Friday February 14, 2020	START: 2:10pm	DURATION: 110 mins

University of Toronto
Department of Mathematics

Term Test 1
MAT224H1S
Linear Algebra II

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Instructions.

1. There are **54** possible marks to be earned in this exam. The examination booklet contains a total of 9 pages. It is your responsibility to ensure that *no pages are missing from your examination.* DO NOT DETACH ANY PAGES FROM YOUR EXAMINATION.
2. DO NOT WRITE ON THE QR CODE AT THE TOP RIGHT-HAND CORNER OF EACH PAGE OF YOUR EXAMINATION.
3. For the full answer questions, WRITE YOUR SOLUTIONS ON THE FRONT OF THE QUESTION PAGES THEMSELVES. THE BACK OF EVERY PAGE WILL **NOT** BE SCANNED NOR SEEN BY THE GRADERS.
4. Ensure that your solutions are LEGIBLE.
5. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
6. Have your student card ready for inspection.
7. You may use the two blank pages at the end for rough work. The last two pages of the examination WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
8. **Show all of your work and justify your answers** but do not include extraneous information.

1. Let V be the set of all 2×2 matrices with real entries of the form $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. Define vector addition in V as $A + B = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ (ordinary matrix multiplication), and scalar multiplication in V as $cA = \begin{bmatrix} 1 & 0 \\ ca & 1 \end{bmatrix}$ for all $c \in \mathbb{R}$.

(a) What is the zero vector in V? Give a brief justification for your answer. [2 marks]

The zero vector is given by $\mathbf{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Observe that for any $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in V$,

$$A + \mathbf{0} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = A$$

Hence, $\mathbf{0}$ satisfies the property that $\mathbf{0} + A = A$ for all $A \in V$, and hence is the zero vector of this vector space.

(b) What is the additive inverse of $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in V$? Give a brief justification for your answer. [2 marks]

The inverse is given by $A^{-1} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$. Observe that A^{-1} satisfies the property that $A + A^{-1} = \mathbf{0}$:

$$A + A^{-1} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{0}$$

(c) Show that $c(A + B) = cA + cB$ for all $A, B \in V$ and $c \in \mathbb{R}$. [4 marks]

Let $c \in \mathbb{R}$ be arbitrary and pick any $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in V$ and $B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$. We proceed with direct computation:

$$c(A + B) = c \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ a + b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca + cb & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ cb & 1 \end{bmatrix} = cA + cB$$

2. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in vector space V.

(a) Define what it means for the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ to be linearly independent. [2 marks]

If whenever we have for $a_1, \ldots, a_n \in \mathbb{R}$ that $a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n = 0$, then $a_i = 0$ for all i.

(b) Define span$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. [2 marks]

span$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is the set of all all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.

Alternatively, may write span$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k | a_1, \ldots, a_k \in \mathbb{R}\}$

(c) Let V be a vector space and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2 \in V$. Suppose that both the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and the list $\mathbf{y}_1, \mathbf{y}_2$ are linearly independent. Prove that if span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \cap$ span$\{\mathbf{y}_1, \mathbf{y}_2\} = \{\mathbf{0}\}$, then the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2$ is linearly independent.
[4 marks]

For the sake of contradiction, suppose that the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2$ is linearly dependent. Then there exists $c_1, \ldots, c_5$ where not all $c_i = 0$ and $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{y}_1 + c_5\mathbf{y}_2 = \mathbf{0}$. Note that at least one of c_4, c_5 must be non-zero, as otherwise $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$ provides a non trivial linear combination of the $\mathbf{x}_i$ which sum to $\mathbf{0}$, which is a contradiction since the $\mathbf{x}_i$ are linearly independent. By a similar argument, at least one of c_1, c_2, c_3 must be non-zero since the $\mathbf{y}_j$ are linearly independent set.

However, we now have that the vector $\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = -c_4\mathbf{y}_1 - c_5\mathbf{y_2}$ is a non-zero vector and $\mathbf{v} \in$ span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \cap$ span$\{\mathbf{y}_1, \mathbf{y}_2\} = \{\mathbf{0}\}$, which is a contradiction.

3. (a) Let V be a finite dimensional vector space. Define the *dimension* of V. [2 marks]

The dimension of V is the number of vectors in any basis of V.

3. (b) A *magic square* is an $n \times n$ matrix with real entries in which each row, each column, and the two diagonals have the same sum; the sum is called the *weight* of the matrix. Let $\mathbb{M}_n$ denote the vector space of all such matrices. (Note that the weight does not have to be the same for all matrices.)

(i) What is the dimension of $\mathbb{M}_2$? Explain. [4 marks]

Note that for a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to also be a magic square, we must have, in particular, $a+b = a+d = a+c = a+b = c+d$ which means $a = b = c = d$. Thus, a basis for $\mathbb{M}_2$ is given by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We conclude that the dimension of $\mathbb{M}_2$ is 1.

(ii) What is the dimension of the subspace of $\mathbb{M}_2$ consisting of the set of all magic squares whose weight is 0? Explain [2 marks]

Let W denote this subspace. Since $W \subset \mathbb{M}_2$ (notice that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is in $\mathbb{M}_2$ but not in W), $\dim W < \dim \mathbb{M}_2 = 1$, we have $\dim W = 0$. In other words, the only matrix in this set is the zero matrix.

4. (a) Let U and W be subspaces of a finite dimensional vector space V.

(i) Define what it means for V to be the *direct sum* of U and W. [2 marks]

$V = U + W$ and $U \cap W = \{0\}$.
Equivalent definition: For all $\mathbf{v} \in V$, there exists unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

(ii) Suppose $V = U + W$. Prove $V = U \oplus W$ if and only if $\dim V = \dim U + \dim W$. [4 marks]

First, suppose that $V = U \oplus W$. i.e. $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. Then,

$$\begin{aligned}\dim\, V &= \dim(U + W)\\ &= \dim\, U + \dim\, W + \dim(U \cap W)\\ &= \dim\, U + \dim\, W\end{aligned}$$

since $\dim(U \cap W) = \dim\{\mathbf{0}\} = 0$.

Now suppose that $\dim V = \dim U + \dim W$. We need only show $U \cap W = \{\mathbf{0}\}$ since we're given $V = U + W$. Since

$$\begin{aligned}\dim(U \cap W) &= \dim\, U + \dim W - \dim(U + W)\\ &= \dim\, U + \dim W - \dim\, V\\ &= 0\end{aligned}$$

we have $U \cap W = \{\mathbf{0}\}$.

4. (b) Let P be the plane $\operatorname{span}\{(1,1,1),(1,-1,1)\}$ in $\mathbb{R}^3$, and let L be the line $\operatorname{span}\{(3,-1,3)\}$. Is $\mathbb{R}^3 = P \oplus L$? Why or why not? [2 marks]

No. Notice that $(3,-1,3) = (1,1,1) + 2(1,-1,1) \in P$ so $L \subset P$. Hence, $L \cap P \neq \{0\}$. Additionally, $L + P = P \neq \mathbb{R}^3$.

5. (a). Define what it means for a function $T : V \to W$ to be a *linear transformation.* [2 marks]

A function $T : V \to W$ is a linear transformation if:
(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
(ii) $T(a\mathbf{v}) = aT(\mathbf{v})$ for all $a \in \mathbb{R}$ and $\mathbf{v} \in V$.

5. (b) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. For each statement below, either prove it is true or find a counter-example to show it is false.

(i) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is a list of linearly dependent vectors in V then $T\mathbf{x}_1, T\mathbf{x}_2, T\mathbf{x}_3$ is a linearly dependent list of vectors in W. [4 marks]

True. If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is a list of linearly dependent vectors, there there exists $c_1, c_2, c_3 \in \mathbb{R}$ where not all c_i are zero and $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$. But then we have $\mathbf{0} = T(\mathbf{0}) = T(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3) = c_1T\mathbf{x_1} + c_2T\mathbf{x}_2 + c_3T\mathbf{x}_3$ which gives a non-trivial linear combination of the $T\mathbf{x}_i$ which sum to $\mathbf{0}$.

(ii) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is a list of linearly independent vectors in V then $T\mathbf{x}_1, T\mathbf{x}_2, T\mathbf{x}_3$ is a list of linearly independent vectors in W. [4 marks]

False. Consider the case when T is the zero transformation. Then $\{T\mathbf{x}_1, T\mathbf{x}_2, T\mathbf{x}_3\}$ contains just the zero vector, and hence cannot be linearly independent.

6. Determine if each statement below is True or False and *indicate your answer by circling one of the options.* No explanation is necessary. Each correct answer is worth 2 marks. Each incorrect answer will be worth 0 marks. [12 marks]

(i) Let V be the set of all $n \times n$ matrices with real entries. Define vector addition in V as ordinary matrix addition, and scalar multiplication in V by $cA = cA^{\mathrm{T}}$ for all $c \in \mathbb{R}$. Then V is a vector space.

(**True**) (**False**)

False. Notice that $(cd)A \neq c(dA)$ when A is not a symmetric matrix.

(ii) A list of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ in a vector space V is linearly dependent *iff* $\mathbf{x}_j \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_k\}$ for each $j = 1, 2 \ldots, k$.

(**True**) (**False**)

False. Let $V = \mathbb{R}^2$, $k = 2$, $\mathbf{x}_1 = (1, 0)$ and $\mathbf{x}_2 = \mathbf{0}$. Then the list $\mathbf{x}_1, \mathbf{x}_2$ is linearly dependent and $\mathbf{x}_1 \notin \operatorname{span}\{\mathbf{x}_2\}$

(iii) If U and W are subspaces of a vector space V then $\operatorname{span}(U \cup W) = \operatorname{span} U + \operatorname{span} W$.

(**True**) (**False**)

True. Note that $\mathbf{v} \in \operatorname{span}(U \cup W) \iff \mathbf{v} = c_1\mathbf{u} + c_2\mathbf{w}$ for $c_1, c_2 \in \mathbb{R}$, $\mathbf{u} \in U$ and $\mathbf{w} \in W \iff \mathbf{v} \in \operatorname{span} U + \operatorname{span} W$

(iv) If W_1, W_2, U are subspaces of a vector space V and $W_1 \cap W_2 = \{\mathbf{0}\}$ then $U \cap (W_1 \oplus W_2) = (U \cap W_1) \oplus (U \cap W_2)$.

(**True**) (**False**)

False. Let $V = \mathbb{R}^2$, $U = \operatorname{span}\{(1, 1)\}$, $W_1 = \operatorname{span}\{(1, 0)\}$ and $W_2 = \operatorname{span}\{(0, 1)\}$. Then $(U \cap W_1) \oplus (U \cap W_2) = \{0\} \oplus \{0\} = \{0\}$ but $U \cap (W_1 \oplus W_2) = U \cap V = U$.

(v) The function $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ defined by $T(p(x)) = p(x) + x^2 p''(x)$ is a linear transformation.

(**True**) (**False**)

True. Let $c \in \mathbb{R}$ and $p(x), q(x) \in P_n(\mathbb{R})$. Then $T(cp(x) + q(x)) = cp(x) + q(x) + x^2(cp''(x) + q''(x)) = c(p(x) + x^2 p''(x)) + (q(x) + x^2 q''(x)) = cT(p(x)) + T(q(x))$.

(vi) Let V and W be vector spaces, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be a linearly independent list of vectors in V. Then for any $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ in W there exists a linear transformation $T : V \to W$ with $T\mathbf{x}_j = \mathbf{y}_j$ for each $j = 1, 2, \ldots, k$.

(**True**) (**False**)

True. Extend $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ to a basis for V, and define T by defining where it sends each basis vector to. For example, T could map $\mathbf{x}_i$ to $\mathbf{y}_i$ and each of the extended basis vectors to $\mathbf{0}$.

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