MATH 458- ASSIGNMENT 1 - DUE FEBRUARY 10TH, 2017.

1. Classical theorems. One has in calculus textbooks the classical Stokes theorem: for M a surface in $\mathbb{R}^3$ with boundary ∂M, one has for F a vector field on M:

$$\int_M (\nabla \times F) \cdot dA = \int_{\partial M} F \cdot dr$$

Likewise the divergence theorem, for a vector field F on the interior M of a closed surface ∂M in $\mathbb{R}^3$:

$$\int_M \nabla \cdot F dV = \int_{\partial M} F \cdot dA$$

In $\mathbb{R}^3$ one has the standard Euclidean metric, which identifies vector fields with 1-forms, and even 2-forms:

$$\frac{\partial}{\partial x_i} \leftrightarrow dx_i \leftrightarrow dx_j \wedge dx_k,$$

where (i, j, k) is some cyclic permutation of $(1, 2, 3)$. One also has standard volume forms $dV = dx_1 \wedge dx_2 \wedge dx_3$, and surface elements dA, thought of as an outward normal vector. You are asked to reinterpret the quantities defined above in terms of differential forms, and show that the two theorems above are simply manifestations of the invariant Stokes theorem. (This is of course in books; the aim is to digest this yourself, then provide a reasonably coherent account. A soft skills question, in educational parlance).

2. Spherical area form and winding number One has on $\mathbb{R}^3$ the spherical area form, centred at (a, b, c):

$$\omega_{(a,b,c)} = \frac{(x-a)dy \wedge dz + (y-b)dz \wedge dx + (z-c)dx \wedge dy}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}}$$

a) Show that this form (as a form in (x, y, z)) is closed.

b) Let M be a sphere centred at (a, b, c), with the standard orientation. Show that integral of $\omega_{(a,b,c)}$ on M is -4π (you might want to write the form in spherical polars centred at (a, b, c)). Show that if M is the boundary surface in $\mathbb{R}^3$ of a volume N, then the integral on M is also -4π if (a, b, c) lies in N, and is zero otherwise.

c) Now let M be compact, oriented surface with boundary in $\mathbb{R}^3$, and (a, b, c) not in M. Define

$$\Omega(a, b, c) = \int_M \omega_{(a,b,c)}.$$

This integral, roughly, is the size of the radial shadow (with signs) that M casts on a sphere centred at (a, b, c), times -4π.

Show that if there is a path $p(t)$ crossing M transversally, such that if μ is an orientation form on M and $p'(t) \wedge \mu$ is an orientation form on $\mathbb{R}^3$ (for $p'(t)$, using the metric to identify tangent and cotangent bundles) , then

$$\lim_{t_1\to 0^-, t_2\to 0^+} \Omega(p(t_2)) - \Omega(p(t_1)) = -4\pi$$

(Hint: part b))

d) Now, fixing M, think of $\Omega(a,b,c)$ as a smooth function on the complement of M in $\mathbb{R}^3$. Suppose that $\partial M = f(S^1)$. Show that

$$\frac{\partial\Omega}{\partial a}(a,b,c) = \int_{S^1} f^*\Big(\frac{(y-b)dz-(z-c)dy}{|(x,y,z)-(a,b,c)|^3}\Big)$$

$$\frac{\partial\Omega}{\partial b}(a,b,c) = \int_{S^1} f^*\Big(\frac{(z-c)dx-(x-a)dz}{|(x,y,z)-(a,b,c)|^3}\Big)$$

$$\frac{\partial\Omega}{\partial c}(a,b,c) = \int_{S^1} f^*\Big(\frac{(x-a)dy-(y-b)dx}{|(x,y,z)-(a,b,c)|^3}\Big)$$

Hint: compare $\frac{\partial\omega}{\partial a,b,c}$ with the exterior derivative (in x,y,z) of the integrands.

d) Now let $\partial M = f(S^1)$, with compatible orientations, and suppose that a curve $g: S^1 \to \mathbb{R}^3$, $t \mapsto g(t) = ((a(t), b(t), c(t))$ whose image does not intersect that of f, $g(t)$ crosses the interior of M transversally, i.e., at the intersection points, $dg(t)\wedge\mu$ is non-zero. Let n_+ be the number of points of intersection for which $dg(t)\wedge\mu$ is a positive multiple of the orientation form on $\mathbb{R}^3$, and n_- be the number of points of intersection for which $dg(t)\wedge\mu$ is a negative multiple. Show that

$$n_+ - n_- = \frac{-1}{4\pi}\int_{S^1} g^*(d\Omega)$$

This is the linking number of the two curves defined by f, g.

3. The Gauss map. Consider the torus $T^2 = S^1\times S^1$, embedded into $\mathbb{R}^3$ by

$$I(\theta,\varphi) = \Big(\cos(\theta)\big(R + r\cos(\varphi)\big), \sin(\theta)\big(R + r\cos(\varphi)\big), r\sin(\varphi)\Big)$$

where $0 < r < R$.

a) Compute the differential I_*, mapping $T_p(T^2)$ to $T_{I(p)}(\mathbb{R}^3)$, in the coordinates θ,φ on the torus and the standard coordinate x,y,z on $\mathbb{R}^3$. Compute the induced map I^* on T_p^*, and the map on the second exterior power $\Lambda^2 T_p^*$.

b) One has the Gauss map $G: T^2 \to S^2$, which associates to each point p on the torus the unit outward normal vector to $I_*(T_p(T^2))$ in $T_{I(p)}\mathbb{R}^3$. Parametrise the upper half of the two-sphere by $(x,y)\mapsto(x,y,\sqrt{1-x^2-y^2})$

Now compute the Gauss map $T^2\to S^2$, for the upper half of the torus, in the coordinates θ,φ on the torus and the coordinates x,y on the two-sphere; compute the differential $dG: T_p(T^2)\to T_{G(p)}S^2$.