Professor Kindred
Math 104, Graph Theory
Homework 6 Solutions
March 7, 2013

Introduction to Graph Theory, West
Section 4.2 4, 14
Section 4.3 7, 10, 14

Problems you should be able to do: 4.2.12, 4.3.8

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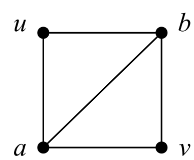
4.2.4 Prove or disprove: if P is a u,v-path in a 2-connected graph G, then there is a u,v-path Q that is internally disjoint from P.

The given statement is **false**. The graph $G = K_4 - uv$ with $V(G) = \{u, v, a, b\}$ is 2-connected (connected and has no cut-vertex). But it has no u,v-path internally disjoint from the u,v-path P that visits vertices u, a, b, v in that order.



Notice that for *every* path P there is not necessarily a path Q that is internally disjoint. Menger's theorem only guarantees that for two vertices u and v, there exist two such paths P and Q. ■

4.2.14 A u,v-*necklace* is a list of cycles $C_1, C_2, \ldots, C_k$ such that $u \in C_1$, $v \in C_k$, consecutive cycles share one vertex, and nonconsecutive cycles are disjoint. Prove that a graph G is 2-edge-connected if and only if for all $u, v \in V(G)$, there is a u,v-necklace in G.

($\Leftarrow$) Suppose G has a u,v-necklace for every pair of vertices $u, v \in V(G)$. Such a necklace has two edge-disjoint u,v-paths. Thus, for any vertices u, v, $\lambda'(u,v) \geq 2$ and we know that $\kappa'(u,v) = \lambda'(u,v)$ by Menger's theorem (edge, undirected, local version), so G is 2-edge-connected.

($\Rightarrow$, induction) Suppose G is 2-edge-connected, and consider two distinct vertices u and v in G. We prove by induction on $d(u,v)$, the length of a shortest u,v-path in G, that G has a u,v-necklace.

Base case ($d(u,v) = 1$: If $d(u,v) = 1$, then $u \sim v$. Since G is 2-edge-connected, it follows that $G - uv$ is connected, and a u,v-path in $G - uv$ concatenated with the edge uv forms a u, v-necklace in G.

Induction hypothesis: Assume that if $d(u, v) = k$ for some positive integer k, then G has a u, v-necklace.

Now consider a pair of vertices u and v in G such that $d(u, v) = k + 1$. Let w be the vertex adjacent to v on a shortest u, v-path. Note that $d(u, w) = d(u, v) - 1 = k$. By the induction hypothesis, G has a u, w-necklace. If v lies on this u, w-necklace, then the cycles up to the one containing v form a u, v-necklace. See Figure 3 for an example of this situation.

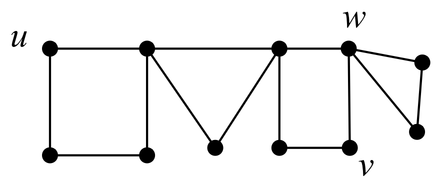


Figure 1: A 2-edge-connected graph and vertices u, v, w such that v is on a u, w-necklace.

Otherwise, we have that v is not on the u, w-necklace. Since G is 2-edge-connected, $G - wv$ is connected, so let P be a u, v-path in $G - wv$. Let z be the last vertex of P on the u, w-necklace, and let C_j be the last cycle containing z in the necklace.

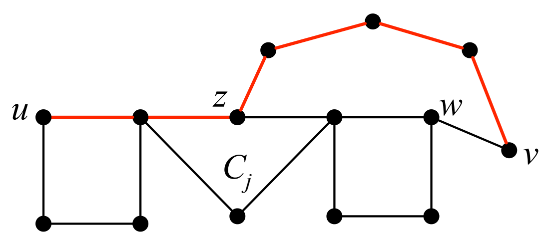


Figure 2: A 2-edge-connected graph with edges of P colored red.

The desired u, v-necklace can be constructed by combining the cycles before C_j in the u, w-necklace with the cycle consisting of

- portion of P from z to v,
- edge vw,
- path from w to cycle C_j in u, w-necklace,
- and path on C_j from there to z that contains the vertex of $C_{j-1} \cap C_j$.

The choice of z guarantees this is a cycle.

Therefore, the result holds by induction on $d(u, v)$.

($\Rightarrow$, extremality) Suppose G is 2-edge-connected, and consider $u, v \in V(G)$. Then there exist two edge-disjoint u, v-paths. Among all such pairs of paths, choose a pair P_1, P_2 whose lengths have the minimum sum. Let S be the set of common vertices among P_1 and P_2. If the vertices of S occur in the same order on P_1 and P_2, then the union of P_1 and P_2 is a u, v-necklace.

Otherwise, let x and y be the first vertices of P_1 in S that occur in the opposite order on P_2, with x before y in P_1 and vice-versa in P_2. We can form two new u, v-paths in the following way:

- Q_1 consists of portion of P_1 up to x and portion of P_2 after x,
- and Q_2 consists of portion of P_2 up to y and portion of P_1 after y.

Neither of Q_1, Q_2 uses any portion of P_1 or P_2 between x and y, so we have found edge-disjoint paths between u and v with shorter total length than P_1 and P_2. $\Rightarrow\Leftarrow$ It must be that no vertices of S occur in opposite orders on P_1 and P_2, so we have a u, v-necklace formed from P_1 and P_2.

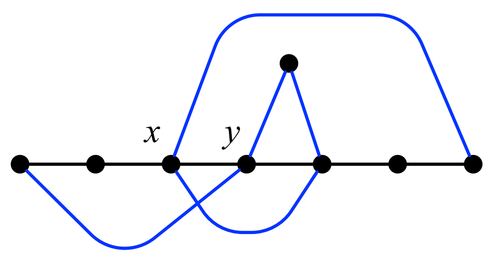


Figure 3: Edge-disjoint paths P_1 and P_2 with edges of P_2 colored blue.

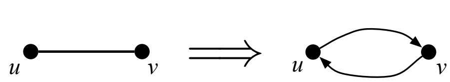
■

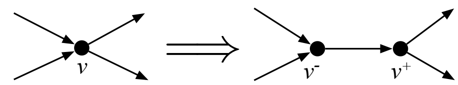
4.3.7 Use network flows to prove Menger's Theorem for nonadjacent vertices in graphs: $\kappa(x, y) = \lambda(x, y)$. *(Hint: Use both transformations suggested in Remark 4.3.15.)*

Let x and y be nonadjacent vertices in a graph G with n vertices.

A set of vertices whose removal disconnects all x, y-paths contains a vertex of each path in a set of pairwise internally-disjoint x, y-paths, so $\kappa(x, y) \geq \lambda(x, y)$. It suffices to show that some set of vertices whose removal disconnects all x, y-paths and some set of pairwise internally-disjoint x, y-paths have the same size.

Construct a network N from the undirected graph G.
Starting with G, first replace each edge uv with two directed edges (u, v) and (v, u), as on the left below. Next, replace each vertex $v \in V(G) - \{x, y\}$ with two vertices v^- and v^+ and an edge of unit capacity from v^- to v^+; call these internal edges. Every edge that had the form (u, v) before this split now is replaced with the edge (u^+, v^-), having

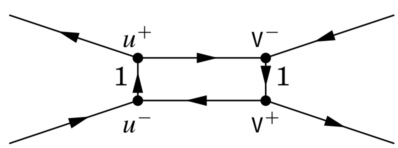




capacity n. (We often write ∞ as a capacity to mean a sufficiently large capacity to keep those edges out of minimum cuts. Here n is enough.) Let N be the resulting network, with source x and sink y.

By the max flow-min cut theorem, the maximum value of a flow in N equals the minimum value of a source-sink cut in N. Let k be the common value.

Claim: G **has** k **pairwise internally disjoint** x, y**-paths.**
By the integrality theorem, there is a flow of value k that has integer flow on each edge. Since only the internal edge leaves v^- , with capacity 1, at most one edge into v^- has nonzero flow, and that flow would be 1. Since only the internal edge enters v^+, with capacity 1, at most one edge leaving v has nonzero flow, and that flow would be 1. Hence the k units of flow transform back into k (x, y)-paths in G, and the restriction of capacity 1 on (v^-, v^+) ensures that these paths are internally disjoint. (This includes the observation that we cannot have one path use the edge from u to v and another from v to u ; one can see explicitly that the capacity of 1 on the internal edges directly prevents this, as illustrated below.)



Claim: G **has a set of** k **vertices whose removal disconnects all** x, y**-paths.**
Since the capacity of every edge of the form (v^+, w^-) is n, every source-sink cut $[S, \overline{S}]$ that has such an edge has capacity at least n. On the other hand, the cut that has x and all internal vertices of the form u^- in S and has y and all internal vertices of the form u^+ in $\overline{S}$ has capacity $n-2$. Therefore, in every cut with minimum capacity, the only edges from S to $\overline{S}$ are edges of the form (u^-, u^+) . If such a set of edges $[S, \overline{S}]$ breaks all (x, y)-paths in N, then $\{u \in V(G) : (u^-, u^+) \in [S, \overline{S}]\}$ is a set of k vertices in G that disconnects all (x, y)-paths in G.

Thus,

$$\kappa(x, y) \leq \text{max flow value on } N = k = \text{min cut capacity on } N \leq \lambda(x, y),$$

as desired. ∎

4.3.10 Use network flows to prove the König-Egerváry Theorem ($\alpha'(G) = \beta(G)$ if G is bipartite).

Let G be a bipartite graph with bipartition X, Y. Construct a network N by adding a source s and sink t, with edges of capacity 1 from s to each $x \in X$ and from each $y \in Y$ to t. Orient each edge of G from X to Y in N with infinite capacity.

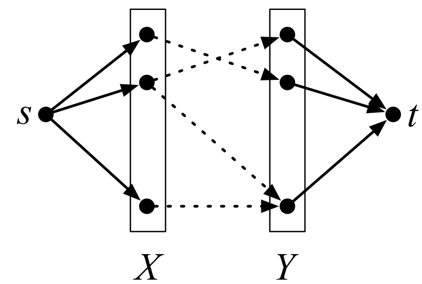


Figure 4: Network N (dashed arcs have ∞ capacities and solid arcs have capacities of 1).

By the integrality theorem (Corollary 4.3.12), there is a maximum flow f with integer value at each edge. The edges of capacity 1 then force the edges between X and Y receiving nonzero flow in f to be a matching. Furthermore, the flow value of f is the number of these edges, since the conservation constraints require the flow along each such edge to extend by edges of capacity 1 from s and to t. We have constructed a matching of size equal to the flow value of f, so $\alpha'(G) \geq$ max flow value on N.

A minimum cut must have finite capacity, since $[s, V(N) - s]$ is a cut of finite capacity. Let $[S, \overline{S}]$ be a minimum cut in N. A cut of finite capacity has no edge of infinite capacity

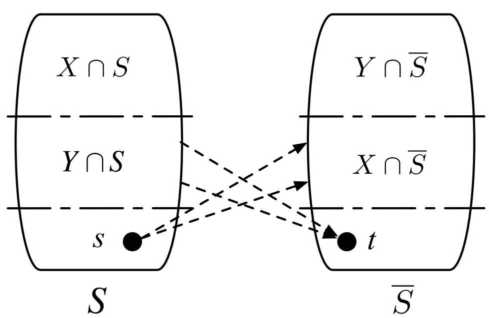


Figure 5: A minimum $[S, \overline{S}]$ cut on network N.

from S to $\overline{S}$. Hence, G has no edge from $X \cap S$ to $Y \cap \overline{S}$.

Consider any edge $e = \{a, b\}$ in the bipartite graph G with $a \in X, b \in Y$. Since the directed arc (a, b) has infinite capacity in N, it must be that $(a, b) \notin [S, \overline{S}]$, which means $a \in \overline{S}$ or $b \in S$ (or both). If $a \in \overline{S}$, then $a \in X - S$, and likewise, if f $b \in S$, then $b \in Y - \overline{S}$. So $(X - S) \cup (Y - \overline{S})$ is a set of vertices in G that form a vertex cover of G.

Furthermore, the only edges in the cut $[S, \overline{S}]$ are edges from s to $X \cap \overline{S} = X - S$ and from $Y \cap S = Y - \overline{S}$ to t. The capacity of the cut is the number of these edges, which equals $|(X - S) \cup (Y - \overline{S})|$. We have constructed a vertex cover of size cap$[S, \overline{S}]$, so $\beta(G) \leq$ cap$[S, \overline{S}]$.

By the max flow-min cut theorem, we now have

$$\beta(G) \leq \text{cap}[S, \overline{S}] = \text{ max flow value of } N \ \leq \alpha'(G).$$

But $\alpha'(G) \leq \beta(G)$ in every graph, so equality holds throughout, and we have $\alpha'(G) = \beta(G)$ for every bipartite graph G. ∎

4.3.14 In a large university with k academic departments, we must appoint an important committee. One professor will be chosen from each department. Some professors have joint appointments in two or more departments, but each must be the designated representative of at most one department. We must use equally many assistant professors, associate professors, and full professors among the chosen representatives (assume that k is divisible by 3). How can the committee be found?

(Hint: Build a network in which units of flow correspond to professors chosen for the committee and capacities enforce the various constraints. Explain how to use the network to test whether such a committee exists and find it if it does.)

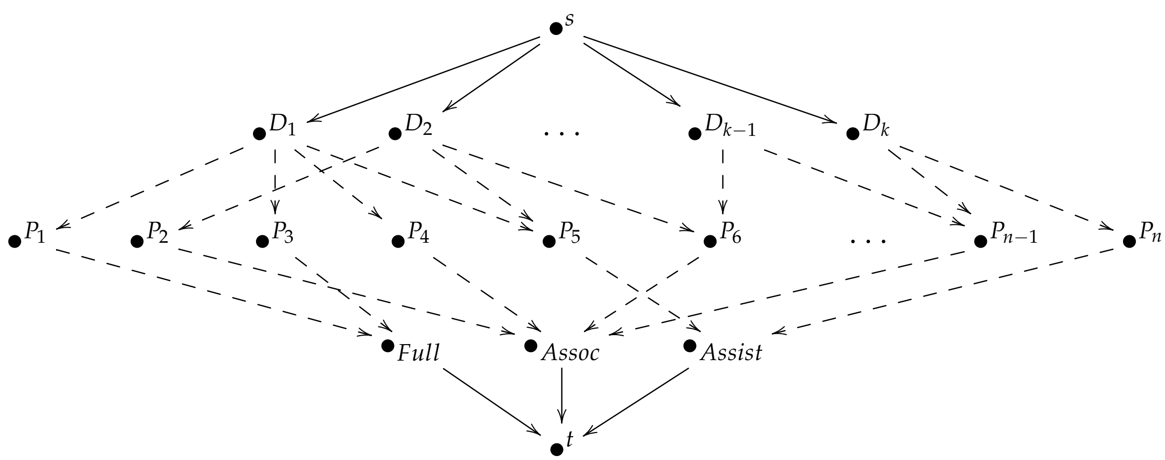


Figure 6: Network for forming committee of department representatives (dashed arcs are examples of possible arcs but are not required arcs)

We form a network model of the given problem. Starting with a source vertex s and a sink vertex t, we add vertices $D_1, \ldots, D_k$ to represent the k academic departments, vertices $P_1, \ldots, P_n$ to represent each of the n professors, and vertices $Full, Assoc, Assist$ to represent the three classifications of professors.

For $i = 1, \ldots, k$, we construct an arc (s, D_i). For $j = 1, \ldots, n$, we add the arc (D_i, P_j) if the professor represented by P_j is a member of the ith department D_i. (Note that since a professor may have joint appointments, we may have more than one incoming arc to any particular vertex P_j.) Moving on, for $j = 1, \ldots, n$, we construct an arc (P_j, x) if professor P_j has classification x, where $x \in \{Full, Assoc, Assist\}$. Since each professor has a unique classification, we know there will be exactly one outgoing arc from each P_j. Assign capacity 1 to all previously-mentioned arcs in the network. Finally, add arcs (x, t) with capacity $k/3$ for each $x \in \{Full, Assoc, Assist\}$. (See Figure 6.)

We now prove that this network enforces the constraints of the original problem.

- *Each dept is represented by at most one professor.*
  This is guaranteed by the arcs (s, D_i) of capacity 1.

- *Each professor is designated as representative for at most one department.*
  This is upheld by the arcs (P_j, x) of capacity 1 (where $x \in \{Full, Assoc, Assist\}$).

- *Each professor is eligible to represent any dept to which he has an appointment.*
  This is ensured by the fact that there exist arcs (D_i, P_j) for every department D_i to which professor P_j is appointed.

- *There are $k/3$ professors on the committee from any particular class of professors (full, assoc, assist).*
  We know this holds since the arcs (x, t) for $x \in \{Full, Assoc, Assist\}$ each have capacity $k/3$.

We have shown the network correctly models the problem of forming a committee under the specified constraints. Hence, we conclude that a valid committee exists precisely when the max flow of the constructed network has value k. In this case, we can determine the make-up of the committee by examining the arcs (D_i, P_j). For $i = 1, \ldots, k$ and $j = 1, \ldots, n$, if the arc (D_i, P_j) has a flow of 1 on it, we know that professor P_j was chosen to represent the ith department D_i on the committee. ∎

4.3.8 Let G be a directed graph with $x, y \in V(G)$. Suppose that capacities are specified *not* on the edges of G, but rather on the *vertices* (other than x, y); for each vertex there is a fixed limit on the total flow through it. There is no restriction on flows in edges. Show how to use ordinary network flow theory to determine the maximum value of a feasible flow from x to y in the vertex-capacitated graph G.

Let $G = (V, E)$ be a directed graph with $x, y \in V$ and vertex capacities $c'(v)$ for all $v \in V(G) - \{x, y\}$. We define a network N based on G in which the capacities are on the edges (not the vertices). For any vertex $v \in V \backslash \{x, y\}$ with capacity $c'(v)$, "split" v into two uncapacitated vertices v_1 and v_2, and let $c'(v)$ be the capacity of the directed edge (v_1, v_2). (See Figure 7.) All incoming edges to v in G become incoming edges to v_1 in N with infinite capacities, and all outgoing edges from v in G become outgoing edges out of v_2 in N with infinite capacities. Repeat this process for all capacitated vertices $v \in V \backslash \{x, y\}$. Since N is an edge-capacitated digraph, we can find a max flow from x to y in N in the usual way.

Consider a maximum flow from x to y in N. Contracting all edges of the form (v_1, v_2) in N transforms a feasible flow in N into a vertex-feasible flow in G with the same flow value. Similarly, any feasible flow in the vertex-capacitated network G "expands" into a feasible flow in N with the same value. Therefore, the max flow algorithm in N solves the original problem.

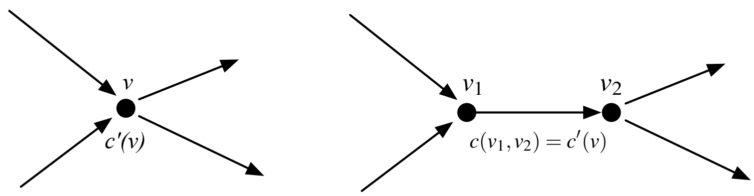


Figure 7: "Splitting" vertex v of capacity $c'(v)$ into two vertices v_1, v_2 with edge (v_1, v_2) assigned capacity $c'(v)$.

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