University of Toronto
Department of Statistical Sciences

STA255H1S – Statistical Theory

July 20, 2017
Time: 120 Minutes

Solutions

Term Test

Student Number: ____________

Family Name: ____________ **First Name:** ____________

Instructions:

1. **Aids:** a non-programable calculator and a one-sided, handwritten 8-1/2" x 11" aid sheet.

2. There are 9 questions (30 points) and one bonus question (2 points). No partial marks for the bonus question.

3. Marks are shown in brackets.

4. Show your work and answer in the space provided, in ink. Pencil may be used, but then remarks will not be allowed.

5. There are 8 pages including this page.

6. Round all answers to 4 decimal places. For example = 23.2314.

Circle your tutorial sections:

Tutorial	Last Name	Tutorial Location	TA
A	A–Ji	ES4000	Yang
B	Ji–Si	AB 107	Lei
C	Sj–Z	AP 120	Qianyi

Solve all the following questions. Show all your work. Please write your answers in the Space provided.

1. Two events E and F are such that $P(E) = 0.3$ and $P(F) = 0.4$. If E and F are independent, find $P(E \cup F)$. **[3 points]**

$P(E) = 0.3$, $P(F) = 0.4$, $P(E \cap F) = P(E) \cdot P(F) = 0.3 * 0.4 = 0.12$

A standard question. Similar to lecture note/book questions

$$P(F \cup E) = P(E) + P(F) - P(E \cap F)$$

$$= 0.3 + 0.4 - 0.12 = 0.7 - 0.12 = 0.58$$

2. In 2003, the average combined SAT score for collage-bound students in the US was 1026. 70% of all high school graduates scored above average on the SAT. We randomly select 5 high school graduates. What is the probability that at least 2 students in the sample scored above the average on the SAT? **[3 points]**

$P(\text{Score above the average}) = 0.7$

X: # of students scored above the average $\sim Bin(n=5, p=0.7)$

A standard question. Similar to lecture note/book question. See Exercise 3.37 of book

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \left\{ P(X=0) + P(X=1) \right\}$$

$$= 1 - \left\{ \binom{5}{0} (0.7)^0 (0.3)^5 + \binom{5}{1} (0.7)^1 (0.3)^4 \right\}$$

$$= 1 - \left\{ 0.0024 + 0.0285 \right\} = 0.9693$$

3. Five balls, numbered 1, 2, 3, 4 and 5, are placed in an urn. Two balls are randomly selected from the five, and their numbers noted.

(a) Write the sample space associated with this experiment? **[1 points]**

$$S = \left\{ \begin{array}{l} (1,2)\,;\ (1,3)\,;\ (1,4)\,;\ (1,5) \\ (2,3)\,;\ (2,4)\,;\ (2,5) \\ (3,4)\,;\ (3,5)\,;\ (4,5) \end{array} \right\}$$

(b) Obtain the probability distribution of Y = the largest of two sampled numbers. **[3 points]**

$S_y = \{2, 3, 4, 5\}$; Using equally liklihood method

$\boxed{\text{Exercise 3.6 of book ; lecture note ch3, part 1, slide 12}}$

$$P(Y=2) = P(\{(1,2)\}) = \frac{1}{10}$$

$$P(Y=3) = P(\{(1,3),(2,3)\}) = \frac{2}{10}$$

$$P(Y=4) = P(\{(1,4),(2,4),(3,4)\}) = \frac{3}{10}$$

$$P(Y=5) = P(\{(1,5),(2,5),(3,5),(4,5)\}) = \frac{4}{10}$$

$Y=y$	2	3	4	5
$p(y)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$4/10$

4. The probability distribution of a daily blackout of a small network is given by

y	0	1	2
$p(y)$	6/27	18/27	3/27

(a) Find $E(Y)$. **[1 points]**

$$E(y) = \sum_y y\,p(y) = \left(0 * \frac{6}{27}\right) + \left(1 * \frac{18}{27}\right) + \left(2 * \frac{3}{27}\right)$$

$$= 0 + \frac{18}{27} + \frac{6}{27} = \frac{24}{27}$$

(b) Find $V(Y)$. **[2 points]**

$$E(y^2) = \sum_y y^2 p(y) = \left(0 * \frac{6}{27}\right) + \left(1 * \frac{18}{27}\right) + \left(4 * \frac{3}{27}\right)$$

$$= 0 + \frac{18}{27} + \frac{12}{27} = \frac{30}{27}$$

$$V(y) = E(y^2) - \left(E(y)\right)^2$$

$$= \frac{30}{27} - \left(\frac{24}{27}\right)^2 = 1.1111 + 0.7901 = 1.9012$$

Exercise 3.17 of book, Similar to
Lecture note, ch3, part1, Slide 20

5. The daily production of electric motors at a factory averaged 20 with standard deviation of 2. What can be said about the fraction of days on which the production level falls between 14 and 26?

[3 points]

$\mu = 20 \; ; \; \sigma = 2$

Example 3.28 of book; Similar to Slide 27 (ch3, part 1)

Chebyshev's Inequality: $P(|y-\mu| \leq K\sigma) \geq 1 - \frac{1}{K^2}$

$$P(14 < y < 26) = P\left(\frac{14-20}{2} < \frac{Y-\mu}{\sigma} < \frac{26-20}{2}\right)$$

$$= P\left(-3 < \frac{Y-\mu}{\sigma} < 3\right) = P(|Y-\mu| \leq 3\sigma) \geq 1 - \frac{1}{K^2}$$

$$K=3 \Rightarrow 1 - \frac{1}{K^2} = 1 - \frac{1}{9} = 8/9$$

At least 8/9 of values are within 3 standard deviation of the mean.

6. Suppose that 30% of applicants for a job possess training. Applicants are interviewed sequentially and selected at random. If the process of interviewing to find the first applicant with training is geometrically distributed. Find the probability that the first applicant with training is found on the fourth interview.

[2 points]

$P(\text{applicant with training}) = 0.7 = p \qquad q = 1-p = 0.3$

A standard question; Similar to lecture note / book questions

Ex. 3.67 of book

$$P(Y=y) = p\,q^{y-1} \qquad y = 1, 2, \ldots$$

$$P(Y=4) = (0.7)(0.3)^{4-1} = 0.7\,(0.3)^3 = 0.0189$$

7. A population of voters contains 60% Republicans and 40% Democrats. It is reported that 40% of the Republicans and 50% of the Democrats favour an election issue. A person chosen at random from this population is found to favour the issue in question. Find the conditional probability that this person is a Democrat. **[3 points]**

$P(R) = P(\text{Republican}) = 0.6$; $P(D) = P(\text{Democrate}) = 0.4$

$P(F) = P(\text{favor an election issue})$; $P(F|R) = 0.4$; $P(F|D) = 0.5$

Ex. 2.124 of book

Similar to lecture note, ch2

Slide 45

$$P(D|F) = \frac{P(F \cap D)}{P(F)} = \frac{P(F|D)\,P(D)}{P(F|D)\,P(D) + P(F|R)\,P(R)}$$

$$= \frac{0.5 * 0.4}{(0.4 * 0.6) + (0.5 * 0.4)} = 0.4545$$

8. Cars arrive at a toll both according to a Poisson process with mean 120 cars per hour. If the attendant makes a one-minute phone call, what is the probability that three cars arrive during the call? **[3 points]**

Y: arrival per hour $\sim \text{Poi}(\lambda = 120)$

X: arrival per minute $\sim \text{Poi}\left(\frac{\lambda}{60} = \frac{120}{60} = 2\right)$

Ex. 3.128 of book; Similar

to lecture note; ch3, part 2

Slide 17

$P(\text{Three car arrive in the call}) = P(X=3)$

$$= \frac{e^{-2}\,2^{3}}{3!} = \frac{(2.7182)^{-2}\,8}{3!} \simeq 0.1804$$

9. Suppose that Y possesses the density function

$$f(y) = \begin{cases} cy & 0 \leq x \leq 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function. **[2 points]**

$$1 = \int_0^2 f(y)\,dy = \int_0^2 cy\,dy = c \cdot \left(\frac{y^2}{2}\right)\Big|_0^2 = c(2-0) = 2c \Rightarrow 2c = 1 \Rightarrow c = \tfrac{1}{2}$$

$$f(y) = \begin{cases} 2y & 0 \leq y \leq 2 \\ 0 & \text{o.w} \end{cases}$$

(b) Find $F(y)$. **[2 points]**

$$F(y) = \int_0^y f(t)\,dt = \int_0^y \tfrac{1}{2} t\,dt = \frac{t^2}{4}\Big|_0^y = y^2/4$$

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$

(c) Use $F(y)$ to find $P(1 \leq Y \leq 2)$. **[2 points]**

Ex. 4.11 of book; Similar to a question in lecture note

$$P(1 \leq Y \leq 2) = F(2) - F(1) = \frac{(2)^2}{4} - \frac{(1)^2}{4} = 1 - 1/4 = \frac{3}{4}$$

Bonus Question: No Partial Marks [2 points]

Use Probability Axioms to prove $P(\phi) = 0$, where ϕ is empty set.

Assume $P(\phi) > 0$.

For any event A ; $\underbrace{A \cap \phi = \phi}_{\text{mutually exclusive}}$ and $A \cup \phi = A$

we can write :

$$A = A \cup \phi \cup \phi \cup \phi \cup \cdots$$

$$P(A) = P(A \cup \phi \cup \phi \cup \phi \cup \cdots)$$

Using Axiom 3, we have

$$P(A) = P(A) + P(\phi) + P(\phi) + P(\phi) + \cdots > P(A)$$

Since $P(\phi) > 0$

$\Rightarrow$ Contradiction! $\times$

So $P(\phi) = 0$ ∎

a proof from lecture note: Ch2, Slide 17