Professor Kindred
Math 104, Graph Theory
Homework 10 Solutions
April 25, 2013

Introduction to Graph Theory, West
Section 7.2   10, 26, 38
Schur's theorem
Section 8.3   17, 22

Problems you should be able to do: 7.2.14, 5.1.43, monotone subsequences problem

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7.2.10

(a) Find a 2-connected non-Eulerian graph whose line graph is Hamiltonian.

(b) Prove that $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one endpoint of each edge.

Note: Recall that an *Eulerian graph* is a graph G that has a closed walk which contains every edge of G exactly once (an *Eulerian tour*). A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices are all of even degree.

(a) The graph G shown in Figure 1 is connected and has no cut vertex, so it is 2-connected. In addition, it has two vertices of odd degree and so cannot be Eulerian. Its line graph has a Hamiltonian cycle, indicated in Figure 1 by the red edges.

G $L(G)$

Figure 1: A 2-connected non-Eulerian graph is shown, along with its line graph that is Hamiltonian.

(b) ($\Leftarrow$) Suppose G has a closed trail T that contains at least one endpoint of each edge. Let T have vertices $v_1, v_2, \ldots, v_t$ in that order. (If G has a vertex cover of size 1, then this special case is handled separately below; otherwise it has no such vertex cover, so the closed trail T must be nontrivial and have at least one edge.) Note that consecutive edges in T share an endpoint in G, so vertices of $L(G)$ associated with

$E(T)$ from a cycle C in $L(G)$.

For each edge $e \in E(G) - E(T)$, select an endpoint v of e that occurs in $V(T)$. Although v may occur more than once in T, select a particular occurrence of v in T arbitrarily, say v_i. Between the vertices of C (in $L(G)$) representing edges $v_{i-1}v_i$ and v_iv_{i+1} of T, insert the vertices of $L(G)$ for all edges of $E(G) - E(T)$ whose selected vertex occurrence is v_i. Since these edges all share endpoint v_i, the corresponding vertices in $L(G)$ replace an edge of C with a path. Every vertex of $L(G)$ is in the original cycle C or in exactly one of the paths used to enlarge it, so the result is a Hamiltonian cycle of $L(G)$.

Remark: We should handle a special case separately. If G is a star graph, then it has a trivial closed trail of length 0 (namely the vertex of max degree) that includes an endpoint of every vertex. Then its line graph is the complete graph on $\Delta(G)$ vertices, which is clearly Hamiltonian.

($\Rightarrow$) Suppose $L(G)$ is Hamiltonian, and let C be a Hamiltonian cycle of $L(G)$. If there are three successive vertices in the cycle that correspond to edges e_{i-1}, e_i, and e_{i+1} of G which have a common endpoint, delete the vertex representing e_i from the cycle in $L(G)$. Since e_{i-1} and e_{i+1} share a common endpoint in G, what remains is still a cycle in $L(G)$. We repeat this step until all triples described above are removed.

Each deletion preserves the property that the cycle is still a cycle and that the vertices of the cycle correspond to edges of G that include an endpoint of every edge in G. When no more deletions are possible, every successive three vertices in the cycle in $L(G)$ correspond to edges in G with no common endpoint but every pair of successive vertices in the cycle correspond to incident edges in G. Thus, the edges in the cycle in $L(G)$ correspond to a closed trail in G that contains a vertex of every edge in G.

■

7.2.26 Prove that if G fails Chvátal's condition, then $\overline{G}$ has at least $n-2$ edges. Conclude from this that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2} + 1$.

Suppose G fails Chvátal's condition. Then there exists some $i < \frac{n}{2}$ such that $d_i \leq i$ and $d_{n-i} < n - i$. Let u be a vertex with degree d_i, and let v be a vertex with degree d_{n-i}. Thus, in $\overline{G}$, we have

$$\begin{aligned} d_{\overline{G}}(u) + d_{\overline{G}}(v) &= (n-1) - d_G(u) + (n-1) - d_G(v) \\ &= 2n - 2 - d_i - d_{n-i} \\ &> 2n - 2 - i - (n-i) = n - 2 \end{aligned}$$

so $d_{\overline{G}}(u) + d_{\overline{G}}(v) \geq n - 1$.

Since u and v have degree sum at least $n-1$ in $\overline{G}$, and since a simple graph has at most one edge joining them (counted twice in the degree sum), there must be at least $n-2$ distinct edges in G incident to $\{u, v\}$. So $|E(\overline{G})| \geq n-2$.

Therefore, the number of edges in a simple non-Hamiltonian graph G on n vertices is

$$\begin{aligned} |E(G)| = \binom{n}{2} - |E(\overline{G})| &\leq \frac{n(n-1)}{2} - (n-2) \\ &= \frac{n^2-3n+4}{2} = \frac{(n-1)(n-2)}{2} + 1 = \binom{n-1}{2} + 1 \end{aligned}$$

so the maximum number of edges is $\binom{n-1}{2} + 1$. ■

7.2.38 Let G be a connected simple graph with $\delta(G) = k \geq 2$ and $|V(G)| > 2k$.

(a) Let P be a maximal path in G (not a subgraph of any longer path). If $|V(P)| \leq 2k$, prove that the induced subgraph $G[V(P)]$ has a spanning cycle (this cycle need not have its vertices in the same order as P).

(b) Use part (a) to prove that G has a path with at least $2k+1$ vertices. Give an example for each odd value of n to show that G need not have a cycle with more than $k+1$ vertices.

Suppose G is a connected simple graph with $\delta(G) = k \geq 2$ and $|V(G)| > 2k$.

(a) Assume P is a maximal path in G with at most $2k$ vertices, and let u, v be the endpoints of P. If $u \sim v$, then we are done, since this means that P along with the edge uv is a spanning cycle of $G[V[P]]$. So, going forward, assume $u \not\sim v$. Let $H = G[V[P]]$.

   We know that $H + uv$ is Hamiltonian (by previous argument). Furthermore, since P is a maximal path, an endpoint of P, either u or v, cannot have a neighbor in G that is not a vertex in $V(P) = V(H)$. (If such a neighbor w did exist, then we could extend path P to a longer path by adding in w and an edge from w to an endpoint of P.) Thus,

$$d_H(u) + d_H(v) = d_G(u) + d_G(v) \geq 2\delta(G) = 2k \geq |V(H)|.$$

   Using Ore's theorem (Lemma 7.2.9), since u and v are nonadjacent vertices with $d_H(u) + d_H(v) \geq |V(H)|$ and $H + uv$ is Hamiltonian, it follows that H is Hamiltonian.

(b) Let P be a longest path in G. Suppose, for sake of contradiction, that $|V(P)| \leq 2k$. Then, by part (a), we know there exists a cycle containing the vertices of P. Since G is connected, we know there must exist an edge uv with $u \in V(P)$ and $v \in V(G) - V(P)$. But now adding the vertex v and the edge uv to the path P produces a longer path than P. $\Longrightarrow\Longleftarrow$ Therefore, it must be the case that G has a path with at least $2k+1$ vertices.

■

Schur's theorem We describe an interesting application of Ramsey's theorem to combinatorial number theory. Consider the partition $\{\{1,4,10,13\},\{2,3,11,12\},\{5,6,7,8,9\}\}$ of the set of integers $\{1,2,\ldots,13\}$. Notice that no subset of the partition contains three integers x, y, and z (not necessarily distinct) such that

$$x + y = z. \tag{1}$$

Yet, no matter how we partition $\{1,2,\ldots,14\}$ into three subsets, there will always exist a subset of the partition which contains a solution to (1).

Prove Schur's theorem, which generalizes the example above:

> Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the set of integers $\{1,2,\ldots,R_n(3)\}$, where $R_n(3) = R(\underbrace{3,3,\ldots,3}_{n \text{ terms}})$, into n subsets. Then some A_i contains three integers x, y, and z satisfying the equation $x + y = z$.

Consider the complete graph whose vertex set is $\{1,2,\ldots,R_n(3)\}$. Color the edges of this graph with colors $1,2,\ldots,n$ in the following way:

$$\text{edge } uv \text{ is assigned color } i \text{ if } |u - v| \in A_i.$$

By the definition of $R_n(3)$, there exists a monochromatic triangle in the graph. That is, there are three vertices a, b, and c such that the edges ab, bc, and ac all have the same color i. Without loss of generality, assume $a > b > c$, and let $x = a - b, y = b - c$, and $z = a - c$. Then $x, y, z \in A_i$ and we have that

$$x + y = (a - b) + (b - c) = a - c = z.$$

■

8.3.17 *Ramsey numbers for multiple colors.*

(a) Let $\mathbf{p} = (p_1, \ldots, p_k)$ and let $\mathbf{q_i}$ be obtained from $\mathbf{p}$ by subtracting 1 from p_i but leaving the other coordinates unchanged. Prove that $R(\mathbf{p}) \leq \sum_{i=1}^{k} R(\mathbf{q_i}) - k + 2$, i.e.,

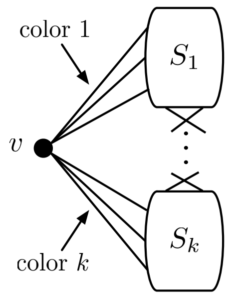
$$R(p_1, p_2, \ldots, p_k) \leq R(p_1 - 1, p_2, \ldots, p_k) + R(p_1, p_2 - 1, \ldots, p_k) + \cdots \\ + R(p_1, p_2, \ldots, p_k - 1) - k + 2.$$

(b) Prove that $R(p_1 + 1, \ldots, p_k + 1) \leq \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}$.

(a) Let

$$n = \sum_{i=1}^{k} R(q_i) - k + 2 = \sum_{i=1}^{k} R(p_1, \ldots, p_{i-1}, p_i - 1, p_{i+1}, \ldots, p_k) - k + 2,$$

and consider any k-edge coloring of K_n. Let v be a fixed vertex in K_n. Partition the other vertices into sets $S_1, \ldots, S_k$ by the color of the edges they share with v, so S_i contains any vertex that shares an edge with v that is assigned color i.



If $|S_i| \leq R(q_i) - 1$ for all i, then we would have

$$n = 1 + \sum_i |S_i| \leq 1 + \sum_i (\mathbf{q_i} - 1) = 1 + \sum_i R(\mathbf{q_i}) - k < n,$$

which is clearly a contradiction. So there must exist i such that $|S_i| \geq R(\mathbf{q_i})$. Then we consider two cases.

Case 1 There exists a K_{p_i-1} in S_i in which all edges are assigned color i. Then combining these vertices with vertex v, which shares color i edges with all vertices in S_i, yields a K_{p_i} with all color i edges.

Case 2 There exists a K_{p_j} in S_j in which all edges are assigned color j, where $j \neq i$.

Either case gives the desired result that $R(p_1, \ldots, p_k) \leq n = \sum_{i=1}^k R(\mathbf{q_i}) - k + 2$.

(b) First, it is assumed that $k \geq 2$.

We give a proof by induction on $\sum_{i=1}^k (p_i + 1)$. Note that since $p_i \geq 0$ for all i, the minimum value of $\sum_{i=1}^k (p_i + 1)$ is k.

Base case: Suppose $\sum_{i=1}^k (p_i + 1) = k$. Then $p_i = 0$ for all i. It is trivially true that $R(1, 1, \ldots, 1) \leq 1$ since there are no edges in K_1.

Induction hypothesis: Assume that for $\sum_{i=1}^k (p_i + 1) = \ell$, where $\ell \geq 0$, it is true that $R(p_1 + 1, \ldots, p_k + 1) \leq \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}$.

Inductive step: Consider $p_1, \ldots, p_k$ such that $\sum_{i=1}^k (p_i + 1) = \ell + 1$. Then, by using part (a), we have

$$\begin{aligned}
R(p_1 + 1, \ldots, p_k + 1) &\leq \sum_i R(\underbrace{p_1 + 1, \ldots, p_{i-1} + 1, p_i, p_{i+1} + 1, \ldots, p_k}_{\text{sum to } \ell}) - k + 2 \\
&\leq \sum_i \frac{\left(p_i - 1 + \sum_{j \neq i} p_j\right)!}{(p_i - 1)! \prod_{j \neq i} p_j!} - k + 2 \qquad \text{by induction hypothesis} \\
&= \sum_i \frac{(\ell - k)!}{(p_i - 1)! \prod_{j \neq i} p_j!} - k + 2.
\end{aligned}$$

Thus,

$$
\begin{aligned}
R(p_1+1,\ldots,p_k+1) &\le \sum_i \frac{(\ell-k)!}{(p_i-1)!\prod_{j\ne i} p_j!} && \text{since } k\ge 2\\
&= \frac{\left(\sum_i p_i\right)(\ell-k)!}{\prod_j p_j!} && \text{by finding common denominator}\\
&= \frac{(\ell-k+1)\,(\ell-k)!}{\prod_j p_j!} = \frac{(\ell-k+1)!}{\prod_j p_j!} = \frac{\left(\sum_j p_j\right)!}{\prod_j p_j!}.
\end{aligned}
$$

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8.3.22 (modified)

(a) Using induction, prove that if T is a tree with m edges and G is a simple graph with $\delta(G)\ge m$, then T is a subgraph of G.

(b) Let T be a tree with k vertices. Given that $k-1$ divides $n-1$, determine the Ramsey number $R(T,K_{1,n})$.

(a) We prove by induction on $m=|E(T)|$.

Base case: Assume $m=0$. Then T must be K_1, and every (nonempty) simple graph contains K_1, so the result holds.

Induction hypothesis: Assume that if T is a tree with $m\ge 0$ edges and G is a simple graph with $\delta(G)\ge m$, then T is a subgraph of G.

Let T be a tree with $m+1$ edges, and suppose G is a simple graph with $\delta(G)\ge m+1$. The tree T has at least one edge, so it must have a leaf v. Let u be the unique neighbor of v in T. Consider the smaller tree $T'=T-v$.

By the induction hypothesis, since $\delta(G)\ge m+1>m$, we know that G must contain T' as a subgraph. Let x be the vertex in this copy of T' that corresponds to u. Because $|V(T')|=|E(T')|+1=m+1$, we know T' has only m other vertices besides u. Furthermore, since $\delta(G)\ge m+1$, there must be at least one neighbor of x in G, call it y, that is not in the copy of T'. Adding the edge xy and the vertex y expands the copy of T' into a copy of T in G, with y playing the role of v. Thus, the result holds by induction.

(b) Suppose T is a tree with k vertices, and assume $k-1$ divides $n-1$. We claim that $R(T,K_{1,n})=n+k-1$.

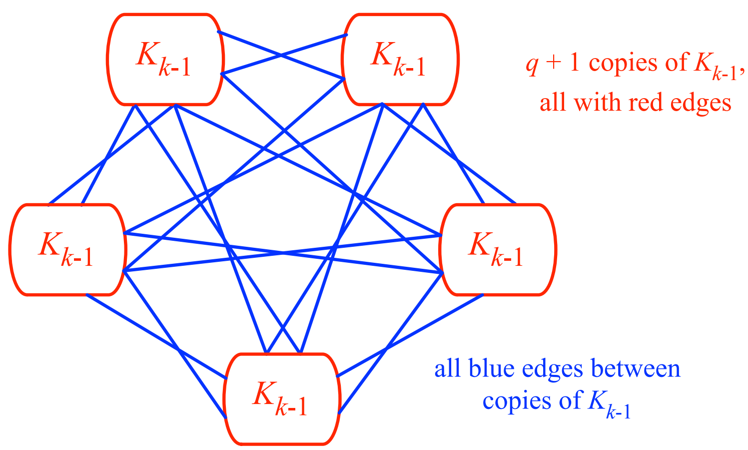
Consider any red/blue edge-coloring of the complete graph K_{n+k-1}. If there exists a vertex v with at least n blue edges incident to it, then we have a blue copy of $K_{1,n}$.

Otherwise, every vertex has at most $n-1$ blue edges incident to it, which means that every vertex has at least $(n+k-2)-(n-1)=k-1$ red edges incident to it, i.e., in the red subgraph H (the subgraph consisting of only the red edges), we have that $\delta(H) \geq k-1$. By the result of part (a), this implies that any tree with $k-1$ edges is contained in H, which means that T is contained in H. Thus, we have shown that either there is a red copy of T or a blue copy of $K_{1,n}$, so $R(T, K_{1,n}) \leq n+k-1$.

For the lower bound, we consider a particular red/blue edge-coloring of K_{n+k-2}. Because $k-1$ divides $n-1$, we know there exists a positive integer q such that $n-1=q(k-1)$. Thus,

$$n+k-2=(n-1)+(k-1)=q(k-1)+(k-1)=(q+1)(k-1),$$

so we can partition the vertices into $q+1$ groups of size $k-1$. Make the components of the red subgraph $q+1$ copies of K_{k-1}, and make all other edges of K_{n+k-2} blue. Then this edge-coloring of the complete graph does not contain a red copy of T since T is a connected graph with k vertices. Furthermore, each vertex has $k-2$ neighbors along red edges, so each must have $n-1$ neighbors along blue edges. This means there is no vertex of degree n in the blue subgraph. Thus, there is no blue copy of $K_{1,n}$. It follows that $R(T, K_{1,n}) > n+k-2$.



a red/blue edge-coloring of K_{n+k-2}

By the two arguments above (for the upper and lower bounds), we have that

$$R(T, K_{1,n}) = n+k-1.$$

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