MAT224 Lecture 9&10

Wednesday, February 5, 2020 9:10 AM

MAT224H1S - Linear Algebra II
Winter 2020

Notes on Kernel & Image and the Dimension Theorem:

Definition: Let $T : V \to W$ be a function.

Let $A \subseteq V$. The *image* of the set A under T is

$$T(A) = \{T\mathbf{x} \mid \mathbf{x} \in A\}$$ a set

Let $B \subseteq W$. The *pre-image* of the set B under T is

$$T^{-1}(B) = \{\mathbf{x} \in V \mid T\mathbf{x} \in B\}$$ a set.

Exercise and Discussion: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the *sort function* defined by $T\mathbf{x}$ = the vector whose entries are the entries of $\mathbf{x}$ sorted in descending order. For example, in $\mathbb{R}^3$, $T(1, -2, 3) = (3, 1, -2)$.

(a) Is the sort function a linear transformation?

(b) What is $T(\mathbb{R}^n)$?

(c) In $\mathbb{R}^3$ what is $T^{-1}(\{(1, 0, 0)\})$?

(d) What is the difference between writing $T^{-1}(\{(1, 0, 0)\})$ and $T^{-1}(1, 0, 0)$? Is one more valid than the other?

(a) $T(1,-2,3) = (3,1,-2)$.
$\Rightarrow -T(1,-2,3) = (-3,-1,2)$

$T(-1,2,-3) = (2,-1,-3)$. $\neq -T(1,-2,3)$.

$\therefore$ not a linear transformation

(b) $T(\mathbb{R}^n)$ is set of $a_1, \cdots, a_n \in \mathbb{R}^n$ where $a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1$.

(c) $T^{-1}(\{(1,0,0)\})$ is $\{\{0,0,1\}, \{0,1,0\}, \{1,0,0\}\}$

(d). $T^{-1}(1,0,0)$ is not well-defined. because the output is a set and the input also has to be a set.

Exercise and Discussion: Let V and W be vector spaces and $T \in \mathfrak{L}(V, W)$

(a) Show that if A is a subspace of V, its image $T(A)$ is a subspace of W.

(b) Show that if B is a subspace of W, its pre-image is a subspace of V.

(a) $T(A)$ is non-empty and $cx+y \in T(A)$ whenever $x, y \in T(A)$ and $c \in \mathbb{R}$.

1° $0 \in T(A)$ since $0 = T(0)$.
$\Rightarrow T(A)$ is non-empty.

2° $x \in T(A) \Rightarrow x = Ta$, where $a \in A$.
$y \in T(A) \Rightarrow y = Tb$, where $b \in A$.
$cx+y \in T(A) \Rightarrow cx+y = Tz$, where $z \in A$.

$$\Rightarrow cx+y = cTa + Tb = T(\underbrace{ca+b}_{z})$$

why $z \in A$? because A is a subspace.

Definition: Let V and W be vector spaces and let $T \in \mathfrak{L}(V, W)$.

The *kernel* of T is

$$\ker T = T^{-1}(\{\mathbf{0}\})$$

The *image* of T is

$$\operatorname{im} T = T(V)$$

Exercise and Discussion: Express $\ker T$ and $\operatorname{im} T$ using set notation.

$\ker T = \{x \in V \mid Tx = 0\}$ a subspace of V.
$\operatorname{im} T = \{Tx \in W \mid x \in V\}$ a subspace of W.

Notice that by parts (b) and (a) of the previous exercise, $\ker T$ and $\operatorname{im} T$ are subspaces of V and W respectively. If V and W are finite dimensional, then so too are $\ker T$ and $\operatorname{im} T$, and $\dim(\ker T) \leq \dim V$, and $\dim(\operatorname{im} T) \leq \dim W$.

Exercise and Discussion: Let V and W be vector spaces. Suppose that V is finite dimensional and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for V. Show that $\operatorname{im} T = \operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n\}$.

show 1° $\operatorname{im} T \subseteq \operatorname{span}\{Tv_1, Tv_2, \cdots, Tv_n\}$. $y \in \operatorname{im} T \Rightarrow y = Tx$, $x \in V$. < x can be written as a linear combination of the basis vectors.

$\operatorname{im} T = T(V)$
$= T(\operatorname{span}\{v_1, v_2, \cdots, v_n\})$

$T(c_1v_1 + \cdots + c_nv_n)$.
$= c_1Tv_1 + \cdots + c_nTv_n$.

$\text{im}T = T(V)$
$= T(\text{span}\{v_1, v_2, \cdots, v_n\})$

show 2° $\text{span}\{Tv_1, Tv_2, \cdots, Tv_n\} \subseteq \text{im}T$.
in $\text{im}T$.
Since $Tv_j \in \text{im}T$, where $j \in \{1, 2, \cdots n\}$,
any Linear combination of Tv_j is also in $\text{im}T$.

$T(c_1v_1 + \cdots + c_nv_n)$. as a Linear combination of the basis vectors.
$= c_1Tv_1 + \cdots + c_nTv_n$.
$= \text{span}\{Tv_1, Tv_2, \cdots, Tv_n\}$

As a consequence, $\operatorname{im} T$ is finite dimensional and $\dim(\operatorname{im} T) \le n = \dim V$.

Exercise and Discussion: Let $T \in \mathfrak{L}(V, W)$ and let $\mathbf{x} \in V$ and $\mathbf{y} \in W$ be such that $T\mathbf{x} = \mathbf{y}$.

(a) Show that $T^{-1}(\{\mathbf{y}\}) = \{\mathbf{x}\} + \ker T$.

(b) Can you find a condition that ensures for every $\mathbf{y} \in \operatorname{im} T$ there exists a unique $\mathbf{x} \in V$ such that $T\mathbf{x} = \mathbf{y}$?

(a) show 1° $T^{-1}(\{y\}) \subseteq \{x\} + \ker T$.
show if $\underbrace{z \in T^{-1}(\{y\})}_{T(z) = y}$, then $\underbrace{z \in \{x\} + \ker T}_{z = x + u,\text{ where } u \in \ker T}$.
if $Tz = y$, $u = x - z \in \ker T$.

2° $\{x\} + \ker T \subseteq T^{-1}(\{y\})$.

if $z = x + u$.
$Tz = Tx + Tu = y + 0 = y$.

b) iff $\ker T = \{0\}$

If $Ax = b$. has a unique solution
nullity $A = 0$.

Definition: Let $T : V \to W$ be a function

T is *injective* if $T\mathbf{x} = T\mathbf{y}$ then $\mathbf{x} = \mathbf{y}$. one to one

T is *surjective* if for all $\mathbf{y} \in W$, there exists an $\mathbf{x} \in V$ such that $\mathbf{y} = T\mathbf{x}$. onto

T is *bijective* if it both injective and surjective.

$T: V \to W$ every $y \in W$ is the image of some unique vector in V

$\forall y \in W\ \exists! x \in V$.
s.t. $Tx = y$.

Example: For any given basis α of an n-dimensional vector space V, the α-basis representation function $[\ ]_\alpha : V \to \mathbb{R}^n$ is bijective.

basis $\alpha = \{v_1, \cdots, v_n\}$ x uniquely represented $x = c_1v_1 + \cdots + c_nv_n$ $[x]_\alpha = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ $\left([\ ]_\alpha : V \to R^n \right)$

Exercise and Discussion: Let V and W be finite dimensional vector spaces and $T \in \mathfrak{L}(V, W)$

(a) If $\dim(\ker T) = 0$ can you conclude T is injective? What about surjective? need more info.

(b) If $\dim(\ker T) \ne 0$ can you conclude whether or not T is injective? What about surjective?

(c) If $\dim(\operatorname{im} T) = \dim W$ can you conclude whether or not T is injective? What about surjective?

(d) If $\dim V = \dim W$ and $\dim(\ker T) = 0$ can you conclude T is surjective?

(e) If $\dim V < \dim W$ and $\dim(\ker T) = 0$ can you conclude T is surjective? not surjective.

injective. $[x]_\alpha = [y]_\alpha$.
$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_2 \end{bmatrix}$ since $c_1, \ldots, c_n$ is a unique representation of x
and $d_1, \ldots, d_2$ is a unique representation of y
x and y must be the same

(a) $\dim(\ker T) = 0$.
$\Updownarrow$
$\ker T = T^{-1}(\{0\}) = \{0\}$
nullity $= 0$.

T is injective iff $\ker T = \{0\}$
If $Tx = Ty$, then $Tx - Ty = 0$.
$\Rightarrow T$ is linear $\Rightarrow T(x - y) = 0$
$\Rightarrow x - y \in \ker T = \{0\}$
$\Rightarrow x - y = 0 \Rightarrow x = y \Rightarrow T$ is injective.

(b) If $\dim(\ker T) \ne 0$, there exists more than one vector $x \in V$ such that $0 = Tx \in W$
$\Rightarrow T$ is not surjective.

(c) $T: V \to W$
$\forall y \in W$, $\exists! x \in V$ s.t. $Tx = y$.
$\dim(\text{im}T) = \dim W$
$\text{im}T = W$.

(d) If $\dim(\ker T) = 0$, $\ker T = \{0\} \Rightarrow T$ is injective.
Suppose $\{v_1, \cdots, v_n\}$ is a basis for V
if $c_1Tv_1 + \cdots + c_nTv_n = 0$.
then $T(c_1v_1 + \cdots + c_nv_n) = 0$. $\Rightarrow c_1v_1 + \cdots + c_nv_n \in \ker T = \{0\} \Rightarrow c_1v_1 + \cdots + c_nv_n = 0$.
and $\{v_1, \cdots, v_n\}$ is a basis for V.
$\Rightarrow c_1 = \cdots = c_n = 0. \Rightarrow W = \text{span}\{\underbrace{Tv_1, \cdots, Tv_n}_{\text{a basis for } W}\}$

Theorem: (Dimension Theorem) Let V and W be vector spaces. Suppose V is finite dimensional and let $T \in \mathfrak{L}(V, W)$. Then

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim V$$

Theorem: (Dimension Theorem) Let V and W be vector spaces. Suppose V is finite dimensional and let $T \in \mathfrak{L}(V, W)$. Then

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim V$$

and $\{v_1, \cdots, v_n\}$ is a basis for V.

$\Rightarrow c_1 = \cdots = c_n = 0. \Rightarrow W = \operatorname{span}\{\underbrace{Tv_1, \cdots, Tv_n}_{\text{a basis for } W}\}$

$\Rightarrow \dim(\operatorname{im} T) = \dim V = \dim W.$

Defintion: If $T \in \mathfrak{L}(V, W)$ is bijective, T is called an *isomorphism* and we say V and W are *isomorphic* vector spaces.

Example: Any n-dimensional vector space V is isomorphic to $\mathbb{R}^n$. For any given basis α for V, the α-basis representation function $[\ ]_\alpha : V \to \mathbb{R}^n$ is an isomorphism.

Exercise and Discussion: Suppose V and W are finite dimensional vector spaces, and $T \in \mathfrak{L}(V, W)$.

(a) Suppose T is an isomorphism. Must $\dim V = \dim W$?

(b) Suppose $\dim V = \dim W$. If T is injective is it also surjective? If T is surjective is it also injective?

(c) Suppose $\dim V = \dim W$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for V, and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is a basis for W. if T is defined by $T\mathbf{v}_k = \mathbf{w}_k$ for each $k = 1, 2, \ldots, n$, is T an isomorphism?