Exercises - Graph Theory
SOLUTIONS

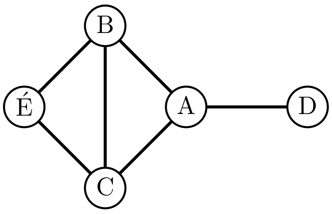
Question 1
Model the following situations as (possibly weighted, possibly directed) graphs. Draw each graph, and give the corresponding adjacency matrices.

(a) Ada and Bertrand are friends. Ada is also friends with Cecilia and David. Bertrand, Cecilia and Évariste are all friends of each other.

(b) Wikipedia has five particularly interesting articles: Animal, Burrow, Chile, Desert, and Elephant. Some of them even link to each other!

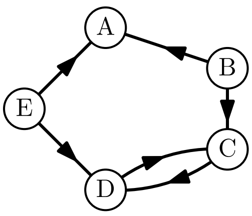
(c) It is well-known that in the Netherlands, there is a 2-lane highway from Amsterdam to Breda, another 2-lane highway from Amsterdam to Cappele aan den IJssel, a 3-lane highway from Breda to Dordrecht, a 1-lane road from Breda to Ede and another one from Dordrecht to Ede, and a 5-lane superhighway from Cappele aan den IJssel to Ede.

Solution

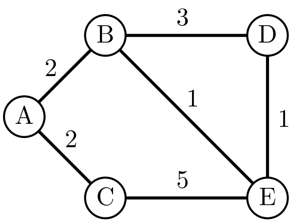
(a) 

	A	B	C	D	É
A	0	1	1	1	0
B	1	0	1	0	1
C	1	1	0	0	1
D	1	0	0	0	0
É	0	1	1	0	0

(b) Checking Wikipedia, we find that the Animal article links to none of the others, the Burrow article links to Animal and Chile, the Chile article links to Desert, the Desert article links to Chile, and the Elephant article links to Animal and Desert. Strangely, no relation between Burrow and Elephant seems to be present. We encode this information with a directed graph.

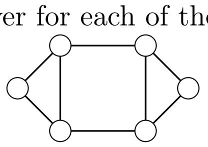


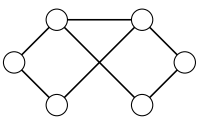
	A	B	C	D	E
A	0	0	0	0	0
B	1	0	1	0	0
C	0	0	0	1	0
D	0	0	1	0	0
E	1	0	0	1	0

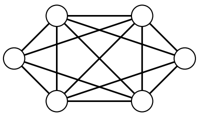
(c) 

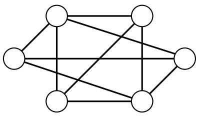
	A	B	C	D	E
A	0	2	2	0	0
B	2	0	0	3	1
C	2	0	0	0	5
D	0	3	0	0	1
E	0	1	5	1	0

Question 2
Answer for each of these graphs: Is it planar? Is it bipartite?

(a) 

(b) 

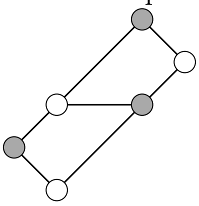
(c) 

(d) 

Solution

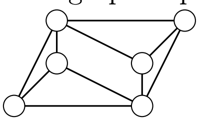
(a) This graph is planar, since there are no edge crossings in its drawing. It is not bipartite, since it has a cycle of odd length.

(b) This graph is also planar: we can "flip" part of the drawing to obtain a planar drawing. It is also bipartite, since we can colour all vertices with two colours.



(c) This graph is not planar: it has 6 vertices and 14 edges, and by Euler's formula a planar graph with 6 vertices can have at most $3V - 6 = 12$ edges. It is also not bipartite, since it contains triangles.

(d) This graph is planer, as can be seen in this drawing of the same graph:



It is not bipartite, since it contains triangles.

Question 3

Draw an example graph for each of these.

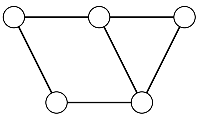
(a) A planar graph has 5 vertices and 3 faces. How many edges does it have?

(b) A planar graph has 7 edges and 5 faces. How many vertices does it have?

Solution

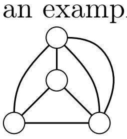
We use Euler's formula: $V + F = E + 2$.

(a) There are $E = V + F - 2 = 6$ edges. Here's an example:



(Note that the outer face is also counted!)

(b) There should be $V = E - F + 2 = 4$ vertices. However, this is not possible without creating duplicate edges. With duplicate edges, it is possible, and the formula gives the correct answer if we count the space between two duplicate edges also as a face. Here's an example:



Note, however, that this is *not* a graph, but a multigraph.

Question 4

What is the maximum number of edges in a bipartite planar graph with n vertices?

Solution

Many different proofs are possible. Here is a proof based on counting and Euler's formula. It is also possible to prove the statement using induction.

In a bipartite graph, every cycle has even length, so in particular, every face has an even number of edges. A face cannot have only 2 edges, because then there would be a double edge. A face with 6 edges can always be subdivided into two faces with 4 edges each (and the same is true for faces with even more edges), so in any planar bipartite graph with a maximum number of edges, every face has length 4. Since every edge is used in two faces, we have $4F = 2E$. Plugging this into Euler's formula, we find that $V + \frac{1}{2}E = E + 2$, which we rewrite as $E = 2V - 4$. Since $n = V$, the answer is $2n - 4$ edges.