Due: April 1, beginning of lecture; NOTE change of original date

1. (15 points)

   Consider the carpet production problem in section 7.1.2 of the text. Explain the objective function; that is, explain each of the coefficients preceding the 5 summations in the linear objective.

2. (15 points)

   Consider the "{0,1} IP" decision problem; namely, given a set of linear constraints, is there a {0,1} setting of the IP variables satisfying all the constraints. Show that the {0,1} IP decision problem is NP-complete. That is, for some known NP-hard or complete problem L, show that $L \leq_p IP$.

   Note: For the {0,1} case it is obvious that the problem is in NP. More generally, the IP decision problem is also in NP but this is not immediately obvious as we have to argue (using linear algebra) that if a set of linear constraints is satisfied by integers, then there is a solution where all the IP variables are not too big (i.e. have length polynomial in the length of the input representation).

3. (20 points) **This problem was incorrectly answered by many students.**

   Consider the following partial vertex cover problem. We are given a graph $G = (V, E)$ which has both node weights $w : V \rightarrow \Re^{\geq 0}$ and edge penalties $p : E \rightarrow \Re^{\geq 0}$. Let $V' \subseteq V$ cover some set edges $E' \subseteq E$; that is, for every $e = (u, v) \in E'$, either u or v (or both) are in V'. The total cost of such a partial cover (V', E') is the sum of node weights in V' plus the sum of edge penalties for edges not in E'.

   (a) Express the partial vertex cover problem as an IP.

   NON-SOLUTIONS: The following are two common incorrect answers.

   - The IP is min $\sum_{i=1}^{n} w_i x_i + \sum_{(i,j)\in E} p_{i,j}(1 - x_i)(1 - x_j)$
     subj to $x^i \in \{0, 1\}$.
     NOTE: The objective being minimized is not linear.
   - Let $E' \subseteq E$ be those edges not covered by a vertex in an optimal solution. Then min $\sum_{i=1}^{n} w_i x_i + \sum_{(i,j)\in E'} p_{i,j}$
     subj to $x_i + x_j \geq 1$ for all $(i, j) \in E \setminus E'$.
     NOTE: We do not know a-priori what E' is. That is what the IP is solving.

   SOLUTION: min $\sum_{i=1}^{n} w_i x_i + \sum_{(i,j)\in E} p_{i,j} y_{i,j}$
   subj to $x_i + x_j + y_{i,j} \geq 1$ for every $(i, j) \in E$
   $x_i \in \{0, 1\}$ for all $i \in V$ $\quad$ $y_{i,j} \in \{0, 1\}$ for all $(i, j) \in E$

(b) Provide an LP relaxation and rounding that yields a constant approximation algorithm for computing the total cost of a partial vertex cover. You must justify the approximation bound.

NON-SOLUTION: Some students (using the correct IP above) simply rounded by saying $\bar{x}_i = 1$ if $x_i \geq \frac{1}{2}$ and 0 otherwise; $\bar{y}_{i,j} = 1$ if $y_{i,j} \geq \frac{1}{2}$ and 0 otherwise. This rounded solution is then a 2-approximation.

This is not a correct rounding since it might be that for some edge (i,j), the optimal LP sets $x_i = x_j = y_{i,j} = \frac{1}{3}$. When rounded, the integral solution will not cover this edge.

SOLUTION: A correct rounding (for the correct IP above) would be
$\bar{x}_i = 1$ if $x_i \geq \frac{1}{3}$ and 0 otherwise; $\bar{y}_{i,j} = 1$ if $y_{i,j} \geq \frac{1}{3}$ and 0 otherwise.
This gives a 3-approximation and the rounded solution is a valid partial cover.

4. (10 points) Consider the Max-Cut problem (as in lecture 22).

   Show that in some instance, it helps to use the neighbourhood $N_2(A,B)$. Specifically, give an unweighted graph having a locality ratio $\frac{1}{2}$ for the $N_1(A,B)$ neighbourhood but where the local search algorithm using neighbourhood $N_2(A,B)$ is optimal when applied to graph G.
   Hint: The graph G is a small graph.

5. (20 points) Prove the key observation in slide 11 of L23. Note: this observation leads to the non-oblivious $\frac{3}{4}$ locality ratio for Exact Max-2-Sat as suggested on slide 10.

6. (10 points) Consider a propositional formula F in exact 3 CNF form; that is every clause has exactly three literals involving 3 distinct variables. Using ideas from the naive randomized algorithm, show that F is satisfiable if F has at most 7 clauses.

   SOLUTION: We know that in the naive randomized algorithm, the expected number of satisfied clauses in an exact 3CNF formula with m clauses is $\frac{7}{8} \cdot m$. When there are 7 clauses, this expectation (wrt to a random truth assignment) is $\frac{49}{8} \approx 6.125$. For any truth assignment, the number of satisfied clauses must be intergral and hence there must be at least one assignment satisfying more than 6.125 claues; that is at least 7 clauses implying that the formual is satisfied. (It also follows that there is at least one assignment satisfying at most 6 clauses.)

7. (20 points) You are to generate a random 2CNF formula over 4 variables as follows: Consider the $\binom{4}{2} = 6$ ways to select 2 of the 4 variables. Now for each such choice (say x_1, x_2), randomly define two (different) clauses by setting each literal to be the variable (say x_1) or its complement ($\bar{x}_1$). This will result in 12 clauses, each with two literals. Randomly choose the order in which you will set variables and apply the method of conditional expectations to derive a truth assignment satisfying at least a fraction $\frac{3}{4} = 9$ clauses.

8. Consider the following variant of the Max-Sat problem:
We are given a clause weighted CNF Formula $F = C_1 \wedge C_2 \ldots \wedge C_m$ with $w_j = w(C_j)$, the weight of clause C_j.
Goal: to find a truth assignment that maximizes the weight of clauses that are satsified by at least two literals.

   (a) (10 points) Represent this problem as a {0,1} IP.

   Note: (Without loss of generality we can assume each clause has at least two literals but your formulation should work without this assumption.

   SOLUTION: $\max w_j z_j$
   subj to $\sum_{y_i \text{ occurs positively} \in C_j} y_i + \sum_{y_i \text{ occurs negatively } \in C_j} (1 - y_i) \geq 2 \cdot z_j$ for all clauses C_j
   $y_i \in \{0, 1\}$ for all prop variables x_i; $z_j \in \{0, 1\}$ for all clauses C_j.

   (b) **Bonus**

   We showed how to relax the IP for the standard Max-Sat problem. Relax the IP above and see what approximation ratio you can derive.
   Note: When I wrote down this problem, I thought there could be an easy modification of the analysis for the standard Max-Sat problem but that is not the case. It is instructive to see where the analysis becomes complicated in this variant.