Proof

① Show x_1, x_2, x_3, y_1, y_2 is linearly independent $\Rightarrow \text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} = \{0\}$

suppose x_1, x_2, x_3, y_1, y_2 are linearly independent

$\Rightarrow$ only $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ such that $c_1x_1 + c_2x_2 + c_3x_3 + c_4y_1 + c_5y_2 = 0$ (*)

WTS $\text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} = \{0\}$

$0 = 0x_1 + 0x_2 + 0x_3 \in \text{span}\{x_1, x_2, x_3\}$

$0 = 0y_1 + 0y_2 \in \text{span}\{y_1, y_2\}$

$\Rightarrow 0 \in \text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\}$

assume $\text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} \neq \{0\}$

$\Rightarrow$ there are not all zero scalar $c_1^*, c_2^*, \dots c_5^*$

such that $c_1^*x_1 + c_2^*x_2 + c_3^*x_3 = c_4^*y_1 + c_5^*y_2$

$c_1^*x_1 + c_2^*x_2 + c_3^*x_3 - c_4^*y_1 - c_5^*y_2 = 0$

which is contradiction to (*)

thus $\text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} = \{0\}$

② Show $\text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} = \{0\} \Rightarrow x_1, x_2, x_3, y_1, y_2$ are linearly independent

suppose $\text{span}\{x_1, x_2, x_3) \cap \text{span}\{y_1, y_2\} = \{0\}$

$\Rightarrow$ only $c_1 = c_2 = c_3 = c_4 = c_5 = 0$

such that $c_1x_1 + c_2x_2 + c_3x_3 = c_4y_1 + c_5y_2$

$c_1x_1 + c_2x_2 + c_3x_3 - c_4y_1 - c_5y_2 = 0$

$\Rightarrow x_1, x_2, x_3, y_1, y_2$ are linearly independent.

$\square$

Thus list x_1, x_2, x_3, y_1, y_2 is linearly independent $\Leftrightarrow$ $\text{span}\{x_1, x_2, x_3\} \cap \text{span}\{y_1, y_2\} = \{0\}$