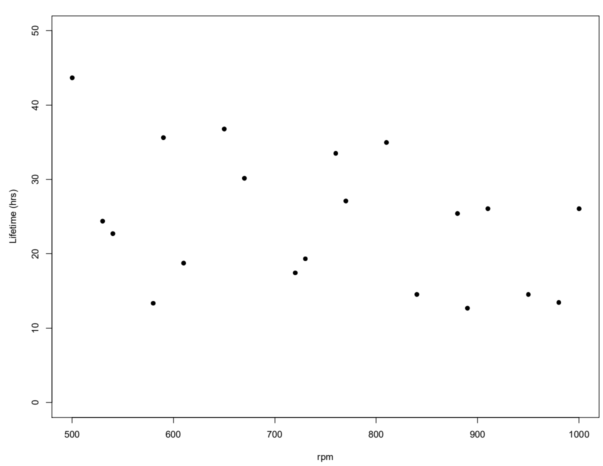
Multiple Regression with Factor Predictors

Example: Tool lifetime data

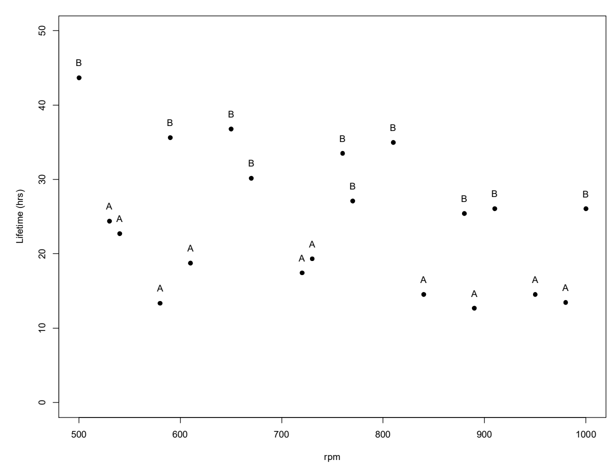
20 observations of machine tools operating lifetimes:

- x_{i1} - operating speed (`rpm`): continuous;
- x_{i2} - tool type (`tool`):
  - factor predictor,
  - $M_2 = 2$ levels (Type: A, B);
- x_{i3} - oil type (`oil`):
  - factor predictor,
  - $M_3 = 4$ levels (Type: 1, 2, 3, 4);
- y_i - lifetime in hours, outcome.

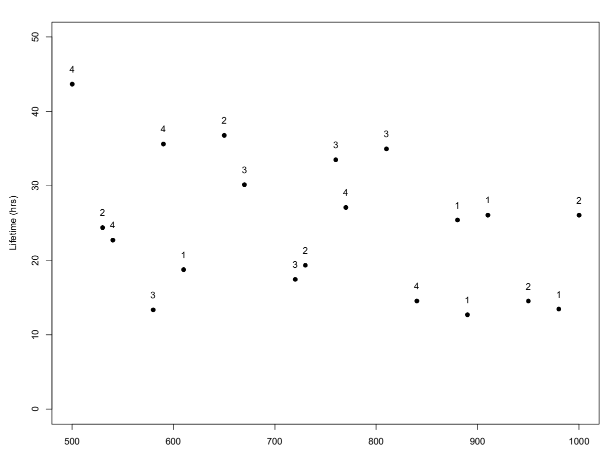
Data



Data



Data



Analysis

```
> Tools<-read.csv('Tools.csv')
> Tools$oil<-as.factor(Tools$oil)
> str(Tools)
'data.frame':   20 obs. of  5 variables:
 $ i   : int  1 2 3 4 5 6 7 8 9 10 ...
 $ y   : num  18.7 14.5 17.4 14.5 13.4 ...
 $ rpm : int  610 950 720 840 980 530 580 540 890 730 ...
 $ tool: Factor w/ 2 levels "A","B": 1 1 1 1 1 1 1 1 1 1 ...
 $ oil : Factor w/ 4 levels "1","2","3","4": 1 2 3 4 1 2 3 4 1 2 ...
> head(Tools)
  i     y rpm tool oil
1 1 18.73 610    A   1
2 2 14.52 950    A   2
3 3 17.43 720    A   3
4 4 14.54 840    A   4
5 5 13.44 980    A   1
6 6 24.39 530    A   2
```

Analysis

Fit Model 1:

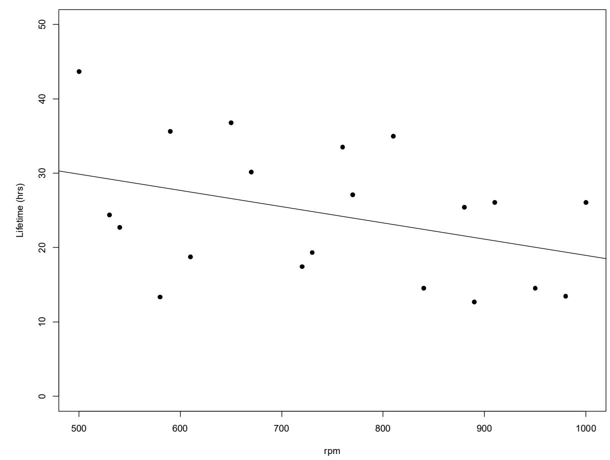
$$Y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i$$

or X_1, or `rpm`.

```
> fit1<-lm(y ~ rpm,data=Tools)
> summary(fit1)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 40.79865    9.54829   4.273 0.000458 ***
rpm         -0.02184    0.01254  -1.741 0.098729 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 8.654 on 18 degrees of freedom
Multiple R-squared:  0.1441,    Adjusted R-squared:  0.09659
F-statistic: 3.031 on 1 and 18 DF,  p-value: 0.09873
```


Model 1 fit



Model 2

Fit Model 2:

$$X_1 + X_2$$

that is,

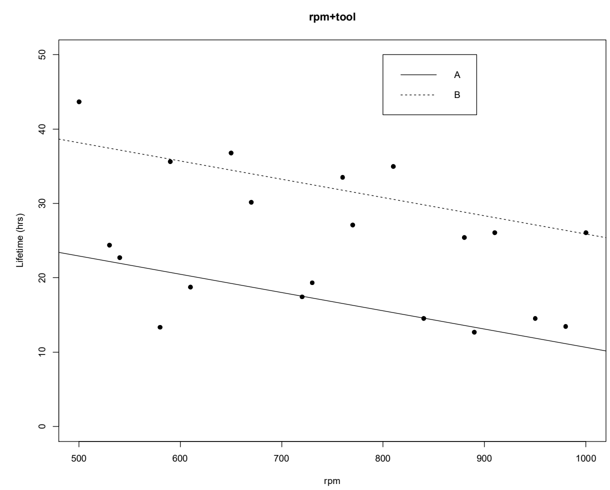
$$Y_i = \beta_0 + \beta_1 x_{i1} + \sum_{j=1}^{M_2-1} \beta_{2j}^{C} \mathbb{1}_j(x_{i2}) + \epsilon_i$$

or `rpm+tool`.

```
> fit2<-lm(y ~ rpm+tool,data=Tools)
> summary(fit2)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 35.208726   3.738882   9.417 3.71e-08 ***
rpm         -0.024557   0.004865  -5.048 9.92e-05 ***
toolB       15.235474   1.501220  10.149 1.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.352 on 17 degrees of freedom
Multiple R-squared:  0.8787,    Adjusted R-squared:  0.8645
F-statistic:  61.6 on 2 and 17 DF,  p-value: 1.627e-08
```


Model 2 fit



Model 2

In this case, the factor predictors has $M_2 = 2$ levels, to there is only one non-baseline group. In `R`, the default action sets the baseline group by considering the factor level names alphabetically; here level `A` is the baseline group.

$$\mathbb{E}_{Y_i|X_i}[Y_i|\mathbf{x}_i] = \begin{cases} \beta_0 + \beta_1 x_{i1} & x_{i2} = 0 \quad \text{(Type A)} \\ \beta_0 + \beta_1 x_{i1} + \beta_{21}^{C} & x_{i2} = 1 \quad \text{(Type B)} \end{cases}$$

The parameter β_{21}^{C} measures the difference in the intercept between the Type A and Type B tools.

The estimate is $\widehat{\beta}_{21}^{C} = 15.235$ (line 36); the associated t-test of the null hypothesis

$$\mathrm{H}_0 \ : \ \beta_{21}^{C} = 0$$

reveals that the hypothesis is rejected (line 35, p-value 1.25e-08).

Comparing Model 2 to Model 1

```
> drop1(fit2,test='F')
Single term deletions

Model:
y ~ rpm + tool
       Df Sum of Sq     RSS    AIC F value    Pr(>F)
<none>               190.98 51.129
rpm     1    286.24  477.22 67.445   25.48 9.917e-05 ***
tool    1   1157.08 1348.06 88.214  103.00 1.246e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

In terms of single term deletions, we see that if either term is omitted from the model `rpm+tool` ($X_1 + X_2$), then the test statistic is highly significant.

- Line 50: compares the model `rpm+tool` ($X_1 + X_2$) with the model `tool` (X_2), and tests the hypothesis $\mathrm{H}_0 \ : \ \beta_1 = 0$; this hypothesis is rejected ($p = 9.917 \times 10^{-5}$).
- Line 51: compares the model `rpm+tool` ($X_1 + X_2$) with the model `rpm` (X_1), and tests the hypothesis $\mathrm{H}_0 \ : \ \beta_{21}^{C} = 0$; this hypothesis is rejected ($p = 1.246 \times 10^{-8}$).

Comparing Model 2 to Model 1

```
> anova(lm(y~rpm+tool,data=Tools))
Analysis of Variance Table

Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
rpm        1  227.03  227.03  20.209 0.0003188 ***
tool       1 1157.08 1157.08 102.997 1.246e-08 ***
Residuals 17  190.98   11.23
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> anova(lm(y~tool+rpm,data=Tools))
Analysis of Variance Table

Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
tool       1 1097.87 1097.87  97.726 1.832e-08 ***
rpm        1  286.24  286.24  25.480 9.917e-05 ***
Residuals 17  190.98   11.23
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Comparing Model 2 to Model 1 (cont.)

Partial F-tests reveal the same conclusions:

- Line 54: adds `rpm` (X_1) first, then `tool` (X_2)
- Line 65: adds `tool` (X_2) first, then `rpm` (X_1)

Note that the sequence of adding terms makes a difference to the sums of squares terms and the significance test results; lines 59 and 60 give the decomposition

$$\overline{\text{SS}}_{\text{R}}(\beta_1, \beta_{21}^{\text{C}}|\beta_0) = \overline{\text{SS}}_{\text{R}}(\beta_1|\beta_0) + \overline{\text{SS}}_{\text{R}}(\beta_{21}^{\text{C}}|\beta_0, \beta_1)$$

whereas lines 70 and 71 give the decomposition

$$\overline{\text{SS}}_{\text{R}}(\beta_1, \beta_{21}^{\text{C}}|\beta_0) = \overline{\text{SS}}_{\text{R}}(\beta_{21}^{\text{C}}|\beta_0) + \overline{\text{SS}}_{\text{R}}(\beta_1|\beta_0, \beta_{21}^{\text{C}})$$

We conclude that both predictors are helpful in predicting the response.

Model 3

Fit Model 3:

$$X_1 + X_2 + X_1 : X_2$$

that is,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \sum_{j=1}^{M_2-1} \beta_{2j}^{\text{C}} \mathbb{1}_j(x_{i2}) + \sum_{j=1}^{M_2-1} \beta_{12j}^{\text{C}} x_{i1} \mathbb{1}_j(x_{i2}) + \epsilon_i$$

or

`rpm+tool+rpm:tool`.

In `R`, this model can also be specified as

`rpm*tool`

Model 3 (cont.)

```
> fit3<-lm(y~rpm+tool+rpm:tool,data=Tools)
> summary(fit3)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 30.176013   4.724895   6.387 9.01e-06 ***
rpm         -0.017729   0.006262  -2.831  0.01204 *
toolB       26.569340   7.115681   3.734  0.00181 **
rpm:toolB   -0.015186   0.009338  -1.626  0.12345
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.201 on 16 degrees of freedom
Multiple R-squared:  0.8959,    Adjusted R-squared:  0.8764
F-statistic: 45.92 on 3 and 16 DF,  p-value: 4.37e-08
```


Model 3 (cont.)

The model fitted here is

$$\mathbb{E}_{Y_i|X_i}[Y_i|\mathbf{x}_i] = \begin{cases} \beta_0 + \beta_1 x_{i1} & x_{i2} = 0 \quad \text{(Type A)} \\ \beta_0 + \beta_1 x_{i1} + \beta_{21}^{\text{C}} + \beta_{121}^{\text{C}} x_{i1} & x_{i2} = 1 \quad \text{(Type B)} \end{cases}$$

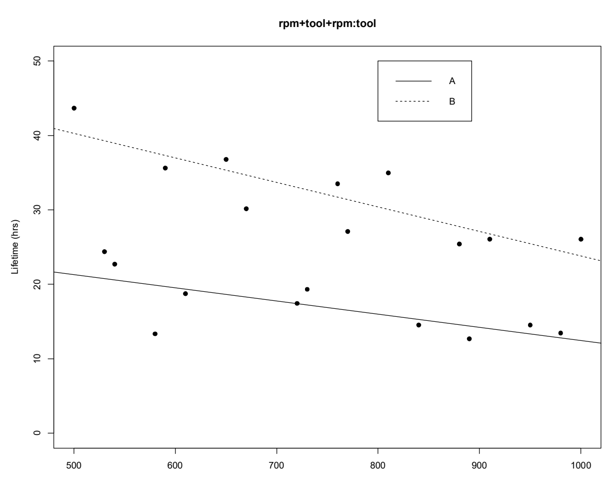
- The parameter β_{21}^{C} measures the difference in the **intercept** between the Type A and Type B tools.
- The parameter β_{121}^{C} measures the difference in the **slope** between the Type A and Type B tools.

Lines 79 – 82 give inference and testing details for

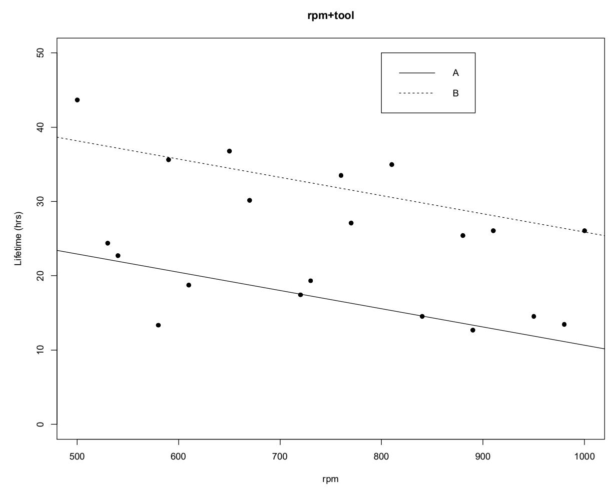
$$\beta_0, \beta_1, \beta_{21}^{\text{C}}, \beta_{121}^{\text{C}}$$

respectively.

Model 3 fit



Recall Model 2 fit



Comparing Model 3 to Model 2

```
> drop1(fit3,test='F')
Single term deletions

Model:
y ~ rpm + tool + rpm:tool
         Df Sum of Sq    RSS    AIC F value Pr(>F)
<none>                163.89 50.070
rpm:tool  1    27.087 190.98 51.129  2.6443 0.1235
```

The only single term deletion that is considered is the interaction term `rpm:tool`; the null hypothesis is

$$H_0 \;:\; \beta^{C}_{121} = 0$$

that is, whether there is a change in slope between the two groups.

Comparing Model 3 to Model 2 (cont.)

The test being carried out is a standard F-test using the test statistic

$$F = \frac{(SS_{Res}(\text{Model 2}) - SS_{Res}(\text{Model 3}))/r}{SS_{Res}(\text{Model 3})/(n-p)}$$

where here

- $r = 1$ (the number of parameters set to zero by the null hypothesis)
- $n - p = n - 4$, as there are four parameters in Model 3.

Line 96 reveals that this null hypothesis is not rejected ($p = 0.1235$).

Comparing Model 3 to Model 2 (cont.)

```
> anova(fit3)
Analysis of Variance Table

Response: y
          Df  Sum Sq Mean Sq  F value    Pr(>F)
rpm        1  227.03  227.03  22.1640  0.000237 ***
tool       1 1157.08 1157.08 112.9591 1.169e-08 ***
rpm:tool   1   27.09   27.09   2.6443  0.123451
Residuals 16  163.89   10.24
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> anova(fit2,fit3)
Analysis of Variance Table

Model 1: y ~ rpm + tool
Model 2: y ~ rpm + tool + rpm:tool
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     17 190.98
2     16 163.89  1    27.087 2.6443 0.1235
```


Model 4

Fit Model 4:

$$X_1 + X_2 + X_3$$

that is,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \sum_{j=1}^{M_2-1} \beta^{C}_{2j} \mathbb{1}_j(x_{i2}) + \sum_{l=1}^{M_3-1} \beta^{C}_{3l} \mathbb{1}_l(x_{i3}) + \epsilon_i$$

or

`rpm+tool+oil`.

Model 4 (cont.)

```
> fit4<-lm(y~rpm+tool+oil,data=Tools)
> summary(fit4)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 33.521061   5.071841   6.609 1.17e-05 ***
rpm         -0.023894   0.005744  -4.160 0.000962 ***
toolB       15.371825   1.571087   9.784 1.22e-07 ***
oil2         2.988662   2.197253   1.360 0.195273
oil3         0.047057   2.344421   0.020 0.984269
oil4         1.465395   2.495856   0.587 0.566465
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.393 on 14 degrees of freedom
Multiple R-squared:  0.8976,    Adjusted R-squared:  0.8611
F-statistic: 24.56 on 5 and 14 DF,  p-value: 1.82e-06
```

Model 4 (cont.)

The model fitted for $\mathbb{E}_{Y_i|X_i}[Y_i|\mathbf{x}_i]$ is

$\beta_0 + \beta_1 x_{i1}$	$x_{i2} = 0, x_{i3} = 0$	(Type A, Oil 1)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{31}$	$x_{i2} = 0, x_{i3} = 1$	(Type A, Oil 2)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{32}$	$x_{i2} = 0, x_{i3} = 2$	(Type A, Oil 3)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{33}$	$x_{i2} = 0, x_{i3} = 3$	(Type A, Oil 4)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{21}$	$x_{i2} = 1, x_{i3} = 0$	(Type B, Oil 1)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{21} + \beta^C_{31}$	$x_{i2} = 1, x_{i3} = 1$	(Type B, Oil 2)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{21} + \beta^C_{32}$	$x_{i2} = 1, x_{i3} = 2$	(Type B, Oil 3)
$\beta_0 + \beta_1 x_{i1} + \beta^C_{21} + \beta^C_{33}$	$x_{i2} = 1, x_{i3} = 3$	(Type B, Oil 4)

There are eight subgroups of data defined by the $M_2 \times M_3 = 2 \times 4$ combinations of factor levels. There are six parameters in total, including the intercept β_0.

Model 4 (cont.)

The dependence on continuous predictor X_1 is the same in all subgroups, that is, the slope is the same.

- The parameter β^C_{21} measures the difference in the **intercept** between the Type A and Type B tools, for every Oil type.
- The parameters $\beta^C_{31}, \beta^C_{33}, \beta^C_{33}$ measure the difference in the **intercepts** between Oil Types 2, 3 and 4 and Oil Type 1.

Lines 121 – 126 give inference and testing details for

$$\beta_0, \beta_1, \beta^C_{21}, \beta^C_{31}, \beta^C_{32}, \beta^C_{33}$$

respectively.

Comparing Model 4 with Model 2

```
> anova(fit2,fit4)
Analysis of Variance Table

Model 1: y ~ rpm + tool
Model 2: y ~ rpm + tool + oil
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     17 190.98
2     14 161.21  3    29.766 0.8616 0.4838
```

The comparison between models

`rpm+tool+oil`

and

`rpm+tool`

is a test of the null hypothesis

$$\mathrm{H}_0 \ : \ \beta^C_{31} = \beta^C_{32} = \beta^C_{33} = 0$$

Comparing Model 4 with Model 2 (cont.)

The test uses the F-statistic

$$F = \frac{(\mathrm{SS}_{\mathrm{Res}}(\text{Model 2}) - \mathrm{SS}_{\mathrm{Res}}(\text{Model 4}))/r}{\mathrm{SS}_{\mathrm{Res}}(\text{Model 4})/(n-p)}$$

- $r = 3$ (the number of parameters set to zero by the null hypothesis)
- $n - p = n - 6$, as there are six parameters in Model 4.

The result on line 140 indicates that the null hypothesis is not rejected, so Model 2 is an adequate simplification of Model 4 ($p = 0.4838$).

Model 5

Fit Model 5:

$$X_2 + X_3$$

that is,

$$Y_i = \beta_0 + \sum_{j=1}^{M_2-1} \beta^C_{2j} \mathbb{1}_j(x_{i2}) + \sum_{l=1}^{M_3-1} \beta^C_{3l} \mathbb{1}_l(x_{i3}) + \epsilon_i$$

or

`tool+oil.`

Model 5 (cont.)

```
> fit5<-lm(y~tool+oil,data=Tools)
> summary(fit5)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   13.553      2.368   5.723 4.03e-05 ***
toolB         14.277      2.238   6.380 1.24e-05 ***
oil2           4.948      3.101   1.596   0.1314
oil3           3.755      3.133   1.199   0.2493
oil4           6.607      3.133   2.109   0.0522 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.902 on 15 degrees of freedom
Multiple R-squared:  0.7711,    Adjusted R-squared:  0.7101
F-statistic: 12.63 on 4 and 15 DF,  p-value: 0.0001068
```

Model 5 (cont.)

The model fitted for $\mathbb{E}_{Y_i|X_i}[Y_i|\mathbf{x}_i]$ is

$$\begin{array}{lll}
\beta_0 & x_{i2}=0, x_{i3}=0 & \text{(Type A, Oil 1)} \\
\beta_0+\beta_{31}^{C} & x_{i2}=0, x_{i3}=1 & \text{(Type A, Oil 2)} \\
\beta_0+\beta_{32}^{C} & x_{i2}=0, x_{i3}=2 & \text{(Type A, Oil 3)} \\
\beta_0+\beta_{33}^{C} & x_{i2}=0, x_{i3}=3 & \text{(Type A, Oil 4)} \\
\beta_0+\beta_{21}^{C} & x_{i2}=1, x_{i3}=0 & \text{(Type B, Oil 1)} \\
\beta_0+\beta_{21}^{C}+\beta_{31}^{C} & x_{i2}=1, x_{i3}=1 & \text{(Type B, Oil 2)} \\
\beta_0+\beta_{21}^{C}+\beta_{32}^{C} & x_{i2}=1, x_{i3}=2 & \text{(Type B, Oil 3)} \\
\beta_0+\beta_{21}^{C}+\beta_{33}^{C} & x_{i2}=1, x_{i3}=3 & \text{(Type B, Oil 4)}
\end{array}$$

There are eight subgroups of data defined by the $M_2 \times M_3 = 2 \times 4$ combinations of factor levels. There are five parameters in total, including the intercept β_0.

Model 5 (cont.)

- The parameter β_{21}^{C} measures the difference in the **intercept** between the Type A and Type B tools, for every Oil type.
- The parameters $\beta_{31}^{C}, \beta_{33}^{C}, \beta_{33}^{C}$ measure the difference in the **intercepts** between Oil Types 2, 3 and 4 and Oil Type 1.

Lines 145 – 149 give inference and testing details for

$$\beta_0, \beta_{21}^{C}, \beta_{31}^{C}, \beta_{32}^{C}, \beta_{33}^{C}$$

respectively.

Model 5 (cont.)

```
> anova(lm(y~tool+oil,data=Tools))
Analysis of Variance Table

Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
tool       1 1097.87 1097.87 45.6789 6.429e-06 ***
oil        3  116.71   38.90  1.6186     0.227
Residuals 15  360.52   24.03
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> anova(lm(y~oil+tool,data=Tools))
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
oil        3 236.22   78.74  3.2761   0.05047 .
tool       1 978.35  978.35 40.7063 1.236e-05 ***
Residuals 15 360.52   24.03
```


Model 5 (cont.)

The order of fitting again makes a difference to the sums of squares decomposition. This is because the design is *unbalanced*: there are different numbers of observations in the eight factor level combinations.

```
> table(Tools$tool,Tools$oil)
    1 2 3 4
  A 3 3 2 2
  B 2 2 3 3
```


Model 5 (cont.)

```
> drop1(lm(y~tool+oil,data=Tools),test='F')
Single term deletions

Model:
y ~ tool + oil
       Df Sum of Sq     RSS    AIC F value    Pr(>F)
<none>               360.52 67.836
tool    1    978.35 1338.87 92.077 40.7063 1.236e-05 ***
oil     3    116.71  477.22 67.445  1.6186     0.227
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Note that dropping the single term `oil` removes three parameters: all contrasts corresponding to that variable are omitted.

Balanced Designs

Example: Two factor design

- Factor A: 3 levels, labelled $j = 1, 2, 3$;
- Factor B: 5 levels, labelled $l = 1, 2, 3, 4, 5$;
- $n_{jl} = 4$ replicate observations for each factor level combination;
- total sample size $n = 3 \times 5 \times 4 = 60$.

```
> table(A,B)
   B
A   1 2 3 4 5
  1 4 4 4 4 4
  2 4 4 4 4 4
  3 4 4 4 4 4
```

We fit the *full factorial* model

`A*B = A + B + A:B`

Example: Two factor design (cont.)

In this model, there is a different assumed mean response for each of the 3×5 factor level combinations; if x_{iA} and x_{iB} represent the indicators of levels for factors A and B, the modelled mean is

$$\beta_0 + \sum_{j=1}^{2} \beta^{C}_{Aj} \mathbb{1}_j(x_{iA}) + \sum_{l=1}^{4} \beta^{C}_{Bj} \mathbb{1}_j(x_{iB}) + \sum_{j=1}^{2} \sum_{l=1}^{4} \beta^{C}_{ABjl} \mathbb{1}_j(x_{iA}) \mathbb{1}_l(x_{iB})$$

Example: Two factor design (cont.)

```
> fit.bal1<-lm(Y~A*B); summary(fit.bal1)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.2006     0.9809   2.243   0.0298 *
A2            1.7808     1.3872   1.284   0.2058
A3           -2.7656     1.3872  -1.994   0.0523 .
B2           -2.2448     1.3872  -1.618   0.1126
B3            1.9327     1.3872   1.393   0.1704
B4            0.5620     1.3872   0.405   0.6873
B5           -0.4287     1.3872  -0.309   0.7587
A2:B2         0.4066     1.9618   0.207   0.8368
A3:B2         1.6549     1.9618   0.844   0.4034
A2:B3        -1.1205     1.9618  -0.571   0.5707
A3:B3         0.9232     1.9618   0.471   0.6402
A2:B4        -0.8099     1.9618  -0.413   0.6817
A3:B4         2.0317     1.9618   1.036   0.3059
A2:B5        -1.1250     1.9618  -0.573   0.5692
A3:B5         0.9885     1.9618   0.504   0.6168
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 1.962 on 45 degrees of freedom
Multiple R-squared:  0.5094,    Adjusted R-squared:  0.3568
F-statistic: 3.338 on 14 and 45 DF,  p-value: 0.001058
```


Example: Two factor design (cont.)

```
> anova(lm(Y~A*B))
Analysis of Variance Table

Response: Y
          Df  Sum Sq Mean Sq F value    Pr(>F)
A          2  84.443  42.221 10.9703 0.0001316 ***
B          4  83.362  20.840  5.4149 0.0012023 **
A:B        8  12.054   1.507  0.3915 0.9194592
Residuals 45 173.191   3.849
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> anova(lm(Y~B*A))
Analysis of Variance Table

Response: Y
          Df  Sum Sq Mean Sq F value    Pr(>F)
B          4  83.362  20.840  5.4149 0.0012023 **
A          2  84.443  42.221 10.9703 0.0001316 ***
B:A        8  12.054   1.507  0.3915 0.9194592
Residuals 45 173.191   3.849
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Example: Two factor design (cont.)

In the balanced case, the order in which the factors are included in the model does not change the sum of squares decomposition, or the assessment of statistical significance.

For example in assessing the significance of the interaction term `A:B`, we obtain the same test result from the two `anova` calculations as from `drop1`: see lines 37, 49 and 60.

```
> drop1(fit.bal1,test='F')
Single term deletions

Model:
Y ~ A * B
       Df Sum of Sq    RSS    AIC F value Pr(>F)
<none>              173.19 93.603
A:B     8    12.054 185.25 81.640  0.3915 0.9195
```