AR(p) Model

$X_t-\phi_1X_{t-1}-\phi_2X_{t-2}-\cdots-\phi_pX_{t-p}=a_t,\ a_t\sim WN(0,\sigma^2)$

$(1-\phi_1B-\phi_2B^2-\cdots-\phi_pB^p)X_t=a_t$

$\phi(B)X_t=a_t$

ARMA(p,q) Model

$X_t-\phi_1X_{t-1}-\phi_2X_{t-2}-\cdots-\phi_pX_{t-p}=a_t+\theta_1a_{t-1}+\cdots+\theta_qa_{t-q}$

$a_t\sim WN(0,\sigma^2)$

$(1-\phi_1B-\phi_2B^2-\cdots-\phi_pB^p)X_t=(1+\theta_1B+\theta_2B^2+\cdots+\theta_qB^q)a_t$

$\phi(B)X_t=\theta(B)a_t$

$\theta(B)=1$ → AR(p); $\phi(B)=1$ → MA(q)

MA(q) Model

$X_t=a_t+\theta_1a_{t-1}+\theta_2a_{t-2}+\cdots+\theta_qa_{t-q},\ a_t\sim WN(0,\sigma^2)$

$X_t=(1+\theta_1B+\theta_2B^2+\cdots+\theta_qB^q)a_t$

$X_t=\theta(B)a_t$

- Stationarity:

AR:

Method1: $\{X_t\}$ stationary if $\phi(B)=0\Rightarrow|B|>1$ for all roots of $\phi(B)$

If $\{X_t\}$ is stationary, then $\{X_t\}$ can be written as a MA(∞) process

$$X_t=\frac{a_t}{\phi(B)}=(1+B\psi_1+B^2\psi_2+\cdots)a_t$$

$$\Rightarrow 1=\phi(B)(1+\psi_1B+\psi_2B^2+\cdots)$$

Method2: By defn of stationary ①②③

ARMA:

Method1: $\{X_t\}$ stationary if $\phi(B)=0\Rightarrow|B|>1\ \forall B$.

If $\{X_t\}$ is stationary, then $\{X_t\}$ can be written as MA(∞) process

$$X_t=\frac{\theta(B)a_t}{\phi(B)}=(1+\psi_1B+\psi_2B^2+\cdots)a_t$$

$$\Rightarrow\theta(B)=\phi(B)(1+\psi_1B+\psi_2B^2+\cdots)$$

若有常数则此过程忽略常数

Method2: Defn of stationary

MA:

MA model always stationary.

since $\phi(B)=1$ has no roots

- Invertibility:

AR:

AR model always invertible

since $\theta(B)=1$ has no roots

ARMA:

$\{X_t\}$ invertible if $\theta(B)=0\Rightarrow|B|>1\ \forall B$

If $\{X_t\}$ is invertible, then it can be written as AR(∞) process

$$X_t=\frac{\theta(B)a_t}{\phi(B)}=\frac{a_t}{(1-\pi_1B-\pi_2B^2-\cdots)}$$

$$\Rightarrow\theta(B)=\phi(B)(1-\pi_1B-\pi_2B^2-\cdots)$$

若有常数则此过程忽略常数.

MA:

$\{X_t\}$ invertible if $\theta(B)=0\Rightarrow|B|>1\ \forall B$

If $\{X_t\}$ is invertible, then it can be written as AR(∞) process

$$X_t=\theta(B)a_t=\frac{a_t}{(1-\pi_1B-\pi_2B^2-\cdots)}$$

$$\Rightarrow\theta(B)(1-\pi_1B-\pi_2B^2-\cdots)=1$$

Autocorrelation Function (ACF):

AR:

ACF tail off

Walker Equation: 注意符号, 公式中为⊖ 此处为⊕

$\rho(s)=\phi_1\rho(s-1)+\phi_2\rho(s-2)+\cdots+\phi_p\rho(s-p)$ $s\neq0$

or $\gamma(s)=\phi_1\gamma(s-1)+\phi_2\gamma(s-2)+\cdots+\phi_p\gamma(s-p)$ $s\neq0$

$\Rightarrow\rho(k)/\gamma(k)$

ARMA:

ACF tail off

If $\{X_t\}$ stationary $\Rightarrow$ transform to MA(∞) then find ACF

If $\{X_t\}$ invertible $\Rightarrow$ AR(∞) then find ACF

MA:

ACF cut off after lag q Note: $\theta_0=1$

$\gamma(0)=Var(X_t)=\sigma^2+\theta_1^2\sigma^2+\cdots+\theta_q^2\sigma^2=(1+\theta_1^2+\cdots+\theta_q^2)\sigma^2$

since $Cov(a_i,a_j)=0$ for $i\neq j$.

$$\begin{cases}\gamma(k)=Cov(X_t,X_{t+k})=Cov(a_t+\theta_1a_{t-1}+\cdots+\theta_qa_{t-q},\ a_{t+k}+\theta_1a_{t+k-1}+\cdots+\theta_qa_{t+k-q})\\ \quad=(\theta_k+\theta_1\theta_{k+1}+\theta_2\theta_{k+2}+\cdots+\theta_{q-k}\theta_q)\sigma^2 & \text{if } k\le q\\ \gamma(k)=0 & \text{if } k>q\end{cases}$$

VAR Model

- a concise way of summarizing interrelationship among data
- good forecast results
- testing for Granger causality among time series

VAR(1) Model

$$\underset{k\times1}{Y_t} = \underset{k\times k}{A}\,\underset{k\times1}{Y_{t-1}} + \underset{k\times1}{U_t}$$

Companion Form

$$\xi_t = A\,\xi_{t-1} + V_t$$

$$\xi_t = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}_{kp\times1} \quad A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}_{kp\times kp} \quad V_t = \begin{pmatrix} U_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{kp\times1}$$

VAR(p) Model

$$\underset{k\times1}{Y_t} = \underset{k\times1}{A_0} + \underset{k\times k}{A_1}\,\underset{k\times1}{Y_{t-1}} + \cdots + \underset{k\times k}{A_p}\,\underset{k\times1}{Y_{t-p}} + \underset{k\times1}{U_t}$$

$$\Rightarrow (I - A_1B - A_2B^2 - \cdots - A_pB^p)\,Y_t = A_0 + U_t$$

Check Stationary.

Method 1:

only apply to VAR(1) model.
if VAR(p) model, use companion form to transfer to VAR(1) model, then use this method.

ALL eigenvalues of A are within 1.
$\forall i,\ |\lambda_i| < 1$

Method 2: $Y_t = AY_{t-1} + U_t \Rightarrow (I-AB)Y_t = U_t$

$\det(I-AB) = 0 \Rightarrow$ if $|B|>1$, then stationary

Method 3:

$\det(I - A_1B - A_2B^2 - \cdots - A_pB^p) = 0$

- if $|B|>1$, then stationary

if stationary,

State Space Model (SSM)

useful tools for expressing DYNAMIC systems that involve with unobserved state variables.

- **Observation Equation / Measurement Equation:**
  describes the relationship between observed variables and UNobserved state variable.

$$y_t = F'X_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$$

- **State Equation / Transition Equation:**
  describes the dynamics of the state variables.
  has the form of a FIRST-ORDER DIFFERENCE equation in state vector.

$$X_t = G X_{t-1} + W_t$$

- **AR(p) Model in state eqn.**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + a_t$$

method 1:

$$F = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{p\times 1} \quad X_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}_{p\times 1} \quad \varepsilon_t = 0$$

$$G = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{p\times p} \quad W_t = \begin{pmatrix} a_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{p\times 1}$$

$$y_t = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix} + 0$$

$$\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} a_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Method 2:

X_t = constant vector

F : involves state variables in terms of t. $\quad \varepsilon_t = a_t$.

$G = I \quad W_t = 0$

- **ARIMA(p,d,q) model in state eq^n.**

$$y_t = \Delta^d Z_t \text{ s.t. } y_t \sim ARMA(p,q)$$

$$y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} = a_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}$$

Let $r = \max(p, q+1)$

$$y_t = \sum_{j=1}^{r} \phi_j y_{t-j} + \sum_{j=0}^{r-1} \theta_j a_{t-j}. \quad \theta_0 = 1, \ \theta_j = 0 \text{ if } j > q$$
$$\phi_j = 0 \text{ if } j > p$$

$$F = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{r\times 1} \quad X_t = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \cdots + \phi_r y_{t-r+1} + \theta_1 a_t + \cdots + \theta_{r-1} a_{t-r+2} \\ \phi_3 y_{t-1} + \cdots + \phi_r y_{t-r+2} + \theta_2 a_t + \cdots + \theta_{r-1} a_{t-r+3} \\ \vdots \\ \phi_r y_{t-1} + \theta_{r-1} a_t \end{pmatrix}_{r\times 1}$$

$$G = \begin{pmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \phi_{r-1} & 0 & 0 & \cdots & 1 \\ \phi_r & 0 & 0 & \cdots & 0 \end{pmatrix}_{r\times r} \quad W_t = \begin{pmatrix} a_t \\ \theta_1 a_t \\ \vdots \\ \theta_{r-1} a_t \end{pmatrix}_{r\times 1}$$

- **MA(q) Model in state eqn.**

$$y_t = a_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}$$

$$F = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(q+1)\times 1} \quad X_t = \begin{pmatrix} y_t \\ \theta_1 a_t + \theta_2 a_{t-1} + \cdots + \theta_q a_{t-q+1} \\ \theta_2 a_t + \theta_3 a_{t-1} + \cdots + \theta_q a_{t-q+2} \\ \vdots \\ \theta_q a_t \end{pmatrix}_{(q+1)\times 1} \quad \varepsilon_t = 0$$

$$G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(q+1)\times(q+1)} \quad W_t = \begin{pmatrix} a_t \\ \theta_1 a_t \\ \theta_2 a_t \\ \vdots \\ \theta_{q-1} a_t \\ \theta_q a_t \end{pmatrix}$$

$$y_t = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \begin{pmatrix} y_t \\ \theta_1 a_t + \cdots + \theta_q a_{t-q+1} \\ \vdots \\ \theta_q a_t \end{pmatrix} + 0$$

$$\begin{pmatrix} y_t \\ \theta_1 a_t + \theta_2 a_{t-1} + \cdots + \theta_q a_{t-q+1} \\ \vdots \\ \theta_q a_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q} \\ \theta_2 a_{t-1} + \theta_3 a_{t-2} + \cdots + \theta_q a_{t-q+1} \\ \vdots \\ \theta_q a_{t-1} \end{pmatrix} + \begin{pmatrix} a_t \\ \theta_1 a_t \\ \theta_2 a_t \\ \vdots \\ \theta_q a_t \end{pmatrix}$$

- **Combined SSM** (2个式子)

$$y_t = F_1' X_t + F_2' Z_t + \varepsilon_t$$
$$X_t = G_1 X_{t-1} + W_t$$
$$Z_t = G_2 Z_{t-1} + U_t$$

$$\Rightarrow y_t = \begin{bmatrix} F_1' & F_2' \end{bmatrix} \begin{bmatrix} X_t \\ Z_t \end{bmatrix} + \varepsilon_t$$

and

$$\begin{bmatrix} X_t \\ Z_t \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} W_t \\ U_t \end{bmatrix}$$

VAR Model

Stationary?

Method 1: (VAR(1) Model)

$Y_t = AY_{t-1} + U_t$

all eigenvalues in A has $|\lambda_i| < 1 \Rightarrow$ stationary

Method 2/3:

(VAR(1) model)

$Y_t = AY_{t-1} + U_t \Rightarrow (I - AB)Y_t = U_t$

$\det(I - AB) = 0$ if $|B| > 1 \Rightarrow$ stationary

(VAR(p) model)

$Y_t = A_0 + A_1 Y_{t-1} + \cdots + A_p Y_{t-p} + U_t$

$\Rightarrow (I - A_1 B - \cdots - A_p B^p) Y_t = A_0 + U_t$

$\det(I - A_1 B - \cdots - A_p B^p) = 0$

if $|B| > 1$ then stationary.

if yes

Model Identification (Order Selection)

Method 1: Sequential Likelihood Ratio Test (LRT)

VAR(p) V.S. VAR(p-1)

Deviance $= D_R - D_F$

p-value

Method 2: Information Criteria

AIC(n)

HQ(n)

SC(n)

FPE(n)

BIC(n)

Method 3: BigVAR

Test Model Adequacy — portmanteau tests: $H_0: \rho = 0$ no autocorrelation

$$Q_{BP} = T \sum_{j=1}^{m} tr(\hat{C}_j^T \hat{C}_0^{-1} \hat{C}_j \hat{C}_0^{-1}) \sim \chi^2_{k^2(m-n)}$$

$$Q_{LB} = T^2 \sum_{j=1}^{m} \frac{1}{T-j} tr(\hat{C}_j^T \hat{C}_0^{-1} \hat{C}_j \hat{C}_0^{-1}) \sim \chi^2_{k^2(m-n)}$$

where $\hat{C}_i = \frac{1}{T} \sum_{t=i+1}^{T} \hat{u}_t \hat{u}_{t-i}$

n = # of coefficients excluding deterministic terms of VAR(p) model.

if accepted

Granger Causality

VAR(p) model with $k=2$:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \sum_{j=1}^{p} \begin{pmatrix} \phi_{11}^{(j)} & \phi_{12}^{(j)} \\ \phi_{21}^{(j)} & \phi_{22}^{(j)} \end{pmatrix} \begin{pmatrix} y_{1t-j} \\ y_{2t-j} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

- if y_{2t} does not granger cause y_{1t}, then $\phi_{12}^{(j)} = 0 \ \forall j$.
- if y_{1t} does not granger cause y_{2t}, then $\phi_{21}^{(j)} = 0 \ \forall j$

Method 1: Likelihood Ratio Test (LRT)

Step 1: obtain OLS/ML estimates of eqns

$$y_{1t} = a_1 + \sum_{j=1}^{p} \phi_{11}^{(j)} y_{1t-j} + e_{1t} \qquad y_{2t} = a_2 + \sum_{j=1}^{p} \phi_{22}^{(j)} y_{2t-j} + e_{2t}$$

$$y_{1t} = a_1 + \sum_{j=1}^{p} \phi_{11}^{(j)} y_{1t-j} + \sum_{j=1}^{p} \phi_{12}^{(j)} y_{2t-j} + \varepsilon_{1t} \qquad y_{2t} = a_2 + \sum_{j=1}^{p} \phi_{22}^{(j)} y_{2t-j} + \sum_{j=1}^{p} \phi_{21}^{(j)} y_{1t-j} + \varepsilon_{2t}$$

Step 2: calculate log likelihood functions and LR statistics is

$$\text{Deviance} = n(\log|\tilde{\Sigma}| - \log|\hat{\Sigma}|) \sim \chi^2_p$$

where $\tilde{\Sigma}$ and $\hat{\Sigma}$ denote the residual covariance matrix

Method 2: Portmanteau Test for Granger causality / Univariate

Let $\{X_t\}$ and $\{Y_t\}$ be stationary and invertible univariate ARMA processes

$$\Phi_X(B)(X_t - \mu_X) = \Theta_X(B) U_t, \quad U_t \sim WN(0, \sigma_u^2)$$

$$\Phi_Y(B)(Y_t - \mu_Y) = \Theta_Y(B) V_t, \quad V_t \sim WN(0, \sigma_v^2)$$

(Then correlation of $X_t, Y_t \Rightarrow$ correlation of U_t, V_t)

The crosscorrelation function at lag k between U_t, V_t is

$$\rho_{uv}(k) = \frac{E(U_t, V_{t+k})}{\sqrt{E(U_t^2)\,E(V_t^2)}}$$

H_0: X does not cause Y

$$Q_L = n^2 \sum_{k=0}^{L} (n-k)^{-1} r_{uv}(k)$$

$$\sim \chi^2_{L+1}$$

$\Rightarrow$

- $\rho_{uv}(k) \neq 0$ for some $k > 0$ $\Rightarrow$ X cause Y
- $\rho_{uv}(k) \neq 0$ for some $k < 0$ $\Rightarrow$ Y cause X
- $\rho_{uv}(0) \neq 0$ $\Rightarrow$ Instantaneous causality
- $\rho_{uv}(k) \neq 0$ for some $k > 0$ and for some $k < 0$ $\Rightarrow$ feedback
- $\rho_{uv}(k) \neq 0$ for some $k > 0$ and $\rho_{uv}(0) = 0$ $\Rightarrow$ X cause Y but not instantaneously
- $\rho_{uv}(k) = 0$ for all $k < 0$ $\Rightarrow$ Y does not cause X
- $\rho_{uv}(k) = 0$ for all $k \leq 0$ $\Rightarrow$ Y does not cause X at all
- $\rho_{uv}(k) \neq 0$ for some $k > 0$ and $\rho_{uv}(k) = 0$ for all $k < 0$ or all $k \leq 0$ $\Rightarrow$ Unidirectional causality from X to Y
- $\rho_{uv}(0) \neq 0$ and $\rho_{uv}(k) = 0$ for all $k \neq 0$ $\Rightarrow$ X and Y are only related instantaneously
- $\rho_{uv}(k) = 0$ for all k. $\Rightarrow$ X and Y are independent.

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$\gamma(k)=0 \quad$ if $k>q$

$$\Rightarrow \rho(k)=\frac{\gamma(k)}{\gamma(0)}$$

$$\Rightarrow \rho(k)=\begin{cases}\dfrac{\theta_k+\theta_1\theta_{k+1}+\theta_2\theta_{k+2}+\cdots+\theta_q\theta_{q+k}}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2} & \text{if } k\le q\\ 0 & \text{if } k>q\end{cases}$$

$$\gamma(0)=\phi_1\gamma(1)+\phi_2\gamma(2)+\cdots+\phi_p\gamma(p)+\underline{\underline{Var(a_t)}}$$

$$\rho(0)=1$$

Note:

$MA(\infty)$ has

- $\gamma(k)=\sigma^2\sum_{j=0}^{\infty}\psi_j\psi_{j+|k|}$
- $\rho(k)=\dfrac{\gamma(k)}{\gamma(0)}$

- Partial Autocorrelation Function (PACF):

PACF cut off after lag p

$$\phi_{11}=\rho(1)=\frac{\phi_1}{1-\phi_2}$$

$$\phi_{22}=\frac{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}}=\phi_2$$

$\vdots$

$$\phi_{pp}=\phi_p$$

$$\phi_{(p+1)(p+1)}=\phi_{(p+2)(p+2)}=\cdots=0$$

PACF tail off

PACF tail off

$$\phi_{11}=\rho(1)$$

$$\phi_{22}=\frac{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}}$$

$$\phi_{33}=\frac{\det\begin{pmatrix}1 & \rho_1 & \rho_1\\ \rho_1 & 1 & \rho_2\\ \rho_2 & \rho_1 & \rho_3\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_1\\ \rho_2 & \rho_1 & 1\end{pmatrix}}$$

$\vdots$

$$\phi_{kk}=\frac{\det\begin{pmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_1\\ \rho_1 & 1 & \rho_1 & \cdots & \rho_2\\ \rho_2 & \rho_1 & 1 & \cdots & \vdots\\ \vdots & & & \ddots & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_k\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1}\\ \rho_1 & 1 & \rho_1 & \rho_2\cdots & \rho_{k-2}\\ \rho_2 & \rho_1 & 1 & & \rho_{k-3}\\ \vdots & \vdots & & \ddots & \vdots\\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1\end{pmatrix}}$$

Given Time Series

$\phi(B) X_t = \theta(B) a_t$

where $a_t \sim WN(0, \sigma^2)$

$\{X_t\}$ nonstationary

- nonstationary in variance
  $Var(X_t) \neq$ constant $\xrightarrow{X_t \geq 0\ \forall t}$ Power Transformation by Box and Cox (before ANY analysis)

  $$\boxed{T(X_t) = \frac{X_t^{\lambda} - 1}{\lambda}} \Rightarrow \boxed{\phi(B)\left(X_t^{(\lambda)} - \mu\right) = \theta(B) a_t \quad a_t \sim NID(0, \sigma^2)}$$

  - $\lambda = 0$

- Nonstationary in mean
  $E(X_t) \neq$ constant $\Rightarrow$ Differencing $\overset{d \text{ times}}{\underset{I(d)}{\Rightarrow}}$ ARIMA(p, d, q)

  $$\boxed{(1-B)^d \phi(B) X_t = \theta(B) a_t \quad a_t \sim N(0, \sigma^2)}$$

  Note: $\nabla^d = (1-B)^d$

  $\nabla_d = (1 - B^d)$ deal with SEASONALITY of period d

  $\nabla_d X_t = X_t - X_{t-d} = (1 - B^d) X_t$.

Step 1: remove deterministic time trend

Step 2: carry out statistical inference using DF test

Unit Root Test for identifying I(1) process:

I. Dickey-Fuller Test:

$X_t = \phi X_{t-1} + a_t,\ a_t \sim NID(0, \sigma^2)$

$\Rightarrow X_t - X_{t-1} = \phi X_{t-1} - X_{t-1} + a_t$

$$\Rightarrow \boxed{\begin{aligned} \Delta X_t &= (\phi - 1) X_{t-1} + a_t \\ &= \pi X_{t-1} + a_t. \end{aligned}}$$

AR(1) model

$(1 - \phi B) X_t = a_t$

$\phi(B) = 1 - \phi B = 0$

if $|B| = 1 \Rightarrow \phi = 1$

$H_0: \pi = 0$ (or $\phi = 1$) V.S. $H_a: \pi > 0$ (or $\phi > 1$)

- p-value $< \alpha \Rightarrow$ reject $H_0 \Rightarrow \{X_t\}$ is stationary
- p-value $> \alpha \Rightarrow$ fail to reject $H_0 \Rightarrow \{X_t\}$ is I(1) process (i.e X_t has a unit root)

II. General Dickey-Fulles Test:

(contain an intercept and a deterministic time trend)

$\Delta X_t = a + \tau^T DR_t + \pi X_{t-1} + a_t,\ a_t \sim NID(0, \sigma^2)$

where a denotes the regression intercept

$DR_t = (t, t^2, t^3, \ldots)$ are deterministic independent variables

$\tau = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ is the corresponding coefficient vector

$H_0: \pi = 0$ (i.e. $\{X_t\}$ is I(1) process) $H_a: \pi \neq 0$ (i.e. $\{X_t\}$ is stationary)

DF test for two unit roots

$\Delta^2 X_t = a_0 + \pi_1 \Delta X_{t-1} + \varepsilon_t$

$H_0: \pi_1 = 0$ (i.e. $\{X_t\}$ is $I(2)$ process)

$H_a: \pi_1 \neq 0$

$\Downarrow$

Continue to test whether there is a single unit root

$\Delta^2 X_t = a_0 + \pi_1 \Delta X_{t-1} + \pi_2 X_{t-1} + \varepsilon_t$.

$H_0: \pi_1 < 0$ and $\pi_2 = 0$ (i.e. $\{X_t\}$ is $I(1)$ process) has a single unit root

$H_a: \{X_t\}$ stationary

Solution

III. Issues on DF test:

① consider only a single unit root

② Assume correct model specification
- correct specification of time trend and intercept
- DGP may contain both AR and MA terms
- might have structural breaks in the data.

③ two-step procedure introduces the error in variable (EIV) problem for testing the presense of unit root

④ one-step procedure doesn't consider the presence of an autocorrelated error process.

IV. Augmented Dickey Fuller Test (ADF): (use autoregression to take into account the presence of serial correlated errors

$$\boxed{\Delta X_t = \tau' DR_t + \pi X_{t-1} + \sum_{j=1}^{k} \gamma_j \cdot \Delta X_{t-j} + a_t, \quad a_t \sim NID(0, \sigma^2) \quad \text{where } k = p-1}$$

$H_0: \pi = 0$ (i.e. $\{X_t\}$ contains a unit root, is a $I(1)$ process)

$H_a: \pi \neq 0$ (i.e. $\{X_t\}$ is stationary)

V. How to Select k?

① Autoregression Approximation:

Said and Dickey show that an unknown ARIMA(p,1,q) process can be approximated by an ARIMA(n,1,0) process with $n \leq T^{1/3}$, T denotes the length of time series

② General-to-Specific Methodology:

a) Start with a relatively long lag length and pare down the model by usual t-test or F-test.

b) Repeat the process until the last lag is significantly different from 0.

c) Once a tentative lag length has been determined, diagnostic checking should be used (Portmanteau test on residuals and residual autocorrelation plot) to ensure that the choice of the lag length is correct.

$\{X_t\}$ stationary $\Rightarrow$

Model Identification (ACF/PACF) 利用PACF/ACF图确定 p,q order

$\downarrow$

Model Estimation (MLE, method of moments, Kalmen Filter on model parameters) + AIC/BIC select model

$\downarrow$

Model Evaluation/Adequacy (portmanteau test
- ① $Q_{BP} = \sum_{k=1}^{m} n \hat{\rho}_k^2 \sim \chi^2_{m-(p+q)}$
- ② $Q_{LB} = \sum_{k=1}^{m} \frac{n(n+2)}{n-k} \hat{\rho}_k^2 \sim \chi^2_{m-(p+q)}$