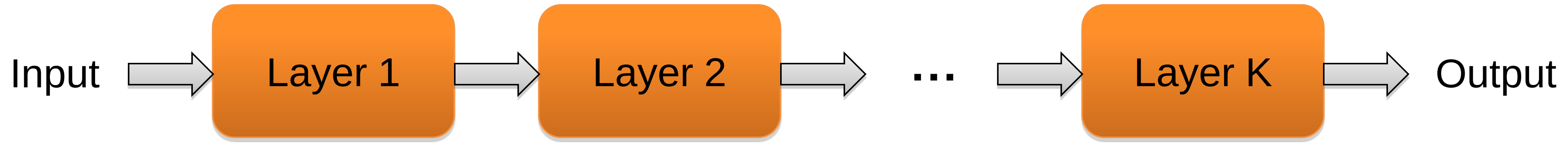
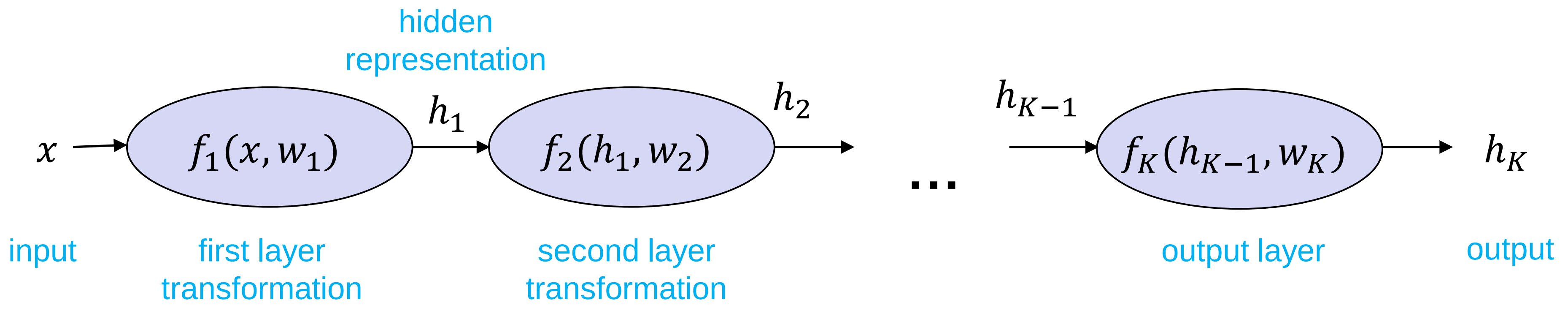
How to train a multi-layer network?

Recall: Multi-layer neural networks

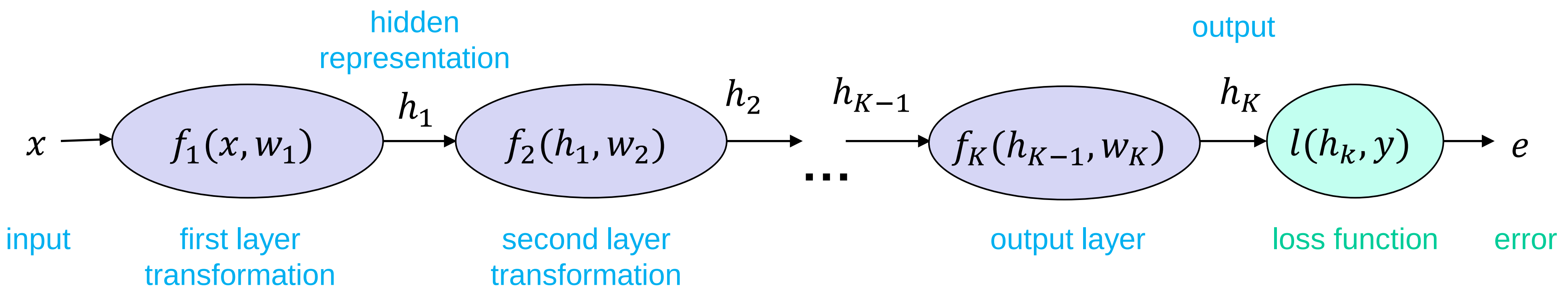
- The function computed by the network is a composition of the functions computed by individual layers (e.g., linear layers and nonlinearities):



- More precisely:



Training a multi-layer network

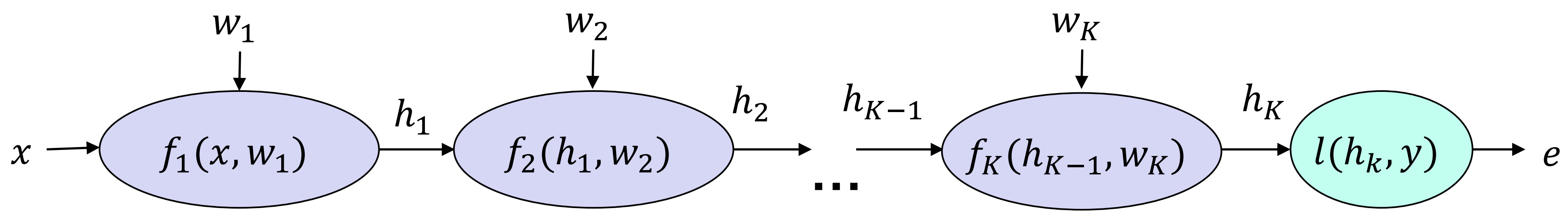


- What is the SGD update for the parameters w_k of the kth layer?

$$w_k \leftarrow w_k - \eta \frac{\partial e}{\partial w_k}$$

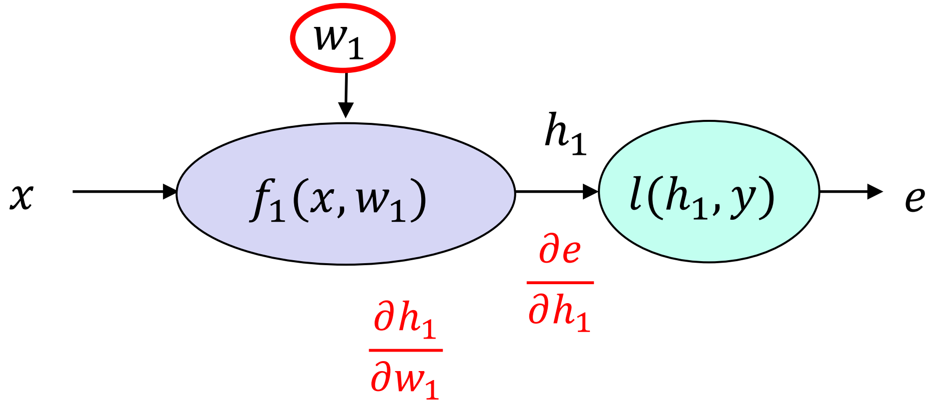
- To train the network, we need to find the gradient of the error w.r.t. the parameters of each layer, $\frac{\partial e}{\partial w_k}$

Computation graph



Chain rule

Let's start with $k = 1$



$$e = l(f_1(x, w_1), y)$$

$$\frac{\partial}{\partial w_1} l(f_1(x, w_1), y) =$$

Example: $e = (y - w_1^T x)^2$

$$h_1 = f_1(x, w_1) = w_1^T x$$

$$e = l(h_1, y) = (y - h_1)^2$$

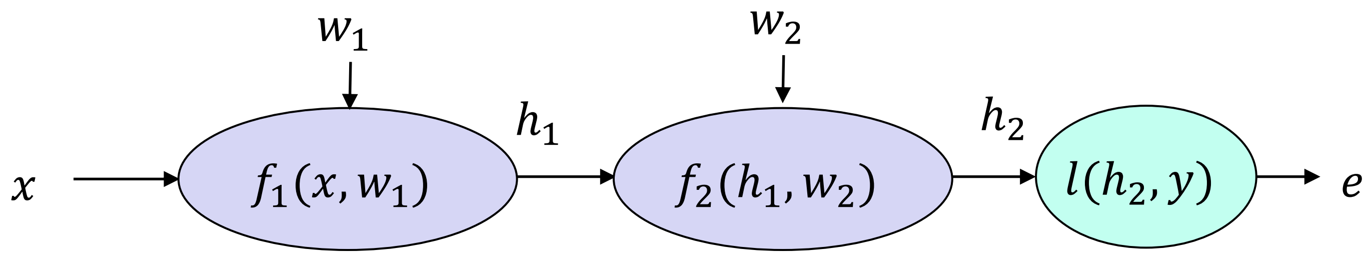
$$\frac{\partial h_1}{\partial w_1} =$$

$$\frac{\partial e}{\partial h_1} =$$

$$\frac{\partial e}{\partial w_1} = \frac{\partial e}{\partial h_1} \frac{\partial h_1}{\partial w_1}$$

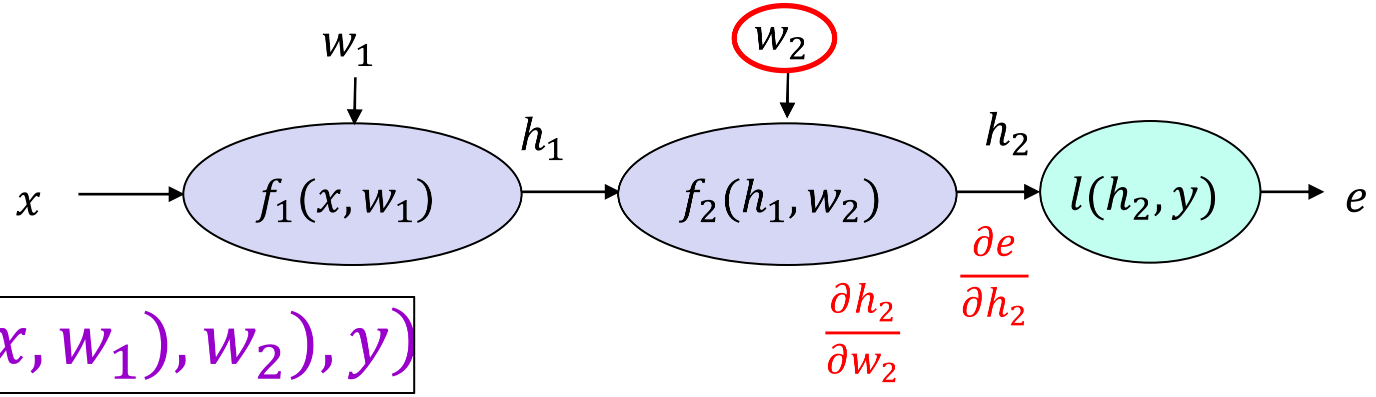
Chain rule

$k = 2$



Chain rule

$k = 2$

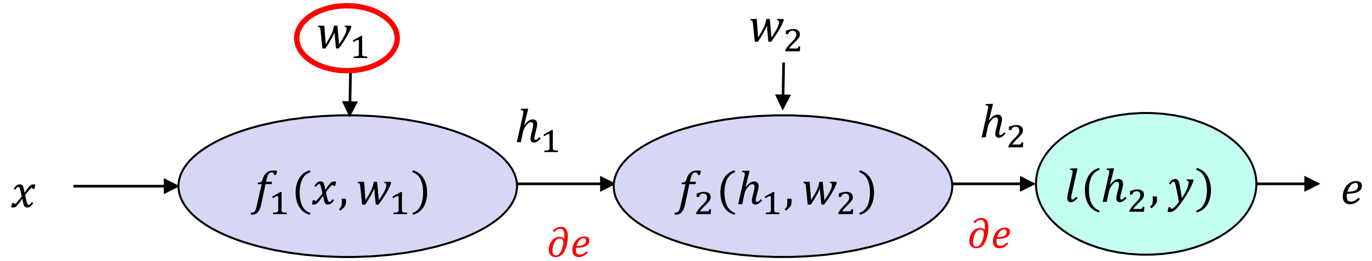


$$\boxed{e = l(f_2(f_1(x, w_1), w_2), y)}$$

$$\frac{\partial e}{\partial w_2} =$$

Chain rule

$k = 2$



$\boxed{e = l(f_2(f_1(x, w_1), w_2), y)}$ $\frac{\partial h_1}{\partial w_1}$ $\frac{\partial e}{\partial h_1}$ $\frac{\partial h_2}{\partial h_1}$ $\frac{\partial e}{\partial h_2}$

$$\frac{\partial e}{\partial w_2} = \frac{\partial e}{\partial h_2} \frac{\partial h_2}{\partial w_2}$$

Example: $e = -\log\left(\sigma(w_1^T x)\right)$ (assume $y = 1$)

$$h_1 = f_1(x, w_1) = w_1^T x$$
$$h_2 = f_2(h_1) = \sigma(h_1)$$
$$e = l(h_2, 1) = -\log(h_2)$$

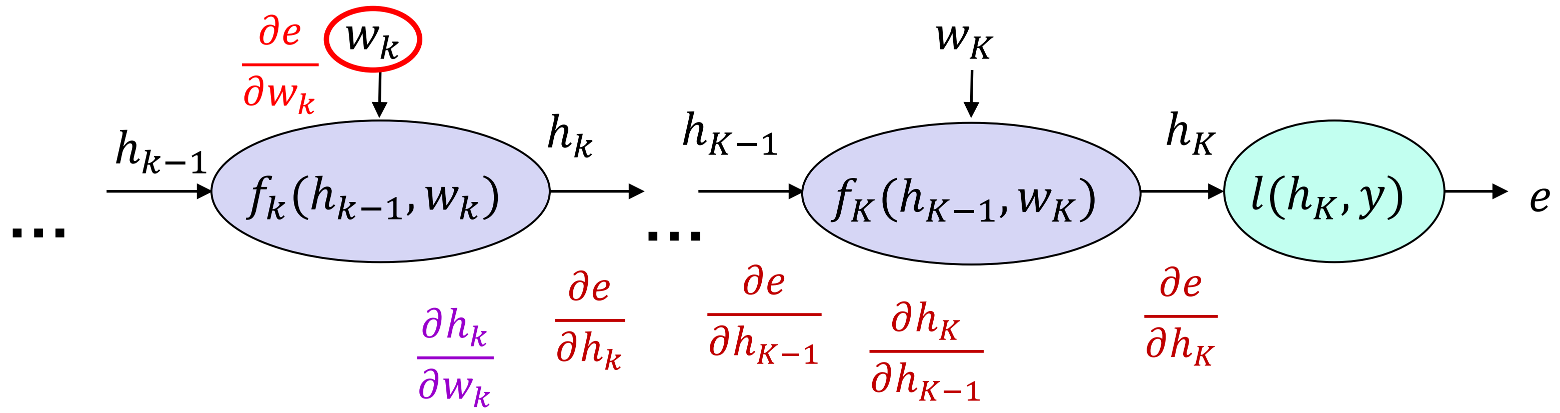
$$\frac{\partial h_1}{\partial w_1} =$$

$$\frac{\partial h_2}{\partial h_1} =$$

$$\frac{\partial e}{\partial h_2} =$$

$$\frac{\partial e}{\partial w_1} = \frac{\partial e}{\partial h_2} \frac{\partial h_2}{\partial h_1} \frac{\partial h_1}{\partial w_1} =$$

Chain rule



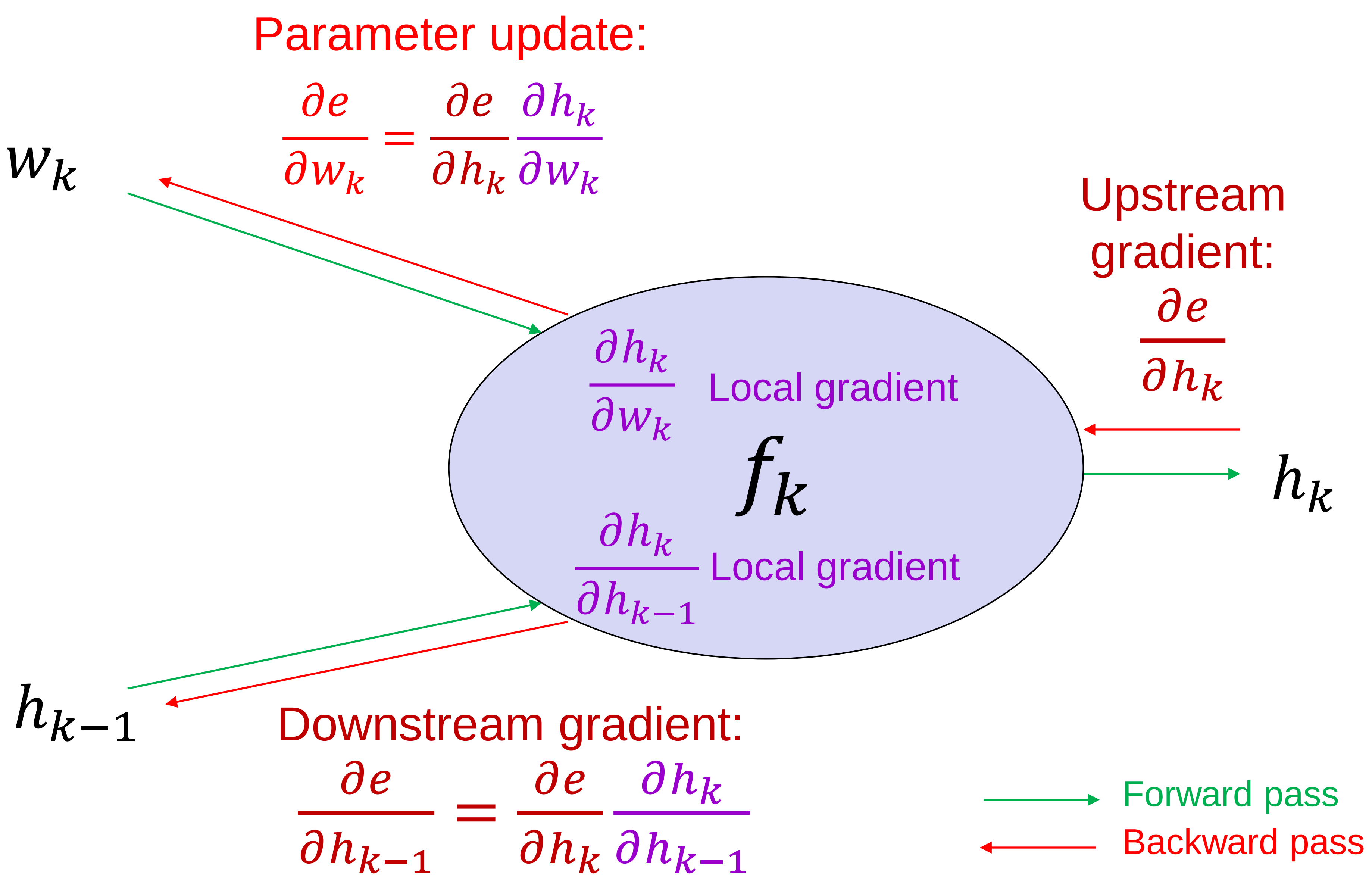
General case:

$$\frac{\partial e}{\partial w_k} = \boxed{\frac{\partial e}{\partial h_K} \frac{\partial h_K}{\partial h_{K-1}} \cdots \frac{\partial h_{k+1}}{\partial h_k}} \frac{\partial h_k}{\partial w_k}$$

Upstream gradient $\frac{\partial e}{\partial h_k}$

Local gradient

Backpropagation summary



What about more general computation graphs?

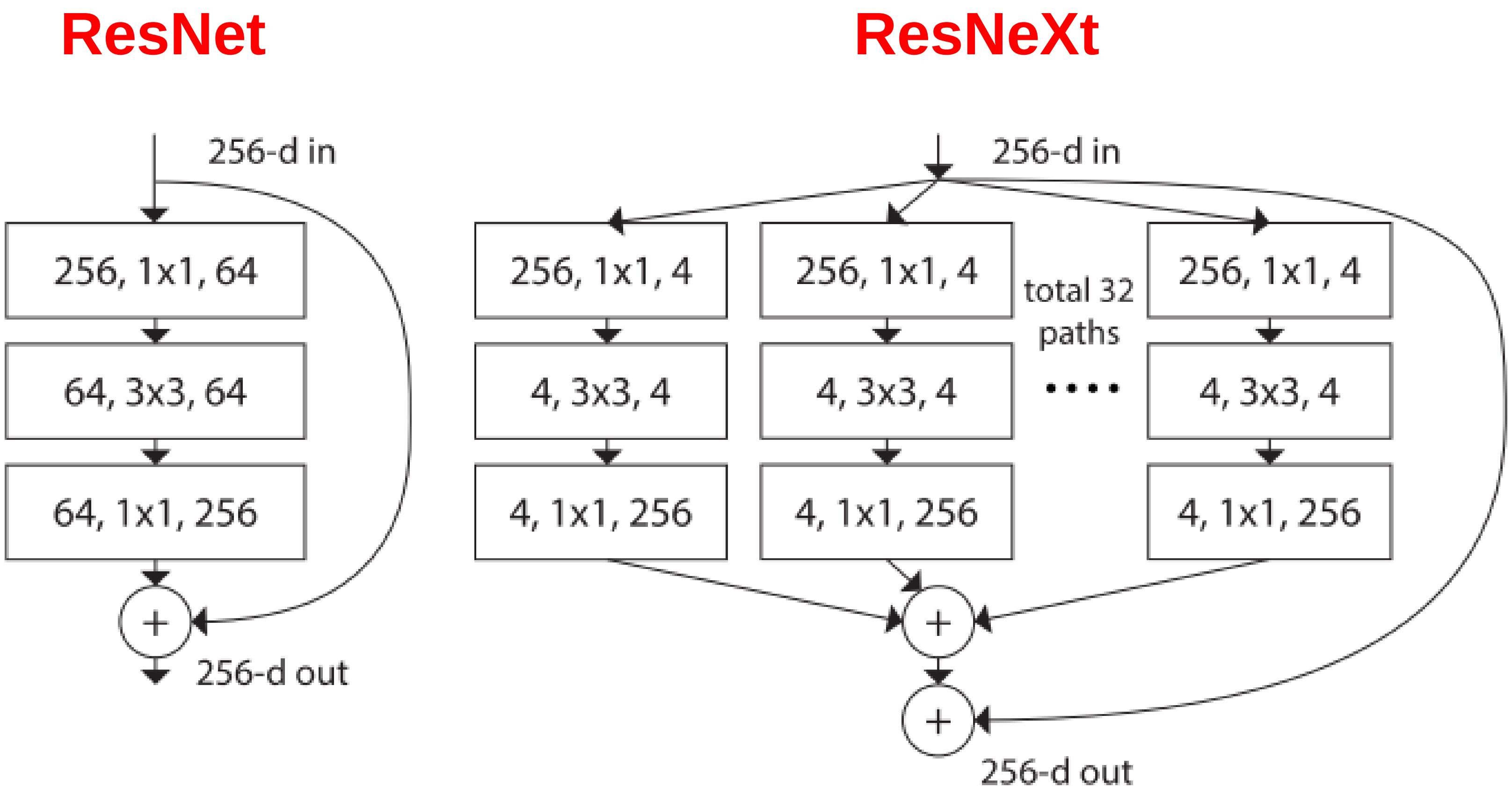
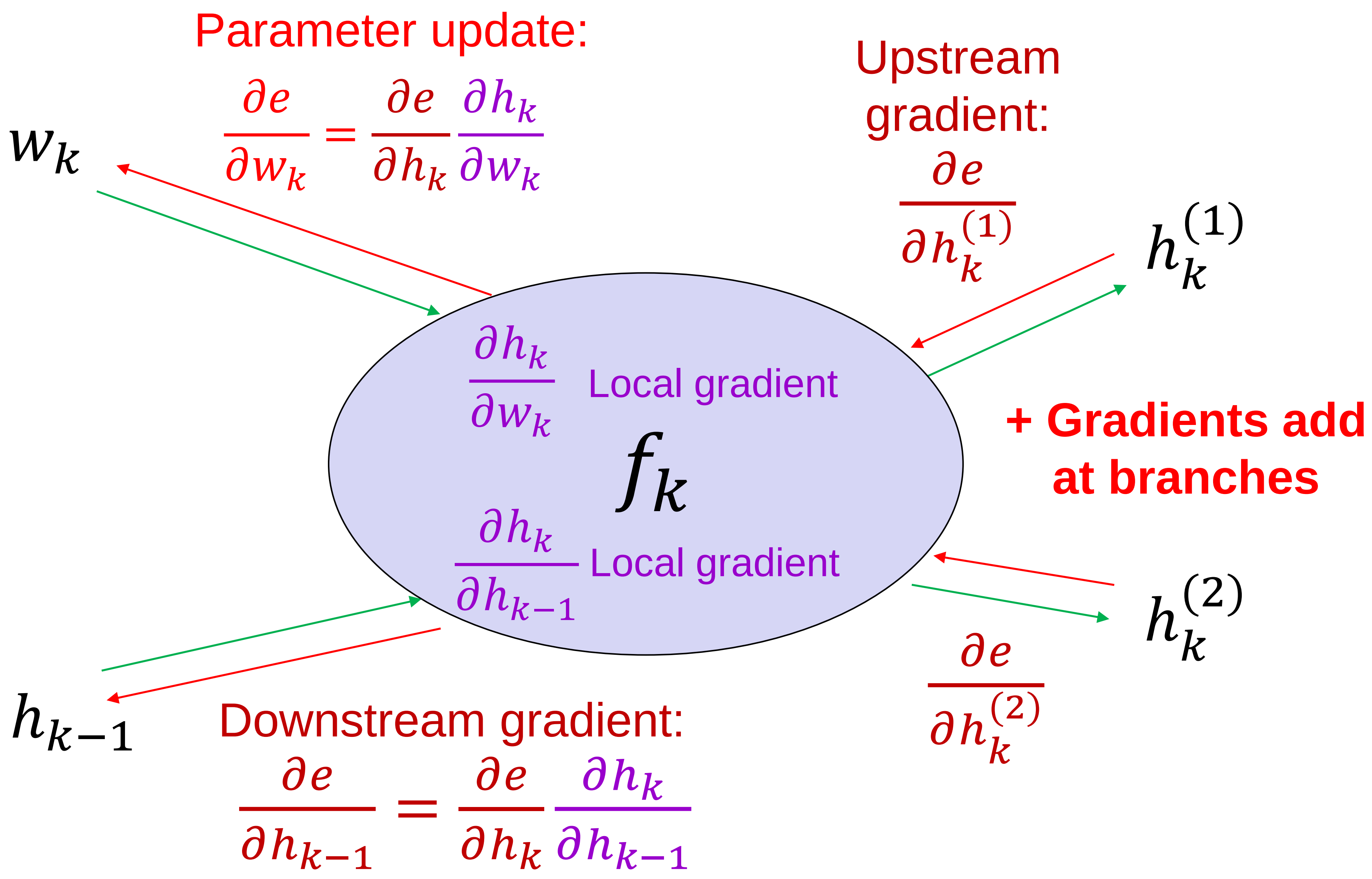


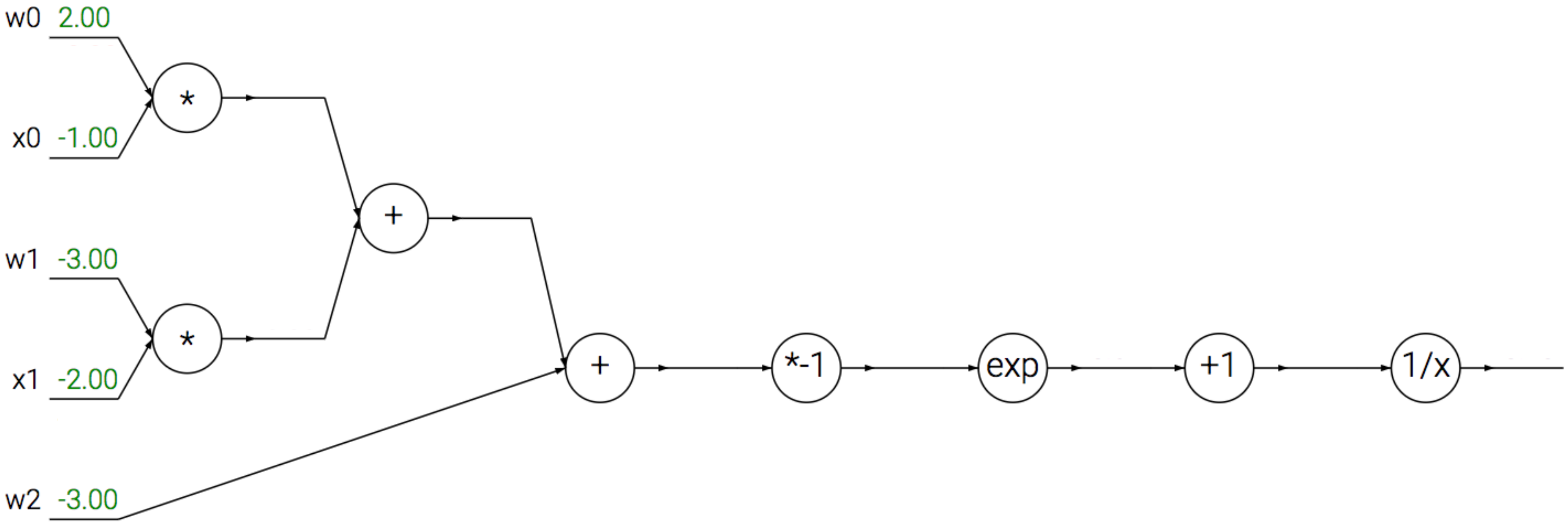
Figure source

What about more general computation graphs?



A detailed example

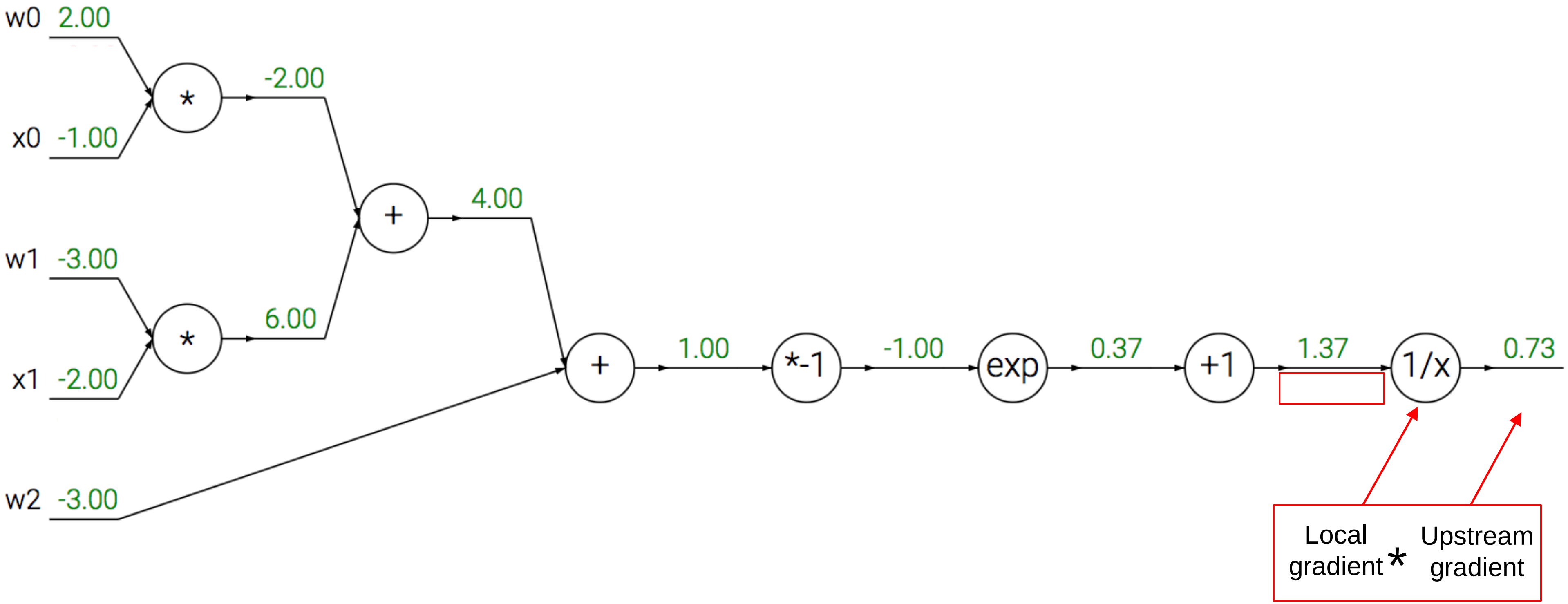
$$f(x,w) = \frac{1}{1+\exp[-(w_0x_0 + w_1x_1 + w_2)]}$$



Source: [Stanford 231n]

A detailed example

$$f(x,w) = \frac{1}{1 + \exp[-(w_0 x_0 + w_1 x_1 + w_2)]}$$



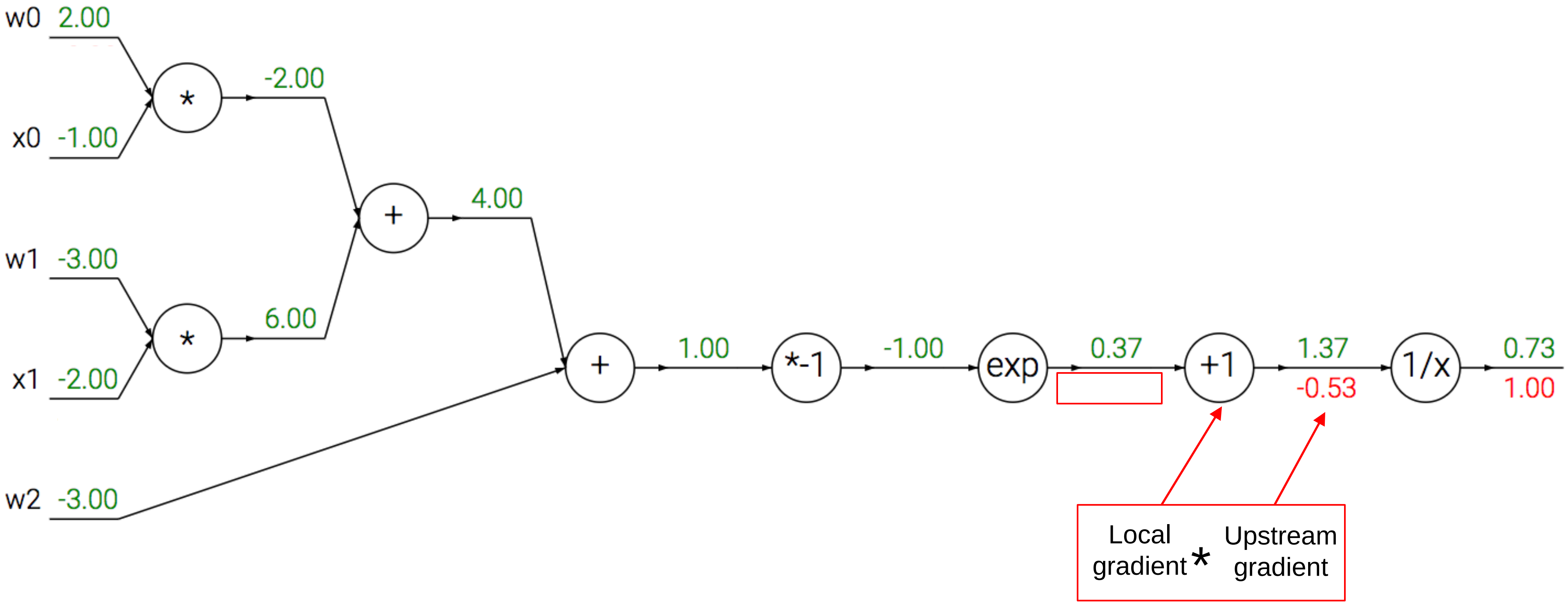
$$(1/x)' = -1/x^2$$

$$-\frac{1}{1.37^2} * 1 = -0.53$$

Source: [Stanford 231n]

A detailed example

$$f(x,w) = \frac{1}{1+\exp[-(w_0x_0 + w_1x_1 + w_2)]}$$



Source: [Stanford 231n]()

A detailed example

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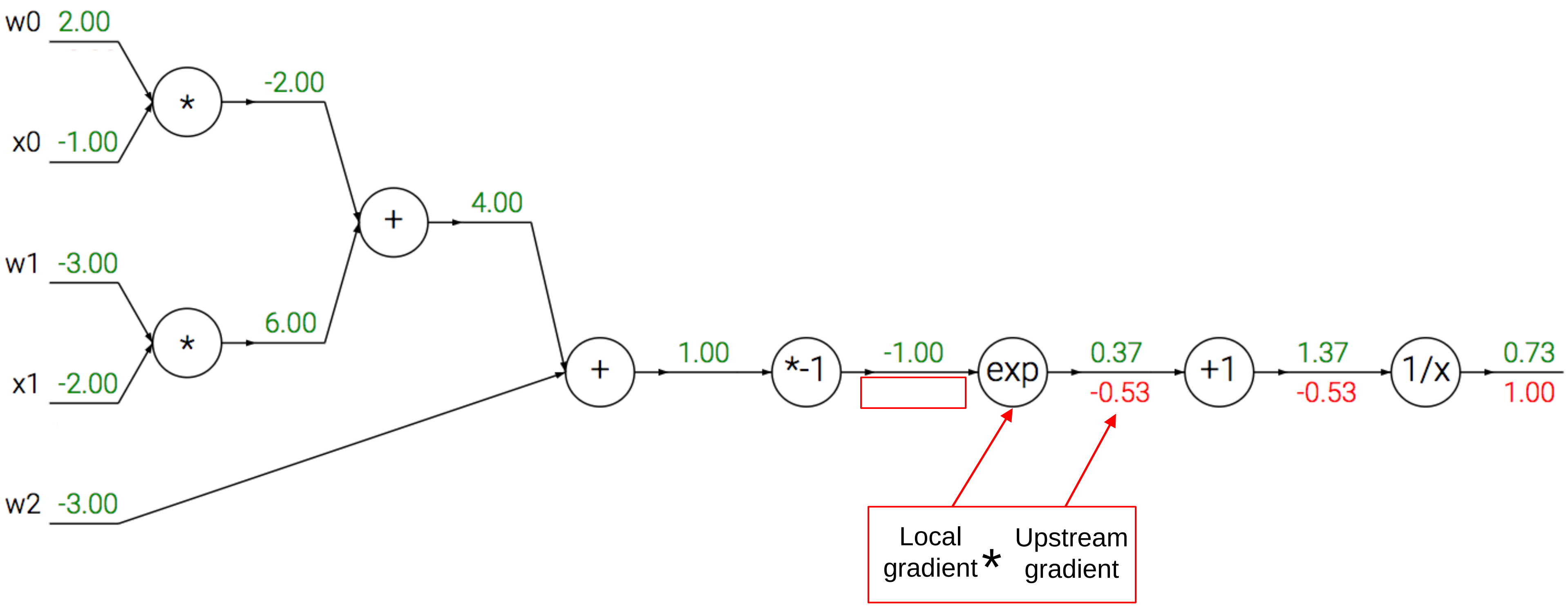
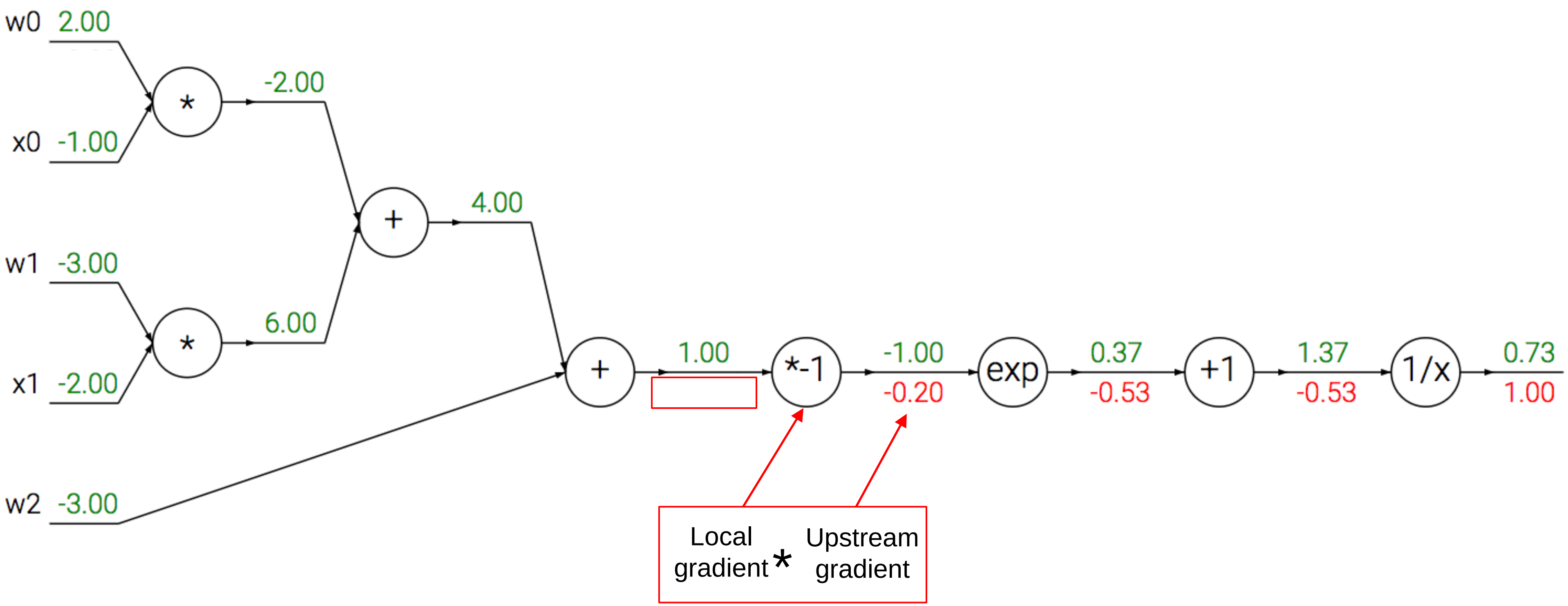


$$\exp(-1) * (-0.53) = -0.20$$

Source: [Stanford 231n]()

A detailed example

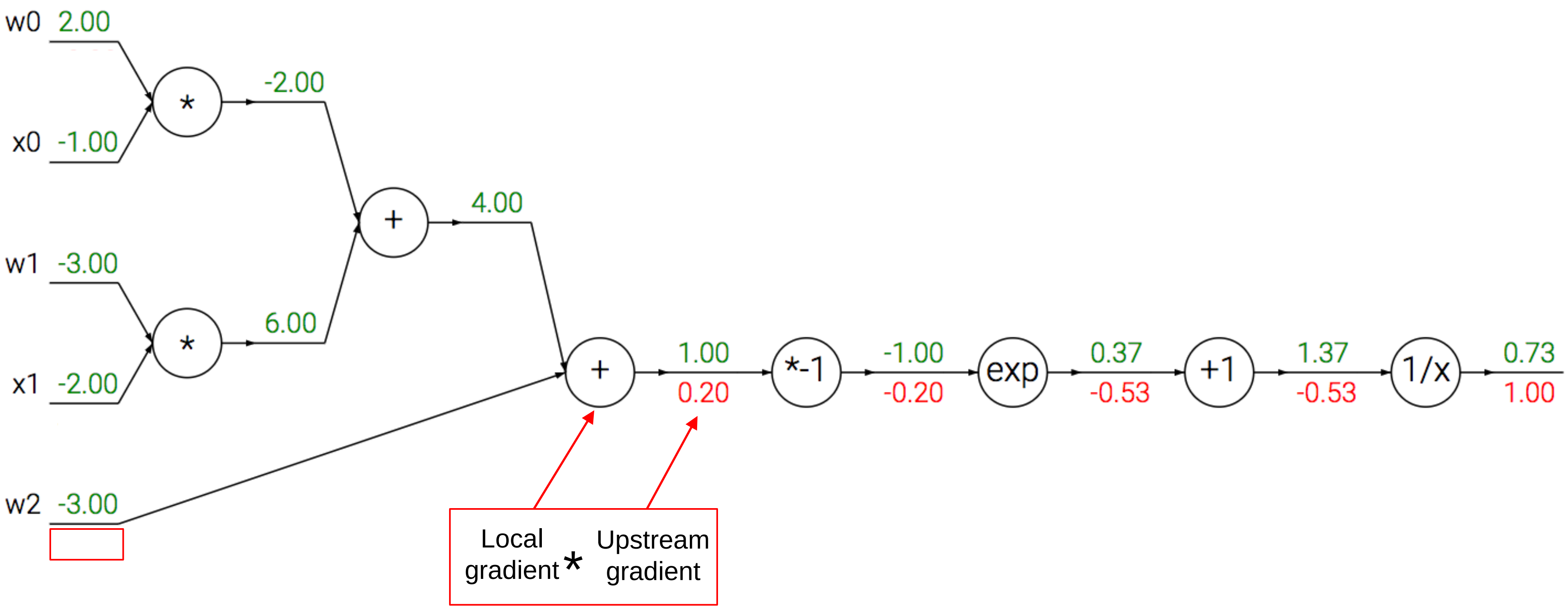
$$f(x,w) = \frac{1}{1+\exp[-(w_0x_0 + w_1x_1 + w_2)]}$$



Source: [Stanford 231n]()

A detailed example

$$f(x,w) = \frac{1}{1 + \exp[-(w_0 x_0 + w_1 x_1 + w_2)]}$$



Source: [Stanford 231n]

A detailed example

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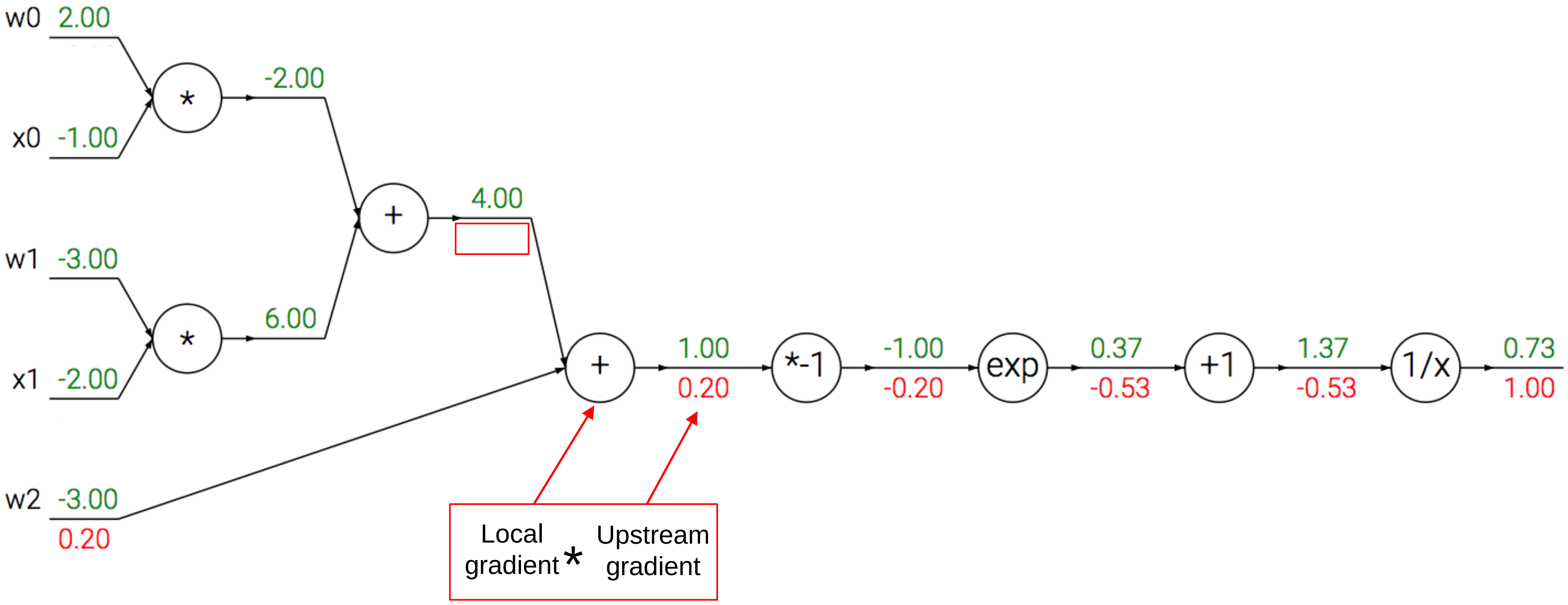

Source: [Stanford 231n]

A detailed example

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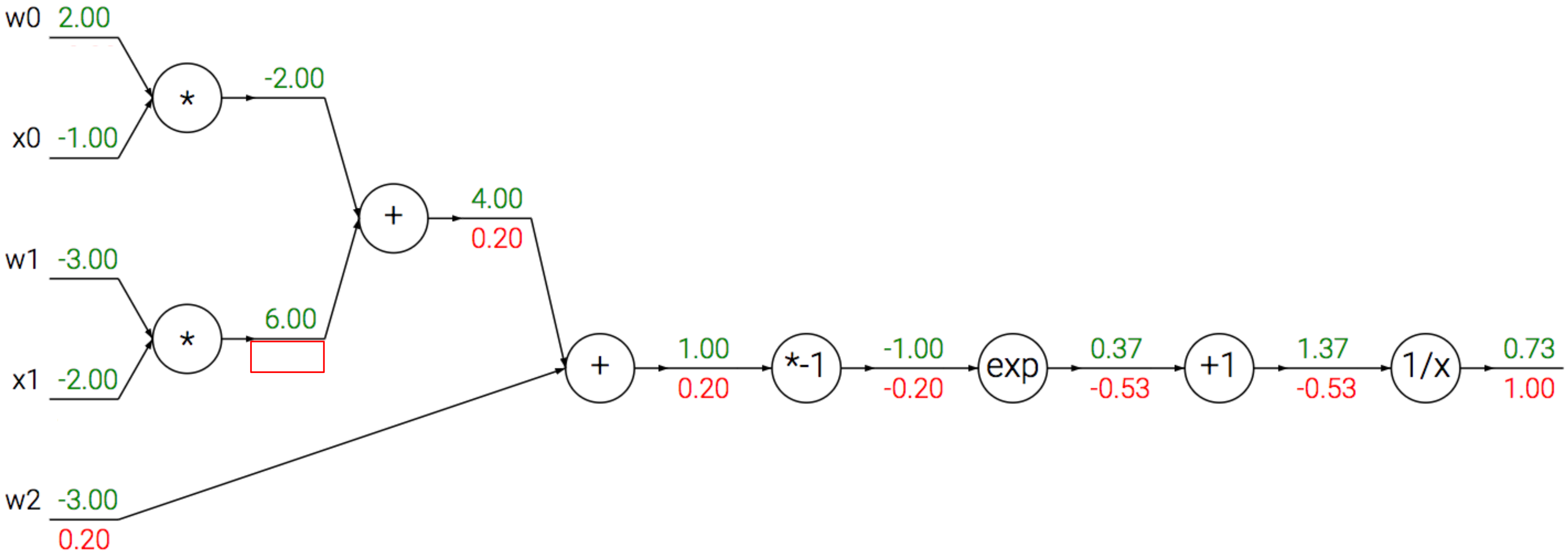
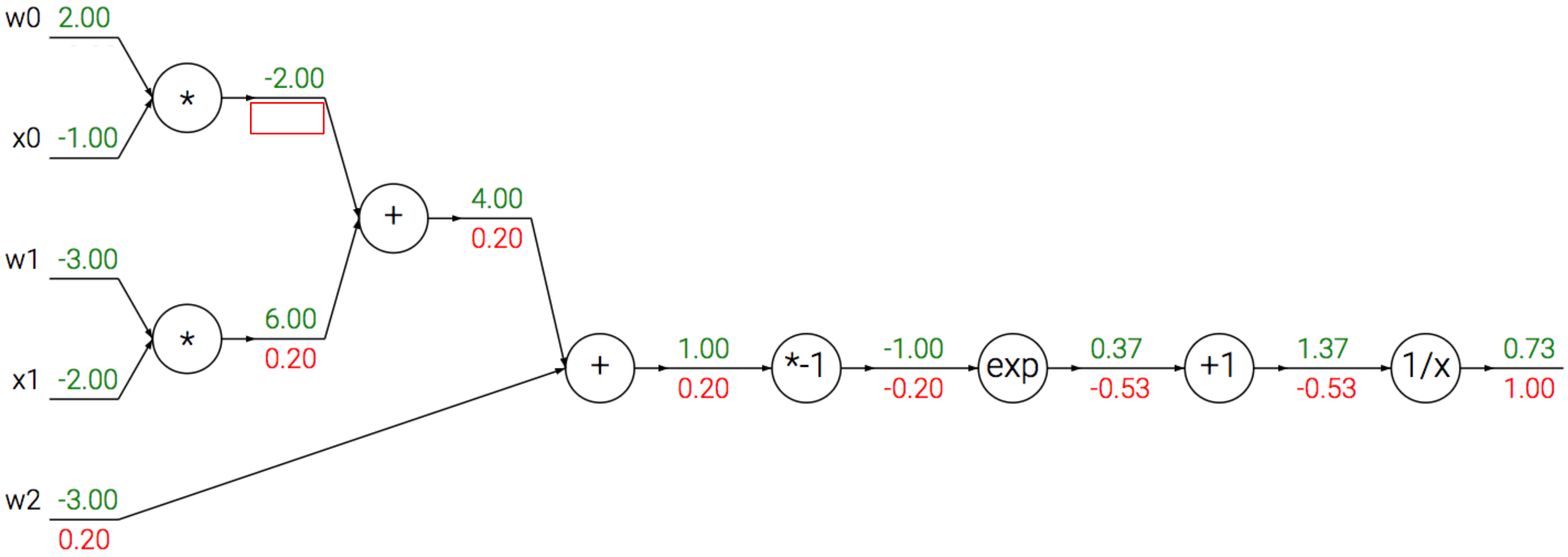


Source: [Stanford 231n]

A detailed example

$$f(x,w) = \frac{1}{1 + \exp[-(w_0 x_0 + w_1 x_1 + w_2)]}$$



Source: [Stanford 231n]

A detailed example

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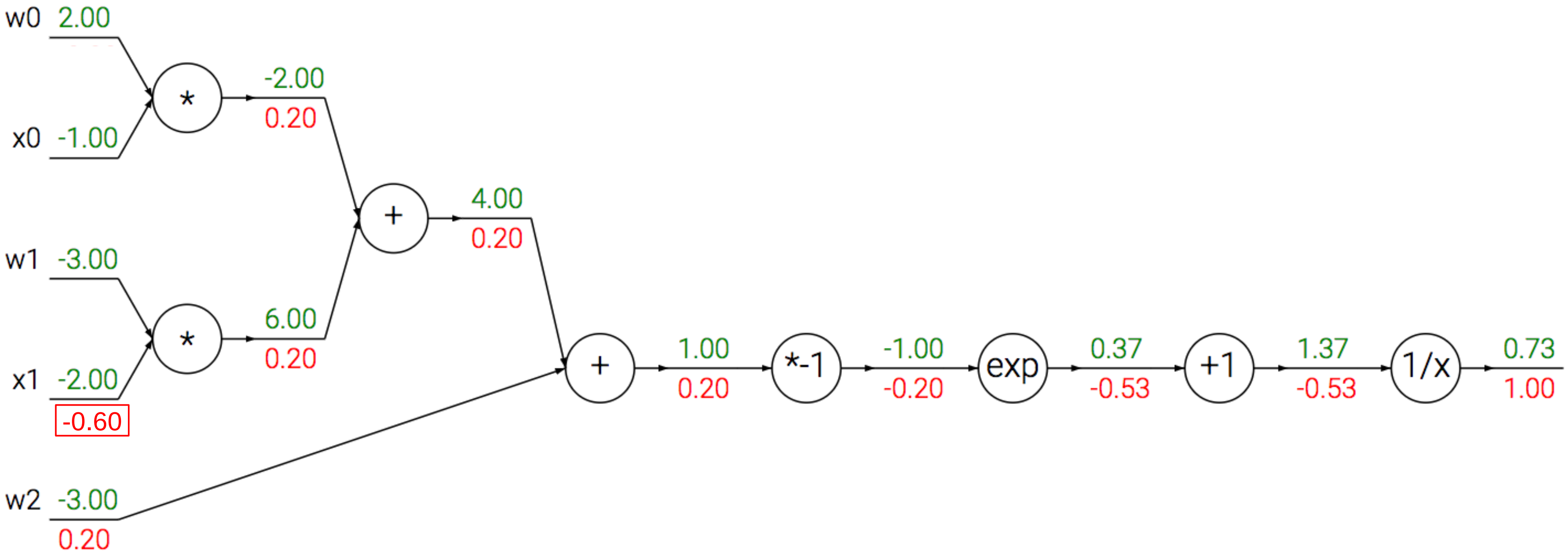

Source: [Stanford 231n]()

A detailed example

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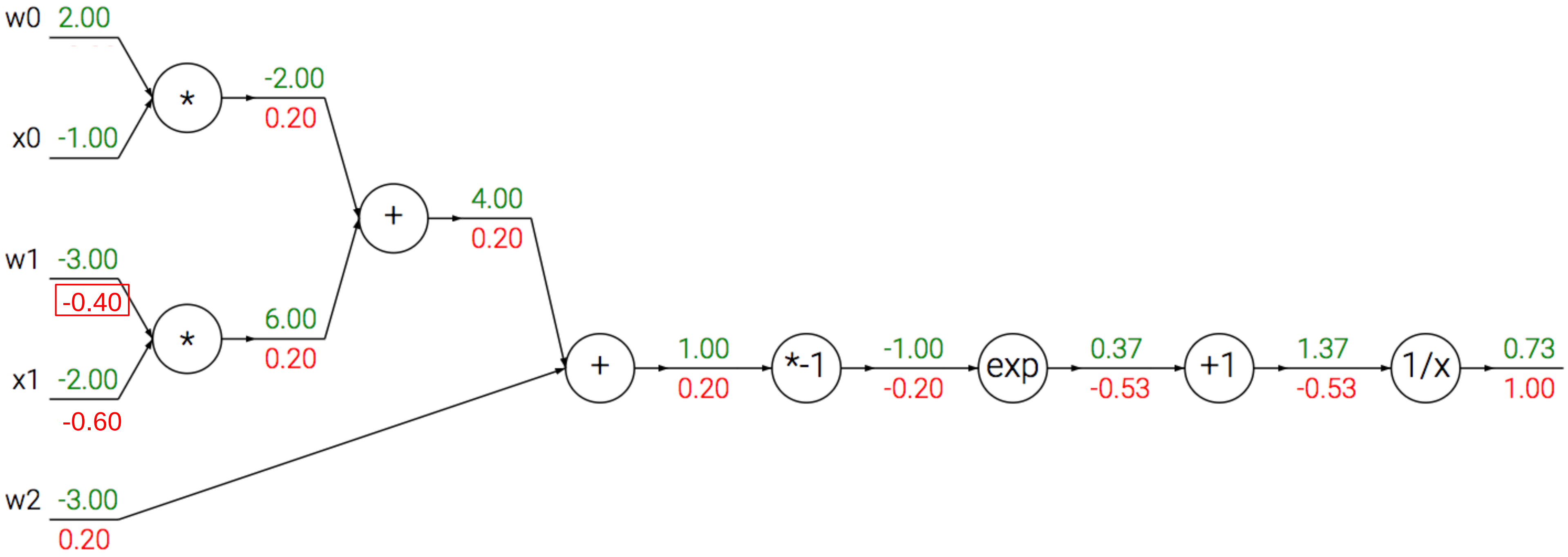

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A detailed example

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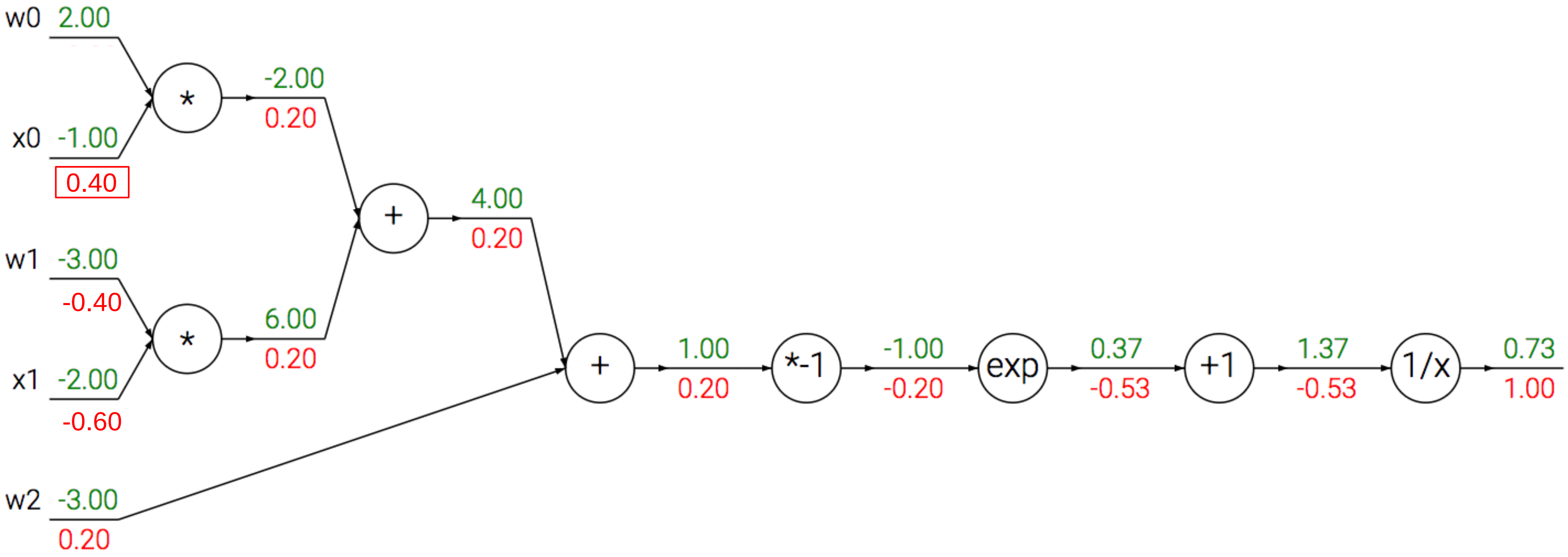

Source: [Stanford 231n]()

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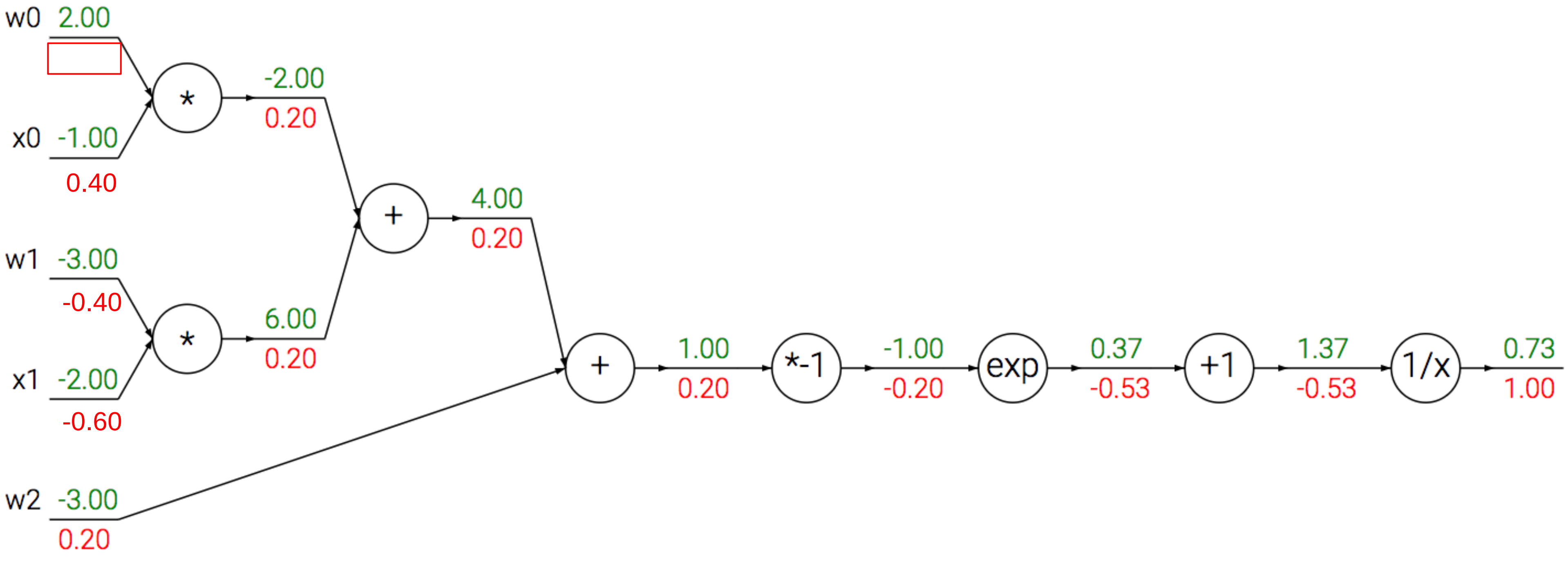

Source: [Stanford 231n]

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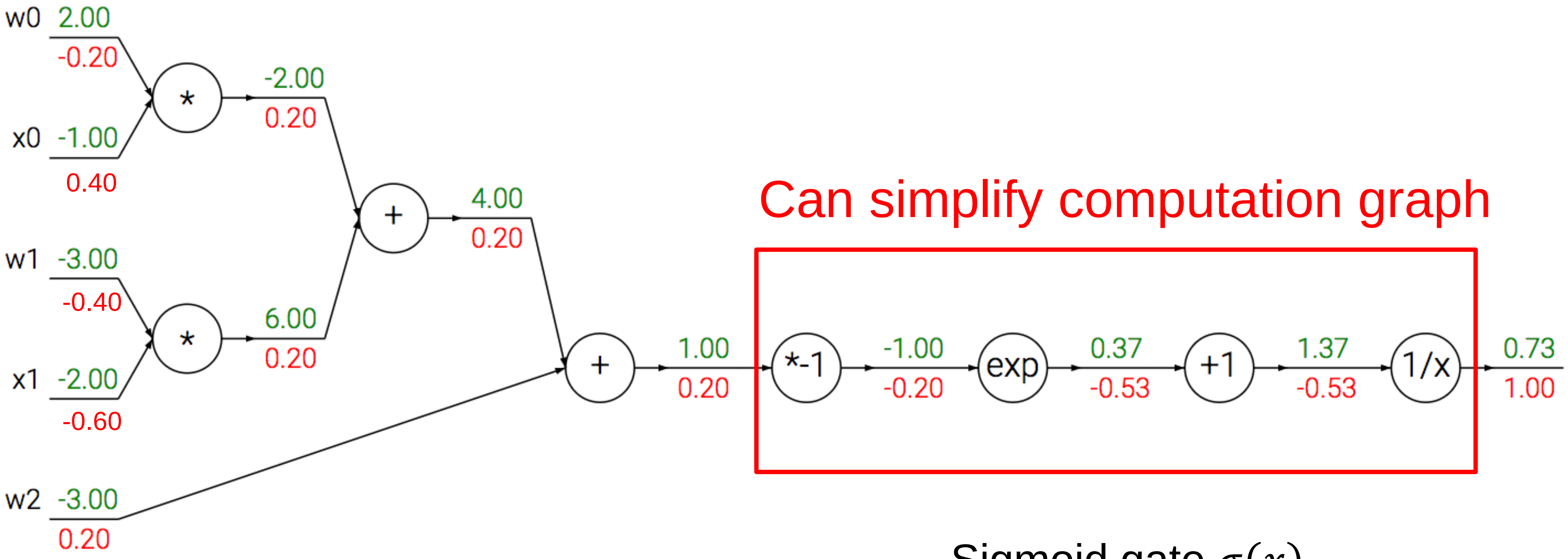


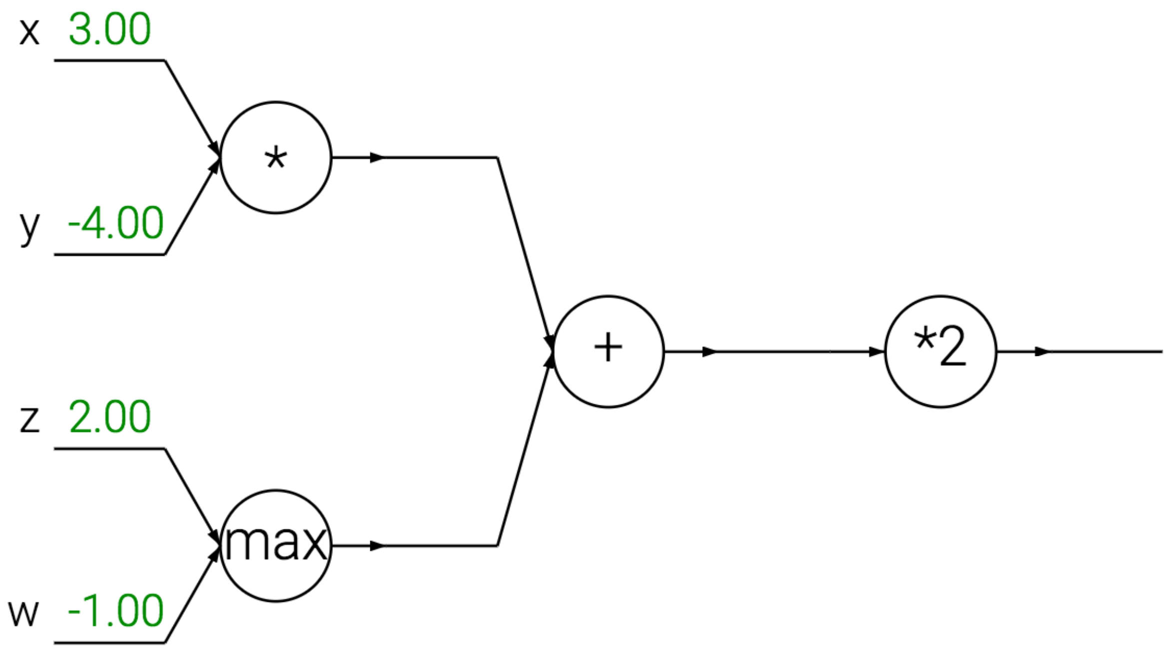
Sigmoid gate $\sigma(x)$

$$\sigma'(x) = \sigma(x)\big(1-\sigma(x)\big)$$

$$\sigma(1)\big(1-\sigma(1)\big) = 0.73 * (1-0.73) = 0.20$$

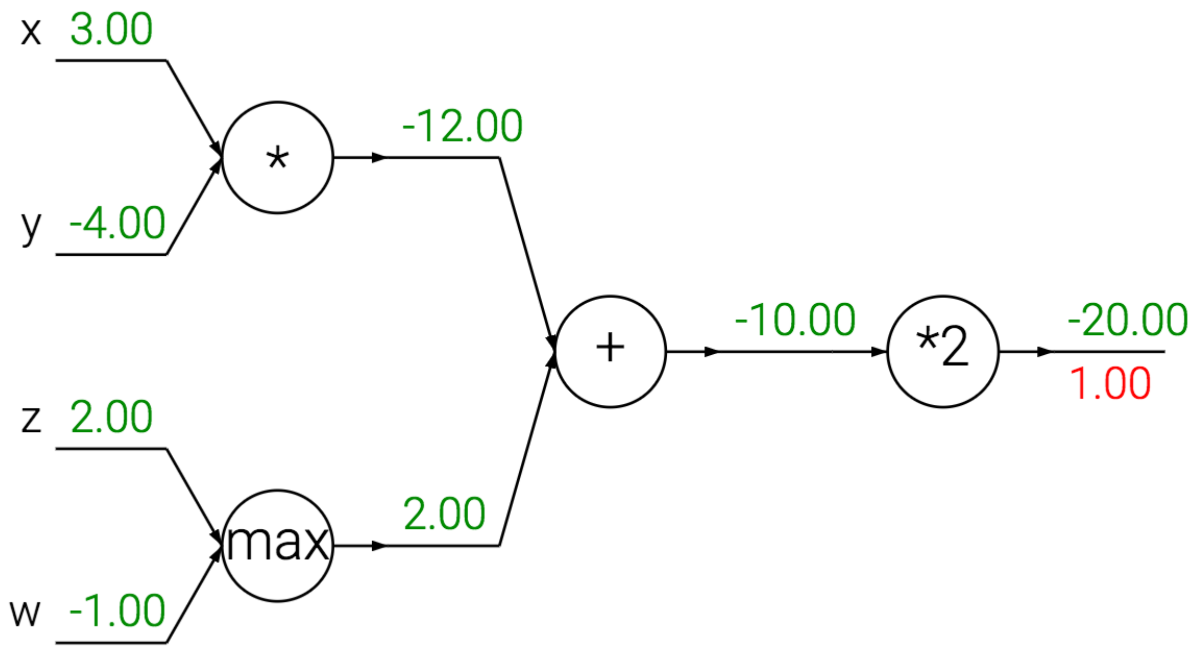
Source: [Stanford 231n]

Patterns in gradient flow



Source: [Stanford 231n]

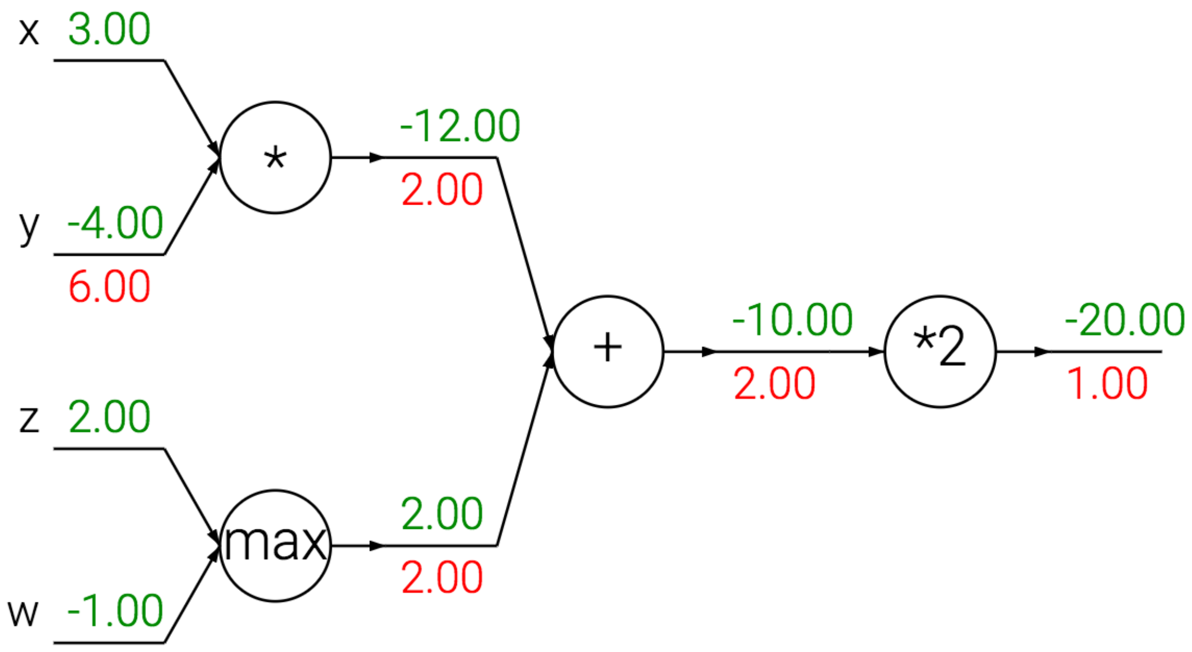
Patterns in gradient flow



Add gate: “gradient distributor”

Source: [Stanford 231n]()

Patterns in gradient flow

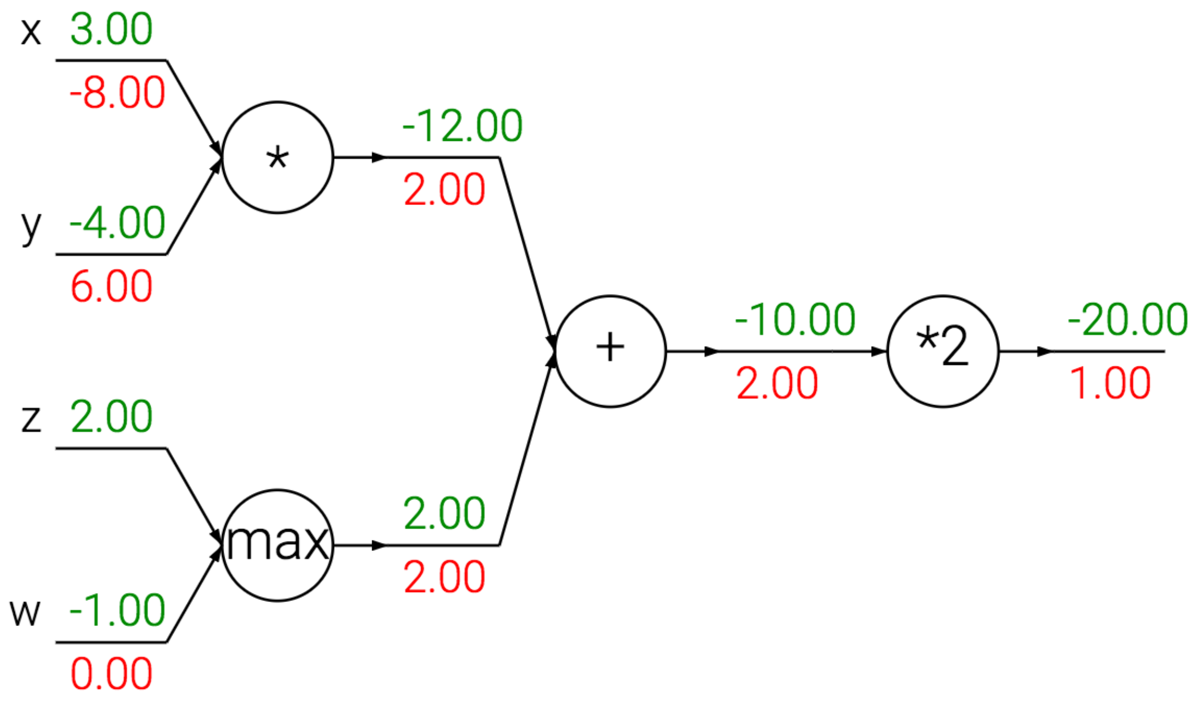


Add gate: “gradient distributor”

Multiply gate: “gradient switcher”

Source: [Stanford 231n]

Patterns in gradient flow



Add gate: “gradient distributor”

Multiply gate: “gradient switcher”

Max gate: “gradient router”

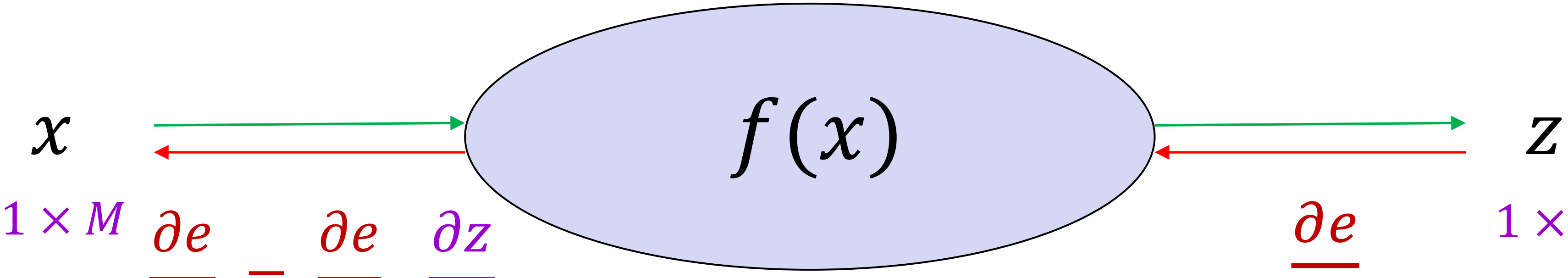
Source: [Stanford 231n]()

Dealing with vectors

$$\frac{\partial z}{\partial x} = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_N}{\partial x_1} & \cdots & \frac{\partial z_N}{\partial x_M} \end{pmatrix}$$

$N \times M$

Jacobian



$1 \times M$

$1 \times N$

$$\frac{\partial e}{\partial x} = \frac{\partial e}{\partial z} \frac{\partial z}{\partial x}$$

$1 \times M$ $\quad$ $1 \times N N \times M$

$$\frac{\partial e}{\partial z}$$

$1 \times N$

Simple case: Elementwise operation

Simple case: Elementwise operation (ReLU layer)

$$\frac{\partial z}{\partial x} = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_M}{\partial x_1} & \cdots & \frac{\partial z_M}{\partial x_M} \end{pmatrix}$$

$M \times M$

Jacobian

x

$1 \times M$

$$f(x) = \max(0, x)$$

z

$1 \times M$

$$\frac{\partial e}{\partial z}$$

$1 \times M$

Simple case: Elementwise operation (ReLU layer)

$$\frac{\partial z}{\partial x} = \begin{pmatrix} \mathbb{I}[x_1 > 0] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{I}[x_M > 0] \end{pmatrix}$$

$M \times M$
Jacobian

x $\quad f(x) = \max(0, x) \quad$ z

$1 \times M$ $\qquad$ $1 \times M$

$$\frac{\partial e}{\partial z}$$

$1 \times M$

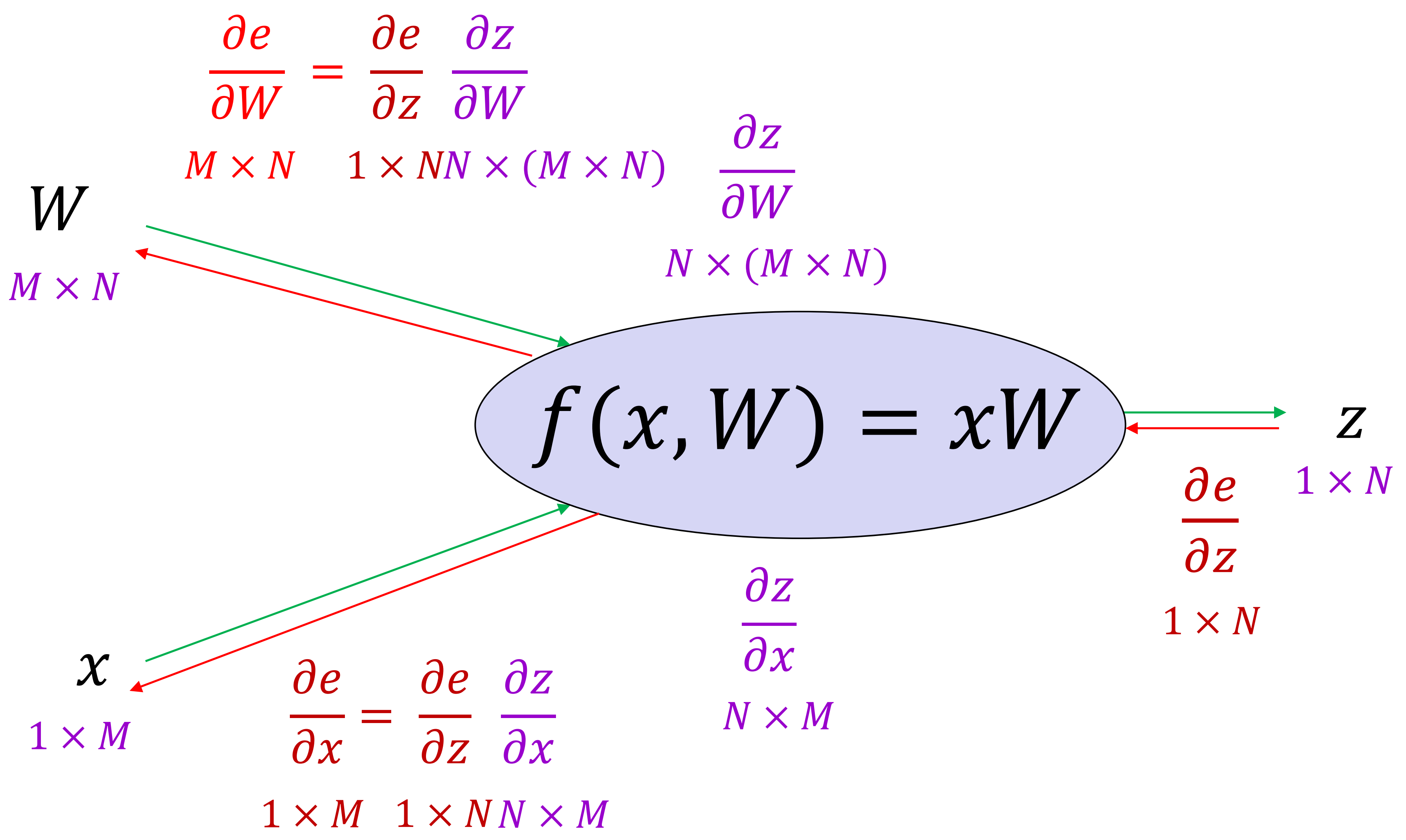
$$\frac{\partial e}{\partial x} = \frac{\partial e}{\partial z} \frac{\partial z}{\partial x}$$

$1 \times M \quad 1 \times M \quad M \times M$

$$\frac{\partial e}{\partial x_i} = \frac{\partial e}{\partial z_i} \mathbb{I}[x_i > 0]$$

$$\frac{\partial e}{\partial x} = \frac{\partial e}{\partial z} .* \ \mathbb{I}[x > 0]$$

Matrix-vector multiplication (linear layer)



Matrix-vector multiplication (linear layer)

$$(z_1 \quad \cdots \quad z_N) = (x_1 \quad \cdots \quad x_M)\begin{pmatrix} W_{11} & \cdots & W_{1N} \\ \vdots & \ddots & \vdots \\ W_{M1} & \cdots & W_{MN} \end{pmatrix} \qquad z_j = \sum_{i=1}^{M} x_i W_{ij}$$

Want: $\underset{1\times M}{\frac{\partial e}{\partial x}} = \underset{1\times N}{\frac{\partial e}{\partial z}} \underset{N\times M}{\boxed{\frac{\partial z}{\partial x}}}$

$$\frac{\partial z_j}{\partial x_i} =$$

jth row, ith column of Jacobian

$$\frac{\partial z}{\partial x} = W^T$$

Matrix-vector multiplication (linear layer)

$$(z_1 \quad \dots \quad z_N) = (x_1 \quad \dots \quad x_M)\begin{pmatrix} W_{11} & \dots & W_{1N} \\ \vdots & \ddots & \vdots \\ W_{M1} & \dots & W_{MN} \end{pmatrix}$$

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$$\frac{\partial z_j}{\partial x_i} = W_{ij}$$

jth row, ith column of Jacobian

$$\frac{\partial z}{\partial x} = W^T$$

$$\boxed{\frac{\partial e}{\partial x} = \frac{\partial e}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial e}{\partial z} W^T}$$

Matrix-vector multiplication (linear layer)

$$(z_1 \quad \cdots \quad z_N) = (x_1 \quad \cdots \quad x_M)\begin{pmatrix} W_{11} & \cdots & W_{1N} \\ \vdots & \ddots & \vdots \\ W_{M1} & \cdots & W_{MN} \end{pmatrix}$$

$$z_j = \sum_{i=1}^{M} x_i W_{ij}$$

Want: $\dfrac{\partial e}{\partial W} = \dfrac{\partial e}{\partial z} \boxed{\dfrac{\partial z}{\partial W}}$

$M \times N$ $\quad 1 \times N \quad N \times (M \times N)$

$$\frac{\partial z_k}{\partial W_{ij}}$$

z_k depends only on kth column of W

Matrix-vector multiplication (linear layer)

$$(z_1 \quad \cdots \quad z_N) = (x_1 \quad \cdots \quad x_M)\begin{pmatrix} W_{11} & \cdots & W_{1N} \\ \vdots & \ddots & \vdots \\ W_{M1} & \cdots & W_{MN} \end{pmatrix} \qquad z_j = \sum_{i=1}^{M} x_i W_{ij}$$

Want: $$\underset{M \times N}{\frac{\partial e}{\partial W}} = \underset{1 \times N}{\frac{\partial e}{\partial z}} \underset{N \times (M \times N)}{\boxed{\frac{\partial z}{\partial W}}}$$

$$\frac{\partial z_k}{\partial W_{ij}} = \mathbb{I}[k = j] x_i$$

z_k depends only on kth column of W

$$\frac{\partial e}{\partial W_{ij}} = \frac{\partial e}{\partial z} \frac{\partial z}{\partial W_{ij}}$$

Matrix-vector multiplication (linear layer)

$$(z_1 \quad \dots \quad z_N) = (x_1 \quad \dots \quad x_M)\begin{pmatrix} W_{11} & \dots & W_{1N} \\ \vdots & \ddots & \vdots \\ W_{M1} & \dots & W_{MN} \end{pmatrix} \qquad z_j = \sum_{i=1}^{M} x_i W_{ij}$$

Want: $\underset{M \times N}{\dfrac{\partial e}{\partial W}} = \underset{1 \times N \quad N \times (M \times N)}{\dfrac{\partial e}{\partial z} \boxed{\dfrac{\partial z}{\partial W}}}$

$$\frac{\partial z_k}{\partial W_{ij}} = \mathbb{I}[k = j] x_i$$

z_k depends only on kth column of W

$$\frac{\partial e}{\partial W_{ij}} = \frac{\partial e}{\partial z} \frac{\partial z}{\partial W_{ij}} = \sum_{k=1}^{N} \frac{\partial e}{\partial z_k} \frac{\partial z_k}{\partial W_{ij}} = \frac{\partial e}{\partial z_j} \frac{\partial z_j}{\partial W_{ij}} = \frac{\partial e}{\partial z_j} x_i$$

$$\boxed{\frac{\partial e}{\partial W} = x^T \frac{\partial e}{\partial z}}$$

General tips

- Derive error signal (upstream gradient) directly, avoid explicit computation of huge local derivatives
- Write out expression for a single element of the Jacobian, then deduce the overall formula
- Keep consistent indexing conventions, order of operations
- Use dimension analysis

- **For further reading:**
  - Lecture 4 of Stanford 231n
  - Yes you should understand backprop by Andrej Karpathy

Acknowledgement

- Deep Learning, Stanford University
- Introduction to Deep Learning, University of Illinois at Urbana-Champaign
- Introduction to Deep Learning, Carnegie Mellon University
- Convolutional Neural Networks for Visual Recognition, Stanford University
- Natural Language Processing with Deep Learning, Stanford University