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Q1. (a) $\frac{\partial C}{\partial A_i} = a_i(A_i - r_i)$

$$A_i^{(t+1)} = A_i^{(t)} - \alpha \cdot \frac{\partial C}{\partial A_i^{(t)}}$$

$$= (1 - \alpha a_i) A_i^{(t)} + \alpha a_i r_i$$

(b) From the result of part (a), we have

$$A_i^{(t+1)} = (1 - \alpha a_i) A_i^{(t)} + \alpha a_i r_i$$

$\because e_i^{(t)} = A_i^{(t)} - r_i$

$\therefore e_i^{(t+1)} = A_i^{(t+1)} - r_i$

$$\Rightarrow e_i^{(t+1)} = (1 - \alpha a_i) A_i^{(t)} + \alpha a_i r_i - r_i$$

$$= (1 - \alpha a_i) A_i^{(t)} + (\alpha a_i - 1) r_i$$

$$= (1 - \alpha a_i)(A_i^{(t)} - r_i)$$

$$= (1 - \alpha a_i) e_i^{(t)}$$

(c)

$$e_i^{(t)} = (1 - \alpha a_i) e_i^{(t-1)}$$

$$= (1 - \alpha a_i)(1 - \alpha a_i) e_i^{(t-2)}$$

$$= (1 - \alpha a_i)^3 e_i^{(t-3)}$$

$$\vdots$$

$$= (1 - \alpha a_i)^t e_i^{(0)}$$

Conclusion:

When $0 < \alpha < \frac{1}{a_i}$, the procedure is stabe and when $\alpha > \frac{1}{a_i}$, the procedure is unstable.

(d) We have $e_i^{(t)} = (1-\alpha a_i)^t e_i^{(0)}$ from previous question.

$$\Rightarrow C(A^{(t)}) = \frac{1}{2}\sum_{i=1}^{N} a_i (e_i^{(t)})^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} a_i ((1-\alpha a_i)^t e_i^{(0)})^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} a_i (1-\alpha a_i)^{2t} (A_i^{(0)} - r_i)^2$$

As $t \to \infty$, the term whose $(1-\alpha a_i)^{2t}$ is largest starts to dominate.

Q2 (a) $E[y] = E[\sum_j m_j w_j x_j]$

$$= \sum_j x_j w_j E[m_j]$$

$$= \sum_j x_j w_j \cdot \frac{1}{2}$$

$$= \frac{1}{2} w^T x$$

$$Var[y] = Var[\sum_j m_j w_j x_j]$$

$$= \sum_j Var[m_j w_j x_j]$$

$$= \sum_j w_j^2 x_j^2 \, Var[m_j]$$

$$= \frac{1}{4} \sum_j w_j^2 x_j^2$$

$$= \frac{1}{4} (w^T x)^2$$

(b) Since $E[y] = \sum_j \frac{1}{2} x_j w_j$

$$= \sum_j (\frac{1}{2} w_j) x_j$$

$$= \sum_j \tilde{w}_j x_j$$

$\Rightarrow \forall j, \tilde{w}_j = \frac{1}{2} w_j$

$\Rightarrow \tilde{w} = \frac{1}{2} w$

(c) $\mathcal{E} = \frac{1}{2N}\sum_{i=1}^{N} E[(y^{(i)} - t^{(i)})^2]$

$$= \frac{1}{2N}\sum_{i=1}^{N} E\left((y^{(i)})^2 - 2y^{(i)}t^{(i)} + (t^{(i)})^2\right)$$

$$= \frac{1}{2N}\sum_{i=1}^{N} E[(y^{(i)})^2] - 2E(y^{(i)}t^{(i)}) + E[(t^{(i)})^2]$$

$$= \frac{1}{2N}\sum_{i=1}^{N} E[(y^{(i)})^2] - 2t^{(i)}E(y^{(i)}) + (t^{(i)})^2$$

$$= \frac{1}{2N}\sum_{i=1}^{N} Var(y^{(i)}) + (E(y^{(i)}))^2 - 2t^{(i)}E(y^{(i)}) + (t^{(i)})^2$$

$$= \frac{1}{2N}\sum_{i=1}^{N} Var(y^{(i)}) + \frac{1}{2N}\sum_{i=1}^{N}(E(y^{(i)}) - t^{(i)})^2$$

$$= \frac{1}{2N}\sum_{i=1}^{N}\frac{1}{4}(w^Tx)^2 + \frac{1}{2N}\sum_{i=1}^{N}(E(y^{(i)}) - t^{(i)})^2 \qquad (Var(y) = \frac{1}{4}(w^Tx)^2)$$

$$= \frac{1}{2N}\sum_{i=1}^{N}(\tilde{w}^Tx)^2 + \frac{1}{2N}\sum_{i=1}^{N}(\tilde{y}^{(i)} - t^{(i)})^2 \qquad (w = 2\tilde{w})$$

$$= \frac{1}{2}(\tilde{w}^Tx)^2 + \frac{1}{2N}\sum_{i=1}^{N}(\tilde{y}^{(i)} - t^{(i)})^2$$

$$= \frac{1}{2N}\sum_{i=1}^{N}(\tilde{y}^{(i)} - t^{(i)})^2 + R(\tilde{w}_1, \cdots, \tilde{w}_D)$$

where $R(\hat{w}_1, \cdots, \hat{w}_D) = \frac{1}{2}(\tilde{w}^Tx)^2 = \frac{1}{2}\sum_{i=1}^{D}(\tilde{w}_i x_i)^2$