第二讲 Principle of Well-Ordering, Structural Induction

1. Principle of Well-Ordering

Every non-empty subset of N has a smallest element.

Principle of well ordering 与 induction 等价

如果见到一道题要用 principle of well ordering 证明，如果你这道题会用 induction 做那么证明步骤如下。

- 定义 predicate P(n)
- Assume there is some m such that $\neg P(m)$
- 定义$S = \{n \in N | \neg P(n)\}$
- 说明 S 是 non-empty by assumption (m in S)
- By principle of well-ordering, there is a smallest element in S, say $a \in S$
- 那么 a-1<a(或者 a-k for some k)
- 证明 a-1 在 natural number 里(相当于 induction 里的 base case)
- P(a-1) since a is the smallest element of S
- 由 P(a-1)推出 P(a)
- Contradiction

Example 1: 利用 principle of well-ordering 证明$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \boldsymbol{N}$

Example 2: 利用 principle of well-ordering 证明$\sqrt{2}$是无理数（$\sqrt{2}$ is irrational）

Hint 1: $1.4 < \sqrt{2} < 1.5$

Hint 2: $(\sqrt{2}+1)(\sqrt{2}-1) = 1$

2. Structural Induction

(i) Recursively Defined Sets

- Smallest element
- How to construct complex elements from simpler ones.

Recursive definition of ***N***

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Recursive definition of the set of well-formed algebraic expressions ***E***

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Recursive definition of the set of rooted binary trees T

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(ii) Structural Induction

定义 predicate

Base Case: 对应 definition 中的 base case

Induction Step: IH

Example:

Let E be a set of expressions defined as follows:

- `two` $\in E$;
- if e_1 and e_2 are in E, then so are (e_1 `plus` e_2) and (e_1 `times` e_2);

For $e \in E$, let $L(e)$ be the total number of `two`'s, `plus`'s, and `times`'s in e, and $T(e)$ be the number of `two`'s in e. For example

$$L((\texttt{two plus two})) = 3, \qquad T((\texttt{two plus two})) = 2,$$
$$L((\texttt{two plus (two plus two)})) = 5, \qquad T((\texttt{two plus (two plus two)})) = 3.$$

Prove by Structural Induction that for all $e \in E$, $T(e) = \frac{L(e)+1}{2}$.

Example 2:

Let $\Sigma = \{a, b\}$ be a set of characters. Let A be a set of strings of characters in Σ. Assume A is defined as follows:

- $a \in A$;
- if $s \in A$, then $s \cdot a \in A$ and $s \cdot b \in A$, where $\cdot$ denotes string concatenation;

Use structural induction to prove that every string in A begins with an a.