Friday October 11, 2019 **START: 3:10pm** **DURATION: 110 mins**

University of Toronto
Department of Mathematics

Term Test 1
MAT224H1F
Linear Algebra II

EXAMINERS: H. Horowitz, S. Uppal, I. Varma

Last Name (PRINT): ____________________

Given Name(s) (PRINT): ____________________

Student NUMBER: ____________________

EMAIL: @mail.utoronto.ca ____________________

Student SIGNATURE: ____________________

Instructions.

1. There are **53** possible marks to be earned in this exam. The examination booklet contains a total of 9 pages. It is your responsibility to ensure that *no pages are missing from your examination.* DO NOT DETACH ANY PAGES FROM YOUR EXAMINATION.
2. DO NOT WRITE ON THE QR CODE AT THE TOP RIGHT-HAND CORNER OF EACH PAGE OF YOUR EXAMINATION.
3. For the full answer questions, WRITE YOUR SOLUTIONS ON THE FRONT OF THE QUESTION PAGES THEMSELVES. THE BACK OF EVERY PAGE WILL **NOT** BE SCANNED NOR SEEN BY THE GRADERS.
4. Ensure that your solutions are LEGIBLE.
5. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
6. Have your student card ready for inspection.
7. You may use the two blank pages at the end for rough work. The last two pages of the examination WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
8. **Show all of your work and justify your answers** but do not include extraneous information.

1. (a) Let $V = \{\mathbf{x}, \mathbf{y}\}$ be a set with exactly two vectors. Define vector addition and scalar multiplication in V by the following rules:

Vector addition: $\mathbf{x} + \mathbf{x} = \mathbf{x}$, $\mathbf{y} + \mathbf{y} = \mathbf{x}$, $\mathbf{x} + \mathbf{y} = \mathbf{y}$, and $\mathbf{y} + \mathbf{x} = \mathbf{y}$.
Scalar multiplication: $c\mathbf{x} = \mathbf{x}$, and $c\mathbf{y} = \mathbf{y}$ for all $c \in \mathbb{R}$.

Show that V is **not** a vector space by citing one axiom in the definition of a vector space that fails to hold. You must both state the axiom clearly and show it does not hold. [4 marks]

1. (b) Define what it means for a subset W of a vector space V to be a *subspace* of V. [2 marks]

2. A vector $\mathbf{x} \in \mathbb{R}^n$ is *symmetric* if $x_k = x_{n-k+1}$ for $k = 1, 2, \ldots, n$. It is *anti-symmetric* if $x_k = -x_{n-k+1}$ for $k = 1, 2, \ldots, n$. Let

$$U = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is symmetric}\}$$
$$W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is anti-symmetric}\}$$

(a) Both U and W are subspaces of $\mathbb{R}^n$ but pick only one (your choice) and show it is a subspace. [5 marks]

(b) Is $\mathbb{R}^n = U \oplus W$? Explain your answer. [5 marks]

3. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in vector space V.

(a) Define what it means for the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ to be linearly dependent. [2 marks]

(b) Define $\mathrm{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. [2 marks]

(c) Either prove the following statement is true or find a counterexample to show it is false: Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be a linearly independent list of vectors in a vector space V. If the pair $\mathbf{u}, \mathbf{v} \in V$ is linearly independent and both $\mathbf{u}, \mathbf{v} \notin \mathrm{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, then the list $\mathbf{u}, \mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly independent. [4 marks]

4. (a) Let V be a finite dimensional vector space. Define the *dimension* of V. [2 marks]

4. (b) Let V be an n-dimensional vector space. Let W_1 and W_2 be unequal subspaces of V, each with dimension $(n-1)$. Prove that $V = W_1 + W_2$, and that $\dim(W_1 \cap W_2) = n - 2$. [5 marks]

5. (a) Define what it means for a function $T : V \to W$ to be a *linear transformation*. [2 marks]

5. (b) Suppose $\mathbf{a} \in \mathbb{R}^n$ be a fixed vector. Show $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T\mathbf{x} = \mathbf{a}^{\mathrm{T}}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ is a linear transformation. [4 marks]

5. (c) Suppose you are given that the function $T : P_2(\mathbb{R}) \to \mathbb{R}$ satisfies

$$T(1+x) = 1,\ T(-1+x+x^2) = -1, \text{and } T(1-x-x^2) = -1$$

Choose one of the following (i) T must be linear, (ii) T might be linear, or (iii) T cannot be linear, and justify your choice. [4 marks]

6. Determine if each statement below is True or False and indicate your answer by circling one of the options. No explanation is necessary. Each correct answer is worth 2 marks and each incorrect answer is worth 0 marks.

(i) Let V and W be vector spaces. For $\mathbf{x}, \mathbf{y} \in V$, the *line segment* joining $\mathbf{x}$ and $\mathbf{y}$ in V is $L = \{\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \mid 0 \leq t \leq 1\}$. If $T : V \to W$ is a linear transformation, then $T(L)$ is a line segment in W.

(**True**) (**False**)

(ii) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is a list of linearly independent vectors in a vector space V, then $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} \cap \operatorname{span}\{\mathbf{x}_3, \mathbf{x}_4\} = \{\mathbf{0}\}$.

(**True**) (**False**)

(iii) If $\mathbf{x}_1, \mathbf{x}_2$ are vectors in a vector space V, then $\operatorname{span}\{\mathbf{x}_1\} + \operatorname{span}\{\mathbf{x}_2\} = \operatorname{span}\{\mathbf{x}_1 + \mathbf{x}_2\}$.

(**True**) (**False**)

(iv) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for a vector space V and $S_1, S_2, \ldots, S_n$ are subsets of V such that $\mathbf{x}_i \in S_i$ for each $i = 1, 2, \ldots, n$, then $V = \operatorname{span} S_1 + \operatorname{span} S_2 + \cdots + \operatorname{span} S_n$.

(**True**) (**False**)

(v) If W_1, W_2, W_3 are subspaces of a vector space V such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$ then $W_2 = W_3$.

(**True**) (**False**)

(vi) Let k be a positive integer. If W_1, W_2 are subspaces of a finite dimensional vector space V such that $\dim W_1 + \dim W_2 \geq \dim V + k$, then $W_1 \cap W_2$ contains at least k linearly independent vectors.

(**True**) (**False**)

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