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October 6, 2016

MATH 417/487 - Mathematical Programming

Homework Set No. 5 - hints for soulution

5.1 We first transform the given LP in standard form:

$$\min\ c^T x \quad \text{s.t.} \quad Ax = b \quad \iff \quad \min \begin{pmatrix} c \\ -c \end{pmatrix}^T \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \quad \text{s.t.} \quad [A, -A]\begin{pmatrix} x^+ \\ x^- \end{pmatrix} = b, \ \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \geq 0.$$

The dual of this LP in standard form is given by

$$\max b^T y \quad \text{s.t.} \quad \begin{pmatrix} A^T \\ -A^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -c \end{pmatrix},$$

which can be simplified to

$$\max b^T y \quad \text{s.t.} \quad A^T y = c.$$

5.2 a) If (P) is unbounded, i.e. $\inf(P) = -\infty$ there exists $\{x_k\}$ with x_k feasible (P) $k \in \mathbb{N}$) and $c^T x_k \to -\infty$. Due to the weak duality theorem (Th. 2.5.1) there hence cannot exist a point $y \in \mathbb{R}^m$ feasible for (D).

b) Analogous to a).

We have

	$\inf(P) = -\infty$	$\inf(P) \in \mathbb{R}$	$\inf(P) = +\infty$
$\sup(D) = -\infty$	$\checkmark^{\alpha}$	$\times^{i)}$	$\checkmark^{\delta)}$
$\sup(D) \in \mathbb{R}$	$\times^{ii)}$	$\checkmark^{\beta}$	$\times^{iii)}$
$\sup(D) = +\infty$	$\times^{iv)}$	$\times^{v)}$	$\checkmark^{\gamma)}$

i) Since $\inf(P) \in \mathbb{R}$, by Theorem 2.5.7, (P) has a solution. By the strong duality theorem also (D) has a solution, in particular a feasible point.

ii) Follows from a).

iii) Analogous to i).

iv) Follows from a) and b), resp.

v) Follows from b).

α) Choose $A = (0\ 1)$, $b = 0$, $c = (-1, 1)^T$. then

$$(P) \quad \min -x_1 + x_2 \quad \text{s.t.} \quad x_2 = 0,\ x_1, x_2 \geq 0$$

is unbounded (choose the sequence $\{x^k\} = \{(k, 0)\}$). In turn,

$$(D) \quad \max 0 \cdot y \quad \text{s.t.} \quad 0 \cdot y \leq -1, y \leq 1$$

has no feasible point.

β) Choose $A = (0, 1)$, $b = 1$, $c = (1, 1)^T$. Then

$$(P) \quad \min x_1 + x_2 \quad \text{s.t.} \quad x_2 = 1,\ x_1, x_2 \geq 0$$

has the solution $\bar{x} = (0, 1)^T$ and

$$(D) \quad \max y \quad \text{s.t.} \quad y \leq 1$$

has the solution $\bar{y} = 1$.

γ) Choose $A = (0, 1)$, $b = -1$, $c = (1, 1)^T$.

$$(P) \quad \min x_1 + x_2 \mid x_2 = -1,\ x_1, x_2 \geq 0$$

is infeasible.

$$(D) \quad \max -y \mid y \leq 1$$

is unbounded (choose $\{y^k\} = \{-k\}$).

δ) Choose $A = (0, 1)$, $b = -1$, $c = (-1, 1)^T$.

$$(P) \quad \min -x_1 + x_2 \quad \text{s.t.} \quad x_2 = -1,\ x_1, x_2 \geq 0$$

is infeasible. This holds also for

$$(D) \quad \max -y \quad \text{s.t.} \quad 0 \cdot y \leq -1, y \leq 1.$$

5.3 By (3.11) and (3.14) in the notes and the fact that and $x_J \geq 0$ we have $z(t) \geq 0$ for all $t \geq 0$, i.e., by its construction $z(t)$ is feasible for (P) for all $t \geq 0$. By (3.12), for the corresponding ojective values, we observe that

$$c^T z(t) = c^T x + t u_r \quad (t \geq 0).$$

By assumption (3.10) we hence have

$$\inf(P) \leq \inf_{t \geq 0} c^T z(t) = -\infty,$$

i.e. (P) is unbounded.

5.4 First we compute y by equation (3.6) in the notes and obtain $y = (0, 0, -2, 0)^T$. Then (3.7) gives $u_2 = -3$, $u_3 = -4$, $u_6 = 2$. For $r \in K$ we may hence choose $r = 3$ or $r = 2$.

Choosing $r = 3$ the vector d determined by the linear system (3.13) is given by $d = (0, 1, 1, 1)^T$. The scalars $\hat{t}$ and s determined by (3.18) then read $\hat{t} = 2$, $s = 4$. Hence we obtain

$$\hat{J} = \{1, \underline{3}, 5, 7\}, \quad \hat{x} = (2, 0, 2, 0, 4, 0, 1)^T, \quad c^T\hat{x} = c^Tx - 8 = -12.$$

Choosing $r = 2$, in turn, gives $d = (0, 1, 3, 0)^T$ and $\hat{t} = 2$ as well as $s = 4$ <u>or</u> $s = 5$. This nonuniqueness in the choice of s leads to degeneracy of $\hat{x}$! For $s = 4$ we obtain

$$\hat{J} = \{1, \underline{2}, 5, 7\}, \quad \hat{x} = (2, 2, 0, 0, 0, 0, 3)^T, \quad c^T\hat{x} = c^Tx - 6 = -10.$$

Choosing $s = 5$ leads to the same basic feasible point $\hat{x}$, with a different index set $\hat{J} = \{1, 4, \underline{2}, 7\}$.