COINTEGRATION

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REVIEW I(D) PROCESS

- In econometrics, a time series z_t is said to be an integrated process of order one, that is, an $I(1)$ process, if $(1-B)z_t$ is stationary and invertible.
  - A stationary and invertible time series is said to be an $I(0)$ process.
- In general, a univariate time series z_t is an $I(d)$ process if $(1-B)^d z_t$ is stationary and invertible, where $d > 0$ and order d is referred to as the order of integration or the multiplicity of a unit root.

MOTIVATION OF COINTEGRATION

- It is incorrect to analyze nonstationary time serie using standard statistical inference techniques.
- In this course, we have learned that the Box-Jenkins approach uses differencing to solve the problem.
- Cointegration is another technique to model nonstationary (multivariate) time series.
- What is the intuition behind cointegration?
    1. Balance of the (linear) regression equation
    2. If time series share the same source of the $I(1)$'ness, or time series move together in the long-run.

COINTEGRATION

- Consider a multivariate time series $\boldsymbol{z}_t$. If z_{it} $\forall i$ are $I(1)$ processes but a nontrivial linear combination $\boldsymbol{\beta}'\boldsymbol{z}_t$ is $I(0)$, then $\boldsymbol{z}_t$ is said to be cointegrated of order one.
  - The linear combination vector $\boldsymbol{\beta}$ is called a cointegrating vector.
- In general, if z_{it} are $I(d)$ nonstationary and $\boldsymbol{\beta}'\boldsymbol{z}_t$ is $I(h)$ with $h < d$, then $\boldsymbol{z}_t$ is cointegrated. In practice, the case of $d = 1$ and $h = 0$ is of major interest.
- Thus, cointegration often means that a linear combination of individually unit-root nonstationary time series becomes a stationary and invertible series.

USEFUL RESULTS FOR THE LINEAR COMBINATION OF STOCHASTIC PROCESS

Linear combinations of $I(0)$ and $I(1)$ processes

1. $X_t \rightarrow I(0) \Rightarrow a + bX_t \rightarrow I(0)$

   $X_t \rightarrow I(1) \Rightarrow a + bX_t \rightarrow I(1)$

2. $X_t, Y_t \rightarrow I(0) \Rightarrow aX_t + bY_t \rightarrow I(0)$

3. $X_t \rightarrow I(0), Y_t \rightarrow I(1) \Rightarrow aX_t + bY_t \rightarrow I(1)$

4. $X_t, Y_t \rightarrow I(1) \Rightarrow aX_t + bY_t \rightarrow I(1),\ \text{in general}$

COMMON TRENDS

- The idea of Stock and Watson (1988) provides a very useful way to understand cointegration relationships.
  - Cointegrated variables sharing common stochastic trends
- A naive example: X_t and Y_t are $I(1)$ processes and satisfy:

$$X_t \equiv \alpha \cdot W_t + \widetilde{X}_t$$

$$Y_t \equiv W_t + \widetilde{Y}_t$$

Stationary ARMA(p,q) process, or I(0) process

X_t and Y_t share the same nonstationary sources W_t--an ARIMA(p,1,q) process, or I(1) process

COMMON TRENDS

- X_t and Y_t have a common $I(1)$ trend, W_t.
- Consider a linear combination Z_t as follows:

$$Z_t \equiv X_t - \alpha \cdot Y_t = \cancel{\alpha \cdot W_t} \boxed{+\widetilde{X}_t} \cancel{-\alpha \cdot W_t} \boxed{-\alpha \cdot \widetilde{Y}_t}$$

$$Z_t \equiv \widetilde{X}_t - \alpha \cdot \widetilde{Y}_t \rightarrow I(0) \quad (\text{rule } 2)$$

If two $I(1)$ process have a common $I(1)$ trend (factor) and $I(0)$ idiosyncratic components, then they are cointegrated.

In the case, we say that $(1, -\alpha)$ as the cointegrating vector.

MORE COMMON TRENDS

Example 1.

$$\left.\begin{aligned} Y_t &\equiv W_t + u_t \\ X_t &\equiv W_t + v_t \\ Z_t &\equiv W_t + s_t \end{aligned}\right\} W_t \to I(1) \quad u_t, v_t, s_t \to I(0)$$

$$1 \ \text{common stochastic trend} \to W_t$$

$$2 \ \text{cointegrating vectors}: (1 \quad -1 \quad 0)' \quad (0 \quad 1 \quad -1)'$$

Example 2.

$$\left.\begin{aligned} Y_t &\equiv W_t + u_t \\ X_t &\equiv W_t + R_t + v_t \\ Z_t &\equiv R_t + s_t \end{aligned}\right\} W_t, R_t \to I(1) \quad u_t, v_t, s_t \to I(0)$$

$$2 \ \text{common stochastic trends} \to W_t, R_t$$

$$1 \ \text{cointegrating vector}: (1 \quad -1 \quad 1)'$$

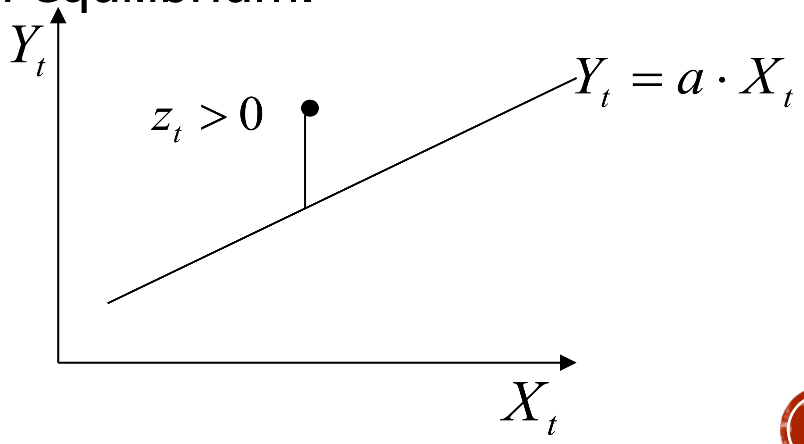
What conclusion can we draw from the above examples

ERROR CORRECTION MODEL

- Let $z_t = Y_t - aX_t$ denote the deviation from the long-run equilibrium.

- If the system is going to return to long-run equilibrium, the short-run movements of the variables (at least some of them) must be respond to the magnitude of disequilibrium.

- Hence, the path of a cointegrated system is influenced by the extend of deviation from the long-run equilibrium.



ERROR CORRECTION MODEL

- Consider the example in "Applied Econometric Time Series"

$$\Delta r_{S,t} = a_{10} + \alpha_s (r_{L,t-1} - \beta \cdot r_{S,t-1}) + \sum a_{11}(i)\Delta r_{s,t-i} + \sum a_{12}(i)\Delta r_{L,t-i} + \varepsilon_{S,t}$$

$$\Delta r_{L,t} = a_{20} - \alpha_L (r_{L,t-1} - \beta \cdot r_{S,t-1}) + \sum a_{21}(i)\Delta r_{s,t-i} + \sum a_{22}(i)\Delta r_{L,t-i} + \varepsilon_{L,t}$$

$$\alpha_s, \alpha_L > 0, \quad \varepsilon_{i,t} \sim \mathrm{WN}(0, \sigma_i^2), \quad i = s, L$$

- This two variable error-correction model is a bivariate VAR in first differences augmented by the error-correction terms. Need to understand the following concepts
  1. Speed of adjustment parameters
  2. Granger representation theorem
  3. Co-integration coefficient restrictions in a VAR model

GRANGER REPRESENTATION THEOREM AND ECM

<u>Granger Representation Theorem</u>: If X_t and Y_t are co-integrated, then there exists an ECM representation. Co-integration is a necessary condition for ECM and vice versa.

1. Vector autoregressions on differenced $I(1)$ processes will be a misspecification if the component series are cointegrated.
2. Engle and Granger (1987) showed that an equilibrium specification is missing from a VAR representation.
3. However, when lagged disequilibrium terms are included as explanatory variables, the model becomes well specified.
4. Such a model is called an error correction model (ECM) because the model is structured so that short-run deviation from the long-run equilibrium will be corrected.

THE PROCEDURE OF ENGLE AND GRANGER (1987)

1) Test wherer X_t and Y_t are $I(1)$ using a unit root test.
2) If both series are $I(1)$, regress one series against the other using least squares.
3) Run a unit root test on regression residuals. If residuals are stationary, these two series are cointegrated.
   - The regression line indicates the long-run equilibrium relationship between two variables. The disequilibrium term is simply the regression residuals.
4) Finally, we consider the following ECM

$$\Delta X_t = c_1 + \rho_1(Y_{t-1} - \hat{\alpha}X_{t-1}) + \beta_{x1}\Delta X_{t-1} + \ldots\ldots + \beta_{y1}\Delta Y_{t-1} + \ldots\ldots + \varepsilon_{xt}$$

$$\Delta Y_t = c_2 + \rho_2(Y_{t-1} - \hat{\alpha}X_{t-1}) + \gamma_{x1}\Delta X_{t-1} + \ldots\ldots + \gamma_{y1}\Delta Y_{t-1} + \ldots\ldots + \varepsilon_{yt}$$

WHY USE ENGLE-GRANGER METHOD

- It is very straightforward to implement and to interpret the Engle-Granger procedure.
- From the risk management point of view, the Engle-Granger criterion that minimizes variance is usually more important than the Johansen criterion that maximizes stationarity.
- Sometimes there is a natural choice of dependent variables in the cointegrating regressions, for example, in equity index tracking.

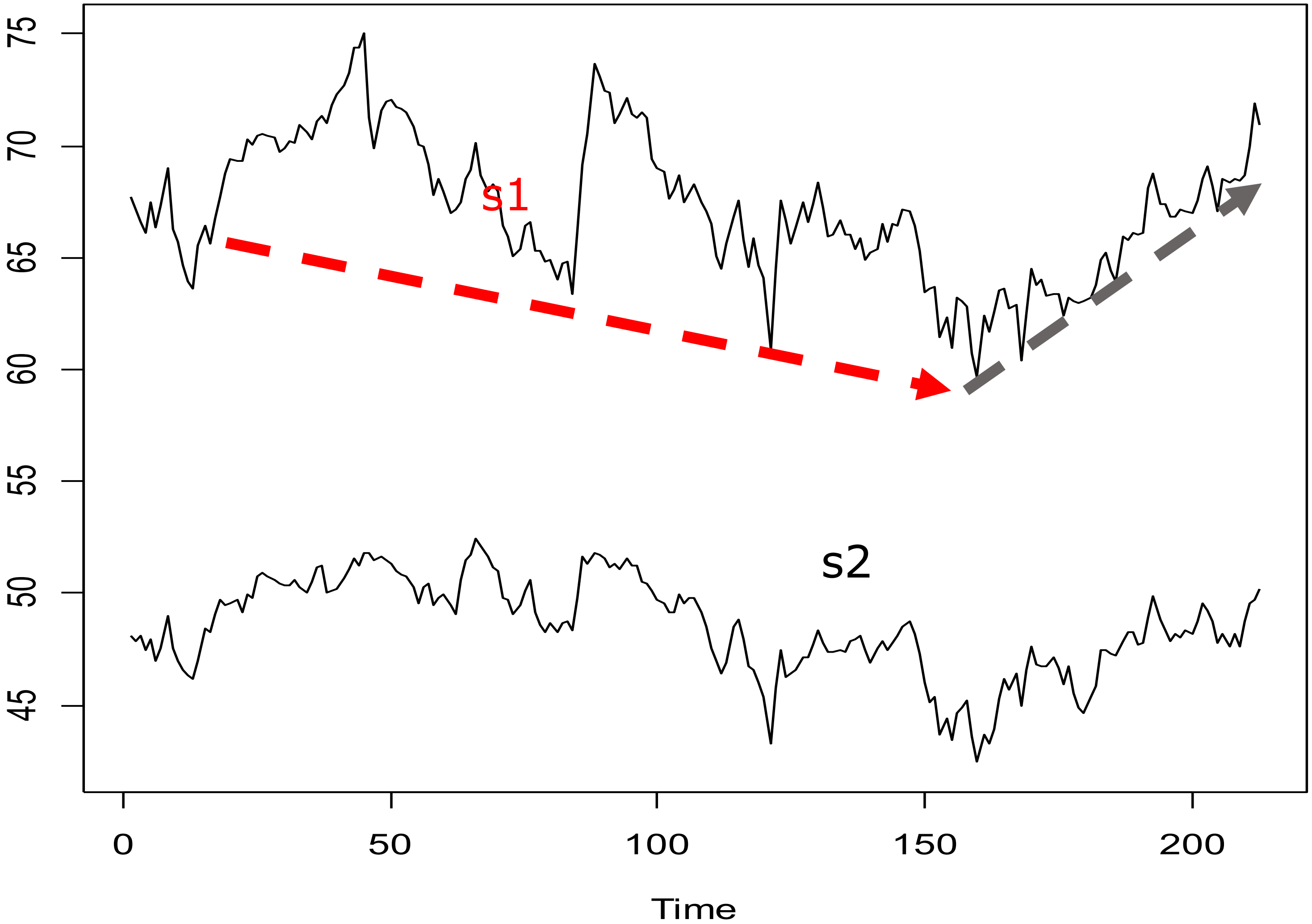
REMARKS

- What's the assumption implicitly imposed in this approach?
  - The Engle-Granger procedure is only applicable to systems with more than two variables in a very special circumstances.--Carol Alexander (2001)
- Question: Is there another way to test (model) co-integration?
  - The Johansen procedure (1988) seeks the linear combination which is most stationary whereas the Engle-Granger tests seek the linear combination having minimum variance.
  - The Johansen tests are a multivariate generalization of the unit root tests.
- The presence of change points will affect the effectiveness of cointegration analysis.

PAIRS TRADING BASED ON COINTEGRATION

- If two asset prices are cointegrated, the value of a wisely built portfolio (spread) between these 2 assets is stationary/ mean-reversion
- What should we do if something is mean-reverting?
  - Buy low and sell high (above its mean)
  - Pairs trading is executed when spread diverges too much from its mean
- Warning: Cointegration is the long-run relationship so the constructed spread may diverge substantially from these relationship in the short run

STOCK PRICE MOVEMENT IS HARD TO PREDICT



RANDOM WALK AND MAKET EFFICIENT

- What do we know about a random walk process?

$$X_t = X_{t-1} + a_t, \qquad a_t \sim \mathrm{WN}(0, \sigma^2)$$

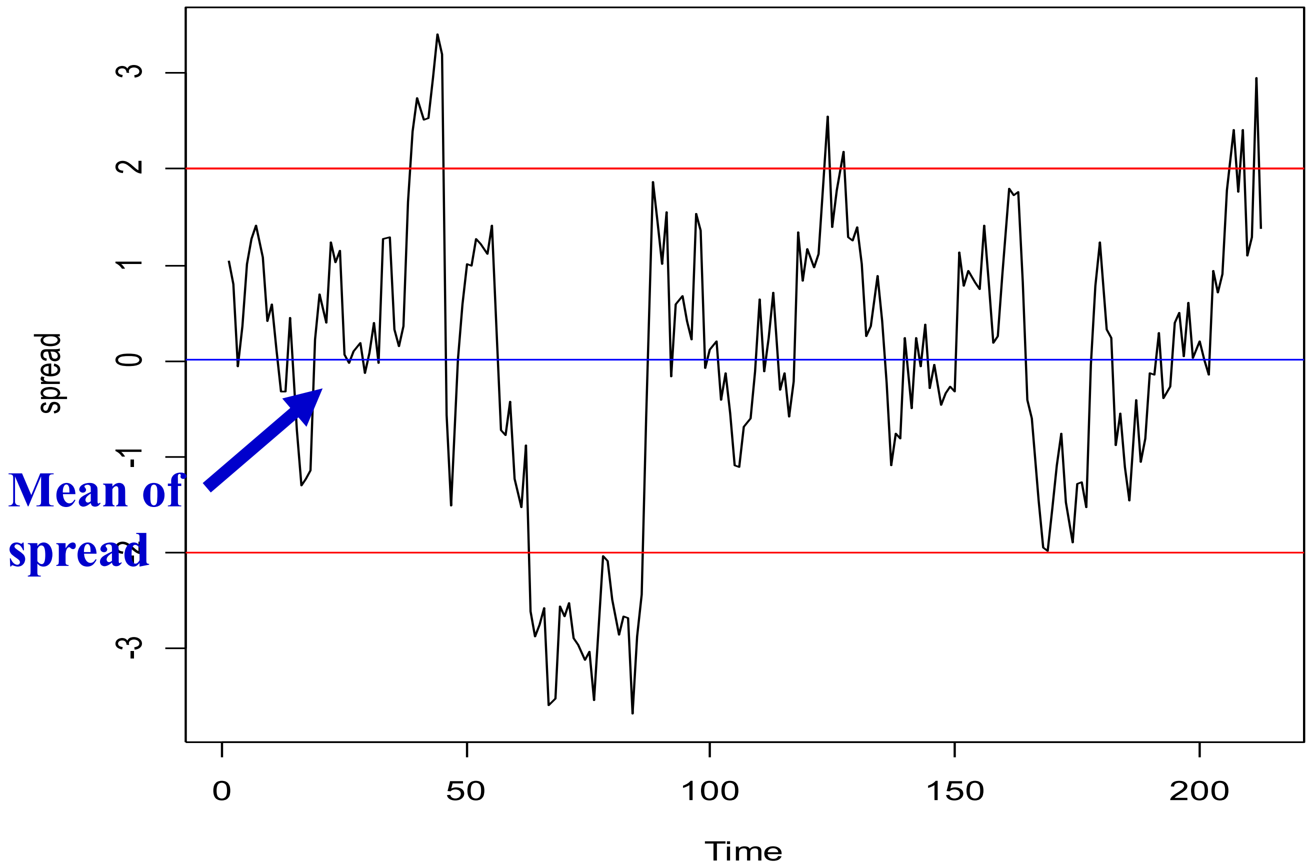
- What's the best forecast of X_{t+1} at original t ?
- The best forecast about tomorrow's price is today's price

Unpredictability

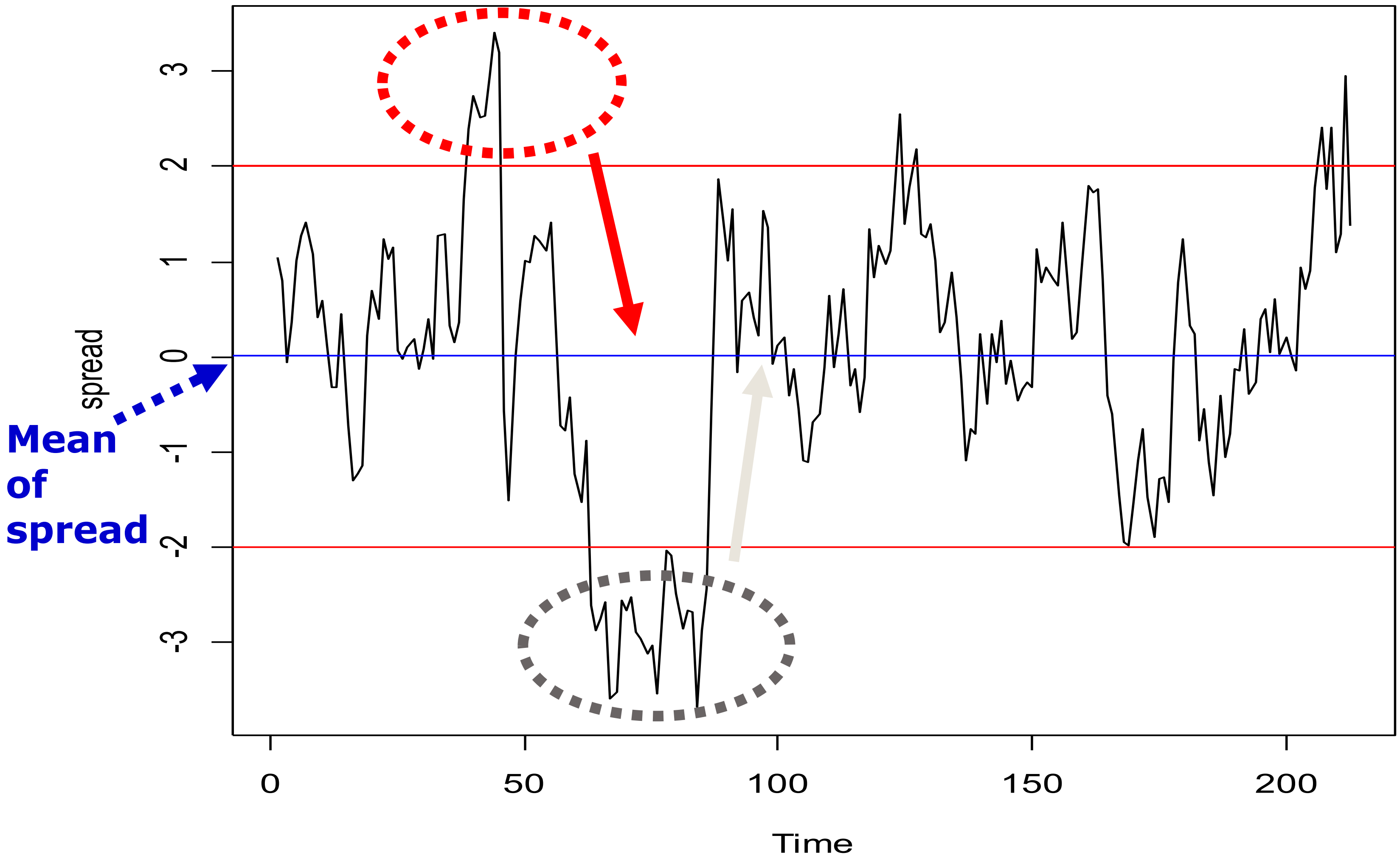
PAIRS TRADING IN STOCKS

- Construct a portfolio consisting of two stocks, S1 and S2
- The value of such a portfolio may be referred to as
- Spread=$S1 - b \cdot S2$
- Avoid guessing trends and explore the market inefficiency in a statistical sense
- Different methods for constructing spreads are summarized in the supplement course note

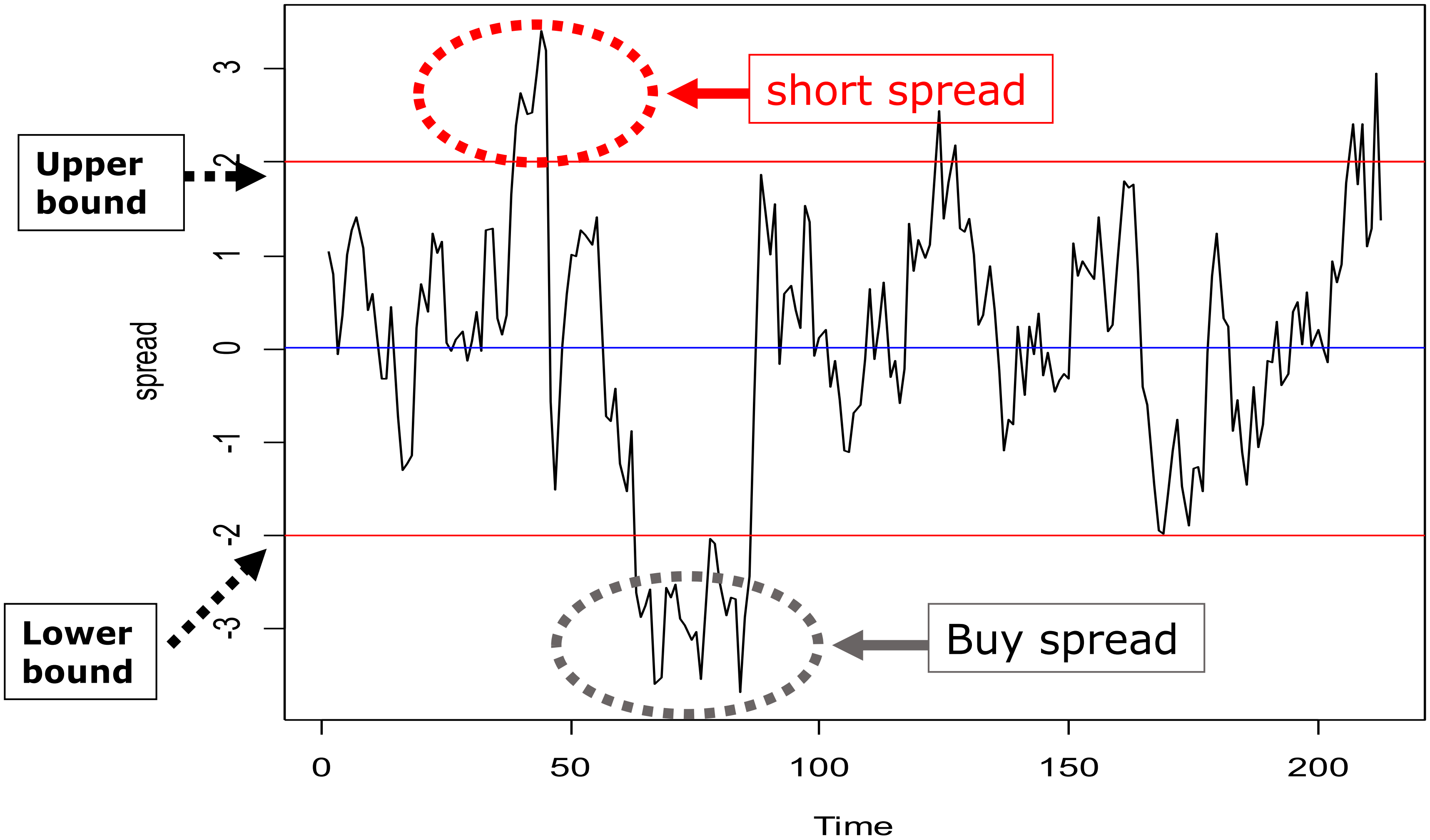
MOVEMENTS OF SPREADS



MEAN REVERSION



TRADING THRESHOLD



WHAT IS A GOOD SPREAD

- It is is mean-reversion.
  1. There exists a constant mean
  2. A good spread is bounded.
     - Some people think that mean-reversion requires the second moments of the spread process to be a finite constant. (e.g. Carol Alexander, 2001)
- What does this sound like?

WEAK STATIONARITY

PAIRS TRADING

- Find pairs of stocks
- Determine the trading signal
- Determine how to size your position (as well as portfolio construction.
- Risk management and capital management (positon sizing)
- Take into account the transaction costs
- Backtest if your strategy works
- The better implementations and less assumptions used by your model, the more successful your trading strategy.
- **Warning: Even though you carry out perfect analysis, you could lose money using statistical arbitrage.**