CSC373 Assignment 2 Eric Bannatyne
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1. (a) Define the decision problem D_0 as follows.
Input: A string $x \in \Sigma^*$, where $\Sigma = \{0, 1\}$.
Output: Does $x = 0$?
To show that $D_0 \leq_p \overline{D_0}$, we wish to construct in polynomial time a function $f : \Sigma^* \to \Sigma^*$ such that x is a yes-instance of D_0 if and only if $f(x)$ is a yes-instance of $\overline{D_0}$ (or equivalently a no-instance of D_0). The reduction function f is simply defined as follows.

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

This function can trivially be computed in constant time, which means that it is computable in polynomial time. Moreover, if x is a yes-instance of D_0, then $x = 0$, so $f(x) = 1$, which is a no-instance of D_0. Conversely, if x is a no-instance of D_0, then $f(x) = 0$, which is the only yes-instance of D_0. Therefore $D_0 \leq_p \overline{D_0}$.

(b) The problem of determining whether a binary input string x is equal to the string 0 is clearly decidable in constant time, and in particular $D_0 \in$ P. Since P $\subseteq$ NP, it follows that $D_0 \in$ NP.

(c) Let D_1 be NP-complete and suppose $D_1 \leq_p \overline{D_1}$. Then there is some reduction function $f : \Sigma^* \to \Sigma^*$ computable in polynomial time such that x is a yes-instance of D_1 if and only if $f(x)$ is a yes-instance of $\overline{D_1}$. This implies that we also have $\overline{D_1} \leq_p D_1$, by using the exact same reduction function f.
Since $D_1 \in$ NP, it follows that $\overline{D_1} \in$ coNP. Moreover, a language L is NP if and only if $\overline{L}$ is coNP. Since D_1 is NP-complete, we have $L \leq_p D_1$, and equivalently (by the same reasoning as above), $\overline{L} \leq_p \overline{D_1}$. Therefore $\overline{D_1}$ is coNP-complete.
Let $A \in$ NP. Then $A \leq_p D_1 \leq_p \overline{D_1}$. Since $\overline{D_1} \in$ coNP it follows that $A \in$ coNP. Therefore NP $\subseteq$ coNP.
Similarly, let $B \in$ coNP. Then $B \leq_p \overline{D_1} \leq_p D_1$, since $\overline{D_1}$ is coNP-complete. Then $B \in$ NP. Hence coNP $\subseteq$ NP.
Therefore, combining these two statements, it follows that NP = coNP, under the assumption that such a problem D_1 exists.