第一讲 Simple Induction, Complete Induction, Principle of Well-Ordering

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此份讲义提供完整答案。报名后可在 Easy 4.0 学员系统 Moodle 下载。

0.General:
简介：
课程内容：4 parts
part1: induction and wop
part2: recurrence relation and complexity
part3: program correctness
part4: Formal language and automata

逻辑顺序： 这门课名字为 intro to theory of computation， 就是计算理论的导论， 计算理论有两个重要的议题：程序是否正确 还有时间复杂度，为了证明我们写的程序是正确的，我们需要先了解 induction：证明程序正确性的方法。165 中讲到了 iterative program complexity anaylsis， 236 中我们将进一步探讨 recursive program analysis，进一步提高分析程序复杂度的能力。 最后，学习 formal language 和 automata 是为了证明程序 syntax 正确。总而言之，这门课内容承前启后，为日后学习算法 / 数据结构打下了理论上的基础。
评分标准： 3 assignment 1 midterm 1 final and 10 quiz

1.Simple Induction
Simple induction 也叫 ordinary induction，我们大部分同学在 CSC165 中都有所接触。
回顾：Predicate: a Boolean function
$P(n)$: 1+2+⋯+n=n(1+n)/2
$P(n)$: $\exists k \in \mathbf{N}, n = 2k$
在 induction 中通常研究与 natural number 有关的 predicate.
Simple induction 通常用于证明 $P(n)$ is true for all natural numbers n.

遇到一道 induction 有关的题目，首先要定义一个跟 natural number 有关的 predicate，这步非常关键，通常在作业或考试中占 1 分，另外定义出这个 predicate 的好处是可以帮助你理解 what you induction on

定义 predicate! 不要忘

格式: To prove: $\forall n \in \boldsymbol{N}, P(n)$
Proof: Base Case: Prove P(0) is true.
Induction Step: Induction Hypothesis: Let $\mathrm{k} \in \mathrm{N}$. Assume P(k) is true.
Want to prove: P(k+1)
…
… some step by induction hypothesis
…(some steps)

P(k+1) is true.

对于 Base case 不是从 0 开始的情况

格式: To prove $\forall n \in \boldsymbol{N}, n \geq a \rightarrow P(n)$ for some $a \in \boldsymbol{N}$

Proof: Base Case: Prove P(a) is true.

Induction Step: Induction Hypothesis: Let $k \in \boldsymbol{N}$ and $k \geq a$. Assume P(k) is true.

Want to prove: P(k+1)

…

… some step by induction hypothesis

…(some steps)

P(k+1) is true.

Example:

A binary string is a (possibly empty) sequence of 0's and 1's. Let $B(n)$ be the number of binary strings of length n. Use simple induction to prove that for all $n \in \mathbb{N}$, $B(n) = 2^n$.

Exercise: Prove $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for all n≥0

2. Complete Induction

Complete induction 也叫 strong induction，在 winter 2016 或 summer 2016 学习 CSC165 的同学，以及学习 MAT137/157 的同学有所接触。

Complete induction 和 simple induction 唯一的区别就在 induction hypothesis 上，complete induction 有更强的 induction hypothesis.

To prove P(n) for all $n \geq a$ *(a is some natural number)*

Proof: Base Case: Prove P(a) is true.

Induction Step: Induction Hypothesis: Let $k \in \boldsymbol{N}$ and $k \geq a$. Assume P(*j*) is true for all $a \leq j < k$.

Want to prove: P(k)

...

⋯ some step by induction hypothesis

⋯(some steps)

P(k) is true.

Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$\begin{aligned} h_0 &= 1, \\ h_1 &= 2, \\ h_2 &= 3, \\ h_k &= h_{k-1} + h_{k-2} + h_{k-3} \qquad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

Example 2:

Prove that any natural number $n \geq 2$ has a prime factorization.

5. More Exercise（取自往年的作业及练习，附有答案，建议多练习，练习时不必写完整步骤，有思路即可，在真正写作业时，一定要格式完整）

(1) .

Use induction to prove that $3^{2n} - 1$ is divisible by 8, for all $n \in \mathbb{N}$.

(2) .

Assume $x \in \mathbb{R}$ and $(x + \frac{1}{x}) \in \mathbb{Z}$. Use induction to prove that for all $n \in \mathbb{N}$

$$(x^n + \frac{1}{x^n}) \in \mathbb{Z}.$$

(3) A **ternary tree** is a tree where each node has no more than 3 children, and the **height** of a tree is defined as the number of nodes in the longest path[1] from the root to any leaf. Use Complete Induction to prove that if a[2] ternary tree has height n, it has no more than $3^n - 2$ nodes.

(4) Use Complete Induction or Mathematical Induction[3] to prove that any binary string that begins and ends with the same bit has an even number of occurrences of substrings from {01, 10}, e.g. 010 has two: 01 and 10. You may find it useful to combine this claim with a similar claim about binary strings that begin and end with different bits, and then prove the combined claims simultaneously.

(5) Use the well-ordering principle to prove that every natural number greater than 1 is divisible by a prime number.