First we derive cdf of $U = Y^2$ when $Y \sim f_Y(y)$

$$F_U(u) = P(U \leqslant u) = P(Y^2 \leqslant u)$$

if $u < 0$; then $F_U(u) = 0$

if $u \geqslant 0$; $F_U(u) = P(Y^2 \leqslant u) = P\left(-\sqrt{u} \leqslant Y \leqslant \sqrt{u}\right)$

$$= \int_{-\sqrt{u}}^{\sqrt{u}} f_Y(y)\,dy = F_Y\left(\sqrt{u}\right) - F_Y\left(-\sqrt{u}\right)$$

So $F_U(u) = F_Y\left(\sqrt{u}\right) - F_Y\left(-\sqrt{u}\right)$

$$f_U(u) = \frac{dF_Y(\sqrt{u})}{du} \cdot \frac{1}{2\sqrt{u}} + \frac{dF_Y(-\sqrt{u})}{du} \cdot \frac{1}{2\sqrt{u}}$$

$$= f_Y(\sqrt{u}) \cdot \frac{1}{2\sqrt{u}} + f_Y(-\sqrt{u}) \cdot \frac{1}{2\sqrt{u}}$$

So $f_U(u) = \begin{cases} f_Y(\sqrt{u})\left(\frac{1}{2\sqrt{u}}\right) + f_Y(-\sqrt{u})\left(\frac{1}{2\sqrt{u}}\right) & u > 0 \\ 0 & \text{o.w} \end{cases}$

Since $Y \sim U(-1,1) \Rightarrow f_Y(y) = \frac{1}{2} \quad -1 < y < 1$

$U = Y^2 \qquad y: -1 \longrightarrow 1$;

$\qquad\qquad u: 0 \longrightarrow 1$

$$f_U(u) = \begin{cases} \frac{1}{2}\left(\frac{1}{2\sqrt{u}}\right) + \frac{1}{2}\left(\frac{1}{2\sqrt{u}}\right) & u > 0 \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & \text{o.w} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & o.w \end{cases}$$

$U = -4Y + 3$;

$$h(y) = -4y + 3 \Rightarrow u = -4y + 3 \Rightarrow u - 3 = -4y \Rightarrow 4y = 3 - u$$

$$\Rightarrow y = \frac{3-u}{4} = h^{-1}(u) \Rightarrow \frac{dh^{-1}(u)}{du} = -1/4$$

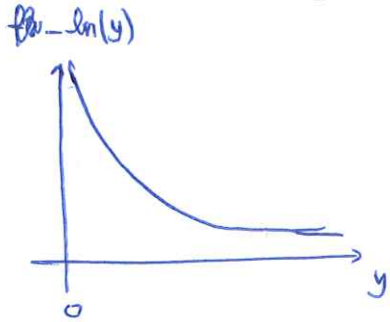
$$f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}}{du} \right|$$

$$\Rightarrow f_U(u) = 2\left(\frac{3-u}{4}\right) \cdot \left| -\frac{1}{4} \right| = 2\left(\frac{3-u}{4}\right) \cdot \frac{1}{4} = \frac{3-u}{8} \qquad 0 \leq \frac{3-u}{4} \leq 1$$

$$0 \leq \frac{3-u}{4} \leq 1 \Rightarrow 0 \leq 3-u \leq 4 \Rightarrow -4 \leq u-3 \leq 0 \Rightarrow -1 \leq u \leq 3$$

$$= \begin{cases} \frac{3-u}{8} & -1 \leq u \leq 3 \\ 0 & o.w \end{cases}$$

Chapter 6 ; Slide 13 (# 6.28)



$Y \sim U(0,1)$; $U = -2\ln(Y)$

$y = 0 \to 1$
$-\ln(y) : 0 \to \infty$

$f_Y(y) = 1 \qquad 0 < y < 1$

$$u = -2\ln(y) \Rightarrow -\frac{u}{2} = \ln(y) \Rightarrow y = e^{-u/2} ; \quad \frac{\partial y}{\partial u} = -e^{-u/2} \cdot \frac{1}{2}$$

$$|J| = \left| \frac{\partial y}{\partial u} \right| = e^{-u/2} \cdot \frac{1}{2}$$

$$f_U(u) = f_Y(y) \cdot |J| = f_Y\left(e^{-u}\right) \cdot e^{-u/2} \cdot \frac{1}{2}$$

$$= 1/2 \cdot e^{-u/2} = \frac{1}{2} e^{-u/2} \qquad u \geq 0 \Rightarrow U \sim Exp(2).$$

$Y \sim Bin(n, p) \Rightarrow$ mgf of Y: $m_Y(t) = \left(1-p+pe^{t}\right)^{n}$

$X = n - Y \Rightarrow$

$$m_X(t) = E\left(e^{tX}\right) = E\left(e^{t(n-Y)}\right) = e^{tn} E\left(e^{-tY}\right)$$

$$= e^{nt} M_Y(-t) = e^{nt}\left(1-p+pe^{-t}\right)^{n} = \left(e^{t} - pe^{t} + p\right)^{n}$$

$$= \left(p + (1-p)e^{t}\right)^{n} \Rightarrow X \sim B(n, 1-p)$$

So X is # of failures observed in the expriment.

From Slide 7 and page 305 of textbook, we have

$Y \sim f_Y(y)$ Then $U = Y^2$

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}}\left(f_Y(\sqrt{u}) + f_Y(-\sqrt{u})\right) & u>0 \\ 0 & o.w \end{cases}$$

Here $Z = \frac{X-\mu}{\sigma} \sim N(0,1) \Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

Now $U = Z^2$

$$f_U(u) = \frac{1}{2\sqrt{u}}\left(f_Z(\sqrt{u}) + f_Z(-\sqrt{u})\right)$$

$$= \frac{1}{2\sqrt{u}}\left(\frac{1}{\sqrt{2\pi}} e^{-u/2} + \frac{1}{\sqrt{2\pi}} e^{-u/2}\right)$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi u}} e^{-u/2} & u \geq 0 \\ 0 & o.w \end{cases} = \frac{u^{-1/2} e^{-u/2}}{2^{1/2}\sqrt{\pi}}$$

$X \sim Gamma(\alpha, \beta)$ $\quad f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}$

$\Rightarrow U \sim Gamma\left(\frac{1}{2}, 2\right)$; $\Gamma(1/2) = \sqrt{\pi}$

$\quad = \chi_1^2$

$$Z \sim N(0,1) \Rightarrow m_{Z^2}(t) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} f(z)\,dz = \int_{-\infty}^{\infty} e^{tz^2} \cdot \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

?

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)(1-2t)}\,dz \quad ; \text{ if } \quad 1-2t>0 \Rightarrow t<\tfrac{1}{2}$$

$$= \frac{\exp\left(-\left(\frac{z^2}{2}\right)(1-2t)\right)}{\sqrt{2\pi}} = \frac{\exp\left(-\left(\frac{z^2}{2}\right)\Big/(1-2t)^{-1}\right)}{\sqrt{2\pi}}$$

$$\Rightarrow m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(1-2t)^{-1/2}} \exp\left(-\left(\frac{z^2}{2}\right)\Big/(1-2t)^{-1}\right) dz$$

But this integral equals to 1; if $t<\frac{1}{2}$

$$m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} = (1-2t)^{-1/2}$$

$$X \sim \text{Gamma}(\alpha,\beta) \; ; \quad m_X(t) = (1-\beta t)^{-\alpha}$$

$$\text{So } \; Z^2 = U \sim \text{Gamma}\left(\alpha=\tfrac{1}{2}, \beta=2\right) \equiv \chi^2_{(1)}$$

$$\chi^2_{(1)} = \text{Gamma}\left(\nu/2, 2\right)$$

$$Y_1 \sim Poi(\mu_1) \Rightarrow m_{y_1}(t) = \exp\left(\mu_1(e^t - 1)\right)$$

$$Y_2 \sim Poi(\mu_2) \Rightarrow m_{y_2}(t) = \exp\left(\mu_2(e^t - 1)\right)$$

$$m_u(t) = E\left(e^{tU}\right) = E\left(e^{t(Y_1+Y_2)}\right) = E\left(e^{tY_1 + tY_2}\right) = E\left(e^{tY_1}\right) \cdot E\left(e^{tY_2}\right)$$

$$= m_{y_1}(t) \cdot m_{y_2}(t) = \exp\left(\mu_1(e^t - 1)\right) \exp\left(\mu_2(e^t - 1)\right)$$

$$= \exp\left((e^t - 1)(\mu_1 + \mu_2)\right) \Rightarrow U \sim Poi(\mu_1 + \mu_2)$$

Proof ; chapter 6 , Slide 21

$$Y_i \sim N(\mu_i, \sigma_i^2) \qquad m_{y_i}(t) = E\left(e^{ty_i}\right) = e^{\mu_i t + \frac{t^2\sigma_i^2}{2}}$$

$$m_{a_i y_i} = E\left(e^{ta_i y_i}\right) = m_{y_i}(ta_i)$$

$$m_U(t) \underset{\uparrow}{=} m_{a_1y_1}(t) * m_{a_2y_2}(t) * \cdots * m_{a_ny_n}(t)$$

independent

$$= m_{y_1}(ta_1) \cdot m_{y_2}(ta_2) * \cdots * m_{y_n}(ta_n)$$

$$= \prod_{i=1}^{n} e^{\mu_i a_i t + \frac{t^2 a_i^2 \sigma_i^2}{2}}$$

$$= e^{t\sum_{i=1}^{n} a_i\mu_i + \frac{t^2}{2} \cdot \sum_{i=1}^{n} a_i^2\sigma_i^2}$$

So $U \sim N\left(\sum_{i=1}^{n} a_i\mu_i , \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

$$Y_i \sim \chi^2_{(v_i)} \equiv \text{Gamma}\left(\frac{v_i}{2}, 2\right) \Rightarrow m_{y_i}(t) = (1-2t)^{-v_i/2}$$

$$U = \sum_{i=1}^{n} Y_i \Rightarrow m_U(t) = E\left(e^{t\sum_{i=1}^{n} Y_i}\right) \underset{\substack{\downarrow \\ \text{Slide 15}}}{=} m_{y_1}(t) * m_{y_2}(t) * \cdots m_{y_n}(t)$$

$$= \prod_{i=1}^{n} m_{y_i}(t) = \prod_{i=1}^{n} (1-2t)^{-v_i/2} = (1-2t)^{-\sum_{i=1}^{n} v_i/2}$$

$$\Rightarrow U \sim \cancel{g}\ \text{Gamma}\left(\sum_{i=1}^{n} v_i/2, 2\right) \equiv \chi^2_{\left(\sum_{i=1}^{n} v_i\right)}$$