TRANSFER FUNCTION NOISE MODEL

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Transfer function noise (TFN) model

- A TFN model is a time series regression that predict current values of a dependent variable based on both the current values as well as the lagged values of one or more explanatory variables.
- A distributed lag model in statistics and econometrics*
- E.g. sales and advertisement are the example of the dependent variable and the input or explanatory variable in a TFN model

- Mathematical form of a TFN model

$$y_t = \alpha + v_0 x_t + v_1 x_{t-1} + v_2 x_{t-2} + \cdots + \varepsilon_t = \alpha + \sum_{i}^{\infty} v_i x_{t-i} + \varepsilon_t$$

- The coefficients $\nu_0, \nu_1, \cdots$ are referred to as the impulse response function of the system.*
  - *Distributed lag models require all impulse response functions of the same sign.*
- For the above equation to be *meaningful*, the impulse responses must be absolutely summable, i.e., $\sum_{j=0}^{\infty} |\nu_j| < \infty$.
  - *In this case, the system is said to be stable.*
- The value $g = \sum_{j=0}^{\infty} \nu_j$ is called the stead-state gain
  - *It represents the impact on Y when X_{t-j} are held constant over time.*

Model infinite number of parameters in practice

- Unstructured estimation
- Structured estimation/approximation
  - *Finite distributed lag model, e.g. Almon distributed lag model*

$$v_j = \sum_{j=0}^{n} a_j i^j \,,$$

where $i = 0, \ldots, k$ *and* $n < k$.

  - *Rational (infinite) distributed lag model, e.g. Koyck distributed lag model*

Koyck distributed lag model

$$y_t = \alpha + \sum_{i=0}^{\infty} \beta\lambda^i x_{t-i} + \sum_{i=0}^{\infty} \lambda^i \xi_{t-i} \, .$$

- That is, we have $v_i = \beta\lambda^i$. Suppose that $|\lambda| < 1$
- We can approximate the Koyck distributed lag model using the following ARX model
$$y_t = a + \lambda y_{t-1} + \beta x_t + \xi_t.$$

Rational distributed lag model

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + \varepsilon_t$$

- Jorgenson (1966, Econometrica) proves that $v(B) = \sum_{i=0}^{\infty} v_i B^i$

can be approximated by a ratio of two polynomials

$$v(B) = \frac{\delta_0 + \delta_1 B + \cdots + \delta_r B^r}{1 - \vartheta_1 B - \cdots - \vartheta_s B^s} = \frac{\delta(B)}{\vartheta(B)}.$$

- Using the rational distributed lag function, we cam approximate y_t as

$$y_t = \frac{\delta(B)}{\vartheta(B)} x_t + \varepsilon.$$

where we allows the error term ε_t to follow a stationary ARMA process.
The above equation satisfies the form of a transfer function noise model.

Model building process

- The procedure of building the single input TFN model includes

1. Preliminary identification of the impulse response coefficients ν_i's;
2. Specification of the noise term ε_t;
3. Specification of the transfer function using a rational polynomial in *B if necessary*;
4. Estimation of the TFN model specified in Step 2 and 3;
5. Model diagnostic checks.

- See the supplement materials for Model diagnostic checks and estimation using the Box and Tiao approach.
- In practice, we may model the multiple inputs TFN model using vector autoregression.

Preliminary identification (prewhitening)

- Suppose that x follows an $ARMA$ model

$$\phi_x(B)x_t = \theta_x(B)\alpha_t, \qquad \alpha_t \sim NID(0, \sigma_\alpha^2).$$

- Apply the operator $\phi_x(B)/\theta_x(B)$ on both sides of the above equation

$$\tau_t = \frac{\phi_x(B)}{\theta_x(B)} y_t = \nu(B) \underbrace{\frac{\phi_x(B)}{\theta_x(B)} x_t}_{\alpha_t} + \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t = \nu(B)\alpha_t + n_t, \qquad (*)$$

  where $\tau_t = \frac{\phi_x(B)}{\theta_x(B)} Y_t$ and $n_t = \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t$

- By design, $\{n_t\}$ is independent of $\{\alpha_t\}$.

Preliminary identification (prewhitening)

- Multiplying both sides of eqn. $(*)$ by α_{t-j} for $j \geq 0$, we have
$$\tau_t \alpha_{t-j} = \nu(B)\alpha_t \alpha_{t-j} + n_t \alpha_{t-j}.$$
- Taking expectation, we have
$$cov(\tau_t, \alpha_{t-j}) = \nu_j \cdot var(\alpha_{t-j}).$$
- By definition
$$\nu_j = \frac{cov(\tau_t, \alpha_{t-j})}{var(\alpha_t)} = corr(\tau_t, \alpha_{t-j}) \cdot \frac{se(\tau_t)}{se(\alpha_t)}.$$
- Thus, we can test the statistical significance of ν_j by examining the statistical significance of $corr(\tau_t, \alpha_{t-j})$.