

CSC236: Assignment 2

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10th, July, 2017

1.

a)

Let $T(n) = S(n) + 1$

$$= (S(n-1))^2 + 2 \times S(n-1) + 1$$

$$= (S(n-1) + 1)^2$$

$$= ((S(n-2))^2 + 2 \times S(n-2) + 1)^2$$

$$= ((S(n-2) + 1)^2)^2$$

$$= (S(n-2) + 1)^2 \times (S(n-2) + 1)^2$$

$$= (S(n-2) + 1)^{2^2}$$

$$= (S(n-3)^2 + 2 \times S(n-3) + 1)^{2^2}$$

$$= (S(n-3) + 1)^{2^2}$$

$$= (S(n-3) + 1)^{2^3}$$

...

$$= (S(n-k) + 1)^{2^k}$$

...

$$= (S(n - n) + 1)^{2^n} \quad (\text{When } k=n.)$$

$$= (S(0) + 1)^{2^n}$$

$$= 2^{2^n}$$

$$S(n) = T(n) - 1$$

$$= 2^{2^n} - 1 \quad (\text{This is the closed form of } S(n))$$

Prove for the closed form.

Defining predicate:

$P(n)$: “ $S(n) = 2^{2^n} - 1$ ”, where $n \in \mathbb{N}$.

Base Case:

Let $n = 0$. Proof for $P(0)$.

$$\text{LHS} = S(0) = 1$$

$$\text{RHS} = 2^{2^0} - 1 = 2 - 1 = 1$$

Then, $\text{LHS} = \text{RHS}$. $P(0)$ holds.

Inductive steps:

Let n be an arbitrary natural number, and suppose that $P(n)$ holds, i.e., $S(n) = 2^{2^n} - 1$.

WTS: $P(n+1)$ holds, i.e., $S(n+1) = 2^{2^{n+1}} - 1$

$$S(n+1) = (S(n))^2 + 2 \times S(n) \quad (\text{By given recurrence relationship.})$$

$$= (2^{2^n} - 1)^2 + 2 \times (2^{2^n} - 1) \quad (\text{By inductive hypothesis})$$

$$= (2^{2^n})^2 - 2 \times 2^{2^n} + 1 + 2 \times 2^{2^n} - 2$$

$$= (2^{2^n})^2 - 1$$

$$= 2^{2^{n+1}} - 1$$

Therefore, $P(n+1)$ holds. The closed form is proven.

$$\text{Let } f(n) = 2^{2^n}$$

Prove that $S(n) \in \Theta(f(n))$

In order to show that $S(n) \in \Theta(f(n))$, we have to show that as n goes to infinity asymptotically $S(n)$ and $f(n)$ grow the same.

$$\lim_{n \rightarrow \infty} \left(\frac{S(n)}{f(n)} \right) = \lim_{n \rightarrow \infty} \frac{2^{2^n} - 1}{2^{2^n}} = 1$$

Since, the above limit is less than infinity and more than 0, we have that $S(n) \in \Theta(f(n))$.

Therefore, the bound is correct.

b)

Case 1: n is even and n > 1

$$S(n) = S(n-2) + 2n - 1$$

$$= S(n-4) + 2(n-2) - 1 + 2n - 1$$

$$= S(n-6) + 2(n-4) - 1 + 2(n-2) - 1 + 2n - 1$$

$$= S(n-6) + 2((n-4) + (n-2) + n) - 3$$

...

$$= S(n-k) + 2((n-k+2) + (n-k+4) + \dots + n) - \frac{k}{2}$$

...

$$= S(n-n) + 2((n-n+2) + (n-n+4) + \dots + n) - \frac{n}{2}$$

$$= S(0) + 2(2 + 4 + \dots + n) - \frac{n}{2}$$

$$= 0 + 2 \times \frac{(2+n) \times \frac{n}{2}}{2} - \frac{n}{2}$$

$$= \frac{n^2}{2} + \frac{n}{2}$$

Case 2: n is odd and n > 1:

$$S(n) = S(n-2) + 3n$$

$$= S(n-4) + 3(n-2) + 3n$$

$$= S(n-6) + 3(n-4) + 3(n-2) + 3n$$

...

$$= S(n-k) + 3[(n-k+2) + (n-k+4) + \dots + n]$$

Since LHS = RHS, $P(1)$ holds.

Inductive steps:

Let $n \in \mathbb{N}$, $n > 1$. Assume $H(n) : \forall i \in \mathbb{N}, 0 \leq i \leq n, P(i)$

We want to show $P(n)$.

Case 1: n is even.

Since $n > 1$, then $n - 2 \geq 0$.

Then $0 \leq n-2 < n$, by Induction hypothesis, $P(n-2)$ holds.

Since $(n-2)$ is even, we have $S(n-2) = \frac{(n-2)^2}{2} + \frac{n-2}{2}$.

$$S(n) = S(n-2) + 2n - 1 \quad (\text{By given recursive relationship})$$

$$= \frac{(n-2)^2}{2} + \frac{n-2}{2} + 2n - 1 \quad (\text{By inductive hypothesis})$$

$$= \frac{n^2 - 4n + 4}{2} + \frac{5n - 4}{2}$$

$$= \frac{n^2 + n}{2}$$

Then $P(n)$ holds.

Case 2: n is odd

Since $n > 1$, then $n - 2 \geq 0$.

Then $0 \leq n-2 < n$, by Induction hypothesis, $P(n-2)$ holds.

Since $(n-2)$ is odd, $S(n-2) = \frac{3}{4}(n-2)^2 + \frac{3}{2}(n-2) - \frac{5}{4}$.

$S(n) = S(n-2) + 3n$ (By given recursive relationship)

$$\begin{aligned} &= \frac{3}{4}(n-2)^2 + \frac{3}{2}(n-2) - \frac{5}{4} + 3n \\ &= \frac{3}{4}(n^2 - 4n + 4) + \frac{3}{2}n - 3 - \frac{5}{4} + 3n \\ &= \frac{3}{4}n^2 + \frac{3}{2}n - \frac{5}{4} \end{aligned}$$

Then $P(n)$ holds.

Therefore, the closed form is correct.

Let $f(n) = n^2$

Prove that $S(n) \in \Theta(f(n))$

In order to show that $S(n) \in \Theta(f(n))$, we have to show that as n goes to infinity asymptotically $S(n)$ and $f(n)$ grow the same.

Case 1: n is even

$$\lim_{n \rightarrow \infty} \left(\frac{S(n)}{f(n)} \right) = \lim_{n \rightarrow \infty} \frac{\frac{n^2+n}{2}}{n^2} = \frac{1}{2}$$

Since, the above limit is less than infinity and more than 0, we have that $S(n) \in \Theta(f(n))$.

Therefore, the bound is correct for all even natural number.

Case 2 : n is odd

$$\lim_{n \rightarrow \infty} \left(\frac{S(n)}{f(n)} \right) = \lim_{n \rightarrow \infty} \frac{\frac{3}{4}n^2 + \frac{3}{2}n - \frac{5}{4}}{n^2} = \frac{3}{4}$$

Since, the above limit is less than infinity and more than 0, we have that $S(n) \in \Theta(f(n))$.

Therefore, the bound is correct for all odd natural number.

Therefore, the bound is correct for all natural number.

2.

Prove for special case: $n = 2^k$, for some $k \in \mathbb{N}$.

$$S(n) = n \log n + 3n - 5 \quad (\text{for this particular special case})$$

$$\text{Let } f(n) = n \log n$$

In order to show that $S(n) \in \Theta(f(n))$ for this special case, we have to show that as n goes to infinity asymptotically S(n) and f(n) grow the same.

$$\lim_{n \rightarrow \infty} \left(\frac{S(n)}{f(n)} \right) = \lim_{n \rightarrow \infty} \frac{n \log n + 3n - 5}{n \log n} = 1$$

Since, the above limit is less than infinity and more than 0, we have that $S(n) \in \Theta(f(n))$ for this special case. Therefore, the bound is correct for this special case when $n = 2^k$, for some $k \in \mathbb{N}$.

Proving for general:

Let $n \in \mathbb{N}$, $n > 1$.

Case 1: $\exists k \in \mathbb{N}$, $n = 2^k$

Then $S(n) \in \Theta(n \log n)$ (we have proved it above)

Case 2: $\forall k \in \mathbb{N}$, $n \neq 2^k$

By Fact 1, $\exists k \in \mathbb{N}$, $2^{k-1} \leq n \leq 2^k$.

Then $2^k \leq 2n$, and $\frac{n}{2} \leq 2^{k-1}$.

Since $S(n)$ is monotonic non-decreasing,

then $S(2^{k-1}) \leq S(n) \leq S(2^k)$.

In order to prove $S(n) \in \Theta(n \log n)$, we need to prove $S(n) \in \mathbf{O}(n \log n)$ **and** $S(n) \in \mathbf{\Omega}(n \log n)$

Prove $S(n) \in \mathbf{O}(n \log n)$: i.e, $\exists c_1 \in \mathbb{R}^+$, $\exists n_0 \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $n \geq n_0 \implies S(n) \leq c_1 (n \log n)$.

Let $n_0 = 64$, $c_1 = 4$

Let $n \in \mathbb{N}$, $n \geq n_0$

$$S(n) \leq S(2^k)$$

$$= 2^k \times \log(2^k) + 3 \times 2^k - 5 \quad (\text{by given information})$$

$$\leq 2n \times \log(2n) + 3 \times (2n) - 5 \quad (\text{since } 2^k \leq 2n)$$

$$\leq 2n \times \log(2n) + 6n$$

$$= 2n \times \log(2) + 2n \log(n) + 6n$$

$$= 2n \times \log(n) + n(2 \times \log(2) + 6)$$

$$\begin{aligned}
&= 2n \times \log(n) + n(\log 4 + 6 \times \log(2)) && (\text{since } \log 2 = 1) \\
&\leq 2n \times \log(n) + n(\log n + \log(2^6)) && (\text{because } n \geq 64, \text{ so } n \geq 4) \\
&\leq 2n \times \log(n) + n(\log n + \log(n)) && (\text{because } n \geq 64) \\
&= 4 \times (n \log n) \\
&= c_1 (n \log n) && (\text{since } c_1 = 4)
\end{aligned}$$

Then, $S(n) \in O(n \log n)$.

Prove $S(n) \in \Omega(n \log n)$: i.e, $\exists c_2 \in R^+, \exists n_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow S(n) \geq c_2 (n \log n)$.

Let $n_1 = 20, c_2 = \frac{1}{2}$

Let $n \in \mathbb{N}, n \geq n_1$

$$\begin{aligned}
S(n) &\geq S(2^{k-1}) = 2^{k-1} \times \log(2^{k-1}) + 3 \times 2^{k-1} - 5 && (\text{by given information}) \\
&\geq \frac{n}{2} \times \log\left(\frac{n}{2}\right) + 3 \times \frac{n}{2} - 5 \times \log(n) && (\text{since } \frac{n}{2} \leq 2^{k-1}) \\
&= \left(\frac{n}{2} \times \log(n) - \frac{n}{2} \times \log(2)\right) + 3 \times \frac{n}{2} - 5 \times \log(n) \\
&= \frac{1}{2} (n \times \log(n) - n \times \log(2) + 3n - 10 \times \log(n)) \\
&= \frac{1}{2} (n \times \log(n) - n \times \log(2) + 3n \times \log(2) - 10 \times \log(n)) \\
&= \frac{1}{2} (n \times \log(n) + 2n \times \log(2) - 10 \times \log(n)) \\
&\geq \frac{1}{2} (n \times \log(n) - 10 \times \log(n)) && (\text{since } 2n \times \log(2) > 0) \\
&\geq \frac{1}{2} (n \log(n) - \frac{1}{2} \times n \log(n)) && (\text{since } n > 20, \text{ then } \frac{1}{2}n > 10) \\
&\geq \frac{1}{2} \left(\frac{1}{2} \times (n \log(n))\right) \\
&\geq \frac{1}{4} (n \log(n)) \\
&= c_2 (n \log(n))
\end{aligned}$$

Then $S(n) \in \Omega(n \log n)$

Therefore, $S(n) \in \Theta(n \log n)$

3.

a)

```
def brute_max(A):
    curr_max = A[0]
    n = len(A)
    for i in range(n):
        curr_sum = A[i]
        if curr_sum > curr_max:
            curr_max = curr_sum
        for j in range(i+1, n):
            curr_sum += A[j]
            if curr_sum > curr_max:
                curr_max = curr_sum
    return curr_max
```

Let n be the length of the input A , which is a natural number.

For a fixed iteration of the outer loop, the inner loop runs $n-1-i$ iteration(s), with each iteration taking constant time. The outer loop runs n iterations, for i going from 0 to $n-1$. The total cost is

$$\sum_{i=0}^{n-1} (n-1-i) = n(n-1) - \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}, \text{ which } \in \mathbf{O}(n^2).$$

b)

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n + 1 & \text{if } n > 1 \end{cases}$$

where $n \in \mathbb{N}$.

a = 2, b=2

c)

```
def find_max(A,b,e):
    if len(A) == 0:
        return None
    elif b == e:
        return A[b]
    else:
        m = (b + e) // 2
        sum_l, sum_r = 0, 0
        curr_max_l, curr_max_r = A[m], A[m+1]
        for i in range(m,b-1,-1):
            sum_l += A[i]
            if sum_l > curr_max_l:
                curr_max_l = sum_l
        for j in range(m+1,e+1):
            sum_r += A[j]
            if sum_r > curr_max_r:
                curr_max_r = sum_r
        max_middle = curr_max_l + curr_max_r
        return max(max_middle, find_max(A,b,m),find_max(A,m+1,e))
```

d)

Using master theorem to prove that $T(n) \in \Theta(n \log n)$.

Cost of splitting and recombining: $f(n) = n + 1$

The cost of splitting and combining $\in \Theta(n^d)$, which is $\in \Theta(n)$ for $T(n)$. This means, $d = 1$.

Since $a=2$, $b=2$, $d=1$, $a = b^d$, so $T(n) \in \Theta(n^d \log n)$.

Therefore, $T(n) \in \Theta(n \log n)$.

4.

a)

```
def f(x):  
    if x == 4:  
        return 1  
    elif x == 6:  
        return 1  
    elif x == 8:  
        return 1  
    else:  
        if x % 4 == 0: # if x could be divided by 4  
            return f(x-6) + 1 # 1 means the number of way that is all 4-cent,  
                               # f(x-6) means the number of ways that have at least one 6-cent.  
        else:  
            return f(x-6)
```

Let $n=4$, $n=6$, $n=8$ as base cases.

For all even number which is greater than 6, we classify it by whether or not it can be divided by 4 with no remainder. If it can be divided by four with no remainder, we can divide the ways of construction into two different group with no overlap. The first group is the way that this number is composed only by 4-cent. The second group is the set that each element of the set has at least one 6-cent. In other words, the element number of group 2 is $f(n-6)$. If it can not be divided by four with no remainder, it must have at least one 6-cent, so the number of construction is $f(n-6)$.

b)

```
def s(n):  
    if n == 2:  
        return 0  
    elif n == 4:  
        return 1  
    elif n == 6:  
        return 1  
    elif n == 8:  
        return 1  
    elif n == 10:  
        return 2  
    else:  
        return f(n) + s(n-10)  
    # f(n) means the number of ways that n is composed only by 4-cent and 6-cent,  
    # s(n-10) means the number of ways that have at least one 10-cent.
```

Let $H(n)$ denote the number of way that a postage of n cents, where $n \geq 4$ and n is even, can be only made by 4-cent stamps.

$$H(n) = \begin{cases} 1 & \text{if } (n \bmod 4) = 0 \\ 0 & \text{if } (n \bmod 4) \neq 0 \end{cases}$$

Let $T(n)$ denote the number of distinct ways that a postage of n cents, where $n \geq 4$ and n is even, can be made by 4-cent and 6-cent stamps.

$$T(n) = \begin{cases} 1 & \text{if } n = 4, 6, 8 \\ T(n-6) + H(n) & \text{if } n \geq 10 \end{cases}$$

Let $S(n)$ denote the number of distinct ways that a postage of n cents, where $n \geq 4$ and n is even, can be made by 4-cent, 6-cent and 10-cent stamps.

$$S(n) = \begin{cases} 1 & \text{if } n = 4, \text{ or } n = 6, \text{ or } n = 8 \\ 2 & \text{if } n = 10 \\ 2 & \text{if } n = 12 \\ T(n) + S(n-10) & \text{if } n > 12 \end{cases}$$

c)

Let $Q(n)$ denote the number of way that a postage of n cents, where $n \geq 4$ and n is even, can be only made by 6-cent stamps.

$$Q(n) = \begin{cases} 1 & \text{if } (n \bmod 6) = 0 \\ 0 & \text{if } (n \bmod 6) \neq 0 \end{cases}$$

Define $M(n)$ which is equivalent to $T(n)$:

$$M(n) = \begin{cases} 1 & \text{if } n = 4, 6, 8 \\ M(n-4) + Q(n) & \text{if } n \geq 10 \end{cases}$$

Defining predicate:

$P(n)$: “ $S(n) \geq S(n-2)$ ”, where $n \in \mathbb{N}$, $n \geq 4$ and n is even.

Base Case 1: $n = 4$

$P(4)$ is vacuous true since there is no $S(2)$ to compare.

Base Case 2: $n = 6$

$$S(6) = 1$$

$$S(4) = 1$$

$S(6) = S(4)$, then $P(6)$ holds.

Base Case 3: $n = 8$

$$S(8) = 1$$

$$S(6) = 1$$

$S(8) = S(6)$, then $P(8)$ holds.

Base Case 4: $n = 10$

$$S(10) = 2$$

$$S(8) = 1$$

$S(10) > S(8)$, then $P(10)$ holds.

Base Case 5: $n = 12$

$$S(12) = 2$$

$$S(10) = 2$$

$S(12) = S(10) = 2$, then $P(12)$ holds.

Base Case 6: $n = 14$

$$S(14) = (1+1) = 2$$

$$S(12) = 2$$

$S(14) = S(12)$, then $P(14)$ holds.

Base Case 6: $n = 16$

$$S(16) = (2+1) = 3$$

$$S(14) = 2$$

$S(16) > S(14)$, then $P(16)$ holds.

Base Case 6: $n = 18$

$$S(18) = 3$$

$$S(16) = 3$$

$S(18) = S(16)$, then $P(18)$ holds.

Base Case 6: $n = 20$

$$S(20) = 4$$

$$S(18) = 3$$

$S(20) > S(18)$, then $P(20)$ holds.

Base Case 6: $n = 22$

$$S(22) = 4$$

$$S(20) = 4$$

$S(22) = S(20)$, then $P(22)$ holds.

Base Case 6: $n = 24$

$$S(24) = 5$$

$$S(22) = 4$$

$S(24) = S(22)$, then $P(24)$ holds.

Inductive Steps:

Let $n \in \mathbb{N}$, $n > 24$ and n is an even number.

Assume $H(n)$: $\forall i \in \mathbb{N}$, $4 \leq i < n$ and i is even number, then $P(i)$, i.e., $S(i) \geq S(i-2)$

WTS: $P(n)$.

$$\begin{aligned}
S(n) - S(n-2) &= (T(n) + S(n-10)) - (T(n-2) + S(n-12)) \\
&= (T(n) - T(n-2)) + (S(n-10) - S(n-12)) \\
&= (T(n) - T(n-2)) + (T(n-10) + S(n-20) - T(n-12) - S(n-22)) \\
&\geq T(n) - T(n-2) + T(n-10) - T(n-12)
\end{aligned}$$

(By Inductive hypothesis, $4 \leq (n-22) < (n-20)$, and both $(n-22)$ and $(n-20)$ are even number, then $S(n-20) \geq S(n-22)$).

$$\begin{aligned}
&= T(n) + T(n-10) - T(n-2) - T(n-12) \\
&= H(n) + T(n-6) + H(n-10) + T(n-16) - T(n-2) - T(n-12) \\
&\quad \text{(By the definition of } T(n)) \\
&= H(n) + T(n-6) + H(n-10) + T(n-16) - M(n-2) - M(n-12) \\
&\quad \text{(Since } M(n) \text{ is equivalent to } T(n)) \\
&= H(n) + T(n-6) + H(n-10) + T(n-16) - Q(n-2) - M(n-6) - Q(n-12) - M(n-16) \\
&= H(n) + H(n-10) - Q(n-2) - Q(n-12) + (T(n-6) - M(n-6)) + (T(n-16) - M(n-16)) \\
&= H(n) + H(n-10) - Q(n-2) - Q(n-12) \\
&\quad \text{(Since } T(n) \text{ is equivalent to } M(n)) \\
&= H(n) + H(n-10) - (Q(n-2) + Q(n-12))
\end{aligned}$$

Case 1: $(n \bmod 4) = 2$ and $(n \bmod 6) = 0$

$$H(n) + H(n-10) - (Q(n-2) + Q(n-12)) = 0 + 1 - (0 + 1) = 0$$

Then, case 1 holds.

Case 2: $(n \bmod 4) = 2$ and $(n \bmod 6) \neq 0$

$$H(n) + H(n-10) - (Q(n-2) + Q(n-12)) = 0 + 1 - (Q(n-2) + 0) = 1 - Q(n-2) \geq 0$$

(Since $Q(n-2) = 1$ or 0)

Then, case 2 holds.

Case 3: $(n \bmod 4) = 0$ and $(n \bmod 6) = 0$

$$H(n) + H(n-10) - (Q(n-2) + Q(n-12)) = 1 + 0 - (0 + 1) = 0$$

Then, case 3 holds.

Case 4: $(n \bmod 4) = 0$ and $(n \bmod 6) \neq 0$

$$H(n) + H(n-10) - (Q(n-2) + Q(n-12)) = 1 + 0 - (Q(n-2) + 0) = 1 - Q(n-2) \geq 0$$

(Since $Q(n-2) = 1$ or 0)

Then, case 4 holds.

All cases hold, so $P(n)$ holds.

Therefore, $S(n)$ is non-decreasing.

5.

a)

```
def distance(a,b):  
    return ((a[0]-b[0])**2 + (a[1]-b[1])**2)**0.5
```

```
def find_closest(A):  
    n = len(A)  
    res = (A[0],A[1])  
    dis = distance(A[0],A[1])  
  
    i = 0  
  
    while i < n:  
        for j in range(i+1,n):  
            if distance(A[i],A[j]) < dis:  
                res = (A[i],A[j])  
                dis = distance(A[i],A[j])  
            i += 1  
  
    return res
```

b)

Let n be the length of the input A , which is a natural number.

For a fixed iteration of the outer loop, the inner loop runs $n-1-i$ iteration(s), with each iteration taking constant time. The outer loop runs n iterations, for i going from 0 to $n-1$. The total cost is

$$\sum_{i=0}^{n-1} (n-1-i) = n(n-1) - \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}, \text{ which } \in O(n^2).$$

c)

Let n be the length of input A and $n \in \mathbb{N}$.

Outer Loop Invariant:

- $i \leq n$,
- $\forall m \in \mathbb{N}, 0 \leq m \leq i-1, \forall s \in \mathbb{N}, 0 \leq s \leq n-1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$

Inner Loop Invariant:

- $i \leq j \leq n$
- $\forall p \in \mathbb{N}, i \leq p \leq j-1, \text{res} \leq \text{distance}(A[i-1], A[p])$

Prove for inner loop's partial correctness.

For a fixed i -th outer loop, assume the outer loop invariant holds for $(i-1)$ -th iteration.

Base Case:

At the end of 0-th iteration of inner loop,

$$j_0 = i, \text{ then } i \leq j_0 \leq n$$

Since $j_0 - 1 = i - 1 < i$, then $\forall p \in \mathbb{N}, i \leq p \leq j-1, \text{res} \leq \text{distance}(A[i-1], A[p])$ is vacuous true.

Inductive Steps:

Let $j_k \in \mathbb{N}$, where $k \in \mathbb{N}$

At the end of k -th iteration of inner loop,

assume $i \leq j_k \leq n$ and $\forall p \in \mathbb{N}, i \leq p \leq j_k - 1$, $\text{res} \leq \text{distance}(A[i-1], A[p])$ is true.

We want to show that at the end of $(k+1)$ -th iteration of inner loop, $\forall p \in \mathbb{N}, i \leq p \leq j_{k+1} - 1$, $\text{res} \leq \text{distance}(A[i-1], A[p])$ is true.

Case 1: there is no $(k+1)$ -th iteration of inner loop, then $j_{k+1} = j_k$, then $i \leq j_{k+1} \leq n$ and $\forall p \in \mathbb{N}, i \leq p \leq j_{k+1} - 1$, $\text{res} \leq \text{distance}(A[i-1], A[p])$ is true.

Case 2: there is $(k+1)$ -th iteration of inner loop, then $i \leq j_k < n$, then $j_{k+1} = j_k + 1$, then $i \leq j_{k+1} \leq n$.

Subcase 1: res updates because the condition of the if-statement in the inner loop is triggered, i.e., $\text{distance}(A[i-1], A[j_{k+1}-1]) < \text{dis}$ # dis is the previous minimum distance of two points which we have passed.

Subcase 2: res doesn't update because $\text{distance}(A[i-1], A[j_{k+1}-1]) \geq \text{dis}$.

By inductive hypothesis, $\forall p \in \mathbb{N}, i \leq p \leq j_k - 1$, $\text{res} \leq \text{distance}(A[i-1], A[p])$, combining with the result from subcase 1 and subcase 2,

then $i \leq j_{k+1} \leq n$ and $\forall p \in \mathbb{N}, i \leq p \leq j_{k+1} - 1$, $\text{res} \leq \text{distance}(A[i-1], A[p])$.

Prove for termination of inner loop:

Let $E_k = n - j_k$. By loop invariant, $j_k \leq n$. So $E_k \geq 0 \Rightarrow E_k \in \mathbb{N}, \forall k$.

If there is a $k+1$ iteration, then $E_{k+1} = n - j_{k+1} = n - (j_k + 1) < n - j_k = E_k$.

Thus, the inner loop terminates.

Prove for outer loop's partial correctness.

Base Case:

Since, $i_0 = 0$

Then $i_0 \leq n$,

Since, $i_0 - 1 < 0$, then,

$\forall m \in \mathbb{N}, 0 \leq m \leq i_0 - 1, \forall s \in \mathbb{N}, 0 \leq s \leq n - 1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$ is vacuous true.

Inductive Steps:

Let $i_k \in \mathbb{N}$, where $k \in \mathbb{N}$.

At the end of k -th iteration of outer loop,

assume that $i_k \leq n$ and

$\forall m \in \mathbb{N}, 0 \leq m \leq i_k - 1, \forall s \in \mathbb{N}, 0 \leq s \leq n - 1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$

WTS: at the end of $(k+1)$ th iteration of outer loop, $i_{k+1} \leq n$ and $\forall m \in \mathbb{N}, 0 \leq m \leq i_{k+1} - 1, \forall s \in \mathbb{N}, 0 \leq s \leq n - 1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$

Case 1: there is no $(k+1)$ th iteration of outer loop, then $i_{k+1} = i_k \leq n$ and

$\forall m \in \mathbb{N}, 0 \leq m \leq i_{k+1} - 1, \forall s \in \mathbb{N}, 0 \leq s \leq n - 1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$

(By Induction Hypothesis)

Case 2: there is $(k+1)$ th iteration of outer loop, it means $i_k < n$, then $i_{k+1} \leq n$.

Assume that at the end of $(k+1)$ -th iteration of outer loop, the post-condition of inner loop is correct, i.e., $\forall p \in \mathbb{N}, i_{k+1} \leq p \leq n - 1, \text{res} \leq \text{distance}(A[i_{k+1} - 1], A[p])$.

Combining with Induction Hypothesis, we can get:

$\forall m \in \mathbb{N}, 0 \leq m \leq i_{k+1} - 1, \forall s \in \mathbb{N}, 0 \leq s \leq n - 1, m \neq s \Rightarrow \text{res} \leq \text{distance}(A[m], A[s])$

Prove for outer loop's termination.

Let $E_k = n - i_k$. By loop invariant, $i_k \leq n$. So $E_k \geq 0 \Rightarrow E_k \in \mathbb{N}, \forall k$.

If there is a $k+1$ iteration, then $E_{k+1} = n - i_{k+1} = n - (i_k + 1) < n - j_k = E_k$.

Thus, the outer loop terminates.

Since, the partial correctness of inner loop and outer loop are both correct, and the inner loop and outer loop both terminate, the algorithm is correct.