

**MATH 350: Graph Theory and Combinatorics. Fall 2016.**  
**Assignment #2: Bipartite graphs, Matchings, Connectivity**

Due Wednesday, October 19th, 2016, 14:30

1. Recall that a graph  $G$  is called  $d$ -regular if every vertex of  $G$  has degree equal to  $d$ .

- a) Construct a 3-regular graph that does not contain a perfect matching. You have to prove that the constructed graph does not contain a perfect matching.  
*Hey, I bet your graph doesn't contain a 2-factor either! A coincidence?*

**Solution:**

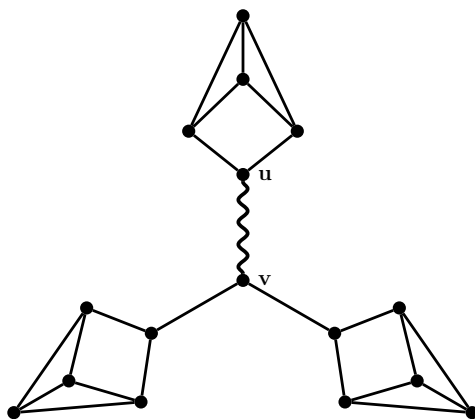


Figure 1: A 3-regular graph with no perfect matching.

Let  $G$  be the 3-regular graph depicted in Figure 1. Suppose  $G$  has a perfect matching  $M$ , and let  $e$  be the edge of  $M$  incident to the vertex  $v$ . By symmetry, we may assume  $e = \{u, v\}$ . Let  $C$  be any of the two connected components of  $G - v$  that does not contain  $u$ . Every vertex from  $C$  is incident to one edge in  $M$  and the edges of  $M$  must lie completely inside  $C$ . But that is impossible, because  $|C|$  is odd.

- b) Prove the following statement: Let  $G$  be a 3-regular graph.  $G$  contains a perfect matching  $\iff G$  contains a 2-factor.  
*... Ah, so no coincidences on this sheet!*

**Solution:** First, suppose  $G$  has a perfect matching  $M$ . Then the spanning subgraph of  $G$  that contains the edges  $E(G) \setminus M$  is a 2-factor in  $G$ .

On the other hand, if  $H$  is a 2-factor in  $G$ , then every vertex  $v \in V(G)$  is incident to exactly one edge from the set  $M := E(G) \setminus E(H)$ . Therefore,  $M$  is a perfect matching in  $G$ .

**2.** Prove that every graph  $G = (V, E)$  contains a subgraph  $H$  that is bipartite and has at least  $|E|/2$  edges.

**Solution:** Let  $H$  be a bipartite subgraph of  $G$  that has maximum number of edges, and let  $A \subseteq V$  and  $B \subseteq V$  be the parts of the bipartition. Clearly,  $H$  contains every edge of  $G$  that has one endpoint in  $A$  (and hence also one endpoint in  $B$ ). It is enough to show that  $\deg_H(v) \geq \frac{1}{2} \deg_G(v)$  for every  $v \in V$ .

Suppose for contradiction there exists a vertex  $v \in V$  such that  $\deg_H(v) < \frac{1}{2} \deg_G(v)$ . Without loss of generality,  $v \in A$ . But then the vertex  $v$  has in  $G$  more neighbors inside  $A$  than inside  $B$ . Therefore, the bipartite graph  $H'$  between the parts  $A' := A \setminus \{v\}$  and  $B' := B \cup \{v\}$  contains more edges than  $H$ , a contradiction.

**3.** Let  $G$  be a connected graph. We say that  $F \subseteq E(G)$  is *even-degree*, if every vertex of  $G$  is incident with an even number of edges in  $F$ . Let  $T$  be an arbitrary spanning tree of  $G$ . Prove that there exists an even-degree set  $F_T \subseteq E(G)$  such that  $F_T \cup E(T) = E(G)$ .

**Solution:** We claim that if  $F_1$  and  $F_2$  are both even-degree then so is  $F_1 \Delta F_2 := (F_1 - F_2) \cup (F_2 - F_1)$ . Indeed, if  $E_1$  and  $E_2$  are the sets of edges in  $F_1$  and  $F_2$ , respectively, incident to the vertex  $v$ , then  $|E_1 \Delta E_2| = |E_1| + |E_2| - 2|E_1 \cap E_2|$ , which is even if  $|E_1|$  and  $|E_2|$  are even.

For  $e \in E(G) \setminus E(T)$ , let  $F(e)$  be the edge set of the cycle formed by  $e$  and the unique path between the endpoints of  $e$  in  $T$ . Clearly,  $F(e)$  is even-degree. Let

$$F_T := F(e_1) \Delta F(e_2) \dots \Delta F(e_k),$$

where  $E(G) \setminus E(T) = \{e_1, e_2, \dots, e_k\}$ . Then  $F$  is an even-degree set, by the claim above, and  $F \cup E(T) = E(G)$ , as  $e_i \in F(e_i)$  and  $e_i \notin F(e_j)$  for  $i, j \in \{1, 2, \dots, k\}$ ,  $i \neq j$ .

**4.** Let  $G$  be a 3-regular graph. Show that the edge connectivity  $\kappa_e(G)$  is equal to the vertex connectivity  $\kappa_v(G)$ .

**Solution:** First observe that for an arbitrary graph  $G$ , it holds that  $\kappa_v(G) \leq \kappa_e(G)$ . Indeed, any internally  $k$  vertex-disjoint paths between any two vertices form also  $k$  edge-disjoint paths between the two vertices. So it is enough to show  $\kappa_v(G) \geq \kappa_e(G)$ . Also, since the graph  $G$  is 3-regular,  $\kappa_e(G) \leq 3$ . Therefore, if  $\kappa_v(G) = 3$ , then there is nothing to prove. On the other hand, if  $\kappa_v(G) = 0$ , then  $G$  is disconnected and so  $\kappa_e(G) = 0$  as well. It remains to analyze  $\kappa_v(G) \in \{1, 2\}$ .

If  $\kappa_v(G) = 1$ , then by Menger's theorem there exists a cut-vertex  $v \in V(G)$  such that  $G - v$  is disconnected. Let  $C_1$  and  $C_2$  be any two different connected components of  $G - v$ . Since the degree of  $v$  is only three, there must be  $i \in \{1, 2\}$  such that  $v$  is adjacent to only one vertex  $w \in V(C_i)$ . But then  $\{v, w\}$  is a cut-edge in  $G$  and hence  $\kappa_e(G) \leq 1$ .

Finally, consider  $\kappa_v(G) = 2$ . Again, by Menger's theorem there exists two vertices  $u$  and  $v$  so that  $G - u - v$  is disconnected. And again, let  $C_1$  and  $C_2$  be any two different connected components of  $G - u - v$ . Clearly, both  $u$  and  $v$  have at least one neighbor in  $C_1$  and at least one neighbor in  $C_2$  (as otherwise  $\kappa_v(G) = 1$ ). We now consider the following two cases:

- If  $\{u, v\} \in E(G)$ , then since  $\deg_G(u) = 3$ , it must be that  $u$  has exactly one neighbor in  $C_1$ , call it  $u_1$ , and exactly one neighbor in  $C_2$ , call it  $u_2$ . Analogously,  $v$  has exactly one neighbor in  $C_1$ , say  $v_1$ , and exactly one neighbor in  $C_2$ , call it  $v_2$ . But then both  $\{\{u, u_1\}, \{u, u_2\}\}$  and  $\{\{v, v_1\}, \{v, v_2\}\}$  are edge cuts in  $G$ , and hence  $\kappa_e(G) \leq 2$ .
- If  $\{u, v\} \notin E(G)$ , then there exists  $i \in \{1, 2\}$  such that  $u$  has exactly one neighbor in  $C_i$  (this time, we do not claim anything about the number of its neighbors in  $C_{3-i}$ ). Let  $u_i$  be the neighbor of  $u$  in  $C_i$ . There also exists  $j \in \{1, 2\}$  such that  $v$  has exactly one neighbor in  $C_j$ ; we denote this neighbor by  $v_j$ . It follows that  $\{\{u, u_i\}, \{v, v_j\}\}$  is an edge cut in  $G$  of size 2.

**5.** Let  $G$  be an  $n$ -vertex bipartite graph such that every degree of  $G$  is between 10 and 20. Show that  $G$  contains a matching of size at least  $n/3$ .

**Solution:** By König's theorem, it is enough to show that the size of any vertex cover of  $G$  is at least  $n/3$ . Let  $X$  be a vertex cover of  $G$ . Every vertex  $v \in X$  is incident to  $\deg_G(v)$  edges, but every edge of  $G$  is incident to at least one  $v \in X$ , hence

$$|E(G)| \leq \sum_{v \in X} \deg_G(v).$$

On the other hand,

$$2|E(G)| = \sum_{v \in V(G)} \deg_G(v) = \left( \sum_{v \in X} \deg_G(v) \right) + \left( \sum_{v \in V(G) \setminus X} \deg_G(v) \right).$$

Therefore, combining the two estimates on  $|E(G)|$  together and using the fact that each degree is between 10 and 20 yield

$$20 \cdot |X| \geq \sum_{v \in X} \deg_G(v) \geq \sum_{v \in V(G) \setminus X} \deg_G(v) \geq 10 \cdot (n - |X|).$$

But this immediately implies  $|X| \geq n/3$ .