Math350

Graph theory

Homework IV

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We construct two graphs, G and F with the same vertex set such that the union of their edges gives us K_8. We want that G has no subgraph isomorphic to K_3. Let $G' := C_8$ and label its edges in an ordered way $V(G') = \{v_1, v_2, \ldots, v_8\}$. Construct G the following way:

$$G = G' + \{v_1, v_5\} + \{v_2, v_6\}$$

Call the added edges e and f respectively. Now it is quite obvious that we can partition the graph G' in two as it is a cycle of even length. So we have $\alpha(G') = 4$. The two vertex sets thus formed are the even and the odd numbered vertices. By adding the edges e and f we have made two vectors in each independent set adjacent, thus we have $\alpha(G) = 3$.
Claim : G has no triangle
Proof : Assume there exists a subgraph L of G isomorphic to K_3. Clearly either $v_1, v_5 \in L$ or $v_2, v_6 \in L$ since those are the only two pairs of adjacent vertices with degree 3. However $|N(v_1) \cap N(v_5)| = |N(v_2) \cap N(v_6)| = 0$. Clearly $L \ncong K_3$.
■ The edges of our graph G are our red coloring of $E(K_8)$.
Now let :

$$F = \overline{G}$$

Claim : There exists no subgraph of F isomorphic to K_4.
Proof : We prove the claim by assuming there exists such a subgraph, we look at the different cases and for each derive a contradiction. We do so by reducing the set of possible vertices in F such that we must have two consecutive vertices. They are obviously adjacent in G, so not in F, therefore they cannot be in a graph isomorphic to a complete graph. Let $S := \{v_1, v_2, v_5, v_6\}$

(i) $|V(L) \cap V(S)| = 0$. Then we have $V(L) = \{v_3, v_4, v_7, v_8\}$ ✓

(ii) $|V(L) \cap V(S)| = 1$. We can assume WLOG (by obvious symmetry) that the vertex in this intersection is v_1. $N_F(G_1) - v_2 - v_6 - v_5 = \{v_7, v_3, v_4\}$ ✓

Moreover there cannot be more than one vertex from S in L as we would have at least two non F-adjacent vertices in L so there is no way L could be isomorphic to K_4.
■

We just proved $R(3,4) > 8$ so we only have to show $R(3,4) \leq 9$ to prove equality. Let G be a graph with $|V(G)| = 9$.
Claim: Either $\alpha(G) \geq 3$ or $\omega(G) \geq 4$.
Proof : We already know that $R(3,3) = 6$. So whenever G has a vertex with at least 6 neighbors the claim is true. Moreover we know that $R(2,4) = 4$ so that whenever G has a vertex with at least 4 non neighbors the claim is true. We then only have to consider the following case:

$$\forall\ v \in G \qquad |N(v)| \geq 5 \qquad \text{and} \qquad |G - N(v)| \leq 3 \tag{1}$$

These are actually equalities since $N(v) \cup (G - N(v)) = G$, the order of both sets need to add up to 8. This means that $\deg(v) = 5$ for all $v \in G$. Hence $\sum_{v \in V(G)} \deg(v) = 45 = 2|E|$. This is a contradiction and so the proof is complete.
■

Claim: $R(4,4) \leq 18$
Proof : We know from b) that $R(3,4) = 9$. Moreover Ramsey's theorem says that $R(s,t) \leq R(s-1,t) + R(s,t-1)$. Finally $R(3,4) = R(4,3)$ by obvious symmetry. Putting those 3 facts together proves the claim.
■

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Let G be a 3-regular simple graph with no cut edge.
Claim: For all edges in G there exists a perfect matching that doesn't contain e.

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Let G be a simple graph s.t. there exists no disjoint odd cycles. Let $\chi(G)$ denote the chromatic index of G.
<u>Claim</u>: If G is as specified above we have $\chi(G) \leq 5$
<u>Lemma</u>: If a graph contains no odd cycle it is bipartite. Let $v_0 \in G$ be any vertex. Use two colors to color every vertex in the same component as v_0. Color them blue if they are at even distance from v and red otherwise. Do the same for every component. No edge can have both endpoints in different colors as this would create an odd cycle. Hence the coloring is a partition in two.
<u>Proof</u>: If there are no disjoint cycles of odd length there is at least one $v \in G$ such that it is in every cycle of odd length. This means that $G - v$ has no cycle of odd length. Let C be any cycle in G of odd length. We now have two facts:

1. $\chi(G - C) \leq 2$ since this graph is bipartite

2. $\chi(C) = 3$, this is a well known fact.

We can thus find a 3-coloring of C and a 2-coloring of $G - C$. Those two coloring together (assuming we took different colors in both cases, quite obviously) form a coloring G with at most 5 colors.
$\blacksquare$

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Let G be a triangle free simple graph with n vertices. Let $\alpha(G)$ denote the size of a maximum independent set in G.
Claim: $\alpha(G) \geq \lfloor \sqrt{n} \rfloor$
Proof: We can assume that there exists no vertex, call it v, with degree higher than $\lfloor \sqrt{n} \rfloor$ (i.e. $\Delta G \geq \sqrt{n} - 1$). If there were such a vertex the independent set of the required size is the set of neighbors of v. They have to be independent otherwise this would form a triangle. A small lemma with this bound on the maximum degree completes the proof.
Lemma:

$$\alpha(G) \geq \frac{|V(G)|}{\Delta G + 1} \tag{2}$$

To see this apply the following greedy algorithm. Start with an empty set S. Pick a vertex, put S and remove its neighbors from G. We thus remove at most $\Delta G + 1$ vertices from G at each iteration. The number of times we can iterate this algorithm is the RHS of 2 and the resulting set S is clearly independent. Substituting the bound on ΔG in the lower bound α completes the proof.