STA255: Statistical Theory

Chapter 5: Multivariate Probability Distributions

Summer 2017

Joint Probability Distribution: Discrete Case

- We often deal with several random variables simultaneously.
- We may look at the size of RAM and the speed of a CPU, temperature and humidity, ...

Definition

Let Y_1 and Y_2 be discrete random variables. The **joint** (or bivariate) probability mass function (pmf) for Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2), -\infty < y_1 < -\infty, -\infty < y_2 < -\infty.$$

such that

(a) $p(y_1, y_2) \geq 0$ for all y_1, y_2.

(b) $\sum_{y_2} \sum_{y_1} p(y_1, y_2) = 1$, where the sum is over all values (y_1, y_2) that are assigned nonzero probabilities.

Joint Distribution Function: Continuous Case

Definition

Let Y_1 and Y_2 be continuous random variables. The joint (or bivariate) probability density function (pdf) for Y_1 and Y_2 is given by

$$f(y_1, y_2), -\infty < y_1 < -\infty, -\infty < y_2 < -\infty.$$

such that

(a) $f(y_1, y_2) \geq 0$ for all y_1, y_2.

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 dy_1 = 1$.

Example 1

The joint probability mass function of bivariate discrete random variables Y_1 and Y_2 is given by the following table:

	y_1		
y_2	0	1	2
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find $P(Y_1 + Y_2 < 4)$.

Example 2: #5.8

Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} ky_1y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the values of k that makes this a probability density function.

Example 2: #5.8

(b) Find $P(Y_1 \leq 1/2, Y_2 \leq 3/4)$.

Example 3: #5.16

Let Y_1 and Y_2 denote the proportions of time (out of one workday) during which employees I and II, respectively, perform their assigned tasks. The joint relative frequency behaviour of Y_1 and Y_2 is modelled by the density function

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $P(Y_1 \leq 1/2, Y_2 > 1/4)$.

Example 3: #5.16

(b) Find $P(Y_1 + Y_2 \leq 1)$.

Marginal Distributions

Definition (Marginal Distributions)

- Let Y_1 and Y_2 be two discrete random variables with joint probability function $p(y_1, y_2)$. Then the marginal probability functions of Y_1 and Y_2, respectively, are given by
$$p_1(y_1) = \sum_{y_2} p(y_1, y2) \;\; \& \;\; p_2(y_2) = \sum_{y_1} p(y_1, y_2).$$
- Let Y_1 and Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$. Then the marginal density functions of Y_1 and Y_2, respectively, are given by
$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y2) dy_2 \;\; \& \;\; f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1.$$

Example 1-Continued

Recall: The joint pmf of (Y_1, Y_2) is given by

y_2 \ y_1	0	1	2
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find the marginal pmf of Y_1 and Y_2.

Solution:

y_2 \ y_1	0	1	2	$p_2(y_2)$
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
$p_1(y_1)$	0.3	0.4	0.3	1

Example 3: #5.16- Continued

Recall: the joint density function

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the marginal density functions of Y_1 and Y_2.

Conditional Distributions

Conditional Distributions: Discrete

Let Y_1 and Y_2 be two discrete random variables with joint probability distribution $p(y_1, y_2)$, and marginal distributions, $p_1(y_1)$, $p_2(y_2)$. Then:

- *The conditional discrete probability function of Y_1 given Y_2 is*

$$p(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p(y_1, y_2)}{p_2(y_2)}$$

  provide that $p_2(y_2) > 0$.

- *The conditional discrete probability function of Y_2 given Y_1 is*

$$p(y_2|y_1) = P(Y_2 = y_2|Y_1 = y_1) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_1 = y_1)} = \frac{p(y_1, y_2)}{p_1(y_1)}$$

  provide that $p_1(y_1) > 0$.

Conditional Distributions

Conditional Distributions: Continuous

Let Y_1 and Y_2 be two continuous random variables with joint density $f(y_1, y_2)$, and marginal distributions, $f_1(y_1)$, $f_2(y_2)$. Then:

- *The conditional density Y_1 given $Y_2 = y_2$ is*

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)},$$

  provide that $f_2(y_2) > 0$.

- *The conditional density Y_2 given $Y_1 = y_1$ is*

$$f(y_2|y_1) = \frac{f(y_1, y_2)}{f_1(y_1)},$$

  provide that $f_1(y_1) > 0$.

Example 4: #5.26

$$f(y_1, y_2) = \begin{cases} 4y_1y_2 & 0 \le y_1 \le 1, 0 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $f_1(y_1)$ and $\mathrm{f}_2(y_2)$.

Example 4: #5.26

(b) Find $P(Y_1 \leq 1/2 | Y_2 \geq 3/4)$.

(c) Find the conditional density of Y_1 given $Y_2 = y_2$.

Example 4: #5.26

(d) Find the conditional density of Y_2 given $Y_1 = y_1$.

(e) Find $P(Y_1 \leq 3/4 | Y_2 = 1/2)$.

Independent Random Variables

Recall: A and B are independent if $P(A \cap B) = P(A)P(B)$.

Theorem (Independent Random Variables)

- *If Y_1 and Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then*
  Y_1 and Y_2 are independent iff $p(y_1, y_2) = p_1(y_1)p_2(y_2)$
  for all pairs of real numbers (y_1, y_2).
- *If Y_1 and Y_2 are continuous random variables with joint density function $f(y_1, y_2)$ and marginal density functions $f_1(y_1)$ and $f_2(y_2)$, respectively, then*
  Y_1 and Y_2 are independent iff $f(y_1, y_2) = f_1(y_1)f_2(y_2)$
  for all pairs of real numbers (y_1, y_2).

Example 1-Continued

Recall: the joint pmf of (Y_1, Y_2) is given by

	y_1			
y_2	0	1	2	$p_2(y_2)$
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
$p_1(y_1)$	0.3	0.4	0.3	1

Are Y_1 and Y_2 independent?

Example 5

$$f(y_1, y_2) = \begin{cases} 10y_1^2 y_2 & 0 \leq y_2 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Are the random variables Y_1 and Y_2 independent?

Independent Random Variables

Theorem (Independent Random Variables)

Let Y_1 and Y_2 have a joint density $f(y_1, y_2)$ that is positive if and only if $a \leq y1 \leq b$ and $c \leq y2 \leq d$, for constants a,b,c, and d; and $f(y_1, y_2) = 0$ otherwise. Then

Y_1 and Y_2 are independent iff $f(y_1, y_2) = g(y_1)h(y_2)$

where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone.

Note: *$g(y_1)$ and $h(y_2)$ need not to be density functions.*

Example 6: #5.52

$$f(y_1, y_2) = \begin{cases} 4y_1y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

The Expected Value of a Function of Random Variables

Definition (Expected Value of a Function)

- Let $g(Y_1, Y_2, \ldots, Y_k)$ be a function of the discrete random variables $Y_1, Y_2, \ldots, Y_k$, which have probability function $p(y_1, y_2, \ldots, y_k)$. Then the expected value of $g(Y_1, Y_2, \ldots, Y_k)$ is

$$E[g(Y_1, Y_2, \ldots, Y_k)] =$$
$$\sum_{y_k} \cdots \sum_{y_2} \sum_{y_1} g(y_1, y_2, \ldots, y_k) p(y_1, y_2, \ldots, y_k).$$

- If $Y_1, Y_2, \ldots, Y_k$ are continuous random variables with joint density function $f(y_1, y_2, \ldots, y_k)$, then

$$E[g(Y_1, Y_2, \ldots, Y_k)] =$$
$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1, y_2, \ldots, y_k) f(y_1, y_2, \ldots, y_k) dy_1 dy_2 \ldots dy_k.$$

Example 7

Let

$$f(y_1, y_2) = \begin{cases} 2y_1 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $E(Y_1 Y_2)$

Example

(b) Find $E(Y_1)$.

(c) Find $V(Y_1)$

Theorem (Special Theorems)

1. *Let c be a constant. Then $E(c) = c$.*
2. *Let $g(Y_1, Y_2)$ be a function of the random variables Y_1 and Y_2. Then*
$$E[cg(Y_1, Y_2)] = cE[g(Y_1, Y_2)]$$
3. *Let $g_1(Y_1, Y_2), \ldots, g_k(Y_1, Y_2)$ be functions of Y_1 and Y_2. Then*
$$E[g_1(Y_1, Y_2) + \ldots + g_k(Y_1, Y_2)] = E[g_1(Y_1, Y_2))] + \ldots + E[g_k(Y_1, Y_2)]$$
4. *If Y_1 and Y_2 are independent, then*
$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$$

where $g(Y_1)$ and $h(Y_2)$ are functions of only Y_1 and Y_2, respectively.

The Covariance of Two Random Variables

Definition (Covariance)

The covariance between Y_1 and Y_2, denoted by $Cov(Y_1, Y_2)$ or $\sigma_{Y_1 Y_2}$, is given by

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)]$$

Definition (Correlation)

The correlation between Y_1 and Y_2, denoted by $Corr(Y_1, Y_2)$ or ρ, is defined by

$$Corr(Y_1, Y_2) = \frac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2}$$

where σ_1 and σ_2 are the standard deviations of Y_1 and Y_2, respectively.

Fact: For any two random variables Y_1, Y_2,

$$-1 \leq \rho \leq 1$$

Theorem

Theorem (Covariance)

$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

Theorem (Independence and Covariance)

If Y_1 and Y_2 are independent random variables, then $Cov(Y_1, Y_2) = 0$.

Example 1-Continued

Recall: the joint pmf of (X, Y) is given by

	y_1			
y_2	0	1	2	$p_2(y_2)$
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
$p_1(y_1)$	0.3	0.4	0.3	1

Find $Cov(Y_1, Y_2)$ and $Corr(Y_1, Y_2)$.

The Expected Value and Variance of Linear Functions of Random Variables

Theorem (Linear Functions of Random Variables)

Let $Y_1, Y_2, \ldots, Y_n$ *and* $X_1, X_2, \ldots, X_m$ *be random variables with* $E(Y_i) = \mu_i$ *and* $E(X_j) = \xi_j$. *Define*

$$U_1 = \sum_{i=1}^{n} a_i Y_i \quad \& \quad U_2 = \sum_{j=1}^{m} b_j X_j$$

for constants $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_m$. *Then the following hold:*

(a) $E(U_1) = \sum_{i=1}^{n} a_i \mu_i$

(b) $V(U_1) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i \leq j \leq n} a_i a_j Cov(Y_i, Y_j)$

(c) $Cov(U_1, U_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(Y_i, X_j)$

Properties of Covariance

Special Cases

- $Cov(Y, Y) = Var(Y)$
- $Cov(aY_1 + b, cY_2 + d) = acCov(Y_1, Y_2)$.
- $Cov(Y_1, Y_2) = Cov(Y_2, Y_1)$.
- $V(aY_1 + bY_2 + c) = V(aY_1 + bY_2) =$ $a^2 Var(Y_1) + b^2 Var(Y_2) + 2abCov(Y_1, Y_2)$.
- *If Y_1 and Y_2 are independent, then* $V(aY_1 + bY_2) = a^2 V(Y_1) + b^2 V(Y_2)$.

Example 8

Let Y_1, Y_2, and Y_3 be random variables, where $E(Y1) = 1, E(Y_2) = 2, E(Y_3) = 1, V(Y_1) = 1, V(Y_2) = 3, V(Y_3) = 5, Cov(Y_1, Y_2) = 0.4, Cov(Y_1, Y_3) = 1/2$, and $Cov(Y_2, Y_3) = 2$. Let $U = Y_1 - 2Y_2 + Y_3$ and $W = 3Y_1 + Y_2$. Find $E(U)$, $V(U)$ and $CCov(U, W)$.

Multinomial Distribution

The binomial experiment becomes a multinomial experiment if we let each trial to have more than two possible outcomes.

- The classification of a manufactured product as being light, heavy, or acceptable.
- Recording of accidents at certain intersection: fatal, major injury, minor injury.

Multinomial Distribution

Multinomial Distribution

- *If a given trial can result in the k outcomes* $E_1, E_2, \ldots, E_k$ *with probabilities* $p_1, p_2, \ldots, p_k$*, then the probability distribution of the random variables* $Y_1, Y_2, \ldots, Y_k$*, representing the number of occurrences for* $E_1, E_2, \ldots, E_k$ *in n independent trials, is given by*

$$p(y_1, y_2, \ldots, y_k) = \frac{n!}{y_1!y_2!...y_k!} p_1^{y_1} p_2^{y_2} \ldots p_k^{y_k},$$

  with $y_1 + y_2 + \ldots + y_k = n$*,* $p_1 + p_2 + \ldots + p_k = 1$ *and* $y_i = 0, 1, 2, \ldots, n.$

- *Theorem:*

  $E(Y_i) = np_i; V(Y_i) = np_i(1 - p_i); Cov(Y_i, Y_j) = np_i p_j, i \neq j.$

Example 9: #120

A sample of size n is selected from a large lot of items in which a proportion p_1 contains exactly one defect and a proportion p_2 contains more than one defect (with $p_1 + p_2 < 1$). The cost of repairing the defective items in the sample is $C = Y_1 + 3Y_2$, where Y_1 denotes the number of items with one defect and Y_2 denotes the number with two or more defects. Find the expected value and variance of C.

Example 10: #122

The weights of a population of mice fed on a certain diet since birth are assumed to be normally distributed with $\mu = 100$ and $\sigma = 20$ (measurement in grams). Suppose that a random sample of $n = 4$ mice is taken from this population. Find the probability that

(a) exactly two weigh between 80 and 100 grams and exactly one weighs more than 100 grams.

(b) all four mice weigh more than 100 grams.

Additional Exercises

(1) Let Y_1 and Y_2 have the joint pdf

$$f(y_1, y_2) = \begin{cases} 2 & 0 < y_1 < y_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal probability density functions.
(b) Find the conditional pdf of Y_1 given $Y_2 = y_2$.
(c) Find the conditional mean of Y_1 given $Y_2 = y_2$.
(d) $P(0 < Y_1 < 1/2 | y_2 = 3/4)$.
(e) Find the correlation coefficient.

Additional Exercises

(2) Let Y_1 and Y_2 have the joint pdf

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 < y_1 < 1, 0 < y_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal probability density functions.

(b) Are Y_1 and Y_2 independent? Justify your answer.

(c) Find $P(0 < Y_1 < 1/2, 0 < Y_2 < 1/2), P(0 < Y_1 < 1/2)$, and $P(0 < Y_2 < 1/2)$

Additional Exercises

(3) If $Y_1 \sim N(\mu = 10, \sigma^2 = 4)$ and $Y_2 \sim N(\mu = 8, \sigma^2 = 16)$. Assume that Y_1 and Y_2 are independent.

   (a) Find $P(Y_1 \geq 11 | Y_2 \leq 7)$
   (b) Find $P(Y_1 \geq 11, Y_2 \leq 7)$.
   (c) Find $E(2Y_1^2 - 3Y_1Y_2 + 10)$

(4) Show that $V(aY_1 + bY_2) = a^2V(Y_1) + b^2V(Y_2) + 2abCov(Y_1, Y_2)$, for any real numbers a and b.

(5) Show that $Cov(aY_1 + b, cY_2 + d) = acCov(Y_1, Y_2)$, for any real numbers a and b.

Additional Exercises

(6) Suppose that Y_1 and Y_2 are discrete independent random variables with the following moment generating functions:

$$m_{Y_1}(t) = E(e^{tY_1}) = \frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$$

$$m_{Y_2}(t) = E(e^{tY_2}) = \frac{1}{10}e^{-t} + \frac{4}{10}e^{2t} + \frac{1}{2}e^{3t}.$$

(a) Find the mean and variance of Y_1.

(b) Find the moment generating function of $U = Y_1 - Y_2$.

(c) Find the probability mass function of $U = Y_1 - Y_2$.