1.

2.

3. Show how to polynomial time reduce the 3-colouring search problem to the 3-colouring decision problem. Hint: Introduce three new nodes to represent the three colours and then try to colour one node at a time.

   Solution Sketch We first ask if G can be 3-coloured; if not, we are done. So now we assume that G can be 3-coloured. To actually find a colouring of G, the idea is (as in the hint) to colour one node at a time so that the partial solution at any time can be extended to a 3-colouring of all nodes in the given graph G. To that end, we introduce three nodes (call them R, B, Y) (for red, blue,yellow) and add edges between them to form a triangle and add that triangle to the given graph G. Let's call the new graph G'. To colour a node v we will add edges (to G') between v to exactly 2 of the nodes in $\{R, B, Y\}$. The intention is that if (for example) we add edges (v, R) and (v, Y), then we are colouring v by B (i.e. blue). As we consider each node v, we check (using the 3-colouring decision problem) if a suggested colouring of v (by adding 2 edges as above) still makes it possible to colourG' with 3 colours. coloured to any adjacent nodes. We do this by adding an edge to R Then we will add edges to

4. Show that the following decision problems are NP-complete. To establish the NP hardness, you can use a polynomial time transformation from any problem in the tree of transformations on slide 14 of lecture 15.

   (a) Half independent set (HIS) where
   HIS = $\{G | G = (V, E)$ and G has an independent set of size at least $\lceil |V|/2 \rceil$ $\}$.

   Solution Sketch. To simplify the notation let me assume that $|V|$ is even so we don't have to worry about floors and ceilings. We will describe a transformation of IS $\leq_p HISwhereIS = \{(G, k) | G$ has an independent set of size at least $k\}$. We have three cases to consider depending on whether $k = |V|/2, k < |V|/2, k > |V|/2$. Let $n = |V|$.

   - The first case $k = n/2$ is simple being the transformation $(G, k) \rightarrow G$.
   - $k < 2$. We want to add an independent set of r new nodes to G forming a graph G' with $n' = n + r$ nodes. We choose r so that $k + r = (n + r)/2$. That is, we choose $r = n - 2k$. We claim that G has a size k indepedent set iff G' has a size $n'/2$ independent set.
   - $k > n/2$. We basically do a simple construction as in the previous case but now we add a size $r = n - 2k$ clique and connect each node in this clique to every node in the initial graph G. Alternatively, we can just add a r cliqueand choose r so that $k + 1 + r = (n + r)/2$.

   (b)