CSC236: Assignment 1

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1.

Defining predicate:

P(n): "$\sum_{i=0}^{n} f(2i)$ = f(2n+1)", where n ∈ ℕ.

Base Case:

Let n = 0. Proof for P(0).

LHS = f(0) = 1 (due to given Fibonacci function)

RHS = f(2×0+1) = f(1) = 1 (due to given Fibonacci function)

LHS = RHS = 1

Therefore, P(0) holds.

Inductive steps:

Let k be an arbitrary natural number, and suppose that P(k) holds, i.e., $\sum_{i=0}^{k} f(2i)$ = f(2k+1). We must show that P(k+1) holds as well, i.e., $\sum_{i=0}^{k+1} f(2i)$ = f(2(k+1)+1) = f(2k+3).

$$\sum_{i=0}^{k+1} f(2i) = (\sum_{i=0}^{k} f(2i)) + \mathrm{f}(2(\mathrm{k}+1))$$

$$\sum_{i=0}^{k+1} f(2i) = (\sum_{i=0}^{k} f(2i)) + \mathrm{f}(2\mathrm{k}+2)$$

$= f(2k+1) + \mathrm{f}(2\mathrm{k}+2)$ (by inductive hypothesis)

$= f((2k+3)-2) + \mathrm{f}\big((2\mathrm{k}+3)-1\big)$

$= f(2k+3)$ (since 2k+3>1 and given Fibonacci function.)

Then, P(k+1) holds.

Therefore, if n is a natural number, then f(0)+ f (2) + · · · + f (2n) = f (2n + 1).

2.

Defining predicate:

P(n) : "∃k ∈ $\mathbb{Z}, (2n+1)^2 - 1$ = 8k", where n ∈ ℕ.

Base Case:

Let n = 0, k = 0

LHS = $(2\times 0+1)^2$ – 1 = 0

RHS = 8 × 0 = 0

LHS = RHS

Then, P(0) holds.

Inductive steps:

Let i be an arbitrary natural number, and suppose that P(i) holds, i.e., $\exists k_1 \in \mathbb{Z}, (2i+1)^2 - 1 = 8k_1$, which is $4i^2 + 4i = 8k_1$

We must show that P(i+1) holds as well, i.e., $\exists k_2 \in \mathbb{Z}, (2(i+1)+1)^2 - 1 = 8k_2$.

Let $k_2 = k_1 + i + 1$

$(2(i+1)+1)^2 - 1 = 4i^2 + 12i + 8$

$= (4i^2 + 4i) + (8i + 8)$

$= 8k_1 + 8i + 8$ (by inductive hypothesis)

$= 8(k_1 + i + 1)$

$= 8\ k_2$

Then, P(i+1) holds.

Therefore, $x^2 - 1\ is\ divisible\ by\ \ 8\ for\ any\ odd\ natural\ number\ x.$

3.

Defining predicate:

P(n): "$\exists a, b \in \mathbb{N}, n = 3a + 5b$ ", where $n \in \mathbb{N} \wedge n \geq 8$

Base Case:

Let n = 8

Let a = 1, b = 1

$8 = 3 \times 1 + 5 \times 1$

Then, P(8) holds.

Inductive Steps:

Let k be an arbitrary natural number which is greater than 7, and suppose that P(k) holds,

i.e., $\exists a_1, b_1 \in \mathbb{N}, \boldsymbol{k = 3}a_1 + \boldsymbol{5}b_1$.

We must show that P(k+1), i.e., $\exists a_2, b_2 \in \mathbb{N}, k + 1 = 3a_2 + 5b_2$

Case 1, $\boldsymbol{b_1 > 0}$:

Let $a_2 = a_1 + 2,\ b_2 = b_1 - 1$

k + 1 = $(3a_1 + 5b_1) + 1$ (By inductive hypothesis)

= $(3a_1 + 6) + (5b_1 - 5)$

= $3a_2 + 5b_2$

Case 2, $\boldsymbol{b_1 = 0}$:

Let $a_2 = a_1 - 3,\ b_2 = 2$

k + 1 = $3a_1 + 1$ (By inductive hypothesis)

= $3a_1 - 9 + 10$

= $3\ (a_1 - 3) + 5 \times 2$

= $3a_2 + 5b_2$

Then, P(k+1) holds.

Therefore, if a number is greater than 7, we can represent it with 3 and 5 coins.

4.

Let n$\in \mathbb{N}$, n≥ 1.

Define set A as follows:

A = $\{m \in \mathbb{Z}: \exists k \in \mathbb{N}, n = 2^k \times m\}$

Prove that A is a non-empty set:

Let k=0, and m=n.

Then, n=$2^0 \times n = n$.

Then, n is a non-empty set.

Prove that A is a non-empty subset of $\mathbb{N}$.

Since n≥ 1, $2^k \geq 1$, and n = $2^k \times$ m,

then, m≥ 1.

Now, we know m$\in \mathbb{Z}$ and m≥ 1. This means A is a non-empty subset of $\mathbb{N}$.

By well-ordering principal, there is a minimum element i* in A,

i.e. $\exists k_1 \in \mathbb{N}$, n $= 2^{k_1} \times i^*$.

We want to prove i* is odd by using contradiction.

We firstly assume i* is even, i.e., $\exists s \in \mathbb{N}$, i*=2s.

Then, n = $2^{k_1} \times i^*$.

$= 2^{k_1} \times (2s)$

$= 2^{k_1+1} \times s$

Then, s∈A and s<i*. This contradicts i* is the minimum element in A. It means our assumption that i* is even is false, and thus i* is odd.

Therefore, given any natural number n ≥ 1, there exists an odd integer m and a natural number k such that n = $2^k \times$ m.

5.

Defining predicate:

P((x, y)): "$\exists k \in \mathbb{N}, (\boldsymbol{x}, \boldsymbol{y}) = (2^{k+1} + 1,\ 2^k + 1)$", where x, y∈ $\mathbb{Z}$.

Base Case:

Prove for P((3, 2)).

Let k = 0.

$(2^{k+1} + 1,\ 2^k + 1)$ = $(2^{0+1} + 1, 2^0 + 1)$ = (3, 2)

Therefore, P((3, 2)) holds.

Inductive step: Let (x, y) ∈ M.

Assume P((x, y)) holds, i.e., $\exists k_1 \in \mathbb{N}, (\boldsymbol{x}, \boldsymbol{y}) = (2^{k_1+1} + 1,\ 2^{k_1} + 1)$. We want to prove for P((3x-2y, x)), i.e., $\exists k_2 \in \mathbb{N}, (\boldsymbol{3x - 2y}, \boldsymbol{x}) = (2^{k_2+1} + 1,\ 2^{k_2} + 1)$.

Let $k_2 = k_1 + 1$.

$(3x - 2y, x) = (3(2^{k_1+1} + 1) - 2(2^{k_1} + 1), 2^{k_1+1} + 1)$ (by inductive hypothesis)

$= (3\times 2^{k_1+1} + 3 - 2^{k_1+1} - 2, 2^{k_1+1} + 1)$

$= (2\times 2^{k_1+1} + 1, 2^{k_1+1} + 1)$

$= (2^{(k_1+1)+1} + 1, 2^{(k_1+1)} + 1)$

$= (2^{k_2+1} + 1, 2^{k_2} + 1)$ (since $k_2 = k_1 + 1$)

Therefore, P((3x-2y, x)) holds.

Therefore, $\forall (x,y) \in M, \exists k \in N, (x,y) = (2^{k+1} + 1, 2^k + 1)$.

6.

a)

There are $\frac{n}{2}$ pairs of people. In each pair, each person is the unique nearest neighbor of the other one in the pair. Then, everyone is hit by a water balloon from the other person in the pair, and then no one is dry.

b)

Defining predicate:

P(n): "There is at least one dry person when (2n+1) people are positioned such that each person has a unique nearest neighbor and each person has a single water balloon that they throw at their nearest neighbor. ", where $n \in \mathbb{N}$, $n \geq 1$(because if n = 0, the only one person will not have neighbor)

Base Cases:

Case 1: Let n=0.

Then, there is 1 person in total in the play. This means no one throw water balloon to the only person in the play. So, there is a person left dry. P(0) holds.

Case 2: Let n=1.

Then, 2n+1 = 3, which means there are 3 people in total in the play.

Given that each person only has one unique nearest neighbor, there must be two people who are the unique closest neighbor to each other among the three people.

Let P1 and P2 be the two people who are the unique closest neighbor to each other among the three people. Let P3 be the third person in the play other than P1 and P2.

Then, P1 and P2 will throw water balloon to each other and they are both wet.

Then, P3 will hit either P1 or P2, then P3 will be the dry person.

Then P(1) holds.

Induction Steps:

Let $k \in \mathbb{N}$, $k \geq 1$

Assume P(k), i.e., There is at least one dry person when (2k+1) people are positioned such that each person has a unique nearest neighbor and each person has a single water balloon that they throw at their nearest neighbor.

We want to prove that P(k+1), i.e., There is at least one dry person when (2k+3) people are positioned such that each person has a unique nearest neighbor and each person has a single water balloon that they throw at their nearest neighbor.

We separate the (2k+3) people into two groups.

Group 1: The group only contains 2 people, S1 and S2, such that S1 and S2 are the unique closest neighbor to each other among the (2k+3) people. So they will throw water balloon to each other and then they are both wet.

Group 2: the (2k+1) people in the (2k+3) people other than S1 and S2.

Case 1: No one in Group 2 will throw water balloon to people in Group 1.

Then, there will be at least one dry person in Group 2. (By Inductive hypothesis)

Case 2: At least one person in Group 2 will throw water balloon to people in Group 1.

It leaves at most 2k water balloons to be thrown in the 2k+1 people in Group 2. This means there is at least 1 dry person in Group 2.

Therefore, P(k+1) holds.

Therefore, if n is odd, then there is always at least one dry person.

7.

Defining Predicate:

P(n): "P is a convex polygon with consecutive vertices v_1, v_2, ..., v_n, P can be triangulated into n−2 triangles, the n−2 triangles can be numbered 1,2,..., n−2 so that v_i is a vertex of triangle i for i = 1, 2, ..., n − 2. ", where n $\in \mathbb{N}$, n ≥ 3.

Base Case:

Let n = 3, P is a triangle and the vertices can be numbered 1, 2, 3. The triangle can be numbered 1.

Therefore P(3) holds.

Inductive Steps:

Let $k \in \mathbb{N}$, $k \geq 3$.

Assume H(k) : $\forall i \in \mathbb{N}, 3 \leq i < k \Rightarrow P(i)$

We want to show P(k) is true.

Let P be a convex polygon with consecutive vertices, v_1, v_2, ..., v_k. Let $a \in \mathbb{N}$, $2 \leq a \leq k - 2$.

Then, we separate P into two parts, P1 and P2.

P1 contains the vertices: v_1, v_2, ..., v_a, v_k.

P2 contains the vertices: v_a, v_{a+1}, ..., v_k.

P1 has (a + 1) vertices and because $2 \leq a \leq k - 2$, then $3 \leq a + 1 \leq k - 1 < k$. By inductive hypothesis, P1 can be triangulated into $(a - 1)$ triangles, the $(a - 1)$ triangles can be numbered 1,2,..., $(a - 1)$, so that v_i is a vertex of triangle i for i = 1, 2, ..., $(a - 1)$.

P2 has $(k - a + 1)$ vertices and because $2 \leq a \leq k - 2$, then $3 \leq (k - a + 1) \leq k - 1 < k$.

Next, we rename the vertices on P2.

By adding (1- a) to each sub index for each vertex, we make P2 has vertices: v_1, v_2, ..., v_{k-a}, v_{k-a+1}.

By inductive hypothesis, P2 can be triangulated into $(k - a - 1)$ triangles, the $(k - a - 1)$ triangles can be numbered 1,2,..., $(k - a - 1)$, so that v_i is a vertex of triangle i for i = 1, 2, ..., $(k - a - 1)$.

Then, we add $(a - 1)$ to the sub index of each vertex and each labelled triangle in P2. Now, we have P2 with vertices: v_a, v_{a+1}, ..., v_{k-1}, v_k. The triangles in P2 are labelled as $a, a + 1, \ldots, k - 3, k - 2$.

In total, now P is a convex polygon with consecutive vertices v_1, v_2, ..., v_k, and P has (k−2) triangles, the (k−2) triangles can be numbered 1,2,..., k−2 so that v_i is a vertex of triangle i for i = 1, 2, ..., k − 2.

Therefore, P(k) holds.

In conclusion, for every convex polygon with consecutive vertices v_1, v_2, ..., v_n, the polygon can be triangulated into n−2 triangles, the n−2 triangles can be numbered 1,2,..., n−2 so that v_i is a vertex of triangle i for i = 1, 2, ..., n − 2. ", where n $\in \mathbb{N}$, n ≥ 3.