Solve all the following questions. Show all your work. Please write your answers in the Space provided.

1. Suppose A, B, C are events that $P(A|C) \geq P(B|C)$ and $P(A|C^c) \geq P(B|C^c)$. Show that $P(A) \geq P(B)$.

2. The probability density function for the lifetime distribution of a particular type of component, in years, has the following form, with an unknown constant k, in it:

$$f(y) = \begin{cases} kx^2 e^{\frac{-x^3}{10}} & \text{for} x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

where k is a constant.

(a) Find the constant k that makes it a valid density function.

(b) Determine the cumulative distribution function for this distribution.

(c) Half of components fail before what time?

2. Suppose a couple of your friends go to 'Sushi on Bloor' on either Monday or Friday each week, not both. 26% of the time they go on Monday. On Mondays, the probability of receiving a good service is 0.72. On Fridays, the probability of receiving a good service is only 0.13.

(a) What is the probability that they went to that restaurant on Monday and received a good service?

(b) What is the probability that they received good service at that restaurant last week?

(c) Suppose that you don't know which day they went last week, but they tell you they received good service. What is the probability that they went on Monday?

3. Suppose $X \sim \text{Poisson}(\lambda)$. Find $E(\frac{1}{1+X})$.

4. The random variables X and Y have joint probability density function

$$f(y) = \begin{cases} 2 & x + y > 1, x < 1, y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

a) Find the marginal density function of X.

b) Find the density function of Y conditional on $x = 3/4$

c) Find $P(Y > 1/2 | x = 3/4)$

5. Let $Y_1, Y_2, \dots, Y_n$ denote independent and identically distributed random variables from following distribution

$$f(y;\beta) = \begin{cases} \alpha\beta^{\alpha}y^{-(\alpha+1)} & y \geq \beta \\ 0 & \text{elsewhere.} \end{cases}$$

if $\alpha, \beta > 0$ and α is known. Find a sufficient statistic for parameter β.

6. Let $X_1, \ldots, X_n$ denote independent and identically distributed random variables from the following distribution with α

$$f(y;\alpha) = \begin{cases} \alpha y^{\alpha-1}/3^{\alpha} & 0 < y < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Then if $\alpha > 0$, show that $E(Y) = 3\alpha/(\alpha+1)$ and derive the method of moment estimator for α.

7. Suppose that Z has a standard normal distribution. Find density function of $U = Z^2$.

8. Let the random variable Y follow

$$f(y) = \begin{cases} (3/2)y^2 + y & 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Use method of transformation to find the probability density function of $U = 5 - (y/2)$.

9. If Z_1, Z_2, Z_3 are iid variables that are normally distributed with mean 0 and variance 1 then state the distribution of

a) $W = Z_1^2 + Z_2^2 + Z_3^2$.

b) $U = \frac{(Z_1^2+Z_2^2)/2}{Z_3^2}$

10. A study by Children's Hospital in Boston indicates that about 67% of American adults and about 15% of children and adolescents are overweight. Thirteen children in random sample of size 100 were found to be overweight. Is there sufficient to indicate that the percentage reported by Children's hospital is too high? Test at $\alpha = 0.05$ level of significance.

11. For a comparison of the rates of defectives produced by two assembly lines, independent random samples of 100 items were selected from each line. Line A yielded 18 defectives in the sample, and line B yielded 12 defectives.

a) Find 98% confidence interval for the true difference in proportions of defectives for the two lines.

b) Is there evidence here to suggest that one line produces a higher proportion of defectives than the other?

12. Let Y be a continuous random variable with probability density function:

$$f(y) = \begin{cases} \alpha y^{\alpha+1} & 0 \leq y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

What is the maximum likelihood estimator of based on a random sample of size n?