Handout 5: Recurrence

Oct, 23, 2016

1. Prove

$$f(n) = \begin{cases} c, & n = 1 \\ T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + dn, & n > 1 \end{cases}$$

   is non-decreasing.

2. Linear recurrence relation

$$f(n) = c_1 f(n-1) + c_2 f(n-2) + c_3 f(n-3) + \dots + c_k f(n-k)$$

   To get the closed form of $f(n)$, try to solve the equation

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + c_3 r^{n-3} + \dots + c_k r^{n-k}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + c_3 r^{k-3} + \dots + c_k$$

   This equation is called the characteristic equation.
   Theorem: $f(n) = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \dots + \alpha_k r_k^n$ satisfies the recurrence relation.

   (a) Fibonacci

$$f(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ f(n-1) + f(n-2), & n > 1 \end{cases}$$

   (b)

$$T(n) = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 3T(n-1) - 2T(n-2), & n > 1 \end{cases}$$

3. Let $a, b \in \mathbb{N}$. Consider the following function $f : \mathbb{N} \to \mathbb{N}$.

$$f(n) = \begin{cases} a, & n = 0 \\ b, & n = 1 \\ 2f(n-1) - f(n-2) + 1, & n \geq 2 \end{cases}$$

   Find a closed-form expression for $f(n)$ by unwinding (repeated substitution).

Master Theorem

$$T(n) = \begin{cases} k & \text{if} \quad n \leq B \\ a_1 T(\lceil \frac{n}{b} \rceil) + a_2 T(\lfloor \frac{n}{b} \rfloor) + f(n) & \text{if} \quad n > B \end{cases}$$

If $a = a_1 + a_2$ and $f \in \Theta(n^d)$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if} \quad a < b^d \\ \Theta(n^d \log n) & \text{if} \quad a = b^d \\ \Theta(n^{\log_b a}) & \text{if} \quad a > b^d \end{cases}$$