3.1 Eigensets

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

a. Find $\det(A) \Rightarrow \det(A) = (1)(1) - (3)(3)$

$$= 1 - 9$$
$$= -8$$

b. Find $tr(A) \Rightarrow tr(A) = 1 + 1$

$= 2$ (i.e., sum of the diagonal elements of A)

c. Find the eigenvalues of A

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12})$$
$$= \lambda^2 - (1+1)\lambda + ((1)(1) - (3)(3))$$
$$= \lambda^2 - 2\lambda + (1 - 9)$$
$$= \lambda^2 - 2\lambda - 8$$

$\Rightarrow \lambda^2 - 2\lambda - 8 = \cancel{0} \Rightarrow (\lambda - 4)(\lambda + 2) = 0$

$\Rightarrow$ eigenvalues $= 4$ and -2

d. What is the relationship of det(A) to the eigenvalues of A?

The product of the eigenvalues $[(4) \times (-2) = -8]$ is equal to $\det(A) = (1)(1) - (3)(3) = -8$

e. What is the relationship of tr(A) to the eigenvalues of A?

The sum of the eigenvalues $[(4) + (-2) = 2]$ is equal to $tr(A) = 1 + 1 = 2$

3.1 Eigenset (continued)

f. Find the normalized eigenvectors corresponding to the eigenvalue?

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

* $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

* $A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix}$

* $\det \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - (3)(3) = 1 - 2\lambda + \lambda^2 - 9$

$= \lambda^2 - 2\lambda - 8$

* $\lambda^2 - 2\lambda - 8 = 0 \Rightarrow (\lambda - 4)(\lambda + 2) = 0 \Rightarrow \lambda_1 = 4$

$\lambda_2 = -2$

Solving for eigenvectors

For $\lambda_1 = 4$

$(A - \lambda_1 I)u = \emptyset \Rightarrow \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \emptyset$

$\Rightarrow \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{Bmatrix} -3x + 3y = 0 \\ 3x - 3y = 0 \end{Bmatrix} \; x = y$

Starting w/ $u_1 = [1 \;\; 1]^t$ and normalizing by dividing by $|u|$, the unit

* eigenvector $u_1 = \left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}} \right]^t = \left[\frac{1}{1.41421} \;\; \frac{1}{1.41421} \right]^t = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$

For $\lambda_2 = -2 \Rightarrow \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \left(-2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\Rightarrow \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{Bmatrix} 3x + 3y = 0 \\ 3x + 3y = 0 \end{Bmatrix} \; x = -y$

From $u_2 = [1 \;\; -1]^t \Rightarrow u_2 = \left[\frac{1}{\sqrt{2}} \;\; \frac{-1}{\sqrt{2}} \right]^t = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$

*

3.2 Eigenset of a 3x3 exchangeable correlation matrix

$$A = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix}$$

a. Find det(A) $\Rightarrow$

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= (1)(1)(1) - (1)(0.2)(0.2) - (0.2)(0.2)(1) + (0.2)(0.2)(0.2) + (0.2)(0.2)(0.2) - (0.2)(1)(0.2)$$

$$= 1 - 0.04 - 0.04 + 0.008 + 0.008 - 0.04$$

$$\ast \quad = 0.896$$

b. The eigenvalues of A are 1.4, 0.8 and 0.8. Show they satisfy the characteristic equation for A.

$$A - \lambda I = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 0.2 & 0.2 \\ 0.2 & (1-\lambda) & 0.2 \\ 0.2 & 0.2 & (1-\lambda) \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda)(0.2)(0.2) - (0.2)(0.2)(1-\lambda) + (0.2)(0.2)(0.2) + (0.2)(0.2)(0.2) - (0.2)(1-\lambda)(0.2)$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 - (0.04)(1-\lambda) - (0.04)(1-\lambda) + 0.008 + 0.008 - 0.04(1-\lambda)$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 - \left[3(0.04)(1-\lambda)\right] + 0.016$$

$$= 1.016 - 3\lambda + 3\lambda^2 - \lambda^3 - \left[0.12(1-\lambda)\right] = 1.016 - 3\lambda + 3\lambda^2 - \lambda^3 - 0.12 + 0.12\lambda$$

$$= 0.896 - 3\lambda + 3\lambda^2 - \lambda^3 + 0.12\lambda = 0.896 - 2.88\lambda + 3\lambda^2 - \lambda^3 \ \ast$$

For eigenvalue of 1.4

$$0.896 - 2.88\lambda + 3\lambda^2 - \lambda^3 = 0 \Rightarrow 0.896 - 2.88(1.4) + 3(1.4)^2 - (1.4)^3$$

$$= 0.896 - 4.032 + 5.88 - 2.744$$

$$\ast \quad = 0$$

For either/both eigenvalue of 0.8

$$0.896 - 2.88(0.8) + 3(0.8)^2 - (0.8)^3 = 0$$

$$0.896 - 2.304 + 1.92 - 0.512 = 0 \quad \ast$$

3.2 Eigenset of a 3×3 exchangeable correlation matrix (continued)

c. What is the relationship of det (A) to the eigenvalues of A?

The product of the eigenvalues of A $[(1.4)\times(0.8)\times(0.8)=0.896]$

is equal to $\det(A)=0.896$ as shown in 3.2 part a.

d. If $v=\begin{bmatrix}0.57735\\0.57735\\0.57735\end{bmatrix}$ Show that v is a normalized ($|v|=1$) eigenvector corresponding to the eigenvalue 1.4

Verify $|v|^2=v^t v=1$

$$\Rightarrow \begin{bmatrix}0.57735 & 0.57735 & 0.57735\end{bmatrix}\begin{bmatrix}0.57735\\0.57735\\0.57735\end{bmatrix}$$

$$=(0.57735)(0.57735)+(0.57735)(0.57735)+(0.57735)(0.57735)$$

$$=0.\overline{33333}+0.\overline{33333}+0.\overline{33333}$$

$$* \quad =1.0$$

verify $Av-\lambda v=0$

$$\Rightarrow Av-1.4v=0 \Rightarrow \begin{bmatrix}1 & 0.2 & 0.2\\0.2 & 1 & 0.2\\0.2 & 0.2 & 1\end{bmatrix}\begin{bmatrix}0.57735\\0.57735\\0.57735\end{bmatrix}-1.4\begin{bmatrix}0.57735\\0.57735\\0.57735\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}(1)(0.57735)+(0.2)(0.57735)+(0.2)(0.57735)\\(0.2)(0.57735)+(1)(0.57735)+(0.2)(0.57735)\\(0.2)(0.57735)+(0.2)(0.57735)+(1)(0.57735)\end{bmatrix}-\begin{bmatrix}0.80829\\0.80829\\0.80829\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0.80829\\0.80829\\0.80829\end{bmatrix}-\begin{bmatrix}0.80829\\0.80829\\0.80829\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

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