2.1 Matrix Inverse

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

a. Find $A^{-1} = \text{inverse}(A)$

1st form the (block) matrix $M = [A, I]$ and row reduce M to echelon form:

$$M = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -2 \end{array}\right] \Rightarrow$$

In echelon form, the left half of M is in triangular form; hence A has an inverse. Next I further row reduce M to its row canonical form:

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

b. Solve $Ax = b$ using A^{-1}

$$Ax = b \Rightarrow x = A^{-1}b = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} (\frac{1}{3})(1) + (-\frac{1}{3})(0) + (\frac{1}{3})(1) \\ (-\frac{1}{3})(1) + (\frac{1}{3})(0) + (\frac{2}{3})(1) \\ (\frac{1}{3})(1) + (\frac{2}{3})(0) + (-\frac{1}{3})(1) \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

2.2 RANK

$$A = \begin{bmatrix} 2 & 2 & 6 \\ -1 & 1 & -1 \end{bmatrix}$$

a. Based only on the number of rows of A, what is the maximum the row rank of A can be?

2

b. Based only on the number of columns of A, what is the maximum the column rank of A can be?

3

c. Based only on the order (number of rows and columns) of A, What is the maximum the rank of A can be?

2 per page 20 of Lesson 2 Notes $\text{rank}(A) \leq \min(n, m)$

d. Find the row rank of A.

2

e. Find the column rank of A.

2 per page 20 of Lesson 2 notes (i.e. "the column rank will always be equal to the row rank")

f. What is rank(A)?

2

2.3 Generalized Inverse

$$A = \begin{bmatrix} 2 & 2 & 6 \\ -1 & 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a. Find a generalized inverse of A.

a right inverse of a non-square matrix A is given by $A_R^{-1} = A^T(AA^T)^{-1}$

so $A^T = \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 6 & -1 \end{bmatrix}$ and $AA^T = \begin{bmatrix} 2 & 2 & 6 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 6 & -1 \end{bmatrix}$

1st calculate $AA^T = \begin{bmatrix} 2 & 2 & 6 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 44 & -6 \\ -6 & 3 \end{bmatrix}$

2nd Calculate $(AA^T)^{-1}$

$$M = \left[\begin{array}{cc|cc} 44 & -6 & 1 & 0 \\ -6 & 3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} -6 & 3 & 0 & 1 \\ 44 & -6 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & -1/2 & 0 & -1/6 \\ 44 & -6 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & -1/2 & 0 & -1/6 \\ 0 & 16 & 1 & 44/6 \end{array}\right]$$

$$\Rightarrow M = \left[\begin{array}{cc|cc} 1 & 0 & 0.03125 & 0.0625 \\ 0 & 1 & 0.0625 & 0.458\overline{3333} \end{array}\right] \Rightarrow (AA^T)^{-1} = \begin{bmatrix} 0.03125 & 0.0625 \\ 0.0625 & 0.458\overline{3333} \end{bmatrix}$$

Finally $A^T(AA^T)^{-1} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 0.03125 & 0.0625 \\ 0.0625 & 0.458\overline{3333} \end{bmatrix}$

$$= \begin{bmatrix} 0 & -0.\overline{3333} \\ 0.125 & 0.58\overline{3333} \\ 0.125 & -0.08\overline{3333} \end{bmatrix}$$

2.3 Generalized Inverse

b. Find the generalized solution to $Ax = 0$

$\Rightarrow \quad x = A^{-1} \cdot \varnothing = \varnothing$

-OR- Since A^{-1} is a 3×2 matrix, x is a column vector

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

c. Find the general solution to $Ax = b$

$$\Rightarrow \quad x = A^{-1} \cdot b = \begin{bmatrix} \varnothing & -\varnothing.\overline{3333} \\ \varnothing.125 & \varnothing.58\overline{3333} \\ 0.125 & -\varnothing.\varnothing8\overline{3333} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} \varnothing \\ \varnothing.125 \\ \varnothing.125 \end{bmatrix}$$

2.4 Determinants

$$A = \begin{bmatrix} \varnothing & 1 & \varnothing \\ 2 & \varnothing & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

a. Find $\det(A)$

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$|A| = (0)(0)(1) - (0)(1)(2) - (1)(2)(1) + (1)(1)(1) + (0)(2)(2) - (0)(0)(1)$$

$$|A| = \varnothing - \varnothing - 2 + 1 + \varnothing - \varnothing$$

$$|A| = -1$$

b. Is A invertible?

1st Rearrange + form block matrix M

$$M = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow M = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] *$$

* Thus in echelon form the left half of M is triangular so A * has an inverse.

c. What is rank(A)?

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2.4 Determinants

d. What is $\det(A^t)$?

Since $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$|A^t| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$|A^t| = (0)(0)(1) - (0)(2)(1) - (2)(1)(1) + (2)(2)(0) + (1)(1)(1) - (1)(0)(0)$$

$$|A^t| = \emptyset - \emptyset - 2 + \emptyset + 1 - \emptyset$$

$$\boxed{|A^t| = -1}$$