4.1 Eigenset $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

a. Find det(A) $\Rightarrow$ $\det(A) = (3)(3) - (2)(2) = 9 - 4 = 5$

b. Find the eigenvalue of A.

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}$$
$$= a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{21}a_{12}$$
$$= \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{21}a_{12})$$
$$= \lambda^2 - (3+3)\lambda + ((3)(3) - (2)(2))$$
$$= \lambda^2 - 6\lambda + 5$$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda - 1)(\lambda - 5) = \emptyset$

$\Rightarrow$ eigenvalue are 1, 5

c. What is the relationship of det(A) to the eigenvalue of A?

The product of the eigenvalue of A $[1 \times 5 = 5]$ is equal to the $\det(A) = (3)(3) - (2)(2) = 5$

d. Is A positive definite, positive semi-definite or non-negative definite or none of these?

A is "positive definite" because it is a symmetric matrix in which all of its eigenvalue (i.e. $\lambda = 1, 5$) are greater than zero.

4.1 Eigenset (continued)

e. Find the normalized eigenvectors corresponding to the eigenvalue.

For $\lambda_1 = 1$: $(A-\lambda_1 I)u = \emptyset = \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \emptyset$

$$\Rightarrow \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \emptyset \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix}$$

$$\Rightarrow \left\{\begin{matrix} 2x+2y=\emptyset \\ 2x+2y=\emptyset \end{matrix}\right\} \quad x=-y$$

From $u_1 = [1 \ -1]^t$ and normalizing by dividing by $|u|$, the unit eigenvector $u_1 = \left[\frac{1}{\sqrt{2}} \ \frac{-1}{\sqrt{2}}\right]^t = \left[\frac{1}{1.41421} \ \frac{-1}{1.41421}\right]^t$

$$u_1 = \begin{bmatrix} \emptyset.7\emptyset711 \\ -\emptyset.7\emptyset711 \end{bmatrix}$$

For $\lambda_2 = 5$: $(A-\lambda_2 I)u = \emptyset \Rightarrow \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \emptyset$

$$\Rightarrow \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{matrix} -2x+2y=\emptyset \\ 2x-2y=\emptyset \end{matrix}\right\} \quad x=y$$

From $u_2 = [1 \ 1]^t$ and normalizing by dividing by $|u|$, the unit eigenvector $u_2 = \left[\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right]^t = \left[\frac{1}{1.41421} \ \frac{1}{1.41421}\right]^t$

$$u_2 = \begin{bmatrix} \emptyset.7\emptyset711 \\ \emptyset.7\emptyset711 \end{bmatrix}$$

4.1 Eigenset (continued)

f. Show how A is formed by its spectral decomposition,

Knowing the unit eigenvectors v_1 and v_2, I find find the eigenvector matrices

$$E_1 = v_1 v_1^t = \begin{bmatrix} 0.70711 \\ -0.70711 \end{bmatrix} \begin{bmatrix} 0.70711 & -0.70711 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50000455 & -0.50000455 \\ -0.50000455 & 0.50000455 \end{bmatrix} \approx \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$E_2 = v_2 v_2^t = \begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix} \begin{bmatrix} 0.70711 & 0.70711 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50000455 & 0.50000455 \\ 0.50000455 & 0.50000455 \end{bmatrix} \approx \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$A = \lambda_1 v_1 v_1^t + \lambda_2 v_2 v_2^t = \lambda_1 E_1 + \lambda_1 E_2 \quad \text{where } \lambda_1 = 1, \lambda_2 = 5$$

$$A = 1 \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} + 5 \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} + \begin{bmatrix} 2.5 & 2.5 \\ 2.5 & 2.5 \end{bmatrix}$$

$$* \quad A = \begin{bmatrix} 3.0 & 2.0 \\ 2.0 & 3.0 \end{bmatrix}$$

Thus A is equal to its spectral decomposition

4.2 Eigenset of a 3x3 exchangeable correlation matrix

$$A = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix}$$ with eigenvalues $\lambda = 1.4, 0.8, 0.8$

a. Is A positive definite, positive semi-definite or non-negative definite or none of these?

A is "positive definite" because it is a symmetric matrix in which all of its eigenvalues (i.e. $\lambda = 1.4, 0.8, 0.8$) are greater than zero.

b. The normalized eigenvectors corresponding to the eigenvalues 1.4, 0.8 and 0.8, respectively, are

$$V_1 = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0.66057 \\ 0.08532 \\ -0.74590 \end{bmatrix} \quad V_3 = \begin{bmatrix} 0.47991 \\ -0.81203 \\ 0.33212 \end{bmatrix}$$

Show how A is formed by its spectral decomposition

Using equation XIX.10 $A = \lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t + \lambda_3 u_3 u_3^t$

where $\lambda_1 = 1.4$, $\lambda_2 = 0.8$ and $\lambda_3 = 0.8$ and

where $u_1 = V_1$, $u_2 = V_2$ and $u_3 = V_3$

$$E_1 = V_1 V_1^t = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix} \begin{bmatrix} 0.57735 & 0.57735 & 0.57735 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.\overline{33333} & 0.\overline{33333} & 0.\overline{33333} \\ 0.\overline{33333} & 0.\overline{33333} & 0.\overline{33333} \\ 0.\overline{33333} & 0.\overline{33333} & 0.\overline{33333} \end{bmatrix}$$

$$E_2 = V_2 V_2^t = \begin{bmatrix} 0.66057 \\ 0.08532 \\ -0.74590 \end{bmatrix} \begin{bmatrix} 0.66057 & 0.08532 & -0.74590 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0.43635272 & 0.05635983 & -0.49271916 \\ 0.05635983 & 0.00772793 & -0.06364019 \\ -0.49271916 & -0.06364019 & 0.55636681 \end{bmatrix}$$

4.2 Eigenset of a 3x3 exchangeable correlation matrix (continued)

$$E_3 = V_3 V_3^* = \begin{bmatrix} 0.47991 \\ -0.81203 \\ 0.33212 \end{bmatrix} \begin{bmatrix} 0.47991 & -0.81203 & 0.33212 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0.23031361 & -0.38970132 & 0.1593877 1 \\ -0.38970132 & 0.65939272 & -0.2696914 \\ 0.15938771 & -0.2696914 & 0.11030369 \end{bmatrix}$$

$$A = \lambda_1 v_1 v_1^* + \lambda_2 v_2 v_2^* + \lambda_3 v_3 v_3^* = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

$$\Rightarrow \lambda_1 E_1 = 1.4 \begin{bmatrix} 0.\overline{33333} & 0.\overline{33333} & 0.\overline{33333} \\ 0.\overline{33323} & 0.\overline{33333} & 0.\overline{33333} \\ 0.\overline{33333} & 0.\overline{33333} & 0.\overline{33333} \end{bmatrix}$$

$$= \begin{bmatrix} 0.4666623 & 0.46666623 & 0.46666623 \\ 0.46666623 & 0.466623 & 0.46666623 \\ 0.46666623 & 0.46666623 & 0.46666623 \end{bmatrix}$$

$$\Rightarrow \lambda_2 E_2 = 0.8 \begin{bmatrix} 0.43635272 & 0.05635983 & -0.49271916 \\ 0.05635983 & 0.00072795 & -0.06364019 \\ -0.49271916 & -0.06364019 & 0.55636681 \end{bmatrix}$$

$$= \begin{bmatrix} 0.34908218 & 0.04508787 & -0.39417533 \\ 0.04508787 & 0.00058236 & -0.05091215 \\ -0.39417523 & -0.05091215 & 0.44509345 \end{bmatrix}$$

$$\Rightarrow \lambda_3 E_3 = 0.8 \begin{bmatrix} 0.23031361 & -0.38970132 & 0.15938771 \\ -0.38970132 & 0.65939272 & -0.2696914 \\ 0.15938771 & -0.2696914 & 0.11030369 \end{bmatrix}$$

$$= \begin{bmatrix} 0.18425089 & -0.31176105 & 0.12751017 \\ -0.31176105 & 0.52751418 & -0.21575312 \\ 0.12751017 & -0.21575312 & 0.08824291 \end{bmatrix}$$

4.2 Eigenset of a 3x3 exchangeable correlation matrix (continued)

$$A = \begin{bmatrix} 0.4666623 & 0.4666623 & 0.4666623 \\ 0.4666623 & 0.4666623 & 0.4666623 \\ 0.4666623 & 0.4666623 & 0.4666623 \end{bmatrix} + \begin{bmatrix} 0.37408218 & 0.04508787 & -0.39417537 \\ 0.04508787 & 0.40558236 & -0.05091215 \\ -0.39417523 & -0.05091215 & 0.44509345 \end{bmatrix} + \begin{bmatrix} 0.18425087 & -0.31176105 & 0.12751017 \\ -0.31176105 & 0.52751418 & -0.21575312 \\ 0.12751017 & -0.21575312 & 0.08824216 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.9999993 & 0.19999304 & 0.20000107 \\ 0.19999304 & 1.00000401 & 0.20000096 \\ 0.20000107 & 0.20000096 & 1.00000264 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix}$$

Thus A is equal to its spectral decomposition