FAQ for G1BCOF: Complex Functions

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1 Questions asked in 2003-4

1.1 How useful will it be to look at the exams for G12CAN?

Answer Obviously I can't give too much away, but I think that it is a good idea to look at questions from both subsections of the G12CAN exams. You can expect parts of G1BCOF questions to be in a 'similar style' to some parts of G12CAN questions, but each G1BCOF question will be worth 30 marks.
Note added 2004-5 The 2003-4 G1BCOF examination paper is now available on the web, and you may find it useful to look at that.

1.2 I've forgotten some of the basic theory of complex numbers: what should I do?

Answer One thing that you could do is to read through Section 1.1 of the full lecture notes for G1BCOF. This runs through the facts and theory about complex numbers that you are expected to know before the start of G1BCOF. Alternatively, you could look at Chapter 1 of the Autumn Semester G11LMA notes.

1.3 Why are we looking for the coefficient of z^{-1} in the function $f(z)$ in order to find the integral of f around a closed curve around 0?

Answer All other integer powers of z have an antiderivative: for $n \neq -1$, z^n is the derivative of $z^{n+1}/(n+1)$. Functions which have antiderivatives will integrate to zero when integrated round a closed curve. As a result, we can throw away all the other powers of z and concentrate on the coefficient of z^{-1} (assuming that the series we have found is valid on an annulus which contains the curve). If we are integrating once anticlockwise around a simple closed curve around 0 in such an annulus, the integral is $2\pi i$ times the coefficient of $1/z$ in the series.
If instead we are working with powers of $z - a$ (so our annulus is centred on the point a) then we need to find the coefficient of $1/(z-a)$. If we are integrating once anticlockwise round a simple closed curve around a inside such an annulus, the integral is $2\pi i$ times the coefficient of $1/(z-a)$ in the series.
For more information on this, see Laurent's Theorem, Cauchy's Residue Theorem and the Cauchy Integral Formula.

1.4 I don't understand the manipulations involved in finding the Laurent series for $1/(z-3)$ in the annulus $2 < |z-1| < \infty$.

Answer A fairly general principle applies here. We usually use some form of geometric series argument. In general, if we are looking at $1/(1-g(z))$ or $1/(1+g(z))$ then we can use the geometric series expansions **provided that** $|g(z)| < 1$. **Given that** $|g(z)| < 1$ **in the region under consideration**, we have

$$\frac{1}{1-g(z)} = 1 + g(z) + g(z)^2 + g(z)^3 + \cdots$$

and

$$\frac{1}{1+g(z)} = 1 - g(z) + g(z)^2 - g(z)^3 + \cdots.$$

In the case of $1/(z-3)$ in the annulus $2 < |z-1| < \infty$, we want an expansion in powers of $(z-1)$, and we know $|z-1| > 2$ in our region. With the geometric series method in mind, and writing everything in terms of $(z-1)$, we rearrange as follows

$$\frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{1}{z-1}\left(\frac{1}{1-2/(z-1)}\right).$$

The second fraction is now in the form we want, with $g(z) = 2/(z-1)$, as $|g(z)| < 1$ in the given region ($|z-1| > 2$). We can expand $\dfrac{1}{1-2/(z-1)}$ as a geometric series, and then multiply by $1/(z-1)$ afterwards, giving us

$$\begin{aligned}\frac{1}{z-3} &= \frac{1}{z-1}\left(1 + \frac{2}{z-1} + \frac{4}{(z-1)^2} + \cdots\right)\\ &= \frac{1}{z-1} + \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3} + \cdots\end{aligned}$$

(as in the solutions to Coursework 4).

1.5 In the formula for the Laurent series coefficients,

$$a_k = \frac{1}{2\pi i}\int_{|z-a|=T} f(z)(z-a)^{-k-1}\,\mathrm{d}z\,,$$

why do we have the power $-k-1$ instead of $-k$?

Answer This is related to the earlier question about the coefficient of $1/z$ when finding integrals. You may well be thinking of examples like Fourier series, where you would use the power $-k$ instead of $-k-1$. There is in fact a connection: when you integrate round a circle centred on a with respect to the angle t, using the parametrization $z = a + R\exp(it)$, we have $\mathrm{d}z = iR\exp(it)\mathrm{d}t = i(z-a)\mathrm{d}t$, and so in some sense you can think of $\mathrm{d}z$ as already including an extra power of $(z-a)$ in it.
In terms of the earlier question, for an analytic function g on an annulus containing the circle $|z-a| = T$ we know that the coefficient of $1/(z-a)$ in the Laurent series for g is

$$\frac{1}{2\pi i}\int_{|z-a|=T} g(z)\,\mathrm{d}z\,.$$

Setting $g(z) = f(z)(z-a)^{-k-1}$, the coefficient of $1/(z-a)$ in g will be the same as the coefficient of $(z-a)^k$ in f, which is a_k. Thus

$$a_k = \frac{1}{2\pi i}\int_{|z-a|=T} g(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{|z-a|=T} f(z)(z-a)^{-k-1}\,\mathrm{d}z\,,$$

as in the notes.

1.6 When finding the integral once anticlockwise around $|z| = 1$ of $\dfrac{1}{z^4(1-\sin z)}$, we expanded $1/(1-\sin z)$ as a geometric series,

$$\frac{1}{1-\sin(z)} = 1 + \sin(z) + \sin^2(z) + \sin^3(z) + \cdots.$$

Why can we do this?

Answer This is a very interesting question, especially when you realize that $|\sin(z)|$ can be bigger than 1 in our region. For example, $|\sin(i)| = \sinh(1) = 1.1752\ldots > 1$. Nevertheless, **the series found** by this apparently dubious procedure of substituting $\sin z$ into the geometric series and collecting the powers of z gives the correct answer. Why is this?
We know that $g(z) = 1/(1-\sin(z))$ is in fact analytic for $|z| < \pi/2$ (remember that we know all the solutions of $\sin(z) = 1$). This means that g must have a Taylor series valid in $|z| < \pi/2$. However, Taylor series in powers of z are unique, so this Taylor series must be the same as the Taylor series for g on a very small disc centred on 0 (e.g. $|z| < 10^{-6}$). On such a small disk, we do have $|\sin(z)| < 1$ and so the expansion of $1/(1-\sin(z))$ gives the right answer there. But this means that **the series found is correct** on the bigger disk too, even though **the geometric series calculations themselves would not be valid** on the bigger disk.

1.7 When we used the Fundamental Estimate to estimate the modulus of the integral of $1/(z^n + \overline{z})$ along the straight line from 2 to $3+i$, I did not understand how we obtained the given bound M for $|1/(z^n + \overline{z})|$.

Answer Let γ be the straight line from 2 to $3+i$. We want a bound that makes it clear that $\int_\gamma 1/(z^n + \overline{z})\,\mathrm{d}z$ is small when n is large. If you look at γ, it is clear that the modulus of z increases as z moves from 2 to $3+i$ (you are moving away from the origin). So, for z on γ, we have

$$2 \le |z| \le |3+i| = \sqrt{10}.$$

We next give a rather crude estimate for $|z^n + \overline{z}|$ on γ. We have $|z| \ge 2$ and so $|z^n| \ge 2^n$ on γ. Also $|\overline{z}| = |z| \le \sqrt{10}$ on γ, so by the triangle inequality $|z^n + \overline{z}| \ge 2^n - \sqrt{10}$ for z on γ. Thus, for $n \ge 2$, we have

$$\left|\frac{1}{z^n + \overline{z}}\right| = \frac{1}{|z^n + \overline{z}|} \le \frac{1}{2^n - \sqrt{10}}.$$

Since the length of γ is a constant, namely $\sqrt{2}$, we now have from the Fundamental Estimate that (for $n \ge 2$)

$$\left|\int_\gamma \frac{1}{z^n + \overline{z}}\,\mathrm{d}z\right| \le \frac{\sqrt{2}}{2^n - \sqrt{10}} \to 0$$

as $n \to \infty$.

1.8 I understand that if a function has an antiderivative then the integral of the function round a closed curve is 0. But why doesn't this apply to the function $1/z$? Surely $1/z$ is the derivative of $\log(z)$?

Answer This is another interesting question (closely connected to Question and Answer 1.3). The answer is that the reasoning is almost correct. Remember that there is a problem defining $\log(z)$ because of the fact that, for all non-zero complex numbers z there are infinitely many solutions w to the equation $\exp(w) = z$. In the notes we defined the Principal Logarithm $\mathrm{Log}(z)$, and it is true that $\mathrm{d}/\mathrm{d}z\,\mathrm{Log}(z) = 1/z$ **where** $\mathrm{Log}(z)$ **is defined**. But $\mathrm{Log}(z)$ is undefined for z on the non-positive real axis. So we can not use $\mathrm{Log}(z)$ as an antiderivative for $1/z$ on a closed curve which goes around 0. In fact, when you cross the negative real axis there is a discontinuity of $2\pi i$ in the value of $\mathrm{Log}(z)$. This is what you should expect, since the integral of $1/z$ once anticlockwise round a circle centred on 0 is $2\pi i$.

1.9 How can I estimate $|e^{iz}|$?

Answer If $z = x + iy$ then $iz = -y + ix$ and $e^{iz} = e^{-y}e^{ix}$. Remember also that $|e^{ix}| = 1$, as e^{ix} is just a number on the unit circle (where x measures the angle round from the positive real axis). From this we see that $|e^{iz}| = |e^{-y}| = e^{-y} = \exp(-\mathrm{Im}(z))$. In particular, if $\mathrm{Im}(z) \geq 0$ then $|e^{iz}| \leq 1$.

1.10 How can I estimate the modulus of a contour integral?

Answer You need to use the **Fundamental Estimate**. This says that when you are integrating a function f along a contour γ, you can estimate $\left|\int_\gamma f(z)\,\mathrm{d}z\right|$ in terms of the length of γ and the maximum value of $|f(z)|$ on γ. If $M > 0$ is a constant such that $|f(z)| \leq M$ for all z on the contour γ, then $\left|\int_\gamma f(z)\,\mathrm{d}z\right|$ is at most equal to M times the length of the contour γ.

For a semicircular arc from R to $-R$ via iR, the length of the contour is πR. If we want to use the Fundamental Estimate to show that the integral along this arc has small modulus, we need to show that the function integrated has maximum modulus much smaller than $1/R$ on this semicircular arc. This is true for all of the examples we have looked at.

1.11 Which standard series should I remember?

Answer The more the better! Having said that, you can usually use Taylor's theorem to work out the series if necessary. Also, many of them can be deduced from each other by differentiating or integrating term by term.
Please note that the following is NOT a comprehensive list.
I would certainly expect everyone to remember or be able to work out for themselves standard series **such as** those for $\exp(z)$, $\sin(z)$, $\cos(z)$, $1/(1-z)$, $1/(1+z)$ etc. and to be able to differentiate or integrate term by term to find other series, e.g. for $1/(1-z)^2$. You should also be able to multiply series and substitute series into each other as in the notes and the module questions.

1.12 How do I find the Taylor series for $\mathrm{Log}(1+z)$ in powers of z?

Answer The first thing here is to remember some properties of $\mathrm{Log}(z)$. This is the principal logarithm, and is defined for all complex numbers EXCEPT for those which lie on the non-positive real axis. Where it is defined, we have $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$, $\mathrm{Log}(z)$ is analytic, $\exp(\mathrm{Log}(z)) = z$, and $\mathrm{d}/\mathrm{d}z(\mathrm{Log}(z)) = 1/z$.
This means that $\mathrm{Log}(1+z)$ is analytic where it is defined, which is for all z except for numbers on the real axis which are ≤ -1. By the chain rule, $\mathrm{d}/\mathrm{d}z(\mathrm{Log}(1+z)) = 1/(1+z)$. There are now (at least) two ways to proceed.
(1) We know that the Taylor series for $1/(1+z)$ in powers of z is $1 - z + z^2 - z^3 + \cdots$. We can integrate this series term by term to find the series for $\mathrm{Log}(1+z)$ (which is an antiderivative for $1/(1+z)$). This gives us that $\mathrm{Log}(1+z) = C + (z - z^2/2 + z^3/3 - z^4/4 + \cdots)$ and setting $z = 0$ gives us $C = \mathrm{Log}(1) = 0$, so

$$\mathrm{Log}(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots$$

is the Taylor series we want (valid for $|z| < 1$).

(2) We can use Taylor's theorem to find the coefficients. With $f(z) = \mathrm{Log}(1+z)$, $f'(z) = 1/(1+z)$, $f^{(2)}(z) = -1/(1+z)^2$, $f^{(3)}(z) = 2/(1+z)^3$ and generally, for $k \geq 1$, $f^{(k)}(z) = (-1)^{k-1}(k-1)!/(1+z)^k$. We have $f^{(0)}(0) = \mathrm{Log}(1) = 0$ and, for $k \geq 1$, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. We know that the kth Taylor coefficient here is $f^{(k)}(0)/k!$ and this leads to the same answer as above.

1.13 Why don't I need to find the residue at 1 when calculating $\displaystyle\int_{|z|=4} z^{-6}(1-z)^{-1}\,\mathrm{d}z$?

Answer Actually there are (at least) two methods to answer this question, one of which does not need the residue at 1 and the other method does.

The first method is to find the Laurent series in powers of z valid on an annulus including $|z| = 4$: then you only need the coefficient of $1/z$ and the singularity at 1 is in some sense irrelevant. The Laurent series you want (valid for $|z| > 1$) is $-(1/z^7 + 1/z^8 + 1/z^9 + \cdots)$. The integral is just $2\pi i$ times the coefficient of $1/z$, and this is 0.

The second method uses the Cauchy residue theorem. In this case, to calculate the residues at 0 and 1 you find another two Laurent series, one valid near 0 in powers of z, one valid near 1 in powers of $(z-1)$. Neither of these two series is the same as the Laurent series in powers of z valid when $|z| = 4$.

The residue at 0 is 1 (found using the Taylor series of $1/(1-z)$ in powers of z, which is then divided by z^6).

The function has a simple pole at 1, and the residue there is -1 (found using your favourite method for simple poles). The integral is $2\pi i$ times the sum of the residues at the singularities inside the contour, and this is 0.

1.14 Can I expand $1/(1+\cos z)$ as a geometric series $1 - \cos z + \cos^2 z - \cos^3 z + \cdots$?

Answer This is very risky! One problem here is when $z = 0$, then $\cos z = 1$ and the attempted geometric series expansion would be $1 - 1 + 1 - 1 + \cdots$, which is nonsense. (It is, however, OK if you work near a point like $\pi/2$ where $\cos z = 0$). If you want to deal with $1/(1+\cos z)$ in powers of z, valid for small z, you should use the method given in lectures:

$$\cos z = 1 - z^2/2 + z^4/4! - \cdots$$

so

$$1 + \cos z = 2 - z^2/2 + z^4/4! - \cdots.$$

Now write this as $2 + g(z)$, where $g(0) = 0$, and you want to find $1/(2 + g(z))$ in powers of z. Since $g(0) = 0$, near 0 we can use the geometric series expansion to find

$$\frac{1}{2+g(z)} = (1/2)\frac{1}{1+g(z)/2} = (1/2)(1 - g(z)/2 + g(z)^2/4 - g(z)^3/8 + \cdots).$$

Manipulation of series then gives the series you want. These manipulations are valid at least near 0, but in fact the series you find is valid all the way out to the first singularity. (See the earlier question about $1/(1-\sin z)$.)

1.15 What is the connection between the semicircular contour and the integral along the real axis?

Answer Let R be a large positive real number, let γ_R be the semicircular arc from R to $-R$ via iR, let L_R be the straight line from $-R$ to R on the real axis, and let Γ_R be the closed semicircular contour consisting of L_R followed by γ_R.
Now suppose that you have a function f which is analytic on Γ_R. (Or at least continuous there.) Then

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{L_R} f(z)\,\mathrm{d}z + \int_{\gamma_R} f(z)\,\mathrm{d}z.$$

On the right hand side, the first term can be re-expressed by parametrizing the straight line L_R as $z = x$, $-R \leq x \leq R$. Then $f(z) = f(x)$, $\mathrm{d}z = \mathrm{d}x$, and we have

$$\int_{L_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\ \mathrm{d}x,$$

and this gives us

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\ \mathrm{d}x + \int_{\gamma_R} f(z)\,\mathrm{d}z.$$

For most of the examples we have looked at, the left hand side here is constant once R is large enough (using the residue calculus, i.e. Cauchy's residue theorem). Call this constant A. Then we have

$$A = \int_{-R}^{R} f(x)\ \mathrm{d}x + \int_{\gamma_R} f(z)\,\mathrm{d}z.$$

Again, for most of the examples we have looked at, the second integral on the right is small and tends to 0 as $R \to \infty$. In this case, taking limits as $R \to \infty$ gives us

$$A = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x.$$

2 Questions asked in 2004-5

2.1 Why is the modulus of $x+iy$ equal to $\sqrt{x^2+y^2}$? Since $i^2=-1$, shouldn't it be $\sqrt{x^2-y^2}$?

Answer (24/1/05) It is certainly true that $x^2+(iy)^2=x^2-y^2$. Nevertheless, $|x+iy|^2=x^2+y^2$ and **not** $x^2+(iy)^2$. Why is this? The modulus of the complex number $z=x+iy$ is the distance from the origin to z. If you look at the relevant right-angled triangle and use Pythagoras' theorem to find this distance, the lengths of the two shorter sides are $|x|$ and $|y|$: i does not appear here, because our two coordinates are real numbers. The distance we want is

$$|z|=\sqrt{|x|^2+|y|^2}=\sqrt{x^2+y^2}.$$

2.2 I don't understand why $|z+w|\geq|z|-|w|$

Answer (8/2/05) The standard triangle inequality for complex numbers tells us that $|z+w|\leq|z|+|w|$. To obtain $|z+w|\geq|z|-|w|$, write $z=(z+w)+(-w)$. Then (using the standard triangle inequality)

$$|z|=|(z+w)+(-w)|\leq|z+w|+|(-w)|=|z+w|+|w|.$$

We have $|z+w|+|w|\geq|z|$. Subtracting $|w|$ from both sides gives the result.

2.3 How can I tell which complex numbers z satisfy $|z+i|>|z-i|$?

Answer (8/2/05) Geometrically, for a complex number w, $|w|$ is the distance from the origin to w. More generally, for complex numbers z and a, $|z-a|$ can be interpreted either as the distance from $z-a$ to 0 **or** as the distance from z to a. Indeed there are often many possible ways to interpret these numbers.

In this question, we are concerned with $|z-i|$ and $|z+i|$. We may interpret $|z-i|$ as the distance from z to i. Similarly, $|z+i|=|z-(-i)|$ is the distance from z to $-i$. We need to know which complex numbers are further from $-i$ than from i (i.e. $|z+i|>|z-i|$). It may help to think first about which points are at the same distance from i and $-i$. These are the points on the perpendicular bisector, which in this case is the real axis. It is now clear that points strictly above the real axis will be closer to i than to $-i$ (as required).

2.4 Why are we calculating contour integrals? What does the contour integral mean?

Answer (8/2/05) Contour integrals have various interpretations, some related to 2-dimensional fluid flow (circulation and flux: see Section 3.9 of the full lecture notes). More directly, we will see that for continuous functions f which have an antiderivative F (in the complex sense) on the whole of a contour γ, the value of the contour integral $\int_\gamma f(z)\mathrm{d}z$ is the difference in the values of F (the antiderivative) at the endpoints of the contour γ.

Contour integrals have applications throughout pure and applied mathematics. For example, contour integration methods can help to find the inverse Laplace transforms of functions. At the end of this module we will use contour integration methods to find the exact values of some real integrals that are otherwise rather hard to calculate, for example $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,\mathrm{d}x$.

2.5 I thought that the interval $(0, 1)$ was open. Why isn't it a domain?

Answer (30/3/05) There is a subtle question here: which sense of **open** is appropriate? Obviously, open intervals such as $(0, 1)$ were examples in G1BMAN of **open subsets of** $\mathbb{R}$. Moreover, as subsets of $\mathbb{C}$, you can get from one point to another in such a set using a (horizontal) stepwise curve. So why is it not a domain?
The answer is that such an interval is **open in** $\mathbb{R}$ but is **not open in** $\mathbb{C}$. For a non-empty set U to be open in $\mathbb{C}$ requires at least one small disc of positive radius to be a subset of U (In fact, every point of U must be at the centre of such a disc, though the range of radii available usually depends on the point.) Obviously no such disc can be contained in a line segment, so in particular intervals such as $(0, 1)$ on the real axis are NOT open in $\mathbb{C}$.
The issue of which sets are open in which is discussed in more depth in the module G13MTS Metric and Topological Spaces.

2.6 Why do you only have to differentiate with respect to x in order to find f'?

It says in the notes that, with $z = x + iy$ and, as usual, $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, then, at points where f is complex-differentiable,

$$f' = u_x + iv_x = \partial f/\partial x.$$

But why do we only use the horizontal direction of approach?
Answer (30/3/05) Suppose that we are given that f is complex-differentiable at a point $a = A + iB$ (where A and B are in $\mathbb{R}$) and with derivative $f'(a)$. By definition, this means that

$$\lim_{w \to a} \frac{f(w) - f(a)}{w - a} = f'(a),$$

no matter how w approaches a in $\mathbb{C}$ (except that w must approach a through values which are not actually equal to a). This is a very strong condition, because it tells us (in particular) that we will get the same answer **whichever** direction we approach a from.
Note that checking one direction does NOT tell us that the condition of complex-differentiability is satisfied. Instead, here, we are GIVEN that f is complex differentiable at a, and are allowed to use this fact.
Setting $w = a + h = (A + h) + iB$ for $h \in \mathbb{R}$ $(h \neq 0)$ gives us a horizontal direction of approach. The equalities

$$\frac{f(w) - f(a)}{w - a} = \frac{f((A + h) + iB) - f(A + iB)}{h} = \frac{u(A + h, B) - u(A, B)}{h} + i\frac{v(A + h, B) - v(A, B)}{h}$$

quickly lead to the desired result when you let h tend to zero.

2.7 Do I need to learn all the proofs of theorems for the G1BCOF exam?

Answer (30/3/05) The general answer in these cases is usually the same: The more you understand, the better! Ideas and methods from the proofs reinforce those involved in individual problems, and a good understanding of what is going on will enable you to decide which methods are appropriate when solving particular problems.
Having said this, you can see some of the most common styles of exam question for G1BCOF by looking at last year's exam paper. Taking FAQ Question and Answer 1.1 from 2003-4 into account, you could also look at previous exam papers for G12CAN.

For this module (at least as I give it), you will not be expected to write out PROOFS of theorems in full in the exam. You will, however, be expected to be familiar with the statements of standard definitions and results from the notes, as well as associated ideas, conditions and typical applications.

2.8 How do I cube a series?

Answer (19/4/05) In Problem Class 4, many people had problems cubing the series $z^2 - z^6/3! + z^{10}/5! - \cdots$. The suggestion in the problem class was to first take out the factor z^2 (which cubes to give z^6) and then cube the remaining series. Interestingly, one of the most common problems in the class appeared to be with the binomial expansions

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

So here we can use

$$z^6(1 - z^4/3! + \cdots)^3 = z^6(1 - 3z^4/3! + \cdots) = z^6 - 3z^{10}/3! + \cdots$$

or directly (perhaps harder)

$$(z^2 - z^6/3! + z^{10}/5! - \cdots)^3 = z^6 - 3(z^2)^2 z^6/3! + \cdots = z^6 - 3z^{10}/3! + \cdots .$$

You should convince yourselves that the $\cdots$ terms do not contribute to the terms we are interested in, which are the ones up to and including the term in z^{10}.

2.9 Why do we look at $1/g'(a)$ when investigating the residue at a of the function $1/g(z)$?

Answer (5/5/05) In fact this only works if both $g(a) = 0$ and $g'(a) \neq 0$. It is one of the short cuts available when dealing with simple poles. For more details on these short cuts, see the **Solutions to Section 5.3, examples 3-8**, available from the module web page. The idea is that, by definition,

$$g'(a) = \lim_{z \to a} \frac{g(z) - g(a)}{z - a} = \lim_{z \to a} \frac{g(z)}{z - a}$$

because $g(a) = 0$. Assuming this limit is non-zero, we can take the reciprocal and obtain

$$\frac{1}{g'(a)} = \lim_{z \to a} \frac{z - a}{g(z)} = \lim_{z \to a} (z - a) \frac{1}{g(z)} .$$

Since this limit exists in $\mathbb{C}$, it follows that the function $1/g(z)$ must have a simple pole at a with residue $1/g'(a)$. (Remember that the residue is the coefficient of $1/(z - a)$ in the Laurent series expansion in a small punctured disc centred on a.)

2.10 What is the connection between the Cauchy Residue Theorem and the Cauchy Integral Formula?

Answer (9/5/05) Recall that the Cauchy Integral Formula tells you that if f is analytic on and inside the simple, closed, piecewise-smooth curve γ (as usual, described once counter-clockwise), then

$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z$$

is equal to $2\pi i f(w)$ if w is inside γ, and is 0 if w is outside γ.

Let us look at this in terms of singularities and residues. Set $g(z) = f(z)/z - w$. If w is outside γ then $g(z)$ has no singularities inside γ, so the integral is zero. If w is inside γ, then g has one singularity inside γ, when $z = w$. By Cauchy's Residue Theorem, the integral will be $2\pi i$ times the residue of g at this singularity, i.e. $2\pi i\mathrm{Res}(g, w)$.

It is clear that $\lim_{z\to w}(z - w)g(z) = f(w)$. Since this limit exists and is finite, it must be equal to the residue we want: $\mathrm{Res}(g, w) = f(w)$. Thus

$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = 2\pi i f(w),$$

as predicted by the Cauchy Integral Formula.

Further information: note that if $f(w) = 0$, then g has a removable singularity at w, while if $f(w) \neq 0$ then g has a simple pole at w.

2.11 How do I find the Taylor series for $\sin(1 + z)$ in powers of z?

Am I allowed to substitute $u = 1 + z$ into the standard series $\sin u = u - u^3/3! + u^5/5! - \cdots$?

Answer (11/5/05) If you take $f(u) = \sin u$, expanded as above, and $g(z) = 1 + z$, this situation does not fit the standard substitution conditions because $g(0) \neq 0$. If $g(z)$ is expanded in powers of $z - a$ and $f(u)$ is expanded in powers of $u - b$, then you usually require $g(a) = b$ to justify the substitution. Here $a = b = 0$.

Nevertheless, since $\sin$ is entire, it is true that $\sin u = u - u^3/3! + u^5/5! - \cdots$ for ALL u, and so we do have

$$\sin(1 + z) = (1 + z) - \frac{(1 + z)^3}{3!} + \frac{(1 + z)^5}{5!} - \cdots .$$

This is, of course, the correct expansion of $\sin(1 + z)$ in powers of $1 + z$, but we want an expansion in powers of z instead. Unfortunately, the above expansion is not very helpful for this purpose. We can just about see that the constant coefficient must be $1 - 1/3! + 1/5! - \cdots = \sin 1$, as expected, but the remaining coefficients require more work (optional exercise). So what should we do instead? There are two approaches.

One method that is fairly well guaranteed to work is to use Taylor's theorem, centred on the point 1, to find a series for $\sin u$ in powers of $(u - 1)$ (instead of powers of u). In terms of a and b above, this would mean that we now had $b = 1$ and so $g(a) = b$. This allows a successful substitution (note that when $u = 1 + z$, we have $u - 1 = z$).

Alternatively (related to the first method, but less general), note that $\sin(1+z)$ is itself an entire function, and so is equal to its Taylor series expanded in powers of z. Writing $h(z) = \sin(1+z)$, we have

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \,,$$

where $a_k = h^{(k)}(0)/k!$. Usually it is a very bad idea to try to find the coefficients a_k using this formula, but here, for once, it is not too bad. We have $h(z) = \sin(1+z)$, $h'(z) = \cos(1+z)$, $h^{(2)}(z) = -\sin(1+z)$, etc, and so $h^{(k)}(0)$ is easy to evaluate. This gives us the answer:

$$\sin(1+z) = h(z) = \sin 1 + (\cos 1)z - \frac{\sin 1}{2!}z^2 - \frac{\cos 1}{3!}z^3 + \frac{\sin 1}{4!}z^4 + \cdots .$$

If we had used the first method, we would have found that

$$\sin u = \sin 1 + (\cos 1)(u-1) - \frac{\sin 1}{2!}(u-1)^2 - \frac{\cos 1}{3!}(u-1)^3 + \frac{\sin 1}{4!}(u-1)^4 + \cdots .$$

(this time the coefficients are $f^{(k)}(1)/k!$ where $f(u) = \sin u$) and then the substitution $u = z+1$ would have led to the same result.

2.12 When changing variables from x to z why does $\cos x$ change to $\exp(iz)$?

For example, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+16)}\,\mathrm{d}x$ **is found by calculating**

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{(z^2+4)(z^2+16)} \mathrm{d}z \,,$$

where Γ_R is the usual closed semicircular contour of radius R.

Answer (17/5/05) You might well expect to see $\cos z$ there instead, and that would work except that $|\cos z|$ gets big in the upper half plane: e.g. look at $\cos(iR)$ when R is large and positive. That spoils the estimate for the integral on the arc of the semicircle when you try the semicircular contour argument.

The only reason you can use e^{iz} instead, is that, working on the real axis,

$$e^{ix} = \cos x + i \sin x,$$

so (working on the real axis only, see below) $\cos x = \mathrm{Re}(e^{ix})$. This allows you to first work out the integral with e^{ix} in, and then take real parts at the end. So the substitution should really be regarded as saying that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+16)}\,\mathrm{d}x = \lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{(z^2+4)(z^2+16)} \mathrm{d}z \,.$$

Warning: although

$$e^{iz} = \cos z + i \sin z$$

is true for **all** complex numbers z, for z off the real axis you can show that $\cos z$ and $\sin z$ are never real numbers (exercise!). So it is NOT correct to say that $\cos z = \mathrm{Re}(e^{iz})$ when z is off the real axis. For more details on why you use e^{iz} instead of cos z, see **Solutions to Section 5.3, examples 3-8** available from

`http://www.maths.nott.ac.uk/personal/jff/G1BCOF/#handouts`