Worth: 2% **Due:** By 9:59pm on **Tuesday 11 November**

Remember to write your *full name* and *student number* prominently on your submission.

---

*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission **every** source of information you used to complete this homework (other than your own lecture and tutorial notes, and materials available directly on the course webpage). For example, indicate clearly the **name** of every student with whom you had discussions, the **title** of every additional textbook you consulted, the **source** of every additional web document you used, etc.*

*For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.*

A *dominating set* in an undirected graph $G = (V, E)$ is a subset of vertices $D \subseteq V$ with the property that all vertices outside of D have some edge to a vertex within D ($\forall u \in V - D, \exists v \in D, (u, v) \in E$).

Consider the following DOMINATINGSET ("DS" for short) decision problem.

- **Input:** Undirected graph $G = (V, E)$ and positive integer k.
- **Output:** Does G contain some dominating set of size k?

Below you will find different "proofs" that DOMINATINGSET is *NP*-complete.

(a) Each of these "proofs" is missing one key part of the argument (the same part is missing from each "proof"). State clearly what part is missing and then prove it (you only have to prove it once as it is the same for each "proof").

(b) State clearly whether or not each "proof" is correct (except for the missing part you have already identified) and, if it is not correct, explain *every* error committed in the "proof."

1. DS is *NP*-complete because DS $\leqslant_p$ VERTEXCOVER:

   On input (G, k), let D be a dominating set of size k in G and let ℓ be the number of edges in G that do *not* have at least one endpoint in D; output $(G, k + \ell)$.

   Clearly, this can be computed in polytime: it requires examining at most every edge of G once, and for each one looking through the set D of size k.

   Also, if D is a dominating set of size k in G, then D together with one endpoint from every edge that does not already have one endpoint in D is a vertex cover of size $k + \ell$ in G.

   Finally, if C is a vertex cover in G of size $k + \ell$, then we can simply remove from C the ℓ extra vertices to reconstruct the dominating set D of size k.

2. DS is *NP*-complete because VERTEXCOVER $\leqslant_p$ DS:

   On input (G, k), output (G, k).

   Clearly, this can be computed in polytime.

   Also, D is a vertex cover in G iff D is a dominating set in G.

3. DS is *NP*-complete because VERTEXCOVER $\leqslant_p$ DS:

On input (G,k), where $G=(V,E)$ with $V=\{v_1,\ldots,v_n\}$ and $E=\{e_1,\ldots,e_m\}$, let $I(G)=\{v_i : v_i \text{ is isolated}\}$ (a vertex is *isolated* when it has *no* adjacent edge). Output (G',k'), where $G'=(V',E')$ with $V'=V\cup\{w_1,\ldots,w_m\}$ (add one new vertex for each edge in G), $E'=E\cup\{(u,w_i),(w_i,v) : e_i=(u,v)\}$ (connect each new vertex w_i to both endpoints of edge e_i), and $k'=\min\left(k+|I(G)|,n+m\right)$ (increase k by the number of isolated vertices in G, up to the maximum number of vertices in G').

Clearly, this can be computed in polytime: it requires going through each vertex and each edge of G once and executing a constant amount of work for each one.

Also, if G contains a vertex cover C of size k, then every vertex outside of C either is isolated or has one or more edges to vertices within C. In addition, each new vertex w_i is connected to some vertex in C (because C contains at least one endpoint from every edge in G). Hence, $C\cup I(G)$ is a dominating set of size k' in G'.

Finally, if G' contains a dominating set D of size k', then start with $C=D-I(G)$. If $|C|<k$, then this means $k>n+m-|I(G)|$. In particular, k is greater than the number of non-isolated vertices in G so {all non-isolated vertices plus enough isolated vertices to reach size k} is a vertex cover of size k in G. Else, $|C|=k$. Consider each of the new vertices $w_1,\ldots,w_m$. For each $w_i\in C$, remove w_i from C and replace it with one of the two endpoints of edge e_i (if both endpoints of e_i are already in C, replace w_i with any other vertex not already in C). The resulting C is a vertex cover in G because each of the vertices w_i is connected to some vertex in C, so every edge of G has at least one endpoint in C.

4. DS is *NP*-complete because VERTEXCOVER $\leqslant_p$ DS.

   On input (G,k), if G contains some vertex cover C of size k, then every vertex outside of C is either an isolated vertex (with no edge attached to it) or it belongs to some edges whose other endpoint is in C (by definition of "vertex cover"). If we move all of the isolated vertices into C, we get a dominating set. So output $(G,k+\ell)$, where ℓ is the number of isolated vertices in G. This can be done in polynomial time.