, b \geq 0$, that $2ab \leq a^2 + b^2$. Hence

$$2\mathrm{Re}\,(z_1 w_1 \overline{z_2 w_2}) \leq 2|z_1 w_2||z_2 w_1| \leq |z_1 w_2|^2 + |z_2 w_1|^2 .$$

The result for n terms is proved similarly.

Exercises

1. Find all the third roots of $3i$.

2. Find all the sixth roots of -1.

3. Find all the fourth roots of $-5i$.

4. Find all the fifth roots of $-1 + i$.

5. Find all third roots of $3 - 6i$.

6. Find all arguments of each of these complex numbers: i, $1+i$, $-1+i\sqrt{3}$, $-2-2i$, $\sqrt{3}-i$.

7. If z is any complex number then explain why

$$|z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|\,.$$

8. If z is any complex number then explain why

$$|\mathrm{Re}\, z| \leq |z| \quad \text{and} \quad |\mathrm{Im}\, z| \leq |z|\,.$$

9. If z, w are any complex numbers then explain why

$$|z+w| \geq |z| - |w|\,.$$

10. If $\sum_n |z_n|^2 < \infty$ and $\sum_n |w_n|^2 < \infty$ then explain why $\sum_n |z_n w_n| < \infty$.

11. Use `MatLab` to find all cube roots of i. Now calculate those roots by hand. [**Hint:** Use a fractional power, together with `^`, to determine the roots of any number.] Use `MatLab` to take suitable third powers to check your work.

12. Use `MatLab` to find all the square roots and all the fourth roots of $1+i$. Now perform the same calculation by hand. Use `MatLab` to take suitable second and fourth powers to check your work.

13. Use `MatLab` to calculate

$$\sqrt{1-4i+\sqrt[3]{3-i}}\,.$$

It would be quite complicated to calculate this number in the form $a+ib$ by hand, but you may wish to try. [**Hint:** There is a complication lurking in the background here. Any complex number except 0 has multiple roots. This is because of a built-in ambiguity in the definition of the logarithm—see Section 2.5. You need not worry about this subtlety now, but it may affect the answer(s) that `MatLab` gives you.]

14. Use `MatLab` to calculate the square root of

$$z = e^{i\pi/3} + 2e^{-i\pi/4}\,.$$

15. Find the polar form of the complex number $z = -1$. Find all fourth roots of -1.

16. The Cauchy-Schwarz inequality has an interpretation in terms of vectors. What is it? What does the inequality say about the cosine of an angle?

Chapter 2

The Relationship of Holomorphic and Harmonic Functions

2.1 Holomorphic Functions

2.1.1 Continuously Differentiable and C^k Functions

Holomorphic functions are a generalization of complex polynomials. But they are more flexible objects than polynomials. The collection of all polynomials is closed under addition and multiplication. However, the collection of all holomorphic functions is closed under reciprocals, division, inverses, exponentiation, logarithms, square roots, and many other operations as well.

There are several different ways to introduce the concept of holomorphic function. They can be defined by way of power series, or using the complex derivative, or using partial differential equations. We shall touch on all these approaches; but our initial definition will be by way of partial differential equations.

If $U \subseteq \mathbb{R}^2$ is a region and $f : U \to \mathbb{R}$ is a continuous function, then f is called C^1 (or *continuously differentiable*) on U if $\partial f/\partial x$ and $\partial f/\partial y$ exist and are *continuous* on U. We write $f \in C^1(U)$ for short.

More generally, if $k \in \{0, 1, 2, ...\}$, then a real-valued function f on U is called C^k (k times continuously differentiable) if all partial derivatives of f up to and including order k exist and are continuous on U. We write in this case $f \in C^k(U)$. In particular, a C^0 function is just a continuous function.

We say that a function is C^∞ if it is C^k for every k. Such a function is called *infinitely differentiable*.

EXAMPLE 15 Let $D \subseteq \mathbb{C}$ be the unit disc, $D = \{z \in \mathbb{C} : |z| < 1\}$. The function $\varphi(z) = |z|^2 = x^2 + y^2$ is C^k for every k. This is so just because we may differentiate φ as many times as we please, and the result is continuous. In this circumstance we sometimes write $\varphi \in C^\infty$.

By contrast, the function $\psi(z) = |z|$ is not even C^1. For the restriction of ψ to the real axis is $\widetilde{\psi}(x) = |x|$, and this function is well known not to be differentiable at $x = 0$. $\square$

A function $f = u + iv : U \to \mathbb{C}$ is called C^k if both u and v are C^k.

2.1.2 The Cauchy-Riemann Equations

If f is *any* complex-valued function, then we may write $f = u + iv$, where u and v are real-valued functions.

EXAMPLE 16 Consider

$$f(z) = z^2 = (x^2 - y^2) + i(2xy); \tag{2.1}$$

in this example $u = x^2 - y^2$ and $v = 2xy$. We refer to u as the *real part* of f and denote it by $\operatorname{Re} f$; we refer to v as the *imaginary part* of f and denote it by $\operatorname{Im} f$. $\square$

Now we formulate the notion of "holomorphic function" in terms of the real and imaginary parts of f :

Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ a C^1 function. Write

$$f(z) = u(x, y) + iv(x, y), \tag{2.2}$$

with u and v real-valued functions. Of course $z = x + iy$ as usual. If u and v satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.3}$$

at every point of U, then the function f is said to be *holomorphic* (see Section 2.1.4, where a more formal definition of "holomorphic" is provided). The first order, linear partial differential equations in (2.3) are called the *Cauchy-Riemann equations*. A practical method for checking whether a given function is holomorphic is to check whether it satisfies the Cauchy-Riemann equations. Another practical method is to check that the function can be expressed in terms of z alone, with no $\overline{z}$'s present (see Section 2.1.3).

EXAMPLE 17 Let $f(z) = z^2 - z$. Then we may write

$$f(z) = (x+iy)^2 - (x+iy) = (x^2 - y^2 - x) + i(2xy - y) \equiv u(x,y) + iv(x,y)\,.$$

Then we may check directly that

$$\frac{\partial u}{\partial x} = 2x - 1 = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}\,.$$

We see, then, that f satisfies the Cauchy-Riemann equations so it is holomorphic. Also observe that f may be expressed in terms of z alone, with no $\overline{z}$s. □

EXAMPLE 18 Define

$$g(z) = |z|^2 - 4z + 2\overline{z} = z\cdot\overline{z} - 4z + 2\overline{z} = (x^2 + y^2 - 2x) + i(-6y) \equiv u(x,y) + iv(x,y)\,.$$

Then

$$\frac{\partial u}{\partial x} = 2x - 2 \neq -6 = \frac{\partial v}{\partial y}\,.$$

Also

$$\frac{\partial v}{\partial x} = 0 \neq -2y = -\frac{\partial u}{\partial y}\,.$$

We see that *both* Cauchy-Riemann equations fail. So g is not holomorphic. We may also observe that g is expressed both in terms of z and $\overline{z}$—another sure indicator that this function is not holomorphic. □

2.1.3 Derivatives

We define, for $f = u + iv : U \to \mathbb{C}$ a C^1 function,

$$\frac{\partial}{\partial z} f \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{2.4}$$

and

$$\frac{\partial}{\partial \overline{z}} f \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{2.5}$$

If $z = x + iy$, $\overline{z} = x - iy$, then one can check directly that

$$\frac{\partial}{\partial z} z = 1\,, \qquad \frac{\partial}{\partial z}\overline{z} = 0\,, \tag{2.6}$$

$$\frac{\partial}{\partial \overline{z}} z = 0\,, \qquad \frac{\partial}{\partial \overline{z}}\overline{z} = 1. \tag{2.7}$$

In traditional multivariable calculus, the partial derivatives $\partial/\partial x$ and $\partial/\partial y$ span all directions in the plane: *any* directional derivative can be expressed in terms of $\partial/\partial x$ and $\partial/\partial y$. Put in other words, if f is a continously differentiable function in the plane, if $\partial f/\partial x \equiv 0$ and $\partial f/\partial y \equiv 0$, then *all* directional derivatives of f are identically 0. Hence f is constant. So it is with $\partial/\partial z$ and $\partial/\partial \overline{z}$. If $\partial f/\partial z \equiv 0$ and $\partial f/\partial \overline{z} \equiv 0$ then *all* directional derivatives of f are identically 0. Hence f is constant.

The partial derivatives $\partial/\partial z$ and $\partial/\partial \overline{z}$ are most convenient for complex analysis because they interact naturally with the complex coordinate functions z and $\overline{z}$ (as noted above). And, because of the Cauchy-Riemann equations, they characterize holomorphic functions. Just as a function that satisfies $\partial f/\partial x \equiv 0$ is a function that is independent of x, so it is the case that a function that satisfies $\partial f/\partial \overline{z} \equiv 0$ is independent of $\overline{z}$; it only depends on z. Thus it is holomorphic.

Of course

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}\,.$$

We may use this information, together with

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\cdot\frac{\partial}{\partial y}\,,$$

to derive the formula for $\partial/\partial z$ and likewise for $\partial/\partial \overline{z}$.

If a C^1 function f satisfies $\partial f/\partial z \equiv 0$ on an open set U, then f does not depend on z (but it *can* depend on $\overline{z}$). If instead f satisfies $\partial f/\partial \overline{z} \equiv 0$ on an open set U, then f does not depend on $\overline{z}$ (but it *does* depend on z). The condition $\partial f/\partial \overline{z} \equiv 0$ is just a reformulation of the Cauchy-Riemann equations—see Section 2.1.2. Thus $\partial f/\partial \overline{z} \equiv 0$ if and only if f is holomorphic. We work out the details of this claim in Section 2.1.4. Now we look at some examples to illustrate the new ideas.

Example 19 Review Example 17. Now let us examine that same function using our new criterion with the operator $\partial/\partial \overline{z}$. We have

$$\frac{\partial}{\partial \overline{z}} f(z) = \frac{\partial}{\partial \overline{z}}\bigl(z^2 - z\bigr) = 2z\frac{\partial z}{\partial \overline{z}} - \frac{\partial z}{\partial \overline{z}} = 0 - 0 = 0\,.$$

We conclude that f is holomorphic. □

Example 20 Review Example 18. Now let us examine that same function using our new criterion with the operator $\partial/\partial \overline{z}$. We have

$$\frac{\partial}{\partial \overline{z}} g(z) = \frac{\partial}{\partial \overline{z}}\bigl(|z|^2 - 4z + 2\overline{z}\bigr) = \frac{\partial}{\partial \overline{z}}\bigl(z \cdot \overline{z} - 4z + 2\overline{z}\bigr) = z + 2 \neq 0\,.$$

We conclude that g is *not* holomorphic. □

It is sometimes useful to express the derivatives $\partial/\partial z$ and $\partial/\partial \overline{z}$ in polar coordinates. Recall that

$$r^2 = x^2 + y^2\ ,\ x = r\cos\theta\ ,\ y = r\sin\theta\,.$$

Now notices that

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\cdot\frac{\partial}{\partial \theta} = \frac{x}{r}\cdot\frac{\partial}{\partial r} - \frac{y}{r^2}\cdot\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\,.$$

A similar calculation shows that

$$\frac{\partial}{\partial y} = \sin\theta\cdot\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\,.$$

As a result, we see that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right) - \frac{i}{2}\left(\sin\theta\cdot\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right) + \frac{i}{2}\left(\sin\theta\cdot\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\,.$$

We invite the reader to write $z = re^{i\theta} = r\cos\theta + ir\sin\theta$ and check directly (in polar coordinates) that $\partial z/\partial z \equiv 1$. Likewise verify that $\partial\overline{z}/\partial\overline{z} \equiv 1$.

2.1.4 Definition of a Holomorphic Function

Functions f that satisfy $(\partial/\partial\overline{z})f \equiv 0$ are the main concern of complex analysis. A continuously differentiable (C^1) function $f : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{C}$ is said to be *holomorphic* if

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{2.8}$$

at every point of U. Note that this last equation is just a reformulation of the Cauchy-Riemann equations (Section 2.1.2). To see this, we calculate:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \overline{z}} f(z) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[u(z) + iv(z)] \\ &= \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \ i \ \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]. \end{aligned} \tag{2.9}$$

Of course the far right-hand side cannot be identically zero unless each of its real and imaginary parts is identically zero. It follows that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{2.10}$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \tag{2.11}$$

These are the Cauchy-Riemann equations (2.3).

Example 21 The function $h(z) = z^3 - 4z^2 + z$ is holomorphic because

$$\frac{\partial}{\partial \overline{z}} h(z) = 3z^2 \frac{\partial z}{\partial \overline{z}} - 4 \cdot 2z \frac{\partial z}{\partial \overline{z}} + \frac{\partial z}{\partial \overline{z}} = 0\,.$$

□

2.1.5 Examples of Holomorphic Functions

Certainly any polynomial in z (*without* $\overline{z}$) is holomorphic. And the reciprocal of any polynomial is holomorphic, as long as we restrict attention to a region where the polynomial does not vanish.

Earlier in this book we have discussed the complex function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,.$$

One may calculate directly, just differentiating the power series term-by-term, that

$$\frac{\partial}{\partial z} e^z = e^z \,.$$

In addition,

$$\frac{\partial}{\partial \overline{z}} e^z = 0 \,,$$

so the exponential function is holomorphic.

Of course we know, and we have already noted, that

$$e^{x+iy} = e^x(\cos y + i \sin y) \,.$$

When $x = 0$ this gives Euler's famous formula

$$e^{iy} = \cos y + i \sin y \,.$$

It follows immediately that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

and

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \,.$$

We explore other derivations of Euler's formula in the exercises.

In analogy with these basic formulas from calculus, we now define complex-analytic versions of the trigonometric functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iy} - e^{-iy}}{2i} \,.$$

The other trigonometric functions are defined in the usual way. For example,

$$\tan z = \frac{\sin z}{\cos z} \,.$$

We may calculate directly that

(a) $\frac{\partial}{\partial z} \sin z = \cos z;$

(b) $\frac{\partial}{\partial z} \cos z = -\sin z;$

(c) $\frac{\partial}{\partial z} \tan z = \frac{1}{\cos^2 z} \equiv \sec^2 z \quad .$

All of the trigonometric functions are holomorphic on their domains of definition. We invite the reader to verify this assertion.

It is straightforward to check that sums, products, and quotients of holomorphic functions are holomorphic (provided that we do not divide by 0). Any convergent power series—in powers of z only—defines a holomorphic function (just differentiate under the summation sign). We shall see later that holomorphic functions may be defined with integrals as well. So we now have a considerable panorama of holomorphic functions.

2.1.6 The Complex Derivative

Let $U \subseteq \mathbb{C}$ be open, $P \in U$, and $g : U \setminus \{P\} \to \mathbb{C}$ a function. We say that

$$\lim_{z \to P} g(z) = \ell \,, \quad \ell \in \mathbb{C} \,, \tag{2.12}$$

if, for any $\epsilon > 0$ there is a $\delta > 0$ such that when $z \in U$ and $0 < |z - P| < \delta$ then $|g(z) - \ell| < \epsilon$. Notice that, in this definition of limit, the point z may approach P in an arbitrary manner—from any direction. See Figure 2.1. Of course the function g is *continuous* at $P \in U$ if $\lim_{z \to P} g(z) = g(P)$.

We say that f possesses the *complex derivative* at P if

$$\lim_{z \to P} \frac{f(z) - f(P)}{z - P} \tag{2.13}$$

exists. In that case we denote the limit by $f'(P)$ or sometimes by

$$\frac{df}{dz}(P) \qquad \text{or} \qquad \frac{\partial f}{\partial z}(P). \tag{2.14}$$

This notation is consistent with that introduced in Section 2.1.3: for a *holomorphic function*, the complex derivative calculated according to formula (2.13) or according to formula (2.4) is just the same. We shall say more about the complex derivative in Section 2.2.1 and Section 2.2.2.

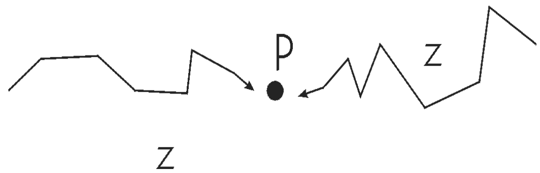

Figure 2.1: The point z may approach P arbitrarily.

We repeat that, in calculating the limit in (2.13), z must be allowed to approach P from *any* direction (refer to Figure 2.1). As an example, the function $g(x, y) = x - iy$—equivalently, $g(z) = \overline{z}$—does *not* possess the complex derivative at 0. To see this, calculate the limit

$$\lim_{z \to P} \frac{g(z) - g(P)}{z - P} \tag{2.15}$$

with z approaching $P = 0$ through values $z = x + i0$. The answer is

$$\lim_{x \to 0} \frac{x - 0}{x - 0} = 1. \tag{2.16}$$

If instead z is allowed to approach $P = 0$ through values $z = iy$, then the value is

$$\lim_{z \to P} \frac{g(z) - g(P)}{z - P} = \lim_{y \to 0} \frac{-iy - 0}{iy - 0} = -1. \tag{2.17}$$

Observe that the two answers do not agree. *In order for the complex derivative to exist, the limit must exist and assume only one value no matter how z approaches P.* Therefore this example g does not possess the complex derivative at $P = 0$. In fact a similar calculation shows that this function g does *not* possess the complex derivative at any point.

If a function f possesses the complex derivative at every point of its open domain U, then f is holomorphic. This definition is equivalent to definitions given in Section 2.1.4. We repeat some of these ideas in Section 2.2. In fact, from an historical perspective, it is important to recall a theorem of Goursat (see the Appendix in [GRK]). Goursat's theorem has great historical and philosophical significance, though it rarely comes up as a practical matter in complex function theory. We present it here in order to give the student some perspective. Goursat's result says that if a function f possesses the complex derivative at each point of an open region $U \subseteq \mathbb{C}$ then f is in fact continuously differentiable[1] on U. One may then verify the Cauchy-Riemann equations, and it follows that f is holomorphic by any of our definitions thus far.

2.1.7 Alternative Terminology for Holomorphic Functions

Some books use the word "analytic" instead of "holomorphic." Still others say "differentiable" or "complex differentiable" instead of "holomorphic." The use of the term "analytic" derives from the fact that a holomorphic function has a local power series expansion about each point of its domain (see Section 4.1.6). In fact this power series property is a complete characterization of holomorphic functions; we shall discuss it in detail below. The use of "differentiable" derives from properties related to the complex derivative. These pieces of terminology and their significance will all be sorted out as the book develops. Somewhat archaic terminology for holomorphic functions, which may be found in older texts, are "regular" and "monogenic."

Another piece of terminology that is applied to holomorphic functions is "conformal" or "conformal mapping." "Conformality" is an important geometric property of holomorphic functions that make these functions useful for modeling incompressible fluid flow (Sections 8.2.2 and 8.3.3) and other physical phenomena. We shall discuss conformality in Section 2.4.1 and Chapter 7. We shall treat physical applications of conformality in Chapter 8.

Exercises

1. Verify that each of these functions is holomorphic whereever it is defined:

(a) $f(z) = \sin z - \dfrac{z^2}{z+1}$

[1] A more classical formulation of the result is this. If f possesses the complex derivative at each point of the region U, then f satisfies the Cauchy integral theoreom (see Section 3.1.1 below). This is sometimes called the *Cauchy-Goursat theorem.* That in turn implies the Cauchy integral formula (Section 3.1.4). And this result allows us to prove that f is continuously differentiable (indeed infinitely differentiable).

(b) $g(z) = e^{2z-z^3} - z^2$

(c) $h(z) = \dfrac{\cos z}{z^2+1}$

(d) $k(z) = z(\tan z + z)$

2. Verify that each of these functions is *not* holomorphic:

(a) $f(z) = |z|^4 - |z|^2$

(b) $g(z) = \dfrac{\overline{z}}{z^2+1}$

(c) $h(z) = z(\overline{z}^2 - z)$

(d) $k(z) = \overline{z} \cdot (\sin z) \cdot (\cos \overline{z})$

3. For each function f, calculate $\partial f/\partial z$:

(a) $2z(1-z^3)$

(b) $(\cos z) \cdot (1 + \sin^2 z)$

(c) $(\sin \overline{z})(1 + \overline{z} \cos z)$

(d) $|z|^4 - |z|^2$

4. For each function g, calculate $\partial g/\partial \overline{z}$:

(a) $2\overline{z}(1-z^3)$

(b) $(\sin \overline{z}) \cdot (1 + \sin^2 \overline{z})$

(c) $(\cos z) \cdot (1 + z \cos \overline{z})$

(d) $|z|^2 - |z|^4$

5. Verify the equations

$$\frac{\partial}{\partial z} z = 1\,, \qquad \frac{\partial}{\partial z} \overline{z} = 0\,,$$

$$\frac{\partial}{\partial \overline{z}} z = 0\,, \qquad \frac{\partial}{\partial \overline{z}} \overline{z} = 1.$$

6. Show that, in polar coordinates, the Cauchy-Riemann equations take the form

$$r \cdot u_r = v_\theta \quad \text{and} \quad r v_r = -u_\theta\,.$$

Here, of course, subscripts denote derivatives.

7. It is known that the solution y of a second order, linear ordinary differential equation with constant coefficients and satisfying $y(0) = 1$ and $y'(0) = i$ is unique. Let the differential equation be $y'' = -y$. Verify that the function $f(x) = e^{ix}$ satisfies all three conditions. Also verify that the function $g(x) = \cos x + i \sin x$ satisfies all three conditions. By uniqueness, $f(x) \equiv g(x)$. That gives another proof of Euler's formula.

8. Both of the expressions $f(x) = e^{ix}$ and $g(x) = \cos x + i \sin x$ take the value 1 at 0. Also both expressions are invariant under rotations in a certain sense. From this it must follow that $f \equiv g$. This gives another proof of Euler's formula. Fill in the details of this argument.

9. Calculate the derivative

$$\frac{\partial}{\partial z}[\tan z - e^{3z}] .$$

10. Calculate the derivative

$$\frac{\partial}{\partial \overline{z}}[\sin \overline{z} - z\overline{z}^2] .$$

11. Find a function g such that

$$\frac{\partial g}{\partial z} = z\overline{z}^2 - \sin z .$$

12. Find a function h such that

$$\frac{\partial h}{\partial \overline{z}} = \overline{z}^2 z^3 + \cos \overline{z} .$$

13. Find a function k such that

$$\frac{\partial^2 k}{\partial z \partial \overline{z}} = |z|^2 - \sin z + \overline{z}^3 .$$

14. From the definition (line (2.13)), calculate

$$\frac{d}{dz}(z^3 - z^2) .$$

15. From the definition (line (2.13)), calculate

$$\frac{d}{dz}(\sin z - e^z) \, .$$

16. The software MatLab does not know the partial differential operators

$$\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} \, .$$

But you may define MatLab functions (see [PRA, p. 35]) to calculate them as follows:

```
function [zderiv] =  ddz(f,x,y,z)

syms x y real;
syms z complex;

z = x + i*y;

z_deriv = (diff(f, 'x'))/2 - (diff(f, 'y'))*i/2
```

and

```
function [zbarderiv] =  ddzbar(f)

syms x y real;
syms z complex;

z = x + i*y;

zbar_deriv = (diff(f, 'x'))/2 + (diff(f, 'y'))*i/2
```

You must give the first macro file the name `ddz.m` and the second macro file the name `ddzbar.m`. With these macros in place you can proceed as follows. At the MatLab prompt >>, type these commands (following each one by <`Enter`>):

```
>> syms x y real
>> syms z complex
>> z = x + i*y
```

This gives `MatLab` the information it needs in order to do complex calculus. Now let us define a function:

```
>> f = z^2
```

Finally type `ddz(f)` and press <`Enter`> . `MatLab` will produce an answer (that is equivalent to) `2*(x + iy)`. What you have just done is differentiated z^2 with respect to z and obtained the answer $2z$. If instead you type, at the `MatLab` prompt, `ddzbar(f)`, you will obtain an answer (that is equivalent to) `0`. That is because the macro `ddzbar` performs differentiation with respect to $\overline{z}$.

For practice, use your new `MatLab` macros to calculate several other complex derivatives. [Remember that the `MatLab` command for $\overline{z}$ is `conj(z)`.] For example, try

$$\frac{\partial}{\partial z} z^2 \cdot \overline{z}^3 \ , \ \frac{\partial}{\partial z} \sin(z \cdot \overline{z}) \ , \ \frac{\partial}{\partial \overline{z}} \cos(z^2 \cdot \overline{z}^3) \ , \ \frac{\partial}{\partial \overline{z}} e^{z \cdot \overline{z}^2} \, .$$

17. The function $f(z) = z^2 - z^3$ is holomorphic. Why? It has real part u that describes a steady state flow of heat on the unit disc. Calculate this real part. Verify that u satisfies the partial differential equation

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} u(z) \equiv 0 \, .$$

This is the *Laplace equation*. We shall study it in greater detail as the book progresses.

18. Do the last exercise with "real part" u replaced by "imaginary part" v.

2.2 The Relationship of Holomorphic and Harmonic Functions

2.2.1 Harmonic Functions

A C^2 (twice continuously differentiable) function u is said to be *harmonic* if it satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0. \tag{2.18}$$

This partial differential equation is called *Laplace's equation*, and is frequently abbreviated as

$$\triangle u = 0. \tag{2.19}$$

Example 22 The function $u(x, y) = x^2 - y^2$ is harmonic. This assertion may be verified directly:

$$\triangle u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2}\right) x^2 - \left(\frac{\partial^2}{\partial y^2}\right) y^2 = 2 - 2 = 0\,.$$

A similar calculation shows that $v(x, y) = 2xy$ is harmonic. For

$$\triangle v = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) 2xy = 0 + 0 = 0\,.$$

□

Example 23 The function $\widetilde{u}(x, y) = x^3$ is *not* harmonic. For

$$\triangle \widetilde{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) x^3 = 6x \neq 0\,.$$

Likewise, the function $\widetilde{v}(x, y) = \sin x - \cos y$ is *not* harmonic. For

$$\triangle \widetilde{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \widetilde{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) [\sin x - \cos y] = -\sin x + \cos y \neq 0\,.$$

□

2.2.2 Holomorphic and Harmonic Functions

If f is a holomorphic function and $f = u + iv$ is the expression of f in terms of its real and imaginary parts, then both u and v are harmonic. The easiest way to see this is to begin with the equation

$$\frac{\partial}{\partial \overline{z}} f = 0 \tag{2.20}$$

and to apply $\partial/\partial z$ to both sides. The result is

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0 \tag{2.21}$$

or

$$\left(\frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]\right)\left(\frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]\right)[u + iv] = 0\,. \tag{2.22}$$

Multiplying through by 4, and then multiplying out the derivatives, we find that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[u + iv] = 0\,. \tag{2.23}$$

We may now distribute the differentiation and write this as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \quad + \quad i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0\,. \tag{2.24}$$

The only way that the left-hand side can be zero is if its real part is zero and its imaginary part is zero. We conclude then that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \tag{2.25}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0\,. \tag{2.26}$$

Thus u and v are each harmonic.

EXAMPLE 24 Let $f(z) = (z + z^2)^2$. Then f is certainly holomorphic because it is defined using only zs, and no $\overline{z}$s. Notice that

$$\begin{aligned} f(z) &= z^4 + 2z^3 + z^2 \\ &= [x^4 - 6x^2y^2 + y^4 + 2x^3 - 6xy^2 + x^2 - y^2] \\ &\qquad + i[-4xy^3 + 4x^3y + 6x^2y - 2y^3 + 2xy] \\ &\equiv u + iv\,. \end{aligned}$$

We may check directly that

$$\triangle u = 0 \qquad \text{and} \qquad \triangle\, v = 0\,.$$

Hence the real and imaginary parts of f are each harmonic. $\square$

A sort of converse to (2.25) and (2.26) is true provided the functions involved are defined on a domain with no holes:

> If $\mathcal{R}$ is an open rectangle (or open disc) and if u is a real-valued harmonic function on $\mathcal{R}$, then there is a holomorphic function F on $\mathcal{R}$ such that $\operatorname{Re} F = u$. In other words, for such a function u there exists another harmonic function v defined on $\mathcal{R}$ such that $F \equiv u + iv$ is holomorphic on $\mathcal{R}$. Any two such functions v must differ by a real constant.
>
> More generally, if U is a region with no holes (a *simply connected* region—see Section 3.1.4), and if u is harmonic on U, then there is a holomorphic function F on U with $\operatorname{Re} F = u$. In other words, for such a function u there exists a harmonic function v defined on U such that $F \equiv u + iv$ is holomorphic on U. Any two such functions v must differ by a constant. We call the function v a *harmonic conjugate* for u.

The displayed statement is false on a domain with a hole, such as an annulus. For example, the harmonic function $u = \log(x^2 + y^2)$, defined on the annulus $U = \{z : 1 < |z| < 2\}$, has no harmonic conjugate on U. See also Section 2.2.2. Let us give an example to illustrate the notion of harmonic conjugate, and then we shall discuss why the displayed statement is true.

EXAMPLE 25 Consider the function $u(x, y) = x^2 - y^2 - x$ on the square $U = \{(x, y) : |x| < 1, |y| < 1\}$. Certainly U is simply connected. And one may verify directly that $\triangle u \equiv 0$ on U. To solve for v a harmonic conjugate of u, we use the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 2x - 1\,, \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = 2y\,. \end{aligned}$$

The first of these equations indicates that $v(x,y) = 2xy - y + \varphi(x)$, for some unknown function $\varphi(x)$. Then

$$2y = \frac{\partial v}{\partial x} = 2y - \varphi'(x) .$$

It follows that $\varphi'(x) = 0$ so that $\varphi(x) \equiv C$ for some real constant C.

In conclusion,

$$v(x,y) = 2xy - y + C .$$

In other words, $h(x,y) = u(x,y) + iv(x,y) = [x^2 - y^2 - x] + i[2xy - y + C]$ should be holomorphic. We may verify this claim immediately by writing h as

$$h(z) = z^2 - z + iC .$$

□

You may also verify that the function h in the last example is holomorphic by checking the Cauchy-Riemann equations.

We may verify the displayed statement above just by using multivariable calculus. Suppose that U is a region with no holes and u is a harmonic function on U. We wish to solve the system of equations

$$\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} . \end{aligned} \tag{2.27}$$

These are the Cauchy-Riemann equations.

Now we know from calculus that this system of equations can be solved on U precisely when

$$\frac{\partial}{\partial y}\left[-\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] ,$$

that is, when

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

Thus we see that we can solve the required system of equations (2.27) provided only that u is harmonic. Of course we are *assuming* that u is harmonic. Thus the system (2.27) gives us the needed function v, and (2.27) also guarantees that $F = u + iv$ is holomorphic as desired.

Exercises

1. Verify that each of these functions is harmonic:

(a) $f(z) = \operatorname{Re} z$

(b) $g(z) = x^3 - 3xy^2$

(c) $h(z) = |z|^2 - 2x^2$

(d) $k(z) = e^x \cos y$

2. Verify that each of these functions is *not* harmonic:

(a) $f(z) = |z|^2$

(b) $g(z) = |z|^4 - |z|^2$

(c) $h(z) = \overline{z} \sin z$

(d) $k(z) = e^{\overline{z} \cos z}$

3. For each of these (real-valued) harmonic functions u, find a (real-valued) harmonic function v such that $u + iv$ is holomorphic.

(a) $u(z) = e^x \sin y$

(b) $u(z) = 3x^2 y - y^3$

(c) $u(z) = e^{2y} \sin x \cos x$

(d) $u(z) = x - y$

4. Use the chain rule to express the Laplace operator $\triangle$ in terms of polar coordinates (r, θ).

5. Let $\rho(x, y)$ be a rotation of the plane. Thus ρ is given by a 2×2 matrix with each row a unit vector and the two rows orthogonal to each other. Further, the determinant of the matrix is 1. Prove that, for any C^2 function f,

$$\triangle(f \circ \rho) = (\triangle f) \circ \rho \,.$$

6. Let $a \in \mathbb{R}^2$ and let λ_a be the operator $\lambda_a(x, y) = (x, y) + a$. This is translation by a. Verify that, for any C^2 function f,

$$\triangle(f \circ \lambda_a) = (\triangle f) \circ \lambda_a \,.$$

7. A function u is *biharmonic* if $\triangle^2 u = 0$. Verify that the function $x^4 - y^4$ is biharmonic. Give two distinct other examples of non-constant biharmonic functions. [Note that biharmonic functions are useful in the study of charge-transfer reactions in physics.]

8. Calculate the real and imaginary parts of the holomorphic function

$$f(z) = z^2 \cos z - e^{z^3 - z}$$

and verify directly that each of these functions is harmonic.

9. Create a `MatLab` function, called `lapl`, that will calculate the Laplacian of a given function. [**Hint:** You will find it useful to know that the `MatLab` command `diff(f, 'x', 2)` differentiates the function f two times in the x variable.] Your macro should calculate the Laplacian of a function whether it is expressed in terms of x, y or $z, \overline{z}$. Use your macro to calculate the Laplacians of these functions

$$f(x,y) = x^2 + y^2 \ , \ f(x,y) = x^2 - y^2 \ , \ f(x,y) = e^x \cdot \cos y \ , \ f(x,y) = e^{-y} \cdot \sin x \ ,$$

$$g(z) = z \cdot \overline{z}^2 \ , \ g(z) = \frac{z}{\overline{z}} \ , \ g(z) = z^2 - \overline{z}^2 \ .$$

10. Consider a unit disc made of some heat-conducting metal like aluminum. Imagine an initial heat distribution φ on the boundary of this disc, and let the heat flow to the interior of the disc. The *steady state heat distribution* turns out to be a harmonic function $u(x, y)$ with boundary function φ. We shall study this matter in greater detail in Chapter 8. See also Chapter 9.

Suppose that $\varphi(e^{it}) = \cos 2t$. Determine what u must be. [**Hint:** Consider the function $\Phi(e^{it}) = \cos 2t + i \sin 2t = e^{2it}$.]

Now answer the same question for $\varphi(e^{it}) = \sin 3t$.

2.3 Real and Complex Line Integrals

In this section we shall recast the line integral from multivariable calculus in complex notation. The result will be the complex line integral.

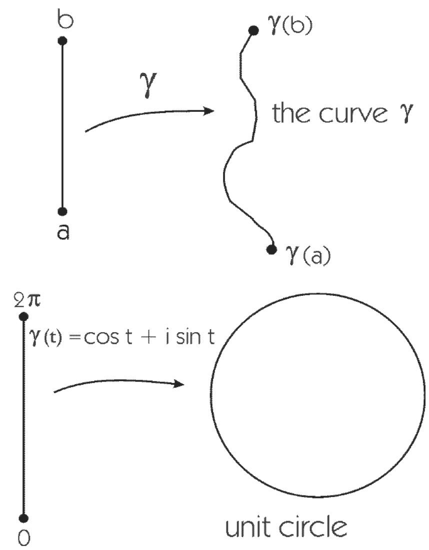

Figure 2.2: Two curves in the plane, one closed.

2.3.1 Curves

It is convenient to think of a *curve* as a (continuous) function γ from a closed interval $[a, b] \subseteq \mathbb{R}$ into $\mathbb{R}^2 \approx \mathbb{C}$. In practice it is useful *not* to distinguish between the *function* γ and the image (or set of points that make up the curve) given by $\{\gamma(t) : t \in [a, b]\}$. In the case that $\gamma(a) = \gamma(b)$, then we say that the curve is *closed*. Refer to Figure 2.2.

It is often convenient to write

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \qquad \text{or} \qquad \gamma(t) = \gamma_1(t) + i\gamma_2(t). \tag{2.28}$$

For example, $\gamma(t) = (\cos t, \sin t) = \cos t + i \sin t$, $t \in [0, 2\pi]$, describes the unit circle in the plane. The circle is traversed in a counterclockwise manner as t increases from 0 to 2π. This curve is closed. Refer to Figure 2.3.

2.3.2 Closed Curves

We have already noted that the curve $\gamma : [a, b] \to \mathbb{C}$ is called *closed* if $\gamma(a) = \gamma(b)$. It is called *simple, closed* (or Jordan) if the restriction of γ to the interval $[a, b)$ (which is commonly written $\gamma\big|_{[a,b)}$) is one-to-one *and*

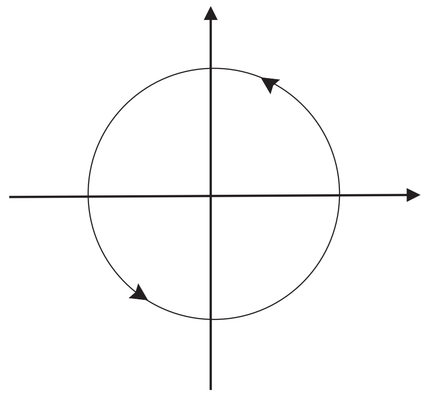

Figure 2.3: A simple, closed curve.

$\gamma(a) = \gamma(b)$ (Figures 2.3, 2.4). Intuitively, a simple, closed curve is a curve with no self-intersections, except of course for the closing up at $t = a, b$.

In order to work effectively with γ we need to impose on it some differentiability properties.

2.3.3 Differentiable and C^k Curves

A function $\varphi : [a, b] \to \mathbb{R}$ is called *continuously differentiable* (or C^1), and we write $\varphi \in C^1([a, b])$, if

(2.29) φ is continuous on $[a, b]$;
(2.30) φ' exists on (a, b);
(2.31) φ' has a continuous extension to $[a, b]$.

In other words, we require that

$$\lim_{t \to a^+} \varphi'(t) \quad \text{and} \quad \lim_{t \to b^-} \varphi'(t) \tag{2.32}$$

both exist.

Note that, under these circumstances,

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\, dt, \tag{2.33}$$

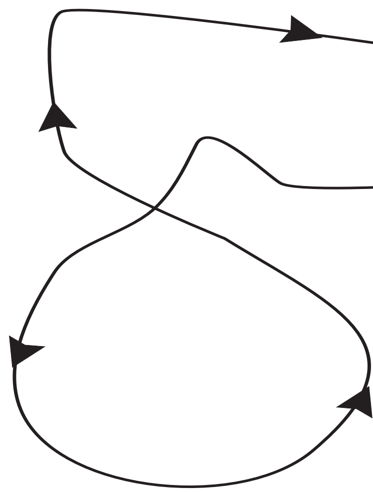

Figure 2.4: A closed curve that is not simple.

so that the Fundamental Theorem of Calculus holds for $\varphi \in C^1([a,b])$.

A curve $\gamma : [a,b] \to \mathbb{C}$, with $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ is said to be *continuous* on $[a,b]$ if both γ_1 and γ_2 are. The curve is *continuously differentiable* (or C^1) on $[a,b]$, and we write

$$\gamma \in C^1([a,b]), \tag{2.34}$$

if γ_1, γ_2 are continuously differentiable on $[a,b]$. Under these circumstances we will write

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}. \tag{2.35}$$

We also sometimes write $\gamma'(t)$ or $\dot{\gamma}(t)$ for $d\gamma/dt$.

2.3.4 Integrals on Curves

Let $\psi : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$. Write $\psi(t) = \psi_1(t) + i\psi_2(t)$. Then we define

$$\int_a^b \psi(t)\,dt \equiv \int_a^b \psi_1(t)\,dt + i\int_a^b \psi_2(t)\,dt \tag{2.36}$$

We summarize the ideas presented thus far by noting that, if $\gamma \in C^1([a,b])$

is complex-valued, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\, dt. \tag{2.37}$$

2.3.5 The Fundamental Theorem of Calculus along Curves

Now we state the Fundamental Theorem of Calculus (see [BLK]) along curves.

Let $U \subseteq \mathbb{C}$ be a domain and let $\gamma : [a, b] \to U$ be a C^1 curve. If $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t)) \cdot \frac{d\gamma_1}{dt} + \frac{\partial f}{\partial y}(\gamma(t)) \cdot \frac{d\gamma_2}{dt} \right) dt. \tag{2.38}$$

Note that this formula is a part of calculus, *not* complex analysis.

2.3.6 The Complex Line Integral

When f is holomorphic, then formula (2.38) may be rewritten (using the Cauchy-Riemann equations) as

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial z}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\, dt, \tag{2.39}$$

where, as earlier, we have taken $d\gamma/dt$ to be $d\gamma_1/dt + id\gamma_2/dt$. The reader may write out the right-hand side of (2.39) and see that it agrees with (2.38).

This latter result plays much the same role for holomorphic functions as does the Fundamental Theorem of Calculus for functions from $\mathbb{R}$ to $\mathbb{R}$. The expression on the right of (2.39) is called the *complex line integral of $\partial f/\partial z$ along γ* and is denoted

$$\oint_\gamma \frac{\partial f}{\partial z}(z)\, dz\,. \tag{2.40}$$

The small circle through the integral sign $\int$ tells us that this is a complex line integral, and has the meaning (2.39).

More generally, if g is *any* continuous function (not neessarily holomorphic) whose domain contains the curve γ, then the complex line integral of

g along γ is defined to be

$$\oint_\gamma g(z)\,dz \equiv \int_a^b g(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\,dt. \tag{2.41}$$

This is the *complex line integral of* g *along* γ. Compare with line (2.39).

Example 26 Let $f(z) = z^2 - 2z$ and let $\gamma(t) = (\cos t, \sin t) = \cos t + i \sin t$, $0 \le t \le \pi$. Then $\gamma'(t) = -\sin t + i\cos t$. This curve γ traverses the upper half of the unit circle from the initial point $(1, 0)$ to the terminal point $(-1, 0)$. We may calculate that

$$\begin{aligned}
\oint_\gamma f(z)\,dz &= \int_0^\pi f(\cos t + i\sin t) \cdot (-\sin t + i\cos t)\,dt \\
&= \int_0^\pi \left[(\cos t + i\sin t)^2 - 2(\cos t + i\sin t)\right] \cdot (-\sin t + i\cos t)\,dt \\
&= \int_0^\pi 4\cos t\sin t - 3\sin t\cos^2 t - 2i\cos 2t \\
&\qquad -3i\sin^2 t\cos t + \sin^3 t + i\cos^3 t\,dt \\
&= \Bigg[2\sin^2 t + \cos^3 t - i\sin 2t - i\sin^3 t \\
&\qquad - \cos t + i\sin t + \frac{\cos^3 t}{3} - i\frac{\sin^3 t}{3}\Bigg]_0^\pi \\
&= -\frac{2}{3}.
\end{aligned}$$

□

Example 27 If we integrate the holomorphic function f from the last example around the closed curve $\eta(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$, then we obtain

$$\begin{aligned}
\oint_\eta f(z)\,dz &= \int_0^{2\pi} f(\cos t + i\sin t)\cdot(-\sin t + i\cos t)\,dt \\
&= \int_0^{2\pi} \big[(\cos t + i\sin t)^2 - 2(\cos t + i\sin t)\big]\cdot(-\sin t + i\cos t)\,dt \\
&= \int_0^{2\pi} 4\cos t\sin t - 3\sin t\cos^2 t - 2i\cos 2t \\
&\qquad -3i\sin^2 t\cos t + \sin^3 t + i\cos^3 t\,dt \\
&= \Bigg[2\sin^2 t + \cos^3 t - i\sin 2t - i\sin^3 t \\
&\qquad - \cos t + i\sin t + \frac{\cos^3 t}{3} - \frac{\sin^3 t}{3}\Bigg]_0^{2\pi} \\
&= 0\,.
\end{aligned}$$

□

The whole concept of complex line integral is central to our further considerations in later sections. We shall use integrals like the one on the right of (2.39) or (2.41) even when f is not holomorphic; but we can be sure that the equality (2.39) holds *only when f is* holomorphic.

EXAMPLE 28 Let $g(z) = |z|^2$ and let $\mu(t) = t + it$, $0 \le t \le 1$. Let us calculate

$$\oint_\mu g(z)\,dz\,.$$

We have

$$\oint_\mu g(z)\,dz = \int_0^1 g(t+it)\cdot\mu'(t)\,dt = \int_0^1 2t^2\cdot(1+i)dt = \frac{2t^3}{3}(1+i)\bigg|_0^1 = \frac{2+2i}{3}\,.$$

□

2.3.7 Properties of Integrals

We conclude this section with some easy but useful facts about integrals.

(A) If $\varphi : [a, b] \to \mathbb{C}$ is continuous, then

$$\left| \int_a^b \varphi(t)\, dt \right| \leq \int_a^b |\varphi(t)|\, dt.$$

(B) If $\gamma : [a, b] \to \mathbb{C}$ is a C^1 curve and φ is a continuous function on the curve γ, then

$$\left| \oint_\gamma \varphi(z)\, dz \right| \leq \left[\max_{t\in[a,b]} |\varphi(t)| \right] \cdot \ell(\gamma)\,, \tag{2.42}$$

where

$$\ell(\gamma) \equiv \int_a^b |\varphi'(t)|\, dt$$

is the *length* of γ.

(C) The calculation of a complex line integral is independent of the way in which we parametrize the path:

> Let $U \subseteq \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a, b] \to U$ be a C^1 curve. Suppose that $\varphi : [c, d] \to [a, b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\widetilde{\gamma} = \gamma \circ \varphi$. Then
>
> $$\oint_{\widetilde{\gamma}} f\, dz = \oint_\gamma f\, dz.$$

This last statement implies that one can use the idea of the integral of a function f along a curve γ when the curve γ is described geometrically but without reference to a specific parametrization. For instance, "the integral of $\overline{z}$ *counterclockwise* around the unit circle $\{z \in \mathbb{C} : |z| = 1\}$" is now a phrase that makes sense, even though we have not indicated a specific parametrization of the unit circle. Note, however, that the direction counts: The integral of $\overline{z}$ counterclockwise around the unit circle is $2\pi i$. If the direction is reversed, then the integral changes sign: The integral of $\overline{z}$ *clockwise* around the unit circle is $-2\pi i$.

EXAMPLE 29 Let $g(z) = z^2 - z$ and $\gamma(t) = t^2 - it$, $0 \le t \le 1$. Then

$$\begin{aligned}
\oint_\gamma g(z)\,dz &= \int_0^1 g(t^2 - it)\cdot\gamma'(t)\,dt \\
&= \int_0^1 [(t^2 - it)^2 - (t^2 - it)]\cdot(2t - i)\,dt \\
&= \int_0^1 [t^4 - 2it^3 - 2t^2 + it]\cdot(2t - i)\,dt \\
&= \int_0^1 2t^5 - 5it^4 - 6t^3 + 4it^2 + t\,dt \\
&= \left[\frac{2t^6}{6} - \frac{5it^5}{5} - \frac{6t^4}{4} + \frac{4it^3}{3} + \frac{t^2}{2}\right]_0^1 \\
&= \frac{1}{3} - i - \frac{3}{2} + \frac{4i}{3} + \frac{1}{2} \\
&= -\frac{2}{3} + \frac{i}{3}\,.
\end{aligned}$$

If instead we replace γ by $-\gamma$ (which amounts to parametrizing the curve from 1 to 0 instead of from 0 to 1) then we obtain

$$\begin{aligned}
\oint_{-\gamma} g(z)\,dz &= \int_1^0 g(t^2 - it)\cdot\gamma'(t)\,dt \\
&= \int_1^0 [(t^2 - it)^2 - (t^2 - it)]\cdot(2t - i)\,dt \\
&= \int_1^0 [t^4 - 2it^3 - 2t^2 + it]\cdot(2t - i)\,dt \\
&= \int_1^0 2t^5 - 5it^4 - 6t^3 + 4it^2 + t\,dt \\
&= \left[\frac{2t^6}{6} - \frac{5it^5}{5} - \frac{6t^4}{4} + \frac{4it^3}{3} + \frac{t^2}{2}\right]_1^0 \\
&= -\left(\frac{1}{3} - i - \frac{3}{2} + \frac{4i}{3} + \frac{1}{2}\right) \\
&= \frac{2}{3} - \frac{i}{3}\,.
\end{aligned}$$

$\square$

Exercises

1. In each of the following problems, calculate the complex line integral of the given function f along the given curve γ:

(a) $f(z) = z\overline{z}^2 - \cos z \quad , \quad \gamma(t) = \cos 2t + i \sin 2t \quad , \quad 0 \le t \le \pi/2$

(b) $f(z) = \overline{z}^2 - \sin z \quad , \quad \gamma(t) = t + it^2 \quad , \quad 0 \le t \le 1$

(c) $f(z) = z^3 + \frac{z}{z+1} \quad , \quad \gamma(t) = e^t + ie^{2t} \quad , \quad 1 \le t \le 2$

(d) $f(z) = e^z - e^{-z} \quad , \quad \gamma(t) = t - i \log t \quad , \quad 1 \le t \le e$

2. Calculate the complex line integral of the holomorphic function $f(z) = z^2$ along the counterclockwise-oriented square of side 2, with sides parallel to the axes, centered at the origin.

3. Calculate the complex line integral of the function $g(z) = 1/z$ along the counterclockwise-oriented square of side 2, with sides parallel to the axes, centered at the origin.

4. Calculate the complex line integral of the holomorphic function $f(z) = z^k$, $k = 0, 1, 2, \ldots$, along the curve $\gamma(t) = \cos t + i \sin t$, $0 \le t \le \pi$. Now calculate the complex line integral of the same function along the curve $\mu(t) = \cos t - i \sin t$, $0 \le t \le \pi$. Verify that, for each fixed k, the two answers are the same.

5. Verify that the conclusion of the last exercise is *false* if we take $k = -1$.

6. Verify that the conclusion of Exercise 4 is still true if we take $k = -2, -3, -4, \ldots$.

7. Suppose that f is a continuous function with complex antiderivative F. This means that $\partial F/\partial z = f$ on the domain of definition. Let γ be a continuously differentiable, closed curve in the domain of f. Prove that

$$\oint_\gamma f(z)\, dz = 0\,.$$

8. If f is a function and γ is a curve and $\oint_\gamma f(z)\, dz = 0$ then does it follow that $\oint f^2(z)\, dz = 0$?

9. Use the script

```
function [w] = cplxln(f,g,a,b)

syms t real;
syms z complex;

gd = diff(g, 't');

fg = subs(f, z, g);

xyz = fg*gd;

cplxlineint = int(xyz,t,a,b)
```

to create a function that calculates the complex line integral of the complex function f over the curve parametrized by g. Notice the following:

- The complex function is called f;
- The curve is $g : [a, b] \to \mathbb{C}$.
- The file must be called `cplxln.m`.

After you have this code entered and the file installed, test it out by entering

```
>> syms t real;
>> syms z complex;
>> f = z^2
>> g = cos(t) + i*sin(t)
>> a = 0
>> b = 2*pi
>> cplxln(f,g,a,b)
```

Notice that we are entering the function $f(z) = z^2$ and integrating over the curve $g : [0, 2\pi] \to \mathbb{C}$ given by $g(t) = \cos t + i \sin t$. You should obtain the answer 0 because the f that you have entered is holomorphic.

Now try `f = conj(z)`. This time you will obtain the answer `2*pi*i` because f is now the conjugate holomorphic function $\overline{z}$. Finally, apply the function `cplxln` to the function `f = 1/z` on the same curve. What answer do you obtain? Why?

10. If **F** is a vector field in the plane and γ a curve then $\oint_\gamma \mathbf{F}\, d\mathbf{r}$ represents the work performed while traveling along the curve and resisting the force **F**. Interpret the complex line integral in this language.

2.4 Complex Differentiability and Conformality

2.4.1 Conformality

Now we make some remarks about "conformality." Stated loosely, a function is *conformal* at a point $P \in \mathbb{C}$ if the function "preserves angles" at P and "stretches equally in all directions" at P. Both of these statements must be interpreted infinitesimally; we shall learn to do so in the discussion below. Holomorphic functions enjoy both properties:

> Let f be holomorphic in a neighborhood of $P \in \mathbb{C}$. Let w_1, w_2 be complex numbers of unit modulus. Consider the directional derivatives
>
> $$D_{w_1} f(P) \equiv \lim_{t \to 0} \frac{f(P + tw_1) - f(P)}{t} \tag{2.43}$$
>
> and
>
> $$D_{w_2} f(P) \equiv \lim_{t \to 0} \frac{f(P + tw_2) - f(P)}{t}. \tag{2.44}$$
>
> Then
>
> (2.45) $|D_{w_1} f(P)| = |D_{w_2} f(P)|$.
>
> (2.46) If $|f'(P)| \neq 0$, then the directed angle from w_1 to w_2 equals the directed angle from $D_{w_1} f(P)$ to $D_{w_2} f(P)$.

Statement (2.45) is the analytical formulation of "stretching equally in all directions." Statement (2.46) is the analytical formulation of "preserves angles."

In fact let us now give a discursive description of why conformality works. Either of these two properties actually characterizes holomorphic functions.

It is worthwhile to picture the matter in the following manner: Let f be holomorphic on the open set $U \subseteq \mathbb{C}$. Fix a point $P \in U$. Write $f = u + iv$ as usual. Thus we may write the mapping f as $(x, y) \mapsto (u, v)$. Then the (real) Jacobian matrix of the mapping is

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix},$$

where subscripts denote derivatives. We may use the Cauchy-Riemann equations to rewrite this matrix as

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ -u_y(P) & u_x(P) \end{pmatrix}$$

Factoring out a numerical coefficient, we finally write this two-dimensional derivative as

$$\begin{aligned} J(P) &= \sqrt{u_x(P)^2 + u_y(P)^2} \cdot \begin{pmatrix} \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \\ \frac{-u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \end{pmatrix} \\ &\equiv h(P) \cdot \mathcal{J}(P) \,. \end{aligned}$$

The matrix $\mathcal{J}(P)$ is of course a special orthogonal matrix (that is, its rows form an orthonormal basis of $\mathbb{R}^2$, and it is oriented positively—so it has determinant 1). Of course a special orthogonal matrix represents a *rotation*. Thus we see that the derivative of our mapping is a rotation $\mathcal{J}(P)$ (which preserves angles) followed by a positive "stretching factor" $h(P)$ (which also preserves angles). Of course a rotation stretches equally in all directions (in fact it does not stretch at all); and our stretching factor, or dilation, stretches equally in all directions (it simply multiplies by a positive factor). So we have established (2.45) and (2.46).

In fact the second characterization of conformality (in terms of preservation of directed angles) has an important converse: If (2.46) holds at points near P, then f has a complex derivative at P. If (2.45) holds at points near P, then either f or $\overline{f}$ has a complex derivative at P. Thus a function that is conformal (in either sense) at all points of an open set U must possess the complex derivative at each point of U. By the discussion in Section 2.1.6, the function f is therefore holomorphic if it is C^1. Or, by Goursat's theorem, it would then follow that the function is holomorphic on U, with the C^1 condition being automatic.

Exercises

1. Consider the holomorphic function $f(z) = z^2$. Calculate the derivative of f at the point $P = 1+i$. Write down the Jacobian matrix of f at P, thought of as a 2×2 real matrix operator. Verify directly (by imitating the calculations presented in this section) that this Jacobian matrix is the composition of a special orthogonal matrix and a dilation.

2. Repeat the first exercise with the function $g(z) = \sin z$ and $P = \pi + (\pi/2)i$.

3. Repeat the first exercise with the function $h(z) = e^z$ and $P = 2 - i$.

4. Discuss, in physical language, why the surface motion of an incompressible fluid flow should be conformal.

5. Verify that the function $g(z) = \overline{z}^2$ has the property that (at all points not equal to 0) it stretches equally in all directions, but it reverses angles. We say that such a function is *anticonformal*.

6. The function $h(z) = z + 2\overline{z}$ is *not* conformal. Explain why.

7. If a continuously differentiable function is conformal then it is holomorphic. Explain why.

8. If f is conformal then any positive integer power of f is conformal. Explain why.

9. If f is conformal then e^f is conformal. Explain why.

10. Let $\Omega \subseteq \mathbb{C}$ is a domain and $\varphi : \Omega \to \mathbb{R}$ is a function. Explain why φ, no matter how smooth or otherwise well behaved, could not possibly be conformal.

11. Use the following script to create a `MatLab` function that will detect whether a given complex function is acting conformally:

```
function [conformal_map] = conf(f,v1,v2,P)

syms x y cos1 cos2 real
syms z complex

z = x + i*y;

digits(5)

u = real(f);
v = imag(f);

p = real(P);
q = imag(P);

a11 = diff(u, 'x');
a12 = diff(u, 'y');
a21 = diff(v, 'x');
a22 = diff(v, 'y');

aa11 = subs(a11, {x,y}, {p,q});
aa12 = subs(a12, {x,y}, {p,q});
aa21 = subs(a21, {x,y}, {p,q});
aa22 = subs(a22, {x,y}, {p,q});

A = [aa11 aa12 ; aa21 aa22];

w1 = A*(v1');
w2 = A*(v2');

d1 = dot(v1,w1);
d2 = dot(v2,w2);
n1 = (dot(v1,v1))^(1/2);
n2 = (dot(v2,v2))^(1/2);
m1 = (dot(w1,w1))^(1/2);
m2 = (dot(w2,w2))^(1/2);

ccos1 = d1/(n1 * m1);
```

```
ccos2 = d2/(n2 * m2);

simplify(ccos1)
simplify(ccos2)

disp('The first number is the cosine of the angle')
disp('between the vector v1 and its image under')
disp('the Jacobian of the mapping.')
disp('      ')
disp('The second number is the cosine of the angle')
disp('between the vector v2 and its image under')
disp('the Jacobian of the mapping.')
disp('      ')
disp('If these numbers are equal then the mapping')
disp('is moving each vector v1 and v2 by the same angle.')
disp('Thus the mapping is acting in a conformal manner.')
disp('      ')
disp('If these numbers are unequal then the mapping')
disp('is moving the vectors v1 and v2 by different angles.')
disp('Thus the mapping is NOT acting in a conformal manner.')
```

This macro file must be called `conf.m`.

Your input for this function will be as follows:

```
>> syms x y real
>> syms z complex
>> z = x + i*y
>> f = z^2
>> v1 = [1 1]
>> v2 = [0 1]
>> P = 3 + 2*i
>> conf(f,v1,v2,P)
```

In this sample input we have used the function $f(z) = z^2$ and vectors $v_1 = \langle 1, 1\rangle$ and $v_2 = \langle 0, 1\rangle$. The base point is $P = 3 + 2i$. The `MatLab` output will explain to you how conformality is being measured.

Test this new function macro on these data sets:

- $f(z) = z^3$, $v_1 = \langle 2, 1\rangle$, $v_2 = \langle 1, 3\rangle$, $P = 2 + 4i$;

- $f(z) = z \cdot \overline{z}$, $v_1 = \langle 2, 2\rangle$, $v_2 = \langle 2, 3\rangle$, $P = 2 - 3i$;
- $f(z) = \overline{z}^2$, $v_1 = \langle 1, 1\rangle$, $v_2 = \langle 1, 4\rangle$, $P = 1 - 5i$;
- $f(z) = z$, $v_1 = \langle 1, 1\rangle$, $v_2 = \langle -1, 3\rangle$, $P = 1 + 4i$.

12. Let

$$\Phi(x, y) = (x^2 - y^2, 2xy) .$$

Let P be the point $(1, 0)$. Calculate the directional derivatives at P of Φ in the directions $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (1/\sqrt{2}, 1/\sqrt{2})$. Confirm that the *magnitudes* of these directional derivatives are the same. This is an instance of conformality. What holomorphic mapping is Φ?

13. Refer to the preceding exercise. The angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ is $\pi/4$. Calculate the angle between the directional derivative of Φ at P in the direction $\mathbf{w}_1$ and the directional derivative of Φ at P in the direction $\mathbf{w}_2$. It should also be $\pi/4$.

14. The surface of an incompressible fluid flow represents conformal motion. An air flow does not. Explain why.

2.5 The Logarithm

It is convenient to record here the basic properties of the complex logarithm.

Let $D = D(0, 1)$ be the unit disc and let f be a nonvanishing, holomorphic function on D. We define, for $z \in D$,

$$\mathcal{F}(z) = \int_0^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta .$$

This is understood to be a complex line integral along a path connecting 0 to z. The standard Cauchy theory (see Section 3.1.2) shows that the result is independent of the choice of path. Notice that $\mathcal{F}'(z) = f'(z)/f(z)$.

Now fix attention on the case $f(z) = z + 1$. Let $G(z) = e^z - 1$. And consider $\mathcal{F} \circ G$. We see that

$$(\mathcal{F} \circ G)' = \mathcal{F}'(G(z)) \cdot G'(z) = \frac{1}{e^z} \cdot e^z \equiv 1 .$$

We conclude from this that

$$\mathcal{F} \circ G = z + C .$$

By adding a constant to $\mathcal{F}$ (which is easily arranged by moving the base point from 0 to some other element of the disc), we may arrange that $C = 0$. Thus $\mathcal{F}$ is the inverse function for G. In other words

$$\mathcal{F}(z) = \log(z+1)\,.$$

In sum, we have constructed the logarithm function. It is plainly holomorphic by design.

Another way to think about the logarithm is as follows: Write

$$\log w = \log\left[|w|e^{i\arg w}\right] = \log|w| + i\arg w\,.$$

It follows that

$$\text{Re } \log w = \log|w|$$

and

$$\text{Im } \log w = \arg w\,.$$

This gives us a concrete way to calculate the logarithm. The circle of ideas is best illustrated with some examples.

Example 30 Let us find all complex logarithms of the complex number $z = e$. We have

$$\text{Re } \log e = \log|e| = \log e = 1$$

and

$$\text{Im } \log e = \arg e = 2k\pi\,.$$

Of course, as we know, the argument function has a built-in ambiguity.

In summary,

$$\log e = 1 + 2k\pi i\,.$$

$\square$

Example 31 Let us find all complex logarithms of the complex number $z = 1+i$. We note that $|z| = \sqrt{2}$ and $\arg z = \pi/4 + 2k\pi$. As a result,

$$\log z = \log(1+i) = \log\sqrt{2} + \left[\frac{\pi}{4} + 2k\pi\right] i = \frac{1}{2}\log 2 + \left[\frac{\pi}{4} + 2k\pi\right] i\,.$$

$\square$

It is frequently convenient to select a particular logarithm from among the infinitely many choices provided by the ambiguity in the argument. The *principal branch* of the logarithm is that for which the argument θ satisfies $0 \leq \theta < 2\pi$. We often denote the principal branch of the logarithm by $\operatorname{Log} z$.

EXAMPLE 32 Let us find the principal branch for the logarithm of $z = -3$. We note that $|z| = 3$ and $\arg z = \pi$. We have selected that value for the argument that lies between 0 and 2π so that we may obtain the principal branch. The result is

$$\log z = \log(-3) = \log 3 + i\pi\,.$$

$\square$

Of course the logarithm is a useful device for defining powers. Indeed, if z, w are complex numbers then

$$z^w \equiv e^{w \log z}\,.$$

As an example,

$$i^i = e^i \log i = e^{i(i\pi/2)} = e^{-\pi/2}\,.$$

Note that we have used the principal branch of the logarithm.

We conclude this section by noting that in each of the three examples we may check our work:

$$e^{1+2k\pi i} = e^1 \cdot e^{2k\pi i} = e\,;$$

$$e^{\log\sqrt{2}+i[\pi/4+2k\pi]} = e^{\log\sqrt{2}} \cdot e^{i[\pi/4+2k\pi]} = \sqrt{2}\cdot e^{i\pi/4} = 1+i\,;$$

and

$$e^{\log 3+i\pi} = e^{\log 3} \cdot e^{i\pi} = 3\cdot(-1) = -3\,.$$

Exercises

1. Calculate the complex logarithm of each of the following complex numbers:

(a) $3 - 3i$

(b) $-\sqrt{3} + i$

(c) $-\sqrt{2} - \sqrt{2}i$

(d) $1 - \sqrt{3}i$

(e) $-i$

(f) $\sqrt{3} - \sqrt{3}i$

(g) $-1 + 3i$

(h) $2 + 6i$

2. Calculate the principal branch of the logarithm of each of the following complex numbers:

(a) $2 + 2i$

(b) $3 - 3\sqrt{3}i$

(c) $-4 + 4i$

(d) $-1 - i$

(e) $-i$

(f) -1

(g) $1 + \sqrt{3}i$

(h) $-2 - 2\sqrt{2}i$

3. Calculate $(1 + i)^{1-i}$, i^{1-i}, $(1 - i)^i$, and $(-3)^{4-i}$.

4. Write a `MatLab` routine to find the principle branch of the logarithm of a given complex number. Use it to evaluate $\log(2+2\sqrt{3}i)$, $\log(4-4\sqrt{2}i)$.

5. Explain why there is no well-defined logarithm of the complex number 0.

6. It is not possible to give a succinct, unambiguous definition to the logarithm function on all of $\mathbb{C} \setminus \{0\}$. Explain why. We typically define the logarithm on $\mathbb{C} \setminus \{x + i0 : x \leq 0\}$. Explain why this restricted domain removes any ambiguities.

7. Consider the function $f(z) = \log(\log(\log z))$. For which values of z is this function well defined and holomorphic. Refer to the preceding exercise.

8. Consider the mapping $z \mapsto \log z$ applied to the annulus $A = \{z \in \mathbb{C} : 1 < |z| < e\}$. What is the image of this mapping? What physical interpretation can you give to this mapping? [**Hint:** You may find it useful to consider the inverse mapping, which is an exponential. You may take the domain of the inverse mapping to be an entire vertical strip.]

Chapter 3

The Cauchy Theory

3.1 The Cauchy Integral Theorem and Formula

3.1.1 The Cauchy Integral Theorem, Basic Form

If f is a holomorphic function on an open disc W in the complex plane, and if $\gamma : [a, b] \to W$ is a C^1 curve in W with $\gamma(a) = \gamma(b)$, then

$$\oint_\gamma f(z)\,dz = 0\,. \tag{3.1}$$

This is the *Cauchy integral theorem*. It is central and fundamental to the theory of complex functions. All of the principal results about holomorphic functions stem from this simple integral formula. We shall spend a good deal of our time in this text studying the Cauchy theorem and its consequences.

We now indicate a proof of this result. In fact it turns out that the Cauchy integral theorem, properly construed, is little more than a restatement of Green's theorem from calculus. Recall (see [BLK]) that Green's theorem says that if u, v are continuously differentiable on a bounded region U in the plane having C^2 boundary, then

$$\int_{\partial U} u\,dx + v\,dy = \iint\limits_U \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\,dxdy\,. \tag{3.2}$$

In the proof that we are about to present, we shall for simplicity assume that the curve γ is simple, closed. That is, γ does not cross itself, so it

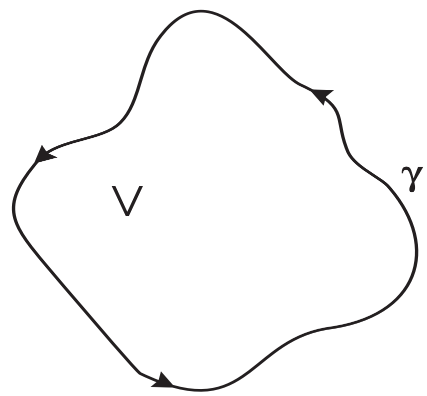

Figure 3.1: The curve γ surrounds the region V.

surrounds a region V. See Figure 3.1. Thus $\gamma = \partial V$. We take γ to be oriented counterclockwise. Let us write

$$\oint_\gamma f\,dz = \oint_\gamma (u+iv)\,[dx+idy] = \left(\oint_\gamma u\,dx - v\,dy\right) + i\left(\oint_\gamma v\,dx + u\,dy\right).$$

Each of these integrals is clearly a candidate for application of Green's theorem (3.2). Thus

$$\oint_\gamma f\,dz = \oint_{\partial V} f\,dz = \iint_V \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right)\,dxdy + i\iint_V \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dxdy\,.$$

But, according to the Cauchy-Riemann equations, each of the integrands vanishes. We learn then that

$$\oint_\gamma f\,dz = 0\,.$$

That is Cauchy's theorem.

An important converse of Cauchy's theorem is called *Morera's theorem*:

> Let f be a continuous function on a connected open set $U \subseteq \mathbb{C}$. If
>
> $$\oint_\gamma f(z)\,dz = 0 \tag{3.3}$$

for every simple, closed curve γ in U, then f is holomorphic on U.

In the statement of Morera's theorem, the phrase "every simple, closed curve" may be replaced by "every triangle" or "every square" or "every circle."

The verification of Morera's theorem also uses Green's theorem. Assume for simplicity that f is continuously differentiable. Then the same calculation as above shows that if

$$\oint_\gamma f(z)\,dz = 0$$

for every simple, closed curve γ, then

$$\iint\limits_U \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0$$

for the region U that γ surrounds. This is true for every possible region U! It follows that the integrand must be identically zero. But this simply says that f satisfies the Cauchy-Riemann equations. So it is holomorphic.[1]

3.1.2 More General Forms of the Cauchy Theorem

Now we present the very useful general statement of the Cauchy integral theorem. First we need a piece of terminology. A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *piecewise* C^k if

$$[a, b] = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{m-1}, a_m] \tag{3.4}$$

with $a = a_0 < a_1 < \cdots < a_m = b$ and the curve $\gamma\big|_{[a_{j-1}, a_j]}$ is C^k for $1 \le j \le m$. In other words, γ is piecewise C^k if it consists of finitely many C^k curves chained end to end. See Figure 3.2.

Piecewise C^k curves will come up both explicitly and implicitly in many of our ensuing discussions. When we deform, and cut and paste, curves then the curves created will often by piecewise C^k. We can be confident that we can integrate along such curves, and that the Cauchy theory is valid for such curves. They are part of our toolkit in basic complex analysis.

[1]For convenience, we have provided this simple proof of Morera's theorem only when the function is continuously differentiable. But it is of definite interest—and useful later—to know that Morera's theorem is true for functions that are only continuous.

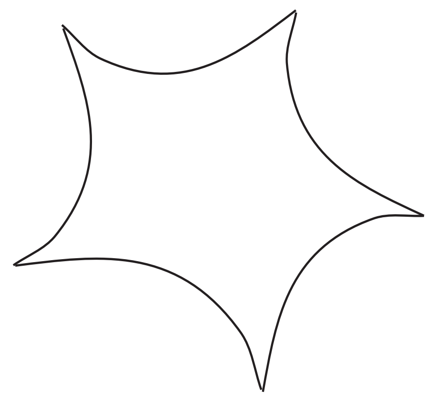

Figure 3.2: A piecewise C^k curve.

Cauchy Integral Theorem: Let $f : U \to \mathbb{C}$ be holomorphic with $U \subseteq \mathbb{C}$ an open set. Then

$$\oint_\gamma f(z)\,dz = 0 \tag{3.5}$$

for each piecewise C^1 closed curve γ in U that can be deformed in U through closed curves to a point in U—see Figure 3.3. We call such a curve *homotopic to 0*. From the topological point of view, such a curve is trivial.

EXAMPLE 33 Let U be the region consisting of the disc $\{z \in \mathbb{C} : |z| < 2\}$ with the closed disc $\{z \in \mathbb{C} : |z - i| < 1/3\}$ removed. Let $\gamma : [0,1] \to U$ be the curve $\gamma(t) = \cos t + [i/4] \sin t$. See Figure 3.4. If f is any holomorphic function on U then

$$\oint_\gamma f(z)\,dz = 0\,.$$

Perhaps more interesting is the following fact. Let P, Q be points of U. Let $\gamma : [0,1] \to U$ be a curve that begins at P and ends at Q. Let $\mu : [0,1] \to U$ be some other curve that begins at P and ends at Q. The requirement that we impose on these curves is that they do not surround any holes in U—in other words, the curve formed with γ followed by (the

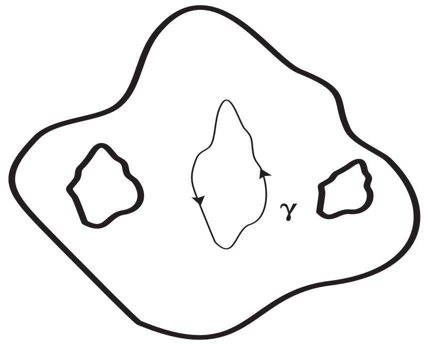

Figure 3.3: A curve γ on which the Cauchy integral theorem is valid.

reverse of) μ is homotopic to 0. Refer to Figure 3.5. If f is any holomorphic function on U then

$$\oint_\gamma f(z)\,dz = \oint_\mu f(z)\,dz\,.$$

The reason is that the curve τ that consists of γ *followed by* the reverse of μ is a closed curve in U. It is homotopic to 0. Thus the Cauchy integral theorem applies and

$$\oint_\tau f(z)\,dz = 0\,.$$

Writing this out gives

$$\oint_\gamma f(z)\,dz - \oint_\mu f(z)\,dz = 0\,.$$

That is our claim. □

3.1.3 Deformability of Curves

A central fact about the complex line integral is the deformability of curves. Let $\gamma : [a,b] \to U$ be a closed, piecewise C^1 curve in a region U of the complex plane. Let f be a holomorphic function on U. The value of the complex line integral

$$\oint_\gamma f(z)\,dz \tag{3.6}$$

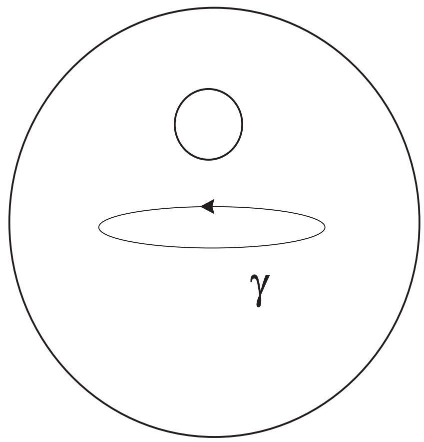

Figure 3.4: A curve γ on which the generalized Cauchy integral theorem is valid.

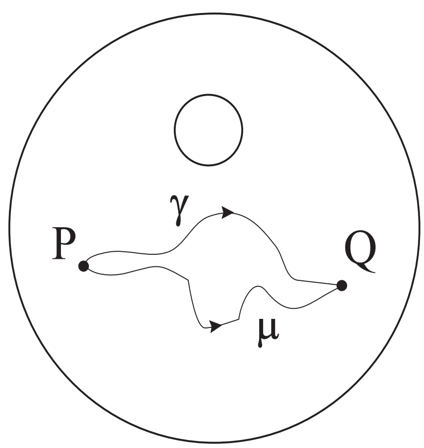

Figure 3.5: Two curves with equal complex line integrals.

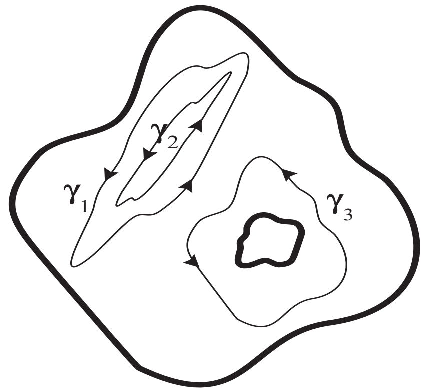

Figure 3.6: Deformation of curves.

does not change if the curve γ is smoothly deformed within the region U. Note that, in order for this statement to be valid, the curve γ must remain inside the region of holomorphicity U of f while it is being deformed, and it must remain a closed curve while it is being deformed. Figure 3.6 shows curves γ_1, γ_2 that *can* be deformed to one another, and a curve γ_3 that can be deformed to neither of the first two (because of the hole inside γ_3).

The reasoning behind the deformability principle is simplicity itself. Examine Figure 3.7. It shows a solid curve γ and a dashed curve $\widetilde{\gamma}$. The latter should be thought of as a deformation of the former. Now let us examine the *difference* of the integrals over the two curves—see Figure 3.8. We see that this difference is in fact the integral of the holomorphic function f over a closed curve that *can be continuously deformed to a point*. Of course, by the Cauchy integral theorem, that integral is equal to 0. Thus the difference of the integral over γ and the integral over $\widetilde{\gamma}$ is 0. That is the deformability principle.

A topological notion that is special to complex analysis is simple connectivity. We say that a domain $U \subseteq \mathbb{C}$ is *simply connected* if any closed curve in U can be continuously deformed to a point. See Figure 3.9. Simple connectivity is a mathematically rigorous condition that corresponds to the intuitive notion that the region U has no holes. Figure 3.10 shows a domain that is *not* simply connected. If U is simply connected, and γ is a closed curve in U, then it follows that γ can be continuously deformed to lie inside a disc in U. It follows that Cauchy's theorem applies to γ. To summarize:

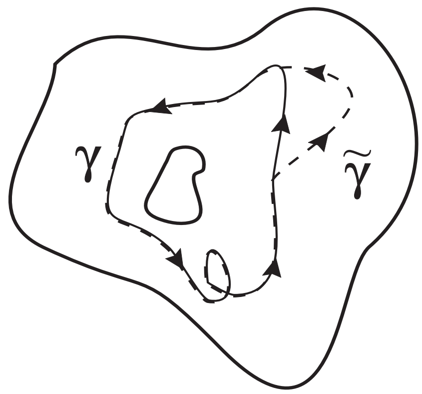

Figure 3.7: Deformation of curves.

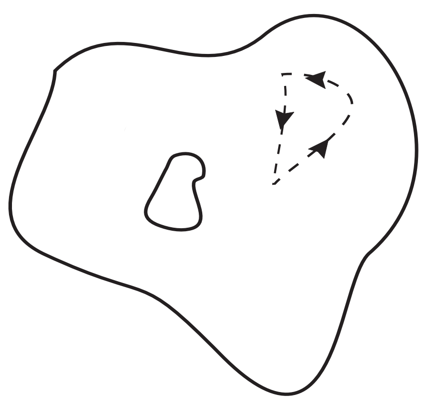

Figure 3.8: The difference of the integrals.

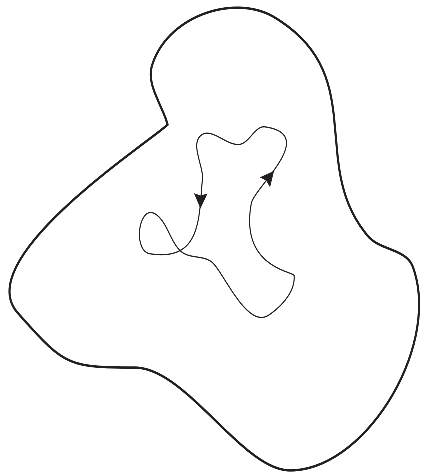

Figure 3.9: A simply connected domain.

on a simply connected region, Cauchy's theorem applies (without any further hypotheses) to any closed curve in U. Likewise, on a simply connected region U, Cauchy's integral formula (to be developed below) applies to any simple, closed curve that is oriented counterclockwise and to any point z that is inside that curve.

3.1.4 Cauchy Integral Formula, Basic Form

The Cauchy integral formula is derived from the Cauchy integral theorem. It tells us that we can express the value of a holomorphic function f in terms of a sort of average of its values around the boundary. This assertion is really quite profound; it turns out that the formula is key to many of the most important properties of holomorphic functions. We begin with a simple enunciation of Cauchy's idea.

Let $U \subseteq \mathbb{C}$ be a domain and suppose that $\overline{D}(P,r) \subseteq U$. Let $\gamma : [0,1] \to \mathbb{C}$ be the C^1 parametrization $\gamma(t) = P + r\cos(2\pi t) + ir\sin(2\pi t)$. Then, for each $z \in D(P,r)$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{3.7}$$

Before we indicate the proof, we impose some simplifications. First, we may as well translate coordinates and assume that $P = 0$. Thus the Cauchy

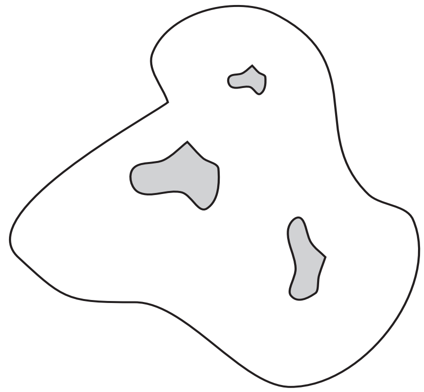

Figure 3.10: A domain that is *not* simply connected.

formula becomes

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \, .$$

Our strategy is to apply the Cauchy integral *theorem* to the function

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z} \, .$$

In fact it can be checked—using Morera's theorem for example—that g is still holomorphic.[2] Thus we may apply Cauchy's theorem to see that

$$\oint_{\partial D(0,r)} g(\zeta) \, d\zeta = 0$$

or

$$\oint_{\partial D(0,r)} \frac{f(\zeta) - z}{\zeta - z} \, d\zeta = 0 \, .$$

But this just says that

$$\frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(z)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \, . \tag{3.8}$$

[2]First, $\lim_{\zeta \to z} g(\zeta)$ exists because f is holomorphic. So g extends to be a continuous function on $D(0, r)$. We know that the integral of g over any curve that *does not* surround z must be zero—by the Cauchy integral theorem. And the integral over a curve that *does* pass through or surround z will therefore also be zero by a simple limiting argument.

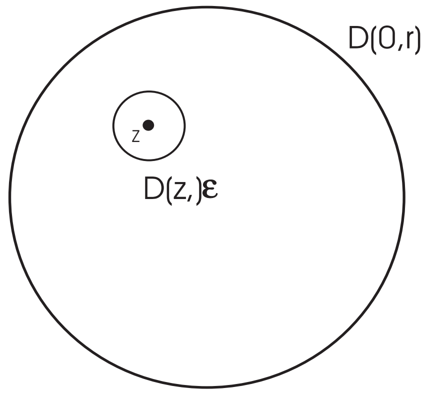

Figure 3.11: The deformation principle.

It remains to examine the left-hand side.

Now
$$\frac{1}{2\pi i}\oint_{\partial D(0,r)} \frac{f(z)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i}\oint_{\partial D(0,r)} \frac{1}{\zeta - z}\, d\zeta \tag{3.9}$$
and we must evaluate the integral. It is convenient to use deformation of curves to move the boundary $\partial D(0,r)$ to $\partial D(z,\epsilon)$, where $\epsilon > 0$ is chosen so small that $D(z,\epsilon) \subseteq D(0,r)$. See Figure 3.11. Then we have

$$\oint_{\partial D(0,r)} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial D(z,\epsilon)} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial D(0,\epsilon)} \frac{1}{\zeta}\, d\zeta\,. \tag{3.10}$$

In the last equality we used a simple change of variable.

Introducing the parametrization $t \mapsto \epsilon e^{it}$, $0 \le t \le 2\pi$, for the curve, we find that our integral is

$$\int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i\,.$$

Putting this information together with (3.8) and (3.9), we find that

$$f(z) = \frac{1}{2\pi i}\oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

That is the Cauchy integral formula when the domain is a disc.

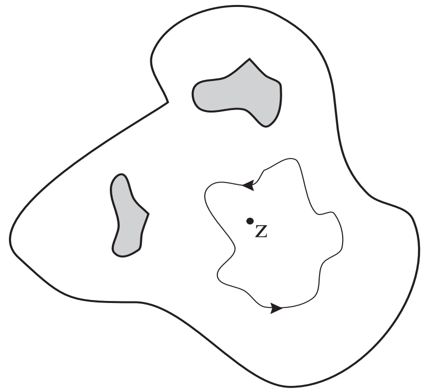

Figure 3.12: A curve that *can* be deformed to a point.

We shall see in Section 4.1.1 that the Cauchy integral formula gives an easy proof that a holomorphic function is infinitely differentiable. Thus the Cauchy-Goursat theorem is swept under the rug: holomorphic functions are as smooth as can be, and we can differentiate them at will.

3.1.5 More General Versions of the Cauchy Formula

A more general version of the Cauchy formula—the one that is typically used in practice—is this:

THEOREM 1 *Let $U \subseteq \mathbb{C}$ be a domain. Let $\gamma : [0,1] \to U$ be a simple, closed curve that can be continuously deformed to a point inside U. See Figure 3.12. If f is holomorphic on U and z lies in the region interior to γ, then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

The proof is nearly identical to the one that we have presented above in the special case. We omit the details.

EXAMPLE 34 Let $U = \{z = x + iy \in \mathbb{C} : -2 < x < 2, 0 < y < 3\} \setminus \overline{D}(-1 + (7/4)i, 1/10)$. Let $\gamma(t) = \cos t + i(3/2 + \sin t)$. Then the curve γ lies in U. The curve γ can certainly be deformed to a point inside U. Thus if f is any

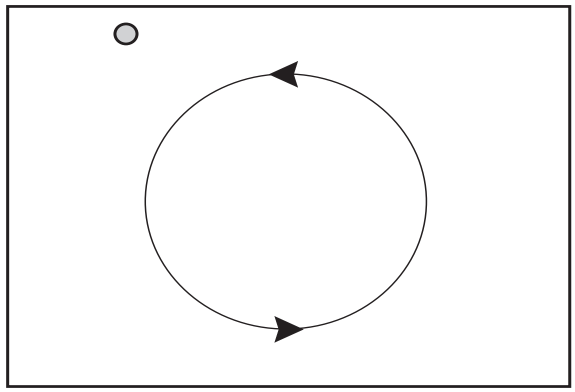

Figure 3.13: Illustration of the Cauchy integral formula.

holomorphic function on U then, for z inside the curve (see Figure 3.13),

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta \, .$$

$\square$

Exercises

1. Let $f(z) = z^2 - z$ and $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. Confirm the statement of the Cauchy integral theorem for this f and this γ by actually calculating the appropriate complex line integral.

2. The points $\pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$ lie on the unit circle. Let $\eta(t)$ be the counterclockwise-oriented, square path for which they are the vertices. Verify the conclusion of the Cauchy integral theorem for *this* path and the function $f(z) = z^2 - z$. Compare with Exercise 1.

3. The Cauchy integral theorem fails for the function $f(z) = \cot z$ on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Calculate the relevant complex line integral and verify that the value of the integral is *not* zero. What hypothesis of the Cauchy integral theorem is lacking?

4. Let u be a harmonic function in a neighborhood of the closed unit disc

$$\overline{D}(0,1) = \{z \in \mathbb{C} : |z| \le 1\}\,.$$

For each $P = (p_1, p_2) \in \partial D(0,1)$, let $\nu(P) = \langle p_1, p_2 \rangle$ be the unit outward normal vector. Use Green's theorem to prove that

$$\int_{\partial D(0,1)} \frac{\partial}{\partial \nu} u(z)\, ds(z) = 0\,.$$

[**Hint:** Be sure to note that this is *not* a complex line integral. It is instead a standard calculus integral with respect to arc length.]

5. It is a fact (Morera's theorem) that if f is a continuously differentiable function on a domain Ω and if $\oint_\gamma f(z)\,dz = 0$ for every continuously differentiable, closed curve in Ω, then f is holomorphic on Ω. Restrict attention to curves that bound closed discs that lie in Ω. Apply Green's theorem to the hypothesis that we have formulated. Conclude that the two-dimensional integral of $\partial f/\partial \overline{z}$ is 0 on any disc in Ω. What does this tell you about $\partial f/\partial \overline{z}$?

6. Let f be holomorphic on a domain Ω and let P, Q be points of Ω. Let γ_1 and γ_2 be continuously differentiable curves in Ω that each begin at P and end at Q. What conditions on γ_1, γ_2, and Ω will guarantee that $\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$?

7. Let Ω be a domain and suppose that γ is a simple, closed curve in Ω that is continuously differentiable. Suppose that $\oint_\gamma f(z)\,dz = 0$ for every holomorphic function f on Ω. What can you conclude about the domain Ω and the curve γ?

8. Let D be the unit disc and suppose that $\gamma : [0,1] \to D$ is a curve that circles the origin *twice* in the counterclockwise direction. Let f be holomorphic on D. What can you say about the value of

$$\oint_\gamma \frac{f(\zeta)}{\zeta - 0}\, d\zeta\,?$$

9. Suppose that the curve in the last exercise circles the origin twice in the clockwise direction. Then what can you say about the value of the integral

$$\oint_\gamma \frac{f(\zeta)}{\zeta - 0}\, d\zeta\,?$$

10. Let the domain D be the unit disc and let g be a *conjugate holomorphic function* on D (that is, $\overline{g}$ is holomorphic). Then there exists a simple, closed, continuously differentiable curve γ in D such that $\oint_\gamma g(\zeta)\, d\zeta \neq 0$. Prove this assertion.

11. Let $U = \{z \in \mathbb{C} : 1 < |z| < 4\}$. Let $\gamma(t) = 3\cos t + 3i \sin t$. Let $f(z) = 1/z$. Let $P = 2 + i0$. Verify with a direct calculation that

$$f(P) \neq \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - P}\, d\zeta\,.$$

12. In the preceding exercise, replace f with $g(\zeta) = \zeta^2$. Now verify that

$$g(P) = \frac{1}{2\pi i} \oint_\gamma \frac{g(\zeta)}{\zeta - P}\, d\zeta\,.$$

Explain why the answer to this exercise is different from the answer to the earlier exercise.

13. Let $U = D(0,2)$ and let $\gamma(t) = \cos t + i \sin t$. Verify by a direct calculation that, for any $z \in D(0,1)$,

$$1 = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - z}\, d\zeta\,.$$

Now derive the same identity immediately using the Cauchy integral formula with the function $f(z) \equiv 1$.

14. Let $U = \{z \in \mathbb{C} : -4 < x < 4, -4 < y < 4\}$. Let $\gamma(t) = \cos t + i \sin t$. Let $\mu(t) = 2\cos t + 3\sin t$. Finally set $f(z) = z^2$. Of course each of the two curves lies in U. Draw a picture. Let $P = 1/2 + i/2$. Calculate

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - P}\, d\zeta$$

and

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - P}\, d\zeta\,.$$

The answers that you obtain should be the same. Explain why.

15. Use the `MatLab` utility `cplxln.m` that you created in Exercise 13 of Section 1.5.7 to test the Cauchy integral theorem and formula in the following ways:

(a) Let $f(z) = z^2$, $g(z) = \overline{z}$, and $h(z) = z \cdot \overline{z}$. Use `cplxln.m` to calculate the complex line integral of each of these functions along the curve $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. How do you account for the answers that you obtain?

(b) Let $k(z) = \overline{z}^2$ and $m(z) = 1/z^2$. Use the curve γ from part **(a)**. Clearly neither of these functions is holomorphic on $\overline{D}(0,1)$. Nonetheless, you can use the utility `cplxln.m` to calculate that $\oint_\gamma k(z)\,dz = 0$ and $\oint_\gamma m(z)\,dz = 0$. How can you account for this?

(c) Let γ be as in part **(a)**. Calculate, using the `MatLab` utility `cplxln.m`, the integrals

- $\displaystyle\oint_\gamma \frac{1}{z}\,dz$,
- $\displaystyle\oint_\gamma \frac{1}{z-1/2}\,dz$,
- $\displaystyle\oint_\gamma \frac{1}{z-(1/3+i/4)}\,dz$,
- $\displaystyle\oint_\gamma \frac{1}{z-0.999999}\,dz$.

You should obtain the same answer in all four cases. Explain why.

(d) Let $p(z) = e^z$. Let the curve γ be as in part **(a)**. Use the `MatLab` utility `cplxln.m` to calculate

- $\frac{1}{2\pi i}\displaystyle\oint_\gamma \frac{p(z)}{z}\,dz$,
- $\frac{1}{2\pi i}\displaystyle\oint_\gamma \frac{p(z)}{z-1/2}\,dz$,
- $\frac{1}{2\pi i}\displaystyle\oint_\gamma \frac{p(z)}{z-(1/3+i/4)}\,dz$.

The answers you get should be, respectively, $p(0)$, $p(1/2)$, and $p(1/3+i/4)$. Verify this assertion.

3.2 Variants of the Cauchy Formula

The Cauchy formula is a remarkably flexible tool that can be applied even when the domain U in question is *not* simply connected. Rather than attempting to formulate a general result, we illustrate the ideas here with some examples.

EXAMPLE 35 Let $U = \{z \in \mathbb{C} : 1 < |z| < 4\}$. Let $\gamma_1(t) = 2\cos t + 2i \sin t$ and $\gamma_2(t) = 3\cos t + 3i \sin t$. See Figure 3.14. If f is any holomorphic function on U and if the point z satisfies $2 < |z| < 3$ (again, see Figure 3.14) then

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta \, . \tag{3.11}$$

The beauty of this result is that it can be established with a simple diagram. Refer to Figure 3.15. We see that integration over γ_2 and $-\gamma_1$, as indicated in formula (3.11), is just the same as integrating over a single contour γ^*. And, with a slight deformation, we see that that contour is equivalent—for the purposes of integration—with integration over a contour $\widetilde{\gamma}^*$ that is homotopic to zero. Thus, with a bit of manipulation, we see that the integrations in (3.11) are equivalent to integration over a curve for which we know that the Cauchy formula holds.

That establishes formula (3.11). □

EXAMPLE 36 Consider the region

$$U = D(0,6) \setminus \left[\overline{D}(-3+0i,2) \cup \overline{D}(3+0i,2)\right] \, .$$

It is depicted in Figure 3.16. We also show in the figure *three* contours of integration: γ_1, γ_2, γ_3. We deliberately do not give formulas for these curves, because we want to stress that the reasoning here is geometric and does *not* depend on formulas.

Now suppose that f is a holomorphic function on U. We want to write a Cauchy integral formula—for the function f and the point z—that will be valid in this situation. It turns out that the correct formula is

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \oint_{\gamma_3} \frac{f(\zeta)}{\zeta - z} \, d\zeta \, .$$

The justification, parallel to that in the last example, is shown in Figure 3.17. □

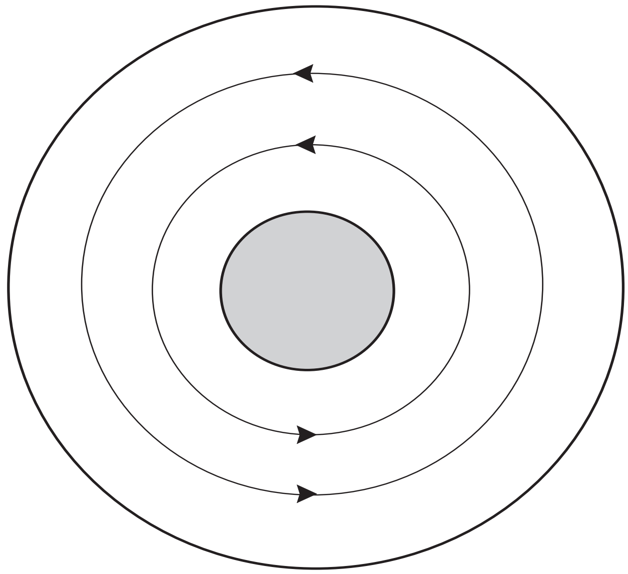

Figure 3.14: A variant of the Cauchy integral formula.

3.3 A Coda on the Limitations of the Cauchy Integral Formula

If f is any continuous function on the boundary of the unit disc $D = D(0,1)$, then the Cauchy integral

$$F(z) \equiv \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{3.12}$$

defines a holomorphic function $F(z)$ on D (use Morera's theorem, for example, to confirm this assertion). What does the new function F have to do with the original function f? In general, not much.

For example, if $f(\zeta) = \overline{\zeta}$, then $F(z) \equiv 0$ (exercise). In no sense is the original function f any kind of "boundary limit" of the new function F. The question of which functions f are "natural boundary functions" for holomorphic functions F (in the sense that F is a continuous extension of f to the closed disc) is rather subtle. Its answer is well understood, but is best formulated in terms of Fourier series and the so-called Hilbert transform. The complete story is given in [KRA1].

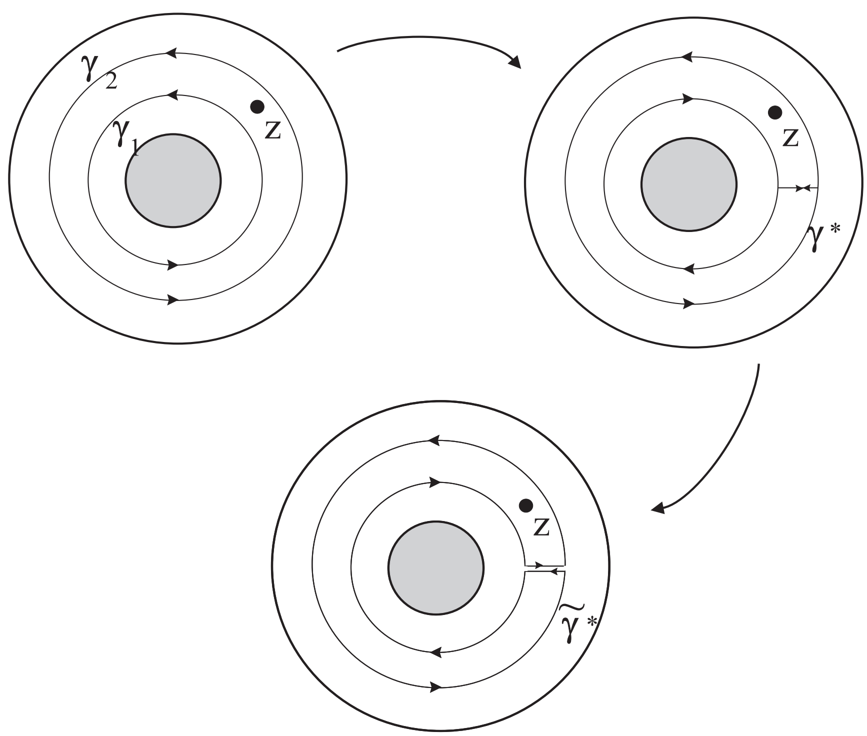

Figure 3.15: Turning two contours into one.

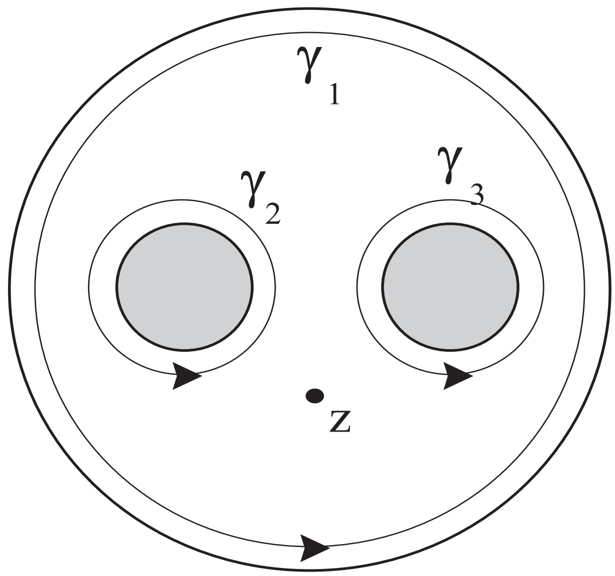

Figure 3.16: A triply connected domain.

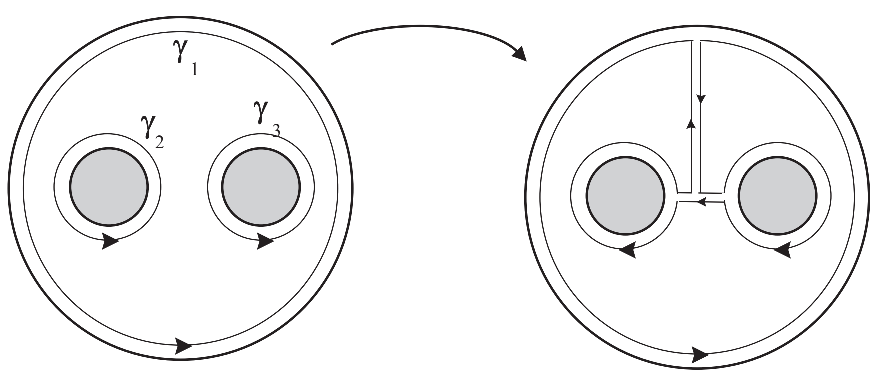

Figure 3.17: Turning three contours into one.

Contrast this situation for holomorphic functions with the much more succinct and clean situation for harmonic functions (Section 9.3).

Exercises

1. Let

$$\varphi(e^{i\theta}) = \begin{cases} 1 & \text{if} \quad 0 \le \theta \le \pi \\ -1 & \text{if} \quad \pi < \theta \le 2\pi \,. \end{cases}$$

Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Use the `MatLab` utility `cplxln.m` to calculate

$$\Phi(a) = \oint_\gamma \frac{\varphi(z)}{z-a}\, dz \,,$$

for $a = 0, 1/2, i/3$. Calculate the value of the integral for **(i)** a sequence of a's tending to 1, **(ii)** a sequence of a's tending to i, and **(iii)** a sequence of a's tending to $1/\sqrt{2}+i/\sqrt{2}$. What can you conclude about the relationship (if any) between the values of the function Φ in the interior of the disc with the values of the function φ on the boundary of the disc?

2. Repeat the first exercise with the function φ replaced by

$$\psi(z) = \overline{z} \,.$$

3. Repeat the first exercise with the function φ replaced by

$$\eta(z) = \frac{1}{z} \,.$$

4. Repeat the first exercise with the function φ replaced by

$$\mu(z) = \frac{1}{z^2} \,.$$

5. Use the `MatLab` utility `cplxln.m` to calculate the Cauchy integral $\frac{1}{2\pi i}\oint_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ for these functions f on the boundary ∂D of the unit disc D:

(a) $f(\zeta) = \frac{1}{\zeta^2}$

(b) $f(\zeta) = \zeta^2$

(c) $f(\zeta) = \zeta \cdot \overline{\zeta}$

(d) $f(\zeta) = \frac{\zeta}{3+\zeta}$

(e) $f(\zeta) = \frac{\zeta}{\overline{\zeta}}$

(f) $f(\zeta) = \frac{\overline{\zeta}^2}{\zeta}$

In each instance, comment on the relationship between the holomorphic function you have created on the interior D of the disc and the original function f on the boundary of the disc.

6. Let f be a continuous, complex-valued function on the boundary of the unit disc D. Let F be its Cauchy integral. Interpret f as a force field. In the case when F agrees with f at the boundary, what does this say about the force field? In the case when F *does not* agree with f at the boundary, what does *that* say about the force field?

Chapter 4

Applications of the Cauchy Theory

4.1 The Derivatives of a Holomorphic Function

One of the remarkable features of holomorphic function theory is that we can express the derivative of a holomorphic function in terms of the function itself. Nothing of the sort is true for real functions. One upshot is that we can obtain powerful estimates for the derivatives of holomorphic functions.

We shall explore this phenomenon in the present section.

EXAMPLE 37 On the real line $\mathbb{R}$, let

$$f_k(x) = \sin(kx)\,.$$

Then of course $|f_k(x)| \leq 1$ for all k and all x. Yet $f_k'(x) = k\cos(kx)$ and $|f_k'(0)| = k$. So there is no sense, and no hope, of bounding the derivative of a function by means of the function itself. We will find matters to be quite different for holomorphic functions. $\square$

4.1.1 A Formula for the Derivative

Let $U \subseteq \mathbb{C}$ be an open set and let f be holomorphic on U. Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P,r) \subseteq U$ and $z \in D(P,r)$, then

$$\left(\frac{d}{dz}\right)^k f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \quad k = 0, 1, 2, \ldots . \tag{4.1}$$

The proof of this new formula is direct. For consider the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \,.$$

We may differentiate both sides of this equation:

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right] .$$

Now we wish to justify passing the derivative on the right under the integral sign. A justification from first principles may be obtained by examining the Newton quotients for the derivative. Alternatively, one can cite a suitable limit theorem as in [RUD1] or [KRA2]. In any event, we obtain

$$\begin{aligned} \frac{d}{dz} f(z) &= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{d}{dz}\left[\frac{f(\zeta)}{\zeta - z}\right] d\zeta \\ &= \frac{1}{2\pi i} \oint_{\partial D(0,r)} f(\zeta) \cdot \frac{d}{dz}\left[\frac{1}{\zeta - z}\right] d\zeta \\ &= \frac{1}{2\pi i} \oint_{\partial D(0,r)} f(\zeta) \cdot \frac{1}{(\zeta - z)^2}\, d\zeta \,. \end{aligned}$$

This is in fact the special instance of formula (4.1) when $k = 1$. The cases of higher k are obtained through additional differentiations, or by induction.

4.1.2 The Cauchy Estimates

If f is a holomorphic on a region containing the closed disc $\overline{D}(P,r)$ and if $|f| \le M$ on $\overline{D}(P,r)$, then

$$\left|\frac{\partial^k}{\partial z^k} f(P)\right| \le \frac{M \cdot k!}{r^k}. \tag{4.2}$$

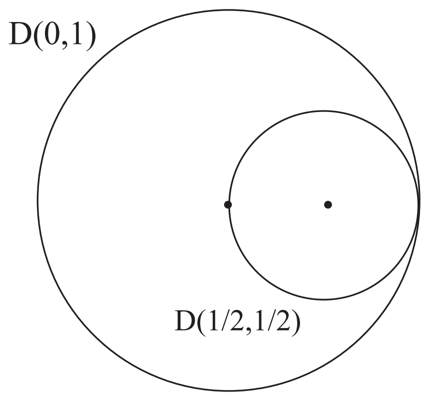

Figure 4.1: The Cauchy estimates.

In fact this formula is a result of direct estimation from (4.1). For we have

$$\left|\frac{\partial^k}{\partial z^k} f(P)\right| = \left|\frac{k!}{2\pi i}\oint_{|\zeta-P|=r}\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta\right| \le \frac{k!}{2\pi}\cdot\frac{M}{r^{k+1}}\cdot 2\pi r = \frac{Mk!}{r^k}\,.$$

EXAMPLE 38 Let $f(z) = (z^3+1)e^{z^2}$ on the unit disc $D(0,1)$. Obviously

$$|f(z)| \le 2\cdot|e^{z^2}| = e^{x^2-y^2} \le e \quad \text{for all } z \in D(0,1)\,.$$

We may then conclude, by the Cauchy estimates applied to f on $D(1/2,1/2) \subseteq D(0,1)$ (see Figure 4.1), that

$$|f'(1/2)| \le \frac{e\cdot 1!}{1/2} = 2e$$

and

$$|f''(1/2)| \le \frac{e\cdot 2!}{(1/2)^2} = 8e\,.$$

Of course one may perform the tedious calculation of these derivatives and determine that $f'(1/2) \approx 1.1235$ and $f''(1/2) \approx 6.2596$. But Cauchy's estimates allow us to estimate the derivatives by way of soft analysis. $\square$

4.1.3 Entire Functions and Liouville's Theorem

A function f is said to be *entire* if it is defined and holomorphic on all of $\mathbb{C}$, that is, $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. For instance, any holomorphic polynomial is entire, e^z is entire, and $\sin z, \cos z$ are entire. The function $f(z) = 1/z$ is not entire because it is undefined at $z = 0$. [In a sense that we shall make precise later (Section 5.1), this last function has a "singularity" at 0.] The question we wish to consider is: "Which entire functions are bounded?" This question has a very elegant and complete answer as follows:

THEOREM 2 (Liouville's Theorem) *A bounded entire function is constant.*

Proof: Let f be entire and assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix a $P \in \mathbb{C}$ and let $r > 0$. We apply the Cauchy estimate (4.2) for $k = 1$ on $\overline{D}(P, r)$. So

$$\left| \frac{\partial}{\partial z} f(P) \right| \leq \frac{M \cdot 1!}{r}. \tag{4.3}$$

Since this inequality is true for every $r > 0$, we conclude (by letting $r \to \infty$) that

$$\frac{\partial f}{\partial z}(P) = 0. \tag{4.4}$$

Since P was arbitrary, we conclude that

$$\frac{\partial f}{\partial z} \equiv 0. \tag{4.5}$$

Of course we also know, since f is holomorphic, that

$$\frac{\partial f}{\partial \bar{z}} \equiv 0. \tag{4.6}$$

It follows from linear algebra then that

$$\frac{\partial f}{\partial x} \equiv 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} \equiv 0\,. \tag{4.7}$$

Therefore f is constant. $\square$

The reasoning that establishes Liouville's theorem can also be used to prove this more general fact: If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and if for some real number C and some positive integer k, it holds that

$$|f(z)| \leq C \cdot (1 + |z|)^k \tag{4.8}$$

for all z, then f is a polynomial in z of degree at most k. We leave the details for the interested reader.

4.1.4 The Fundamental Theorem of Algebra

One of the most elegant applications of Liouville's Theorem is a proof of what is known as the Fundamental Theorem of Algebra (see also Sections 1.2.4 and 6.3.3):

> **The Fundamental Theorem of Algebra:** Let $p(z)$ be a nonconstant (holomorphic) polynomial in z. Then p has a root. That is, there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Proof: Suppose not. Then $g(z) = 1/p(z)$ is entire. Also, when $|z| \to \infty$, then $|p(z)| \to +\infty$. Thus $1/|p(z)| \to 0$ as $|z| \to \infty$; hence g is bounded. By Liouville's Theorem, g is constant, hence p is constant. Contradiction. □

If, in the theorem, p has degree $k \geq 1$, then let α_1 denote the root provided by the Fundamental Theorem. By the Euclidean algorithm (see [HUN]), we may divide $z - \alpha_1$ into p to obtain

$$p(z) = (z - \alpha_1) \cdot p_1(z) + r_1(z)\,. \tag{4.9}$$

Here p_1 is a polynomial of degree $k - 1$ and r_1 is the remainder term of degree 0 (that is, less than 1). Substituting α_1 into this last equation gives $0 = 0 + r_1$, hence we see that $r_1 = 0$. Thus the Euclidean algorithm has taught us that

$$p(z) = (z - \alpha_1) \cdot p_1(z)\,.$$

If $k - 1 \geq 1$, then, reasoning as above with the Fundamental Theorem, p_1 has a root α_2 . Thus p_1 is divisible by $(z - \alpha_2)$ and we have

$$p(z) = (z - \alpha_1) \cdot (z - \alpha_2) \cdot p_2(z) \tag{4.10}$$

for some polynomial $p_2(z)$ of degree $k - 2$. This process can be continued until we arrive at a polynomial p_k of degree 0; that is, p_k is constant. We have derived the following fact: If $p(z)$ is a holomorphic polynomial of degree k, then there are k complex numbers $\alpha_1, \ldots \alpha_k$ (not necessarily distinct) and a nonzero constant C such that

$$p(z) = C \cdot (z - \alpha_1) \cdots (z - \alpha_k). \tag{4.11}$$

If some of the roots of p coincide, then we say that p has *multiple roots*. To be specific, if m of the values $\alpha_{n_1}, \ldots, \alpha_{n_m}$ are equal to some complex number α, then we say that p has a root of order m at α (or that p has a root α of *multiplicity* m). An example will make the idea clear: Let

$$p(z) = (z-5)^3 \cdot (z+2)^8 \cdot (z-7) \cdot (z+6). \tag{4.12}$$

Thus p is a polynomial of degree 13. We say that p has a root of order 3 at 5, a root of order 8 at -2, and it has roots of order 1 at 7 and at -6. We also say that p has *simple roots* at 7 and -6.

4.1.5 Sequences of Holomorphic Functions and Their Derivatives

A sequence of functions g_j defined on a common domain E is said to *converge uniformly* to a limit function g if, for each $\epsilon > 0$, there is a number $N > 0$ such that, for all $j > N$, it holds that $|g_j(x) - g(x)| < \epsilon$ for every $x \in E$. The key point is that the degree of closeness of $g_j(x)$ to $g(x)$ is independent of $x \in E$.

Let $f_j : U \to \mathbb{C}$, $n = 1, 2, 3 \ldots$, be a sequence of holomorphic functions on a region U in $\mathbb{C}$. Suppose that there is a function $f : U \to \mathbb{C}$ such that, for each compact subset E (a compact set is one that is closed and bounded—see Figure 4.2) of U, the restricted sequence $f_j|_E$ converges uniformly to $f|_E$. Then f is holomorphic on U. [In particular, $f \in C^\infty(U)$.]

One may see this last assertion by examining the Cauchy integral formula:

$$f_j(z) = \frac{1}{2\pi i} \oint \frac{f_j(\zeta)}{\zeta - z}\, d\zeta\,.$$

Now we may let $j \to \infty$, and invoke the uniform convergence to pass the limit under the integral sign on the right (see [KRA2] or [RUD1]). The result is

$$\begin{aligned}
\lim_{j\to\infty} f_j(z) &= \lim_{j\to\infty} \frac{1}{2\pi i} \oint \frac{f_j(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \oint \lim_{j\to\infty} \frac{f_j(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \oint \frac{\lim_{j\to\infty} f_j(\zeta)}{\zeta - z}\, d\zeta
\end{aligned}$$

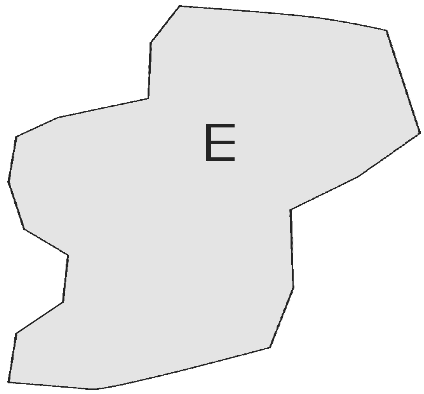

Figure 4.2: A compact set is closed and bounded.

or

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \, d\zeta \, .$$

The right-hand side is plainly a holomorphic function of z (simply differentiate under the integral sign, or apply Morera's theorem). Thus f is holomorphic.

If f_j, f, U are as in the preceding paragraph, then, for any $k \in \{0, 1, 2, \dots\}$, we have

$$\left(\frac{\partial}{\partial z}\right)^k f_j(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z) \tag{4.13}$$

uniformly on compact sets. This again follows from an examination of the Cauchy integral formula (or from the Cauchy estimates). We omit the details.

4.1.6 The Power Series Representation of a Holomorphic Function

The ideas being considered in this section can be used to develop our understanding of power series. A *power series*

$$\sum_{n=0}^{\infty} a_n (z - P)^n \tag{4.14}$$

is defined to be the limit of its *partial sums*

$$S_N(z) = \sum_{n=0}^{N} a_n(z-P)^n. \tag{4.15}$$

We say that the partial sums *converge* to the sum of the entire series.

Any given power series has a *disc of convergence*. More precisely, let

$$r = \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}}. \tag{4.16}$$

The power series (4.15) will then certainly converge on the disc $D(P,r)$; the convergence will be absolute and uniform (by the root test) on any disc $\overline{D}(P,r')$ with $r' < r$.

For clarity, we should point out that in many examples the sequence $|a_j|^{1/j}$ actually converges as $j \to \infty$. Then we may take r to be equal to $1/\lim_{j\to\infty} |a_j|^{1/j}$. The reader should be aware, however, that in case the sequence $\{|a_j|^{1/j}\}$ does not converge, then one must use the more formal definition (4.16) of r. See [KRA2], [RUD1].

Of course the partial sums, being polynomials, are holomorphic on *any* disc $D(P,r)$. If the disc of convergence of the power series is $D(P,r)$, then let f denote the function to which the power series converges. Then, for any $0 < r' < r$, we have that

$$S_N(z) \to f(z), \tag{4.17}$$

uniformly on $\overline{D}(P,r')$. We can conclude immediately that $f(z)$ is holomorphic on $D(P,r)$. Moreover, we know that

$$\left(\frac{\partial}{\partial z}\right)^k S_N(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z). \tag{4.18}$$

This shows that a differentiated power series has a disc of convergence at least as large as the disc of convergence (with the same center) of the original series, and that the differentiated power series converges on that disc to the derivative of the sum of the original series. In fact, the differentiated series has exactly the same radius of convergence as the original.

The most important fact about power series for complex function theory is this: If f is a holomorphic function on a domain $U \subseteq \mathbb{C}$, if $P \in U$, and if the disc $D(P, r)$ lies in U, then f may be represented as a convergent power series on $D(P, r)$. Explicitly, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-P)^n .$$

The reason that any holomorphic f has a power series expansion again relies on the Cauchy formula. If f is holomorphic on U and $\overline{D}(P, r) \subseteq U$, then we write, for $z \in D(P, r)$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - P} \cdot \frac{1}{1 - \frac{z-P}{\zeta-P}}\, d\zeta . \end{aligned} \tag{4.19}$$

Observe that $|(z-P)/(\zeta-P)| < 1$. So we may expand the second fraction in a power series:

$$\frac{1}{1 - \frac{z-P}{\zeta-P}} = \sum_{j=0}^{\infty} \left(\frac{z-P}{\zeta-P} \right)^j .$$

Substituting this information into (4.19) yields

$$\begin{aligned} &= \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - P} \cdot \sum_{j=0}^{\infty} \left(\frac{z-P}{\zeta-P} \right)^j d\zeta \\ &= \sum_{j=0}^{\infty} (z-P)^j \cdot \left[\frac{1}{2\pi i} \oint_{D(P,r)} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta \right] \\ &= \sum_{j=0}^{\infty} (z-P)^j \cdot \frac{f^{(j)}(P)}{j!} . \end{aligned} \tag{4.20}$$

We have used here standard results about switching series and integrals, for which see [KRA2] or [RUD1].

The last formula gives us an explicit power series expansion for the holomorphic function f. It further reveals explicitly that the coefficient of $(z-P)^j$ (that is, the expression in brackets) is $f^{(j)}(P)/j!$.

Let us now examine the question of calculating the power series expansion from a slightly different point of view. If we suppose in advance that f has a convergent power series expansion on the disc $D(P, r)$, then we may write

$$f(z) = a_0 + a_1(z-P) + a_2(z-P)^2 + a_3(z-P)^3 + \cdots . \tag{4.21}$$

Now let us evaluate both sides at $z = P$. We see immediately that $f(P) = a_0$.

Next, differentiate both sides of (4.21). The result is

$$f'(z) = a_1 + 2a_2(z - P) + 3a_3(z - P)^2 + \cdots .$$

Again, evaluate both sides at $z = P$. The result is $f'(P) = a_1$.

We may differentiate one more time and evaluate at $z = P$ to learn that $f''(P) = 2a_2$. Continuing in this manner, we discover that $f^{(k)}(P) = k!a_k$, where the superscript (k) denotes k derivatives.

We have discovered a convenient and elegant formula for the power series coefficients:

$$a_k = \frac{f^{(k)}(P)}{k!} . \tag{4.22}$$

This is consistent with what we learned in (4.20).

EXAMPLE 39 Let us determine the power series for $f(z) = z \sin z$ expanded about the point $P = \pi$. We begin by calculating

$$\begin{aligned} f'(z) &= \sin z + z \cos z \\ f''(z) &= 2\cos z - z \sin z \\ f'''(z) &= -3\sin z - z\cos z \\ f^{(iv)} &= -4\cos z + z \sin z \end{aligned}$$

and, in general,

$$f^{(2\ell+1)}(z) = (-1)^\ell (2\ell + 1) \sin z + (-1)^\ell z \cos z$$

and

$$f^{(2\ell)}(z) = (-1)^{\ell+1}(2\ell) \cos z + (-1)^\ell z \sin z .$$

Evaluating at π, and using formula (4.22), we find that

$$\begin{aligned} a_0 &= 0 \\ a_1 &= -\pi \\ a_2 &= -1 \\ a_3 &= \frac{\pi}{3!} \\ a_4 &= \frac{1}{3!} \\ a_5 &= -\frac{\pi}{5!} \\ a_6 &= -\frac{1}{5!} , \end{aligned}$$

and, in general,

$$a_{2\ell} = \frac{(-1)^\ell}{(2\ell - 1)!}$$

and

$$a_{2\ell+1} = (-1)^{\ell+1} \frac{\pi}{2\ell + 1} .$$

In conclusion, the power series expansion for $f(z) = z \sin z$, expanded about the point $P = \pi$, is

$$\begin{aligned} f(z) &= -\pi(z-\pi) - (z-\pi)^2 + \frac{\pi}{3!} \cdot (z-\pi)^3 + \frac{1}{3!} \cdot (z-\pi)^4 \\ &\quad - \frac{\pi}{5!} \cdot (z-\pi)^5 - \frac{1}{5!} \cdot (z-5)^6 + - \cdots \\ &= \pi \sum_{\ell=0}^{\infty} (-1)^{\ell+1} \frac{(z-\pi)^{2\ell+1}}{(2\ell+1)!} + \sum_{\ell=1}^{\infty} (-1)^\ell \frac{(z-\pi)^{2\ell}}{(2\ell-1)!} . \end{aligned}$$

□

In summary, we have an explicit way of calculating the power series expansion of any holomorphic function f about a point P of its domain, and we have an *a priori* knowledge of the disc on which the power series representation will converge.

Sometimes one can derive a power series expansion by simple algebra and calculus tricks—thereby avoiding the tedious calculation of coefficents that we have just illustrated. An example will illustrate the technique:

Example 40 Let us derive a power series expansion about 0 of the function

$$f(z) = \frac{z^2}{(1 - z^2)^2} .$$

It is a standard fact from calculus that

$$\frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$$

for any $|\alpha| < 1$. Letting $\alpha = z^2$ yields

$$\frac{1}{1 - z^2} = 1 + z^2 + z^4 + z^6 + \cdots .$$

Now a result from real analysis [KRA2] tells us that power series may be differentiated term by term. Thus

$$\frac{2z}{(1-z^2)^2} = 2z + 4z^3 + 6z^5 + \cdots .$$

Finally, multiplying both sides by $z/2$, we find that

$$\frac{z^2}{(1-z^2)^2} = 2z^2 + 4z^4 + 6z^6 + \cdots = \sum_{j=1}^{\infty} 2j \cdot z^{2j} .$$

□

4.1.7 Table of Elementary Power Series

The table below presents a summary of elementary power series expansions.

Table of Elementary Power Series

Function	**Power Series abt. 0**	**Disc of Convergence**
$\frac{1}{1-z}$	$\sum_{n=0}^{\infty} z^n$	$\{z : \|z\| < 1\}$
$\frac{1}{(1-z)^2}$	$\sum_{n=1}^{\infty} nz^{n-1}$	$\{z : \|z\| < 1\}$
$\cos z$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	all z
$\sin z$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$	all z
e^z	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$	all z
$\log(z+1)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1}$	$\{z : \|z\| < 1\}$
$(z+1)^\beta$	$\sum_{n=0}^{\infty} \binom{\beta}{n} z^n$	$\{z : \|z\| < 1\}$

Exercises

1. Calculate the power series expansion about 0 of $f(z) = \sin z^3$. Now calculate the expansion about π.

2. Calculate the power series expansion about $\pi/2$ of $g(z) = \tan[z/2]$. Now calculate the expansion about 0.

3. Calculate the power series expansion about 2 of $h(z) = z/(z^2 - 1)$.

4. Suppose that f is an entire function, k is a positive integer, and

$$|f(z)| \leq C(1 + |z|^k)$$

for all $z \in \mathbb{C}$. Prove that f must be a polynomial of degree at most k.

5. Suppose that f is an entire function, p is a polynomial, and f/p is bounded. What can you conclude about f?

6. Let $0 < m < k$ be integers. Give an example of a polynomial of degree k that has just m distinct roots.

7. Suppose that the polynomial p has a double root at the complex value z_0. Prove that $p(z_0) = 0$ and $p'(z_0) = 0$.

8. Suppose that the polynomial p has a simple zero at z_0 and let γ be a simple closed, continuously differentiable curve that encircles z_0 (oriented in the counterclockwise direction). What can you say about the value of

$$\frac{1}{2\pi i} \oint_\gamma \frac{p'(\zeta)}{p(\zeta)} \, d\zeta \, ?$$

[**Hint:** Try this first with the polynomials $p(z) = z$, $p(z) = z^2$, and $p(z) = z^3$.]

9. Let $\Omega \subseteq \mathbb{C}$ be a domain and let $\{f_n\}$ be holomorphic functions on Ω. Assume that the sequence $\{f_n\}$ converges uniformly on Ω. Prove that, if K is any closed, bounded set in Ω and m is a positive integer, then the sequence $f_n^{(m)}$ will converge uniformly on K.

10. Prove a version of the Cauchy estimates for harmonic functions.

11. For each k, M, r, give an example to show that the Cauchy estimates are sharp. That is, Find a function for which the inequality is an equality.

12. Prove this sharpening of Liouville's theorem: *If f is an entire function and $|f(z)| \leq C|z|^{1/2} + D$ for all z and for some constants C, D then f is constant.* How much can you increase the exponent $1/2$ and still draw the same conclusion?

13. Suppose that $p(z)$ is a polynomial of degree k with leading coefficient 1. Assume that all the zeros of p lie in unit disc. Prove that, for z sufficiently large, $|p(z)| \geq 9|z|^k/10$.

14. Let f be a holomorphic function defined on some open region $U \subseteq \mathbb{C}$. Fix a point $P \in U$. Prove that the power series expansion of f about P will converge absolutely and uniformly on any disc $\overline{D}(P, r)$ with $r < \operatorname{dist}(P, \partial U)$.

15. Let $0 \leq r \leq \infty$. Fix a point $P \in \mathbb{C}$. Give an example of a complex power series, centered at $P \in \mathbb{C}$, with radius of convergence precisely r.

16. We know from the elementary theory of geometric series that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots .$$

Use this model, together with differentiation of series, to find the power series expansion about 0 for

$$\frac{1}{(1-w^2)^2} .$$

17. Use the idea of the last exercise to find the power series expansion about 0 of the function

$$\frac{1-z^2}{(1+z^2)^2} .$$

18. Write a `MatLab` routine to calculate the power series expansion of a given holomorphic function f about a base point P in the complex plane. Your routine should allow you to specify in advance the order of the partial sum (or Taylor polynomial) of the power series that you will generate.

19. Write a second `MatLab` routine to calculate the error term when calculating the Taylor polynomial in the last example. This will necessitate your specifying a disc of convergence on which to work.

20. A simple harmonic oscillator satisfies the differential equation

$$f''(z) + f(z) = 0\,.$$

Guess a solution $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Plug this guess into the differential equation and solve for the a_j. What power series results? Can you recognize this series as a familiar function (or perhaps two functions) in closed form?

21. Apply the technique of the preceding exercise to the differential equation

$$f'(z) - 2f(z) = 0\,.$$

4.2 The Zeros of a Holomorphic Function

4.2.1 The Zero Set of a Holomorphic Function

Let f be a holomorphic function. If f is not identically zero, then it turns out that f cannot vanish at too many points. This once again bears out the dictum that holomorphic functions are a lot like polynomials. To give this notion a precise formulation, we need to recall the topological notion of connectedness (Section 1.2.2). An open set $W \subseteq \mathbb{C}$ is *connected* if it is not possible to find two disjoint, nonempty open sets U, V in $\mathbb{C}$ such that $U \cap W \neq \emptyset$, $V \cap W \neq \emptyset$, and

$$W = (U \cap W) \cup (V \cap W)\,. \tag{4.23}$$

[In the special context of open sets in the plane, it turns out that connectedness is equivalent to the condition that any two points of W may be connected by a curve that lies entirely in W—see the discussion in Section 1.2.3 on path-connectedness.] Now we have:

Discreteness of the Zeros of a Holomorphic Function

> Let $U \subseteq \mathbb{C}$ be a connected (Section 1.2.2) open set and let $f : U \to \mathbb{C}$ be holomorphic. Let the zero set of f be $\mathcal{Z} = \{z \in U : f(z) = 0\}$. If there are a $z_0 \in U$ and $\{z_j\}_{j=1}^{\infty} \subseteq \mathcal{Z} \setminus \{z_0\}$ such that $z_j \to z_0$, then $f \equiv 0$ on U.

A full proof of this remarkable result may be found in [AHL] or [GRK]. The justification is as follows. Of course f must vanish at z_0—say that it vanishes to order[1] $k > 0$. This means that $f(z) = (z - z_0)^k \cdot g(z)$ and g *does not* vanish at z_0. But then observe that $g(z_j) = 0$ for $j = 1, 2, \ldots$. It follows by continuity that $g(z_0) = 0$. That is a contradiction.

Let us formulate the result in topological terms. We recall (see [KRA2], [RUD1]) that a point z_0 is said to be an *accumulation point* of a set $\mathcal{Z}$ if there is a sequence $\{z_j\} \subseteq \mathcal{Z} \setminus \{z_0\}$ with $\lim_{j\to\infty} z_j = z_0$. Then the theorem is equivalent to the statement: If $f : U \to \mathbb{C}$ is a holomorphic function on a connected (Section 1.2.2) open set U and if $\mathcal{Z} = \{z \in U : f(z) = 0\}$ has an accumulation point *in* U, then $f \equiv 0$.

4.2.2 Discrete Sets and Zero Sets

There is still more terminology concerning the discussion of the zero set of a holomorphic function in Section 4.2.1. A set S is said to be *discrete* if for each $s \in S$ there is an $\epsilon > 0$ such that $D(s, \epsilon) \cap S = \{s\}$.

People also say, in a slight abuse of language, that a discrete set has points that are "isolated" or that S contains only "isolated points." The result in Section 4.2.1 thus asserts that if f is a nonconstant holomorphic function on a connected open set, then its zero set is discrete or, less formally, the zeros of f are isolated.

Example 41 It is important to realize that the result in Section 4.2.1 does *not* rule out the possibility that the zero set of f can have accumulation points in $\mathbb{C} \setminus U$; in particular, a nonconstant holomorphic function on an open set U can indeed have zeros accumulating at a point of ∂U. Consider, for instance, the function $f(z) = \sin(1/[1 - z])$ on the unit disc. The zeros

[1] If a holomorphic function vanishes at a point P, then it vanishes to a certain order (see Section 6.1.3). Thus $f(z) = (z - P)^k \cdot g(z)$ for some holomorphic function g that *does not* vanish at P. This claim follows from the theory of power series.

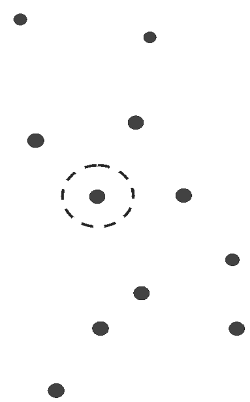

Figure 4.3: A discrete set.

of this f include $\{1 - 1/[n\pi]\}$, and these accumulate at the boundary point 1. Figure 4.3 illustrates a discrete set. Figure 4.4 shows a zero set with a boundary accumulation point. $\square$

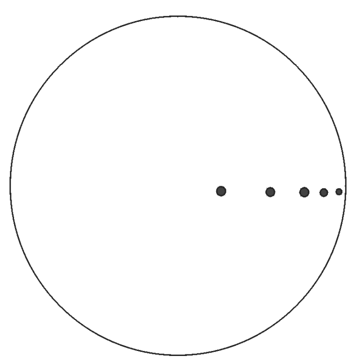

Figure 4.4: A zero set with a boundary accumulation point.

EXAMPLE 42 The function $g(z) = \sin z$ has zeros at $z = k\pi$. Since the domain of g is the entire plane, these infinitely many zeros have no accumulation point so there is no contradiction in that g is not identically zero.

By contrast, the domain $U = \{z = x + iy \in \mathbb{C} : -1 < x < 1, -1 < y < 1\}$ is bounded. If f is holomorphic on U then a holomorphic f can only have finitely many zeros in any compact subset of U. If a holomorphic g has infinitely many zeros, then those zeros can only accumulate at a boundary point. Examples are

$$f(z) = \left(z - \frac{1}{2}\right)^2 \cdot \left(z + \frac{i}{2}\right)^3 ,$$

with zeros at $1/2$ and $-i/2$, and

$$g(z) = \cos\left(\frac{i}{i-z}\right) .$$

Notice that the zeros of g are at $z_k = i\frac{(2k+1)\pi - 2}{(2k+1)\pi}$. There are infinitely many of these zeros, and they accumulate *only* at i. □

4.2.3 Uniqueness of Analytic Continuation

A consequence of the preceding basic fact (Section 4.2.1) about the zeros of a holomorphic function is this: Let $U \subseteq \mathbb{C}$ be a connected open set and $D(P, r) \subseteq U$. If f is holomorphic on U and $f\big|_{D(P,r)} \equiv 0$, then we may conclude that $f \equiv 0$ on U. This is so because the disc $D(P, r)$ certainly contains an interior accumulation point (merely take $z_j = P + r/j$ and $z_j \to z_0 = P$) hence f must be identically equal to 0.

Here are some further corollaries:

1. Let $U \subseteq \mathbb{C}$ be a connected open set. Let f, g be holomorphic on U. If $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U, then $f \equiv g$. For simply apply our uniqueness result to the difference function $h(z) = f(z) - g(z)$.

2. Let $U \subseteq \mathbb{C}$ be a connected open set and let f, g be holomorphic on U. If $f \cdot g \equiv 0$ on U, then either $f \equiv 0$ on U or $g \equiv 0$ on U. To see this, we notice that if neither f nor g is identically 0 then there is either a point p at which $f(p) \neq 0$ or there is a point p' at which $g(p') \neq 0$. Say it is the former. Then, by continuity, $f(p) \neq 0$ on an entire disc centered at p. But then it follows, since $f \cdot g \equiv 0$, that $g \equiv 0$ on that disc. Thus it must be, by the remarks in the first paragraph of this section, that $g \equiv 0$.

3. We have the following powerful result:

> Let $U \subseteq \mathbb{C}$ be connected and open and let f be holomorphic on U. If there is a $P \in U$ such that
>
> $$\left(\frac{\partial}{\partial z}\right)^n f(P) = 0$$
>
> for every $n \in \{0, 1, 2, \dots\}$, then $f \equiv 0$.

The reason for this result is simplicity itself: The power series expansion of f about P will have all zero coefficients. Since the series certainly converges to f on some small disc centered at P, the function is identically equal to 0 on that disc. Now, by our uniqueness result for zero sets, we conclude that f is identically 0.

4. If f and g are entire holomorphic functions and if $f(x) = g(x)$ for all $x \in \mathbb{R} \subseteq \mathbb{C}$, then $f \equiv g$. It also holds that functional identities that are true for all real values of the variable are also true for complex values of the variable (Figure 4.5). For instance,

$$\sin^2 z + \cos^2 z = 1 \qquad \text{for all} \quad z \in \mathbb{C} \tag{4.24}$$

because the identity is true for all $z = x \in \mathbb{R}$. This is an instance of the "principle of persistence of functional relations"—see [GRK].

Of course these statements are true because if U is a connected open set having nontrivial intersection with the x-axis and if f holomorphic on U vanishes on that intersection, then the zero set certainly has an interior accumulation point. Again, see Figure 4.5.

Exercises

1. Let f and g be entire functions and suppose that $f(x+ix^2) = g(x+ix^2)$ whenever x is real. Prove that $f(z) = g(z)$ for all z.

2. Let $p_n \in D$ be defined by $p_n = 1 - 1/n$, $n = 1, 2, \dots$. Suppose that f and g are holomorphic on the disc D and that $f(p_n) = g(p_n)$ for every n. Does it follow that $f \equiv g$?

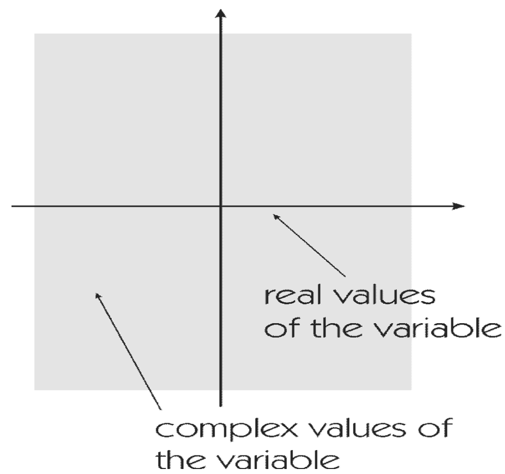

Figure 4.5: The principle of persistence of functional relations.

3. The real axis cannot be the zero set of a not-identically-zero holomorphic function on the entire plane. But it *can* be the zero set of a not-identically-zero harmonic function on the plane. Prove both of these statements.

4. Give an example of a holomorphic function on the disc D that vanishes *on an infinite set in* D but which is not identically zero.

5. Let f and g be holomorphic functions on the disc D. Let $\mathcal{P}$ be the zero set of f and let $\mathcal{Q}$ be the zero set of g. Is $\mathcal{P} \cup \mathcal{Q}$ the zero set of some holomorphic function on D? Is $\mathcal{P} \cap \mathcal{Q}$ the zero set of some holomorphic function on D? Is $\mathcal{P} \setminus \mathcal{Q}$ the zero set of some holomorphic function on D?

6. Give an example of an entire function that vanishes at every point of the form $0 + ik$ and every point of the form $k + i0$, for $k \in \mathbb{Z}$.

7. Let $c \in C$ satisfy $|c| < 1$. The function

$$\varphi_c(z) \equiv \frac{z - c}{1 - \overline{c}z}$$

is called a *Blaschke factor* at the point c. Verify these properties of φ_c:

- $|\varphi_c(z)| = 1$ whenever $|z| = 1$;
- $|\varphi_c(z)| < 1$ whenever $|z| < 1$;
- $\varphi_c(c) = 0$;
- $\varphi_c \circ \varphi_{-c}(z) \equiv z$.

8. Give an example of a holomorphic function on $\Omega \equiv D \setminus \{0\}$ such that $f(1/n) = 0$ for $n = \pm 1, \pm 2, \ldots$, yet f is not identically 0.

9. Suppose that f is a holomorphic function on the disc and $f(z)/z \equiv 1$ for z real (with the meaning of this statement for $z = 0$ suitably interpreted). What can you conclude about f?

10. Let f, g be holomorphic on the disc D and suppose that $[f \cdot g](z) = 0$ for $z = 1/2, 1/3, 1/4, \ldots$. Prove that either $f \equiv 0$ or $g \equiv 0$.

11. Write a `MatLab` routine that will implement Newton's method to find the zeros of a given holomorphic function (see [BLK] for the basic idea of Newton's method). Enumerate the zeros by order of modulus.

12. Refine the `MatLab` routine from the last exercise to calculate the *order* of each zero. You will want to exploit the following simple-minded observations:

(a) The holomorphic function f has a simple zero at P if and only if $f(P) = 0$ but $f'(P) \neq 0$.

(b) The holomorphic function f has a zero of order two at P if $f(P) = 0$, $f'(P) = 0$, yet $f''(P) \neq 0$.

(c) The holomorphic function f has a zero of order k at P if $f(P) = 0$, $f'(P) = 0$, $\ldots$, $f^{(k-1)}(P) = 0$, yet $f^{(k)}(P) \neq 0$.

13. The holomorphic function $f(z) = u(z) + iv(z) \approx (u(x, y), v(x, y))$, describes a fluid flow on the unit disc. The function f is of course conformal. What do the zeros of f signify from a physical point of view? According to our uniqueness theorem, the values $f(x + i0)$ uniquely determine f. What is the physical interpretation of this statement?

14. Interpret the statement that if the zero set of a holomorphic function has an interior accumulation point then it is identically zero from a physical point of view. Refer to the preceding exercise.

Chapter 5

Isolated Singularities and Laurent Series

5.1 The Behavior of a Holomorphic Function near an Isolated Singularity

5.1.1 Isolated Singularities

It is often important to consider a function that is holomorphic on a punctured open set $U \setminus \{P\} \subset \mathbb{C}$. Refer to Figure 5.1.

In this chapter we shall obtain a new kind of infinite series expansion which generalizes the idea of the power series expansion of a holomorphic function about a (nonsingular) point—see Section 4.1.6. We shall in the process completely classify the behavior of holomorphic functions near an isolated singular point (Section 5.1.3).

5.1.2 A Holomorphic Function on a Punctured Domain

Let $U \subseteq \mathbb{C}$ be an open set and $P \in U$. Suppose that $f : U \setminus \{P\} \to \mathbb{C}$ is holomorphic. In this situation we say that f has an *isolated singular point* (or *isolated singularity*) at P. The implication of the phrase is usually just that f is defined and holomorphic on some such "deleted neighborhood" of P. The specification of the set U is of secondary interest; we wish to consider the behavior of f "near P."

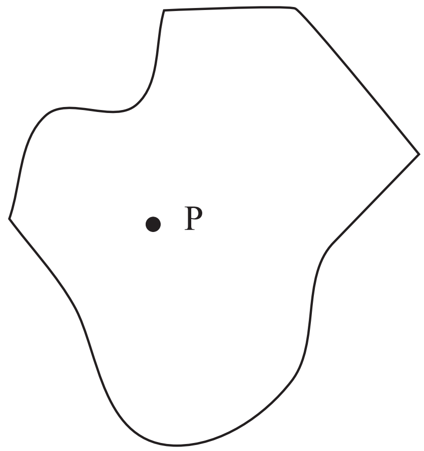

Figure 5.1: A punctured domain.

5.1.3 Classification of Singularities

There are three possibilities for the behavior of f near P that are worth distinguishing:

(1) $|f(z)|$ is bounded on $D(P,r) \setminus \{P\}$ for some $r > 0$ with $D(P,r) \subseteq U$; that is, there is some $r > 0$ and some $M > 0$ such that $|f(z)| \leq M$ for all $z \in U \cap D(P,r) \setminus \{P\}$.

(2) $\lim_{z \to P} |f(z)| = +\infty$.

(3) Neither **(1)** nor **(2)**.

Clearly these three possibilities cover all conceivable situations. It is our job now to identify extrinsically what each of these three situations entails.

5.1.4 Removable Singularities, Poles, and Essential Singularities

We shall see momentarily that, if case **(1)** holds, then f has a limit at P that extends f so that it is holomorphic on all of U. It is commonly said in this circumstance that f has a *removable singularity* at P. In case **(2)**, we will say that f has a *pole* at P. In case **(3)**, the function f will be said to have an *essential singularity* at P. Our goal in this and the next subsection is to understand **(1)**–**(3)** in some further detail.

5.1.5 The Riemann Removable Singularities Theorem

Let $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ be holomorphic and bounded. Then

(a) $\lim_{z \to P} f(z)$ exists.

(b) The function $\widehat{f} : D(P, r) \to \mathbb{C}$ defined by

$$\widehat{f}(z) = \begin{cases} f(z) & \text{if} \quad z \neq P \\ \lim\limits_{\zeta \to P} f(\zeta) & \text{if} \quad z = P \end{cases}$$

is holomorphic.

The reason that this theorem is true is the following. We may assume without loss of generality—by a simple translation of coordinates—that $P = 0$. Now consider the auxiliary function $g(z) = z^2 \cdot f(z)$. Then one may verify by direct application of the derivative that g is continuously differentiable at all points—including the origin. Furthermore, we may calculate with $\partial/\partial\overline{z}$ to see that g satisfies the Cauchy-Riemann equations. Thus g is holomorphic. But the very definition of g shows that g vanishes to order 2 at 0. Thus the power series expansion of g about 0 cannot have a constant term and cannot have a linear term. It follows that

$$g(z) = a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots = z^2(a_2 + a_3 z + a_4 z^2 + \cdots) \equiv z^2 \cdot h(z) .$$

Notice that the function h is holomorphic—we have in fact given its power series expansion explicitly. But now, for $z \neq 0$, $h(z) = g(z)/z^2 = f(z)$. Thus we see that h is the holomorphic continuation of f (across the singularity at 0) that we seek.

5.1.6 The Casorati-Weierstrass Theorem

If $f : D(P, r_0) \setminus \{P\} \to \mathbb{C}$ is holomorphic and P is an essential singularity of f, then $f(D(P, r) \setminus \{P\})$ is dense in $\mathbb{C}$ for any $0 < r < r_0$.

The proof of this result is a nice application of the Riemann removable singularities theorem. For suppose to the contrary that $f(D(P,r) \setminus \{P\})$ is *not* dense in $\mathbb{C}$. This means that there is a disc $D(Q,s)$ that is *not* in the range of f. So consider the function

$$g(z) = \frac{1}{f(z) - Q} .$$

We see that the denominator of this function is bounded away from 0 (by s) hence the function g itself is bounded near P. So we may apply Riemann's theorem and conclude that g continues analytically across the point P. And the value of g near P cannot be 0. But then it follows that

$$f(z) = \frac{1}{g(z)} + Q$$

extends analytically across P. That contradicts the hypothesis that P is an essential singularity for f.

5.1.7 Concluding Remarks

Now we have seen that, at a removable singularity P, a holomorphic function f on $D(P,r_0) \setminus \{P\}$ can be continued to be holomorphic on all of $D(P,r_0)$. And, near an essential singularity at P, a holomorphic function g on $D(P,r_0) \setminus \{P\}$ has image that is dense in $\mathbb{C}$. The third possibility, that h has a *pole* at P, has yet to be described. Suffice it to say that, at a pole (case **(2)**), the limit of modulus the function is $+\infty$ hence the graph of the modulus of the function looks like a pole! See Figure 5.2. This case will be examined further in the next section.

We next develop a new type of doubly infinite series that will serve as a tool for understanding isolated singularities—especially poles.

Exercises

1. Discuss the singularities of these functions at 0:

(a) $f(z) = \dfrac{z^2}{1 - \cos z}$

(b) $g(z) = \dfrac{\sin z}{z}$

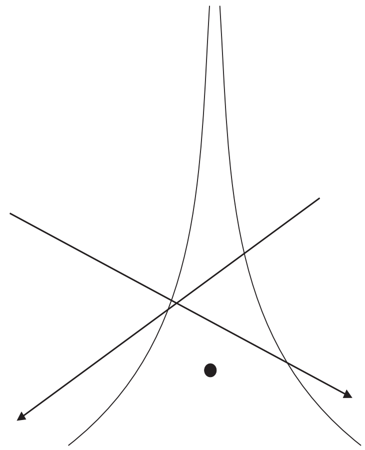

Figure 5.2: A pole.

(c) $h(z) = \dfrac{\sec z - 1}{\sin^2 z}$

(d) $f(z) = \dfrac{\log(1+z)}{z^2}$

(e) $g(z) = \dfrac{z^2}{e^z - 1}$

(f) $h(z) = \dfrac{\sin z - z}{z^2}$

(g) $f(z) = e^{1/z}$

2. If f has a pole at P and g has a pole at P does it then follow that $f \cdot g$ has a pole at P? How about $f + g$?

3. If f has a pole at P and g has an essential singularity at P does it then follow that $f \cdot g$ has an essential singularity at P? How about $f + g$?

4. Suppose that f is holomorphic in a deleted neighborhood $D(P, r) \setminus \{P\}$ of P and that f is not bounded near P. Assume further that $(z - P)^2 \cdot f$ *is* bounded (near P). Prove that f has a pole at p. What happens if the exponent 2 is replaced by some other positive integer?

5. Suppose that f is holomorphic in a deleted neighborhood $D(P, r) \setminus \{P\}$

of P and that $(z-P)^k \cdot f$ is unbounded for very choice of positive integer k. What conclusion can you draw about the singularity of f at P?

6. Write a `MatLab` routine to test whether a holomorphic function defined on a deleted neighborhood $D(0,r) \setminus \{0\}$ of the origin has a holomorphic continuation past 0. Of course use the Riemann removable singularities theorem as a tool.

7. Let f be a holomorphic function defined on a deleted neighborhood $D(0,r) \setminus \{0\}$ of the origin. Devise a `MatLab` routine to test whether f has a pole or an essential singularity at 0. [**Hint:** Bear in mind that a function *blows up* at a pole, whereas (by contrast) the function takes a dense set of values on any neighborhood of 0 when it has an essential singularity there. Use these facts as the basis for your `MatLab` testing routine.]

8. In the Riemann removable singularities theorem, the hypothesis of boundedness is not essential. Describe a weaker hypothesis that will give (with the same proof!) the same conclusion.

9. A differential equation describes an incompressible fluid flow in a deleted neighborhood of the point P in the complex plane. The solution of the equation exhibits a removable singularity at P. What does this tell you about the physical nature of the system?

10. A differential equation describes an incompressible fluid flow in a deleted neighborhood of the point P in the complex plane. The solution of the equation exhibits an essential pole at P. What does this tell you about the physical nature of the system?

11. A differential equation describes an incompressible fluid flow in a deleted neighborhood of the point P in the complex plane. The solution of the equation exhibits an essential singularity at P. What does this tell you about the physical nature of the system?

5.2 Expansion around Singular Points

5.2.1 Laurent Series

A *Laurent series* on $D(P,r)$ is a (formal) expression of the form

$$\sum_{j=-\infty}^{+\infty} a_j(z-P)^j \,. \tag{5.1}$$

Observe that the sum extends from $j=-\infty$ to $j=+\infty$. Further note that the individual summands are each defined for all $z \in D(P,r) \setminus \{P\}$.

5.2.2 Convergence of a Doubly Infinite Series

To discuss convergence of Laurent series, we must first make a general agreement as to the meaning of the convergence of a "doubly infinite" series $\sum_{j=-\infty}^{+\infty} \alpha_j$. We say that such a series *converges* if $\sum_{j=0}^{+\infty} \alpha_j$ and $\sum_{j=1}^{+\infty} \alpha_{-j} = \sum_{j=-\infty}^{-1} \alpha_j$ converge in the usual sense. In this case, we set

$$\sum_{-\infty}^{+\infty} \alpha_j = \left(\sum_{j=0}^{+\infty} \alpha_j\right) + \left(\sum_{j=1}^{+\infty} \alpha_{-j}\right) . \tag{5.2}$$

Thus a doubly infinite series converges precisely when the sum of its "positive part" (that is., the terms of positive index) converges and the sum of its "negative part" (that is, the terms of negative index) converges.

We can now present the analogues for Laurent series of our basic results about power series.

5.2.3 Annulus of Convergence

The set of convergence of a Laurent series is either an open set of the form $\{z : 0 \le r_1 < |z-P| < r_2\}$, together with perhaps some or all of the boundary points of the set, *or* a set of the form $\{z : 0 \le r_1 < |z-P| < +\infty\}$, together with perhaps some or all of the boundary points of the set. Such an open set is called an *annulus* centered at P. We shall let

$$D(P,+\infty) = \{z : |z-P| < +\infty\} = \mathbb{C} \,, \tag{5.3}$$

$$D(P,0) = \{z : |z-P| < 0\} = \emptyset \,, \tag{5.4}$$

and

$$\overline{D}(P,0) = \{P\} \,. \tag{5.5}$$

As a result, all (open) annuli (plural of "annulus") can be written in the form

$$D(P,r_2) \setminus \overline{D}(P,r_1) \,, \quad 0 \le r_1 \le r_2 \le +\infty \,. \tag{5.6}$$

In precise terms, the "domain of convergence" of a Laurent series is given as follows:

Let

$$\sum_{n=-\infty}^{+\infty} a_n(z-P)^n \tag{5.7}$$

be a doubly infinite series. There are (see (5.6)) unique nonnegative extended real numbers r_1 and r_2 (r_1 or r_2 may be $+\infty$) such that the series converges absolutely for all z with $r_1 < |z-P| < r_2$ and diverges for z with $|z-P| < r_1$ or $|z-P| > r_2$. Also, if $r_1 < s_1 \leq s_2 < r_2$, then $\sum_{n=-\infty}^{+\infty} |a_n(z-P)^n|$ converges uniformly on $\{z : s_1 \leq |z-P| \leq s_2\}$ and, consequently, $\sum_{n=-\infty}^{+\infty} a_n(z-P)^n$ converges absolutely and uniformly there.

5.2.4 Uniqueness of the Laurent Expansion

Let $0 \leq r_1 < r_2 \leq \infty$. If the Laurent series $\sum_{n=-\infty}^{+\infty} a_n(z-P)^n$ converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to a function f, then, for any r satisfying $r_1 < r < r_2$, and each $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta-P|=r} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta\,. \tag{5.8}$$

In particular, the a_n's are uniquely determined by f. We prove this result in Section 5.6.

We turn now to establishing that convergent Laurent expansions of functions holomorphic on an annulus do in fact exist.

5.2.5 The Cauchy Integral Formula for an Annulus

Suppose that $0 \leq r_1 < r_2 \leq +\infty$ and that $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic. Then, for each s_1, s_2 such that $r_1 < s_1 < s_2 < r_2$ and each $z \in D(P, s_2) \setminus \overline{D}(P, s_1)$, it holds that

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{5.9}$$

The easiest way to confirm the validity of this formula is to use a little manipulation of the Cauchy formula that we already know. Examine Figure 5.3. It shows a classical Cauchy contour for a holomorphic function with *no singularity* on a neighborhood of the curve and its interior. Now we simply

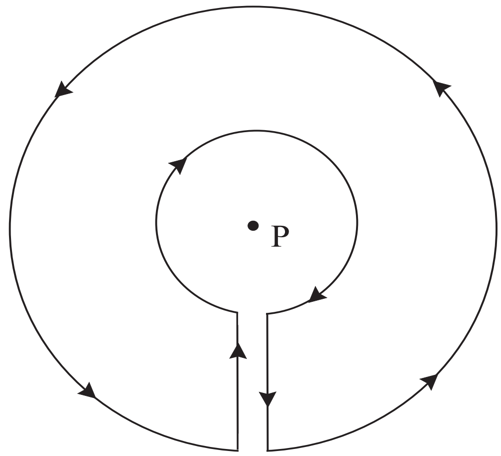

Figure 5.3: The Cauchy integral near an isolated singularity.

let the two vertical edges coalesce to form the Cauchy integral over two circles as in Figure 5.4.

5.2.6 Existence of Laurent Expansions

Now we have our main result:

If $0 \leq r_1 < r_2 \leq \infty$ and $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic, then there exist complex numbers a_j such that

$$\sum_{j=-\infty}^{+\infty} a_j (z-P)^j \tag{5.10}$$

converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to f. If $r_1 < s_1 < s_2 < r_2$, then the series converges absolutely and uniformly on $\overline{D}(P, s_2) \setminus D(P, s_1)$.

The series expansion is independent of s_1 and s_2. In fact, for each fixed $n = 0, \pm 1, \pm 2, \ldots$, the value of

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{n+1}}\, d\zeta \tag{5.11}$$

is independent of r provided that $r_1 < r < r_2$.

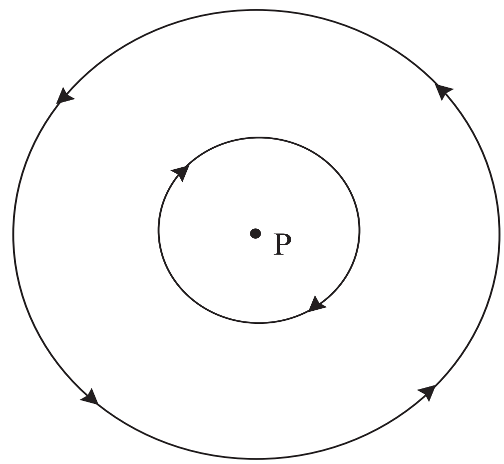

Figure 5.4: Annular Cauchy integral for an isolated singularity.

We may justify the Laurent expansion in the following manner.

If $0 \le r_1 < s_1 < |z-P| < s_2 < r_2$, then the two integrals on the right-hand side of the equation in (5.9) can each be expanded in a series. For the first integral we have

$$\begin{aligned}\oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{1-\frac{z-P}{\zeta-P}} \cdot \frac{1}{\zeta-P}\,d\zeta \\ &= \oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{\zeta-P} \sum_{j=0}^{+\infty} \frac{(z-P)^j}{(\zeta-P)^j}\,d\zeta \\ &= \oint_{|\zeta-P|=s_2} \sum_{j=0}^{+\infty} \frac{f(\zeta)(z-P)^j}{(\zeta-P)^{j+1}}\,d\zeta\,,\end{aligned}$$

where the geometric series expansion of

$$\frac{1}{1-(z-P)/(\zeta-P)}$$

converges because $|z-P|/|\zeta-P| = |z-P|/s_2 < 1$. In fact, since the value of $|(z-P)/(\zeta-P)|$ is independent of ζ, for $|\zeta - P| = s_2$, it follows that the geometric series converges uniformly.

Thus we may switch the order of summation and integration to obtain

$$\oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{+\infty} \left(\oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta \right) (z-P)^j \,.$$

For $s_1 < |z-P|$, similar arguments justify the formula

$$\begin{aligned}
\oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= -\oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{1-\frac{\zeta-P}{z-P}} \cdot \frac{1}{z-P}\, d\zeta \\
&= -\oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{z-P} \sum_{j=0}^{+\infty} \frac{(\zeta-P)^j}{(z-P)^j}\, d\zeta \\
&= -\sum_{j=0}^{+\infty} \left[\oint_{|\zeta-P|=s_1} f(\zeta)\cdot(\zeta-P)^j\, d\zeta \right] (z-P)^{-j-1} \\
&= -\sum_{j=-\infty}^{-1} \left[\oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta \right] (z-P)^j.
\end{aligned}$$

Thus

$$\begin{aligned}
2\pi i f(z) &= \sum_{j=-\infty}^{-1} \left[\oint_{|\zeta-P|=s_1} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta \right] (z-P)^j \\
&\quad + \sum_{j=0}^{+\infty} \left[\oint_{|\zeta-P|=s_2} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta \right] (z-P)^j,
\end{aligned}$$

as desired.

Certainly one of the important benefits of the proof we have just presented is that we have an explicit formula for the coefficients of the Laurent expansion:

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\, d\zeta, \quad \text{any} \quad r_1 < r < r_2 \,.$$

In Section 5.3.2 we shall give an even more practical means, with examples, for the calculation of Laurent coefficients.

5.2.7 Holomorphic Functions with Isolated Singularities

Now let us specialize what we have learned about Laurent series expansions to the case of $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ holomorphic, that is, to a holomorphic function with an isolated singularity. Thus we will be considering the Laurent expansion on a degenerate annulus of the form $D(P, r) \setminus \overline{D}(P, 0)$.

Let us review: If $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ is holomorphic, then f has a unique Laurent series expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - P)^j \tag{5.12}$$

that converges absolutely for $z \in D(P, r) \setminus \{P\}$. The convergence is uniform on compact subsets of $D(P, r) \setminus \{P\}$. The coefficients are given by

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(P,s)} \frac{f(\zeta)}{(\zeta - P)^{j+1}} \, d\zeta, \quad \text{any} \quad 0 < s < r. \tag{5.13}$$

5.2.8 Classification of Singularities in Terms of Laurent Series

There are three mutually exclusive possibilities for the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n (z - P)^n \tag{5.14}$$

about an isolated singularity P:

(5.15) $a_n = 0$ for all $n < 0$.
(5.16) For some $k \geq 1, a_n = 0$ for all $-\infty < n < -k$, but $a_{-k} \neq 0$.
(5.17) Neither **(i)** nor **(ii)** applies.

These three cases correspond exactly to the three types of isolated singularities that we discussed in Section 5.1.3: case (5.15) occurs if and only if P is a removable singularity; case (5.16) occurs if and only if P is a pole (of order k); and case (5.17) occurs if and only if P is an essential singularity.

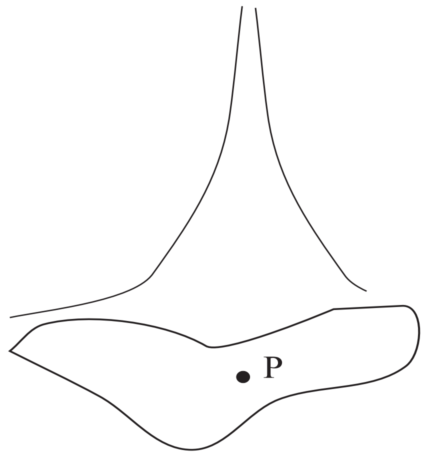

Figure 5.5: A pole at P.

To put this matter in other words: In case (5.15), we have a power series that converges, of course, to a holomorphic function. In case (5.16), our Laurent series has the form

$$\sum_{j=-k}^{\infty} a_j(z-P)^j = (z-P)^{-k}\sum_{j=-k}^{\infty} a_j(z-P)^{j+k} = (z-P)^{-k}\sum_{j=0}^{\infty} a_{j-k}(z-P)^j\,. \tag{5.18}$$

Since $a_{-k} \neq 0$, we see that, for z near P, the function defined by the series behaves like $a_{-k}\cdot(z-P)^{-k}$. In short, the function (in absolute value) blows up like $|z-P|^{-k}$ as $z \to P$. A graph in $(|z|, |f(z)|)$-space would exhibit a "pole-like" singularity. This is the source of the terminology "pole." See Figure 5.5. Case (5.17), corresponding to an essential singularity, is much more complicated; in this case there are infinitely many negative terms in the Laurent expansion and, by Casorati-Weierstrass (Section 5.1.6), they interact in a complicated fashion.

Picard's Great Theorem (see Glossary of Terms) tells us more about the behavior of a holomorphic function near an essential singularity.

Exercises

1. Derive the Laurent expansion for the function $g(z) = e^{1/z}$ about $z = 0$. Use your knowledge of the exponential function plus substitution.

2. Derive the Laurent expansion for the function $h(z) = \frac{\sin z}{z^3}$ about $z = 0$.

3. Derive the Laurent expansion for the function $f(z) = \frac{\sin z}{\cos z}$ about $z = \pi/2$. Use long division.

4. Verify that the functions

$$f(z) = e^{1/z}$$

and

$$g(z) = \cos(1/z)$$

each have an essential singularity at $z = 0$. Now determine the nature of the behavior of f/g at 0.

5. Suppose that the function f has an essential singularity at 0. Does it then follow that $1/f$ has an essential singularity at 0?

6. It is impossible to use a computer to determine whether a given function f has Laurent expansion with infinitely many terms of negative index at a given point P. Discuss other means for using `MatLab` to test f for the various types of singularities at P.

7. Explain using Laurent series why f and g could both have essential singularities at P yet $f - g$ may not have such a singularity at P. Does a similar analysis apply to $f \cdot g$?

8. Explain using Laurent series why f and g could both have poles at P yet $f - g$ may not have such a singularity at P. Does a similar analysis apply to $f \cdot g$?

9. Give an example of functions f and g, each of which has an essential singularity at 0, yet $f + g$ has a pole of order 1 at 0.

10. An incompressible fluid flow has singularity at the origin having the form

$$\frac{\sin z - z}{z^5} .$$

Discuss the nature of this singularity. What will be the behavior of the flow near the origin?

5.3 Examples of Laurent Expansions

5.3.1 Principal Part of a Function

When f has a pole at P, it is customary to call the negative power part of the Laurent expansion of f around P the *principal part* of f at P. (Occasionally we shall also use the terminology "Laurent polynomial.") That is, if

$$f(z) = \sum_{n=-k}^{\infty} a_n(z-P)^n \tag{5.19}$$

for z near P, then the *principal part* of f at P is

$$\sum_{n=-k}^{-1} a_n(z-P)^n. \tag{5.20}$$

EXAMPLE 43 The Laurent expansion about 0 of the function $f(z) = (z^2 + 1)/\sin(z^3)$ is

$$\begin{aligned}
f(z) &= (z^2+1)\cdot\frac{1}{\sin(z^3)} \\
&= (z^2+1)\cdot\frac{1}{z^3 - z^9/3! + z^{15}/5! - + \cdots} \\
&= (z^2+1)\cdot\frac{1}{z^3}\cdot\frac{1}{1 - z^6/3! + z^{12}/5! - + \cdots} \\
&= (z^2+1)\cdot\frac{1}{z^3}\cdot\left(1 + \frac{z^6}{3!} - + \cdots\right) \\
&= \frac{1}{z^3} + \frac{1}{z} + \text{ (a holomorphic function)}.
\end{aligned}$$

The principal part of f is $1/z^3 + 1/z$. □

EXAMPLE 44 For a second example, consider the function $f(z) = (z^2 + 2z +$

2) $\sin(1/(z+1))$. Its Laurent expansion about the point -1 is

$$\begin{aligned} f(z) &= ((z+1)^2+1)\cdot\left[\frac{1}{z+1}-\frac{1}{6(z+1)^3}+\frac{1}{120(z+1)^5}\right. \\ &\qquad \left.-\frac{1}{5040(z+1)^7}+-\cdots\right] \\ &= (z+1)+\frac{5}{6}\frac{1}{(z+1)}-\frac{19}{120}\frac{1}{(z+1)^3}+\frac{41}{5040}\frac{1}{(z+1)^5}-+\cdots\,. \end{aligned}$$

The principal part of f at the point -1 is

$$\frac{5}{6}\frac{1}{(z+1)}-\frac{19}{120}\frac{1}{(z+1)^3}+\frac{41}{5040}\frac{1}{(z+1)^5}-+\cdots\,. \tag{5.21}$$

□

As with power series (see Section 4.1.6), we can sometimes use calculus or algebra tricks to derive a Laurent series expansion. An example illustrates the idea:

Example 45 Let us derive the Laurent series expansion about 0 of the function

$$f(z)=\frac{1}{z^2(z+1)}\,.$$

We use the method of partial fractions (from calculus) to write the function as

$$f(z)=-\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z+1}=-\frac{1}{z}+\frac{1}{z^2}+\frac{1}{1-(-z)}\,.$$

Thus we see that the Laurent expansion of f about 0 is

$$f(z)=\frac{1}{z^2}-\frac{1}{z}+1+(-z)+(-z)^2+(-z)^3+\cdots\,.$$

In particular, the principal part of f at 0 is $1/z^2-1/z$ and the residue is -1. □

5.3.2 Algorithm for Calculating the Coefficients of the Laurent Expansion

Let f be holomorphic on $D(P,r) \setminus \{P\}$ and suppose that f has a pole of order k at P. Then the Laurent series coefficients a_n of f expanded about the point P, for $j = -k, -k+1, -k+2, \ldots$, are given by the formula

$$a_j = \frac{1}{(k+j)!} \left(\frac{\partial}{\partial z}\right)^{k+j} \left((z-P)^k \cdot f\right)\bigg|_{z=P} . \tag{5.22}$$

We begin by illustrating this formula, and provide the justification a bit later.

EXAMPLE 46 Let $f(z) = \cot z$. Let us calculate the Laurent coefficients of negative index for f at the point $P = 0$.

We first notice that

$$\cot z = \frac{\cos z}{\sin z} .$$

Since $\cos z = 1 - z^2/2! + - \cdots$ and $\sin z = z - z^3/3! + - \cdots$, we see immediately that, for $|z|$ small, $\cot z = \cos z/\sin z \approx 1/z$ so that f has a pole of order 1 at 0. Thus, in our formula for the Laurent coefficients, $k = 1$. Also the only Laurent coefficient of negative index is $n = -1$. [We anticipate from this calculation that the coefficient of z^{-1} will be 1. This perception will be borne out in our calculation.]

Now we see, by (5.22), that

$$a_{-1} = \frac{1}{0!} \left(\frac{\partial}{\partial z}\right)^0 \left(z \cdot \frac{\cos z}{\sin z}\right)\bigg|_{z=0} = \left(z \cdot \frac{\cos z}{\sin z}\right)\bigg|_{z=0} .$$

It is appropriate to apply l'Hôpital's Rule to evaluate this last expression. Thus we have

$$\frac{\cos z - z \cdot \sin z}{\cos z}\bigg|_{z=0} = 1 .$$

Not surprisingly, we find that the "pole" term of the Laurent expansion of this function f about 0 is $1/z$. We say "not surprisingly" because $\cos z = 1 - + \cdots$ and $\sin z = z - + \cdots$ and hence we expect that $\cot z = 1/z + \cdots$. □

We invite the reader to use the technique of the last example to calculate a_0 for the given function f. Of course you will find l'Hôpital's rule useful. You should not be surprised to learn that $a_0 = 0$ (and we say "not surprised" because you could have anticipated this result using long division).

EXAMPLE 47 Let us use formula (5.22) to calculate the negative Laurent coefficients of the function $g(z) = z^2/(z-1)^2$ at the point $P = 1$.

It is clear that the pole at $P = 1$ has order $k = 2$. Thus we calculate

$$a_{-2} = \frac{1}{0!} \left(\frac{\partial}{\partial z}\right)^0 \left((z-1)^2 \cdot \frac{z^2}{(z-1)^2}\right)\bigg|_{z=1} = z^2\bigg|_{z=1} = 1$$

and

$$a_{-1} = \frac{1}{1!} \left(\frac{\partial}{\partial z}\right)^1 \left((z-1)^2 \cdot \frac{z^2}{(z-1)^2}\right)\bigg|_{z=1} = \frac{\partial}{\partial z} z^2\bigg|_{z=1} = 2z\bigg|_{z=1} = 2\,.$$

Of course this result may be derived by more elementary means, using just algebra:

$$\frac{z^2}{(z-1)^2} = \frac{(z-1)^2}{(z-1)^2} + \frac{2z-1}{(z-1)^2} = 1 + \frac{2z-2}{(z-1)^2} + \frac{1}{(z-1)^2} = 1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\,.$$

□

The justification for formula (5.22) is simplicity itself. Suppose that f has a pole of order k at the point P. We may write

$$f(z) = (z-P)^{-k} \cdot h(z)\,,$$

where h is holomorphic near P. Writing out the ordinary power series expansion of h, we find that

$$\begin{aligned} f(z) &= (z-P)^{-k} \cdot \big(a_0 + a_1(z-P) + a_2(z-P)^2 + \cdots\big) \\ &= \frac{a_0}{(z-P)^k} + \frac{a_1}{(z-P)^{k-1}} + \frac{a_2}{(z-P)^{k-2}} + \cdots\,. \end{aligned}$$

So the $-(k-j)$th Laurent coefficient of f is just the same as the jth power series coefficient of h. That is the key to our calculation, because

$$h(z) = (z-P)^k \cdot f(z)\,,$$

and thus formula (5.22) is immediate.

Exercises

1. Calculate the Laurent series of the function $f(z) = \frac{z - \sin z}{z^6}$ at $z = \pi/2$.

2. Calculate the Laurent series of the function $g(z) = \frac{\ln z}{(z-1)^3)}$ about the point $z = 1$.

3. Calculate the Laurent series of the function $\sin(1/z)$ about the point $z = 0$.

4. Calculate the Laurent series of the function $\tan z$ about the points $z = 0$, $z = \pi/2$ and $z = \pi$.

5. Suppose that f has a pole of order 1 at $z = 0$. What can you say about the behavior of $g(z) = e^{f(z)}$ at $z = 0$?

6. Suppose that f has an essential singularity at $z = 0$. What can you say about the behavior of $h(z) = e^{f(z)}$ at $z = 0$.

7. Let U be an open region in the plane. Let $\mathcal{M}$ denote the collection of functions on U that has a discrete set of poles and is holomorphic elsewhere (we allow the possibility that the function may have *no* poles). Explain why $\mathcal{M}$ is closed under addition, subtraction, multiplication, and division.

8. Consider Exercise 7 with the word "pole" replaced by "essential singularity." Does any part of the conclusion of that exercise still hold? Why or why not?

9. Let $P = 0$. Classify each of the following as having a removable singularity, a pole, or an essential singularity at P:

(a) $\dfrac{1}{z}$,

(b) $\sin \dfrac{1}{z}$,

(c) $\dfrac{1}{z^3} - \cos z$,

(d) $z \cdot e^{1/z} \cdot e^{-1/z^2}$,

(e) $\dfrac{\sin z}{z}$,

(f) $\dfrac{\cos z}{z}$,

(g) $\dfrac{\sum_{k=2}^{\infty} 2^k z^k}{z^3}$.

10. Prove that

$$\sum_{n=1}^{\infty} 2^{-(2^n)} \cdot z^{-n}$$

converges for $z \neq 0$ and defines a function which has an essential singularity at $P = 0$.

11. A Laurent series converges on an annular region. Give examples to show that the set of convergence for a Laurent series can include some of the boundary, all of the boundary, or none of the boundary.

12. Calculate the annulus of convergence (including any boundary points) for each of the following Laurent series:

(a) $\sum_{n=-\infty}^{\infty} 2^{-n} z^n$,

(b) $\sum_{n=0}^{\infty} 4^{-n} z^n + \sum_{n=-\infty}^{-1} 3^n z^n$,

(c) $\sum_{n=1}^{\infty} z^n / n^2$,

(d) $\sum_{n=-\infty, n\neq 0}^{\infty} z^n / n^n$,

(e) $\sum_{n=-\infty}^{10} z^n / |n|! \quad (0! = 1)$,

(f) $\sum_{n=-20}^{\infty} n^2 z^n$.

13. Use formal algebra to calculate the first four terms of the Laurent series expansion of each of the following functions:

(a) $\tan z \equiv (\sin z / \cos z)$ about $\pi/2$,

(b) $e^z / \sin z$ about 0,

(c) $e^z / (1 - e^z)$ about 0,

(d) $\sin(1/z)$ about 0,

(e) $z(\sin z)^{-2}$ about 0,

(f) $z^2 (\sin z)^{-3}$ about 0.

For each of these functions, identify the type of singularity at the point about which the function has been expanded.

14. An incompressible fluid flow has the form $f(z) = [\cos z - 1]/z^3$. Calculate the principal part at the origin. What does the principal part tell us about the flow?

5.4 The Calculus of Residues

5.4.1 Functions with Multiple Singularities

It turns out to be useful, especially in evaluating various types of integrals, to consider functions that have more than one "singularity." We want to consider the following general question:

> Suppose that $f : U \setminus \{P_1, P_2, \ldots, P_n\} \to \mathbb{C}$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$ with finitely many distinct points $P_1, P_2, \ldots, P_n$ removed. Suppose further that
>
> $$\gamma : [0,1] \to U \setminus \{P_1, P_2, \ldots, P_n\} \tag{5.23}$$
>
> is a piecewise C^1 closed curve (Section 2.3.3) that (typically) "surrounds" some of the points $P_1, \ldots, P_n$ (Figure 5.6). Then how is $\oint_\gamma f$ related to the behavior of f near the points $P_1, P_2, \ldots, P_n$?

The first step is to restrict our attention to open sets U for which $\oint_\gamma f$ is necessarily 0 if $P_1, P_2, \ldots, P_n$ are removable singularities of f. See the next section.

5.4.2 The Concept of Residue

Suppose that U is a domain, $P \in U$, and f is a function holomorphic on $U \setminus \{P\}$ with a pole at P. Let γ be a simple, closed curve in U that surrounds P. And let $D(P, r)$ be a small disc, centered at P, that lies inside γ. Then certainly, by the usual Cauchy theory,

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \frac{1}{2\pi i} \oint_{\partial D(P,r)} f(z)\, dz\,.$$

But more is true. Let a_{-1} be the -1 coefficient of the Laurent expansion of f about P. Then in fact

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma f(z)\,dz &= \frac{1}{2\pi i}\oint_{\partial D(P,r)} f(z)\,dz \\ &= \frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{a_{-1}}{z-P}\,dz = a_{-1}\,. \end{aligned} \tag{5.24}$$

We call the value a_{-1} the *residue* of f at the point P.

The justification for formula (5.24) is the following. Observe that, with the parametrization $\mu(t) = P + re^{it}$ for $\partial D(P,r)$, we see for $n \neq -1$ that

$$\oint_{\partial D(P,r)} (z-P)^n\,dz = \int_0^{2\pi} (re^{it})^n \cdot rie^{it}\,dt = r^{n+1} i \int_0^{2\pi} e^{i(n+1)t}\,dt = 0\,.$$

It is important in this last calculation that $n \neq -1$. If instead $n = -1$ then the integral turns out to be

$$i\int_0^{2\pi} 1\,dt = 2\pi i\,.$$

This information is critical because if we are integrating a meromorphic function $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-P)^n$ around the contour $\partial D(P,r)$ then the result is

$$\begin{aligned}\oint_{\partial D(P,r)} f(z)\,dz &= \oint_{\partial D(P,r)} \sum_{n=-\infty}^{\infty} a_n(z-P)^n = \sum_{n=-\infty}^{\infty} a_n \oint_{\partial D(P,r)} (z-P)^n\,dz \\ &= a_{-1}\oint_{\partial D(P,r)} (z-P)^{-1}\,dz = 2\pi i a_{-1}\,.\end{aligned}$$

In other words,

$$a_{-1} = \frac{1}{2\pi i}\oint_{\partial D(P,r)} f(z)\,dz\,.$$

We will make incisive use of this information in the succeeding sections.

5.4.3 The Residue Theorem

Suppose that $U \subseteq \mathbb{C}$ is a simply connected open set in $\mathbb{C}$, and that $P_1, \ldots, P_n$ are distinct points of U. Suppose that $f : U \setminus \{P_1, \ldots, P_n\} \to \mathbb{C}$ is a holomorphic function and γ is a simple, closed, positively oriented, piecewise C^1 curve in $U \setminus \{P_1, \ldots, P_n\}$. Set

$$R_j \quad = \quad \text{the coefficient of } (z - P_j)^{-1}$$
in the Laurent expansion of f about P_j . (5.25)

Then

$$\frac{1}{2\pi i} \oint_\gamma f = \sum_{j=1}^{n} R_j \cdot \left(\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P_j} d\zeta \right) . \tag{5.26}$$

The rationale behind this residue formula is straightforward from the picture. Examine Figure 5.6. It shows the curve γ and the poles $P_1, \ldots, P_n$. Figure 5.7 exhibits a small circular contour around each pole. And Figure 5.8 shows our usual trick of connecting up the contours. The integral around the big, conglomerate contour in Figure 5.8 (including γ, the integrals around each of the circular arcs, and the integrals along the connecting segments) is equal to 0. This demonstrates that

> The integral of f around γ is equal to the sum of the integrals around each of the circles around the P_n.

If we let C_j be the circle around P_j, oriented in the counterclockwise direction as usual, then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \sum_{j=1}^{n} R_j \left(\frac{1}{2\pi i} \oint_{C_j} \frac{1}{\zeta - P_j}\, d\zeta \right) . \tag{5.27}$$

5.4.4 Residues

The result just stated is used so often that some special terminology is commonly used to simplify its statement. First, the number R_j is usually called the *residue* of f at P_j, written $\mathrm{Res}_f(P_j)$. Note that this terminology of considering the number R_j attached to the point P_j makes sense because $\mathrm{Res}_f(P_j)$ is completely determined by knowing f in a small neighborhood of P_j. In particular, the value of the residue does not depend on what the other points $P_k,\ k \neq j$, might be, or on how f behaves near those points.

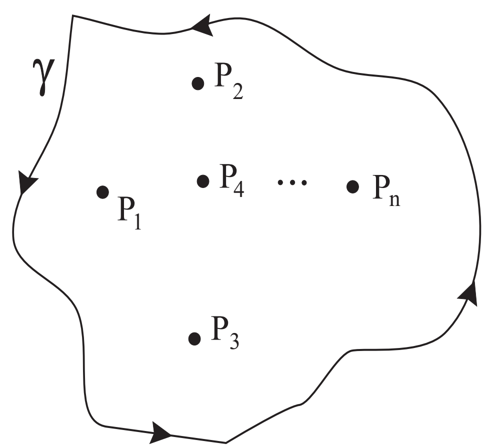

Figure 5.6: A curve γ with poles inside.

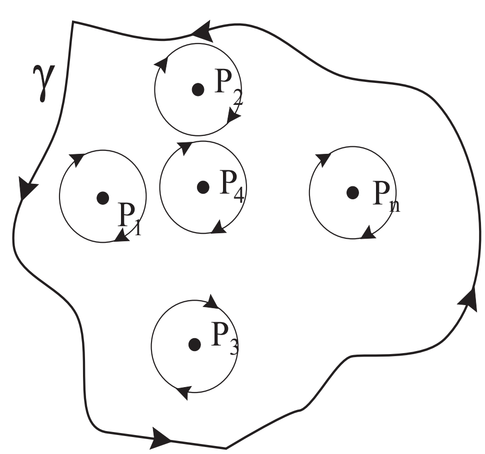

Figure 5.7: A small circle about each pole.

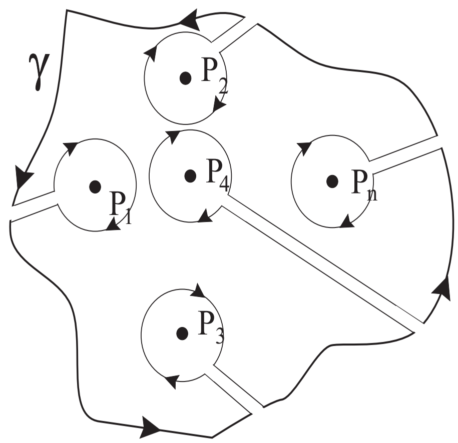

Figure 5.8: Stitching together the circles.

5.4.5 The Index or Winding Number of a Curve about a Point

The second piece of terminology associated to our result deals with the integrals that appear on the right-hand side of equation (5.27).

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1 closed curve and if $P \notin \widetilde{\gamma} \equiv \gamma([a, b])$, then the *index of* γ *with respect to* P, written $\mathrm{Ind}_\gamma(P)$, is defined to be the number

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P} \, d\zeta \, . \tag{5.28}$$

The index is also sometimes called the "winding number of the curve γ about the point P." It is a fact that $\mathrm{Ind}_\gamma(P)$ is always an integer. Figure 5.9 illustrates the index of various curves γ with respect to different points P. Intuitively, the index measures the number of times the curve wraps around P, with counterclockwise being the positive direction of wrapping and clockwise being the negative.

The fact that the index is an integer-valued function suggests that the index counts the topological winding of the curve γ. Note in particular that a curve that traces a circle about the origin k times in a counterclockwise direction has index k with respect to the origin; a curve that traces a circle about the origin k times in a clockwise direction has index $-k$ with respect to the origin. The general fact that the index is integer valued, and counts the

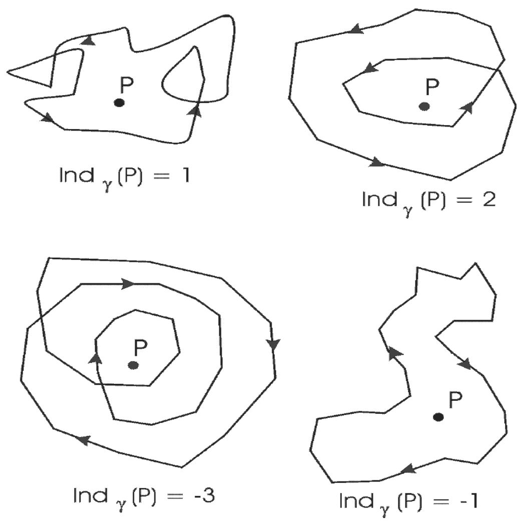

Figure 5.9: Examples of the index of a curve.

winding number, follows from these two simple observations by deformation. The index, or winding number, will prove to be an important geometric device.

5.4.6 Restatement of the Residue Theorem

Using the notation of residue and index, the Residue Theorem's formula becomes

$$\oint_\gamma f = 2\pi i \cdot \sum_{j=1}^{n} \mathrm{Res}_f(P_j) \cdot \mathrm{Ind}_\gamma(P_j)\,. \tag{5.29}$$

People sometimes state this formula informally as "the integral of f around γ equals $2\pi i$ times the sum of the residues counted according to the index of γ about the singularities."

In practice, when we apply the residue theorem, we use a simple, closed, positively-oriented curve γ. Thus the index of γ about any point in its interior is just 1. And therefore we use the ideas of Section 5.4.3 and replace γ with a small circle about each pole of the function (which of course will also have index equal to 1 with respect to the point at its center).

5.4.7 Method for Calculating Residues

We need a method for calculating residues.

Let f be a function with a pole of order k at P. Then

$$\mathrm{Res}_f(P) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\left((z-P)^k f(z)\right)\Big|_{z=P} . \tag{5.30}$$

This is just a special case of the formula (5.22).

5.4.8 Summary Charts of Laurent Series and Residues

We provide two charts, the first of which summarizes key ideas about Laurent coefficients and the second of which contains key ideas about residues.

Poles and Laurent Coefficients

Item	Formula
jth Laurent coefficient of f with pole of order k at P	$\frac{1}{(k+j)!}\frac{d^{k+j}}{dz^{k+j}}\left[(z-P)^k \cdot f\right]\Big\|_{z=P}$
residue of f with a pole of order k at P	$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-P)^k \cdot f\right]\Big\|_{z=P}$
order of pole of f at P	least integer $k \geq 0$ such that $(z-P)^k \cdot f$ is bounded near P
order of pole of f at P	$\lim_{z\to P}\left\|\frac{\log\|f(z)\|}{\log\|z-P\|}\right\|$

Techniques for Finding the Residue at P

Function	Type of Pole	Calculation
$f(z)$	simple	$\lim_{z \to P}(z-P) \cdot f(z)$
$f(z)$	pole of order k k is the least integer such that $\lim_{z \to P} \mu(z)$ exists, where $\mu(z) = (z-P)^k f(z)$	$\displaystyle\lim_{z \to P} \frac{\mu^{(k-1)}(z)}{(k-1)!}$
$\dfrac{m(z)}{n(z)}$	$m(P) \neq 0$, $n(z) = 0$, $n'(P) \neq 0$	$\dfrac{m(P)}{n'(P)}$
$\dfrac{m(z)}{n(z)}$	m has zero of order k at P n has zero of order $(k+1)$ at P	$(k+1) \cdot \dfrac{m^{(k)}(P)}{n^{(k+1)}(P)}$
$\dfrac{m(z)}{n(z)}$	m has zero of order r at P n has zero of order $(k+r)$ at P	$\displaystyle\lim_{z \to P} \frac{\mu^{(k-1)}(z)}{(k-1)!}$, $\mu(z) = (z-P)^k \dfrac{m(z)}{n(z)}$

Exercises

1. Calculate the residue of the function $f(z) = \cot z$ at $z = 0$.

2. Calculate the residue of the function $h(z) = \tan z$ at $z = \pi/2$.

3. Calculate the residue of the function $g(z) = e^{1/z}$ at $z = 0$.

4. Calculate the residue of the function $f(z) = \cot^2 z$ at $z = 0$.

5. Calculate the residue of the function $g(z) = \sin(1/z)$ at $z = 0$.

6. Calculate the residue of the function $h(z) = \tan(1/z)$ at $z = 0$.

7. If the function f has residue a at $z = 0$ and the function g has residue b at $z = 0$ then what can you say about the residue of f/g at $z = 0$? What about the residue of $f \cdot g$ at $z = 0$?

8. Let f and g be as in Exercise 7. Describe the residues of $f + g$ and $f - g$ at $z = 0$.

9. Calculate the residue of $f_k(z) = z^k$ for $k \in \mathbb{Z}$. Explain the different answers for different ranges of k.

10. Is the residue of a function f at an essential singularity always equal to 0? Why or why not?

11. Use the calculus of residues to compute each of the following integrals:

(a) $\displaystyle \frac{1}{2\pi i}\oint_{\partial D(0,5)} f(z)\,dz$ where $f(z) = z/[(z+1)(z+2i)]$,

(b) $\displaystyle \frac{1}{2\pi i}\oint_{\partial D(0,5)} f(z)\,dz$ where $f(z) = e^z/[(z+1)\sin z]$,

(c) $\displaystyle \frac{1}{2\pi i}\oint_{\partial D(0,8)} f(z)\,dz$ where $f(z) = \cot z/[(z-6i)^2+64]$,

(d) $\displaystyle \frac{1}{2\pi i}\oint_{\gamma} f(z)\,dz$ where $f(z) = \dfrac{e^z}{z(z+1)(z+2)}$ and γ is the negatively (clockwise) oriented triangle with vertices $1 \pm i$ and -3,

(e) $\displaystyle \frac{1}{2\pi i}\oint_{\gamma} f(z)\,dz$ where $f(z) = \dfrac{e^z}{(z+3i)^2(z+3)^2(z+4)}$ and γ is the negatively oriented rectangle with vertices $2 \pm i, -8 \pm i$,

(f) $\displaystyle \frac{1}{2\pi i}\oint_{\gamma} f(z)\,dz$ where $f(z) = \dfrac{\cos z}{z^2(z+1)^2(z+i)}$ and γ is as in Figure 5.10.

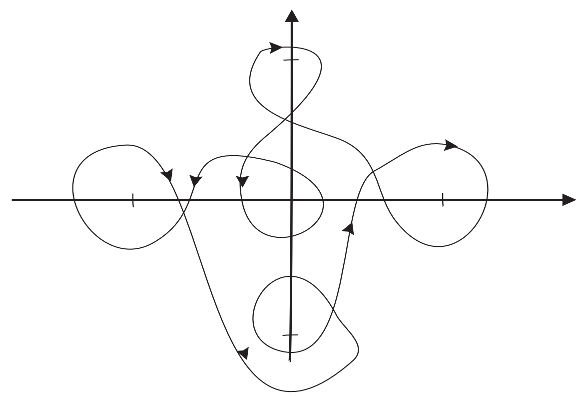

Figure 5.10: The contour in Exercise 11f.

(g) $\dfrac{1}{2\pi i}\oint_\gamma f(z)\,dz$ where $f(z) = \dfrac{\sin z}{z(z+2i)^3}$ and γ is as in Figure 5.11.

(h) $\dfrac{1}{2\pi i}\oint_\gamma f(z)\,dz$ where $f(z) = \dfrac{e^{iz}}{(\sin z)(\cos z)}$ and γ is the positively (counterclockwise) oriented quadrilateral with vertices $\pm 5i, \pm 10$,

(i) $\dfrac{1}{2\pi i}\oint_\gamma f(z)\,dz$ where $f(z) = \tan z$ and γ is the curve in Figure 5.12.

12. Let $R(z)$ be a rational function: $R(z) = p(z)/q(z)$ where p and q are holomorphic polynomials. Let f be holomorphic on $\mathbb{C} \setminus \{P_1, P_2, \dots, P_k\}$ and suppose that f has a pole at each of the points $P_1, P_2, \dots, P_k$. Finally assume that

$$|f(z)| \leq |R(z)|$$

for all z at which $f(z)$ and $R(z)$ are defined. Prove that f is a constant multiple of R. In particular, f is rational. [**Hint:** Think about $f(z)/R(z)$.]

13. Let $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ be holomorphic. Let $U = f(D(P, r) \setminus \{P\})$. Assume that U is open (we shall later see that this is always the case if f is nonconstant). Let $g : U \to \mathbb{C}$ be holomorphic. If f has a removable

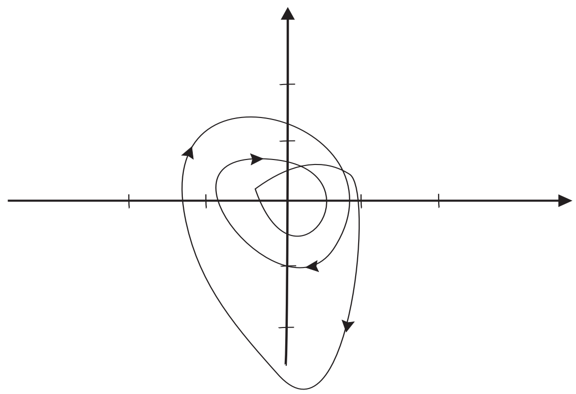

Figure 5.11: The contour in Exercise 11g.

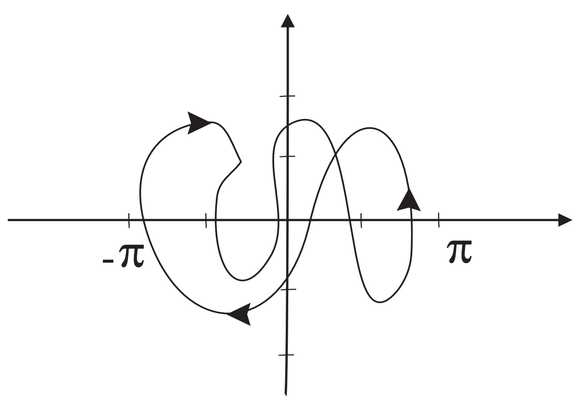

Figure 5.12: The contour in Exercise 11i.

singularity at P, does $g \circ f$ have one also? What about the case of poles and essential singularities?

14. A certain incompressible fluid flow has poles at 0, 1, and i. Each pole is a simple pole, and the respective residues are 3, -5, and 2. Follow along a counterclockwise path consisting of a square of side 4 with center 0 and sides parallel to the axes. What can you say about the flow along that path?

5.5 Applications to the Calculation of Definite Integrals and Sums

5.5.1 The Evaluation of Definite Integrals

One of the most classical and fascinating applications of the calculus of residues is the calculation of definite (usually improper) real integrals. It is an oversimplification to call these calculations, taken together, a "technique": it is more like a *collection* of techniques. We present several instances of the method.

5.5.2 A Basic Example

To evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx\,, \tag{5.31}$$

we "complexify" the integrand to $f(z) = 1/(1+z^4)$ and consider the integral

$$\oint_{\gamma_R} \frac{1}{1+z^4}\, dx\,. \tag{5.32}$$

See Figure 5.13.

Now part of the game here is to choose the right piecewise C^1 curve or "contour" γ_R. The appropriateness of our choice is justified (after the fact) by the calculation that we are about to do. Assume that $R > 1$. Define

$$\begin{aligned} \gamma_R^1(t) &= t + i0 \quad \text{if} \quad -R \le t \le R\,, \\ \gamma_R^2(t) &= Re^{it} \quad \text{if} \quad 0 \le t \le \pi. \end{aligned}$$

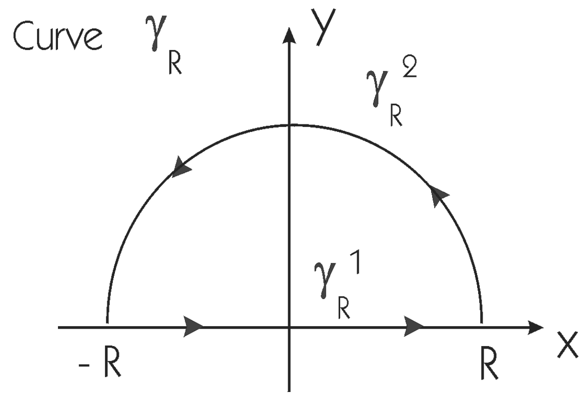

Figure 5.13: The curve γ_R in Section 5.5.2.

Call these two curves, taken together, γ or γ_R.

Now we set $U = \mathbb{C}$, $P_1 = 1/\sqrt{2} + i/\sqrt{2}$, $P_2 = -1/\sqrt{2} + i/\sqrt{2}$, $P_3 = -1/\sqrt{2} - i/\sqrt{2}$, $P_4 = 1/\sqrt{2} - i/\sqrt{2}$; the points P_1, P_2, P_3, P_4 are the poles of $1/[1+z^4]$. Thus $f(z) = 1/(1+z^4)$ is holomorphic on $U \setminus \{P_1, \dots, P_4\}$ and the Residue Theorem applies.

On the one hand,

$$\oint_\gamma \frac{1}{1+z^4}\, dz = 2\pi i \sum_{j=1,2} \mathrm{Ind}_\gamma(P_j) \cdot \mathrm{Res}_f(P_j)\,, \tag{5.33}$$

where we sum only over the poles of f that lie inside γ. These are P_1 and P_2. An easy calculation shows that

$$\mathrm{Res}_f(P_1) = \frac{1}{4(1/\sqrt{2} + i/\sqrt{2})^3} = -\frac{1}{4}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \tag{5.34}$$

and

$$\mathrm{Res}_f(P_2) = \frac{1}{4(-1/\sqrt{2} + i/\sqrt{2})^3} = -\frac{1}{4}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\,. \tag{5.35}$$

Of course the index at each point is 1. So

$$\begin{aligned} \oint_\gamma \frac{1}{1+z^4}\, dz &= 2\pi i \left(-\frac{1}{4}\right)\left[\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right] \\ &= \frac{\pi}{\sqrt{2}}\,. \end{aligned} \tag{5.36}$$

On the other hand,

$$\oint_\gamma \frac{1}{1+z^4}\,dz = \oint_{\gamma_R^1} \frac{1}{1+z^4}\,dz + \oint_{\gamma_R^2} \frac{1}{1+z^4}\,dz\,. \tag{5.37}$$

Trivially,

$$\oint_{\gamma_R^1} \frac{1}{1+z^4}\,dz = \int_{-R}^{R} \frac{1}{1+t^4}\cdot 1\cdot dt \to \int_{-\infty}^{\infty} \frac{1}{1+t^4}dt \tag{5.38}$$

as $R \to +\infty$. That is good, because this last is the integral that we wish to evaluate. Better still,

$$\left|\oint_{\gamma_R^2} \frac{1}{1+z^4}\,dx\right| \le \{\text{length}(\gamma_R^2)\}\cdot \max_{\gamma_R^2}\left|\frac{1}{1+z^4}\right| \le \pi R\cdot \frac{1}{R^4-1}\,. \tag{5.39}$$

[Here we use the inequality $|1+z^4| \ge |z|^4 - 1$, as well as (2.41).] Thus

$$\left|\oint_{\gamma_R^2} \frac{1}{1+z^4}\,dz\right| \to 0 \qquad \text{as} \qquad R \to \infty\,. \tag{5.40}$$

Finally, (5.36), (5.38), (5.40) taken together yield

$$\begin{aligned}
\frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty} \oint_\gamma \frac{1}{1+z^4}\,dz \\
&= \lim_{R\to\infty} \oint_{\gamma_R^1} \frac{1}{1+z^4}\,dz + \lim_{R\to\infty} \oint_{\gamma_R^2} \frac{1}{1+z^4}\,dz \\
&= \int_{-\infty}^{\infty} \frac{1}{1+t^4}dt + 0.
\end{aligned}$$

This solves the problem: the value of the integral is $\pi/\sqrt{2}$.

In other problems, it will not be so easy to pick the contour so that the superfluous parts (in the above example, this would be the integral over γ_R^2) tend to zero, nor is it always so easy to prove that they *do* tend to zero. Sometimes, it is not even obvious how to complexify the integrand.

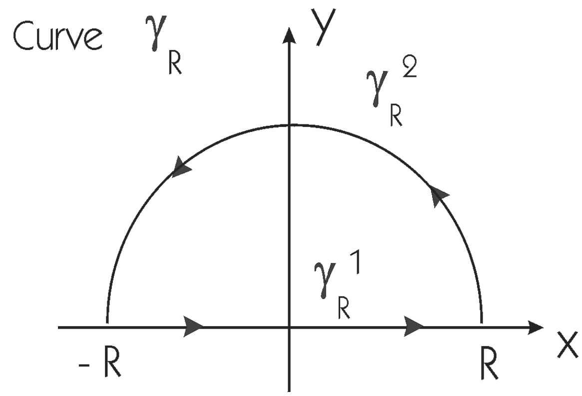

Figure 5.14: The curve γ_R in Section 5.5.3.

5.5.3 Complexification of the Integrand

We evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx \tag{5.41}$$

by using the contour γ_R as in Figure 5.14 (that is, the same contour as in the last example). The obvious choice for the complexification of the integrand is

$$f(z) = \frac{\cos z}{1+z^2} = \frac{[e^{iz}+e^{-iz}]/2}{1+z^2} = \frac{[e^{ix}e^{-y}+e^{-ix}e^{y}]/2}{1+z^2}\,. \tag{5.42}$$

Now $|e^{iz}| = |e^{ix}e^{-y}| = |e^{-y}| \le 1$ on γ_R but $|e^{-iz}| = |e^{-ix}e^{y}| = |e^{y}|$ becomes quite large on γ_R when R is large and positive. There is no evident way to alter the contour so that good estimates result. Instead, we alter the function! Let $g(z) = e^{iz}/(1+z^2)$.

Of course the poles of g are at i and $-i$. Of these two, only i lies inside the contour. On the one hand (for $R > 1$),

$$\begin{aligned}\oint_{\gamma_R} g(z) &= 2\pi i \cdot \mathrm{Res}_g(i) \cdot \mathrm{Ind}_{\gamma_R}(i) \\ &= 2\pi i \left(\frac{1}{2ei}\right) \cdot 1 = \frac{\pi}{e}.\end{aligned}$$

On the other hand, with $\gamma_R^1(t) = t, -R \le t \le R$, and $\gamma_R^2(t) = Re^{it}, 0 \le t \le \pi$, we have

$$\oint_{\gamma_R} g(z)\,dz = \oint_{\gamma_R^1} g(z)\,dz + \oint_{\gamma_R^2} g(z)\,dz\,. \tag{5.43}$$

Of course

$$\oint_{\gamma_R^1} g(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx \quad \text{as} \quad R \to \infty\,. \tag{5.44}$$

And

$$\left|\oint_{\gamma_R^2} g(z)\,dz\right| \le \text{length}(\gamma_R^2)\cdot \max_{\gamma_R^2} |g| \le \pi R \cdot \frac{1}{R^2-1} \to 0 \quad \text{as} \quad R \to \infty\,. \tag{5.45}$$

Here we have again reasoned as in the last section.

Thus

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \text{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \text{Re}\left(\frac{\pi}{e}\right) = \frac{\pi}{e}\,. \tag{5.46}$$

5.5.4 An Example with a More Subtle Choice of Contour

Let us evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx\,. \tag{5.47}$$

Before we begin, we remark that $\sin x/x$ is bounded near zero; also, the integral converges at ∞ (as an improper Riemann integral) by integration by parts. So the problem makes sense. Using the lesson learned from the last example, we consider the function $g(z) = e^{iz}/z$. However, the pole of e^{iz}/z is at $z = 0$ and that lies *on the contour* in Figure 5.14. Thus *that* contour may not be used. We instead use the contour $\mu = \mu_R$ that is depicted in Figure 5.15.

Define

$$\begin{aligned}
\mu_R^1(t) &= t, \qquad -R \le t \le -1/R, \\
\mu_R^2(t) &= e^{it}/R, \ \pi \le t \le 2\pi, \\
\mu_R^3(t) &= t, \qquad 1/R \le t \le R, \\
\mu_R^4(t) &= Re^{it}, \ 0 \le t \le \pi.
\end{aligned}$$

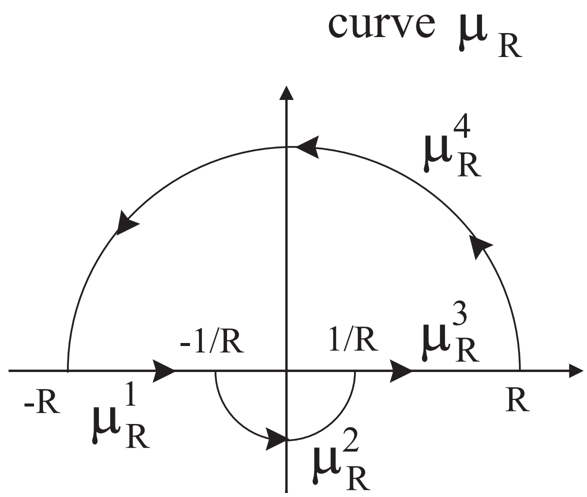

Figure 5.15: The curve μ_R in Section 5.5.4.

Clearly

$$\oint_\mu g(z)\,dz = \sum_{j=1}^{4} \oint_{\mu_R^j} g(z)\,dz\,. \tag{5.48}$$

On the one hand, for $R > 0$,

$$\oint_\mu g(z)\,dz = 2\pi i \mathrm{Res}_g(0) \cdot \mathrm{Ind}_\mu(0) = 2\pi i \cdot 1 \cdot 1 = 2\pi i\,. \tag{5.49}$$

On the other hand,

$$\oint_{\mu_R^1} g(z)\,dz + \oint_{\mu_R^3} g(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx \quad \text{as} \quad R \to \infty\,. \tag{5.50}$$

Furthermore,

$$\left|\oint_{\mu_R^4} g(z)\,dz\right| \leq \left|\oint_{\substack{\mu_R^4 \\ \mathrm{Im}\ y < \sqrt{R}}} g(z)\,dz\right| + \left|\oint_{\substack{\mu_R^4 \\ \mathrm{Im}\ y \geq \sqrt{R}}} g(z)\,dz\right| \tag{5.51}$$

$$\equiv A + B\,. \tag{5.52}$$

Now

$$\begin{aligned} A &\le \text{length}(\mu_R^4 \cap \{z : \text{Im } z < \sqrt{R}\}) \cdot \max\{|g(z)| : z \in \mu_R^4, y < \sqrt{R}\} \\ &\le 4\sqrt{R} \cdot \left(\frac{1}{R}\right) \to 0 \quad \text{as} \quad R \to \infty. \end{aligned}$$

Also

$$\begin{aligned} B &\le \text{length}(\mu_R^4 \cap \{z : \text{Im } z \ge \sqrt{R}\}) \cdot \max\{|g(z)| : z \in \mu_R^4, y \ge \sqrt{R}\} \\ &\le \pi R \cdot \left(\frac{e^{-\sqrt{R}}}{R}\right) \to 0 \quad \text{as} \quad R \to \infty. \end{aligned}$$

So

$$\left| \oint_{\mu_R^4} g(z)\, dz \right| \to 0 \quad \text{as} \quad R \to \infty\,. \tag{5.53}$$

Finally,

$$\begin{aligned} \oint_{\mu_R^2} g(z)\, dz &= \int_\pi^{2\pi} \frac{e^{i(e^{it}/R)}}{e^{it}/R} \cdot \left(\frac{i}{R} e^{it}\right) dt \\ &= i \int_\pi^{2\pi} e^{i(e^{it}/R)} dt. \end{aligned}$$

As $R \to \infty$ this tends to

$$\begin{aligned} &= i \int_\pi^{2\pi} 1\, dt \\ &= \pi i \quad \text{as} \quad R \to \infty\,. \end{aligned} \tag{5.54}$$

In summary, (5.49) through (5.54) yield

$$2\pi i = \oint_\mu g(z)\, dz = \sum_{n=1}^{4} \oint_{\mu_R^n} g(z)\, dz \tag{5.55}$$

$$\to \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx + \pi i \quad \text{as} \quad R \to \infty\,. \tag{5.56}$$

Taking imaginary parts yields

$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx\,. \tag{5.57}$$

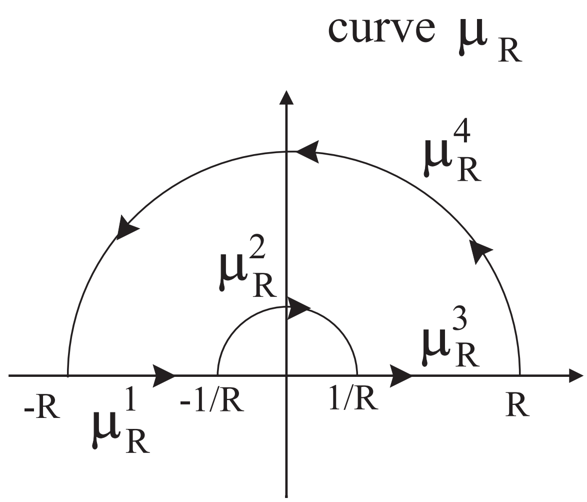

Figure 5.16: The curve μ_R in Section 5.5.5.

5.5.5 Making the Spurious Part of the Integral Disappear

Consider the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx\,. \tag{5.58}$$

We complexify the integrand by setting $f(z) = z^{1/3}/(1+z^2)$. Note that, on the simply connected set $U = \mathbb{C} \setminus \{iy : y < 0\}$, the expression $z^{1/3}$ is unambiguously defined as a holomorphic function by setting $z^{1/3} = r^{1/3}e^{i\theta/3}$ when $z = re^{i\theta}, -\pi/2 < \theta < 3\pi/2$. We use the contour displayed in Figure 5.16.

We must do this since $z^{1/3}$ is not a well-defined holomorphic function in any neighborhood of 0. Let us use the notation from the figure. We refer to the preceding examples for some of the parametrizations that we now use.

Clearly

$$\oint_{\mu_R^3} f(z)\, dz \to \int_0^\infty \frac{t^{1/3}}{1+t^2} dt\,. \tag{5.59}$$

Of course that is good, but what will become of the integral over μ_R^1? We have

$$\begin{aligned}\oint_{\mu_R^1} &= \int_{-R}^{-1/R} \frac{t^{1/3}}{1+t^2} dt \\ &= \int_{1/R}^{R} \frac{(-t)^{1/3}}{1+t^2} dt \\ &= \int_{1/R}^{R} \frac{e^{i\pi/3} t^{1/3}}{1+t^2} dt.\end{aligned}$$

(by our definition of $z^{1/3}$!). Thus

$$\oint_{\mu_R^3} f(z)\,dz + \oint_{\mu_R^1} f(z)\,dz \to \left(1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\right) \int_0^\infty \frac{t^{1/3}}{1+t^2} dt \quad \text{as} \quad R \to +\infty\,. \tag{5.60}$$

On the other hand,

$$\left| \oint_{\mu_R^4} f(z)\,dz \right| \le \pi R \cdot \frac{R^{1/3}}{R^2 - 1} \to 0 \quad \text{as} \quad R \to +\infty \tag{5.61}$$

and

$$\begin{aligned}\oint_{\mu_R^2} f(z)\,dz &= \int_{-\pi}^{-2\pi} \frac{(e^{it}/R)^{1/3}}{1 + e^{2it}/R^2} (i) e^{it}/R\,dt \\ &= R^{-4/3} \int_{-\pi}^{-2\pi} \frac{e^{i4t/3}}{1 + e^{2it}/R^2}\,dt \to 0 \quad \text{as} \quad R \to +\infty.\end{aligned}$$

So, altogether then,

$$\oint_{\mu_R} f(z)\,dz \to \left(\frac{3}{2} + \frac{\sqrt{3}}{2} i\right) \int_0^\infty \frac{t^{1/3}}{1+t^2}\,dt \quad \text{as} \quad R \to +\infty\,. \tag{5.62}$$

The calculus of residues tells us that, for $R > 1$,

$$\begin{aligned}\oint_{\mu_R} f(z)\,dz &= 2\pi i \mathrm{Res}_f(i) \cdot \mathrm{Ind}_{\mu_R}(i) \\ &= 2\pi i \left(\frac{e^{i\pi/6}}{2i}\right) \cdot 1 \\ &= \pi \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\,. \end{aligned} \tag{5.63}$$

Finally, (5.62) and (5.63) taken together yield

$$\int_0^\infty \frac{t^{1/3}}{1+t^2}\,dt = \frac{\pi}{\sqrt{3}}. \qquad \square$$

5.5.6 The Use of the Logarithm

While the integral

$$\int_0^\infty \frac{dx}{x^2+6x+8} \tag{5.64}$$

can be calculated using methods of calculus, it is enlightening to perform the integration by complex variable methods. Note that if we endeavor to use the integrand $f(z) = 1/(z^2+6z+8)$ together with the idea of the last example, then there is no "auxiliary radius" that helps. More precisely, $((re^{i\theta})^2 + 6re^{i\theta} + 8)$ is a constant multiple of $r^2 + 6r + 8$ only if θ is an integer multiple of 2π. The following nonobvious device is often of great utility in problems of this kind. Define $\log z$ on $U \equiv \mathbb{C} \setminus \{x + i0 : x \geq 0\}$ by $\log(re^{i\theta}) = (\log r) + i\theta$ when $0 < \theta < 2\pi, r > 0$. Here $\log r$ is understood to be the standard real logarithm. Then, on U, log is a well-defined holomorphic function. [Observe here that there are infinitely many ways to define the logarithm function on U. One could set $\log(re^{i\theta}) = (\log r) + i(\theta + 2k\pi)$ for any integer choice of k. What we have done here is called "choosing a branch" of the logarithm. See Section 2.5.]

We use the contour η_R displayed in Figure 5.17 and integrate the function $g(z) = \log z/(z^2+6z+8)$. Let

$$\begin{aligned}
\eta_R^1(t) &= t + i/\sqrt{2R}, \quad 1/\sqrt{2R} \leq t \leq R, \\
\eta_R^2(t) &= Re^{it}, \quad \theta_0 \leq t \leq 2\pi - \theta_0,
\end{aligned}$$

where $\theta_0(R) = \tan^{-1}(1/(R\sqrt{2R}))$

$$\begin{aligned}
\eta_R^3(t) &= R - t - i/\sqrt{2R}, \quad 0 \leq t \leq R - 1/\sqrt{2R}, \\
\eta_R^4(t) &= e^{-it}/\sqrt{R}, \quad \pi/4 \leq t \leq 7\pi/4.
\end{aligned}$$

Now

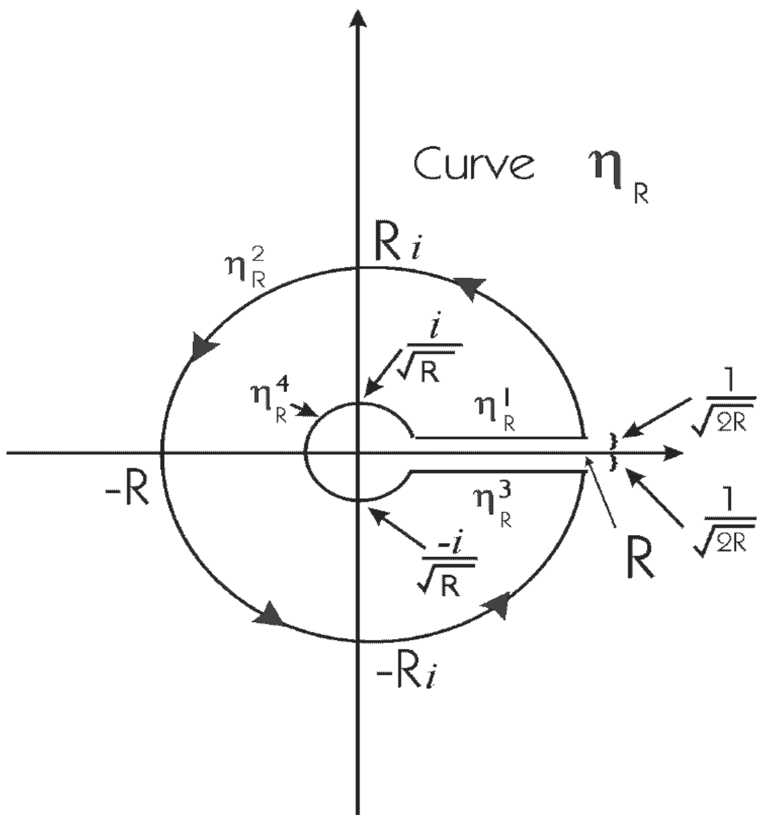

Figure 5.17: The curve μ_R in Section 5.5.6.

$$\begin{aligned}
\oint_{\eta_R} g(z)\,dz &= 2\pi i(\mathrm{Res}_{\eta_R}(-2)\cdot 1 + \mathrm{Res}_{\eta_R}(-4)\cdot 1) \\
&= 2\pi i\left(\frac{\log(-2)}{2} + \frac{\log(-4)}{-2}\right) \\
&= 2\pi i\left(\frac{\log 2 + \pi i}{2} + \frac{\log 4 + \pi i}{-2}\right) \\
&= -\pi i \log 2\,. \qquad (5.65)
\end{aligned}$$

Also, it is straightforward to check that

$$\left|\oint_{\eta_R^2} g(z)\,dz\right| \to 0\,, \qquad (5.66)$$

$$\left|\oint_{\eta_R^4} g(z)\,dz\right| \to 0\,, \qquad (5.67)$$

as $R \to +\infty$. The device that makes this technique work is that, as $R \to +\infty$,

$$\log(x + i/\sqrt{2R}) - \log(x - i/\sqrt{2R}) \to -2\pi i\,. \qquad (5.68)$$

So

$$\oint_{\eta_R^1} g(z)\,dz + \oint_{\eta_R^3} g(z)\,dz \to -2\pi i \int_0^\infty \frac{dt}{t^2+6t+8}\,. \qquad (5.69)$$

Now (5.65) through (5.69) taken together yield

$$\int_0^\infty \frac{dt}{t^2+6t+8} = \frac{1}{2}\log 2\,. \qquad (5.70)$$

5.5.7 Summary Chart of Some Integration Techniques

In what follows we present, in chart form, just a few of the key methods of using residues to evaluate definite integrals.

Use of Residues to Evaluate Integrals

Integral	Properties of	Value of Integral
$I = \int_{-\infty}^{\infty} f(x)\,dx$	No poles of $f(z)$ on real axis. Finite number of poles of $f(z)$ in plane. $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert^2}$ for z large.	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } f \text{ in upper} \\ \text{half-plane} \end{pmatrix}$
$I = \int_{-\infty}^{\infty} f(x)\,dx$	$f(z)$ may have simple poles on real axis. Finite number of poles of $f(z)$ in plane. $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert^2}$ for z large.	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } f \text{ in upper} \\ \text{half-plane} \end{pmatrix} + \pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } f(z) \\ \text{on real axis} \end{pmatrix}$
$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$	p, q polynomials. $[\deg p] + 2 \leq \deg q$. q has no real zeros.	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } p(z)/q(z) \\ \text{in upper half} \\ \text{plane} \end{pmatrix}$

Use of Residues to Evaluate Integrals, Continued

Integral	**Properties of**	**Value of Integral**
$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx$	p, q polynomials. $[\deg p] + 2 \leq \deg q$. $p(z)/q(z)$ may have simple poles on real axis.	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } p(z)/q(z) \\ \text{in upper half} \\ \text{plane} \end{pmatrix} + \pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } p(z)/q(z) \\ \text{on real axis} \end{pmatrix}$
$I = \int_{-\infty}^{\infty} e^{i\alpha x} \cdot f(x)\, dx$	$\alpha > 0$, z large $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert}$ No poles of f on real axis.	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } e^{i\alpha z} f(z) \\ \text{in upper half} \\ \text{plane} \end{pmatrix}$
$I = \int_{-\infty}^{\infty} e^{i\alpha x} \cdot f(x)\, dx$	$\alpha > 0$, z large $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert}$ $f(z)$ may have simple poles on real axis	$I = 2\pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } e^{i\alpha z} f(z) \\ \text{in upper half} \\ \text{plane} \end{pmatrix} + \pi i \times \begin{pmatrix} \text{sum of residues} \\ \text{of } e^{i\alpha z} f(z) \\ \text{on real axis} \end{pmatrix}$

Exercises

Use the calculus of residues to calculate the integrals in Exercises 1 through 13:

1. $$\int_0^{+\infty} \frac{1}{1+x^4}\, dx$$

2. $$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4}\, dx$$

3. $$\int_0^{+\infty} \frac{x^{1/3}}{1+x^2}\, dx$$

4. $$\int_0^{+\infty} \frac{1}{x^3+x+1}\, dx$$

5. $$\int_0^{+\infty} \frac{1}{1+x^3}\, dx$$

6. $$\int_0^{+\infty} \frac{x \sin x}{1+x^2}\, dx$$

7. $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx$$

8. $$\int_{-\infty}^{+\infty} \frac{x}{\sinh x}\, dx$$

9. $$\int_{-\infty}^{+\infty} \frac{x^2}{1+x^6}\, dx$$

10. $$\int_{-\infty}^{\infty} \frac{x^{1/3}}{-1+x^5}\, dx$$

11. $$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx$$

12. Interpret the first two examples in this section in terms of incompressible fluid flow.

5.6 Meromorphic Functions and Singularities at Infinity

5.6.1 Meromorphic Functions

We have considered carefully those functions that are holomorphic on sets of the form $D(P,r) \setminus \{P\}$ or, more generally, of the form $U \setminus \{P\}$, where U is an open set in $\mathbb{C}$ and $P \in U$. As we have seen in our discussion of the calculus of residues, sometimes it is important to consider the possibility that a function could be "singular" at more than just one point. The appropriate precise definition requires a little preliminary consideration of what kinds of sets might be appropriate as "sets of singularities."

5.6.2 Discrete Sets and Isolated Points

We review the concept of discrete. A set S in $\mathbb{C}$ is *discrete* if and only if for each $z \in S$ there is a positive number r (depending on z) such that

$$S \cap D(z,r) = \{z\}\,. \tag{5.71}$$

We also say in this circumstance that S consists of isolated points.

5.6.3 Definition of a Meromorphic Function

Now fix an open set U; we next define the central concept of meromorphic function on U.

A *meromorphic function* f on U *with singular set* S is a function $f : U \setminus S \to \mathbb{C}$ such that

(5.72) S is discrete;
(5.73) f is holomorphic on $U \setminus S$ (note that $U \setminus S$ is necessarily open in $\mathbb{C}$);
(5.74) for each $P \in S$ and $r > 0$ such that $D(P,r) \subseteq U$ and $S \cap D(P,r) = \{P\}$, the function $f\big|_{D(P,r)\setminus\{P\}}$ has a (finite order) pole at P.

For convenience, one often suppresses explicit consideration of the set S and just says that f is a meromorphic function on U. Sometimes we say, informally, that a meromorphic function on U is a function on U that is holomorphic "except for poles." Implicit in this description is the idea that a pole is an "isolated singularity." In other words, a point P is a pole of f if and only if there is a disc $D(P, r)$ around P such that f is holomorphic on $D(P, r) \setminus \{P\}$ and has a pole at P. Back on the level of precise language, we see that our definition of a meromorphic function on U implies that, for each $P \in U$, either there is a disc $D(P, r) \subseteq U$ such that f is holomorphic on $D(P, r)$ *or* there is a disc $D(P, r) \subseteq U$ such that f is holomorphic on $D(P, r) \setminus \{P\}$ and has a pole at P.

5.6.4 Examples of Meromorphic Functions

Meromorphic functions are very natural objects to consider, primarily because they result from considering the (algebraic) reciprocals of holomorphic functions:

If U is a connected open set in $\mathbb{C}$ and if $f : U \to \mathbb{C}$ is a holomorphic function with $f \not\equiv 0$, then the function

$$F : U \setminus \{z : f(z) = 0\} \to \mathbb{C} \tag{5.75}$$

defined by $F(z) = 1/f(z)$ is a meromorphic function on U with singular set (or pole set) equal to $\{z \in U : f(z) = 0\}$. In a sense that can be made precise, all meromorphic functions arise as *quotients* of holomorphic functions.

5.6.5 Meromorphic Functions with Infinitely Many Poles

It is quite possible for a meromorphic function on an open set U to have infinitely many poles in U. The function $1/\sin(1/(1-z))$ is an obvious example on $U = D$. Notice, however, that the poles do not accumulate anywhere in D.

5.6.6 Singularities at Infinity

Our discussion so far of singularities of holomorphic functions can be generalized to include the limit behavior of holomorphic functions as $|z| \to +\infty$. This is a powerful method with many important consequences. Suppose for example that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. We can associate to f a new function $G : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by setting $G(z) = f(1/z)$. The behavior of the function G near 0 reflects, in an obvious sense, the behavior of f as $|z| \to +\infty$. For instance

$$\lim_{|z|\to+\infty} |f(z)| = +\infty \tag{5.76}$$

if and only if G has a pole at 0.

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$ and that, for some $R > 0, U \supseteq \{z : |z| > R\}$. Define $G : \{z : 0 < |z| < 1/R\} \to \mathbb{C}$ by $G(z) = f(1/z)$. Then we say that

(5.77) f has a *removable singularity* at ∞ if G has a removable singularity at 0.

(5.78) f has a *pole at* ∞ if G has a pole at 0.

(5.79) f has an *essential singularity* at ∞ if G has an essential singularity at 0.

5.6.7 The Laurent Expansion at Infinity

The Laurent expansion of G around 0, $G(z) = \sum_{-\infty}^{+\infty} a_j z^j$, yields immediately a series expansion for f which converges for $|z| > R$, namely,

$$f(z) \equiv G(1/z) = \sum_{-\infty}^{+\infty} a_j z^{-j} = \sum_{-\infty}^{+\infty} a_{-j} z^j \,. \tag{5.80}$$

The series $\sum_{-\infty}^{+\infty} a_{-n} z^n$ is called the *Laurent expansion of f around* ∞. It follows from our definitions and from our earlier discussions that f has a removable singularity at ∞ if and only if the Laurent series of f at ∞ has no *positive* powers of z with nonzero coefficients. Also f has a pole at ∞ if and only if the series has only a finite number of positive powers of z with nonzero coefficients. Finally, f has an essential singularity at ∞ if and only if the series has infinitely many positive powers.

5.6.8 Meromorphic at Infinity

Let f be an entire function with a removable singularity at infinity. This means, in particular, that f is bounded near infinity. But then f is a bounded, entire function so it is constant.

Now suppose that f is entire and has a pole at infinity. Then $G(z) = f(1/z)$ has a pole (of some order k) at the origin. Hence $z^k G(z)$ has a removable singularity at the origin. We conclude then that $z^{-k} \cdot f(z)$ has a removable singularity at ∞.

Thus $z^{-k} \cdot f(z)$ is bounded near infinity. Certainly f is bounded on any compact subset of the plane. All told, then,

$$|f(z)| \leq C(1 + |z|)^k .$$

Now examine the Cauchy estimates at the origin, on a disc $D(0, R)$, for the $(k+1)^{\text{st}}$ derivative of f. We find that

$$\left| \frac{\partial^{k+1}}{\partial z^{k+1}} f(0) \right| \leq \frac{(k+1)! C(1+R)^k}{R^{k+1}} .$$

As $R \to +\infty$ we find that the $(k+1)^{\text{st}}$ derivative of f at 0 is 0. In fact the same estimate can be proved at any point P in the plane. We conclude that $f^{(k+1)} \equiv 0$. Thus f must be a polynomial of degree at most k.

We have treated the cases of an entire function f having a removable singularity or a pole at infinity. The only remaining possibility is an essential singularity at infinity. The function $f(z) = e^z$ is an example of such a function. Any transcendental entire function has an essential singularity at infinity.

Suppose that f is a meromorphic function defined on an open set $U \subseteq \mathbb{C}$ such that, for some $R > 0$, we have $U \supseteq \{z : |z| > R\}$. We say that f is *meromorphic* at ∞ if the function $G(z) \equiv f(1/z)$ is meromorphic in the usual sense on $\{z : |z| < 1/R\}$.

5.6.9 Meromorphic Functions in the Extended Plane

The definition of "meromorphic at ∞" as given is equivalent to requiring that, for some $R' > R$, f has no poles in $\{z \in \mathbb{C} : R' < |z| < \infty\}$ *and* that f has a pole at ∞.

A meromorphic function f on $\mathbb{C}$ which is also meromorphic at ∞ must be a rational function (that is, a quotient of polynomials in z). Conversely, every rational function is meromorphic on $\mathbb{C}$ and at ∞.

Remark: It is conventional to rephrase the ideas just presented by saying that the only functions that are meromorphic in the "extended plane" are rational functions. We will say more about the extended plane in Sections 7.3.1 through 7.3.3.

Exercises

1. A holomorphic function f on a set of the form $\{z : |z| > R\}$, some $R > 0$, is said to have a zero at ∞ of order k if $f(1/z)$ has a zero of order k at 0. Using this definition as motivation, give a definition of *pole* of order k at ∞. If g has a pole of order k at ∞, what property does $1/g$ have at ∞? What property does $1/g(1/z)$ have at 0?

2. This exercise develops a notion of residue at ∞.

First, note that if f is holomorphic on a set $D(0,r) \setminus \{0\}$ and if $0 < s < r$, then "the residue at 0" $= \frac{1}{2\pi i} \oint_{\partial D(0,s)} g(z)\, dz$ picks out one particular coefficient of the Laurent expansion of f about 0, namely it equals a_{-1}. If g is defined and holomorphic on $\{z : |z| > R\}$, then the residue at ∞ of g is defined to be the negative of the residue at 0 of $H(z) = z^{-2} \cdot g(1/z)$ (Because a positively oriented circle about ∞ is negatively oriented with respect to the origin and vice versa, we defined the *residue of* g at ∞ to be the *negative* of the residue of H at 0.) Prove that the residue at ∞ of g is the coefficient of z in the Laurent expansion of g on $\{z : |z| > R\}$. Prove also that the definition of residue of g at ∞ remains unchanged if the origin is replaced by some other point in the finite plane.

3. Refer to Exercise 2 for terminology. Let $R(z)$ be a rational function (quotient of polynomials). Prove that the sum of all the residues (including the residue at ∞) of R is zero. Is this true for a more general class of functions than rational functions?

4. Refer to Exercise 2 for terminology. Calculate the residue of the given function at ∞.

(a) $f(z) = z^3 - 7z^2 + 8$

(b) $f(z) = z^2 e^z$

(c) $f(z) = (z+5)^2 e^z$

(d) $f(z) = p(z)e^z$, for p a polynomial

(e) $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomials

(f) $f(z) = \sin z$

(g) $f(z) = \cot z$

(h) $f(z) = \dfrac{e^z}{p(z)}$, where p is a polynomial

5. Give an example of a nontrivial holomorphic function on the upper half-plane that has infinitely many poles.

6. Give an example of an incompressible fluid flow with two poles of order 1. Consider the case where the residues add to zero, and the case where they do not add to zero. How do these situations differ in physical terms?

7. Let f be a meromorphic function on a region $U \subseteq \mathbb{C}$. Prove that the set of poles of f cannot have an interior accumulation point. [**Hint:** Consider the function $g = 1/f$. If the pole set of f has an interior accumulation point then the zero set of g has an interior accumulation point.]

Chapter 6

The Argument Principle

6.1 Counting Zeros and Poles

6.1.1 Local Geometric Behavior of a Holomorphic Function

In this chapter, we shall be concerned with questions that have a geometric, qualitative nature rather than an analytical, quantitative one. These questions center around the issue of the local geometric behavior of a holomorphic function.

6.1.2 Locating the Zeros of a Holomorphic Function

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on a connected, open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P, r) \subseteq U$. We know from the Cauchy integral formula that the values of f on $D(P, r)$ are completely determined by the values of f on $\partial D(P, r)$. In particular, the number and even the location of the zeros of f in $D(P, r)$ are determined in principle by f on $\partial D(P, r)$. But it is nonetheless a pleasant surprise that there is a *simple formula* for the number of zeros of f in $D(P, r)$ in terms of f (and f') on $\partial D(P, r)$. In order to obtain a precise formula, we shall have to agree to count zeros according to multiplicity (see Section 4.1.4). We now explain the precise idea.

Let $f : U \to \mathbb{C}$ be holomorphic as before, and assume that f has *some* zeros in U but that f is not identically zero. Fix $z_0 \in U$ such that $f(z_0) = 0$. Since the zeros of f are isolated, there is an $r > 0$ such that $\overline{D}(z_0, r) \subseteq U$ and such that f does not vanish on $\overline{D}(z_0, r) \setminus \{z_0\}$.

Now the power series expansion of f about z_0 has a first nonzero term determined by the least positive integer n such that $f^{(n)}(z_0) \neq 0$. (Note that $n \geq 1$ since $f(z_0) = 0$ by hypothesis.) Thus the power series expansion of f about z_0 *begins* with the nth term:

$$f(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^j . \tag{6.1}$$

Under these circumstances we say that f has a zero of *order* n (or *multiplicity* n) at z_0. When $n = 1$, then we also say that z_0 is a *simple* zero of f.

The important point to see here is that, near z_0,

$$\frac{f'(z)}{f(z)} \approx \frac{[n/n!] \cdot (\partial^n f/\partial z^n)(z_0)(z - z_0)^{n-1}}{[1/n!] \cdot (\partial^n f/\partial z^n)(z_0)(z - z_0)^n} = \frac{n}{z - z_0} .$$

It follows then that

$$\frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{f'(z)}{f(z)}\, dz \approx \frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{n}{z - z_0}\, dz = n .$$

On the one hand, this is an approximation. On the other hand, the approximation becomes more and more accurate as r shrinks to 0. And the value of the integral—which is a fixed integer!—is independent of r. Thus we may conclude that we have equality. We repeat that the value of the integral is an *integer*.

In short, the complex line integral of f'/f around the boundary of the disc gives the order of the zero at the center. If there are several zeros of f inside the disc $D(z_0, r)$ then we may break the complex line integral up into individual integrals around each of the zeros (see Figure 6.1), so we have the more general result that the integral of f'/f counts *all* the zeros inside the disc, together with their multiplicities. We shall consider this idea further in the discussion that follows.

6.1.3 Zero of Order n

The concept of zero of "order n," or "multiplicity n," for a function f is so important that a variety of terminology has grown up around it (see also Section 4.1.4). It has already been noted that, when the multiplicity $n = 1$, then the zero is sometimes called *simple*. For arbitrary n, we sometimes say

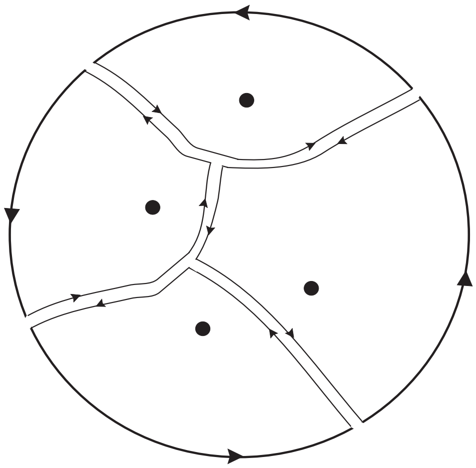

Figure 6.1: Dividing up the complex line integral to count the zeros.

that "n is the order of z_0 as a zero of f" or "f has a zero of order n at z_0." More generally, if $f(z_0) = \beta$ in such a way that, for some $n \geq 1$, the function $f(\cdot) - \beta$ has a zero of order n at z_0, then we say either that "f assumes the value β at z_0 to order n" or that "the order of the value β at z_0 is n." When $n > 1$, then we call z_0 a *multiple point* of the function f and we call β a *multiple value*.

EXAMPLE 48 The function $f(z) = (z-3)^4$ has a zero of order 4 at the point $z_0 = 3$. This is evident by inspection, because the power series for f about the point $z_0 = 3$ begins with the fourth-order term. But we may also note that $f(3) = 0$, $f'(3) = 0$, $f''(3) = 0$, $f'''(3) = 0$ while $f^{(iv)}(3) = 4! \neq 0$. According to our definition, then, f has a zero of order 4 at $z_0 = 3$.

The function $g(z) = 7 + (z-5)^3$ takes the value 7 at the point $z_0 = 5$ with multiplicity 3. This is so because $g(z) - 7 = (z-5)^3$ vanishes to order 3 at the point $z_0 = 5$. $\square$

The next result summarizes our preceding discussion. It provides a method for computing the multiplicity n of the zero at z_0 from the values of f, f' on the boundary of a disc centered at z_0.

6.1.4 Counting the Zeros of a Holomorphic Function

THEOREM 3 *If f is holomorphic on a neighborhood of a disc $\overline{D}(P, r)$ and has a zero of order n at P and no other zeros in the closed disc, then*

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = n. \tag{6.2}$$

More generally, we consider the case that f has several zeros—with different locations and different multiplicities—inside a disc: Suppose that $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P, r) \subseteq U$. Suppose further that f is nonvanishing on $\partial D(P, r)$ and that $z_1, z_2, \ldots, z_k$ are the zeros of f in the interior of the disc. Let n_ℓ be the order of the zero of f at z_ℓ, $\ell = 1, \ldots, k$. Then

$$\frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \sum_{\ell=1}^{k} n_\ell. \tag{6.3}$$

Refer to Figure 6.2 for illustrations of both these situations.

It is worth noting that the particular features of a *circle* play no special role in these considerations. We could as well consider the zeros of a function f that lie inside a simple, closed curve γ. Then it still holds that

$$\text{(number of zeros inside } \gamma \text{, counting multiplicity)} = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \, dz \, . \tag{6.4}$$

Example 49 Use the idea of formula (6.4) to calculate the number of zeros of the function $f(z) = z^2 + z$ inside the disc $D(0, 2)$. □

Solution: Of course we may see by inspection that the function f has precisely two zeros inside the disc (and no zeros on the boundary of the disc). But the point of the exercise is to get some practice with formula (6.4).

We calculate

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{\partial D(0,2)} \frac{f'(z)}{f(z)} \, dz &= \frac{1}{2\pi i} \oint_{\partial D(0,2)} \frac{2z+1}{z^2+z} \, dz \\
&= \frac{1}{2\pi i} \oint_{\partial D(0,2)} \frac{2}{z+1} \, dz + \frac{1}{2\pi i} \oint_{\partial D(0,2)} \frac{1}{z(z+1)} \, dz \, .
\end{aligned}$$

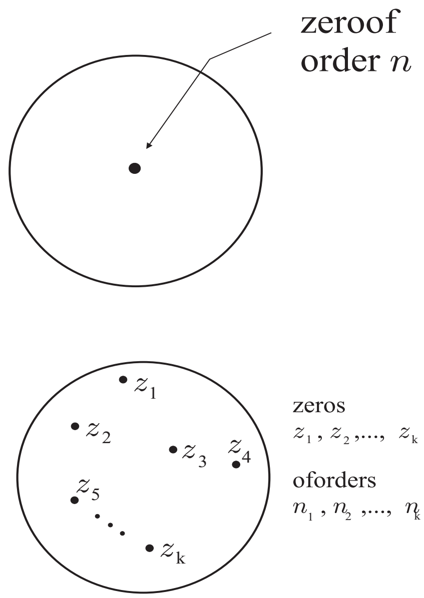

Figure 6.2: Locating the zeros of a holomorphic function.

Now the first integral is a simple Cauchy integral of the function $\phi(z) \equiv 2$, evaluating it at the point $z = -1$. This gives the value 2. The second integral is a double Cauchy integral; here we are integrating the function $\psi(z) \equiv 1/(z+1)$ and evaluating it at the point 0 and then integrating the function $1/z$ and evaluating it at the point -1. The result is $1 - 1 = 0$. Altogether then, the value of our original Cauchy integral is $2 + 0 = 2$. And, indeed, that is the number of zeros of the function f inside the disc $D(0, 2)$. $\square$

Exercise for the Reader: Use formula (6.4) to determine the number of zeros of the function $g(z) = \cos z$ inside the disc $D(0, 4)$.

6.1.5 The Idea of the Argument Principle

This last formula, which is often called the *argument principle*, is both useful and important. For one thing, there is no obvious reason why the integral in the formula should be an integer, much less the crucial integer that it is. Since it is an integer, it is a counting function; and we need to learn more about it.

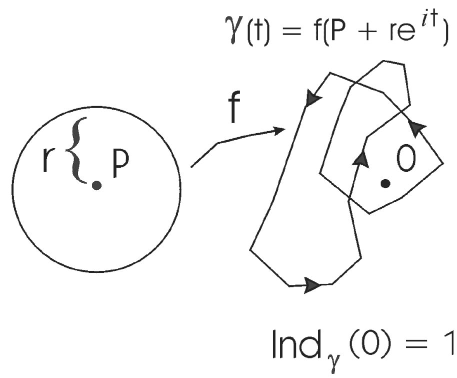

Figure 6.3: The argument principle: counting the zeros.

The integral

$$\frac{1}{2\pi i}\oint_{|\zeta-P|=r}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta \tag{6.5}$$

can be reinterpreted as follows: Consider the C^1 closed curve

$$\gamma(t)=f(P+re^{it})\,,\quad t\in[0,2\pi]. \tag{6.6}$$

Then

$$\frac{1}{2\pi i}\oint_{|\zeta-P|=r}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)}dt, \tag{6.7}$$

as you can check by direct calculation. The expression on the right is just the index of the curve γ with respect to 0 (with the notion of index that we defined earlier—Section 5.4.5). See Figure 6.3. Thus the number of zeros of f (counting multiplicity) inside the circle $\{\zeta : |\zeta - P| = r\}$ is equal to the index of γ with respect to the origin. This, intuitively speaking, is equal to the number of times that the f-image of the boundary circle winds around 0 in $\mathbb{C}$. So we have another way of seeing that the value of the integral must be an integer.

The argument principle can be extended to yield information about meromorphic functions, too. We can see that there is hope for this notion by investigating the analog of the argument principle for a pole.

6.1.6 Location of Poles

If $f : U \setminus \{Q\} \to \mathbb{C}$ is a nowhere-zero holomorphic function on $U \setminus \{Q\}$ with a pole of order n at Q and if $\overline{D}(Q, r) \subseteq U$, then

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = -n. \tag{6.8}$$

The argument is just the same as the calculations we did right after formula (6.1). [Or else think about the fact that if f has a pole of order n at Q then $1/f$ has a zero of order n at Q. In fact notice that $(1/f)'/(1/f) = -f'/f$. That accounts for the minus sign that arises for a pole.] We shall not repeat the details, but we invite the reader to do so.

6.1.7 The Argument Principle for Meromorphic Functions

Just as with the argument principle for holomorphic functions, this new argument principle gives a counting principle for zeros and poles of meromorphic functions:

Suppose that f is a meromorphic function on an open set $U \subseteq \mathbb{C}$, that $\overline{D}(P, r) \subseteq U$, and that f has neither poles nor zeros on $\partial D(P, r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{n=1}^{p} n_n - \sum_{k=1}^{q} m_k, \tag{6.9}$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ of f in $D(P, r)$ and $m_1, m_2, \ldots, m_q$ are the multiplicities of the poles $w_1, w_2, \ldots, w_q$ of f in $D(P, r)$.

Of course the reasoning here is by now familiar. We can break up the complex line integral around the boundary of the disc $D(P, r)$ into integrals around smaller regions, each of which contains just one zero or one pole and no other. Refer again to Figure 6.1. Thus the integral around the disc just sums up $+r$ for each zero of order r and $-s$ for each pole of order s.

Exercises

1. Use the argument principle to give another proof of the Fundamental Theorem of Algebra. [**Hint:** Think about the integral of $p'(z)/p(z)$ over circles centered at the origin of larger and larger radius.]

2. Suppose that f is holomorphic and has n zeros, counting multiplicities, inside U. Can you conclude that f' has $(n-1)$ zeros inside U? Can you conclude anything about the zeros of f'?

3. **Prove:** If f is a polynomial on $\mathbb{C}$, then the zeros of f' are contained in the closed convex hull of the zeros of f. (Here the *closed convex hull* of a set S is the intersection of all closed convex sets that contain S.) [**Hint:** If the zeros of f are contained in a half-plane V, then so are the zeros of f'.]

4. Let $P_t(z)$ be a polynomial in z for each fixed value of $t, 0 \leq t \leq 1$. Suppose that $P_t(z)$ is continuous in t in the sense that

$$P_t(z) = \sum_{n=0}^{N} a_n(t) z^n$$

and each $a_n(t)$ is continuous. Let $\mathcal{Z} = \{(z,t) : P_t(z) = 0\}$. By continuity, $\mathcal{Z}$ is closed in $\mathbb{C} \times [0,1]$. If $P_{t_0}(z_0) = 0$ and $(\partial/\partial z)\, P_{t_0}(z)\Big|_{z=z_0} \neq 0$, then show, using the argument principle, that there is an $\epsilon > 0$ such that for t sufficiently near t_0 there is a unique $z \in D(z_0, \epsilon)$ with $P_t(z) = 0$. What can you say if $P_{t_0}(\cdot)$ vanishes to order k at z_0?

5. Prove that if $f : U \to \mathbb{C}$ is holomorphic, $P \in U$, and $f'(P) = 0$, then f is not one-to-one in any neighborhood of P.

6. **Prove:** If f is holomorphic on a neighborhood of the closed unit disc D and if f is one-to-one on ∂D, then f is one-to-one on $\overline{D}$. [*Note:* Here you may assume any topological notions you need that seem intuitively plausible. Remark on each one as you use it.]

7. Let $p_t(z) = a_0(t) + a_1(t)z + \cdots + a_n(t)z^n$ be a polynomial in which the coefficients depend continuously on a parameter $t \in (-1,1)$. Prove that if the roots of p_{t_0} are distinct (no multiple roots), for some fixed value of the parameter, then the same is true for p_t when t is sufficiently close to t_0—*provided* that the degree of p_t remains the same as the degree of p_{t_0}.

8. Imitate the proof of the argument principle to prove the following formula: If $f : U \to \mathbb{C}$ is holomorphic in U and invertible as a function,

$P \in U$, and if $D(P, r)$ is a sufficiently small disc about P, then

$$f^{-1}(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{\zeta f'(\zeta)}{f(\zeta) - w} \, d\zeta$$

for all w in some disc $D(f(P), r_1)$, $r_1 > 0$ sufficiently small. Derive from this the formula

$$(f^{-1})'(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{\zeta f'(\zeta)}{(f(\zeta) - w)^2} \, d\zeta.$$

Set $Q = f(P)$. Integrate by parts and use some algebra to obtain

$$(f^{-1})'(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \left(\frac{1}{f(\zeta) - Q} \right) \cdot \left(1 - \frac{w - Q}{f(\zeta) - Q} \right)^{-1} d\zeta. \quad (6.10)$$

Let a_k be the k^{th} coefficient of the power series expansion of f^{-1} about the point Q :

$$f^{-1}(w) = \sum_{k=0}^{\infty} a_k (w - Q)^k.$$

Then formula (6.10) may be expanded and integrated term by term (prove this!) to obtain

$$\begin{aligned} n a_n &= \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{1}{[f(\zeta) - Q]^n} \, d\zeta \\ &= \frac{1}{(n-1)!} \left(\frac{\partial}{\partial \zeta} \right)^{n-1} \left. \frac{(\zeta - P)^n}{[f(\zeta) - Q]^n} \right|_{\zeta = P}. \end{aligned}$$

This is called *Lagrange's formula.*

9. Write a `MatLab` routine to calculate the winding number of any given closed curve about a point not on that curve. What can you do to guarantee that your answer will be an integer? [**Hint:** Think about roundoff error.]

10. Let $D(P, r)$ be a disc in the complex plane and let $p(z)$ be a polynomial. Assume that p has no zeros on the boundary of the disc. Write a `MatLab` routine to calculate the complex line integral that will give the number of zeros of p inside the disc.

11. With reference to the last exercise, suppose that $m(z)$ is a quotient of polynomials. Write a `MatLab` routine that will calculate the number of zeros (counting multiplicity) less the number of poles (counting multiplicity).

12. Give a physical interpretation of the argument principle for an incompressible fluid flow. What does a vanishing point of the flow mean? Why should it be true that the vanishing points (together with their multiplicities) can be detected by the behavior of the flow on the boundary of a disc containing the vanishing points?

6.2 The Local Geometry of Holomorphic Functions

6.2.1 The Open Mapping Theorem

The argument principle for holomorphic functions has a consequence that is one of the most important facts about holomorphic functions considered as geometric mappings:

THEOREM 4 *If $f : U \to \mathbb{C}$ is a nonconstant holomorphic function on a connected open set U, then $f(U)$ is an open set in $\mathbb{C}$.*

See Figure 6.4. The result says, in particular, that if $U \subseteq \mathbb{C}$ is connected and open and if $f : U \to \mathbb{C}$ is holomorphic, then either $f(U)$ is a connected open set (the nonconstant case) or $f(U)$ is a single point.

The open mapping principle has some interesting and important consequences. Among them are:

(a) If U is a domain in $\mathbb{C}$ and $f : U \to \mathbb{R}$ is a holomorphic function then f must be constant. For the theorem says that the image of f must be *open* (as a subset of the plane), and the real line contains no planar open sets.

(b) Let U be a domain in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. Suppose that the set E lies in the image of f. Then the image of f must in fact contain a neighborhood of E.

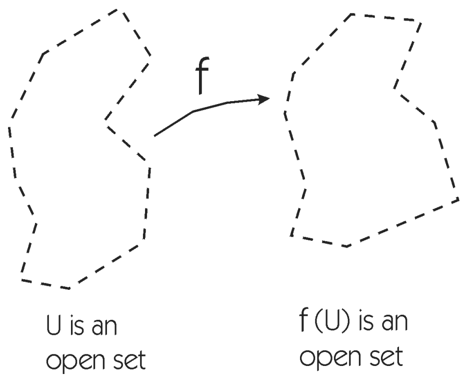

Figure 6.4: The open mapping principle.

(c) Let U be a domain in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. Let $P \in U$ and set $k = |f(P)|$. Then k cannot be the maximum value of $|f|$. For in fact (by part **(b)**) the image of f must contain an entire neighborhood of $f(P)$. So (see Figure 6.5), it will certainly contain points with modulus larger than k. This is a version of the important *maximum principle* which we shall discuss in some detail below.

In fact the open mapping principle is an immediate consequence of the argument principle. For suppose that $f : U \to \mathbb{C}$ is holomorphic and that $P \in U$. Write $f(P) = Q$. We may select an $r > 0$ so that $\overline{D}(P, r) \subseteq U$. Let $g(z) = f(z) - Q$. Then g has a zero at P.

The argument principle now tells us that

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(z)}{g(z)}\, dz \geq 1\,.$$

[We do not write $= 1$ because we do not know the order of vanishing of g—but it is *at least* 1.] In other words,

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(z)}{f(z) - Q}\, dz \geq 1\,.$$

But now the continuity of the integral tells us that, if we perturb Q by a small amount, then the value of the integral—which still must be an integer!—will not change. So it is still ≥ 1. This says that f assumes all values that are

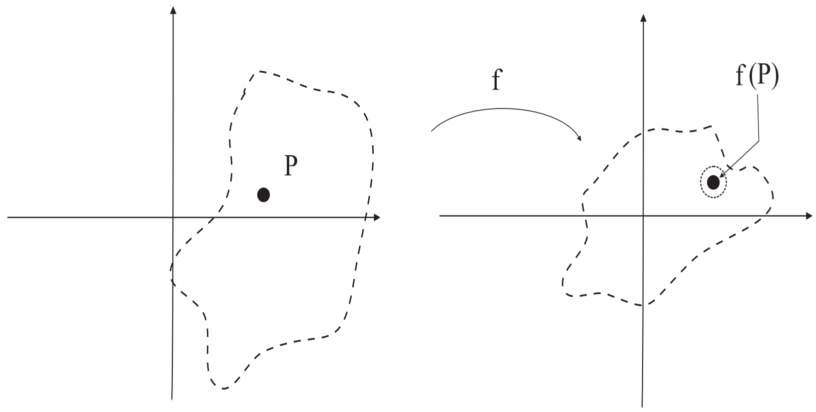

Figure 6.5: The image of f contains a neighborhood of $f(P)$.

near to Q. Which says that the image of f contains a neighborhood of Q; so it is open. That is the assertion of the open mapping principle.

In the subject of topology, a function f is defined to be continuous if the inverse image of any open set under f is also open. In contexts where the $\epsilon - \delta$ definition makes sense, the $\epsilon - \delta$ definition (Section 2.1.6) is equivalent to the inverse-image-of-open-sets definition. By contrast, functions for which the direct image of any open set is open are called "open mappings."

Here is a quantitative, or counting, statement that comes from the proof of the open mapping principle: Suppose that $f : U \to \mathbb{C}$ is a nonconstant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order $k \geq 1$. Then there are numbers $\delta, \epsilon > 0$ such that each $q \in D(Q, \epsilon) \setminus \{Q\}$ has exactly k distinct preimages in $D(P, \delta)$ and each preimage is a simple point of f. This is a striking statement; but all we are saying is that the set of points where f' vanishes cannot have an interior accumulation point. An immediate corollary is that if $f(P) = Q$ and $f'(P) = 0$ then f *cannot* be one-to-one in any neighborhood of P. For $g(z) \equiv f(z) - Q$ vanishes to order at least 2 at P. More generally, if f vanishes to order $k \geq 2$ at P then f is k-to-1 in a deleted neighborhood of P.

The considerations that establish the open mapping principle can also be used to establish the fact that if $f : U \to V$ is a one-to-one and onto holomorphic function, then $f^{-1} : V \to U$ is also holomorphic.

Exercises

1. Let f be holomorphic on a neighborhood of $\overline{D}(P, r)$. Suppose that f is not identically zero on $D(P, r)$. Prove that f has at most finitely many zeros in $D(P, r)$.

2. Let f, g be holomorphic on a neighborhood $\overline{D}(0, 1)$. Assume that f has zeros at $P_1, P_2, \ldots, P_k \in D(0, 1)$ and no zero in $\partial D(0, 1)$. Let γ be the boundary circle of $\overline{D}(0, 1)$, traversed counterclockwise. Compute

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \cdot g(z) dz.$$

3. Without supposing that you have any prior knowledge of the calculus function e^x, prove that

$$e^z \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

never vanishes by computing $(e^z)'/e^z$, and so forth.

4. Let $f_n : D(0, 1) \to \mathbb{C}$ be holomorphic and suppose that each f_n has at least k roots in $D(0, 1)$, counting multiplicities. Suppose that $f_n \to f$ uniformly on compact sets. Show by example that it does *not* follow that f has at least k roots counting multiplicities. In particular, construct examples, for each fixed k and each ℓ, $0 \leq \ell \leq k$, where f has exactly ℓ roots. What simple hypothesis can you add that will guarantee that f *does* have at least k roots? (Cf. Exercise 8.)

5. Let $f : D(0, 1) \to \mathbb{C}$ be holomorphic and nonvanishing. Prove that f has a well-defined holomorphic logarithm on $D(0, 1)$ by showing that the differential equation

$$\frac{\partial}{\partial z} g(z) = \frac{f'(z)}{f(z)}$$

has a suitable solution and checking that this solution g does the job.

6. Let U and V be open subsets of $\mathbb{C}$. Suppose that $f : U \to V$ is holomorphic, one-to-one, and onto. Prove that f^{-1} is a holomorphic function on V.

7. Let $f : U \to \mathbb{C}$ be holomorphic. Assume that $\overline{D}(P,r) \subseteq U$ and that f is nowhere zero on $\partial D(P,r)$. Show that if g is holomorphic on U and g is sufficiently uniformly close to f on $\partial D(P,r)$, then the number of zeros of f in $D(P,r)$ equals the number of zeros of g in $D(P,r)$. (Remember to count zeros according to multiplicity.)

8. What does the open mapping principle say about an incompressible fluid flow? Why does this make good physical sense? Why is it clear that the flow applied to an open region will never have a "boundary?"

9. Suppose that U is a simply connected domain in $\mathbb{C}$. Let f be a non-vanishing holomorphic function on U. Then f will have a holomorphic logarithm. That logarithm may be defined using a complex line integral [**Hint:** Integrate f'/f.] Write a `MatLab` routine to carry out this procedure in the case that f is a holomorphic polynomial.

6.3 Further Results on the Zeros of Holomorphic Functions

6.3.1 Rouché's Theorem

Now we consider global aspects of the argument principle.

Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subseteq \mathbb{C}$. Suppose also that $\overline{D}(P,r) \subseteq U$ and that, for each $\zeta \in \partial D(P,r)$,

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|. \tag{6.11}$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta. \tag{6.12}$$

That is, the number of zeros of f in $D(P,r)$ counting multiplicities equals the number of zeros of g in $D(P,r)$ counting multiplicities. See [GRK] for a more complete discussion and proof of Rouché's theorem.

Remark: Rouché's theorem is often stated with the stronger hypothesis that

$$|f(\zeta) - g(\zeta)| < |g(\zeta)| \tag{6.13}$$

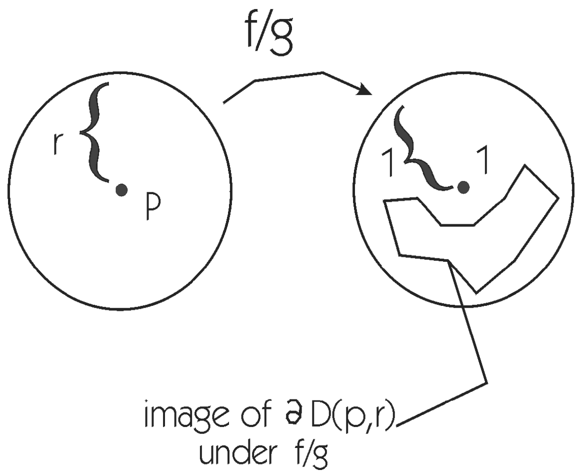

Figure 6.6: Rouché's theorem.

for $\zeta \in \partial D(P, r)$. Rewriting this hypothesis as

$$\left| \frac{f(\zeta)}{g(\zeta)} - 1 \right| < 1, \tag{6.14}$$

we see that it says that the image γ under f/g of the circle $\partial D(P, r)$ lies in the disc $D(1, 1)$. See Figure 6.6. Our weaker hypothesis that $|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|$ has the geometric interpretation that $f(\zeta)/g(\zeta)$ lies in the set $\mathbb{C} \setminus \{x + i0 : x \leq 0\}$. Either hypothesis implies that the image of the circle $\partial D(P, r)$ under f has the same "winding number" around 0 as does the image under g of that circle. And that is the proof of Rouché's theorem.

6.3.2 Typical Application of Rouché's Theorem

EXAMPLE 50 Let us determine the number of roots of the polynomial $f(z) = z^7 + 5z^3 - z - 2$ in the unit disc. We do so by comparing the function f to the holomorphic function $g(z) = 5z^3$ on the unit circle. For $|z| = 1$ we have

$$|f(z) - g(z)| = |z^7 - z - 2| \leq 4 < 5 = |g(z)| \leq |f(z)| + |g(z)|. \tag{6.15}$$

By Rouché's theorem, f and g have the same number of zeros, counting multiplicity, in the unit disc. Since g has three zeros, so does f. $\square$

6.3.3 Rouché's Theorem and the Fundamental Theorem of Algebra

Rouché's theorem provides a useful way to locate approximately the zeros of a holomorphic function that is too complicated for the zeros to be obtained explicitly. As an illustration, we analyze the zeros of a nonconstant polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0. \tag{6.16}$$

If R is sufficiently large (say $R > \max\bigl\{1, n \cdot \max_{0\le n\le n-1} |a_n|\bigr\}$) and $|z| = R$, then

$$\frac{|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0|}{|z^n|} < 1. \tag{6.17}$$

Thus Rouché's theorem applies on $\overline{D}(0, R)$ with $f(z) = z^n$ and $g(z) = p(z)$. We conclude that the number of zeros of $p(z)$ inside $D(0, R)$, counting multiplicities, is the same as the number of zeros of z^n inside $D(0, R)$, counting multiplicities—namely n. Thus we recover the Fundamental Theorem of Algebra. Incidentally, this example underlines the importance of counting zeros with multiplicities: the function z^n has only one root in the naïve sense of counting the number of points where it is zero; but it has n roots when they are counted with multiplicity. So Rouché's theorem teaches us that a polynomial of degree n has n zeros—just as it should.

6.3.4 Hurwitz's Theorem

A second useful consequence of the argument principle is the following result about the limit of a sequence of zero-free holomorphic functions:

THEOREM 5 (Hurwitz's Theorem) *Suppose that* $U \subseteq \mathbb{C}$ *is a connected open set and that* $\{f_j\}$ *is a sequence of nowhere-vanishing holomorphic functions on* U*. If the sequence* $\{f_j\}$ *converges uniformly on compact subsets of* U *to a (necessarily holomorphic) limit function* f_0*, then either* f_0 *is nowhere-vanishing or* $f_0 \equiv 0$.

The justification for Hurwitz's theorem is again the argument principle. For we know that if $\overline{D}(P, r)$ is a closed disc on which all the f_j are zero-free then

$$\frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f_j'(z)}{f_j(z)}\, dz = 0$$

for every j. The limit function f is surely holomorphic. If it is not identically zero, then suppose seeking a contradiction that it has a zero—which is of course isolated—at some point P. Choose $r > 0$ small so that f has no other zeros on $\overline{D}(P, r)$. Since the f_j (and hence the f_j') converge uniformly on $\overline{D}(P, r)$, we can be sure that as $j \to +\infty$ the expression on the left then converges to

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(z)}{f(z)}\, dz\,.$$

And the value of the integral must be zero. We conclude that f has no zeros in the disc, which is clearly a contradiction. Thus f is either identically zero or zero free.

Exercises

1. How many zeros does the function $f(z) = z^3 + z/2$ have in the unit disc?

2. Consider the sequence of functions $f_j(z) = e^{z/j}$. Discuss this sequence in view of Hurwitz's theorem.

3. Consider the sequence of functions $f_j(z) = \sin(jz)$. Discuss in view of Hurwitz's theorem.

4. Consider the sequence of functions $f_j(z) = \cos(z/j)$. Discuss in view of Hurwitz's theorem.

5. Apply Rouché's theorem to see that e^z cannot vanish on the unit disc.

6. Use Rouché's theorem to give yet another proof of the Fundamental Theorem of Algebra. [**Hint:** If the polynomial has degree n, then compare the polynomial with z^n on a large disc.]

7. Estimate the number of zeros of the given function in the given region U.

(a) $f(z) = z^8 + 5z^7 - 20, \qquad U = D(0, 6)$
(b) $f(z) = z^3 - 3z^2 + 2, \qquad U = D(0, 1)$
(c) $f(z) = z^{10} + 10z + 9, \qquad U = D(0, 1)$

(d) $f(z) = z^{10} + 10ze^{z+1} - 9,$	$U = D(0,1)$
(e) $f(z) = z^4 e - z^3 + z^2/6 - 10,$	$U = D(0,2)$
(f) $f(z) = z^2 e^z - z,$	$U = D(0,2)$

8. Each of the partial sums of the power series for the function e^z is a polynomial. Hence it has zeros. But the exponential function has no zeros. Discuss in view of Hurwitz's theorem and the argument principle.

9. Each of the partial sums of the power series for the function $\sin z$ is a polynomial, hence it has finitely many zeros. Yet $\sin z$ has infinitely many zeros. Discuss in view of Hurwitz's theorem and the argument principle.

10. How many zeros does $f(z) = \sin z + \cos z$ have in the unit disc?

11. Let $D(P, r)$ be a disc in the complex plane. Let f and g be holomorphic polynomials. Write a `MatLab` routine to test whether Rouché's theorem applies to f and g. Write the routine so that it declares an appropriate conclusion.

6.4 The Maximum Principle

6.4.1 The Maximum Modulus Principle

A *domain* in $\mathbb{C}$ is a connected open set (Section 2.1.1). A *bounded domain* is a connected open set U such that there is an $R > 0$ with $|z| < R$ for all $z \in U$—or $U \subseteq D(0, R)$.

The Maximum Modulus Principle

> Let $U \subseteq \mathbb{C}$ be a domain. Let f be a holomorphic function on U. If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is constant.

Here is a sharper variant of the theorem:

> Let $U \subseteq \mathbb{C}$ be a domain and let f be a holomorphic function on U. If there is a point $P \in U$ at which $|f|$ has a *local maximum*, then f is constant.

We have already indicated why this result is true; the geometric insight is an important one. Let $k = |f(P)|$. Since $f(P)$ is an *interior point* of the image of f, there will certainly be points—and the proof of the open mapping principle shows that these are nearby points—where f takes values of greater modulus. Hence P cannot be a local maximum.

6.4.2 Boundary Maximum Modulus Theorem

The following version of the maximum principle is intuitively appealing, and is frequently useful.

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must in fact occur on ∂U.
>
> In other words,
>
> $$\max_{\overline{U}} |f| = \max_{\partial U} |f| . \tag{6.18}$$

And the reason for this new assertion is obvious. The maximum must occur somewhere; and it cannot occur in the interior by the previous formulation of the maximum principle. So it must be in the boundary.

6.4.3 The Minimum Modulus Principle

Holomorphic functions (or, more precisely, their moduli) *can* have interior minima. The function $f(z) = z^2$ on $D(0,1)$ has the property that $z = 0$ is a global minimum for $|f|$. However, it is not accidental that this minimum value is 0:

> Let f be holomorphic on a domain $U \subseteq \mathbb{C}$. Assume that f never vanishes. If there is a point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant.

This result is proved by applying the maximum principle to the function $1/f$.

There is also a boundary minimum modulus principle:

Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Assume that f never vanishes on $\overline{U}$. Then the minimum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U.

In other words,

$$\min_{\overline{U}} |f| = \min_{\partial U} |f| . \tag{6.19}$$

Exercises

1. Let $U \subseteq \mathbb{C}$ be a bounded domain. If f, g are continuous functions on $\overline{U}$, holomorphic on U, and if $|f(z)| \leq |g(z)|$ for $z \in \partial U$, then what conclusion can you draw about f and g in the interior of U?

2. Let $f : \overline{D}(0,1) \to \overline{D}(0,1)$ be continuous and holomorphic on the interior. Further assume that f is one-to-one and onto. Explain why the maximum principle guarantees that $f(\partial D(0,1)) \subseteq \partial D(0,1)$.

3. Give an example of a holomorphic function f on $D(0,1)$ so that $|f|$ has three local minima.

4. Give an example of a holomorphic function f on $D(0,1)$, continuous on $\overline{D}(0,1)$, that has precisely three global maxima on $\partial D(0,1)$.

5. The function

$$f(z) = i \cdot \frac{1-z}{1+z}$$

maps the disc $D(0,1)$ to the upper half-plane $\mathcal{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (the upper half-plane) in a one-to-one, onto fashion. Verify this assertion in the following manner:

(a) Use elementary algebra to check that f is one-to-one.

(b) Use just algebra to check that $\partial D(0,1)$ is mapped to ∂U.

(c) Check that 0 is mapped to i.

(d) Invoke the maximum principle to conclude that $D(0,1)$ is mapped to U.

6. Let f be meromorphic on a region $U \subseteq \mathbb{C}$. A version of the maximum principle is still valid for such an f. Explain why.

7. Let $U \subseteq \mathbb{C}$ be a domain and let $f : U \to \mathbb{C}$ be holomorphic. Consider the function $g(z) = e^{f(z)}$. Explain why the maxima of $|g|$ occur precisely at the maxima of $\operatorname{Re} f$. Conclude that a version of the maximum principle holds for $\operatorname{Re} f$. Draw a similar conclusion for $\operatorname{Im} f$.

8. Let $U, V \subseteq \mathbb{C}$ be bounded domains with continuously differentiable boundary. So U and V are open and connected. Let $\varphi : \overline{U} \to \overline{V}$ be continuous, one-to-one, and onto. And suppose that φ is holomorphic on U (and of course φ^{-1} is holomorphic on V). Show that $\varphi(\partial U) \subseteq \partial V$.

9. Let f be holomorphic on the entire plane $\mathbb{C}$. Suppose that

$$|f(z)| \leq C \cdot (1 + |z|^k)$$

for all $z \in \mathbb{C}$, some positive constant C and some integer $k > 0$. Prove that f is a polynomial of degree at most k.

10. Let U be a domain in the complex plane. Let f be a holomorphic polynomial. Write a `MatLab` routine that will find the location of the maximum value of $|f|^2$ in U. Apply this routine to various polynomials to confirm that the maximum never occurs on the boundary.

11. Modify the routine from the last exercise so that it applies to the minimum value of $|f|^2$—in the case that f is nonvanishing on U.

12. Suppose that two incompressible fluid flows are very close together on the boundary of a disc—just as in Rouché's theorem. What might we expect that this will tell us about the two fluid flows inside the disc? Why?

6.5 The Schwarz Lemma

This section treats certain estimates that must be satisfied by bounded holomorphic functions on the unit disc. We present the classical, analytic viewpoint in the subject (instead of the geometric viewpoint—see [KRA3]).

6.5.1 Schwarz's Lemma

THEOREM 6 *Let f be holomorphic on the unit disc. Assume that*

(6.20) $|f(z)| \leq 1$ *for all* z.
(6.21) $f(0) = 0$.

Then $|f(z)| \leq |z|$ *and* $|f'(0)| \leq 1$.

If either $|f(z)| = |z|$ *for some* $z \neq 0$ *or if* $|f'(0)| = 1$, *then* f *is a rotation:* $f(z) \equiv \alpha z$ *for some complex constant* α *of unit modulus.*

Proof: Consider the function $g(z) = f(z)/z$. Since g has a removable singularity at the origin, we see that g is holomorphic on the entire unit disc. On the circle with center 0 and radius $1 - \epsilon$, we see that

$$|g(z)| \leq \frac{1}{1-\epsilon}\,.$$

By the maximum modulus principle, it follows that $|g(z)| \leq 1/(1-\epsilon)$ on all of $\overline{D}(0, 1-\epsilon)$. Since the conclusion is true for all $\epsilon > 0$, we conclude that $|g| \leq 1$ on $D(0,1)$.

For the uniqueness, assume that $|f(z)| = |z|$ for some $z \neq 0$. Then $|g(z)| = 1$. Since $|g| \leq 1$ globally, the maximum modulus principle tells us that g is a constant of modulus 1. Thus $f(z) = \alpha z$ for some unimodular constant α. If instead $|f'(0)| = 1$ then $|[g(0) + g'(0) \cdot 0| = 1$ or $|g(0)| = 1$. Again, the maximum principle tells us that g is a unimodular constant so f is a rotation. $\square$

Schwarz's lemma enables one to classify the invertible holomorphic self-maps of the unit disc (see [GRK]). (Here a *self-map* of a domain U is a mapping $F : U \to U$ of the domain to itself.) These are commonly referred to as the "conformal self-maps" of the disc. The classification is as follows: If $0 \leq \theta < 2\pi$, then define the *rotation through angle* θ to be the function $\rho_\theta(z) = e^{i\theta} z$; if a is a complex number of modulus less than one, then define the associated *Möbius transformation* to be $\varphi_a(z) = [z - a]/[1 - \overline{a}z]$. Any conformal self-map of the disc is the composition of some rotation ρ_θ with some Möbius transformation φ_a. This topic is treated in detail in Sections 7.2.1 and 7.2.2.

We conclude this section by presenting a generalization of the Schwarz lemma, in which we consider holomorphic mappings $f : D \to D$, but we discard the hypothesis that $f(0) = 0$. This result is known as the Schwarz-Pick lemma.

6.5.2 The Schwarz-Pick Lemma

Let f be holomorphic on the unit disc. Assume that

(6.22) $|f(z)| \le 1$ for all z.

(6.23) $f(a) = b$ for some $a, b \in D(0,1)$.

Then

$$|f'(a)| \le \frac{1-|b|^2}{1-|a|^2}\,. \tag{6.24}$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left|\frac{b_2 - b_1}{1-\overline{b}_1 b_2}\right| \le \left|\frac{a_2 - a_1}{1-\overline{a}_1 a_2}\right|\,. \tag{6.25}$$

There is a "uniqueness" result in the Schwarz-Pick Lemma. If either

$$|f'(a)| = \frac{1-|b|^2}{1-|a|^2} \qquad \text{or} \qquad \left|\frac{b_2 - b_1}{1-\overline{b}_1 b_2}\right| = \left|\frac{a_2 - a_1}{1-\overline{a}_1 a_2}\right| \quad \text{with } a_1 \ne a_2\,, \tag{6.26}$$

then the function f is a conformal self-mapping (one-to-one, onto holomorphic function) of $D(0,1)$ to itself.

We cannot discuss the proof of the Schwarz-Pick lemma right now. It depends on knowing the conformal self-maps of the disc—a topic we shall treat later. The reader should at least observe at this time that, in (6.24), if $a = b = 0$ then the result reduces to the classical Schwarz lemma. Further, in (6.25), if $a_1 = b_1 = 0$ and $a_2 = z$, $b_2 = f(z)$, then the result reduces to the Schwarz lemma.

Exercises

1. Let $\mathcal{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (the upper half-plane). Formulate and prove a version of the Schwarz lemma for holomorphic functions $f : U \to U$. [**Hint:** It is useful to note that the mapping $\psi(z) = i(1 - z)/(1 + z)$ maps the unit disc to U in a holomorphic, one-to-one, and onto fashion.]

2. Let U be as in Exercise 1. Formulate and prove a version of the Schwarz lemma for holomorphic functions $f : D(0, 1) \to U$.

3. There is no Schwarz lemma for holomorphic functions $f : \mathbb{C} \to \mathbb{C}$. Give a detailed justification for this statement. Can you suggest why the Schwarz lemma fails in this new context?

4. Give a detailed justification for the formula

$$(f^{-1})'(w) = \frac{1}{f'(z)} .$$

Here $f(z) = w$ and f is a holomorphic function. Part of your job here is to provide suitable hypotheses about the function f.

5. Provide the details of the proof of the Schwarz-Pick lemma. [**Hint:** If $f(a) = b$, then consider $g(z) = \varphi_b \circ f \circ \varphi_{-a}$ and apply the Schwarz lemma.]

6. The expression

$$\rho(z, w) = \frac{|z - w|}{|1 - z\overline{w}|}$$

for z, w in the unit disc is called the *pseudohyperbolic metric*. Prove that ρ is actually a metric, or a sense of distance, on the disc. This means that you should verify these properties:

(i) $\rho(z, w) \geq 0$ for all $z, w \in D(0, 1)$;

(ii) $\rho(z, w) = 0$ if and only if $z = w$;

(iii) $\rho(z, w) = \rho(w, z)$;

(iv) $\rho(z, w) \leq \rho(z, u) + \rho(u, w)$ for all $u, z, w \in D(0, 1)$.

7. Suppose that f is a holomorphic function on a domain $U \subseteq \mathbb{C}$. Assume that $|f(z)| \leq M$ for all $z \in U$ and some $M > 0$. Let $P \in U$. Use the Schwarz lemma to provide an estimate for $|f'(P)|$. [**Hint:** Your estimate will be in terms of M and the distance of P to the boundary of U.]

8. Write a `MatLab` routine that will calculate the pseudohyperbolic metric on the disc. You should be able to input two points from the disc and the routine should output a nonnegative real number that is the distance between them. Use this routine to amass numerical evidence that the distance from any fixed point in the disc to the boundary is infinite.

9. Suppose that $f : D \to D$ is a holomorphic function, that $f(0) = 0$, and that $\lim_{z \to \partial D} |f(z)| = 1$. Then of course Schwarz's lemma guarantees that $|f(z)| \leq |z|$ for all $z \in D$. Write a `MatLab` routine to measure the deviation of $|f(z)|$ from $|z|$. Apply it to various specific examples.

10. What does Schwarz's lemma tell us about the geometric characteristics of a fluid flow? How does this differ from an air flow? Why?

Chapter 7

The Geometric Theory of Holomorphic Functions

7.1 The Idea of a Conformal Mapping

7.1.1 Conformal Mappings

The main objects of study in this chapter are holomorphic functions $h : U \to V$, with U and V open domains in $\mathbb{C}$, that are one-to-one and onto. Such a holomorphic function is called a *conformal* (or *biholomorphic*) mapping. The fact that h is supposed to be one-to-one implies that h' is nowhere zero on U [remember that if h' vanishes to order $k \geq 1$ at a point $P \in U$, then h is $(k+1)$-to-1 in a small neighborhood of P—see Section 6.2.1]. As a result, $h^{-1} : V \to U$ is also holomorphic—as we discussed in Section 6.2.1. A conformal map $h : U \to V$ from one open set to another can be used to transfer holomorphic functions on U to V and vice versa: that is, $f : V \to \mathbb{C}$ is holomorphic if and only if $f \circ h$ is holomorphic on U; and $g : U \to \mathbb{C}$ is holomorphic if and only if $g \circ h^{-1}$ is holomorphic on V.

In fact the word "conformal" has a specific geometric meaning—in terms of infinitesimal preservation of length and infinitesimal preservation of angles. These properties in turn have particularly interesting interpretations in the context of incompressible fluid flow (see Section 8.2). In fact we discussed this way of thinking about conformality in Section 2.4.1. We shall explore other aspects of conformal mappings in the material that follows.

Thus, if there is a conformal mapping from U to V, then U and V are essentially indistinguishable from the viewpoint of complex function theory.

On a practical level, one can often study holomorphic functions on a rather complicated open set by first mapping that open set to some simpler open set, then transferring the holomorphic functions as indicated.

The main point now is that we are going to think of our holomorphic function $f : U \to V$ not as a function but as a mapping. That means that the function is a geometric transformation from the domain U to the domain V. And of course f^{-1} is a geometric transformation from the domain V to the domain U.

7.1.2 Conformal Self-Maps of the Plane

The simplest open subset of $\mathbb{C}$ is $\mathbb{C}$ itself. Thus it is natural to begin our study of conformal mappings by considering the conformal mappings of $\mathbb{C}$ to itself. In fact the conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ can be explicitly described as follows:

> A function $f : \mathbb{C} \to \mathbb{C}$ is a conformal mapping if and only if there are complex numbers a, b with $a \neq 0$ such that

$$f(z) = az + b \quad , \quad z \in \mathbb{C}. \tag{7.1}$$

One aspect of the result is fairly obvious: If $a, b \in \mathbb{C}$ and $a \neq 0$, then the map $z \mapsto az + b$ is certainly a conformal mapping of $\mathbb{C}$ to $\mathbb{C}$. In fact one checks easily that $z \mapsto (z - b)/a$ is the inverse mapping. The interesting part of the assertion is that these are in fact the only conformal maps of $\mathbb{C}$ to $\mathbb{C}$.

A generalization of this result about conformal maps of the plane is the following (consult Section 4.1.3 as well as the detailed explanation in [GRK]):

> If $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that

$$\lim_{|z| \to +\infty} |h(z)| = +\infty \,, \tag{7.2}$$

> then h is a polynomial.

In fact this last assertion is simply a restatement of the fact that if an entire function has a pole at infinity then it is a polynomial. We proved that fact in Section 5.6. Now if $f : \mathbb{C} \to \mathbb{C}$ is conformal then it is easy to see that $\lim_{|z|\to+\infty} |f(z)| = +\infty$—for both f and f^{-1} take bounded sets to bounded sets. So f will be a polynomial. But if f has degree $k > 1$ then it will not be one-to-one: the equation $f(z) = \alpha$ will always have k roots. Thus f is a first-degree polynomial, which is what has been claimed.

Exercises

1. How many points in the plane uniquely determine a conformal self-map of the plane? That is to say, what is the least k such that if $f(p_1) = p_1$, $f(p_2) = p_2$, $\ldots$, $f(p_k) = p_k$ (with $p_1, \ldots, p_k$ distinct) then $f(z) \equiv z$?

2. Let $U = \mathbb{C} \setminus \{0\}$. What are all the conformal self-maps of U to U?

3. Let $U = \mathbb{C} \setminus \{0, 1\}$. What are all the conformal self-maps of U to U?

4. The function $f(z) = e^z$ is an onto mapping from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. Prove this statement. The function is certainly *not* one-to-one. But it is *locally* one-to-one. Explain these assertions.

5. Refer to Exercise 4. The point i is in the image of f. Give an explicit description of the inverse of f near i.

6. The function $g(z) = z^2$ is an onto mapping from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$. It is certainly *not* one-to-one. But it is *locally* one-to-one. Explain these assertions.

7. The function $f(z) = e^z$ maps the strip $S = \{x + iy : 0 < x < 1\}$ conformally onto an annulus. Describe in detail this image annulus. Explain why the mapping is onto but not one-to-one. Explain why it is locally one-to-one.

8. The function $f(z) = z^2$ maps the quarter-disc $Q = \{x + iy : x > 0, y > 0, x^2 + y^2 < 1\}$ conformally onto the half-disc $H = \{x + iy : y > 0, x^2 + y^2 < 1\}$. Explain why this is a one-to-one, onto mapping.

9. Use what you have learned from the preceding exercises to construct a conformal map of the upper half-plane $\mathcal{U} = \{x + iy : y > 0\}$ onto the upper half-disc $H = \{x + iy : y > 0, x^2 + y^2 < 1\}$.

10. A conformal mapping should map a fluid flow to another fluid flow. Discuss why this should be true. Referring to Section 2.4, consider specifically the property of conformality and why that should be preserved.

11. Use `MatLab` to write a utility that will test a given function for conformality. That is, you should input the function itself, a base point, and two directions; the utility will test whether the function stretches equally in each direction. Or you can input the function, a base point, a direction, and an angle; the utility will test whether that angle is preserved.

7.2 Conformal Mappings of the Unit Disc

7.2.1 Conformal Self-Maps of the Disc

In this section we describe the set of all conformal maps of the unit disc to itself. Our first step is to determine those conformal maps of the disc to the disc that fix the origin. Let D denote the unit disc.

Let us begin by examining a conformal mapping $f : D \to D$ of the unit disc to itself such that $f(0) = 0$. We are assuming that f is one-to-one and onto. Then, by Schwarz's lemma (Section 6.5), $|f'(0)| \le 1$. This reasoning applies to f^{-1} as well, so that $|(f^{-1})'(0)| \le 1$ or $|f'(0)| \ge 1$. We conclude that $|f'(0)| = 1$. By the uniqueness part of the Schwarz lemma, f must be a rotation. So there is a complex number ω with $|\omega| = 1$ such that

$$f(z) \equiv \omega z \quad \forall z \in D\,. \tag{7.3}$$

It is often convenient to write a rotation as

$$\rho_\theta(z) \equiv e^{i\theta} z\,, \tag{7.4}$$

where we have set $\omega = e^{i\theta}$ with $0 \le \theta < 2\pi$.

We will next generalize this result to conformal self-maps of the disc that do not necessarily fix the origin.

7.2.2 Möbius Transformations

For $a \in \mathbb{C}, |a| < 1$, we define

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}\,. \tag{7.5}$$

Then each φ_a is a conformal self-map of the unit disc.

To see this assertion, note that if $|z| = 1$, then

$$|\varphi_a(z)| = \left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{\overline{z}(z-a)}{1-\overline{a}z}\right| = \left|\frac{1-a\overline{z}}{1-\overline{a}z}\right| = 1\,. \tag{7.6}$$

Thus φ_a takes the boundary of the unit disc to itself. Since $\varphi_a(0) = -a \in D$, we conclude that φ_a maps the unit disc to itself. The same reasoning applies to $(\varphi_a)^{-1} = \varphi_{-a}$, hence φ_a is a one-to-one conformal map of the disc to the disc.

The biholomorphic self-mappings of D can now be completely characterized.

7.2.3 Self-Maps of the Disc

Let $f : D \to D$ be a holomorphic function. Then f is a conformal self-map of D if and only if there are complex numbers a, ω with $|\omega| = 1, |a| < 1$ such that

$$f(z) = \omega \cdot \varphi_a(z) \quad \forall z \in D\,. \tag{7.7}$$

In other words, any conformal self-map of the unit disc to itself is the composition of a Möbius transformation with a rotation.

It can also be shown that any conformal self-map f of the unit disc can be written in the form

$$f(z) = \varphi_b(\eta \cdot z)\,, \tag{7.8}$$

for some Möbius transformation φ_b and some complex number η with $|\eta| = 1$.

The reasoning is as follows: Let $f : D \to D$ be a conformal self-map of the disc and suppose that $f(0) = a \in D$. Consider the new holomorphic mapping $g = \varphi_a \circ f$. Then $g : D \to D$ is conformal and $g(0) = 0$. By what we learned in Section 7.2.1, $g(z) = \omega \cdot z$ for some unimodular ω. But this says that $f(z) = (\varphi_a)^{-1}(\omega \cdot z)$ or

$$f(z) = \varphi_{-a}(\omega z)\,.$$

That is formulation (7.8) of our result. We invite the reader to find a proof of (7.7).

Example 51 Let us find a conformal map of the disc to the disc that takes $i/2$ to $2/3 - i/4$.

We know that $\varphi_{i/2}$ takes $i/2$ to 0. And we know that $\varphi_{-2/3+i/4}$ takes 0 to $2/3 - i/4$. Thus

$$\psi = \varphi_{-2/3+i/4} \circ \varphi_{i/2}$$

has the desired property. □

Exercises

1. Use the definition of the Möbius transformations in line (7.5) to prove directly that if $|z| < 1$ then $|\varphi_a(z)| < 1$.

2. Give a conformal self-map of the disc that sends $i/4 - 1/2$ to $i/3$.

3. Let a_1, a_2, b_1, b_2 be arbitrary points of the unit disc. Explain why there does not necessarily exists a holomorphic function from $D(0,1)$ to $D(0,1)$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

4. Let $\mathcal{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (the upper half-plane). Calculate all the conformal self-mappings of $\mathcal{U}$ to $\mathcal{U}$. [**Hint:** The function $\psi(z) = i(1-z)/(1+z)$ maps the unit disc D to $\mathcal{U}$ conformally.]

5. Let $\mathcal{U}$ be as in Exercise 4. Calculate all the conformal maps of $D(0,1)$ to $\mathcal{U}$.

6. Let $P \in \mathbb{C}$ and $r > 0$. Calculate all the conformal self-maps of $D(P,r)$ to $D(P,r)$.

7. Let $U = D(0,1) \setminus \{0\}$. Calculate all the conformal self-maps of U to U.

8. Use `MatLab` to write a utility that will construct a conformal self-map of the unit disc that maps a given input point a to another specified point b. Can you refine this utility so that it allows you to make some specifications about the derivative of the function at a?

9. It is a fact that there is a holomorphic function from the disc to the disc that maps two points a_1 and a_2 to two other specified points b_1 and b_2 if and only if the pseudohyperbolic distance of b_1 to b_2 is less than or equal to the pseudohyperbolic distance of a_1 to a_2 (see Exercise 6 in Section 6.5). Write a `MatLab` utility that will test for this condition. Write a more sophisticated utility that will actually produce the function.

10. Describe in the language of Euclidean geometry (that is, using words) what the Möbius transformation $\varphi_{1/2}$ does to the unit disc. What about iterates $\varphi \circ \varphi$, $\varphi \circ \varphi \circ \varphi$, etc.? Can you interpret this geometric action in terms of flows?

7.3 Linear Fractional Transformations

7.3.1 Linear Fractional Mappings

The automorphisms (that is, conformal self-mappings) of the unit disc D are special cases of functions of the form

$$z \mapsto \frac{az+b}{cz+d} \quad , \quad a,b,c,d \in \mathbb{C}\,. \tag{7.9}$$

It is worthwhile to consider functions of this form in generality. One restriction on this generality needs to be imposed, however; if $ad - bc = 0$, then the numerator is a constant multiple of the denominator provided that the denominator is not identically zero. So if $ad - bc = 0$, then the function is either constant or has zero denominator and is nowhere defined. Thus only the case $ad - bc \neq 0$ is worth considering in detail.

A function of the form

$$z \mapsto \frac{az+b}{cz+d} \quad , \quad ad - bc \neq 0\,, \tag{7.10}$$

is called a *linear fractional transformation.*

Note that $(az + b)/(cz + d)$ is not necessarily defined for all $z \in \mathbb{C}$. Specifically, if $c \neq 0$, then it is undefined at $z = -d/c$. In case $c \neq 0$,

$$\lim_{z \to -d/c} \left| \frac{az+b}{cz+d} \right| = \lim_{z \to -d/c} \left| \frac{az/c + b/c}{z + d/c} \right| = +\infty\,. \tag{7.11}$$

This observation suggests that one might well, for linguistic convenience, adjoin formally a "point at ∞" to $\mathbb{C}$ and consider the value of $(az+b)/(cz+d)$ to be ∞ when $z = -d/c\,(c \neq 0)$. Thus we will think of both the domain and the range of our linear fractional transformation to be $\mathbb{C} \cup \{\infty\}$ (we sometimes also use the notation $\widehat{\mathbb{C}}$ instead of $\mathbb{C} \cup \{\infty\}$). Specifically, we are led to the following alternative method for describing a linear fractional transformation.

A function $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a *linear fractional transformation* if there exists $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, such that either

(a) $c = 0, d \neq 0, f(\infty) = \infty$, and $f(z) = (a/d)z + (b/d)$ for all $z \in \mathbb{C}$;

or

(b) $c \neq 0, f(\infty) = a/c, f(-d/c) = \infty$, and $f(z) = (az+b)/(cz+d)$ for all $z \in \mathbb{C}, z \neq -d/c$.

It is important to realize that, as before, the status of the point ∞ is entirely formal: we are just using it as a linguistic convenience, to keep track of the behavior of $f(z)$ both where it is not defined as a map on $\mathbb{C}$ and to keep track of its behavior when $|z| \to +\infty$. The justification for the particular devices used is the fact that

(c) $\lim_{|z|\to+\infty} f(z) = f(\infty)$ [$c = 0$; case **(a)** of the definition];

(d) $\lim_{z\to-d/c} |f(z)| = +\infty$ [$c \neq 0$; case **(b)** of the definition].

7.3.2 The Topology of the Extended Plane

The limit properties of f that we described in Section 7.3.1 can be considered as continuity properties of f from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$ using the definition of continuity that comes from the topology on $\mathbb{C} \cup \{\infty\}$ (which we are about to define). It is easy to formulate that topology in terms of open sets. But it is also convenient to formulate that same topological structure in terms of convergence of sequences:

A sequence $\{j\}$ in $\mathbb{C}\cup\{\infty\}$ *converges to* $p_0 \in \mathbb{C}\cup\{\infty\}$ (notation $\lim_{j\to\infty} p_j = p_0$) if either

(e) $p_0 = \infty$ and $\lim_{j\to+\infty} |p_j| = +\infty$ where the limit is taken for all j such that $p_j \in \mathbb{C}$ (the limit here means that the $|p_j|$ are getting ever larger as $j \to +\infty$);

or

(f) $p_0 \in \mathbb{C}$, all but a finite number of the p_j are in $\mathbb{C}$, and $\lim_{j\to\infty} p_j = p_0$ in the usual sense of convergence in $\mathbb{C}$.