MAT224H1S - Linear Algebra II
Winter 2020

Notes on Dimension:

That every basis for a given vector space V contains exactly the same number of vectors paves the way to the following definition.

Definition: Let V be a vector space and n a positive integer. If there is a list of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of vectors that is a basis for V, then V is *n-dimensional* (or V has *dimension n*). The zero vector space has *dimension zero*. If V has dimension n for some nonnegative integer n, then V is *finite dimensional*; otherwise V is *infinite dimensional*. If V is finite dimensional its dimension is denoted $\dim V$.

Standard Examples: $\dim \mathbb{R}^n = n$, $\dim P_n(\mathbb{R}) = n+1$, $\dim M_{m\times n}(\mathbb{R}) = mn$. → think about 2x2 matrix. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Example: Let $P(\mathbb{R})$ be the set of all polynomials with real coefficients with the same operations of vector addition and scalar multiplication as in $P_n(\mathbb{R})$. Then $P(\mathbb{R})$ is a vector space. As in $P_n(\mathbb{R})$, the zero vector in $P(\mathbb{R})$ is the zero polynomial. The vectors $1, x, x^2, \ldots, x^n$ are linearly independent in $P(\mathbb{R})$ for each $n = 1, 2, \ldots$. The Fundamental Theorem says that if $P(\mathbb{R})$ is finite dimensional, then $\dim P(\mathbb{R}) \geq n+1$. Since this is impossible, $P(\mathbb{R})$ is infinite dimensional.

if $1, x, x^2, \cdots, x^n$ are L.I, $n+1 \leq \dim P_n(R)$

since $1, x, x^2, \cdots, x^n$ are L.I for each $n = 1, 2, 3, \cdots$, → there are infinitely many choices of n.
the dimension of $P(R)$ is

Exercise and Discussion: Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ be vectors in a vector space V. Suppose that $\mathbf{x}_3 = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{x}_4 = 2\mathbf{x}_1 + 3\mathbf{x}_2 - \mathbf{x}_3$, and that $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$.

(a) What are the possible dimensions of U?

(b) Suppose $\dim U = 2$. Must $\{\mathbf{x}_3, \mathbf{x}_4\}$ be linearly independent?

(a) $x_3 = x_1 - x_2 \Rightarrow$ a L.C. of x_1, x_2.

$x_4 = 2x_1 + 3x_2 - x_3$
$= 2x_1 + 3x_2 - (x_1 - x_2)$.
$= x_1 + 4x_2$. $\Rightarrow$ a L.C. of x_1 and x_2.

possible dimensions of U: 0, 1, 2. → x_1 and x_2 are L.I.
$x_1 = x_2 = 0$; x_1 and x_2 are L.D

(b). Suppose $\dim U = 2$, x_1 and x_2 are L.I $\Rightarrow U = \operatorname{span}\{x_1, x_2\}$

$x_3 = x_1 - x_2$
$x_4 = x_1 + 4x_2$.

$a x_3 + b x_4 = 0$.
$a(x_1 - x_2) + b(x_1 + 4x_2) = 0$.
$(a+b)x_1 + (4b-a)x_2 = 0$.
$\because$ x_1 and x_2 are L.I.
$\therefore \begin{cases} a+b=0 \\ 4b-a=0. \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0. \end{cases} \Rightarrow x_3$ and x_4 are L.I.

The following theorem says two things about a finite dimensional vector space V: (a) any list of vectors that span V contains a shorter list that is a basis for V; and (b) any linearly independent list of vectors can be extended to a longer list that is a basis for V.

Theorem: Let V be a nonzero vector space and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r \in V$.

(a) If $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\} = V$, then $n = \dim V \leq r$, and there are indices $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, r\}$ such that the list $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \ldots, \mathbf{x}_{i_n}$ is a basis for V.

(b) If $n = \dim V > r$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ are linearly independent, then there are $n-r$ vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n-r} \in V$ such that the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n-r}$ is a basis for V.

If $V = \operatorname{span}\{x_1\} \Rightarrow \dim V = 1$, $\{x_1\}$ is a basis
If $\{x_1\}$ is not a basis for V, there exists $n-r$ vectors that can
$\Updownarrow$
$V \neq \operatorname{span}\{x_1\}$

any spanning set can be reduced to a basis (by removing redundant vectors until the number of vectors = dimension of V).

a spanning set contains a basis.

independent set can be enlarged into a basis.

Exercise and Discussion: Consider the vectors $p(x) = 1 + x$, and $q(x) = 1 + x + x^2$ in $P_2(\mathbb{R})$. Find a independent set can be enlarged into a basis.

Exercise and Discussion: Consider the vectors $p(x) = 1 + x$, and $q(x) = 1 + x + x^2$ in $P_2(\mathbb{R})$. Find a third vector $r(x)$ such that the list $p(x), q(x), r(x)$ is a basis for $P_2(\mathbb{R})$.

$\{1+x,\ 1+x+x^2,\ 1\}$ is L.I.

because $0 \cdot (1+x) + 0 \cdot (1+x+x^2)$ is the only linear combination to represent 1

dimension of $P_2(\mathbb{R})$ is $n+1 = 2+1 = 3 \Rightarrow \{1+x,\ 1+x+x^2,\ 1\}$ is a basis for $P_2(\mathbb{R})$.

Exercise and Discussion: Let V be an n-dimensional vector space and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in V$. In no more than three sentences, explain why the following statements are true. If it takes you more than three sentences, see if you can find a more elegant explanation.

(a) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ spans V, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V

(b) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is linearly independent, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V.

(a) According to the "Theorem" above,

$\text{span}\{x_1, x_2, \cdots, x_n\} = V$, $n = \dim V$

$\Downarrow$ show $x_1, x_2, \cdots, x_n$ are L.I.

$\{x_1, x_2, \cdots, x_n\}$ is a basis for V.

if - - - L.D, can reduce $\text{span}\{x_1, x_2, \cdots, x_n\}$ to a basis by removing redundant vectors.

The following theorem details the dimensional relationship between a subspace and its parent space.

Theorem: Let U be a subspace of an n-dimensional vector space V. Then U is finite dimensional and $\dim U \leq n$, with equality *iff* $U = V$.

This theorem guarantees that for any two subspaces U and W of a finite dimensional vectors space V, their sum $U + W$ and intersection $U \cap W$ are both finite dimensional since each is a subspace of V. Furthermore, $U \cap W$ is a subspace of U and of W, so any basis for $U \cap W$ can be extended to a basis for U; it can also be extended to a basis for W. Paying close attention to how these bases interact leads to the following result.

Theorem: Let U and W be subspaces of a finite dimensional vector space V. Then

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$