MAT 224 LEC 0501 EXTRA EXCERCISE WEEK 1-2

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1. Linear Combination

1. In each of the following question, write the vector $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ as a linear combination of $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3$ in the following situations.

(1) $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3$ are three vectors and

$$\begin{cases} \vec{\mathbf{v}}_1 = 3\vec{\mathbf{e}}_1 + 2\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{v}}_2 = 2\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{v}}_3 = 2\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \end{cases} \qquad \begin{cases} \vec{\mathbf{w}}_1 = \vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{w}}_2 = \vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{w}}_3 = \vec{\mathbf{e}}_2 \end{cases}$$

(2) $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3$ are three vectors and

$$\begin{cases} \vec{\mathbf{v}}_1 = 3\vec{\mathbf{e}}_1 + 2\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{v}}_2 = 2\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_3 \\ \vec{\mathbf{v}}_3 = 2\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 \end{cases} \qquad \begin{cases} \vec{\mathbf{e}}_1 = \vec{\mathbf{w}}_2 + \vec{\mathbf{w}}_3 \\ \vec{\mathbf{e}}_2 = \vec{\mathbf{w}}_1 + \vec{\mathbf{w}}_2 + \vec{\mathbf{w}}_3 \\ \vec{\mathbf{e}}_3 = \vec{\mathbf{w}}_2 \end{cases}$$

(3) $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3 \in \mathbb{R}^3$ and certain row operation could reduce it to

$$\begin{pmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 & \vec{\mathbf{w}}_1 & \vec{\mathbf{w}}_2 & \vec{\mathbf{w}}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \end{pmatrix}$$

(4) $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3 \in \mathbb{R}^3$ are columns of the product

$$\begin{pmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 & \vec{\mathbf{w}}_1 & \vec{\mathbf{w}}_2 & \vec{\mathbf{w}}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \end{pmatrix}$$

where the left factor is invertible.

(5) $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3 \in P_{2,x} = \{f : f(x) = ax^2 + bx + c : a, b, c \in \mathbb{C}\}$ and it has the following value table

F	$\vec{\mathbf{v}}_1$	$\vec{\mathbf{v}}_2$	$\vec{\mathbf{v}}_3$	$\vec{\mathbf{w}}_1$	$\vec{\mathbf{w}}_2$	$\vec{\mathbf{w}}_3$
$F(0)$	1	2	3	0	1	-1
$F'(0)$	1	1	1	0	0	1
$F''(0)$	0	0	1	1	-1	1

2. Linear Equation

2. Observe the general solution for each equation by eye and write it down. **Do not compute.**

(1)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 9 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

3. After applying some row operation, **Then observe** the general solution for each equation by eye and write it down.

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

3. Missing Entry of a matrix product

4. Fill out the missing entry of the matrix product

$$\begin{pmatrix} 3 & 2 & \square \\ 1 & \square & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} \square & 2 & 2 & 3 \\ \square & 9 & 9 & 9 \\ \square & \square & \square & \square \end{pmatrix}$$

5. Find a matrix P such that

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 5 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 3 & 2 \end{pmatrix} = P \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Linear Spaces

10. A linear space over a field F is defined by the following

Definition. The **linear space over the field F** is a set V, endowed with an **abelian group** structure, which admits action of scalar of F, In other words. V is a set equipped with an operator "+" and scalar multiplication satisfying

(1) (V,+) is an abelian group, which means:
   0.0) For any $v \in V$, $w \in V$, the notation $v + w$ defines an element in V
   0.1) For any $v \in V$, $w \in V$, $v + w = w + v$
   0.2) For any $v \in V$, $u \in V$, $w \in V$,$(u + v) + w = u + (v + w)$
   0.3) There exists $0 \in V$, such that for any $v \in V$, $0 + v = v$
   0.4) For any $v \in V$, There exists $-v \in V$, such that $v + (-v) = 0$
(2) (V,+) admits an action of field from right, which satisfying
   1.0) For any $\vec{\mathbf{v}} \in V$, $\lambda \in F$, the notation $\lambda\vec{\mathbf{v}}$ defines another element in V
   1.1) For any $v \in V$, $1v = v$
   1.2) For any $v \in V$, $\lambda, \mu \in F$,$(\lambda\mu)v = \lambda(\mu v)$
   1.3) For any $v \in V$, $\lambda, \mu \in F$,$(\lambda + \mu)v = \lambda v + \mu v$
   1.4) For any $v, w \in V$, $\lambda \in F$, we have $\lambda(v + w) = \lambda v + \lambda w$.

We often call the element of linear space as **vectors**

For each of the following set, say True if it is a linear space and specify **the zero vector** in such a space, say False if it is not, and specify the code of all axioms where it does not follow.

For all of them we let $F = \mathbb{R}$.

We denote the multiplication in $\mathbb{R}$ as $+$ and $\times$, we denote the addition in V as $\dot{+}$

Example. $(V, \dot{+})$: $V = \mathbb{R}$, $x \dot{+} y := x + y + xy$, $\lambda x := (\lambda \times x)$

False, Axiom 1.4) does not apply to this definition.

(1) $(V, \dot{+}) : V = \mathbb{R}$, $x \dot{+} y = x + y + xy$, $\lambda x := (\lambda \times x) + (\lambda - 1)$.

(2) $(V, \dot{+}) : V = \mathbb{R}$, $x \dot{+} y := x + y$, $\lambda x = \lambda^2 \times x$.

(3) $(V, \dot{+}) : V = \mathbb{R}^2$, $x \dot{+} y := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda \times x_1 \\ \lambda \times x_2 \end{pmatrix}$.

(4) $(V, \dot{+}) : V = \mathbb{R}$, $x \dot{+} y := \sqrt{x^2 + y^2}$, $\lambda x := \sqrt{\lambda} \times x$

(5) $(V, \dot{+}) : V = \mathbb{R}$, $x \dot{+} y := \sqrt[3]{x^3 + y^3}$, $\lambda x := \sqrt[3]{\lambda} \times x$

5. Linearly independence, Span

11. Show that the polynomials $x(x-1)$, $(x+1)(x-1)$, $x(x+1)$ forms a linearly independent set. **Do not use the fact that $\{1, x, x^2\}$ is linearly independent.**

12. Prove $x + 3 \notin \operatorname{span}\{(x-1)(x+1), x^2\}$

13. Find a basis of span$\{(x-1)^2, x^2-1, (x-1)(x-3)\}$(Here you can use the fact $\{1, x, x^2\}$ is linearly independent)

14. Let f_1, f_2, f_3, f_4, f_5 be polynomials of degree at most 2. Given the following value table

F	f_1	f_2	f_3	f_4	f_5	f_6
$F(0)$	1	2	3	0	1	-1
$F(1)$	0	1	1	0	0	1
$F(1)+F(2)$	0	0	0	1	-1	1

Write down at least 5 subset of $S = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ such that it could be a basis for span(S)

6. Change of basis

15. Let $V = P_{2,X} = \{f : f(X) = aX^2 + bX + c : a, b, c \in \mathbb{C}\}$.

(1) Please find 6 quadratic polynomials $f_1(X), f_3(X), f_5(X), g_0(X), g_2(X), g_4(X) \in V$ such that they have the following value table.

F(X)	$f_1(X)$	$f_3(X)$	$f_5(X)$
F(1)	1	0	0
F(3)	0	1	0
F(5)	0	0	1

F(x)	$g_0(X)$	$g_2(X)$	$g_4(X)$
F(0)	1	0	0
F(2)	0	1	0
F(4)	0	0	1

[You can leave into the form like the $f_1(X)$ example.]

$$f_1(X) = \frac{(X-3)(X-5)}{8}$$

$$f_3(X) =$$

$$f_5(X) =$$

$$g_0(X) =$$

$$g_2(X) =$$

$$g_4(X) =$$

(2) Suppose we know $\{f_1, f_3, f_5\}$ and $\{g_0, g_2, g_4\}$ are bases of V. Find the change of basis matrix P such that

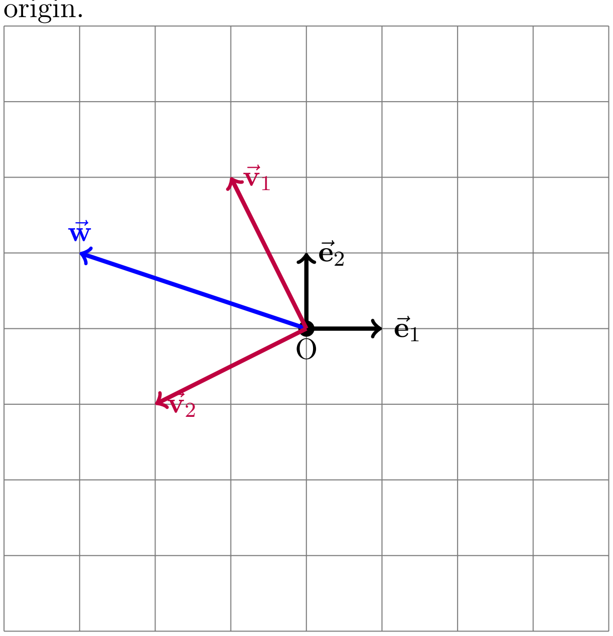
$$\begin{pmatrix} f_1 & f_3 & f_5 \end{pmatrix} = \begin{pmatrix} g_0 & g_2 & g_4 \end{pmatrix} P$$

(3) Write down P^{-1}?

(Hint: Did you see $\begin{pmatrix} f_1 & f_3 & f_5 \end{pmatrix} P^{-1} = \begin{pmatrix} g_0 & g_2 & g_4 \end{pmatrix}$?)

(4) Find the coordinate of X^2 with respect to the basis $\{f_1, f_3, f_5\}$ and basis $\{g_0, g_2, g_4\}$, how is those coordinates related by P?

16. Look at the following picture for 3 vectors $\vec{\mathbf{w}}, \vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ in two dimensional linear space. O is the origin.



(1) (3pt) What is the coordinate of $\vec{\mathbf{w}}$ with respect to basis $\begin{pmatrix} \vec{\mathbf{e}}_1 & \vec{\mathbf{e}}_2 \end{pmatrix}$?

(2) (2pt) Find the change of basis matrix P such that $\begin{pmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{e}}_1 & \vec{\mathbf{e}}_2 \end{pmatrix} P$.

(3) (5pt) Find the coordinate of $\vec{\mathbf{w}}$ in basis $\begin{pmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \end{pmatrix}$

17. Consider the Vandemonde Matrix

$$Van(1,2,3,4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}$$

We set up three basis

The normal basis:

$$\{1, x, x^2, x^3\}$$

The Lagrange interpolation Polynomial Basis:

$$\left\{ \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}, \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}, \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}, \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \right\}$$

The clever Basis:

$$\{1, x-1, (x-1)(x-2), (x-1)(x-2)(x-3)\}$$

(1) Fill in the missing element on ---:

$$\begin{aligned} & \left(\begin{array}{c:c:c:c} 1 & x-1 & (x-1)(x-2) & (x-1)(x-2)(x-3) \end{array} \right) \\ = \quad & \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} U \end{aligned}$$

$$U = \begin{pmatrix} \text{---} & -1 & 2 & -6 \\ \text{---} & \text{---} & -3 & 11 \\ \text{---} & \text{---} & \text{---} & -6 \\ \text{---} & \text{---} & \text{---} & \text{---} \end{pmatrix}$$

(2) What is the change of basis matrix P such that

$$\begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} = \begin{pmatrix} \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} & \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} & \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} & \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \end{pmatrix} P$$

(3) Fill in the missing element on ___:

$$\left(\begin{array}{c:c:c:c} 1 & x-1 & (x-1)(x-2) & (x-1)(x-2)(x-3) \end{array} \right)$$

$$= \left(\begin{array}{cccc} \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} & \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} & \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} & \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \end{array} \right) L$$

$$L = \begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ 1 & \_\_\_ & \_\_\_ & \_\_\_ \\ 1 & 2 & \_\_\_ & \_\_\_ \\ 1 & 3 & 6 & \_\_\_ \end{pmatrix}$$

(4) Which of the following product equal to $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}$?

$A)$ LU

$B)$ UL

$C)$ LU^{-1}

$D)$ UL^{-1}

$E)$ $L^{-1}U$

$F)$ $U^{-1}L$

$G)$ $L^{-1}U^{-1}$

$H)$ $U^{-1}L^{-1}$