PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term Test Sample Solutions

CSC 236H1
Section L5101
Duration — 50 minutes

PLEASE HAND IN

Examination Aids: One 8.5”x11” sheet of paper, handwritten on one side.

Last Name: ______________________________

First Name: ______________________________

Student No: ______________________________

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 4 questions on 6 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided.

Good Luck!

Question 1. [9 MARKS]

Prove that for all $n \in \mathbb{N}$, $n \geq 1$

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution: Let $P(n)$ denote the assertion that $\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

Base Case: Let $k = 1$.
Then $\sum_{i=1}^{k}(2i-1)^2 = (2 \cdot 1 - 1)^2 = 1$.

Also, $\frac{k(2k-1)(2k+1)}{3} = \frac{1(2-1)(2+1)}{3} = 1$.

So $\sum_{i=1}^{k}(2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$.

Induction Step: Let $k \in \mathbb{N}$. Suppose $P(k)$ is true, i.e., $\sum_{i=1}^{k}(2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$. **[IH]**

WTP: $P(k+1)$ holds, i.e., $\sum_{i=1}^{k+1}(2i-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3}$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1)^2 &= \sum_{i=1}^{k}(2i-1)^2 + (2(k+1)-1)^2 \\
&= \sum_{i=1}^{k}(2i-1)^2 + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 && \text{by the IH} \\
&= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\
&= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} && \text{factoring out } (2k+1) \\
&= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3} \\
&= \frac{(2k+1)(2k^2 + 5k + 3)}{3} \\
&= \frac{(2k+1)(2k+3)(k+1)}{3}.
\end{aligned}$$

Question 2. [13 MARKS]

Let $f_1, f_2, ...$ be a sequence of natural numbers defined as follows:

$$
\begin{aligned}
f_1 =& 1, \\
f_2 =& 1, \\
f_n =& f_{n-1} + f_{n-2}, \qquad n \geq 3.
\end{aligned}
$$

Let $a_0, a_1, a_2, ...$ be a sequence of natural numbers defined as follows:

$$
\begin{aligned}
a_0 =& 0, \\
a_1 =& 1, \\
a_n =& a_{n-1} + a_{n-2} + 5 \cdot 3^{n-2}, \qquad n \geq 2.
\end{aligned}
$$

Prove that for all $n \in \mathbb{N}$, $a_n = 3^n - f_{n+2}$.

Solution: Let $P(n)$ denote the assertion that $a_n = 3^n - f_{n+2}$.

Base Case:
Let $k = 0$.
Then by definition, $a_k = 0$.
Also, $3^k - f_{k+2} = 1 - f_2 = 0$, since $f_2 = 1$.
So $a_k = 3^k - f_{k+2}$.

Let $k = 1$.
Then by definition, $a_k = 1$.
Also, $3^k - f_{k+2} = 3 - f_3 = 1$ since $f_3 = 2$.
So $a_k = 3^k - f_{k+2}$.

Induction Step: Let $k \in \mathbb{N}$, and $k \geq 2$. Suppose for all $0 \leq j < k$, $P(j)$ is true, i.e., $a_j = 3^j - f_{j+2}$. **[IH]**
WTP: $P(k)$ holds, i.e., $a_k = 3^k - f_{k+2}$.

$$
\begin{aligned}
a_k &= a_{k-1} + a_{k-2} + 5 \cdot 3^{k-2} && \text{By the definition of } a_k \\
&= 3^{k-1} - f_{k+1} + 3^{k-2} - f_k + 5 \cdot 3^{k-2} && \text{By the IH, and since } 0 \leq k-1, k-2 < k \\
&= 3^{k-2}(3 + 1 + 5) - (f_{k+1} + f_k) && \\
&= 3^2 \cdot 3^{k-2} - f_{k+2} && \text{By the definition of } f_{k+2} \text{ as } k+2 \geq 3 \\
&= 3^k - f_{k+2}.
\end{aligned}
$$

Question 3. [12 MARKS]

Let m, n be integer powers of the same non-zero integer. Let L be a set defined as follows:

- $m, n \in L$;
- if $j, k \in L$, then $j^2 \cdot k^2 \in L$ and $\frac{j^2}{k^2} \in L$.

Prove that all members of L are integer powers of the same non-zero integer.
(Note that powers may also be negative).

Solution: Assume that m and n are powers of a non-zero integer g.
Let $P(r)$ denote the assertion that r is an integer power of g.
(Alternative: $P(r)$ denotes the assertion that if m and n are integer powers of a non-zero integer g, then r is also an integer power of g. Note that if you use the alternative definition for P, you will need to introduce g in the Base Case, Case 1, and Case 2).

Base Case:
Let $r = m$ or $r = n$.
Then by assumption, m and n are powers of g.
So, $P(r)$ holds.

Induction Step: Let j, k be arbitrary members of L. Suppose $P(j)$ and $P(k)$ holds, i.e., both j and k are integer powers of g. **[IH]**
WTP: $P(j^2 \cdot k^2)$ and $P(\frac{j^2}{k^2})$.

Case 1: Let $r = j^2 \cdot k^2$.
By the IH, j and k are integer powers of g, i.e., exists $t_1, t_2 \in \mathbb{Z}$ such that $j = g^{t_1}$ and $k = g^{t_2}$.
Then $j^2 \cdot k^2 = (j \cdot k)^2 = (g^{t_1+t_2})^2 = g^{2(t_1+t_2)}$ and $2(t_1 + t_2) \in \mathbb{Z}$.
So $P(j^2 \cdot k^2)$.

Case 2: Let $r = \frac{j^2}{k^2}$.
By the IH, j and k are integer powers of g, i.e., exists $t_1, t_2 \in \mathbb{Z}$ such that $j = g^{t_1}$ and $k = g^{t_2}$.
Then $\frac{j^2}{k^2} = (\frac{j}{k})^2 = (g^{t_1-t_2})^2 = g^{2(t_1-t_2)}$ and $2(t_1 - t_2) \in \mathbb{Z}$.
So $P(\frac{j^2}{k^2})$.

Question 4. [6 marks]

Find the flaw with the following "proof" that $a^n = 1$ for all non-negative integers n, whenever a is a non-zero real number.
Make sure to identify **all** errors and missing parts in the "proof", and provide enough explanations to justify your answer.
You will lose mark for identifying false errors.

Base Case: $a^0 = 1$ is true by the definition of a^0.

IS: Assume that $a^j = 1$ for all natural numbers j with $j \leq k$.
Then

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

Solution: The induction hypothesis (IH) must include the condition that $k \geq 1$, otherwise it is not possible by apply the (IH) for $k-1$. If we include this condition in the IH, then the IS implies that the claim holds for values greater than or equal to 2, meaning that we must prove the claim for $n = 1$ in the base case. However, for an arbitrary real number a, a^1 is not equal to 1.

This page is left (nearly) blank to accommodate work that wouldn't fit elsewhere and/or scratch work.

1: ______/ 9

2: ______/13

3: ______/12

4: ______/ 6

TOTAL: ______/40