Math776: Graph Theory (I)
Fall, 2013
Homework 5 Solutions

1. [**page 84, #18**] Let $k \geq 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

Solution by Nicholas Stiffler: Let $k \geq 2$ and let G be a k-connected graph with $|G| \geq 2k$. As G is k-connected, it is connected, and as $\delta G \geq \kappa(G) \geq k \geq 2$, it has no leaves, so it is not a tree, so it has a cycle.

Let C be a largest cycle in G. First, as $\delta(G) \geq \kappa(G) \geq k$ and G has a cycle, $|C| \geq k+1$ by Diestel ... Assume for the sake of contradiction that $|C| < 2k$. Then there is a $v \in G \backslash C$. Let $A = N(v)$ and $B = V(C)$. as $\delta(G) \geq \kappa(G) \geq k, |A| \geq k$. Furthermore, any set X of size less than k cannot separate A and B as that would disconnect v and some $c \in C$, contradiction that G is k-connected. Thus the size of a minimum separator is at least k, and by Menger's theorem, there are at least k disjoint AB paths.

By the pigeon-hole principle (with vertices in A as pigeons and edges in C as holes), there are $a, a' \in A$ and $c_1, c_2 \in C$ such that $c_1, c_2 \in E(G)$ there are distinct $a - c_1$ and $a' - c_2$ paths P_a and $P_{a'}$. (Note that these paths may be of length on if a vertex of C is adjacent to v.) Let P be the c_1c_2 path in C of size at least two, Then

$$C' = vP_a\mathring{P}P_{a'}v$$

has size at least one larger than C, contradicting the maximality of C.

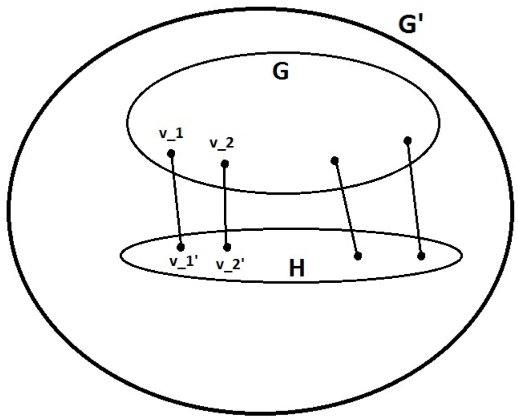
We conclude $|C| \geq 2k$.

2. [**page 84, #19**] Let $k \geq 2$. Show that in a k-connected graph any k vertices lie on a common cycle.

Solution by Michael Laughlin: Let G be a k-connected graph, and let $v_1, \ldots, v_k \in V(G)$. Let C be a cycle containing as many of these specified vertices as possible, without loss of generality say $v_1, \ldots, v_l$, and suppose that $l < k$. Then there exists a v_{l+1} outside of C, and by Menger's Theorem, the minimum number of vertices not equal to v_{l+1} separating v_{l+1} from C is equal to the maximum number of independent $N(v_{l+1})$-C paths. Hence, since G is k-connected, there are at least k paths from v_{l+1} to C, independent save for v_{l+1} as the initial vertex. However, these paths must meet C in between each of the vertices $v_1, \ldots, v_l$ with no two paths meeting in the same portion of the cycle v_iCv_{i+1}, or else there exists a larger cycle containing v_{l+1}. On the other hand, if no such cycle exists, then there are at least k elements from $v_1, \ldots, v_k$ in C (since there are k paths meeting C in this way), a contradiction.

3. **[page 84, #24**] Derive Tutte's 1-factor theorem from Mader's theorem.

Solution by Melissa Bechard: Let $G = (V, E)$ be a graph. For each vertex $v \in V(G)$, add a new vertex v', and connect v to v'. Call this new graph G', and let $H = \{v'\}$. We have the following diagram:



Assume $q_G(S) \leq |S|$ for all $S \subseteq V(G)$. We want to show that G contains a 1-factor.

Notice, there are $\dfrac{|G|}{2}$ many independent H-paths by construction. So, we have $M_{G'}(H) \leq \dfrac{|G|}{2}$. Observe, if $M_{G'}(H) = \dfrac{|G|}{2} = \dfrac{|G'|}{4}$, then G has a 1-factor. So, we need to show

$$\frac{|G|}{2} \leq M_{G'}(H) = |S| + \sum_{C_i \in C_F} \lfloor \frac{1}{2} |\delta C| \rfloor$$

for all $S \subseteq V(G - H)$ and $F \subseteq E(G - S) - E(H)$, where C_F is the set of connected components of F.

Suppose we have r components of $G - H$. We then have $|G| = |S| + |C_1| + \ldots + |C_r|$. So,

$$\begin{aligned} |S| + \lfloor \frac{1}{2} C_1 \rfloor + \ldots + \lfloor \frac{1}{2} C_r \rfloor &= |S| + \frac{1}{2}|C_1| + \ldots + \frac{1}{2}|C_r| - \frac{1}{2} q_G(S) \\ &= \frac{|G|}{2} + \underbrace{\frac{|S|}{2} - \frac{1}{2} q_G(S)}_{\geq 0} \qquad\qquad \text{since } q_G(S) \leq |S| \\ &\geq \frac{|G|}{2} \end{aligned}$$

Therefore, $M_{G'}(H) = \dfrac{|G|}{2}$, hence, we G has a 1-factor.

4. **[page 84, #26**] For every $k \in \mathbb{N}$ find an $l = l(k)$, as large as possible, such that not every l-connected graph is k-linked.

5. **[page 111, #4**] show that every planar graph is a union of three forests.

   Solution by Rade Musulin: Let $G = (V, E)$ be a planar graph. Let $U \subset V(G)$. The subgraph induced by these vertices $G[U]$ is a planar graph because every induced subgraph of a planar graph is planar.

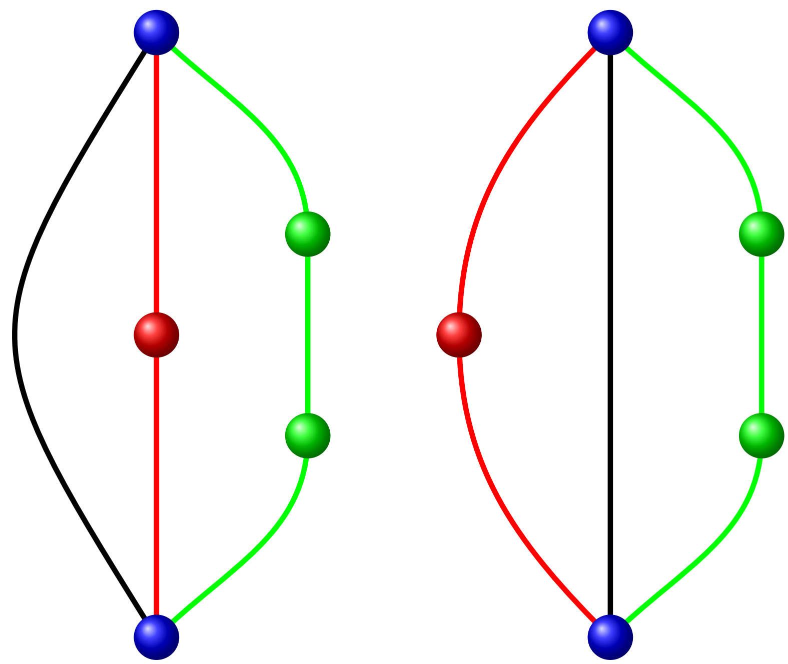
   Let $m = ||G(U)||$ and $n = |G(U)|$. By Theorem 4.2.10, since $G[U]$ is a planar graph, $||G[U]|| = m \leq 3n - 6 = 3(|U| - 2) < 3(|U| - 1)$.

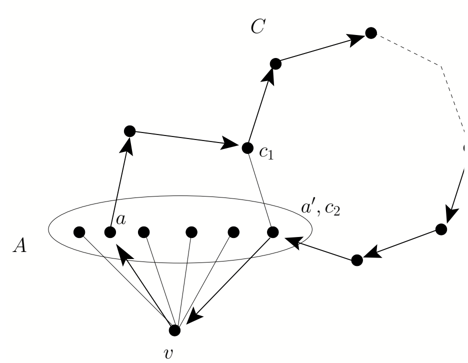
   By Theorem 2.4.4 (Nash-Williams), G can be partitioned into at most 3 forests.

   Therefore, every planar graph is a union of three forests.

6. **[page 111, #13**] Find a 2-connected planar graph whose drawings are all topologically isomorphic but whose planar embeddings are not all equivalent.

   Solution by James Sweeney: The two graphs below are topologically isomorphic because if they are put on a sphere, you will either have the exact same graph, or you will be able to re-orient the graph such that the vertex on the north pole has the same orientation. These graphs are not equivalent embeddings however, because the top vertex is oriented counter-clockwise from smallest to biggest in the left graph, while it is oriented clockwise from smallest to biggest in the right graph.





C
c_1
a', c_2
a
A
v