**Worth:** 7.5% **Due:** Before 10pm on Tuesday 4 December.

**Remember to write the *full name*, *student number*, and *CDF/UTOR email address* of *each group member* prominently on your submission.**

*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission* ***every*** *source of information you used to complete this homework (other than your own lecture and tutorial notes, and materials available directly on the course webpage). For example, indicate clearly the* ***name*** *of every student with whom you had discussions (other than group members), the* ***title*** *of every additional textbook you consulted, the* ***source*** *of every additional web document you used, etc.*

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For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.

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[4] 1. (a) Give a *detailed* argument that for all decision problems $D$ and $E$, if $D \leqslant_p E$ and $E \in NP$, then $D \in NP$.

[3] (b) By analogy with the definition of *NP*-hardness, give a *precise* definition of what it means for a decision problem $D$ to be "*coNP*-hard."

[5] (c) Show that if decision problem $D$ is *coNP*-hard, then $D \in NP$ implies $NP = coNP$.

2. Consider the following "PARTITION" search problem.

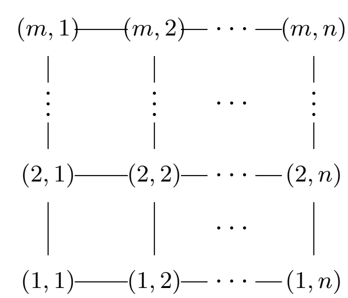
**Input:** A set of integers $S = \{x_1, x_2, \dots, x_n\}$ — each integer can be positive, negative, or zero.

**Output:** A partition of $S$ into subsets $S_1, S_2$ with equal sum, if such a partition is possible; otherwise, return the special value NIL. ($S_1, S_2$ is a "partition" of $S$ if every element of $S$ belongs to one of $S_1$ or $S_2$, but not to both.)

[2] (a) Give a *precise* definition for a decision problem "PART" related to the PARTITION search problem.

[11] (b) Give a *detailed* argument to show that the PARTITION search problem is polynomial-time self-reducible. (WARNING: Remember that the input to the decision problem does **not** contain any information about the partition — if it even exists.)

3. Your friends want to break into the lucrative coffee shop market by opening a new chain called *The Coffee Pot*. They have a map of the street corners in a neighbourhood of Toronto (shown on the right), and estimates $p_{i,j}$ of the profits they can make if they open a shop on corner $(i, j)$, for each corner. However, municipal regulations forbid them from opening shops on corners $(i-1, j)$, $(i+1, j)$, $(i, j-1)$, and $(i, j+1)$ (for those corners that exist) if they open a shop on corner $(i, j)$. As you can guess, they would like to select street corners where to open shops in order to maximize their profits!




[3] (a) Consider the following greedy algorithm to try and select street corners.

$C \leftarrow \{(i,j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ # $C$ is the set of every available corner
$S \leftarrow \varnothing$ # $S$ is the current selection of corners
**while** $C \neq \varnothing$:
  pick $(i,j) \in C$ with the maximum value of $p_{i,j}$
  # Add $(i,j)$ to the selection and remove it (as well as all corners adjacent to it) from $C$.
  $S \leftarrow S \cup \{(i,j)\}$
  $C \leftarrow C - \{(i,j), (i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$
**return** $S$

Give a precise counter-example to show that this greedy algorithm does not always find an optimal solution. State clearly the solution found by the greedy algorithm, and show that it is not optimal by giving another selection with larger profit.

[12] (b) Prove that the greedy algorithm from part (a) has an approximation ratio of 4. (HINT: Let $S$ be the selection returned by the greedy algorithm and let $T$ be any other valid selection of street corners. Show that for all $(i, j) \in T$, either $(i, j) \in S$ or there is an adjacent $(i', j') \in S$ with $p_{i',j'} \geqslant p_{i,j}$. What does this means for all $(i, j) \in S$ and their adjacent corners?)