MAT224H1S - Linear Algebra II
Winter 2020

Homework Problems 2

1. Consider the subset $U = \{x^4 + x + 1, x^3 + x^2 + 2x, x^3 + x^2 + 1, x^4 + 2x\}$ of $P_4(\mathbb{R})$.

(a) Is $5x^4 + 5x^3 + 5x^2 + 13x + 4 \in \operatorname{span} U$? If it is, express it as a linear combination of the polynomials in S.

(b) Is $P_4(\mathbb{R}) = \operatorname{span} U$?

2. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ be vectors in a vector space V where at least one of the vectors is nonzero. Explain why $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\} = \operatorname{span}\{\mathbf{x}_j \mid j = 1, 2, \ldots, r \text{ and } x_j \neq \mathbf{0}\}$.

3. Show that $W = \{p(x) \in P_3(\mathbb{R}) \mid p(0) = 0\}$ is a subspace of $P_3(\mathbb{R})$ and that $W = \operatorname{span}\{x, x^2, x^3\}$.

4. Consider the subspaces $U = \operatorname{span}\{2 + x, 1 - x^2\}$, and $W = \operatorname{span}\{-1 + x + x^2, -x + x^2\}$ of $P_2(\mathbb{R})$. Find $U \cap W$ and $U + W$.

5. For this question, recall the definition of a semimagic square from Homework Problems 1, # 7.

(a) Show that the set $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$ spans the vector space of all 3×3 semimagic squares.

(b) Show that the list $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is linearly independent.

6. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be a list of linearly independent vectors in a vector space V. Show that the list $\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} + \mathbf{x}$ is linearly independent. What about the list $\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}$?

7. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a list of linearly independent vectors in a vector space V. If $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{x}_i$ where each $c_i \in \mathbb{R}$, prove that the list $\mathbf{x} - \mathbf{x}_1, \ldots, \mathbf{x} - \mathbf{x}_n$ is linearly independent if and only if $\sum_{i=1}^{n} c_i \neq 1$.

8. Textbook, Section 1.3, **# 8**.

9. Textbook, Section 1.3, # **9**.

10. Consider the subspaces $W_1 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$, and $W_2 = \left\{ \begin{bmatrix} c & d \\ d & -c \end{bmatrix} \mid c, d \in \mathbb{R} \right\}$ of $M_{2\times 2}(\mathbb{R})$. Show that $M_{2\times 2}(\mathbb{R}) = W_1 \oplus W_2$.

11 (a) Let $W = \text{span}\{1, x\}$, $W_1 = \text{span}\{x^2, x^3\}$, and $W_2 = \text{span}\{1 + x + x^2 + x^3, 1 + x + x^2 - x^3\}$. Show that $\mathbb{P}_3(\mathbb{R}) = W \oplus W_1$ and $\mathbb{P}_3(\mathbb{R}) = W \oplus W_2$.

11 (b) Let $V = P_3(\mathbb{R})$, and $W = \text{span}\{1+x, 1+x^2\}$. Find subspaces W_1 and W_2 of V such that $V = W \oplus W_1$ and $V = W \oplus W_2$ but $W_1 \neq W_2$.

Suggested Textbook Problems:

- Textbook, Section 1.3: # **1-7, 10, 11**.
- Textbook, Section 1.4: # **1-12**.