Math 423
Linear Regression

Homework II

Frédéric Boileau

Prof. David A. Stephens

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a

```
setwd(dirname(rstudioapi::getActiveDocumentContext()$path))

## Error:  RStudio not running

salary<-read.csv("salary.csv",header=TRUE)
x1<-salary$SPENDING/1000
y<-salary$SALARY
```

we want to estimate the parameter β_1 and β_0, namely the slope and the intercept. We use the least square estimators. This is a case of simple linear regression so we can use the following equations:

$$\hat{\beta}_1 = \frac{S_{xx}}{S_{xy}} \tag{1}$$

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{2}$$

$$S_{xy} = \sum_{i=1}^{n} y_i(x_i - \hat{x}) \tag{3}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\hat{x} \tag{4}$$

```
xbar = mean(x1)
ybar = mean(y)
Sxx = sum((x1 - xbar)^2)
Sxy = sum(y*(x1 - xbar))
slope = Sxy/Sxx
intercept = ybar - slope*xbar
print(slope)

## [1] 3307.585

print(intercept)

## [1] 12129.37

fit.RP1 = lm(y~x1)
print(coef(fit.RP1))

## (Intercept)          x1
##   12129.371    3307.585
```

b and c

The residual standard error is given by

$$\hat{\sigma}^2 = \frac{SS_{\text{Res}}}{n-2} \tag{5}$$

MoreoverSS_{Res} is the sum of squares of error:

$$\begin{aligned} SS_{\text{Res}} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 &= \sum_{i=1}^{n} y_i^2 - n\hat{y}^2 - \hat{\beta}_1 S_{xy} \qquad (6) \\ &= \sum_{i=1}^{n}(y_i - \bar{y})^2 - \hat{\beta}_1 S_{xy} \qquad (7) \end{aligned}$$

```
SSRes = sum((y - ybar)^2) - slope*Sxy
n = length(x1)
residualStdError = sqrt(SSRes/(n-2))
print(residualStdError)

## [1] 2324.779
```

d

We wish to compute the standard error with the values in the table already given. This table gives us the degrees of freedom (49) and the t value.

$$t_0 = \frac{\hat{\beta}_0}{\text{se}(\hat{\beta}_0)} \Rightarrow \text{se}(\hat{\beta}_0) = \frac{\hat{\beta}_0}{t_0} = \frac{12129.4}{10.13} \tag{8}$$

Now to do the computation directly from the data we use the actual formula for the standard error which is given by

$$\text{se}(\hat{\beta}_1) = \sqrt{\frac{MS_{\text{Res}}}{S_{xx}}} \qquad MS_{\text{Res}} = \frac{SS_{\text{Res}}}{n-2} = \hat{\sigma}^2 \tag{9}$$

```
MSRes = residualStdError / sqrt(n-2)
print(MSRes)

## [1] 332.1113
```

e

We will derive simple expressions from known relationships

$$
\begin{aligned}
SS_{\text{Res}} &= SS_{\text{T}} - \hat{\beta}_1 S_{xy} \\
SS_{\text{T}} &= SS_{\text{R}} + SS_{\text{Res}}
\end{aligned}
$$

It is easy to see then that

$$
SS_{\text{T}} = SS_{\text{Res}} + \hat{\beta}_1 S_{xy}
$$
$$
SS_{\text{R}} = \hat{\beta}_1 S_{xy}
$$

```
SST = SSRes + slope*Sxy
SSR = slope*Sxy
Rsqrd = SSR/SST
print(Rsqrd)

## [1] 0.6967813
```

f

```
p = 2
Fstat = (SSR/(p-1))/(SSRes/(n-p))
print(Fstat)

## [1] 112.5995
```

g

$$y^\top(I_n H_1)y = y^\top(I_n - H)y + y^\top(H - H_1)y$$

The first statement we want to show is

$$\text{trace}(I_n - H_1) = n - 1$$

Well the matrix I_n has $a_{ii} = 1 \,\forall\, i \in [1, n]$ and $h_{ii} = 1/n \,\forall\, i \in [1, n]$ By definitions:

$$\text{trace}(I_n - H_1) = \sum_{i=1}^{n}(a_{ii} - h_{ii}) \tag{10}$$

$$= \sum_{i=1}^{n}(1 - 1/n) \tag{11}$$

$$= n(1 - 1/n) = n - 1 \tag{12}$$

The second statement we need to prove is that:

$$\text{trace}(H - H_1) = p - 1 \tag{13}$$

We use the properties of the trace operator:

$$\text{trace}(H - H_1) = \text{trace}(H) - \text{trace}(H_1) \tag{14}$$

$$\begin{aligned}
\text{trace}(H) &= \text{trace}(X(X^\top X)^{-1}X^\top) && (15)\\
&= \text{trace}(X^\top X(X^\top X)^{-}1) && (16)\\
&= \text{trace}(I_p) \qquad \text{since} \quad X^\top X \in \mathbb{R}^{p \times p} && (17)\\
&= p && (18)
\end{aligned}$$

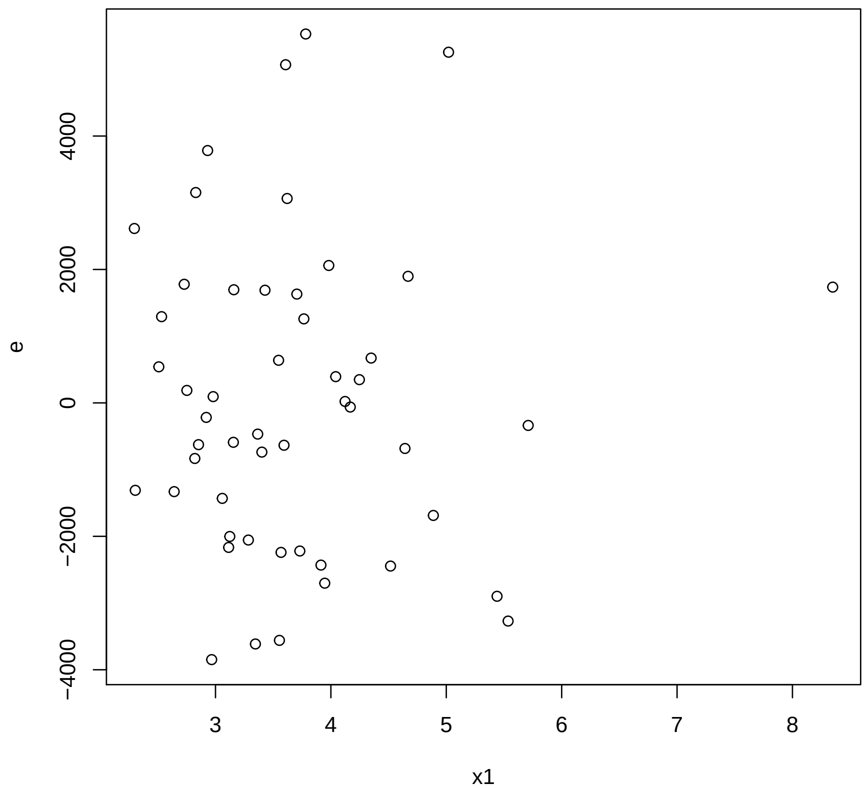
As shown before the trace of H_1 is 1 and this with the previous derivation proves (13).

Numerical part

```
require(MASS)

## Loading required package:  MASS
## Warning:  package 'MASS' was built under R version 3.3.1

bigx = cbind(matrix(1,length(x1)),x1)

n1 =length(x1)
H1 = matrix(1/n1,n1,n1)
sum(diag((diag(n1) - H1)))

## [1] 50

H = bigx %*%  ginv(t(bigx) %*% bigx) %*% t(bigx)
sum(diag(H - H1))

## [1] 1
```

h

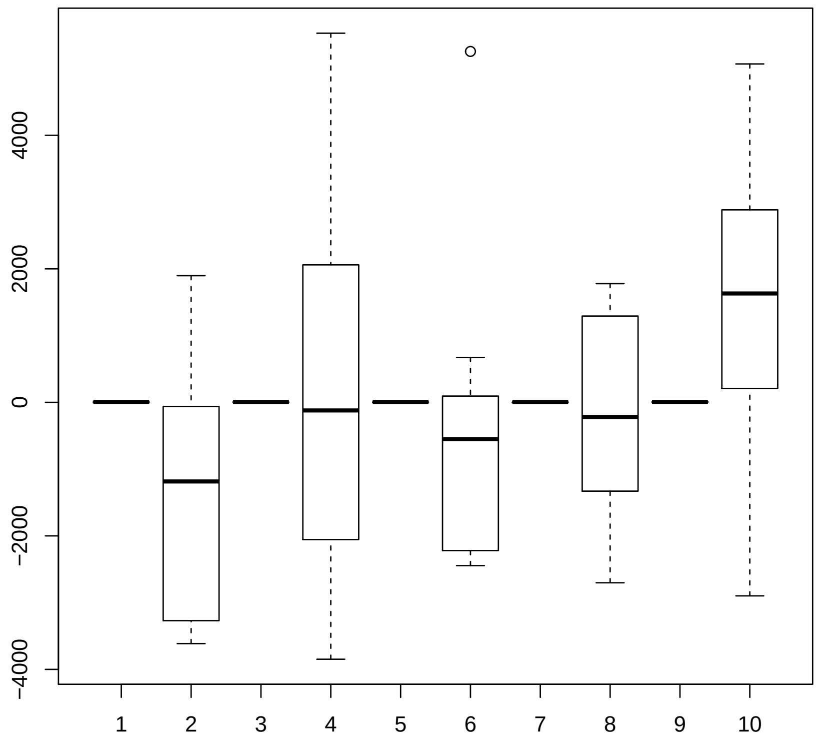
```r
yhat = intercept + slope*x1
e = y - yhat
plot(x1,e)
```




```r
print(mean(x1))

## [1] 3.696608
```

The residuals have zero mean.

```
boxplot(x1[1:10],e[1:10],x1[11:20],e[11:20],x1[21:30],
        e[21:30],x1[31:40],e[31:40],x1[41:41],e[41:51])
```



However a simple box plot shows quite clearly that they do not have constant variance.

```r
print(sum(e))#The sum of the residuals is zero, i.e. they are orthogonal to each other

## [1] -8.731149e-11

bigx = cbind(matrix(1,length(x1)),x1)
print(t(bigx)%*%e)#the residuals are orthogonal to the regressors

##             [,1]
##    -8.731149e-11
## x1 -4.038156e-10

print(t(yhat)%*%e)#the residuals are orthogonal to the fitted values

##               [,1]
## [1,] -2.350658e-06
```

i

```r
prediction = intercept + slope*4.8
print(prediction)

## [1] 28005.78
```

j

$$\hat{Y}^{new} = \hat{\beta}_0 + x_1^{new}\hat{\beta}_1 \tag{19}$$

$$\text{Var}(\hat{Y}^{new}) = \text{Var}(\hat{\beta}_0) + (x_1^{new})^2\text{Var}(\hat{\beta}_1) + 2x_1^{new}\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) \tag{20}$$

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = E[(\hat{\beta}_0 - E(\hat{\beta}_0))(E(\hat{\beta}_1) - \beta_1)] = 0 \tag{21}$$

$$\text{Var}(\hat{Y}^{new}) = \sigma^2 \left(\frac{1}{n} + \frac{1+\bar{x}^2}{S_{xx}}\right) \tag{22}$$