MAT224H1S - Linear Algebra II
Winter 2020

Homework Problems 1 :

1. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ be the set of ordered pairs of real numbers with the operations of vector addition and scalar multiplication defined by

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1 + 1, x_2 + y_2 + 1)\\ c\mathbf{x} &= (cx_1, cx_2)\end{aligned}$$

for all $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in V and $c \in \mathbb{R}$.

Find *two* vector space axioms that fail to hold and conclude that V is not a vector space with respect to the given operations.

2. Decide which of the following sets V are vector spaces with respect to the operations given. If it is not, list *all* the vector space axioms that fail to hold.

(i) $V = M_{2\times 2}(\mathbb{R})$ with operations defined by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} + 1 \\ a_{21} + b_{21} - 1 & a_{22} + b_{22} \end{bmatrix}$$
$$c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

(ii) Let $V = \{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}, a_2 \neq 0\}$ with operations defined by

$$\begin{aligned}(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) &= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)\\ c(a_2x^2 + a_1x + a_0) &= ca_2x^2 + ca_1x + ca_0\end{aligned}$$

3. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ be the set of ordered pairs of real numbers. Suppose in V that vector addition is defined by

$$\mathbf{x} + \mathbf{y} = (a_1x_1 + b_1y_1, a_2x_2 + b_2y_2)$$

for all $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in V$, and a_1, a_2, b_1, b_2 some fixed real numbers; and scalar multiplication by

$$c\mathbf{x} = (cx_1, cx_2)$$

for all $\mathbf{x} = (x_1, x_2) \in V$ and $c \in \mathbb{R}$.

For what values of a_1, a_2, b_1, b_2 is V a vector space?

4. Let V be a vector space, $\mathbf{x} \in V$, and $c \in \mathbb{R}$ be a scalar. Prove that if $c\mathbf{x} = \mathbf{0}$ then either $c = 0$ or $\mathbf{x} = \mathbf{0}$.

5. Let V be a vector space, $\mathbf{x} \in V$. Prove that $-(-\mathbf{x}) = \mathbf{x}$.

6. For each of the following subsets W of a vector space V, determine if W is a subspace of V.

(i) $V = P_n(\mathbb{R})$, and $W = \{xp(x) + (1-x)q(x) \mid p(x), q(x) \in P_n(\mathbb{R})\}$

(ii) $V = P_n(\mathbb{R})$, and $W = \{p(x) \in P_n(\mathbb{R}) \mid p(x) \geq 0 \text{ for all } x\}$

(iii) $V = P_n(\mathbb{R})$, and $W = \{p(x) \in P_n(\mathbb{R}) \mid p(x) \text{ is even }\}$

7. A **semimagic square** is an $n \times n$ matrix with real entries in which every row and every column has the same sum s. For example, the identity matrix is a semimagic square since every row and every column has the same sum $s = 1$. Show that the set of all 3×3 semimagic squares is a subspace of $M_{3\times 3}(\mathbb{R})$.

8. A **magic square** is an $n \times n$ matrix with real entries in which each row, each column, and each diagonal (entries (1,1) to (n,n) and (1, n) to (n,1)) has the same sum s. For example, the $n \times n$ matrix with every entry equal to 1 is a magic square since each row, column, and diagonal has the same sum n. Is the set of all 3×3 magic squares with $s \neq 0$ a subspace of $M_{3\times 3}(\mathbb{R})$? What about if $s = 0$?

9. Textbook, Section 1.2: # **16**.

Textbook Problems:

- Textbook, Section 1.1: # **3-12**.
- Textbook, Section 1.2: # **1-10, 12-15**.