Professor Kindred
Math 104, Graph Theory
Homework 4 Solutions
February 21, 2013

Introduction to Graph Theory, West
Section 3.1 18, 21, 25, 40, 42

Problems you should be able to do: 3.1.8, 3.1.29, 3.1.31

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3.1.18 Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (*Hint: For the second part, the first player should start with a vertex omitted by some maximum matching*.)

Suppose that G has a perfect matching M. Whenever Player 1 chooses a vertex v, Player 2 should choose the vertex "matched" to v by the matching M. (Since M is a perfect matching, it saturates all vertices, so there is exactly one edge in M incident to v.) This vertex must be available, because after each move of Player 2, the set of vertices visited are the endpoints of a set of full edges in M. Thus, with this strategy, Player 2 can always make a move after any move of Player 1 and can never lose.

If G has no perfect matching, then let M be a maximum matching in G. Player 1 should start by choosing a vertex u not saturated by M. Thereafter, whenever Player 2 chooses a vertex x, Player 1 should choose the vertex "matched" to x in M. The vertex x must be saturated by M; otherwise, the path from u to x of vertices visited forms an M-augmenting path, which would imply that M was not a maximum matching by Berge's theorem. Thus, Player 1 always has a move available and cannot lose. ∎

3.1.21 Let G be a bipartite graph with bipartition X, Y such that $|N(S)| > |S|$ whenever $\varnothing \neq S \subset X$. Prove that every edge of G belongs to some matching that saturates X.

Assume G is a bipartite graph with bipartition X, Y and with $|N(S)| > |S|$ whenever $\varnothing \neq S \subset X$. Let $xy \in E(G)$ with $x \in X$ and $y \in Y$, and consider $G' = G - x - y$. Each set $S \subseteq X - \{x\}$ loses at most one neighbor when y is deleted. Combining this with our initial assumption gives

$$|N_{G'}(S)| \geq |N_G(S)| - 1 > |S| - 1 \quad \Longrightarrow \quad |N_{G'}(S)| \geq |S|.$$

Therefore, G' satisfies Hall's condition and so has a matching M that saturates $X - \{x\}$. It follows that $M \cup \{x, y\}$ is a matching in G (since no edge of M has x or y as an endpoint) that saturates X and includes the desired edge. ∎

3.1.25 A **doubly stochastic matrix** Q is a nonnegative real matrix in which every row and every column sum to 1. Prove that a doubly stochastic matrix Q can be expressed as

$$Q = c_1 P_1 + \cdots + c_m P_m,$$

where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(Hint: Use induction on the number of nonzero entries in Q.)

Remark 1: Note that a doubly stochastic matrix Q must be square. Assume it is an $m \times n$ matrix. Then we have

$$m = \sum_{i=1}^{m} 1 = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} Q_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} Q_{ij}\right) = \sum_{j=1}^{n} 1 = n.$$

Remark 2: For the base case for the subsequent induction proof, we need to know the least possible number of nonzero entries in Q. We note that every $n \times n$ doubly stochastic matrix has at least n nonzero entries. (Otherwise, by the pigeonhole principle, there would exist at least one row with all zero entries. $\Rightarrow\Leftarrow$)

Remark 3: Before we begin the proof by induction, we define what we mean by a graph $G_Q = (V, E)$ based on the matrix Q:

$X = \{x_1, \ldots, x_n\}$ represent the n rows of M

$Y = \{y_1, \ldots, y_n\}$ represent the n columns of M

$V = X \cup Y$

edge $x_i y_j \in E \iff M_{ij} > 0$.

We were careful about our definition of the edges in the graph, since the entries of Q may not be integers; morever, the entries may not even be rational.

Suppose Q is an $n \times n$ doubly stochastic matrix. We prove, by strong induction on the number k of nonzero entries in the matrix Q, that Q can be expressed as a convex combination of permutation matrices.

Base case: Suppose $k = n$. Then there must be exactly 1 nonzero entry in each row and 1 in each column, and all of these entries must be exactly 1 so that the row and columns sum are 1. Note that Q is then exactly a permutation matrix, and the result holds.

Strong induction hypothesis: Suppose Q has j nonzero entries where $n \leq j \leq k$. Then Q can be expressed as $Q = c_1 P_1 + \cdots c_m P_m$, where $\sum_{i=1}^{m} c_i = 1$ and $P_1, \ldots, P_m$ are permutation matrices.

Inductive step: Now, suppose Q has $k + 1$ nonzero entries. We construct the simple bipartite graph G_Q from Q as explained previously. We now claim that G_Q has a perfect matching M. It is enough to check

Hall's condition. Given $S \subseteq X$, we have

$$
\begin{aligned}
\overbrace{|S| = \sum_{x_i \in S} 1 = \sum_{x_i \in S} \left(\sum_{j=1}^{n} Q_{ij} \right)}^{\text{sum of rows of } Q \text{ corresp. to vtcs. in } S} &= \sum_{x_i \in S} \left(\sum_{y_j \in N(S)} Q_{ij} \right) \\
&= \sum_{y_j \in N(S)} \left(\sum_{x_i \in S} Q_{ij} \right) \\
&\leq \underbrace{\sum_{y_j \in N(S)} \left(\sum_{x_i \in S} Q_{ij} + \sum_{x_i \in X \setminus S} \underbrace{Q_{ij}}_{\geq 0} \right) = \sum_{y_j \in N(S)} \left(\sum_{i=1}^{n} Q_{ij} \right) = \sum_{y_j \in N(S)} 1 = |N(S)|}_{\text{sum of columns of } Q \text{ corresponding to vtcs. in } N(S)} .
\end{aligned}
$$

Since Hall's condition is satisfied, there must exist a matching M saturating X, but since $|X| = |Y|$, M is a perfect matching. The perfect matching M corresponds to a permutation matrix P where $P_{ij} = 1$ if $x_i y_j \in M$ and $P_{ij} = 0$ otherwise. Furthermore, let

$$\alpha = \min\{Q_{ij} : x_i y_j \in M\}$$

So $0 < \alpha < 1$ since Q_{ij} is nonzero $\forall x_i y_j \in M$ and we assumed $k + 1 > n$, meaning Q is not a permutation matrix and so the minimum value is not 1.

We consider the matrix $Q - \alpha P$ and notice that it has at least one less nonzero entry then Q. It is also a matrix with all row and column sums equal to $1 - \alpha$. In order to have a doubly stochastic matrix (to apply induction hypothesis), we have to normalize $Q - \alpha P$ by a factor of $1 - \alpha$. So $\frac{1}{1-\alpha}(Q - \alpha P)$ is a matrix with at most k nonzero entries and all row/column sums equal to 1. Also, since $\frac{1}{1-\alpha}(Q - \alpha P)$ is a doubly stochastic matrix, by Remark 2, it has least n nonzero entries. We apply the induction hypothesis to obtain

$$\frac{1}{1-\alpha}(Q - \alpha P) = c_1 P_1 + \cdots c_m P_m \text{ with coefficients such that } \sum_{i=1}^{m} c_i = 1.$$

Therefore,

$$
\begin{aligned}
Q = \alpha P + (1-\alpha) \left[\frac{1}{1-\alpha}(Q - \alpha P) \right] &= \alpha P + (1-\alpha)[c_1 P_1 + \cdots c_m P_m] \\
&= \alpha P + (1-\alpha) c_1 P_1 + \cdots + (1-\alpha) c_m P_m
\end{aligned}
$$

where $P, P_1, \ldots, P_m$ are permutation matrices and the coefficients sum to 1 since

$$\alpha + (1-\alpha) \sum_{i=1}^{m} c_i = \alpha + (1-\alpha) = 1.$$

Hence, the desired result holds by induction. ∎

3.1.40 Let G be a bipartite graph with n vertices. Prove that $\alpha(G) = n/2$ if and only if G has a perfect matching.

Suppose $G = (V, E)$ is a bipartite graph with n vertices. We claim that $\alpha(G) = \frac{n}{2}$ if and only if G has a perfect matching.

$$\alpha(G) = \tfrac{n}{2} \underset{\alpha(G)+\beta(G)=n}{\iff} \beta(G) = \tfrac{n}{2} \overset{\text{König-Egerváry Thm}}{\iff} \alpha'(G) = \tfrac{n}{2} \iff G \text{ has max matching } M \text{ of size } \tfrac{n}{2}$$

$$G \text{ has } {}^{\text{max matching}}_{M \text{ of size}}\ \tfrac{n}{2} \underset{\text{since each } e \in M \text{ saturates 2 unique vtcs}}{\iff} M \text{ saturates } |M| \cdot 2 = \tfrac{n}{2} \cdot 2 = n \text{ vtcs} \overset{\text{by def'n of perfect matching}}{\iff} M \text{ is perfect matching}$$

Notice that the argument above can be argued in either direction (with end of first line connected to the start of second line), so the proof of the if and only if statement is complete. ∎

3.1.42 An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$.

If the algorithm runs for s iterations on a graph $G = (V, E)$, then since each iteration places one vertex into the independent set S, we have that $|S| = s$ at the end of the algorithm. Let G_i denote the remaining graph at the start of iteration i and V_i denote the set of vertices removed at iteration i. (So $G_1 = G$.) Since the algorithm does not terminate until all vertices have been removed from the graph, we know $V_1, V_2, \ldots, V_s$ partition V. In addition,

$$\begin{aligned} |V_i| &= |\{\text{vertex } u \text{ of min degree in } G_i\}| + |N_{G_i}(u)| \\ &= 1 + \delta(G_i) \end{aligned}$$

Because vertices are deleted at each iteration, we know that for any $1 \leq i \leq s$ and for any vertex $v \in V(G_i)$,

$$d_G(v) \geq d_{G_i}(v) \geq \delta(G_i) \qquad \Longrightarrow \qquad \frac{1}{1 + d_G(v)} \leq \frac{1}{1 + \delta(G_i)}.$$

Combining the facts mentioned above, we have

$$\begin{aligned} |S| = \frac{|V_1|}{|V_1|} + \cdots + \frac{|V_s|}{|V_s|} &= \sum_{v \in V_1} \frac{1}{|V_1|} + \cdots + \sum_{v \in V_s} \frac{1}{|V_s|} \\ &= \sum_{v \in V_1} \frac{1}{1 + \delta(G_1)} + \cdots + \sum_{v \in V_s} \frac{1}{1 + \delta(G_s)} \\ &\geq \sum_{v \in V_1} \frac{1}{1 + d_G(v)} + \cdots + \sum_{v \in V_s} \frac{1}{1 + d_G(v)} \\ &= \sum_{v \in V} \frac{1}{1 + d_G(v)} \qquad \text{since } V \text{ is partitioned by the } V_i\text{'s.} \end{aligned}$$

Therefore, the algorithm does produce an independent set of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$. ∎