

Assignment #5: Series-parallel and Planar graphs. Solutions.

1. A graph G is *outerplanar* if it can be drawn in the plane so that every vertex is incident with the infinite region. Show that a graph G is outerplanar if and only if G has no K_4 or $K_{2,3}$ minor.

Solution: Let G' be obtained from G by adding an extra vertex to G adjacent to every other vertex. Then G' is planar if and only if G is outerplanar and it is easy to check that G contains a K_4 or $K_{2,3}$ minor if and only if G' contains a K_5 or $K_{3,3}$ minor. Thus the problem follows from Kuratowski's theorem (18.3).

2.

- a) Show that every series-parallel graph is planar.
- b) Is every series-parallel graph outerplanar?
- c) What is the maximum possible number of edges in a simple series-parallel graph with n vertices?

Solution:

- a) A graph is series parallel if and only if it does not contain K_4 as a minor, and a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. As K_4 is a minor of both K_5 and $K_{3,3}$ the result follows.
- b) No, by the first problem in this assignment $K_{2,3}$ is not outerplanar, but it is easy to check that it is series-parallel.
- c) The answer is $2n - 3$ for $n \geq 3$. This bound can be achieved by taking a graph with 2 vertices joined by $n - 1$ parallel edges and subdividing $n - 2$ of those edges once.

We show that $|E(G)| \leq 2|V(G)| - 3$ for every series-parallel graph G with $|V(G)| \geq 3$ by induction on $|V(G)|$. The base case is clear. For

the induction step, by Lemma 16.2, G has a vertex v of degree two. Thus we have

$$|E(G)| \leq |E(G \setminus v)| + 2 \leq 2|V(G \setminus v)| - 3 + 2 = 2|V(G)| - 3,$$

using the induction hypothesis for $G \setminus v$.

3. Let G be a loopless graph, such that G does not contain $K_{2,3}$ as a minor. Show that either $\chi(G) \leq 3$, or G contains K_4 as a subgraph.

Solution: If G does not contain K_4 as a minor, then $\chi(G) \leq 3$ by Corollary 16.3. If G contains K_4 as a minor, but does not contain K_4 as a subgraph, then G contains a graph K_4^+ , obtained from K_4 by subdividing one of its edges, as a minor. (As shown in the proof of Theorem 18.4, if G contains a graph H with maximum degree three as a minor then G contains a subdivision of G as a subgraph.) The graph K_4^+ contains $K_{2,3}$ as a subgraph, and therefore G contains $K_{2,3}$ as a minor, yielding the desired contradiction.

4. Let G be a graph drawn in the plane. Suppose that there exists a vertex v so that v belongs to the boundary of every region. Show that

$$\alpha(G) \geq \frac{1}{2}(|V(G)| - 1).$$

Solution: We have $\text{Reg}(G \setminus v) = 1$ by the assumption. Thus $G \setminus v$ is a forest and 2-colorable. It follows that

$$\alpha(G) \geq \alpha(G \setminus v) \geq \frac{|V(G \setminus v)|}{2} = \frac{1}{2}(|V(G)| - 1).$$

5. Let G be drawn in the plane so that

- the boundary of the infinite region is some cycle C ,
- every other region has boundary a cycle of length 3, and
- every vertex of G not in C has even degree.

Show that $\chi(G) \leq 3$.

Solution: Following the hint, we prove the result by induction on $|V(G)|$. The induction base $|V(G)| = 3$ is trivial. For the induction step, suppose first that some two vertices of C are joined by an edge $e \notin E(C)$. We can express $C + e$ as a union of two cycles C_1 and C_2 . Let G_1 and G_2 be the subgraphs of G bounded by C_1 and C_2 , respectively. Then G_1 and G_2 are 3-colorable by the induction hypothesis and we can combine their colorings to produce a coloring of G , as $G_1 \cap G_2$ consists of a pair of adjacent vertices.

Suppose now that G contains no edge e as above. Consider $v \notin V(C)$. Assume first that there are no parallel edges incident to v . Let u_0, u_1, \dots, u_k be the neighbors of v , listed in the order that the edges incident to v appear around it, with $u_0, u_k \in V(C)$. In a graph $G \setminus v$ the infinite face is bounded by a cycle obtained from C by replacing v with u_0, u_1, \dots, u_k . By the induction hypothesis there exists a 3-coloring $c : V(G \setminus v) \rightarrow \{1, 2, 3\}$. Note that for each vertex $u \in V(G \setminus v)$ the colors of its neighbors alternate as we enumerate these neighbors in the order edges incident to u appear around u . As $\deg(u_i)$ is even for $i = 1, 2, \dots, k-1$ we deduce that $c(u_{i-1}) = c(u_{i+1})$ for each such i . It follows that only two colors are used on u_0, u_1, \dots, u_k and thus c can be extended to v .

Suppose, finally, that some vertex $v \in V(C)$ is joined to another vertex $u \in V(G)$ by a pair of parallel edges e and f . (The graph G does not contain any loops as every finite face is bounded by a cycle of length 3.) Let R be the region of the plane bounded by e and f . Let G_1 be the subgraph of G drawn within R . Let G_2 be obtained from G by deleting the vertices within the interior of R and the edge f . Applying the induction hypothesis to G_1 and G_2 we obtain a valid coloring of G , as in the previous paragraph. The only caveat is that we need to verify that the degree of u is even in G_2 .

Suppose not. Then the degree of u is even in G_1 . Consider the graph G_1^* dual to G_1 . In G_1^* every region is bounded by an even cycle. Thus G_1^* is bipartite, but every vertex of G_1^* has degree 3, except for one vertex of degree 2, corresponding to the infinite face of G_1 . Summing degrees of the vertices on either side of the bipartition we obtain a contradiction, as one of the sums will be divisible by 3, but not the other. This contradiction finishes the proof.