Homework Assignment 1

ECSE 507 / MATH 560 Optimization, Winter 2017

Released: January 15, 2017
Due: January 24, 2017

1. (Nocedal and Wright, Exercise 2.1) Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the Rosenbrock function
$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$
Show that $x^* = (1,1)^\top$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

2. (Nocedal and Wright, Exercise 2.8) We say that a set $X \subseteq \mathbb{R}^n$ is convex if for every $x, y \in X$,
$$\alpha x + (1-\alpha)y \in X, \quad \text{for all } \alpha \in [0,1].$$
Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

3. (Nocedal and Wright, Exercise 2.10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Let $g(y) = f(Ay + b)$ for a given matrix $A \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$. Show that
$$\nabla g(y) = A^\top \nabla f(Ay + b), \quad \text{and} \quad \nabla^2 g(y) = A^\top \nabla^2 f(Ay + b)A.$$
(*Hint:* Use the chain rule to express $\mathrm{d}g / \mathrm{d}y_j$ in terms of $\mathrm{d}f / \mathrm{d}x_i$ and $\mathrm{d}x_i / \mathrm{d}y_j$ for all $i, j = 1, \ldots, n$.

4. Prove the following properties of convex functions:
   (a) Show that $f(x) = e^{ax}$ is convex on $\mathbb{R}$ for any real number $a \in \mathbb{R}$, where $x \in \mathbb{R}$.
   (b) Show that $f(x) = \|x\|_2$ is convex, where $x \in \mathbb{R}^n$.
   (c) Suppose $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are two convex functions. Show that $f(x) = f_1(x) + f_2(x)$ is convex.
   (d) Suppose $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are two convex functions. Show that $f(x) = \max\{f_1(x), f_2(x)\}$ is convex.
   (e) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and let $g(y) = f(Ay + b)$ for a given matrix $A \in R^{n\times m}$ and vector $b \in \mathbb{R}^n$. Show that $g(y)$ is convex.