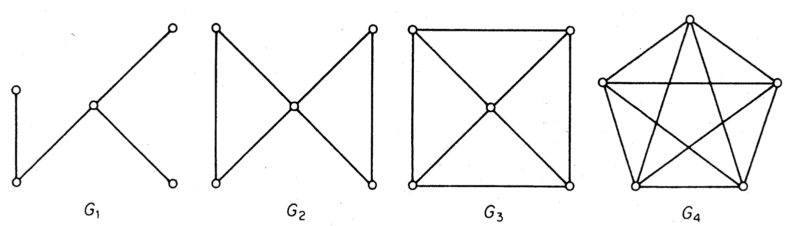
Connectivity and cuts

Math 104, Graph Theory

February 19, 2013

Measure of connectivity

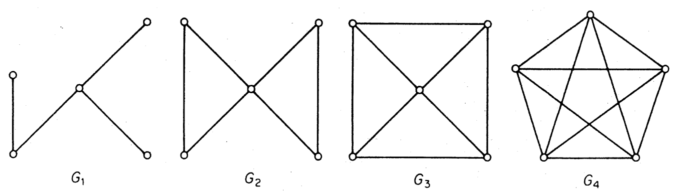


How connected are each of these graphs?

——> increasing connectivity ——>

- G_1 is a tree, so it is a connected graph w/minimum # of edges. Every edge is a cut edge.
- G_2 has no cut edge but can be disconnected by deletion of one vertex. (Vertex of deg 4 is a cut vertex.)
- G_3 has no cut edges and no cut vtcs, but nonetheless, it is not as well connected as G_4, a complete graph.

Measure of connectivity



How connected are each of these graphs?

We would like to have a measure of connectivity of a graph.

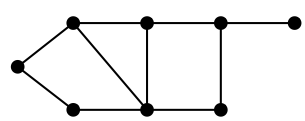
We consider two today: one in terms of vertices ($\kappa(G)$) and one in terms of edges ($\kappa'(G)$).

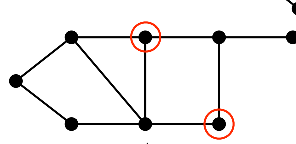
Note that the connectivity of a graph gives an indication of its robustness as a network.

Cut sets and disconnecting sets

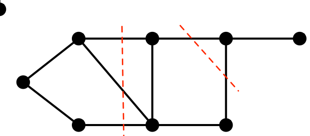
Definition A set of vertices S is a *cut set* if $G - S$ is disconnected.
A set of edges F is a *disconnecting set* of edges if $G - F$ is disconnected.

Example Find a cut set and a disconnecting set in the graph below.





A cut set.



A disconnecting set.

Definition

connectivity of G: $\kappa(G) =$ minimum size of a cut set

G is *k-connected* if its connectivity is at least k

edge connectivity of G: $\kappa'(G) =$ minimum size of a disconnecting set

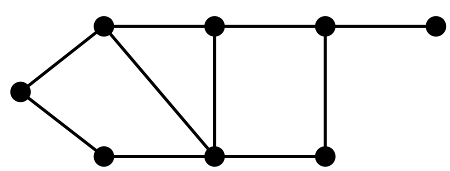
G is *k-edge-connected* if its edge connectivity is at least k

Questions

- When is $\kappa(G) = 0$? When is $\kappa'(G) = 0$?
- When is $\kappa(G) = 1$? When is $\kappa'(G) = 1$?
- What is an example of a graph G for which $\kappa(G) < \kappa'(G)$?
- What is an example of a graph G for which $\kappa'(G) < \kappa(G)$?

An example

Example For our previous example with graph G below

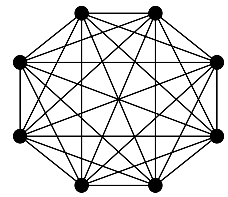


we have $\kappa(G) = 1$ since

- G is connected (which means $\kappa(G) > 0$)
- and G has a cut vertex (so $\kappa(G) \leq 1$).

Also, $\kappa'(G) = 1$ since G is a connected graph with a cut edge.

Are $\kappa(G)$ and $\kappa'(G)$ well-defined for all graphs?



What is $\kappa(K_n)$? $\implies$ No matter how many vtcs we remove from the complete graph, we never disconnect it.

We tweak our definition of connectivity to handle this case.

That is, the *connectivity* of G, denoted $\kappa(G)$, is the minimum size of a set $S \subseteq V(G)$ such that either $G - S$ is disconnected or $G - S$ has one vertex.

Thus, $\boxed{\kappa(K_n) = n - 1.}$

Are $\kappa(G)$ and $\kappa'(G)$ well-defined for all graphs?

Along the same lines as our last remark, are there any graphs G for which $\kappa'(G)$ is not well-defined?

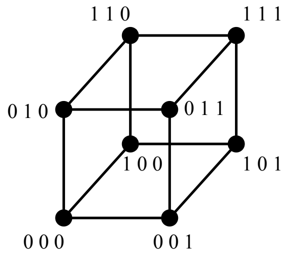
How about $\kappa'(K_1)$? K_1 is connected and there are no edges to remove to try and disconnect the graph.

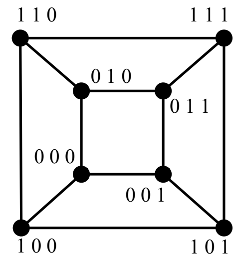
There is no good definition for this value, so we will just define it to be zero, i.e. $\kappa'(K_1) = 0$.

More examples to consider

- What is $\kappa(K_{m,n})$?
- The d-dimensional hypercube Q_d is the simple graph whose vertices are the binary sequences (all entries are 0, 1) of length d and two binary sequences are adjacent if and only if they differ in exactly one position.

  For example, two drawings of Q_3 are shown below:

  What is $\kappa(Q_d)$?

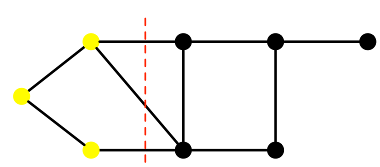
Edge cuts

Definition Given disjoint (nonempty) sets $S, T \subseteq V(G)$ for a graph G, $[S,T]$ denotes the set of edges with one endpoint in S and one endpoint in T,

$$[S,T] = \{uv \in E(G) : u \in S, v \in T\}.$$

In particular, we are often interested in the case where $T = \overline{S} = V(G) - S$. The set of edges in $[S,\overline{S}]$ is called an *edge cut* of G.

Example



An edge cut.

Yellow vtcs are vtcs in S, black vtcs are those in $\overline{S}$, and edges crossed by dashed red line are edges in the edge cut $[S,\overline{S}]$.

Edge cuts

Question What is the relationship between a disconnecting set of edges in G and an edge cut set of G? In other words...

- Is an edge cut always a disconnecting set?

  $\Longrightarrow$ **Yes!** No path exists from vertex in S to vertex in $\overline{S}$.

- Is a disconnecting set always an edge cut?

  $\Longrightarrow$ A **minimal** disconnecting set is an edge cut.

So we can adjust our definition of $\kappa'(G)$, if we like:

$$\kappa'(G) = \text{minimum size of } \underbrace{\cancel{\text{disconnecting set}}}_{\text{edge cut}}$$

Whitney's theorem

What is the relationship between connectivity, edge connectivity, and the degrees of vtcs in a graph?

Theorem (Whitney)

Let G be a graph. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Before we prove this theorem, we mention one basic result and leave the proof as an exercise.

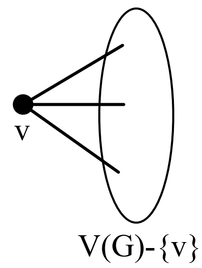
Lemma

Let G be a connected graph with at least three vertices. If G has a cut edge $e = uv$, then at least one of u, v is a cut vertex.

Proof of Whitney's inequality: $\kappa'(G) \le \delta(G)$

The result $\kappa(G) \le \kappa'(G) \le \delta(G)$ is trivially true for graph K_1, so assume G has at least two vtcs.

We first show that $\kappa'(G) \le \delta(G)$. Let v be a vertex of minimum degree.



Note that $[\{v\}, V(G) - \{v\}]$ is an edge cut of size $\delta(G)$.

Thus, $\kappa'(G) \le \delta(G)$.

Proof of Whitney's inequality: $\kappa(G) \le \kappa'(G)$

Next we show that $\kappa(G) \le \kappa'(G)$ via a proof by induction on $\kappa'(G)$.

Base case: If $\kappa'(G) = 0$, then G is disconnected so $\kappa(G) = 0$.

Induction hypothesis: Suppose that for any graph G with $\kappa'(G) = k$, we have $\kappa(G) \le \kappa'(G)$.

Inductive step: Let G be a graph with $\kappa'(G) = k + 1$, and let $F \subseteq E(G)$ be an edge cut of size $k + 1$. Suppose $e \in F$, and let $H = G - e$.

Then $F - e$ is a minimum edge cut of H, so $\kappa'(H) = k$. Apply the induction hypothesis to H to conclude that $\kappa(H) \le \kappa'(H) = k$.

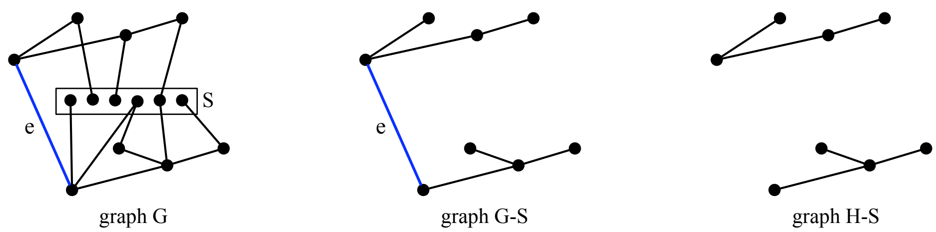
Proof of Whitney's inequality: $\kappa(G) \leq \kappa'(G)$

Let S be an optimal (vertex) cut set of H. Then $H - S$ is disconnected. We consider two cases:

Case 1: $G - S$ is disconnected.
Then
$$\kappa(G) \leq |S| = \kappa(H) \leq k < \kappa'(G).$$

Proof of Whitney's inequality: $\kappa(G) \leq \kappa'(G)$

Case 2: $G - S$ is connected. Since $G - S$ is connected but $H - S$ is not, we know that e is a cut edge of $G - S$.



Since e has endpoints in $V(G) - S$, we have $|V(G) - S| \geq 2$.

- If $|V(G) - S| = 2$, then $|S| = |V(G)| - 2$, so
$$\kappa(G) \leq |V(G)| - 1 = |S| + 1 = \kappa(H) + 1 \leq k + 1 = \kappa'(G).$$

- If $|V(G) - S| > 2$, then an endpoint of the cut edge e is a cut vertex of $G - S$ (by previous lemma). Let v be such a cut vertex. Then $S \cup \{v\}$ is a cut set for G, so
$$\kappa(G) \leq |S| + 1 = \kappa(H) + 1 \leq k + 1 = \kappa'(G).$$

■

Testing the bounds of Whitney's theorem

Find examples of a graph G such that

- $\kappa(G) < \kappa'(G) < \delta(G)$
- $\kappa(G) = \kappa'(G) < \delta(G)$
- $\kappa(G) < \kappa'(G) = \delta(G)$.