CSC 373 - Algorithm Design, Analysis, and Complexity **Summer 2014**

Assignment 3: Due Friday July 25, Midnight

Please follow the instructions provided on the course website to submit your assignment. You may submit the assignments in pairs. Also, if you use **any** sources (textbooks, online notes, friends) please cite them for your own safety.
You can use those data-structures and algorithms discussed in CSC263 (e.g. merge-sort, heaps, etc.) and in the lectures by stating their name. You do not need to provide any explanation or pseudo-code for their implementation. You can also use their running time without proving them: for example, if you are using the merge-sort in your algorithm you can simply state that merge-sorts running time is $\mathcal{O}(n \log n)$.

Every time you are asked to design an efficient algorithm, you should provide both a short high level explanation of how your algorithms works in plain English, and the pseudo-code of your algorithm similar to what we've seen in class. State the running time of your algorithm with a brief argument supporting your claim. You must prove that your algorithm finds an optimal solution!

1. Show how to construct a family of graphs where $G(V, E)$ has an exponential number (in $|V|$) of minimum cuts between the source and the terminal.

2. A function $f : 2^{|V|} \to R$ is **submodular** if and only if for any two subsets $A, B \subseteq V$, we have:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

   Let $G(V, E)$ be a graph, $S \subseteq V$, and let $f(S)$ be the number of edges (u, v) with $u \in S$, $v \in V \backslash S$. Show that f is submodular.

3. Consider the following LP formulation of maximum flow:

$$\begin{aligned} & \text{maximize} \sum f_P \\ \text{s.t.} \sum_{e(u,v) \in P} f_P \leq c(u, v) \qquad & \forall (u, v) \in E \\ & f_P \geq 0 \end{aligned}$$

   In this formulation, we consider the paths that were used to send the flow. f_P denotes the amount of flow sent from s to t along path P. Give the dual of this LP and briefly explain your objective function and constraints.

4. Let P be a polytope represented by a set of inequalities $Ax \leq b$. Give an LP that computes the center of the largest ball you can fit inside P.

5. Consider two polyhedra defined as $P_1 = \{x | A_1 x \leq b_1\}$ and $P_2 = \{x | A_2 x \leq b_2\}$. Show that if P_1 and P_2 do not intersect, then there exists $s, t \geq 0$ where $sA_1 + tA_2 = 0$ but $sb_1 + tb_2 < 0$.

6. Let $G(V, E, c)$ be an edge weighted directed graph with $c : E \to \mathbb{R}^+$. The **minimum mean cycle** problem asks for a cycle $\mathcal{C}(V', E')$ which minimizes the ratio:

$$\frac{\sum\limits_{e \in E'} c(e)}{|E'|}$$

Consider the following LP:

$$\begin{aligned} \text{minimize} \sum c(u, v) f(u, v) & \\ \text{s.t.} \sum_v f(u, v) - \sum_w f(w, u) &= 0 \qquad (\forall u) \\ \sum f(u, v) &= 1 \\ f(u, v) &\geq 0 \end{aligned}$$

Show that this LP captures the minimum mean cycle problem. Give its dual, and show why your dual formulation also captures minimum mean cycles.

Hint: When you formulate your constraint for the dual, think about what it means to sum that inequality over the edges of a given cycle.

7. Alice is deciding how much organic milk and how much conventional milk to order each week. Conventional milk costs Alice \$1 per litre and she sells it at \$2 per litre; organic milk costs Alice \$1.50 per litre and she sells it at \$3 per litre. However, the milk company, CowCo, will only sell a litre of organic milk for each two litres or more of conventional milk that Alice buys. Furthermore, CowCo will not sell Alice more than 3,000 litres per week. Alice knows that she can sell however much milk she has. Formulate a linear program for deciding how much organic and how much conventional milk to buy, so as to maximize Alice's prot.

8. Let $\mathcal{M} = \{x_1, x_2, ..., x_n\}$ be a set of marbles. You have m friends, each with a set $S_i \subseteq \mathcal{M}$ of marbles. The marbles minister asked you whether it's possible to construct a set $X \subseteq \mathcal{M}$ of size $|X| \leq k$ such that $X \cap S_i \neq \emptyset$ for all $S_1, S_2, ..., S_m$. Show that this problem is NP-complete.