MATH-303201

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 303201

No calculators allowed

Examination for the Module MATH 3032

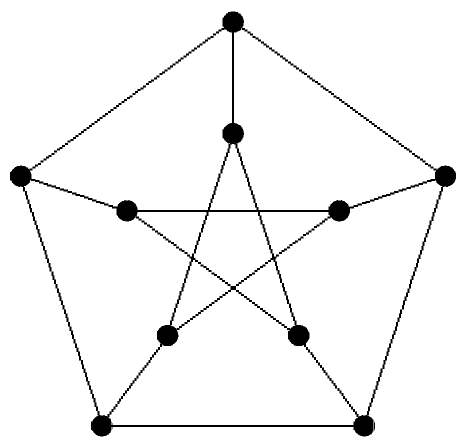
(January 2008)

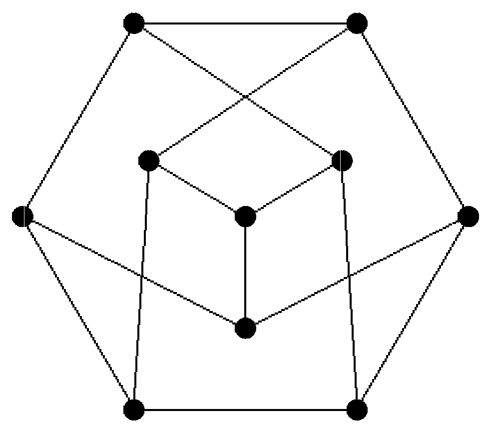
GRAPH THEORY

Time allowed : 2 hours

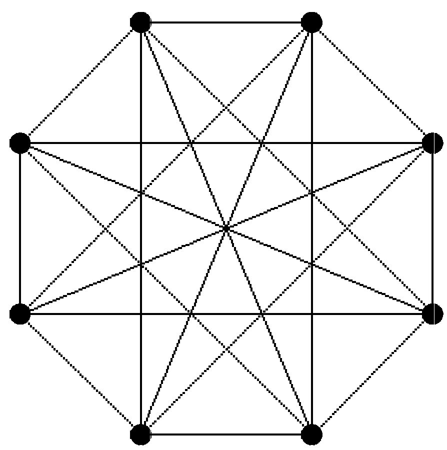
Do not answer more than *FOUR* questions.
All questions carry equal marks.

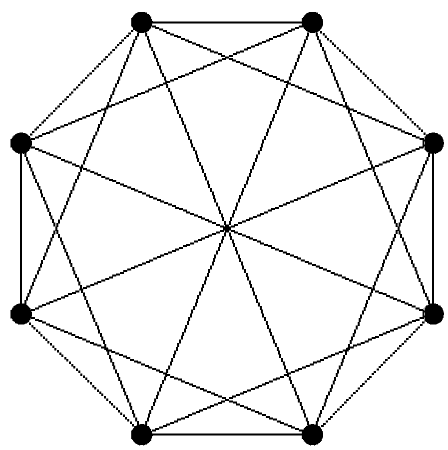
1. (**a**) Show that the two graphs below are isomorphic:





(**b**) Find the complements of the two graphs below. Hence, or otherwise, say, giving reasons, whether the graphs are isomorphic or not.





(**c**) The *girth* of a graph G is the length of a shortest circuit in G (if G has no circuits we define the girth of G to be infinite).

Show that a k-regular graph of girth four has at least $2k$ vertices.

Give, for each $k \geq 2$, an example of a k-regular graph of girth four with *exactly* $2k$ vertices.

Show that a k-regular graph of girth five has at least $k^2 + 1$ vertices.

2. (**a**) Define the terms: *Euler tour*, *Eulerian graph*.

Give (without proof) a necessary and sufficient condition for a graph G to be Eulerian.

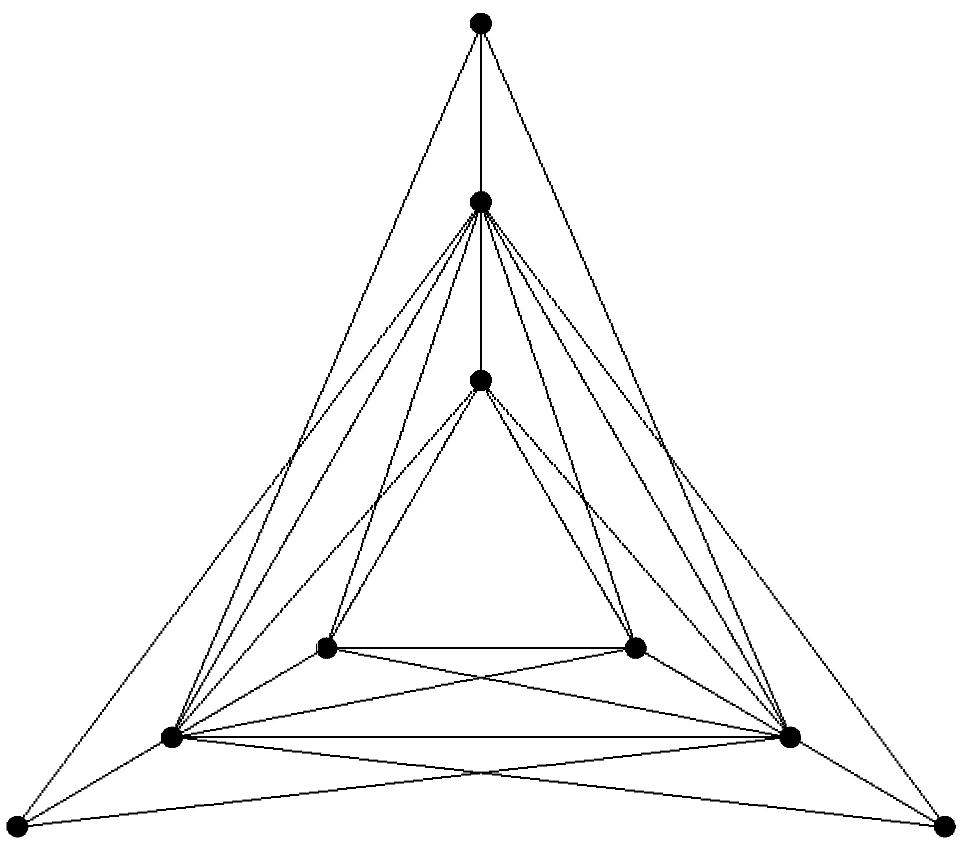
Is it possible for a knight to travel round an 8×8 chessboard in such a way that every possible move (where the direction of the move is not relevant) occurs exactly once?

Give reasons for your answer.

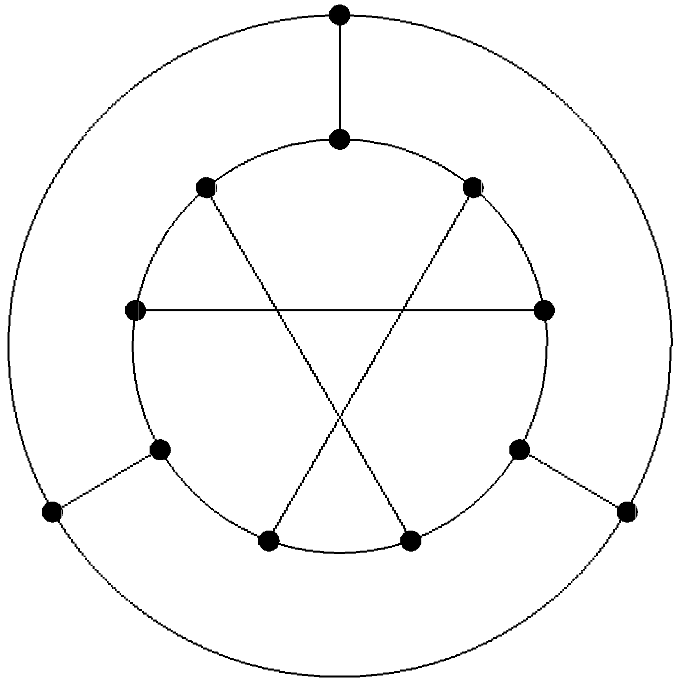
(**b**) Let S be a non-empty set of vertices of a Hamiltonian graph G.

Show that the number $c(G-S)$ of components of $G-S$ is not greater than the number $|S|$ of vertices in S.

Say, giving reasons, whether or not the graph G below is Hamiltonian.



(**c**) Show that the Tietze graph (below) is not Hamiltonian.



3. (**a**) Prove *Euler's Formula* for a connected plane graph G with ν vertices, ε edges and φ faces:

$$\nu - \varepsilon + \varphi = 2\,.$$

(**b**) If G is a plane graph, define the *dual graph* G^* of G.

Deduce that $\nu(G^*) = \varphi(G)$, $\varepsilon(G^*) = \varepsilon(G)$ and $d_{G^*}(f^*) = d_G(f)$ for each face f of G (where f^* is the vertex of G^* corresponding to f).

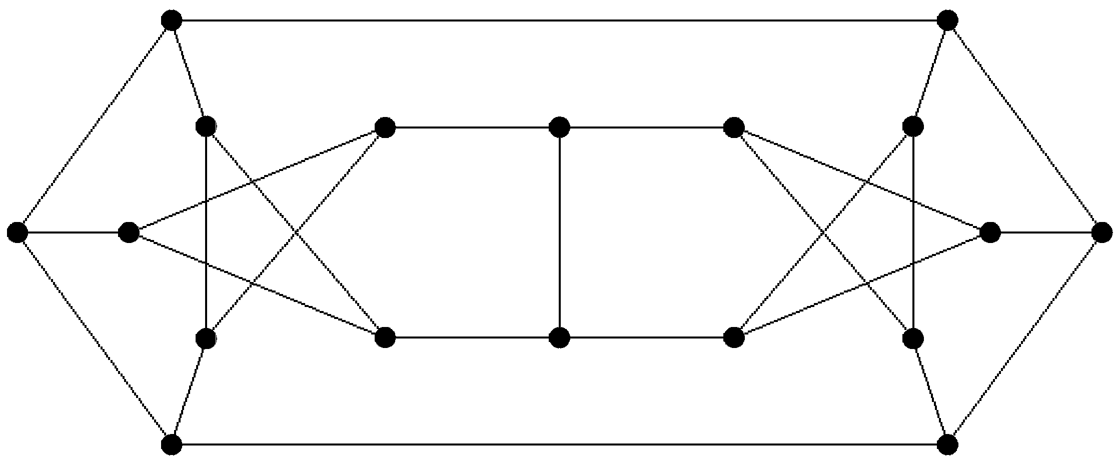
A plane graph is said to be *self-dual* if and only if it is isomorphic to its dual.

(i) Show that if G is self-dual, then $\varepsilon = 2\nu - 2$.

(ii) Find a self-dual plane graph on 6 vertices.

(**c**) *Wagner's Theorem* says that a graph G is planar if, and only if, it contains no subgraph contractible to K_5 or $K_{3,3}$.

Use Wagner's Theorem to show that the *Second Blanusa Snark* (below) is non planar:



4. (**a**) Define: D is a *tournament*.

Show that any tournament contains at most one source and at most one sink.

(**b**) Define: D is *semi-Hamiltonian* for D a digraph.

Prove that every tournament D is semi-Hamiltonian.

(**c**) By considering a vertex of maximum outdegree, or otherwise, prove that:

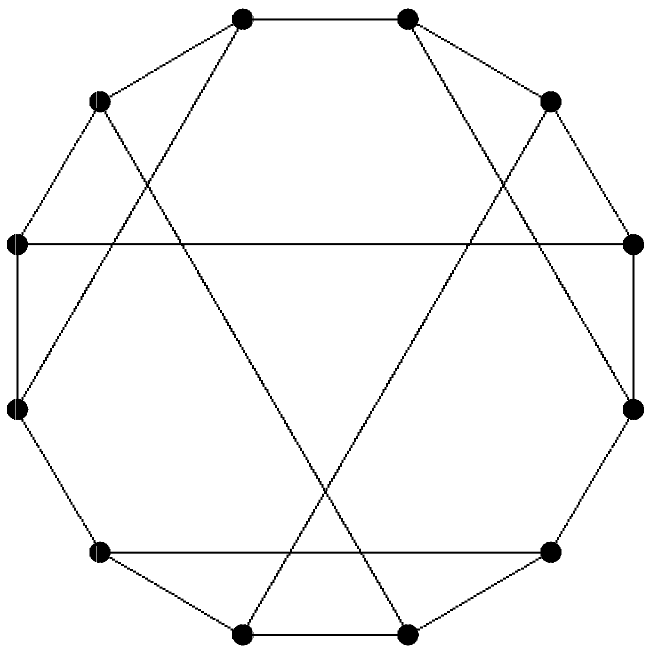
Any tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two.

5. (**a**) Show that any Eulerian map G is 2-face-colourable.

(**b**) Let G be a plane graph in which all the faces are bounded by triangles.

By using part (a), or otherwise, show that if G is Eulerian, then G is 3-colourable.

(**c**) Find an embedding of the Franklin graph (below) on the Klein bottle.



Deduce that the chromatic number $\chi(K)$ of a Klein bottle K is at least 6.

Use Heawood's inequality

$$\chi(S) \leq \frac{1}{2}(7 + \sqrt{49 - 24n})$$

for a surface S of Euler characteristic $n < 2$ to show that $6 \leq \chi(K) \leq 7$ for the Klein bottle K.

END