MAT224H1S - Linear Algebra II
Winter 2020

Notes on Subspaces:

Definition: A *subspace* of a vector space V is a subset W of V that is itself a vector space with the same operations of vector addition and scalar multiplication as in V.

Exercise and Discussion:

1. Why is it that every subspace is non-empty?

2. Which vector space axioms does W automatically inherit from V?

3. What's left then to show that W is a subspace?

Fortunately, there is a streamlined way to do this.

Theorem: A non-empty subset W of a vector space V is a subspace of V *iff* $c\mathbf{x}+\mathbf{y} \in W$ whenever $\mathbf{x}, \mathbf{y} \in W$, and $c \in \mathbb{R}$.

Example: Lines and planes through the origin in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

Example: For $A \in M_{m\times n}(\mathbb{R})$, the null space of A

$$\text{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$; and the column space of A

$$\text{col}(A) = \{A\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} \in \mathbb{R}^n\}$$

is a subspace of $\mathbb{R}^m$.

Exercise and Discussion: Let $n \geq 2$ Which of the following subsets W of $M_{n\times n}(\mathbb{R})$ are subspaces of $M_{n\times n}(\mathbb{R})$?

(a) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ is invertible}\}$

(b) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ is not invertible}\}$

(c) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{ the last column of } A \text{ is zero}\}$

(d) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A^2 = \mathbf{0}\}$