CSC236 Tutorial Exercises, July 19
Suggested Solutions

1. Give a DFA accepting the set of all strings beginning with a 1 that, when interprented as a binary integer, is a multiple of 3. For example the strings 11, 110, and 1001 are in the above language, but the strings, 1, 101, 111, 01 are not. Prove by state invariant that your DFA is accepts the requested language.

   The trick is to realize that reading another bit either multiplies the number seen so far by 2 (if it is a 0), or multiplies by 2 and then adds 1 (if it is a 1). We don't need to remember the entire number seen, just its remainder when divided by 3. That is, if we have any number of the form $3a + b$, where b is the remainder, between 0 and 2, then $2(3a + b) = 6a + 2b$. Since $6a$ is surely divisible by 3, the remainder of $6a + 2b$ is the same as the remainder of $2b$ when divided by 3. Since b, is 0, 1, or 2, we can find the answers easily. The same idea holds if we want to consider what happens to $3a + b$ if we multiply by 2 and add 1.

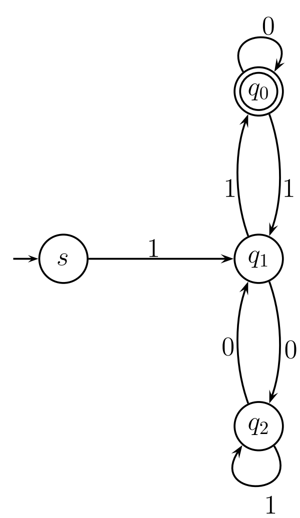


Figure 1: We can see the state q_i as all the number that leave remainder i divided by 3 (in binary).

   Now, we want to prove by state invariant the correctness of the above DFA. Choose the state invariant to be that $\delta^*(s, w)$ is q_i if the binary representation of the string $w \in 1\{0,1\}^*$ is $i \mod 3$.

   Base Case: Let $w = 1$, then by the construction of the above DFA, $\delta(s, w) = q_1$ and the statement is true.

Induction Step: Suppose that the state invariant holds for all $w \in 1\{0,1\}^n$. Let $w = w'a$ for $w' \in\in 1\{0,1\}^n$ and $a \in\in \{0,1\}$. By the induction hypothesis we have that $\delta^*(s, w')$ is q_i if the binary representation of the string $w \in 1\{0,1\}^*$ is $i \mod 3$. Let k_1 be the binary representation of w' and k_2 the binary representation of $w'a$. Notice that for $a = 0$ we have that $k_2 = 2 \cdot k_1$, similarly for $a = 1$ we have that $k_2 = 2 \cdot k_1 + 1$. Then, we have 3 cases:

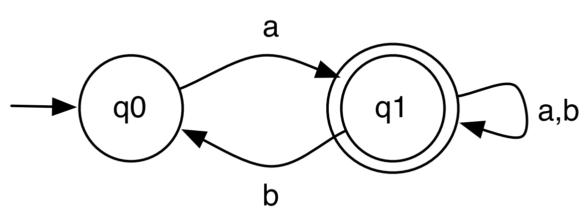
Case 1: The k_1 is $0 \mod 3$, then by the IH we have that $\delta^*(s, w') = q_0$. For $a = 0$, the number $k_2 = 2 \cdot k_1$ is also $0 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_0, a) = q_0$.
For $a = 1$, the number $k_2 = 2 \cdot k_1 + 1$ is $1 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_0, a) = q_1$.

Case 2: The k_1 is $1 \mod 3$, then by the IH we have that $\delta^*(s, w') = q_1$. For $a = 0$, the number $k_2 = 2 \cdot k_1$ is also $2 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_1, a) = q_2$.
For $a = 1$, the number $k_2 = 2 \cdot k_1 + 1$ is $0 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_1, a) = q_0$.

Case 3: The k_1 is $2 \mod 3$, then by the IH we have that $\delta^*(s, w') = q_2$. For $a = 0$, the number $k_2 = 2 \cdot k_1$ is also $1 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_2, a) = q_1$.
For $a = 1$, the number $k_2 = 2 \cdot k_1 + 1$ is $2 \mod 3$, moreover we have that $\delta^*(s, w'a) = \delta(\delta^*(s, w'), a) = \delta(q_2, a) = q_2$.

Hence, for every string $w \in 1\{0,1\}^*$ we have $\delta^*(s, w) = q_i$ if the binary representation of the string w is $i \mod 3$.

2. Let the alphabet be $\Sigma = \{a, b\}$, consider an FSA M_1:



(a) Is this a DFA or an NFA? Why?
Answer: This is an NFA, because $\delta(q_1, b) = \{q_1, q_0\}$, i.e. there is more than one path to take upon observing b in state q_1. Also, the transition $\delta(q_0, b)$ is not determined but sometimes dead states are omitted even in drawing DFSA, so you have to be careful calling FSA an NFA just because dead states are missing.

(b) Write down the language $\mathcal{L}$ that it represents (a sentence describing all strings included in the language $\mathcal{L}$)
Answer: This language contains all strings that start with an a

(c) Write down the complement $\overline{\mathcal{L}}$ of $\mathcal{L}$, i.e. $\overline{\mathcal{L}} = \Sigma^* - \mathcal{L}$ in one sentence
Answer: Language $\overline{\mathcal{L}}$ contains all strings in Σ^* that do not start with an a (including empty string).

(d) Draw an FSA for $\overline{\mathcal{L}}$