Professor Kindred
Math 104 Graph Theory
Homework 8 Solutions
April 11, 2013

Introduction to Graph Theory, West
Section 7.1 21, 26
Section 6.1 25, 26, 30

Problems you should be able to do: 6.1.4, 6.1.17, 6.1.21, 6.1.29, 6.1.36

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7.1.21 (*Algorithmic proof of Thm 7.1.7*) Let G be a bipartite graph with maximum degree k. Let f be a proper k-edge-coloring of a subgraph H of G. Let uv be an edge not in H. By using a path alternating in two colors, show that f can be altered and then extended to a proper k-edge-coloring of $H + uv$. Conclude that $\chi'(G) = \Delta(G)$ when G is bipartite.

Assume G is a bipartite graph with $\Delta(G) = k$ and f is a proper k-edge-coloring of a subgraph H of G. Let $uv \in E(G) - E(H)$.

Suppose that each of the k colors appears on an edge of H incident to u or an edge of H incident to v; if not, assign an unused color to edge uv and we're done.

We have

$$\begin{array}{c} \Delta(G) = k \\ \text{and} \\ uv \notin E(H) \end{array} \implies \begin{array}{l} d_H(u) \leq k-1 \implies \exists \text{ color } i \text{ not assigned to any edge of } H \text{ incident to } u, \\ \\ d_H(v) \leq k-1 \implies \exists \text{ color } j \text{ not assigned to any edge of } H \text{ incident to } v. \end{array}$$

Therefore, by our previous assumption, color i must be assigned to an edge of H incident to v, and likewise, color j must be assigned to an edge of H incident to u. Let

$$\begin{aligned} M_i &= \text{color class } i = \text{matching consisting of edges of color } i \\ \text{and } M_j &= \text{color class } j = \text{matching consisting of edges of color } j. \end{aligned}$$

Then consider the subgraph of H with edge set $M_i \cup M_j$. All vertices have degree at most 2 in this subgraph, and any cycles must have edges that alternate between colors i and j, so the subgraph consists only of even cycles, paths, and isolated vertices.

Note that u must have degree 1 in this subgraph, since it only has an edge of color j incident to it, so it is in a component P that is a path. Note that the other leaf in this path cannot be v, since if it did, the path would have to be of even length and combining P with edge uv would form an odd cycle in C, contradicting the fact that G is bipartite. Thus, we can swap the colors of edges in P to obtain a proper k-edge-coloring of H in which color i is now available. We extend this coloring to $H + uv$ by assigning color i to edge uv. ∎

7.1.26 (!) Let G be a regular graph with a cut vertex. Prove that $\chi'(G) > \Delta(G)$.

Let G be a regular graph with a cut vertex v. Suppose BWOC that $\chi'(G) = \Delta(G)$. We know that any color class forms a matching, and since the degree of every vertex is $\Delta(G)$, it must be that every color class forms a perfect matching of G.

So $n = |V(G)|$ is even. Then $G - v$ has an odd number of vertices (and has at least one more component than G since v is a cut vertex). Let H be an odd component of $G - v$. Let u be a neighbor of v (in G) that is not in H. A perfect matching that contains edge uv must contain a perfect matching of H, which is impossible since H has odd order. Therefore, it must be that $\chi'(G) > \Delta(G)$. ∎

6.1.25 Prove that every n-vertex plane graph isomorphic to its dual has $2n - 2$ edges. For all $n \geq 4$, construct a simple n-vertex plane graph isomorphic to its dual.

Let G be an n-vertex plane graph that is isomorphic to its dual, G^*. Suppose G has n vertices, e edges, f faces, and c components. Because the dual of a plane graph is always connected, G^* has one component, but since $G \cong G^*$, we know that G is connected, i.e., $c = 1$. Also, by definition of the dual, we know that $f = |V(G^*)|$, but since $G \cong G^*$, $|V(G^*)| = |V(G)| = n$, so $f = n$. Thus, using Euler's formula, we have

$$1 = n - e + f - c = n - e + n - 1 \quad \Longrightarrow \quad e = 2n - 2.$$

So G has $2n - 2$ edges.

For $n \geq 4$, a simple n-vertex plane graph G isomorphic to its dual is the wheel graph on n vertices, which can be constructed as follows:

- build a $(n-1)$-cycle
- add 1 vertex, making it adjacent to every vertex in the existing $(n-1)$-cycle

By its construction, it is clear the G is a plane graph with n vertices. We claim that G is isomorphic to its dual G^*.

primal		**dual**
$(n-1)$ faces inside cycle	$\Longrightarrow$	$(n-1)$ vtcs. forming a cycle
1 unbounded face sharing one common boundary edge with each of the $(n-1)$ inner faces	$\Longrightarrow$	1 vertex adjacent to every vertex in the $(n-1)$-cycle

∎

6.1.26 For $n \geq 2$, determine the maximum number of edges in a simple outerplane graph with n vertices, giving three proofs.

(a) By induction on n.

(b) By using Euler's formula.

(c) By adding a vertex in the unbounded face and using Theorem 6.1.23.

The maximum number of edges in a simple outerplanar graph on n vertices is $2n-3$. To show that some simple outerplanar graph has $2n-3$ edges, we provide a construction. A cycle on n vertices together with the chords from one vertex to the $n-3$ vertices not adjacent to it on the cycle forms an outerplanar graph with $2n-3$ edges.

For the upper bound, $|E(G)| \leq 2n-3$, we give three proofs.

(a) (induction on n) When $n=2$, such a graph has at most 1 edge, so the bound of $2n-3$ holds immediately. When $n>2$, recall from the text (Proposition 6.1.20) that every simple outerplanar graph G with n vertices has a vertex v of degree at most two. The graph $G' = G - v$ is an outerplanar graph with $n-1$ vertices. By the induction hypothesis, $|E(G')| \leq 2(n-1)-3$. Replacing vertex v restores at most two edges, so $|E(G)| \leq 2n-3$.

(b) (using Euler's formula) The outer face in an outerplanar graph has length at least n, since each vertex must be visited in the walk traversing it. The bounded faces have length at least 3, since the graph is simple. With $\{f_i\}$ denoting the lengths, or degrees of faces, we have

$$2|E(G)| = \sum_i f_i \geq n + 3(f-1),$$

where f is the number of faces. Substituting $f = |E(G)| - n + 2$ from Euler's formula yields $2|E(G)| \geq n + 3(|E(G)| - n + 1)$, which implies $|E(G)| \leq 2n-3$. *(Comment: If one restricts attention to a maximal outerplanar graph, then equality holds in both bounds: the outer face is a spanning cycle, and the bounded faces are triangles.)*

(c) (graph transformation) Add a new vertex in the outer face and an edge from it to each vertex of G. This produces an $(n+1)$-vertex planar graph G' with n more edges than G. Since $|E(G')| \leq 3(n+1)-6$ edges, we have $|E(G)| \leq 3(n+1) - 6 - n = 2n-3$.

Comment: If G is triangle-free, then the bound becomes $(3n-4)/2$. ■

6.1.30 Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is nonplanar.

Let G be an n-vertex simple planar graph with shortest cycle of length k. Since adding edges to make G connected will not change cycle lengths, i.e., the girth of G, we may assume that G is connected.

Consider an embedding of G in the plane. Each face length is at least k, and each edge contributes twice to boundaries of faces. Therefore, counting the appearances of edges in faces grouped according to the $m = |E(G)|$ edges or according to the f faces yields

$$2m \geq kf. \tag{1}$$

Since G is connected, we can apply Euler's formula to obtain $n - m + f = 2$. Substituting for f in the inequality (1) yields

$$2m \geq k\,(2 - n + m)$$

and thus

$$m \leq (n-2)\frac{k}{k-2}.$$

The Petersen graph has 10 vertices, 15 edges, and girth 5. A simple planar graph with girth 5 has at most $\left\lfloor (10-2)\frac{5}{5-2} \right\rfloor = 13$ edges. Since $15 > 13$, the Petersen graph cannot be planar, and at least two edges must be deleted to obtain a planar subgraph of the Petersen graph. ∎