Question 1

If $P(A|C) \geq P(B|A) \Rightarrow \dfrac{P(A \cap C)}{P(C)} \geq \dfrac{P(B \cap C)}{P(C)}$

$$\Rightarrow P(A \cap B) \geq P(B \cap C) \quad (1)$$

Similarly; $P(A \cap B^c) \geq P(B \cap C^c) \quad (2)$

Since $P(A|C^c) \geq P(B|C^c)$

From (1), (2) Adding: $P(A \cap C) + P(A \cap C^c) \geq P(B \cap C) + P(B \cap C^c)$

$$\Rightarrow P(A) \geq P(B)$$

Question 2

a) Set $\int_0^\infty K x^2 e^{-x^3/10}\,dx = 1$ and solve it.

$$\int_0^\infty K x^2 e^{-x^3/10}\,dx = -\frac{10}{3}K\int_0^\infty \frac{-3x^2}{10} e^{-x^3/10}\,dx$$

$$= -\frac{10}{3}K\left(e^{-x^3/10}\right)\Big|_0^\infty = 0 + \frac{10}{3}K = 1$$

$$\Rightarrow K = 0.3$$

b) $F(x) = P(X \leq x) = \int_0^x 0.3\, y^2 e^{-y^3/10}\,dy$

$$= -e^{-y^3/10}\Big|_0^x = 1 - e^{-x^3/10} \quad \text{for } x > 0$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^3/10} & x > 0 \end{cases}$$

c) $1 - e^{-x^3/10} = \frac{1}{2}$ gives ... $\Rightarrow$ $x = \left(-10 \ln(0.5)\right)^{1/3} = 1.907$ Years

Question 3

a) $P(\text{Monday and good Service}) = P(\text{good Service} \mid \text{Monday})\, P(\text{Monday})$

$$= 0.72 * 0.26 = 0.1872$$

b) $P(\text{good Service}) = P(\text{good service} \mid \text{Monday})\, P(M) + P(\text{good service} \mid F)\, P(F)$

$$= 0.72 * 0.26 + 0.13 * 0.74 = 0.2834$$

c) $$P(M \mid \text{good service}) = \frac{P(\text{M and good service})}{P(\text{good service})} = \frac{0.72 * 0.26}{0.2834}$$

$$= 0.66$$

Question ~~3~~ 4

$$E\left(\frac{1}{1+X}\right) = \sum_{x=0}^{\infty} \frac{1}{1+x} * \frac{\lambda^{x} e^{-\lambda}}{x!}$$

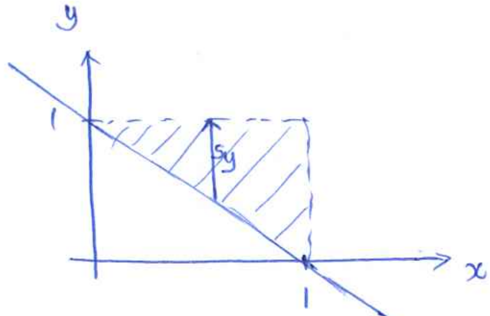
$$= \frac{1}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!}$$

$$= \frac{1}{\lambda} \sum_{y=1}^{\infty} \frac{\lambda^{y} e^{-\lambda}}{y!} =$$

$$= \frac{1}{\lambda}\left(1 - e^{-\lambda}\right)$$

Question 5

(a)



$$f_X(x) = \int_{S_y} f(x,y)\, dy = \int_{1-x}^{1} 2\, dy = 2y \Big|_{1-x}^{1} = 2\left(1-(1-x)\right)$$

$$= 2x \qquad 0 < x < 1$$

(b)

$$f(x,y) = 2 \qquad x+y>1,\ x<1,\ y<1$$

$$f(x) = 2x \qquad 0<x<1$$

$$\Rightarrow f_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x}$$

$y < 1$ (1)

$x + y > 1 \Rightarrow y > 1 - x$ (2)

(1),(2) $\Rightarrow$ $1 - x < y < 1$

$$\Rightarrow f_{y|x}\left(y \,\middle|\, x = \tfrac{3}{4}\right) = \frac{4}{3} \quad ; \quad \frac{1}{4} < y < 1$$

(c) $f_{Y|X}(y|x) = \frac{1}{x}$ $\quad 1-x < y < 1$

$f_{Y|X}(y|x=3/4) = \frac{4}{3}$ $\quad 1/4 < y < 1$

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 3/4\right) = \int_{1/2}^{1} f_{Y|X}(y|x=3/4)\, dy$$

$$= \int_{1/2}^{1} \frac{4}{3}\, dy = \frac{4}{3} \cdot y \Big|_{1/2}^{1}$$

$$= \frac{4}{3}\left(1 - 1/2\right) = \frac{4}{3} * \frac{1}{2} = \frac{2}{3}$$

(d)

Question 6

$$f(y;\alpha,\beta) = \alpha \beta^{\alpha} y^{-(\alpha+1)} \qquad y \geq \beta$$

we introduce indicator function $I_\beta(y) = \begin{cases} 1 & \text{if } y > \beta \\ 0 & \text{o.w} \end{cases}$

Since α is known; we have

$$f(y;\beta) = \alpha \beta^{\alpha} y^{-(\alpha+1)} \cdot I_\beta(y)$$

$$L(\beta) = f_1(y_1) \times \cdots \times f_n(y_n) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \alpha \beta^{\alpha} y_i^{-(\alpha+1)} I_\beta(y_i)$$

$$= \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} y_i\right)^{-(\alpha+1)} \prod_{i=1}^{n} I_\beta(y_i)$$

$$= \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} y_i\right)^{-(\alpha+1)} I_\beta(y_{(1)}) = g(y_{(1)}, \beta)\, h(y_1, \ldots, y_n)$$

where $y_{(1)} = \min(y_1, y_2, \ldots, y_n)$

$$g(y_{(1)}, \beta) = \beta^{n\alpha} I_\beta(y_{(1)})$$

$$h(y_1, \ldots, y_n) = \alpha^n \left(\prod_{i=1}^{n} y_i\right)^{-(\alpha+1)}$$

So $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$ is sufficient for β (when α is known parameter)

$$E(y) = \int_0^3 \alpha y^{\alpha} 3^{-\alpha}\, dy = \alpha 3^{-\alpha} \frac{y^{\alpha+1}}{\alpha+1}\Big|_0^3 = \frac{3\alpha}{\alpha+1}$$

$$\mu_1 = E(y) = \frac{3\alpha}{\alpha+1}$$

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

$$\mu_1 = m_1 \Rightarrow \frac{3\alpha}{\alpha+1} = \bar{y}$$

$$\Rightarrow \alpha\bar{y} + \bar{y} = 3\alpha$$

$0 < y_i < 3$

$$\Rightarrow \alpha(3-\bar{y}) = \bar{y} \Rightarrow \alpha = \frac{\bar{y}}{3-\bar{y}}$$

$$\hat{\alpha}_{MM} = \frac{\bar{y}}{3-\bar{y}}$$

Question 7

Let $F_Z(z)$ and $f_Z(z)$ denote the standard normal distribution and density functions respectively.

$$F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u}) = F_Z(\sqrt{u}) - F_Z(-\sqrt{u})$$

The density function for U is then

$$f_U(u) = F_U'(u) = \frac{1}{2\sqrt{u}} f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_Z(-\sqrt{u})$$

$$= \frac{1}{\sqrt{u}} f_Z(\sqrt{u}) \; ; \quad u \geq 0$$

Evaluating, we find $f_U(u) = \frac{1}{\sqrt{\pi}\sqrt{2}} u^{-1/2} e^{-u/2} \quad u \geq 0$

Question 8

If $U = 5 - \left(\frac{y}{2}\right) \Rightarrow y = 2(5-U)$ $\Rightarrow h^{-1}(u) = y = 2(5-U)$ (1)

Thus $|j| = \left|\frac{\partial y}{\partial u}\right| = 2$

So $f_u(u) = f_y(y)\,|j| = \cancel{4 f 80}$

$= 2 \cdot f_y\left(h^{-1}(u)\right) = 2 \cdot \left(\frac{3}{2}\right)\left(h^{-1}(u)\right)^2 + \left(h^{-1}(u)\right)$

From (1)

$= \cancel{2}\; 4\left(80 - 31u + 3u^2\right)$ $\qquad 4.5 \leq y \leq 5$

$y: 0 \longrightarrow 1$

$u = 5 - \left(\frac{y}{2}\right): \; 4.5 \longrightarrow 5$

Question 9

Pegell

a) $Z_1^2 + Z_2^2 + Z_3^2$

Squared standard normal distribution $\sim \chi^2_{(1)}$

$Z_1^2 \sim \chi^2_{(1)}$

$Z_2^2 \sim \chi^2_{(1)}$

$Z_3^2 \sim \chi^2_{(1)}$

The sum of independent $\chi^2_{(1)}$ and $\chi^2_{(1)}$ and $\chi^2_{(1)}$

has a χ^2_3 distribution with degrees of freedom 3.

b) $U = \dfrac{(Z_1^2 + Z_2^2)/2}{Z_3^2}$

The numerator is a χ^2 distribution with df=2 divided by its ~~df~~ degree of freedom ; the denominator is independent of the numerator and also is a $\chi^2_{(1)}$ divided by df=1

Hence ; $U \sim F_{(2,1)}$

Question 10 (10.24 of textbook)

Let p= proportion of overweight children and adolecents. Then, $H_0: p = .15$ $H_a: p < 0.15$ and the computed large sample test statistic for a proportion is $z = -0.56$. This doesn't not lead to a rejection at the $\alpha = 0.05$ level.

Question 11

Question 8.65 of text book

a) The 98% CI is, with $z_{.01} = 2.326$, is

$$0.18 - 0.12 \pm 2.326 \sqrt{\frac{0.18(0.82) + 0.12(0.88)}{100}}$$

or $\quad 0.06 \pm 0.117 \quad$ or $\quad (-0.057, 0.177)$

b) Since the interval contains zero, it is likely that the two assembly lines produce the same proportion of defectives.

Question 12

$$Y_1, \dots Y_n \overset{iid}{\sim} \alpha y^{\alpha-1}$$

$$L(\alpha) = f_1(y_1) * f_2(y_2) * \dots * f_n(y_n)$$

$$= \left(\alpha y_1^{\alpha+1}\right)\left(\alpha y_2^{\alpha+1}\right) \dots \cdot \left(\alpha y_n^{\alpha+1}\right)$$

$$= \prod_{i=1}^{n} \alpha y_i^{\alpha+1} = \alpha^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha+1}$$

① $$l(\alpha) = \log L(\alpha) = n \ln \alpha + (\alpha+1) \sum_{i=1}^{n} \ln y_i$$

② $$\frac{\partial}{\partial \alpha} l(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln y_i = 0 \quad \Rightarrow \frac{n}{\alpha} = -\sum_{i=1}^{n} \lg y_i$$

$$\Rightarrow \alpha = -\frac{1}{n} \sum_{i=1}^{n} \ln y_i$$

From ② $$\frac{\partial^2}{\partial \alpha^2} l(\alpha) = -\frac{n}{\alpha^2} < 0 \qquad \text{So} \quad \hat{\alpha}_{ML} = -\frac{1}{n} \sum_{i=1}^{n} \ln y_i$$

is maximum likelihood estimate for α