Dept. of Computer Science **Term Test # 2** **CSC 373 H1**
University of Toronto **(Duration: 50 minutes) — Solutions** **Summer 2019**

Note to Students: This file contains sample solutions to the term test together with the marking scheme for each question. Please read the solutions and the marking schemes carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. We will remark your test only if you clearly demonstrate that the marking scheme was not followed correctly.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

Question 1. [6 marks]

Explain how to modify Dijkstra's algorithm so that if there is more than one minimum path from v to w, a path with the fewest number of edges is chosen.

Sample Solution:

Define a new array A of size $|V|$ such that $A[v]$ denotes the minimum number of edges on a path of weight $d[v]$ (recall $d[v]$ denotes the weight of a shortest path from s to v). This array A will be used to break ties when we find two competing shortest paths.

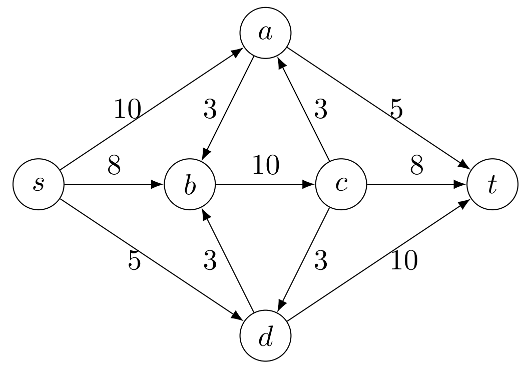
```
procedure Dijkstra(G = (V, E, w), s)
    Define arrays d and A of size |V|
    Set d[v] = A[v] = ∞ for all v ∈ V
    Set d[s] = A[s] = 0
    Define array Π of size |V|
    Set Π[v] = NIL for all v ∈ V
    Q := V
    S := ∅
    while (Q ≠ ∅) do
        u = Extract-Min(Q)
        S = S ∪ {u}
        for (each vertex v ∈ Adj[u]) do
            if (d[v] > d[u] + w[u, v]) then
                d[v] = d[u] + w[u, v]
                Π[v] = u
                A[v] = A[u] + 1
                Decrease-Key(Q, v)
            if (d[v] == d[u] + w[u, v]) then
                if (A[u] + 1 < A[v]) then
                    A[v] = A[u] + 1
                    Π[v] = u
    return d
```

Marking Scheme:

A. 1 mark: for writing a pseudocode
B. 1 mark: correct explanation and definition of the array A
C. 4 marks: 2 marks for each of the two "if" blocks

Question 2. [8 MARKS]

Consider the network N_1 pictured on the right.



Part (a) [3 MARKS]

Compute the *capacity* of the cut $X_1 = (\{s, b, c\}, \{a, d, t\})$.
Show your work: list the components of the network used to obtain your answer.

SAMPLE SOLUTION:

$$\begin{aligned} c(X_1) &= c(s, a) + c(c, a) + c(c, t) + c(c, d) + c(s, d) \\ &= 10 + 3 + 8 + 3 + 5 = 29 \end{aligned}$$

MARKING SCHEME:

A. 2 marks: correct expression (adding up capacities of forward edges across X_1)
B. 1 mark: correct answer

Part (b) [1 MARK]

Let C_1 represent your answer to Part (a). What can you conclude about $|f|$, for *all* valid flows f over N_1?

SAMPLE SOLUTION:

$|f| \leqslant C_1$ (or $|f| \leqslant 29$)

MARKING SCHEME:

A. 1 mark: all-or-nothing

Part (c) [4 MARKS]

Given one network N and one cut $X = (S, T)$ in N, suppose that we **reduce** the capacity of every *forward* edge across X. Does the maximum flow value in N necessarily decrease? Justify your answer.

SAMPLE SOLUTIONS:

No: consider network $N = s \xrightarrow{2} a \xrightarrow{4} t$ and cut $X = (\{s, a\}, \{t\})$. If we reduce the capacity of every forward edge across X by 1, the resulting network $N' = s \xrightarrow{2} a \xrightarrow{3} t$ still has the same maximum flow value as N.

MARKING SCHEME:

A. 1 mark: correct answer ("no"), even if justification is missing of incorrect
B. 1 mark: clear attempt to give a counter-example
C. 2 marks: correct counter-example

Question 3. [11 MARKS]

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported: xantalum, ytterbium, and zastatine. The cargo company may choose to carry any amount of each, up to the maximum available limits given below.

- Xantalum has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue $1,000 per cubic meter.
- Ytterbium has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue $1,200 per cubic meter.
- Zastatine has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue $12,000 per cubic meter.

Part (a) [7 MARKS]

Write a linear program that optimizes revenue for a single plane flight, within the constraints. List all the variables you used and the intuitive meaning of each.

SAMPLE SOLUTION:

Let x_1, x_2, x_3 denote the volume to be transported of the materials xantalum, ytterbium, and zastatine, respectively. Then the total volume is given by $x_1 + x_2 + x3$, the total weight by $2x_1 + x_2 + 3x_3$, and the total revenue by $1000x_1 + 1200x_2 + 12000x_3$. Thus, the corresponding LPP is given by

$$\begin{array}{rl} \text{maximize} & 1000x_1 + 1200x_2 + 12000x_3 \\ \text{subject to} & \end{array}$$
$$\begin{array}{rcl} x_1 & \leqslant & 40 \\ x_2 & \leqslant & 30 \\ x_3 & \leqslant & 20 \\ x_1 + x_2 + x_3 & \leqslant & 60 \\ 2x_1 + x_2 + 3x_3 & \leqslant & 100 \\ x_1, x_2, x_3 & \geqslant & 0. \end{array}$$

MARKING SCHEME:

A. 1 mark: correct definition of the variables
B. 3 marks: 1 mark for each of the correct equations or expressions corresponding to revenue, total weight and total volume
C. 3 marks: 1 mark for the correct upper and lower bounds on each of the three variables

Part (b) [4 MARKS]

Write the corresponding dual program.

SAMPLE SOLUTION: The corresponding dual program is given by

$$\begin{array}{rl} \text{minimize} & 40y_1 + 30y_2 + 20y_3 + 60y_4 + 100y_5 \\ \text{subject to} & \end{array}$$
$$\begin{array}{rcl} y_1 + y_4 + 2y_5 & \geqslant & 1000 \\ y_2 + y_4 + y_5 & \geqslant & 1200 \\ y_3 + y_4 + 3y_5 & \geqslant & 12000 \\ y_1, y_2, y_3, y_4, y_5 & \geqslant & 0. \end{array}$$

MARKING SCHEME:

A. 4 marks: 1 mark for each of the 4 correct expressions/equations.