CSC236 Tutorial Exercises
Sample Solutions

1. Consider the recurrence relation

$$T(n) = \begin{cases} 1 & n = 1 \\ 1 + T\left(\left\lceil \frac{n}{3} \right\rceil\right) & n > 1 \end{cases}$$

Use complete induction to prove that for every positive natural number n, $T(n) \geq c\, lg(n)$, for some positive real constant c.

Sample solution.

Basis step:

$T(1) = 1 \geq c\, lg(1) = 0$

Inductive step:

Assume $T(i) \geq c \lg i$ for $1 \leq i < k$ and an arbitrary $k > 1$. We must show $T(k) \geq c \lg k$.

$T(k) = 1 + T\left(\left\lceil \frac{k}{3} \right\rceil\right)$ *by definition of* T *when* $k > 1$

$\Rightarrow T(k) \geq 1 + c \lg \left\lceil \frac{k}{3} \right\rceil$ *by I.H., since* $1 \leq \left\lceil \frac{k}{3} \right\rceil < k$ *when* $k > 1$

$\Rightarrow T(k) \geq 1 + c \lg \frac{k}{3}$ *since* $\frac{k}{3} \leq \left\lceil \frac{k}{3} \right\rceil$ *and lg is increasing*

$\Rightarrow T(k) \geq 1 + c \lg k - c \lg 3$

$\Rightarrow T(k) \geq c \lg k$ *provided* $1 - c \lg 3 \geq 0 \Rightarrow \frac{1}{\lg 3} \geq c \Rightarrow 0.63 \geq c$ □

2. Consider the recurrence relation

$$T(n) = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 3T(n-1) - 2T(n-2) & n > 1 \end{cases}$$

Find a closed form for $T(n)$, and prove that it is correct using induction.

$$\begin{aligned} T(n) &= \mathbf{3T(n-1) - 2T(n-2)} \\ &= 3\big(3T(n-2) - 2T(n-3)\big) - 2T(n-2) = \mathbf{7T(n-2) - 6T(n-3)} \\ &= 7\big(3T(n-3) - 2T(n-4)\big) - 6T(n-3) = \mathbf{15T(n-3) - 14T(n-4)} \end{aligned}$$

...

$= \mathbf{(2^{k+1} - 1)T(n-k) - (2^{k+1} - 2)T(n-k-1)}$

We must choose (*i.e.*, continue unwinding to) a k such that $T(n-k)$ and $T(n-k-1)$ can be replaced by base cases, *i.e.*, $T(1)$ and $T(0)$. Hence, $\mathbf{k = n - 1}$.

$= \mathbf{(2^n - 1)T(1) - (2^n - 2)T(0)}$

$= \mathbf{(2^n - 1).3 - (2^n - 2).1 = 3.2^n - 3 - 2^n + 2 = 2^{n+1} - 1}$

Now, we use complete induction to prove $T_r(n) = T_c(n)$, where

$$T_r(n) = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 3T_r(n-1) - 2T_r(n-2) & n > 1 \end{cases} \qquad \text{and} \qquad T_c(n) = 2^{n+1} - 1$$

Basis step:

$$T_r(0) = 1 = T_c(0) = 2^{0+1} - 1 = 1$$

$$T_r(1) = 3 = T_c(1) = 2^{1+1} - 1 = 3$$

Inductive step:

Assume $T_r(i) = T_c(i)$ for $0 \le i < k$ and an arbitrary $k > 1$. We must show $T_r(k) = T_c(k)$.

$$\begin{aligned}
T_r(k) &= 3T_r(k-1) - 2T_r(k-2) && \text{by definition of } T_r \text{ when } k > 1 \\
&= 3T_c(k-1) - 2T_c(k-2) && \text{by I.H, since } 0 \le k-1 < k \text{ when } k > 1 \text{ and} \\
& && \qquad 0 \le k-2 < k \text{ when } k > 1 \\
&= 3(2^{k-1+1} - 1) - 2(2^{k-2+1} - 1) && \text{by definition of } T_c \\
&= 3\,.\,2^k - 3 - 2\,.\,2^{k-1} + 2 \\
&= 3\,.\,2^k - 2^k - 1 \\
&= 2\,.\,2^k - 1 \\
&= 2^{k+1} - 1 \\
&= T_c(k)
\end{aligned}$$

$\square$

3. Consider another recurrence relation

$$T(n) = \begin{cases} 1 & n = 0 \\ T(n-1) + n - 2 & n > 0 \end{cases}$$

Unwind the recurrence **carefully**, following the pattern below, for some n that is comfortably greater than 1:

$$\begin{aligned} T(n) &= T(n-1) + n - 2 \\ &= T(n-2) + n - 1 - 2 + n - 2 = T(n-2) + 2n - 5 \\ &= T(n-3) + n - 2 - 2 + 2n - 5 = T(n-3) + 3n - 9 \end{aligned}$$

Continue to see a pattern that leads to a guess at a closed form for $T(n)$.

Sample solution 1.

$$\begin{aligned} T(n) &= \boldsymbol{T(n-1) + n - 2} \\ &= T(n-2) + n - 1 - 2 + n - 2 = \boldsymbol{T(n-2) + 2n - 5} \\ &= T(n-3) + n - 2 - 2 + 2n - 5 = \boldsymbol{T(n-3) + 3n - 9} \\ &= T(n-4) + n - 3 - 2 + 3n - 9 = \boldsymbol{T(n-4) + 4n - 14} \end{aligned}$$

So far, we see the pattern, in a general case, is like

$$T(n) = T(n-k) + kn - \cdots \qquad (*)$$

The question is, in (*), how to express the last term (i.e., 2, 5, 9, 14, etc.) in terms of k. See the following table:

k	***value***
1	2
2	5
3	9
4	14
...	...

We need to manipulate the ***value*** , rewrite it using the corresponding *k*. There are different ways to do so. One is as follows:

k	**value**	
1	2	=1+1
2	5	=2+3
3	9	=3+6
4	14	=4+10
...	...	...

Now the question is what is the relationship of 1,3, 6, 10, etc., with their corresponding k (i.e., 1, 2, 3, 4, etc.)

k	**value**		
1	2	=1+1	=1+(1*2)/2
2	5	=2+3	=2+(2*3)/2
3	9	=3+6	=3+(3*4)/2
4	14	=4+10	=4+(4*5)/2
...	...	..	...

We found the pattern for the last term: *k*+(*k**(*k*+1))/2. So, we can write the (*) as follows:

$$T(n) = T(n-k) + kn - (k + k * (k+1)/2)$$

We must choose (*i.e.*, continue unwinding to) a k such that $T(n-k)$ can be replaced by the base cases, $T(0)$. Hence, $\boldsymbol{k = n}$. Therfore,

$$T(n) = T(n-n) + n.n - (n + \tfrac{n(n+1)}{2})$$
$$T(n) = T(0) + n^2 - (n + \tfrac{n^2+n}{2})$$
$$T(n) = 1 + n^2 - n - \tfrac{n^2+n}{2}$$
$$T(n) = \tfrac{2+2n^2-2n-n^2-n}{2}$$
$$T(n) = \tfrac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}$$

Sample solution 2.

Sometimes, we could faster find the pattern if we do not simplify the unwinding relations a lot:

$$\begin{aligned}
T(n) &= \boldsymbol{T(n-1) + n - 2} \\
&= T(n-2) + n - 1 - 2 + n - 2 = \boldsymbol{T(n-2) + 2n - 2 - 3} \\
&= T(n-3) + n - 2 - 2 + 2n - 2 - 3 = \boldsymbol{T(n-3) + 3n - 2 - 3 - 4} \\
&= T(n-4) + n - 3 - 2 + 3n - 2 - 3 - 4 = \boldsymbol{T(n-4) + 4n - 2 - 3 - 4 - 5} \\
&\dots \\
&= \boldsymbol{T(n-k) + kn - 2 - 3 - 4 - 5 - \cdots (k+1)} \\
&= \boldsymbol{T(n-k) + kn - \frac{(k+1)(k+2)}{2} + 1}
\end{aligned}$$

We must choose (*i.e.*, continue unwinding to) a k such that $T(n-k)$ can be replaced by the base cases, $T(0)$. Hence, $\boldsymbol{k = n}$.

$$\begin{aligned}
&= T(n-n) + nn - \frac{(n+1)(n+2)}{2} + 1 \\
&= T(0) + n^2 - \frac{(n+1)(n+2)}{2} + 1 \\
&= 1 + n^2 - \frac{n^2+3n+2}{2} + 1 \\
&= \tfrac{2n^2-n^2-3n-2+4}{2} \\
&= \tfrac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}
\end{aligned}$$