PROBLEM 4.1: SPECTRAL DECOMPOSITION OF A 2 x 2 MATRIX

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

a. det(**A**) = (3)(3) - (2)(2) = 9 - 4 = 5. tr(**A**) = 3 + 3 = 6.

b. Eigenvalues:

$\det(\mathbf{A} - \lambda\,\mathbf{I}) = (3-\lambda)\,(3-\lambda) - 4 = 0$

$\lambda^2 - 6\,\lambda + 9 - 4 = 0$

$\lambda^2 - 6\,\lambda + 5 = 0 = (\lambda - 5)\,(\lambda - 1)$ => eigenvalues $\lambda = 5, 1$

c. det(**A**) = 5 = (5)(1) = product of eigenvalues. tr(A) = 6 = (5) + (1) = sum of eigenvalues.

d. A is positive definite, because all its eigenvalues are greater than 0.

e. Eigenvectors:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow 3x + 2y = 5x \Rightarrow 2x - 2y = 0 \Rightarrow x = y$$

$$\mathbf{v}_1 = \begin{bmatrix} x \\ x \end{bmatrix} \quad \Rightarrow \quad \mathbf{e}_1 = \begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix} \text{ when normalized}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow 3x + 2y = 1x \Rightarrow 2x + 2y = 0 \Rightarrow x = -y$$

$$\mathbf{v}_2 = \begin{bmatrix} x \\ -x \end{bmatrix} \quad \Rightarrow \quad \mathbf{e}_2 = \begin{bmatrix} 0.70711 \\ -0.70711 \end{bmatrix} \text{ when normalized}$$

f. Spectral decomposition:

$$5\,\mathbf{e}_1\,\mathbf{e}_1^t + 1\,\mathbf{e}_2\,\mathbf{e}_2^t = 5\begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix}[0.70711\ 0.70711] +$$

$$1\begin{bmatrix} 0.70711 \\ -0.70711 \end{bmatrix}[0.70711\ -0.70711]$$

$$= \quad 5\begin{bmatrix} 0.500005 & 0.500005 \\ 0.500005 & 0.500005 \end{bmatrix} + 1\begin{bmatrix} 0.500005 & -0.500005 \\ -0.500005 & 0.500005 \end{bmatrix}$$

$$= \quad \begin{bmatrix} 2.500025 & 2.500025 \\ 2.500025 & 2.500025 \end{bmatrix} + \begin{bmatrix} 0.500005 & -0.500005 \\ -0.500005 & 0.500005 \end{bmatrix}$$

$$= \quad \begin{bmatrix} 3.0000 & 2.0000 \\ 2.0000 & 3.0000 \end{bmatrix}$$

$$= \quad \mathbf{A}$$