MAT224 – Linear Algebra I

Linear Combination and Matrix Multiplication

Matrix Multiplication

Shinchan is operating a coffee shop, making various drinks. For each drink he need the following ingradients.

Milk 🥛 : need 🐄;
Lemon Tea🍵 : need 🍃🍃🍋;
Coffee☕ : need 

Matrix Multiplication

To make it clear, he made it into a table

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
	0	2	0
🐄	1	0	0

Matrix Multiplication

People like those drinks, to sale it better, Shinchan designed the following meal plan

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$$\begin{cases} \text{Meal 1🍱 : 2 milks and 1 tea;} \\ \text{Meal 2🍲 : 1 milk 2 coffee and 1 tea;} \end{cases}$$

Matrix Multiplication

To prepare for each meal, Shinchan need to know how much material is needed, can you combine those two table for him?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		
🍋		
🫘		
🐄		

Matrix Multiplication

Mathematically, we call the table of the ingradients to produce something as the matrix. The combination of two ingradients is called the matrix multiplication.

left factor

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

right factor

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

=

product

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

(1)

Next we will study matrix multiplications from the perspective of columns, rows, and entries. You will see its relation with linear combination.

The column of a matrix

Let's look at the product column by column.

Each column of the ingradients corresponds to how to produce meals by materials

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

🍱 $= \mathbf{2} \times$ 🍃 $+ \mathbf{1} \times$ 🍋 $+ \mathbf{0} \times$ 🫘 $+ \mathbf{2} \times$ 🐄.

This kind of expression is called Linear combinations.

Linear Combination of Column

Ingradients demand of making a meal comes form demand of making semi-finished meals.

Leftfactor

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

Rightfactor

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

=

Product

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

$$🍱 : \begin{bmatrix}2\\1\\0\\2\end{bmatrix} = \underbrace{2 \times \begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{2\times 🥛} \underbrace{+\,0 \times \begin{bmatrix}0\\0\\2\\0\end{bmatrix}}_{0\times ☕} \underbrace{+\,1 \times \begin{bmatrix}2\\1\\0\\0\end{bmatrix}}_{1\times 🍵}$$

Proposition 1

In the matrix product $C = AB$. Each column of C is linear combination of columns of A,with coefficient given by the corresponding column of B.

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐮	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐮	2	1

Linear Combination of Column

Excercise: Find the ingradients list of soup, which coefficent do you use?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

🍲 :

🍃	
🍋	
🫘	
🐄	

$$= \overbrace{\square \times \begin{bmatrix}0\\0\\0\\1\end{bmatrix}}^{\square \times \text{🥛}} + \overbrace{\square \times \begin{bmatrix}0\\0\\2\\0\end{bmatrix}}^{\square \times \text{☕}} + \overbrace{\square \times \begin{bmatrix}2\\1\\0\\0\end{bmatrix}}^{\square \times \text{🍵}}$$

omit the header, and leave only the middle numbers. This is the way to write matrix multiplication in mathematics. For example, the equation 1 can be written as

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 4 \\ 2 & 1 \end{pmatrix}$$

Linear Combination of Column

In the following expression, some element is missing, can you find it out?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} = \quad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ \square & \square & \square \end{pmatrix}$$

Hint: Each column of the product is a linear combination of columns of the left factor, with coefficient coming from the corresponding column on the right factor.

The row of a matrix

Each row of the ingradients corresponds to the demand for each material from each meals.

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

🍋 is need for 1 by 🍱;

🍋 is need for 1 by 🍲.

Linear Combination of Rows

To know Each meal's demand for 🍋. Notice that all meals are made of semi-finished meals 🥛, ☕ and 🍵. It is sufficient to know each semi-finished meal's demand for 🍋.

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

Linear Combination of Rows

	🥛	☕	🍵
🍋	0	0	1

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍋	1	1

To make a 🍱 and a 🍲, we need 2 and 1 🥛 respectively, Each 🥛 need 0 🍋. The 🍱 and 🍲's demand for 🍋 is the sum of all semi-finished meals' demand.

$$\underbrace{\begin{bmatrix}1 & 1\end{bmatrix}}_{\text{🍋 for 🍱🍲}} = \underbrace{0 \times \begin{bmatrix}2 & 1\end{bmatrix}}_{\text{🍋 for 🥛}} + \underbrace{0 \times \begin{bmatrix}0 & 2\end{bmatrix}}_{\text{🍋 for ☕}} + \underbrace{1 \times \begin{bmatrix}1 & 1\end{bmatrix}}_{\text{🍋 for 🍵}}$$

Proposition 2

In the matrix product $C = AB$. Each rows of C is linear combination of rows of B,with coefficient given by the corresponding rows of A.

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

Linear Combination of Rows

Excercise.: Find the need of 🍃. which coefficent do you use?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		

$$= \overbrace{\boxed{} \times \begin{bmatrix} 2 & 1 \end{bmatrix}}^{\text{🍃 } \textit{for} \text{ 🥛}} + \overbrace{\boxed{} \times \begin{bmatrix} 0 & 2 \end{bmatrix}}^{\text{🍃 } \textit{for} \text{ ☕}} + \overbrace{\boxed{} \times \begin{bmatrix} 1 & 1 \end{bmatrix}}^{\text{🍃 } \textit{for} \text{ 🍵}}$$

Linear Combination of Rows

Back to the serious math, some element is missing, can you find it out?

$$\begin{pmatrix} \square & \square \\ \square & \square \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \quad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ \square & \square & \square \end{pmatrix}$$

Hint: Each row of the product is a linear combination of rows of the right factor, with coefficient coming from the corresponding row on the left factor.

Dot Product of rows and columns

Now we combine this understanding of rows and columns. Each entry of the ingradients corresponds to how much the material is needed for a single good.

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

🍲need 1 🍋.

Dot Product of rows and columns

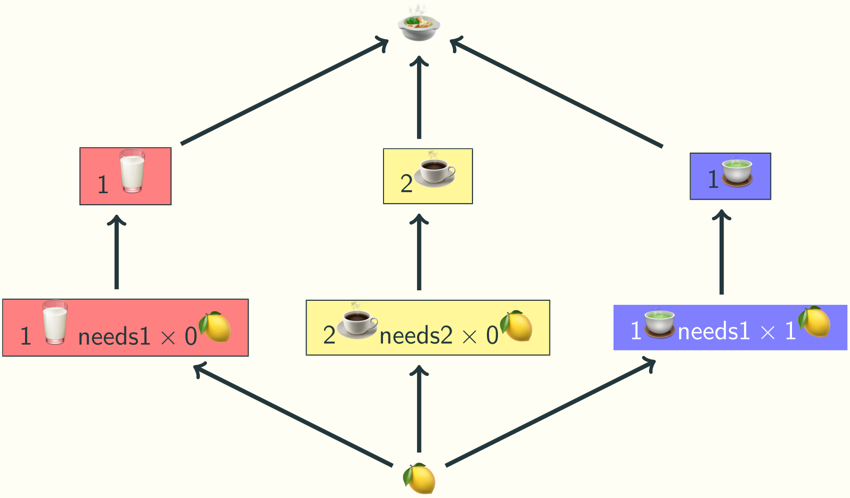
To know how many 🍋 is needed for 🍲. Notice that 🍲 are made of semi-finished meals 🥛, ☕ and 🍵. It is sufficient to know how many 🍋 is needed by those semi-finished meals.

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

Dot Product of rows and columns

We represents the need by the following graph



Total demand: $\boxed{1} \times \boxed{0} + \boxed{2} \times \boxed{0} + \boxed{1} \times \boxed{1} = 1$

Dot Product of rows and columns

Proposition 3

In the matrix product $C = AB$. Each entry of C is given by the corresponding inner product of a row of A and a column of B

A

	0	0	2
	0	0	1
	0	2	0
	1	0	0

B

	2	1
	0	2
	1	1

$=$

C

	2	2
	1	1
	0	4
	2	1

$$1 = \boxed{1} \times \boxed{0} + \boxed{2} \times \boxed{0} + \boxed{1} \times \boxed{1}$$

Dot Product of rows and columns

Come back to serious math. This method is the common method to compute matrix product, try it now.

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

A special request

A custormer requests for a special **new drink** need the following ingradients

Old Drinks ingradients:

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

New Drink requirement:

	🥤
🍃	4
🍋	2
🫘	2
🐄	4

Problem:

	🥤
🥛	?
☕	?
🍵	?

But the chef only have 🥛 , ☕ , 🍵 at the hand, can he produce 🥤 by those materials?

A special request

The chef thought this problem is the same as a matrix product equation, indeed, replace those questionmarks by x, y, z, he need the following equation to be true

Old Drinks ingradients:

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

$\times$

Problem:

	🥤
🥛	x
☕	y
🍵	z

$=$

New Drink requirement:

	🥤
🍃	4
🍋	2
☕	2
🐄	4

A special request

Mathematically, this equation is writting as

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

A special request

By understanding by columns, solving it is the same as asking for

$$\begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{z}$$

This can be write as the following and we call it the **Linear equation**

$$\begin{cases} 0x + 0y + 2z = 4 \\ 0x + 0y + 1z = 2 \\ 0x + 2y + 0z = 2 \\ 1x + 0y + 0z = 4 \end{cases}$$

Changing materials

Observation: The question is only asking for the meal's demand for semi-product meals. It does not asking anything related to the raw material.

	🥤
🥛	x
☕	y
🍵	z

No 🍋, 🍃, 🫘, 🐄 appeared in this question. Therefore we can change materials to **simplify** the problem.

Row reduction

The clever chef changes the **material** so the ingradients table is easier

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

	🥛	☕	🍵	🥤
🐄	1	0	0	4
🫘 🫘	0	1	0	1
🍋 + 🍃 🍃	0	0	1	2
🍃	0	0	0	0

Row reduction

Let's see how he managed to do it. He doubled the material

$$🫘 \mapsto 🫘\,🫘$$

The corresponding row will **multiply by $\frac{1}{2}$**.

	🥛	☕	🍵	🥤
🌿	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

	🥛	☕	🍵	🥤
🌿	0	0	2	4
🍋	0	0	1	2
🫘 🫘	0	1	0	1
🐄	1	0	0	4

This is called **row multiplying**

Row reduction

He **rearrange the order**,

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘 🫘	0	1	0	1
🐄	1	0	0	4

	🥛	☕	🍵	🥤
🐄	1	0	0	4
🫘 🫘	0	1	0	1
🍋	0	0	1	2
🍃	0	0	2	4

This is called **row switching**

Row reduction

He replace 🍋with a package of material 🍋🍃🍃, which means each time using a 🍋will automatically use two more leaves. This means each time the package was used, 2 leaves 🍃🍃is no longer needed. So the demand for 🍃is reduced by 2 times the demand for 🍋after replace 🍋by 🍋🍃🍃

	🥛	☕	🍵	🥤
🐄	1	0	0	4
🫘 🫘	0	1	0	1
🍋$+$🍃🍃	0	0	1	2
🍃	0	0	2-2 $\times$ 1	4-2 $\times$ 2

This is called **row adding**

Row reduction

In one words, row operation is **updating the raw ingradient list** when we change materials by changing its order, amount, or packing them together, which corresponds to row switching, row multiplying and row adding. Our goal is to get a list where there are some column only have a single 1, called piviot. and that the row of those piviot covers all non-zero entries.

	🥛	☕	🍵	🥤
🐄	1	0	0	4
🫘 🫘	0	1	0	1
🍋+🍃🍃	0	0	1	2
🍃	0	0	0	0

Pivot

If we have a matrix with a single 1 at some column, we call it as a **pivot**. This means we can replace a material with some meals. For example,

	☕
🐄	0
🫘 🫘	**1**
🍋+🍃🍃	0
🍃	0

$\Longrightarrow$

☕ $=$ **0** 🐄 $+$ **1** 🫘 🫘 $+$**0**(🍋🍃🍃)$+$**0**🍃

which means 🫘 🫘 $=$ ☕

Pivot

With this observation, we can replace **certain materials** by **certain meals**

	🥛	☕	🍵	🥤
🐄	1	0	0	4
🫘 🫘	0	1	0	1
🍋 + 🍃🍃	0	0	1	2
🍃	0	0	0	0

	🥛	☕	🍵	🥤
🥛	1	0	0	4
☕	0	1	0	1
🍵	0	0	1	2
🍃	0	0	0	0

Pivot

Note that the leaves 🍃 is no longer needed for those packaged materials, we can delete it.

	🥛	☕	🍵	🥤
🥛	1	0	0	4
☕	0	1	0	1
🍵	0	0	1	2

Pivot

which tell us directly the list we want, lets compare the original question

Output

	🥤
🥛	4
☕	1
🍵	2

Original Question

	🥤
🥛	x
☕	y
🍵	z

This tell us directly $x = 4, y = 1, z = 2$.

Pivot

Definition 1

(Only in our slides)A **pivot** in a matrix is an entry valued 1 such that it is the only non-zero entries in its column.

We summarize the chef's method of solving $A\vec{x} = \vec{b}$

1. Combine A and $\vec{b}$ to get the augmented matrix.
2. Change materials(row reductions), reduce until **each non-zero row has a pivot**. Delete zero rows.
3. Use pivot to change the material into meals, this form will tell us the solution $\vec{x}$.

Note: Traditional textbook reduces the matrix into reduced row echelon form. **Echelon is unecessary** for solving equations. Traditional book uses Echelon to garantee uniqueness of so called simplest form.

Pivot

Raw materials can be replaced by meals only when its row have a pivot.

		☕	🥛	🍲	🍱	🥤	🍵	💰
🍃	**1**	4	2	0	2	0	0	1
	0	1	1	0	3	**1**	1	1
🐄	0	0	0	0	0	0	0	0
🍊	0	0	2	**1**	0	0	0	0
🍎	0	0	0	0	0	0	0	0

Pivot

In step (2), the requirement of pivot occupies all non-zero rows is necessary to garateen all raw materials being replaced by meals after deleting all zero rows.

		☕	🥛	🍲	🍱	🥤	🍵	💰
	**1**	4	2	0	2	0	0	1
🥤	0	1	1	0	3	**1**	1	1
🍲	0	0	2	**1**	0	0	0	0

Pivot

Since all raw materials would be finally replaced by products, we do not need to follow the change of raw materials in row reduction. i.e. we only follows how numbers changed but do not need to know how 🍃, 🍋changes. The following table omit the row head. With **pivot** selected, it knows the row head automatically.

	☕	🥛	🍲	🍱	🥤	🍵	💰
1	4	2	0	2	0	0	1
0	1	1	0	3	**1**	1	1
0	0	2	**1**	0	0	0	0

Quick ways of solving linear equation

In step (3), we have to complete, replace, compare, and finally write the answer. now we **omit the header** and provide a quick way to do this. Take the following augmentation matrix as an example, where the pivot is highlighted and the constant part (new product) is bold.

🥛	🍲	☕	🍵	🥤
1	1	0	0	**4**
0	2	1	0	**1**
0	1	0	1	**2**

Quick ways of solving linear equation

We delete the table line, for each number of the augmented part, perform the following operations

1. For each number in the constants, move it horizontally until hit a pivot.
2. Then move it to bottom and record it.
3. read information form it.

Quick ways of solving linear equation

We perform those steps to our example, it looks like the following

🥛	🍲	☕	🍵	🥤
1	1	0	0	4
0	2	**1**	0	1
0	1	0	**1**	2

+4 🥛 +0 🍲 +1 ☕ +2 🍵 = 🥤

Quick ways of solving linear equation

Note that this process is in fact solving the equation

🥛	🍲	☕	🍵	🥤
1	1	0	0	4
0	2	**1**	0	1
0	1	0	**1**	2

$+x$ 🥛 $+y$ 🍲 $+z$ ☕ $+w$ 🍵 $=$ 🥤

$x = 4 \quad y = 0 \quad z = 1 \quad w = 2$

Quick ways of solving linear equation

Mathematically, we are in fact solving the following matrix equation

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

So we can write the process to be

$$\begin{tikzcd}
\mathbf{1} \arrow[ddd] & 1 & 0 & 0 & 4 \arrow[llll] \\
0 & 2 & \mathbf{1} \arrow[dd] & 0 & 1 \arrow[ll] \\
0 & 1 & 0 & \mathbf{1} \arrow[d] & 2 \arrow[l] \\
x = 4 & y = 0 & z = 1 & w = 2 &
\end{tikzcd}$$

Quick ways of solving linear equation

Let us show another example of find the particular solution when pivot occupies all non-zero rows.

$$\begin{pmatrix} \mathbf{1} & 2 & 0 \\ 0 & 2 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix}$$

$$\begin{tikzcd}
\mathbf{1} \arrow[ddd] & 2 & 0 & 2 \arrow[lll] \\
0 & 2 & \mathbf{1} \arrow[dd] & 9 \arrow[l] \\
0 & 0 & 0 & 0 \\
x = 2 & y = 0 & z = 9 &
\end{tikzcd}$$

General Solution for linear equation

A linear equation may have multiple sollution, for example.

🥛	🍲	☕	🍵	🥤
1	1	0	0	**4**
0	2	1	0	**1**
0	1	0	1	**2**

The cola 🥤 can be made by 🥤 $= 4$ 🥛 $+ 1$ ☕ $+ 2$ 🍵.

But it can also be made by

$= 3$ 🥛 $+ (-1)$ ☕ $+ 1$ 🍵 $+ 1$ 🍲.

Why different combination of old meal produce the same new meal 🥤 ?

General Solution for linear equation

Our previous method to produce the 🥤 only considers the pivot meals. Indeed, the way of **producing it by pivot meals is unique**.

All the non-pivot meals can be made of pivot meals, for example, 🍲 $= 1$ 🥛 $+ 2$ ☕ $+ 1$ 🍵. If we have a method of making 🥤 with non-pivot meal 🍲, we use pivot meals to replace all of it by pivot meals.

other bills

	🥤
🥛	3
🍲	1
☕	(-1)
🍵	1

$+$

replace the non-pivot meals.

	🥤
🥛	1
🍲	(-1)
☕	2
🍵	1

General Solution for linear equation

You may find a -1 appears in the non-pivot meal, this means we will throw away it, and then replace it by other pivot meals.

General Solution for linear equation

In general case, suppose we have arbitrary ways of making a 🥤. Then we replace all non-pivot meals to pivot meals

other bills

	🥤
🥛	x
🍲	y
☕	z
🍵	w

$+$

replace all y 🍲.

	🥤
🥛	1
🍲	(-1)
☕	2
🍵	1

$\times\, y$

General Solution for linear equation

After replace all non-pivot meals, it would be the way of making 🥤 by pivot meals. But the way of making it by pivot meals is unique. So

other bills

	🥤
🥛	x
🍲	y
☕	z
🍵	w

$+$

replace all y 🍲.

	🥤
🥛	1
🍲	(-1)
☕	2
🍵	1

$\times\, y =$

The unique way of making by pivot meals

	🥤
🥛	4
🍲	0
☕	1
🍵	2

General Solution for linear equation

This means all solution would have the following property

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} y = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

This tells that all solution would hae the form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{\text{unique combination out of pivot meals}} + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} (-y)}_{\text{replace non-pivot meals}}$$

General Solution for linear equation

Excercise. Solving the following linear equation

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

First step, choose two picot as follows, then find a particular solution.

$$\begin{array}{ccccc} 0 & 1 & 0 & \mathbf{1} & \longleftarrow \mathbf{1} \\ \mathbf{1} \longleftarrow & 1 & 0 & 0 & \mathbf{2} \\ \downarrow & & & \downarrow & \\ x = 2 & y = 0 & z = 0 & w = 1 & \end{array}$$

General Solution for linear equation

Second step, find the two replacement method for two non-pivot columns.

$$\begin{tikzcd}
0 & \mathbf{1} \arrow[rr] & 0 & 1 \arrow[dd] \\
1 \arrow[d] & \mathbf{-1} \arrow[l] \arrow[d, dash] & 0 & 0 \\
x=-1 & y=-1 & z=0 & w=1
\end{tikzcd}
\qquad
\begin{tikzcd}
0 & 1 & \mathbf{0} \arrow[r] & 1 \arrow[dd] \\
1 \arrow[d] & -1 & \mathbf{0} \arrow[ll] \arrow[d, dash] & 0 \\
x=0 & y=0 & z=-1 & w=0
\end{tikzcd}$$

This implies the general solution

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} (-y) + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} (-z).$$

Identity matrix

Filling the following blanks.

	🍵	🥛	☕
?	1	0	0
?	0	1	0
?	0	0	1

	?	?	?
🍵	1	0	0
🥛	0	1	0
☕	0	0	1

Identity matrix

Suppose 🥛☕🍵, any two can not blend to the third drink (called **linealy independent**). Filling the following blanks.

	🍵	🥛	☕
🍵	?	?	?
🥛	?	?	?
☕	?	?	?

Identity matrix

	🍵	🥛	☕
🍵	1	0	0
🥛	0	1	0
☕	0	0	1

This matrix is the **ingradient list of making ingradient**, i.e. do nothing.

Definition 2

The **identity matrix** is a $n \times n$ square matrix with 1 on the diagonal and 0 elsewhere.

Proposition 4

For any $n \times m$ matrix P, $I_n P = P I_m = P$.

Inverse Matrix

The chef is wondering if another guest coming with a special request, so he would like a list to produce the ingradient out of meals.

He has a list

	🥛	☕
🫘	0	2
🐄	1	0

How could he make another list?

	🫘	🐄
🥛	?	?
☕	?	?

He realize this list should have a property, combinging them should be.

	🥛	☕
🫘	0	2
🐄	1	0

	🫘	🐄
🥛	?	?
☕	?	?

=

	🫘	🐄
🫘	1	0
🐄	0	1

Definition 3

For a $n \times n$ matrix A, an inverse is a matrix B, such that

$$AB = BA = I_n.$$

If such a B exists, A is called **invertible** and denote the inverse as A^{-1}.

Simutaneous Row Operation

For the product $C = AB$, when we change materials, the only matrices affected is A and C, they are changed by certain simutaneous row operations. Because B is the list of ingradients to make meals using intermediates, it does not change. Therefore,

Proposition 5

The equality $C = AB$ will still be true if we perform arbitrary simutaneous row operation on A and C

If A is invertible, we can write $B = A^{-1}C$, this actually tells us that

Corollary 1

If A is invertible, the product $A^{-1}C$ does not change if we perform simutaneous row operation on A and C.

Simutaneous Row Operation

Let me show you an application of this in calculation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 2 & 9 \end{pmatrix}$$

$$\xlongequal{r_1 \mapsto r_1 - r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & -6 \\ 1 & 2 & 1 \\ 0 & 2 & 9 \end{pmatrix}$$

$$\xlongequal{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 2 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -6 \\ 0 & 2 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

Simutaneous Column Operation

Similarly, Column operations corresponding to updating list when changing final meals by its order, amount, or packing them together. For the product $C = AB$, when final meals changes, the matrices affected is C and B. A is the list of making intermediates, it does not change.

Proposition 6

The equality $C = AB$ will still be true if we perform arbitrary simutaneous column operation on B and C

If B is invertible, we can write $A = CB^{-1}$, this actually tells us that

Corollary 2

If B is invertible, the product CB^{-1} does not change if we perform simutaneous column operation on B and C.

Let us do Algebra

Come back to previous list

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

Why not think an object 🥛 as created out of 1? just think

	🍃	🍋	🫘	🐄
1	🍃	🍋	🫘	🐄

Indeed, 🍃 = 🍃 $\times$ 1; 🍋 = 🍋 $\times$ 1; ☕ = ☕ $\times$ 1; 🐄 = 🐄 $\times$ 1...

Let us do Algebra

When write a thing made out of 1, it made by multiply 1 with the coefficient as itself. So we have

	🍃	🍋	🫘	🐄
1	🍃	🍋	🫘	🐄

$\times$

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

$=$

	🥛	☕	🍵	🥤
1	🥛	☕	🍵	🥤

This shows the following expression is **valid**

$$\begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} & \text{🥤} \end{pmatrix}$$

This is an important way to represent the ingradients mathematically.

Change of materials

The imporance of this symbol is not only because it shows the materials and products clear. It also represents in a natural way so that **Combination of ingradients is the same as play substitution to the factors**. Go back to our original example

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		
🍋		
🫘		
🐄		

Change of materials

We can simply write this is a question

$$\begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \times ? \tag{4}$$

Change of materials

To get questionmark in (4) is easy, just use (2) to substitute $\begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix}$ part in (3), we got

$$\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

Therefore, the questionmark is gien by the matrix product. This method of **substitution** is a very important strategy and will be **repeatedly used in our course**. Make sure you familiar with it.

Change of materials

We end up this lecture by showing a math example.

Excercise: Suppose we have the following expression

$$\begin{cases} \vec{v}_1 = \vec{e}_2 + 2\vec{e}_3 \\ \vec{v}_2 = \vec{e}_3 \\ \vec{v}_3 = \vec{e}_1 \end{cases} \qquad \begin{cases} \vec{w}_1 = 2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3 \\ \vec{w}_2 = 4\vec{e}_1 + \vec{e}_2 + \vec{e}_3 \\ \vec{w}_3 = 8\vec{e}_1 + \vec{e}_2 \end{cases}$$

Write $\vec{w}_1, \vec{w}_2, \vec{w}_3$ as linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Change of materials

From what given, we write

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Note that the right-side matrix is invertible, therefore

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \tag{5}$$

Also from the given equation, we write

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix} \tag{6}$$

Change of materials

We replace (5) into (6) for $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix}$

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Doing simutaneous row reduction to factors of the form $A^{-1}B$, we have

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 2 & 4 & 8 \end{pmatrix}$$

This means

$$\begin{cases} \vec{w}_1 = \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 \\ \vec{w}_2 = \vec{v}_1 - \vec{v}_2 + 4\vec{v}_3 \\ \vec{w}_3 = \vec{v}_1 - 2\vec{v}_2 + 8\vec{v}_3 \end{cases}$$