CSC236 Winter 2017
Assignment #1: Induction - Sample Solutions
Due February 1st, by 9:00 pm

The aim of this assignment is to give you some practice with various forms of induction. For each question below you will present a proof by induction, using the type of induction specified. For full marks on your proofs, you will need to make it clear to the reader that the base case(s) is/are verified, that the inductive step follows for each element of the domain (typically the natural numbers), where the inductive hypothesis is used and that it is used in a valid case.

Your assignment must be typed to produce a PDF document **a1.pdf** (hand-written submissions are not acceptable). You may work on the assignment in groups of 1, 2, or 3, and submit a single assignment for the entire group on MarkUs.

1. Consider the Fibonacci function f:

$$f(n) = \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1 \\ f(n-2) + f(n-1) & \text{if } n > 1 \end{cases}$$

Use simple induction to prove that if n is a natural number, then $f(0) + f(2) + \cdots + f(2n) = f(2n+1)$.

You may **not** derive or use a closed-form for $f(n)$ in your proof.

Sample solution: Proof, using simple induction.

Inductive step: Let $n \in \mathbb{N}$. Assume $H(n) : f(0) + f(2) + \ldots + f(2n) = f(2n+1)$.

Show that $H(n) \implies C(n)$: $f(0) + f(2) + \ldots + f(2(n+1)) = f(2(n+1)+1)$.

$f(0) + f(2) + \ldots + f(2(n+1))$
$= f(0) + f(2) + \ldots + f(2n) + f(2(n+1))$
$= f(2n+1) + f(2(n+1))$ (by $H(n)$)
$= f(2n+1) + f(2n+2) = f(2n+3)$ (by definition)
$= f(2(n+1)+1$

$C(n)$ follows from our assumptions in this case.

Base cases: $n \in \{0, 1\}$

Let $n = 0$. $f(0) = f(2 * 0 + 1)$. So the claim holds for $n = 0$.

Conclude: $f(0) + f(2) + \ldots + f(2n) = f(2n+1)$, $\forall n \in \mathbb{N}$.

2. Use simple induction to show that $x^2 - 1$ is divisible by 8 for any odd natural number x.

   Sample solution: Proof, using simple induction.

   Inductive step: Let $n \in \mathbb{N}$ ($2n+1$ is the n^{th} odd number).

   Assume $H(n) : ((2n+1)^2 - 1$ is a multiple of 8.

   Show that $H(n) \to C(n) : ((2(n+1)+1)^2 - 1$ is a multiple of 8.

   $((2(n+1)+1)^2 - 1 = (2n+3)^2 - 1 = 4n^2 + 12n + 8 = 4n^2 + 4n + 8n + 8$

   $(2n+1)^2 - 1 = 4n^2 + 4$, and $(2n+1)^2 - 1 = 8 * k$, for some $k \in \mathbb{N}$, by $H(n)$

   So, $4n^2 + 4n + 8n + 8 = 8k + 8n + 8 = 8(k + n + 1)$

   Since $1, k, n \in \mathbb{N}$, then $((2(n+1)+1)^2 - 1$ is a multiple of 8.

   $C(n)$ follows from our assumptions in this case.

   Base case: Let $n = 0$. $(2 * 0 + 1)^2 - 1 = 0$, a multiple of 8. So the claim holds for $n = 0$.

   Conclude: $x^2 - 1$ is a multiple of 8, for all odd $x \in \mathbb{N}$.

3. Use the Well-Ordering Principle to show that given any natural number $n \geq 1$, there exists an odd integer m and a natural number k such that $n = 2^k * m$.

   Sample solution: Proof, using well-ordering.

   Let $S = \{r \in \mathbb{Z} : \exists\, i \in \mathbb{N} \text{ s.t. } n = 2^i * r\}$

   S is non-empty, because when $i = 0$, $2^0 * n = n$, and so $n \in S$.

   $n \geq 1$ and $2^i \geq 1$, so $r \geq 1$.

   Thus, S is a non-empty subset of $\mathbb{N}$.

   By Well Ordering, S has a smallest element m', and we know $n = 2^k * m'$ for some $k \in \mathbb{N}$.

   Claim: m' is odd

   Proof, by contradiction: Assume m' is even. i.e., $m' = 2p$, $p \in \mathbb{N}$.

   Then, $2^k * m' = 2^k * 2p = 2^{k+1} * p$.

   This means $p \in S$ and $p < m'$, and contradicts the choice of m' as smallest.

   Conclude: For $n \in \mathbb{N}, n \geq 1$, there exists an odd integer m and a natural number k s.t. $n = 2^k * m$.

4. Define a set $M \subseteq \mathbb{Z}^2$ as follows:

   (a) $(3,2) \in M$,

   (b) for all $(x,y) \in M$, $(3x-2y, x) \in M$,

   (c) nothing else belongs to M.

   Use structural induction to prove that $\forall (x,y) \in M, \exists k \in \mathbb{N}, (x,y) = (2^{k+1}+1, 2^k+1)$.

   Sample solution: Proof, by structural induction on M.

   $P(x,y)$: $\exists k \in \mathbb{N}, (x,y) = (2^{k+1}+1, 2^k+1)$.

   Inductive step: Let $(x,y) \in M$. Assume $H((x,y)) : P((x,y))$

   Show $H((x,y)) \rightarrow C((x,y))$: $P((3x-2y, x))$

   Let $k' \in \mathbb{N}$ be such that $(x,y) = (2^{k'+1}+1, 2^{k'}+1)$, by $H((x,y))$

   Then
$$\begin{aligned}(3x-2y, x) &= \left(3(2^{k'+1}+1) - 2(2^{k'}+1), 2^{k'+1}+1\right)\\ &= \left(6 \cdot 2^{k'} + 3 - 2 \cdot 2^{k'} - 2, 2^{k'+1}+1\right)\\ &= \left(4 \cdot 2^{k'} + 1, 2^{k'+1}+1\right)\\ &= \left(2^{k'+2}+1, 2^{k'+1}+1\right)\end{aligned}$$

   So $P(3x-2y, x)$, with $k = k'+1$.

   Base Case: $(3,2) = (2^1+1, 2^0+1)$ so $P(3,2)$, with $k=0$.

   Conclusion: $\forall (x,y) \in M, \exists k \in \mathbb{N}, (x,y) = (2^{k+1}+1, 2^k+1)$.

5. Suppose n people are positioned such that each person has a unique nearest neighbour. Each person has a single water balloon that they throw at their nearest neighbour. (We'll assume every throw hits its target.) A dry person is one who is not hit by a water balloon.

   (a) Describe an example that demonstrates than if n is even, there may be no dry person.

   (b) Use simple induction to show that if n is odd, then there is always at least one dry person.

Sample solution:

(a): Consider any example with $n = 2$. Then each person is the other's unique nearest neighbour, and neither is dry.

(b): Proof by simple induction.

$P(n)$: There is one dry person when $2n+1$ people are positioned such that they each have a unique nearest neighbour, and they each throw a water balloon at their nearest neighbour.

We want to show that $\forall n \in \mathbb{N}, n \geq 1,\ P(n)$, since there is no game if there is only 1 person (i.e., when $n = 0$).

Inductive step: Let $n \in \mathbb{N}$, $n \geq 1$.

Assume $H(n) : P(n)$

Show that $H(n) \rightarrow C(n) : P(n+1)$.

Assume that there are $2(n+1)+1 = 2n+3$ people, positioned so they each have a unique nearest neighbour. Let A and B be the closest pair among the $2k+3$ people. So, A and B will throw balloons at each other.

Case 1: Someone other than A or B throws a balloon at either A or B.
Then, at least 3 balloons are thrown at A and B, which leaves $2n$ balloons to be thrown at the remaining $2n+1$ people. Thus, at least one person is dry.

Case 2: No one else throws a balloon at A or B.
Then, the remaining $2n+1$ people are positioned so that they each have a unique nearest neighbour. By $H(n)$, there is at least one dry person among those $2n+1$ people.

$C(n)$ follows from our assumptions.

Base case: Let $n = 1$. There are $2 * 1 + 1 = 3$ people in the game, A, B, and C. Assume A and B are the closest pair. (Note that there cannot be a tie, because each person has a unique nearest neighbour.) The distances between A and C and B and C are greater than the distance between A and B. So, A and B throw balloons at each other, and C throws a balloon at either A or B. Thus, C is dry, and $P(1)$ holds.

Conclude: $\forall n \in \mathbb{N}, n \geq 1,\ P(n)$.

6. Let P be a convex polygon with consecutive vertices $v_1, v_2, ..., v_n$. Use complete induction to show that when P is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, ..., n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., n-2$.

Sample solution: Proof, using complete induction.

Define $P(n)$: A triangulation into $n-2$ triangles of a convex polygon with consecutive vertices $v_1, ..., v_n$ can have its triangles labelled $1, ..., n-2$ such that v_i is a vertex of triangle i.

Inductive step: Let $n \in \mathbb{N}$, $n \geq 3$.

Assume $H(n) : \forall j \in \mathbb{N}, 3 \leq j < n, P(j)$.

Show that $H(n) \rightarrow C(n) : P(n)$

Let T be a triangulation of a polygon with consecutive vertices $v_1, ..., v_n$.

Every T contains a diagonal from v_{n-1} or from v_n. (Otherwise, it is not a triangulation.)

Case: There is a diagonal from v_n in T.

Choose k s.t. there is a diagonal from v_k to v_n in T.

This diagonal divides the polygon into two sub-polygons, P_1 and P_2. P_1 has the vertices $v_1, ..., v_k, v_n$, and P_2 has the vertices $v_k v_{k+1}, ..., v_n$.

Rename the vertex v_n of P_1 as v_{k+1}. Rename each of the vertices of P_2, $v_i' = v_i - k + 1$.

By $H(n)$, P_1 triangulates with triangles numbered as claimed, because $k \leq n-2 < n$.

P_2 has $n-k+1$ vertices, so by $H(n)$, P_2 also triangulates as claimed.

Add $k-1$ to each triangle number in P_2. The original vertices v_i are now part of the triangle numbered i, $i \in k+1, ..., n-2$.

Case: There is a diagonal from v_{n-1} in T.

Choose k s.t. there is a diagonal from v_k to v_{n-1} in T.

This diagonal divides the polygon into two sub-polygons, P_1 and P_2. P_1 has the vertices $v_1, ..., v_k, v_{n-1}, v_n$, and P_2 has the vertices $v_k v_{k+1}, ..., v_{n-1}$.

Rename the vertices v_{n-1}, v_n of P_1 as v_{k+1}, v_{k+2}. Rename each of the vertices of P_2, $v_i' = v_i - k + 1$.

By $H(n)$, P_1 triangulates with triangles numbered as claimed, because $k \leq n-3 < n$.

P_2 has $n-k$ vertices, so by $H(n)$, P_2 also triangulates as claimed.

Add $k-1$ to each triangle number in P_2. The original vertices v_i are now part of the triangle numbered i, $i \in k+1, ..., n-2$.

$C(n)$ follows from our assumptions in this case.

Base case: Let $n = 3$. The vertices are numbered, $1, 2, 3$, and the one triangle can be numbered 1.

Conclude: $P(n)$ holds $\forall n \in \mathbb{N}$, $n \geq 3$