Q2

(a) We know $T^{-1}(W)$ is a subspace of V from lecture.

let $y \in \ker T$

$Ty = 0 \in W$

$\Rightarrow y \in T^{-1}(W)$

$\Rightarrow \ker T \subseteq T^{-1}(W)$

$\square$

(b)

i) Show $T(T^{-1}(W)) \subseteq W \cap T(V)$

let $x \in T(T^{-1}(W))$

by definition, we get $x \in W$

and $T^{-1}(W) \subseteq V$

then $x \in T(V)$

$\Rightarrow x \in W \cap T(V)$

$\Rightarrow T(T^{-1}(W)) \subseteq W \cap T(V)$

ii)

Show $W \cap T(V) \subseteq T(T^{-1}(W))$

let $x \in W \cap T(V)$

$\Rightarrow x \in W$ and $x \in T(V)$

then $x \in T(T^{-1}(W))$ by definition

$\Rightarrow W \cap T(V) \subseteq T(T^{-1}(W))$

c) i) Show $T^{-1}(T(W)) \subseteq W + \ker T$

let $x \in T^{-1}(T(W))$

$\Rightarrow x \in W$ by definition

$\Rightarrow x \in W + \ker T$

$\Rightarrow T^{-1}(T(W)) \subseteq W + \ker T$

ii) Show $W + \ker T \subseteq T^{-1}(T(W))$

let $y \in W + \ker T$

Case 1: $y \notin \ker T$

$\Rightarrow y \in W$

$\Rightarrow y \in T^{-1}(T(W))$ by definition

Case 2 $y \in \ker T$

WTS $T(W)$ is a subspace of V

Since $\vec{0} \in T(W)$

let $x, y \in T(W)$, $c \in \mathbb{R}$

$\Rightarrow$ $x \in T(W) \Leftrightarrow Ta = x$, $a \in W$

$\Rightarrow ca \in W$ (W is a subspace)

$y \in T(W) \Leftrightarrow Tb = y$, $b \in W$

$\Rightarrow$ $ca + b \in W$

$\Rightarrow$ $T(ca+b) \in T(W)$

$\Leftarrow$ $T(ca+b) = cTa + Tb$

$= cx + y \in T(w)$

$\Rightarrow cx+y \in T(w)$, $\forall x, y \in T(w) \in V$
$c \in \mathbb{R}$

$\Rightarrow T(w)$ is a subspace of V

$\Rightarrow \ker T \subseteq T^{-1}(T(w))$ by part (a)

$\Rightarrow y \in T^{-1}(T(w))$

$\Rightarrow T^{-1}(T(w)) = w + \ker T$ $\square$