Math417

Mathematical Programming

Homework IV

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1 Completing the proof of the f. theorem

Let $\{J_i\}$ denote the set of subsets of $\{1, ...n\}$ with $|J| = m$ and such that a_j with $j \in J_i$ are linearly independent $\forall J_i$ Also let A^i be the matrix

$$\begin{pmatrix} a_j^\top \\ . \\ . \\ . \end{pmatrix} \text{ with } \quad j \in J_i$$

Then there exists a unique point, say $\bar{x}$ such that $A^i\bar{x} = b$ because by construction it has full rank. Therefore for all J_i there exists only one associated BFS. Moreover there is only a finite number of J_i so there can only be a finite number of BFS.

2 Derivation of the dual problem

Here we have two cases

i) $v \geq 0$

$$\inf\{v^\top x \,|\, x \geq 0 \quad v \geq 0\} = 0$$

ii) $v_i < 0$ for some i

$$\inf\{v^\top x \,|\, x \geq 0\} = -\infty$$

The first case is quite obvious. The support the second we simply say that we can we make the ith components of x go to infinity and keep the others null. So the infinimum is unbounded.

3 Dual of the dual is the primal

We start with the dual problem as denoted in the course notes.

$$\max b^\mathsf{T} y \qquad A^\mathsf{T} y \leq c \tag{1}$$

$$\iff -\min -b^\mathsf{T} y \qquad A^\mathsf{T} y + s = c \qquad s \geq 0 \tag{2}$$

$$\iff -\min b^\mathsf{T}(y^- - y^+) \qquad A^\mathsf{T}(y^+ - y^-) + s = c \tag{3}$$

Now we have split y to get restrictions on it and introduced a slack variable which must be positive so we have

$$y = y^+ - y^- \qquad y^+, y^-, s \geq 0 \tag{4}$$

Now we want to express the last problem in standard form. We construct the new vectors z and $\hat{b}$ and the new matrix $\hat{A}$

$$z = \begin{pmatrix} y^+ \\ y^- \\ s \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} A^\mathsf{T} & | - A^\mathsf{T} & | I_n \end{pmatrix} \qquad \hat{b}^\mathsf{T} = \begin{pmatrix} -b^\mathsf{T} & b^\mathsf{T} & 0_n \end{pmatrix} \tag{5}$$

We mention the dimensions of these new objects.

$$\hat{A} \in \mathbb{R}^{n \times 2m+n} \qquad s \in \mathbb{R}^{2m+n} \qquad z \in \mathbb{R}^{2m+n} \qquad \hat{b} \in \mathbb{R}^{2n} \tag{6}$$

We can now rexpress (1) as follows

$$-\min \hat{b}^\mathsf{T} z \qquad \hat{A} z = c$$

And we can now take its dual

$$\mathcal{D}: \qquad -\max c^\mathsf{T} \xi \qquad \hat{A}^\mathsf{T} \xi \leq \hat{b}$$

$$\iff \min c^\mathsf{T}(-\xi) \qquad \hat{A}(-\xi) \geq -\hat{b}$$

Now let's take a look at the constraints

$$\begin{pmatrix} A \\ -A \\ I_n \end{pmatrix} - \xi = \begin{pmatrix} b \\ -b \\ 0_n \end{pmatrix} \tag{7}$$

Which we can break down into

$$A(-\xi) \geq b \qquad -A(-\xi) \geq -b \qquad -\xi \geq 0 \tag{8}$$

Now let $x = -\xi \geq 0$. We can see from the above reformulation that the two matrix inequalities yield an equality and so we have the standard form of the primal, namely:

$$\min c^\mathsf{T} x \qquad Ax = b \qquad x \geq 0 \tag{9}$$