Easy 4.0 STA 457 Time Series Analysis
Midterm Review – Part 1
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知识点

1. Basic Definitions
2. Calculate Autocorrelation function and Cross-correlation function
3. Prove weakly stationary time series
4. Calculate Partial Autocorrelation function
5. ARMA model
   - invertible & casual
   - model identification

1. Basic Definitions

1.1 Weakly Stationary

Define weak stationary of a time series
(midterm practice: indtroduction-Q1)

The time series $X_t, t \in Z$, with index set $Z = \{0, \pm 1, \pm 2, \ldots\}$, is said to be stationary if
(1) $E(|X_t|^2) < \infty$ for all $t \in Z$
(2) $E(X_t)$ = constant for all $t \in Z$
(3) $\gamma_x(r, s) = \gamma_x(t + r, t + s)$ for all $t, r, s \in Z$
i.e. $\gamma_x(\cdot)$ is a constant and $r - s$ is independent of t

1.2 Steps of modeling time series

State the steps of modeling time series data
(midterm practice: indtroduction-Q3)

Step1: Plot the time series and check for trend, seasonal and other cyclic components, any apparent
sharp changes in behavior, as well as any outlying observations
Step2: Remove trend and seasonal components to get residuals
Step3: Choose a time series model to fit the residuals
Step4: Forecasting can be carried out by forecasting residual and then inverting the transformation
carried out in Step

1.3 Classical Decomposition

Define Classical Decomposition
(midterm practice: indtroduction-Q2)

Classical decomposition decomposes time series into the following components:
(1) "Trend" refers to a long-term change in the mean level;
(2) Seasonal variation refers to the (regular) variation that is annual in period;
(3) Cyclical variation refers to the (regular) variation at a fixed period;
(4) Irregular variation refers to a weakly stationary noise terms.

1.4 Wold Decomposition

State the Wold decomposition. Discuss the reason why Wold decomposition supports the use of ARMA models for modeling time series
(midterm practice: ARMA models-Q1)

i. A stochastic process $\{V_t\}$ is called deterministic if the values of V_n are perfectly determined by $\{V_t, -\infty < t < n\}$ i.e. $V_n \in \text{span}\{V_t, -\infty < t < n\}$. Thus, a deterministic process has the form of an AR(∞) process.

ii. Wold decomposition: any nondeterministic stationary process $\{X_t\}$ can be expressed as a sum $X_t = U_t + V_t$, where $\{V_t\}$ is a deterministic process, and $\{U_t\}$ is an MA(∞) process, and $\{U_t\}$ and $\{V_t\}$ are uncorrelated.

The Wold decomposition implies that any nondeterministic stationary process can be expressed as the form of an ARMA(∞,∞)process so we may be able to approximate any data/process by an ARMA(p,q) model with sufficiently large p and q.

1.5 Moving Average Model – MA(q)

1.6 Auroregressive Model – AR(p)

2. Calculate Autocorrelation function and Cross-correlation function

2.1 Autocovariance Function and Autocorrelation function (ACF)

Definition

If $\{X_t, t \in T\}$ is a process such that $var(X_t) < \infty$ for each $t \in T$,

then the **autocovariance function** $\gamma_X(\cdot,\cdot)$ of $\{X_t\}$ is defined by

$$\boxed{\gamma_X(r,s)} = cov(X_r, X_s) = E[(X_r - \boxed{EX_r})(X_s - EX_s)], \qquad \forall r, s,$$

and the **autocorrelation function** is given by

$$\rho(r,s) = \gamma_X(r,s)/\sqrt{\boxed{\gamma_X(r,r)}\gamma_X(s,s)}\,.$$

2.1.1 ACF for MA(q) model

Formula

Autocovariance Function: $\gamma(k) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \, \psi_{j+|k|}$

Autocorrelation function: $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$

Steps

Steps1: Rewrite given MA model in terms of $X_t = \sum_{j=0}^{\infty} \psi_j \, a_{t-j}$. In this way, we can get ψ_j

Step2: Apply above formula

Midterm practise question

Introduction - Q4(1)

Consider $x_t = \frac{1}{3}a_t + \frac{1}{3}a_{t-1}$ where $a_t \sim NID(0, \sigma_a^2)$. And $y_t = \frac{1}{2}x_t + \frac{1}{2}x_{t-2}$

(1) Calculate the autocovariance function for x_t and y_t

2.1.2 ACF for AR(p) model

Formula

Autocovariance Function:
Special case: for AR(1) model $x_t - \phi x_t = a_t, \gamma(k) = \phi^k \times \gamma(0)$

Autocorrelation function:
For AR(p) model, $x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \cdots - \phi_p x_{t-p} = a_t$
$\rho(k) = \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}$

Prove

P4. Proof:

1. multiply the equation by X_{t-1} : $X_t X_{t-1} = \phi X_{t-1}^2 + X_{t-1} a_t$ MA(q)

2. Apply expectation: $E(X_t X_{t-1}) = \phi E(X_{t-1}^2) + E(a_t X_{t-1})$.

$E(X_t) = \phi_1 E(X_{t-1}) + \cancel{E(a_t)}^{\to 0}$. Given weakly stationary. we have constant $E(X_t)$ $\forall t$. $\Rightarrow E(X_t) = E(X_{t-1})$. $(1-\phi_1)E(X_t) = 0$.

$\because$ stationary. $\therefore |\phi_1| < 1 \Rightarrow E(X_t) = 0$.

$cov(X_t, X_{t-1}) = E(X_t X_{t-1}) - \cancel{E(X_t)E(X_{t-1})}^{\to 0} \Rightarrow cov(X_t, X_{t-1}) = E(X_t X_{t-1})$

① $E(X_t \cdot X_{t-1}) = E(\phi X_{t-1}^2 + X_{t-1} a_t) = \phi \cdot E(X_{t-1}^2) + E(a_t X_{t-1}) = \phi \cdot var(X_{t-1}) + E(a_t X_{t-1})$.

$\because Var(X_{t-1}) = E(X_{t-1}^2) - \cancel{E(X_{t-1})}^{\to 0}$ $= \phi \cdot \Gamma(0) + \underbrace{E(a_t X_{t-1})}_{=0}$.

Recall: $Var(X_{t-1}) = cov(X_{t-1}, X_{t-1}) = \Gamma(0)$

$E(a_t X_{t-1}) = cov(a_t, X_{t-1})$ Given $X_t = \sum_{j=0}^{\infty} \phi^j a_{t-j}$

$= cov(a_t, \sum_{j=0}^{\infty} \phi^j a_{t-1-j})$ $X_{t-1} = \sum_{j=0}^{\infty} \phi^j a_{t-1-j}$

$= 0$.

① $\Rightarrow E(X_t \cdot X_{t-1}) = cov(X_t, X_{t-1}) = \phi \cdot \Gamma(0) \Rightarrow \Gamma(1) = \phi \cdot \Gamma(0)$

By induction: multiply X_{t-2} & expectation $\cdots \Rightarrow \Gamma(k) = \phi^k \cdot \Gamma(0)$.

Midterm practise question
ARMA Model - Q4(a)

Consider a stationary AR(2) model

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t, \qquad a_t \sim NID(0,1).$$

a. Calculate the autocorrelation functions of x_t for lags 1, 2, and 3.

<u>Past midterm(2017)</u>

Consider $(1-0.5B)(1-0.1B)X_t = a_t$ $\qquad a_t \sim NID(0,1)$

Find the autocorrelation function at lag 1,2,3 for this AR(2) process.

Hint: use systems of ACF

$$(1-0.6B+0.05B^2)X_t = a_t$$

$$X_t - 0.6X_{t-1} + 0.05X_{t-2} = a_t \Rightarrow X_t = \underbrace{0.6}_{\phi_1}X_{t-1} \underbrace{-0.05}_{\phi_2}X_{t-2} + a_t$$

$p=2$.

$$\rho(k) = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} = 0.6\rho_{k-1} - 0.05\rho_{k-2}.$$

- $k=0$. $\quad \rho_0 = \frac{\Gamma(0)}{\Gamma(0)} = 1$ Always.
- $k=1$.

$$\rho_1 = 0.6\rho_0 - 0.05\rho_{-1} \qquad \boxed{\rho_{-1} = \rho_1} \text{ Always.}$$

$$\rho_1 = 0.6\rho_0 - 0.05\rho_1 \Rightarrow 1.05\rho_1 = 0.6\rho_0$$

$$\Rightarrow \rho_1 = \frac{0.6}{1.05} = 0.57.$$

- $k=2$

$$\rho_2 = 0.6\rho_1 - 0.05\rho_0$$

$$= 0.6 \times 0.57 - 0.05 = 0.292$$

- $k=3$.

$$\rho_3 = 0.6\rho_2 - 0.05\rho_1$$

$$= 0.6 \times 0.292 - 0.05 \times 0.57$$

$$= 0.1455.$$

2.2 Cross-correlation function (CCF)

Definition

The **cross-covariance/correlation** between $\{X_t\}$ and $\{Y_t\}$ are

$$\boxed{\gamma_{XY}(r,s)} = E[(X_r - EX_r)(Y_s - EY_s)],$$

$$\rho_{XY}(r,s) = \gamma_{XY}(r,s)/\sqrt{\gamma_X(r,r)\gamma_Y(s,s)}, \forall r,s,$$

Midterm practise question

Introduction - Q4(2)

Consider $x_t = \frac{1}{3}a_t + \frac{1}{3}a_{t-1}$ where $a_t \sim NID(0, \sigma_a^2)$. And $y_t = \frac{1}{2}x_t + \frac{1}{2}x_{t-2}$

(2) Calculate the cross-correlation function between x_t and y_t

3. Prove weakly stationary time series

3.1 Prove MA(q)

Steps

Step1: prove $E\ (X_t)$ = constant $< \infty$ for all $t \in Z$

Step2: prove Var (X_t) = constant $< \infty$ for all $t \in Z$

Step3: prove autocovariance function $\gamma(t, t+k)$ is independent if time t and is a function of k

Past test

Given a MA(1) process: $Y_t = a_t + 0.6\ a_{t-1}$, $a_t \sim NID(0,1)$

Prove this is a weakly stationary time series

$\forall t$ $\quad Var(a_t) = 1 \quad \forall t$

- Prove stationary

step 1: prove $E(Y_t) = m < \infty$ is a constant

2: prove $Var(Y_t) = \sigma_Y^2 < \infty$ is a constant

3: auto-covariance function $\Gamma(t, t+h)$ is independent of time t and is a function of h

① : $E(Y_t) = E(a_t + 0.6 a_{t-1}) = E(a_t) + 0.6 E(a_{t-1}) = 0 + 0 = 0$ constant & $< \infty$

② : $\underset{\Gamma(0)}{Var(Y_t)} = Var(a_t + 0.6 a_{t-1})$ $\because$ indep

$= Var(a_t) + 0.6^2 Var(a_{t-1})$

$= 1 + 0.6^2 \times 1 = 1.36$ constant

③ : $Y_t = a_t + 0.6 a_{t-1} = \sum_{j=0}^{\infty} \psi_j a_{t-j}$

$= \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$

$\Rightarrow \psi_0 = 1, \quad \psi_1 = 0.6. \quad \psi_2 = 0. \quad \psi_3 = 0 \cdots$

$\psi_i = 0$ if $i \geq 2$

$|K| = 1$ $\quad \Gamma(K) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \cdot \psi_{j+1}$

$= \sigma^2 (\psi_0 \cdot \psi_1 + \psi_1 \cdot \psi_2 + \psi_2 \cdot \psi_3 + \cdots)$

$= \sigma^2 (1 \times 0.6 + 0 + 0) = 0.6\sigma^2 = 0.6$

$|K| = 2$ $\quad \Gamma(K) = \sigma^2 (\psi_0 \cdot \psi_2 + \psi_1 \cdot \psi_3 + \cdots) = 0$

for $K = \pm 2 \quad \pm 3 \quad \pm 4 \cdots \quad \Gamma(K) = 0.$

for $K = \pm 1 \quad \Gamma(K) = 0.6$

$\Rightarrow$ auto covariance of Y_t is function of K which is independent of time t. So Y_t is ....

- Auto-correlation

formula: $\rho(K) = \frac{\Gamma(K)}{\Gamma(0)} = \frac{\Gamma(K)}{1.36} = \begin{cases} \frac{0.6}{1.36} & \text{if } |K| = 1 \\ \frac{0}{1.36} & \text{if } |K| \geq 2 \end{cases}$

3.1 Prove AR(p)

The weakly stationary condition of autoregressive processes is that the roots of the following equation $\Phi(B) = 0$ must lie **outside** the unit circle.

Past exam (2017 Midterm)
Consider a AR (2) process, $(1-0.5B)(1-0.1B)X_t = a_t,\ a_t \sim NID(0,1)$
Is the above AR (2) process stationary? Explain your answer.

$(1-0.5B)(1-0.1B) = \Phi(B)$

- get root $\Phi(B) = 0$ $(1-0.5B)(1-0.1B) = 0$ $B_1 = 2$ $B_2 = 10$

$|B| > 1$ $\therefore$ roots lie outside of unit circle

$\Rightarrow$ stationary.

4. Calculate Partial Autocorrelation function

4.1 Yule – Walker equations

Definition:
Yule-Walker equation: mean zero weakly stationary process X_t
$X_t = \phi_{k1}X_{t-1} + \phi_{k2}X_{t-2} + \cdots + \phi_{kk}X_{t-k} + e_t$, k is the given order

Result:
$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \cdots + \phi_{kk}\rho_{j-k}$, j=1,2,…,k
We can write autocovariance and autocorrelation in matrix form.

$$\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(k) \end{bmatrix} = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(k-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(k-1) & \gamma(k-2) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix}$$

$$\begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) & \cdots & \rho(k-1) \\ \rho(1) & 1 & \cdots & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} \quad \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & & 1 \end{pmatrix}$$

if $k=1$. $\rho_1 = [1]\,\phi_{11}$

if $k=2$. $\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix}$

if $k=3$ $\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{pmatrix}$

4.2 Partial Autoregression Function (PACF)

Definition

The conditional correlation

$$\phi_{kk} = corr(X_t, X_{t+k} | X_{t+1}, \dots, X_{t+k-1})$$

<u>Mean</u>

The correlation between X_t and X_{t+k} after mutual linear dependency on the intervening variables, $X_{t+1}, X_{t+2}, \dots,$ and X_{t+k-1} has been removed.

<u>Example</u>

首先理解，要是理解不了就直接背公式。在考试中最多 order=3， 不会计算量很大。

Consider a AR(2) model: $x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t$. In this case order = 2

So we can get $\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$

1. $\phi_{11} = \rho_1 = \dfrac{\phi_1}{1-\phi_2}$

$k=1$: $\rho_1 = [1]\,\phi_{11} \Rightarrow \rho_1 = \phi_{11}$ Always

Now get ρ_1.

$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1} = \phi_1 \rho_0 + \phi_2 \rho_1$

$\Rightarrow \phi_{11} = \rho_1 = \dfrac{\phi_1}{1-\phi_2}$

2. (need to use Cramer's rule)

$$\phi_{22} = \frac{\det\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{bmatrix}}{\det\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = \phi_2$$

$k=2$. $\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix}$

分子: replace second column with $\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$

$\because k=2$.

分母: matrix $\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}$.

$\Rightarrow \phi_{22} = \dfrac{\det(\ )}{\det(\ )}$. recall determine.

3. (need to use Cramer's rule)

$$\phi_{33} = \frac{\det\begin{bmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{bmatrix}}{\det\begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}} = \frac{\det\begin{bmatrix} 1 & \rho_1 & \phi_1+\phi_2\rho_1 \\ \rho_1 & 1 & \phi_1\rho_1+\phi_2 \\ \rho_2 & \rho_1 & \phi_1\rho_2+\phi_2\rho_1 \end{bmatrix}}{\det\begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}} = 0 \quad \phi_{kk} = 0,\ k \geq 3$$

$\rho_1 = \phi_1\rho_0 + \phi_2\rho_1$ dependent on

$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0$ column 1 & 2.

$\rho_3 = \phi_1\rho_2 + \phi_2\rho_2 \Rightarrow \det(\ ) = 0.$

$k = 3.$ replace third column ~~of~~ of $\begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix}$ with $\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$

$\because k = 3.$

<u>Conclusion</u>

For AR(P) model, PACF at lag k is zero if k is greater than order p.

<u>Midterm practise question</u>

ARMA Model - Q4(b)

Consider a stationary AR(2) model

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t, \qquad a_t \sim NID(0,1).$$

b. Calculate the partial autocorrelation functions (PACF) for lags 1, 2, and 3.

<u>Past midterm (2017)</u>

Consider $(1-0.5B)(1-0.1B)X_t = a_t$ $\qquad a_t \sim NID(0,1)$

Find the partial autocorrelation functions at lags 1,2,3

Hint:

What is the order of this AR model ?

Apply the formula and Cramer's rule.

$$X_t - 0.6X_{t-1} + 0.05X_{t-2} = a_t \Rightarrow X_t = \overbrace{0.6}^{\phi_1} X_{t-1} \overbrace{-0.05}^{\phi_2} X_{t-2} + a_t$$

- $k=1$: $\phi_{11} = \rho_1 = \dfrac{\phi_1}{1-\phi_2} = \dfrac{0.6}{1-(-0.05)} = 0.5714.$

- $$\phi_{22} = \frac{\det\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = \phi_2 = -0.05.$$

- $$\phi_{33} = \frac{\det\begin{pmatrix} 1 & \rho_1 & \rho_1 \\ \rho_2 & 1 & \rho_2 \\ \rho_3 & \rho_2 & \rho_3 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix}} = 0 \quad \text{b/c } k > \text{order } 2.$$

no need for calculation.

5. ARMA model

5.1 Definition

A process $\{X_t\}$ is said to be an ARMA(p,q) process if $\{X_t\}$ is stationary and if for every t

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = a_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q},$$

where $a_t \sim WN(0, \sigma^2)$.

We can rewrite this long equation as $\Phi(\mathrm{B})(X_t) = \Theta(\mathrm{B})a_t$, where

$$\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

$$\Theta(z) = 1 + \theta_1 z + \cdots + \theta_p z^p.$$

5.2 Causal Processes

General definition: the process $\{X_t\}$ can be expressed in terms of $\{a_s, s < t\}$.

Remark: 1. Causal also means stationary
2. MA models are always causal

Definition for ARMA model:

An ARMA(p,q) process, defined by the equation $\Phi(B)X_t = \Theta(B)a_t$, is said to be causal if there exists a sequence of constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty}|\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty}\psi_j a_{t-j}$, $t = 0, \pm 1, \pm 2, \ldots$

Notation:

$$\Psi(z) = \sum_{j=0}^{\infty}\psi_j \cdot z^j = \frac{\Theta(z)}{\Phi(z)}. \quad |z| \le 1$$

$$\Phi(B)X_t = \Theta(B)a_t$$

$$X_t = \frac{\Theta(B)}{\Phi(B)}a_t$$

$$\Psi(B) = \frac{\Theta(B)}{\Phi(B)} \Leftarrow$$

$$= \Psi(B)a_t$$

How to prove ARMA model is causal

ARMA model can be written as $\Phi(\mathrm{B})(X_t) = \Theta(\mathrm{B})a_t$. If all roots of $\Phi(\mathrm{B}) = 0$ lie outside of unit circle, then ARMA model is causal

Midterm practise question
ARMA Model – Q2(a)(c)

Consider an ARMA(2,2) model

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}, \qquad a_t \sim NID(0,1).$$

a. Under what condition, the above ARMA(2,2) model is causal/stationary.

c. Suppose that x_t is causal, i.e.

$$x_t = a_t + \sum_{j=1}^{\infty} \psi_j a_{t-j}.$$

Calculate $\psi_j,\ j = 1,2,3,4,5,6.$

5.3 Invertible Processes

General definition:

Invertible processes: In general, no restrictions on the coefficients $\{\theta_i\}$ are required for a MA process to be stationary. The imposition of the inevitability condition ensures that there is a **unique** MA process for a given set of ACF

Remark:

Duality between AR and MA Process: casual or stationary AR(p) corresponds to MA(∞). And invertible MA(q) corresponds to AR(∞).

Definition for ARMA model:

An $ARMA(p,q)$ process is said to be invertible if there exists a sequence of constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty}|\pi_j| < \infty$ and

$$a_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, t = 0, \pm 1, \pm 2, \ldots$$

Notation:

$$\Phi(B)X_t = \Theta(B)a_t$$

$$a_t = \frac{\Phi(B)}{\Theta(B)} X_t$$

$$a_t = \Pi(B) X_t$$

$$\Pi(z) = \sum_{j=0}^{\infty} \pi_j \cdot z^j = \frac{\Phi(z)}{\Theta(z)}, \quad |z| \le 1$$

$$\Pi(B) = \frac{\Phi(B)}{\Theta(B)}$$

How to prove ARMA model is invertible

ARMA model can be written as $\Phi(\mathrm{B})(X_t) = \Theta(\mathrm{B})a_t$. If all roots of $\Theta(\mathrm{B}) = 0$ lie outside of unit circle, then ARMA model is invertible.

Midterm practise question
ARMA Model – Q2(b)(d)

Consider an ARMA(2,2) model

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}, \qquad a_t \sim NID(0,1).$$

b. Under what condition, the above ARMA(2,2) model is invertible. State the reason for us to consider an invertible ARMA model.

d. Suppose that x_t is invertible, i.e.

$$x_t - \sum_{j=1}^{\infty} \pi_j x_{t-j} = a_t.$$

Calculate π_j, $j = 1,2,3,4,5,6$.

Easy4.0 Infinite Dimensions

Notes

Reference

Jen-wen Lin (2018). Lecture note [PowerPoint slides]. Retrieved from University of Toronto Portal site.

Jen-wen Lin (2018). Midterm Practice [PDF]. Retrieved from University of Toronto Portal site.