Chapter 5: Multivariate Probability Distributions

5.1 **a.** The sample space S gives the possible values for Y_1 and Y_2:

S	AA	AB	AC	BA	BB	BC	CA	CB	CC
(y_1, y_2)	(2, 0)	(1, 1)	(1, 0)	(1, 1)	(0, 2)	(1, 0)	(1, 0)	(0, 1)	(0, 0)

Since each sample point is equally likely with probably 1/9, the joint distribution for Y_1 and Y_2 is given by

		y_1		
		0	1	2
y_2	0	1/9	2/9	1/9
	1	2/9	2/9	0
	2	1/9	0	0

b. $F(1, 0) = p(0, 0) + p(1, 0) = 1/9 + 2/9 = 3/9 = 1/3$.

5.2 **a.** The sample space for the toss of three balanced coins w/ probabilities are below:

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
(y_1, y_2)	(3, 1)	(3, 1)	(2, 1)	(1, 1)	(2, 2)	(1, 2)	(1, 3)	(0, –1)
probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

		y_1			
		0	1	2	3
y_2	–1	1/8	0	0	0
	1	0	1/8	2/8	1/8
	2	0	1/8	1/8	0
	3	0	1/8	0	0

b. $F(2, 1) = p(0, -1) + p(1, 1) + p(2, 1) = 1/2$.

5.3 Note that using material from Chapter 3, the joint probability function is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \text{ where } 0 \le y_1,\ 0 \le y_2, \text{ and } y_1 + y_2 \le 3.$$

In table format, this is

		y_1			
		0	1	2	3
y_2	0	0	3/84	6/84	1/84
	1	4/84	24/84	12/84	0
	2	12/84	18/84	0	0
	3	4/84	0	0	0

5.4 **a.** All of the probabilities are at least 0 and sum to 1.
b. $F(1, 2) = P(Y_1 \le 1, Y_2 \le 2) = 1$. Every child in the experiment either survived or didn't and used either 0, 1, or 2 seatbelts.

5.5 **a.** $P(Y_1 \le 1/2, Y_2 \le 1/3) = \int_0^{1/2}\int_0^{1/3} 3y_1 dy_1 dy_2 = .1065$.

b. $P(Y_2 \le Y_1/2) = \int_0^1 \int_0^{y_1/2} 3y_1 dy_1 dy_2 = .5$.

5.6 **a.** $P(Y_1 - Y_2 > .5) = P(Y_1 > .5 + Y_2) = \int_0^{.5}\int_{y_2+.5}^{1} 1 dy_1 dy_2 = \int_0^{.5} [y_1]_{y_2+.5}^{1} dy_2 = \int_0^{.5}(.5 - y_2)dy_2 = .125.$

b. $P(Y_1Y_2 < .5) = 1 - P(Y_1Y_2 > .5) = 1 - P(Y_1 > .5/Y_2) = 1 - \int_{.5}^{1}\int_{.5/y_2}^{1} 1dy_1dy_2 = 1 - \int_{.5}^{1}(1 - .5/y_2)dy_2$

$= 1 – [.5 + .5\ln(.5)] = .8466.$

5.7 **a.** $P(Y_1 < 1, Y_2 > 5) = \int_0^1\int_5^\infty e^{-(y_1+y_2)}dy_1dy_2 = \left[\int_0^1 e^{-y_1}dy_1\right]\left[\int_5^\infty e^{-y_2}dy_2\right] = \left[1 - e^{-1}\right]e^{-5} = .00426.$

b. $P(Y_1 + Y_2 < 3) = P(Y_1 < 3 - Y_2) = \int_0^3\int_0^{3-y_2} e^{-(y_1+y_2)}dy_1dy_2 = 1 - 4e^{-3} = .8009.$

5.8 **a.** Since the density must integrate to 1, evaluate $\int_0^1\int_0^1 ky_1y_2dy_1dy_2 = k/4 = 1$, so $k = 4$.

b. $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = 4\int_0^{y_2}\int_0^{y_1} t_1t_2dt_1dt_2 = y_1^2y_2^2$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$.

c. $P(Y_1 \le 1/2, Y_2 \le 3/4) = (1/2)^2(3/4)^2 = 9/64$.

5.9 **a.** Since the density must integrate to 1, evaluate $\int_0^1\int_0^{y_2} k(1 - y_2)dy_1dy_2 = k/6 = 1$, so $k = 6$.

b. Note that since $Y_1 \le Y_2$, the probability must be found in two parts (drawing a picture is useful):

$P(Y_1 \le 3/4, Y_2 \ge 1/2) = \int_{1/2}^{1}\int_{1/2}^{1} 6(1 - y_2)dy_1dy_2 + \int_{1/2}^{3/4}\int_{y_1}^{1} 6(1 - y_2)dy_2dy_1 = 24/64 + 7/64 = 31/64.$

5.10 **a.** Geometrically, since Y_1 and Y_2 are distributed uniformly over the triangular region, using the area formula for a triangle $k = 1$.

b. This probability can also be calculated using geometric considerations. The area of the triangle specified by $Y_1 \ge 3Y_2$ is 2/3, so this is the probability.

5.11 The area of the triangular region is 1, so with a uniform distribution this is the value of the density function. Again, using geometry (drawing a picture is again useful):

a. $P(Y_1 \le 3/4, Y_2 \le 3/4) = 1 - P(Y_1 > 3/4) - P(Y_2 > 3/4) = 1 - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{29}{32}$.

b. $P(Y_1 - Y_2 \ge 0) = P(Y_1 \ge Y_2)$. The region specified in this probability statement represents 1/4 of the total region of support, so $P(Y_1 \ge Y_2) = 1/4$.

5.12 Similar to Ex. 5.11:

a. $P(Y_1 \le 3/4, Y_2 \le 3/4) = 1 - P(Y_1 > 3/4) - P(Y_2 > 3/4) = 1 - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{7}{8}$.

b. $P(Y_1 \le 1/2, Y_2 \le 1/2) = \int_0^{1/2}\int_0^{1/2} 2\,dy_1 dy_2 = 1/2.$

5.13 **a.** $F(1/2, 1/2) = \int_0^{1/2}\int_{y_1-1}^{1/2} 30y_1y_2^2\,dy_2 dy_1 = \frac{9}{16}$.

b. Note that:

$F(1/2, 2) = F(1/2, 1) = P(Y_1 \le 1/2, Y_2 \le 1) = P(Y_1 \le 1/2, Y_2 \le 1/2) + P(Y_1 \le 1/2, Y_2 > 1/2)$

So, the first probability statement is simply $F(1/2, 1/2)$ from part a. The second probability statement is found by

$$P(Y_1 \le 1/2, Y_2 > 1/2) = \int_{1/2}^{1}\int_0^{1-y_2} 30y_1y_2^2\,dy_2 dy = \frac{4}{16}.$$

Thus, $F(1/2, 2) = \frac{9}{16} + \frac{4}{16} = \frac{13}{16}$.

c. $P(Y_1 > Y_2) = 1 - P(Y_1 \le Y_2) = 1 - \int_0^{1/2}\int_{y_1}^{1-y_1} 30y_1y_2^2\,dy_2 dy_1 = 1 - \frac{11}{32} = \frac{21}{32} = .65625.$

5.14 **a.** Since $f(y_1, y_2) \ge 0$, simply show $\int_0^1\int_{y_1}^{2-y_1} 6y_1^2y_2\,dy_2 dy_1 = 1$.

b. $P(Y_1 + Y_2 < 1) = P(Y_2 < 1 - Y_1) = \int_0^{.5}\int_{y_1}^{1-y_1} 6y_1^2y_2\,dy_2 dy_1 = 1/16$.

5.15 **a.** $P(Y_1 < 2, Y_2 > 1) = \int_1^2\int_1^{y_1} e^{-y_1}\,dy_2 dy_1 = \int_1^2\int_{y_2}^2 e^{-y_1}\,dy_1 dy_2 = e^{-1} - 2e^{-2}$.

b. $P(Y_1 \ge 2Y_2) = \int_0^{\infty}\int_{2y_2}^{\infty} e^{-y_1}\,dy_1 dy_2 = 1/2$.

c. $P(Y_1 - Y_2 \ge 1) = P(Y_1 \ge Y_2 + 1) = \int_0^{\infty}\int_{y_2+1}^{\infty} e^{-y_1}\,dy_1 dy_2 = e^{-1}$.

5.16 **a.** $P(Y_1 < 1/2, Y_2 > 1/4) = \int_{1/4}^{1}\int_{0}^{1/2}(y_1 + y_2)dy_1dy_2 = 21/64 = .328125.$

b. $P(Y_1 + Y_2 \le 1) = P(Y_1 \le 1 - Y_2) = \int_{0}^{1}\int_{0}^{1-y_2}(y_1 + y_2)dy_1dy_2 = 1/3$.

5.17 This can be found using integration (polar coordinates are helpful). But, note that this is a bivariate uniform distribution over a circle of radius 1, and the probability of interest represents 50% of the support. Thus, the probability is .50.

5.18 $P(Y_1 > 1, Y_2 > 1) = \int_1^\infty\int_1^\infty \tfrac{1}{8}y_1e^{-(y_1+y_2)/2}dy_1dy_2 = \left[\int_1^\infty \tfrac{1}{4}y_1e^{-y_1/2}dy_1\right]\left[\int_1^\infty \tfrac{1}{2}e^{-y_2/2}dy_2\right] = \tfrac{3}{2}e^{-\frac{1}{2}}\left(e^{-\frac{1}{2}}\right) = \tfrac{3}{2}e^{-1}$

5.19 **a.** The marginal probability function is given in the table below.

y_1	0	1	2
$p_1(y_1)$	4/9	4/9	1/9

b. No, evaluating binomial probabilities with $n = 3$, $p = 1/3$ yields the same result.

5.20 **a.** The marginal probability function is given in the table below.

y_2	–1	1	2	3
$p_2(y_2)$	1/8	4/8	2/8	1/8

b. $P(Y_1 = 3 \mid Y_2 = 1) = \frac{P(Y_1=3, Y_2=1)}{P(Y_2=1)} = \frac{1/8}{4/8} = 1/4$.

5.21 **a.** The marginal distribution of Y_1 is hypergeometric with $N = 9$, $n = 3$, and $r = 4$.

b. Similar to part a, the marginal distribution of Y_2 is hypergeometric with $N = 9$, $n = 3$, and $r = 3$. Thus,

$$P(Y_1 = 1 \mid Y_2 = 2) = \frac{P(Y_1=1, Y_2=2)}{P(Y_2=2)} = \frac{\binom{4}{1}\binom{3}{2}\binom{2}{0}}{\binom{9}{3}} \Bigg/ \frac{\binom{3}{2}\binom{6}{1}}{\binom{9}{3}} = 2/3.$$

c. Similar to part b,

$$P(Y_3 = 1 \mid Y_2 = 1) = P(Y_1 = 1 \mid Y_2 = 1) = \frac{P(Y_1=1, Y_2=1)}{P(Y_2=1)} = \frac{\binom{3}{1}\binom{2}{1}\binom{4}{1}}{\binom{9}{3}} \Bigg/ \frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} = 8/15.$$

5.22 **a.** The marginal distributions for Y_1 and Y_2 are given in the margins of the table.

b. $P(Y_2 = 0 \mid Y_1 = 0) = .38/.76 = .5$ $\quad P(Y_2 = 1 \mid Y_1 = 0) = .14/.76 = .18$
$P(Y_2 = 2 \mid Y_1 = 0) = .24/.76 = .32$

c. The desired probability is $P(Y_1 = 0 \mid Y_2 = 0) = .38/.55 = .69$.

5.23 **a.** $f_2(y_2) = \int_{y_2}^{1} 3y_1 dy_1 = \frac{3}{2} - \frac{3}{2}y_2^2,\ 0 \le y_2 \le 1$.

b. Defined over $y_2 \le y_1 \le 1$, with the constant $y_2 \ge 0$.

c. First, we have $f_1(y_1) = \int_0^{y_1} 3y_1 dy_2 = 3y_2^2,\ 0 \le y_1 \le 1$. Thus,

$f(y_2 \mid y_1) = 1/y_1,\ 0 \le y_2 \le y_1$. So, conditioned on $Y_1 = y_1$, we see Y_2 has a uniform distribution on the interval $(0, y_1)$. Therefore, the probability is simple:

$$P(Y_2 > 1/2 \mid Y_1 = 3/4) = (3/4 - 1/2)/(3/4) = 1/3.$$

5.24 **a.** $f_1(y_1) = 1,\ 0 \le y_1 \le 1$, $f_2(y_2) = 1,\ 0 \le y_2 \le 1$.

b. Since both Y_1 and Y_2 are uniformly distributed over the interval (0, 1), the probabilities are the same: .2

c. $0 \le y_2 \le 1$.

d. $f(y_1 \mid y_2) = f(y_1) = 1,\ 0 \le y_1 \le 1$

e. $P(.3 < Y_1 < .5 \mid Y_2 = .3) = .2$

f. $P(.3 < Y_2 < .5 \mid Y_2 = .5) = .2$

g. The answers are the same.

5.25 **a.** $f_1(y_1) = e^{-y_1},\ y_1 > 0$, $f_2(y_2) = e^{-y_2},\ y_2 > 0$. These are both exponential density functions with $\beta = 1$.

b. $P(1 < Y_1 < 2.5) = P(1 < Y_2 < 2.5) = e^{-1} - e^{-2.5} = .2858$.

c. $y_2 > 0$.

d. $f(y_1 \mid y_2) = f_1(y_1) = e^{-y_1},\ y_1 > 0$.

e. $f(y_2 \mid y_1) = f_2(y_2) = e^{-y_2},\ y_2 > 0$.

f. The answers are the same.

g. The probabilities are the same.

5.26 **a.** $f_1(y_1) = \int_0^1 4y_1y_2 dy_2 = 2y_1,\ 0 \le y_1 \le 1$; $f(y_2) = 2y_2,\ 0 \le y_2 \le 1$.

b. $P(Y_1 \le 1/2 | Y_2 \ge 3/4) = \dfrac{\int_0^{1/2}\int_{3/4}^{1} 4y_1y_2 dy_1 dy_2}{\int_{3/4}^{1} 2y_2 dy_2} = \int_0^{1/2} 2y_1 dy_1 = 1/4$.

c. $f(y_1 \mid y_2) = f_1(y_1) = 2y_1,\ 0 \le y_1 \le 1$.

d. $f(y_2 \mid y_1) = f_2(y_2) = 2y_2,\ 0 \le y_2 \le 1$.

e. $P(Y_1 \le 3/4 | Y_2 = 1/2) = P(Y_1 \le 3/4) = \int_0^{3/4} 2y_1 dy_1 = 9/16$.

5.27 **a.** $f_1(y_1) = \int_{y_1}^{1} 6(1-y_2)dy_2 = 3(1-y_1)^2$, $0 \le y_1 \le 1$;

$f_2(y_2) = \int_{0}^{y_2} 6(1-y_2)dy_1 = 6y_2(1-y_2)$, $0 \le y_2 \le 1$.

b. $P(Y_2 \le 1/2 | Y_1 \le 3/4) = \dfrac{\int_0^{1/2}\int_0^{y_2} 6(1-y_2)dy_1dy_2}{\int_0^{3/4} 3(1-y_1)^2 dy_1} = 32/63.$

c. $f(y_1 \mid y_2) = 1/y_2$, $0 \le y_1 \le y_2 \le 1$.

d. $f(y_2 \mid y_1) = 2(1-y_2)/(1-y_1)^2$, $0 \le y_1 \le y_2 \le 1$.

e. From part **d**, $f(y_2 \mid 1/2) = 8(1-y_2)$, $1/2 \le y_2 \le 1$. Thus, $P(Y_2 \ge 3/4 \mid Y_1 = 1/2) = 1/4$.

5.28 Referring to Ex. 5.10:

a. First, find $f_2(y_2) = \int_{2y_2}^{2} 1dy_1 = 2(1-y_2)$, $0 \le y_2 \le 1$. Then, $P(Y_2 \ge .5) = .25$.

b. First find $f(y_1 \mid y_2) = \frac{1}{2(1-y_2)}$, $2y_2 \le y_1 \le 2$. Thus, $f(y_1 \mid .5) = 1$, $1 \le y_1 \le 2$ — the conditional distribution is uniform on (1, 2). Therefore, $P(Y_1 \ge 1.5 \mid Y_2 = .5) = .5$

5.29 Referring to Ex. 5.11:

a. $f_2(y_2) = \int_{y_2-1}^{1-y_2} 1dy_1 = 2(1-y_2)$, $0 \le y_2 \le 1$. In order to find $f_1(y_1)$, notice that the limits of integration are different for $0 \le y_1 \le 1$ and $-1 \le y_1 \le 0$. For the first case: $f_1(y_1) = \int_0^{1-y_1} 1dy_2 = 1-y_1$, for $0 \le y_1 \le 1$. For the second case, $f_1(y_1) = \int_0^{1+y_1} 1dy_2 = 1+y_1$, for $-1 \le y_1 \le 0$. This can be written as $f_1(y_1) = 1 - |y_1|$, for $-1 \le y_1 \le 1$.

b. The conditional distribution is $f(y_2 \mid y_1) = \frac{1}{1-|y_1|}$, for $0 \le y_1 \le 1 - |y_1|$. Thus, $f(y_2 \mid 1/4) = 4/3$. Then, $P(Y_2 > 1/2 \mid Y_1 = 1/4) = \int_{1/2}^{3/4} 4/3dy_2 = 1/3$.

5.30 **a.** $P(Y_1 \ge 1/2, Y_2 \le 1/4) = \int_0^{1/4}\int_{1/2}^{1-y_2} 2dy_1dy_2 = \frac{3}{16}$. And, $P(Y_2 \le 1/4) = \int_0^{1/4} 2(1-y_2)dy_2 = \frac{7}{16}$.

Thus, $P(Y_1 \ge 1/2 \mid Y_2 \le 1/4) = \frac{3}{7}$.

b. Note that $f(y_1 \mid y_2) = \frac{1}{1-y_2}$, $0 \le y_1 \le 1-y_2$. Thus, $f(y_1 \mid 1/4) = 4/3$, $0 \le y_1 \le 3/4$.

Thus, $P(Y_2 > 1/2 \mid Y_1 = 1/4) = \int_{1/2}^{3/4} 4/3dy_2 = 1/3$.

5.31 **a.** $f_1(y_1) = \int_{y_1-1}^{1-y_1} 30y_1y_2^2dy_2 = 20y_1(1-y_1)^2$, $0 \le y_1 \le 1$.

b. This marginal density must be constructed in two parts:

$$f_2(y_2) = \begin{cases} \int_0^{1+y_2} 30y_1y_2^2dy_1 = 15y_2^2(1+y_2) & -1 \le y_2 \le 0 \\ \int_0^{1-y_2} 30y_1y_2^2dy_1 = 5y_2^2(1-y_2) & 0 \le y_2 \le 1 \end{cases}.$$

c. $f(y_2 \mid y_1) = \frac{3}{2}y_2^2(1-y_1)^{-3}$, for $y_1 - 1 \le y_2 \le 1 - y_1$.

d. $f(y_2 \mid .75) = \frac{3}{2}y_2^2(.25)^{-3}$, for $-.25 \le y_2 \le .25$, so $P(Y_2 > 0 \mid Y_1 = .75) = .5$.

5.32 **a.** $f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2y_2dy_2 = 12y_1^2(1-y_1)$, $0 \le y_1 \le 1$.

b. This marginal density must be constructed in two parts:

$$f_2(y_2) = \begin{cases} \int_0^{y_2} 6y_1^2y_2dy_1 = 2y_2^4 & 0 \le y_2 \le 1 \\ \int_0^{2-y_2} 6y_1^2y_2dy_1 = 2y_2(2-y_2)^3 & 1 \le y_2 \le 2 \end{cases}.$$

c. $f(y_2 \mid y_1) = \frac{1}{2}y_2/(1-y_1)$, $y_1 \le y_2 \le 2-y_1$.

d. Using

the density found in part **c**, $P(Y_2 < 1.1 \mid Y_1 = .6) = \frac{1}{2}\int_{.6}^{11} y_2/.4dy_2 = .53$

5.33 Refer to Ex. 5.15:

a. $f_1(y_1) = \int_0^{y_1} e^{-y_1}dy_2 = y_1e^{-y_1}$, $y_1 \ge 0$. $f_2(y_2) = \int_{y_2}^{\infty} e^{-y_1}dy_1 = e^{-y_2}$, $y_2 \ge 0$.

b. $f(y_1 \mid y_2) = e^{-(y_1-y_2)}$, $y_1 \ge y_2$.

c. $f(y_2 \mid y_1) = 1/y_1$, $0 \le y_2 \le y_1$.

d. The density functions are different.

e. The marginal and conditional probabilities can be different.

5.34 **a.** Given $Y_1 = y_1$, Y_2 has a uniform distribution on the interval $(0, y_1)$.

b. Since $f_1(y_1) = 1$, $0 \le y_1 \le 1$, $f(y_1, y_2) = f(y_2 \mid y_1)f_1(y_1) = 1/y_1$, $0 \le y_2 \le y_1 \le 1$.

c. $f_2(y_2) = \int_{y_2}^{1} 1/y_1\,dy_1 = -\ln(y_2)$, $0 \le y_2 \le 1$.

5.35 With $Y_1 = 2$, the conditional distribution of Y_2 is uniform on the interval (0, 2). Thus, $P(Y_2 < 1 \mid Y_1 = 2) = .5$.

5.36 **a.** $f_1(y_1) = \int_0^1 (y_1 + y_2)dy_2 = y_1 + \frac{1}{2}$, $0 \le y_1 \le 1$. Similarly $f_2(y_2) = y_2 + \frac{1}{2}$, $0 \le y_2 \le 1$.

b. First, $P(Y_2 \ge \frac{1}{2}) = \int_{1/2}^1 (y_2 + \frac{1}{2}) = \frac{5}{8}$, and $P(Y_1 \ge \frac{1}{2}, Y_2 \ge \frac{1}{2}) = \int_{1/2}^1 \int_{1/2}^1 (y_1 + y_2)dy_1dy_2 = \frac{3}{8}$.

Thus, $P(Y_1 \ge \frac{1}{2} \mid Y_2 \ge \frac{1}{2}) = \frac{3}{5}$.

c. $P(Y_1 > .75 \mid Y_2 = .5) = \dfrac{\int_{.75}^1 \left(y_1 + \frac{1}{2}\right)dy_1}{\frac{1}{2} + \frac{1}{2}} = .34375.$

5.37 Calculate $f_2(y_2) = \int_0^\infty \frac{y_1}{8} e^{-(y_1+y_2)/2} dy_1 = \frac{1}{2} e^{-y_2/2}$, $y_2 > 0$. Thus, Y_2 has an exponential distribution with $\beta = 2$ and $P(Y_2 > 2) = 1 - F(2) = e^{-1}$.

5.38 This is the identical setup as in Ex. 5.34.

a. $f(y_1, y_2) = f(y_2 \mid y_1) f_1(y_1) = 1/y_1$, $0 \le y_2 \le y_1 \le 1$.

b. Note that $f(y_2 \mid 1/2) = 1/2$, $0 \le y_2 \le 1/2$. Thus, $P(Y_2 < 1/4 \mid Y_1 = 1/2) = 1/2$.

c. The probability of interest is $P(Y_1 > 1/2 \mid Y_2 = 1/4)$. So, the necessary conditional density is $f(y_1 \mid y_2) = f(y_1, y_2)/f_2(y_2) = \frac{1}{y_1(-\ln y_2)}$, $0 \le y_2 \le y_1 \le 1$. Thus,

$$P(Y_1 > 1/2 \mid Y_2 = 1/4) = \int_{1/2}^1 \frac{1}{y_1 \ln 4} dy_1 = 1/2.$$

5.39 The result follows from:

$$P(Y_1 = y_1 \mid W = w) = \frac{P(Y_1 = y_1, W = w)}{P(W = w)} = \frac{P(Y_1 = y_1, Y_1 + Y_2 = w)}{P(W = w)} = \frac{P(Y_1 = y_1, Y_2 = w - y_1)}{P(W = w)}.$$

Since Y_1 and Y_2 are independent, this is

$$P(Y_1 = y_1 \mid W = w) = \frac{P(Y_1 = y_1)P(Y_2 = w - y_1)}{P(W = w)} = \frac{\frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!}\left(\frac{\lambda_2^{w-y_1} e^{-\lambda_2}}{(w-y_1)!}\right)}{\frac{(\lambda_1+\lambda_2)^w e^{-(\lambda_1+\lambda_2)}}{w!}}$$

$$= \binom{w}{y_1}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{y_1}\left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{w-y_1}.$$

This is the binomial distribution with $n = w$ and $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

5.40 As the Ex. 5.39 above, the result follows from:

$$P(Y_1 = y_1 \mid W = w) = \frac{P(Y_1 = y_1, W = w)}{P(W = w)} = \frac{P(Y_1 = y_1, Y_1 + Y_2 = w)}{P(W = w)} = \frac{P(Y_1 = y_1, Y_2 = w - y_1)}{P(W = w)}.$$

Since Y_1 and Y_2 are independent, this is (all terms involving p_1 and p_2 drop out)

$$P(Y_1 = y_1 \mid W = w) = \frac{P(Y_1 = y_1)P(Y_2 = w - y_1)}{P(W = w)} = \frac{\binom{n_1}{y_1}\binom{n_2}{w - y_1}}{\binom{n_1 + n_2}{w}}, \qquad \begin{matrix} 0 \le y_1 \le n_1 \\ 0 \le w - y_1 \le n_2 \end{matrix}.$$

5.41 Let Y = # of defectives in a random selection of three items. Conditioned on p, we have

$$P(Y = y \mid p) = \binom{3}{y} p^y (1-p)^{3-y},\ y = 0, 1, 2, 3.$$

We are given that the proportion of defectives follows a uniform distribution on (0, 1), so the unconditional probability that $Y = 2$ can be found by

$$P(Y = 2) = \int_0^1 P(Y = 2, p)dp = \int_0^1 P(Y = 2 \mid p) f(p) dp = \int_0^1 3p^2(1-p)^{3-1} dp = 3\int_0^1 (p^2 - p^3) dp$$
$$= 1/4.$$

5.42 (Similar to Ex. 5.41) Let Y = # of defects per yard. Then,

$$p(y) = \int_0^\infty P(Y = y, \lambda) d\lambda = \int_0^\infty P(Y = y \mid \lambda) f(\lambda) d\lambda = \int_0^\infty \tfrac{\lambda^y e^{-\lambda}}{y!} e^{-\lambda} d\lambda = \left(\tfrac{1}{2}\right)^{y+1},\ y = 0, 1, 2, \dots .$$

Note that this is essentially a geometric distribution (see Ex. 3.88).

5.43 Assume $f(y_1 \mid y_2) = f_1(y_1)$. Then, $f(y_1, y_2) = f(y_1 \mid y_2) f_2(y_2) = f_1(y_1) f_2(y_2)$ so that Y_1 and Y_2 are independent. Now assume that Y_1 and Y_2 are independent. Then, there exists functions g and h such that $f(y_1, y_2) = g(y_1)h(y_2)$ so that

$$1 = \int\int f(y_1, y_2) dy_1 dy_2 = \int g(y_1) dy_1 \times \int h(y_2) dy_2 .$$

Then, the marginals for Y_1 and Y_2 can be defined by

$$f_1(y_1) = \int \frac{g(y_1)h(y_2)}{\int g(y_1)dy_1 \times \int h(y_2)dy_2} dy_2 = \frac{g(y_1)}{\int g(y_1)dy_1},\ \text{so}\ f_2(y_2) = \frac{h(y_2)}{\int h(y_2)dy_2}.$$

Thus, $f(y_1, y_2) = f_1(y_1) f_2(y_2)$. Now it is clear that

$$f(y_1 \mid y_2) = f(y_1, y_2)/f_2(y_2) = f_1(y_1) f_2(y_2)/f_2(y_2) = f_1(y_1),$$

provided that $f_2(y_2) > 0$ as was to be shown.

5.44 The argument follows exactly as Ex. 5.43 with integrals replaced by sums and densities replaced by probability mass functions.

5.45 No. Counterexample: $P(Y_1 = 2, Y_2 = 2) = 0 \ne P(Y_1 = 2)P(Y_2 = 2) = (1/9)(1/9)$.

5.46 No. Counterexample: $P(Y_1 = 3, Y_2 = 1) = 1/8 \ne P(Y_1 = 3)P(Y_2 = 1) = (1/8)(4/8)$.

5.47 Dependent. For example: $P(Y_1 = 1, Y_2 = 2) \neq P(Y_1 = 1)P(Y_2 = 2)$.

5.48 Dependent. For example: $P(Y_1 = 0, Y_2 = 0) \neq P(Y_1 = 0)P(Y_2 = 0)$.

5.49 Note that $f_1(y_1) = \int_0^{y_1} 3y_1 dy_2 = 3y_1^2,\ 0 \le y_1 \le 1$, $f_2(y_2) = \int_{y_1}^{1} 3y_1 dy_1 = \frac{3}{2}[1 - y_2^2],\ 0 \le y_2 \le 1$.
Thus, $f(y_1, y_2) \neq f_1(y_1) f_2(y_2)$ so that Y_1 and Y_2 are dependent.

5.50 **a.** Note that $f_1(y_1) = \int_0^1 1 dy_2 = 1,\ 0 \le y_1 \le 1$ and $f_2(y_2) = \int_0^1 1 dy_1 = 1,\ 0 \le y_2 \le 1$. Thus, $f(y_1, y_2) = f_1(y_1) f_2(y_2)$ so that Y_1 and Y_2 are independent.

b. Yes, the conditional probabilities are the same as the marginal probabilities.

5.51 **a.** Note that $f_1(y_1) = \int_0^\infty e^{-(y_1+y_2)} dy_2 = e^{-y_1},\ y_1 > 0$ and $f_2(y_2) = \int_0^\infty e^{-(y_1+y_2)} dy_1 = e^{-y_2},\ y_2 > 0$.
Thus, $f(y_1, y_2) = f_1(y_1) f_2(y_2)$ so that Y_1 and Y_2 are independent.

b. Yes, the conditional probabilities are the same as the marginal probabilities.

5.52 Note that $f(y_1, y_2)$ can be factored and the ranges of y_1 and y_2 do not depend on each other so by Theorem 5.5 Y_1 and Y_2 are independent.

5.53 The ranges of y_1 and y_2 depend on each other so Y_1 and Y_2 cannot be independent.

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5.58 Following Ex. 5.32, it is seen that $f(y_1, y_2) \neq f_1(y_1) f_2(y_2)$ so that Y_1 and Y_2 are dependent.

5.59 The ranges of y_1 and y_2 depend on each other so Y_1 and Y_2 cannot be independent.

5.60 From Ex. 5.36, $f_1(y_1) = y_1 + \frac{1}{2},\ 0 \le y_1 \le 1$, and $f_2(y_2) = y_2 + \frac{1}{2},\ 0 \le y_2 \le 1$. But, $f(y_1, y_2) \neq f_1(y_1) f_2(y_2)$ so Y_1 and Y_2 are dependent.

5.61 Note that $f(y_1, y_2)$ can be factored and the ranges of y_1 and y_2 do not depend on each other so by Theorem 5.5 Y_1 and Y_2 are independent.

5.62 Let X, Y denote the number on which person A, B flips a head on the coin, respectively. Then, X and Y are geometric random variables and the probability that the stop on the same number toss is:

$$P(X=1,Y=1)+P(X=2,Y=2)+\cdots = P(X=1)P(Y=1)+P(X=2)P(Y=2)+\cdots$$

$$=\sum_{i=1}^{\infty}P(X=i)P(Y=i)=\sum_{i=1}^{\infty}p(1-p)^{i-1}p(1-p)^{i-1}=p^2\sum_{k=0}^{\infty}[(1-p)^2]^k=\frac{p^2}{1-(1-p)^2}.$$

5.63 $P(Y_1>Y_2,Y_1<2Y_2)=\int_0^{\infty}\int_{y_1/2}^{y_1}e^{-(y_1+y_2)}dy_2dy_1=\frac{1}{6}$ and $P(Y_1<2Y_2)=\int_0^{\infty}\int_{y_1/2}^{\infty}e^{-(y_1+y_2)}dy_2dy_1=\frac{2}{3}$. So, $P(Y_1>Y_2\,|\,Y_1<2Y_2)=1/4.$

5.64 $P(Y_1>Y_2,Y_1<2Y_2)=\int_0^{1}\int_{y_1/2}^{y_1}1dy_2dy_1=\frac{1}{4}$, $P(Y_1<2Y_2)=1-P(Y_1\geq 2Y_2)=1-\int_0^{1}\int_0^{y_1/2}1dy_2dy_1=\frac{3}{4}$. So, $P(Y_1>Y_2\,|\,Y_1<2Y_2)=1/3.$

5.65 **a.** The marginal density for Y_1 is $f_1(y_1)=\int_0^{\infty}[(1-\alpha(1-2e^{-y_1})(1-2e^{-y_2})]e^{-y_1-y_2}dy_2$

$$=e^{-y_1}\left[\int_0^{\infty}e^{-y_2}dy_2-\alpha(1-2e^{-y_1})\int_0^{\infty}(e^{-y_2}-2e^{-2y_2})dy_2\right].$$

$$=e^{-y_1}\left[\int_0^{\infty}e^{-y_2}dy_2-\alpha(1-2e^{-y_1})(1-1)\right]=e^{-y_1}.$$

b. By symmetry, the marginal density for Y_2 is also exponential with $\beta = 1$.

c. When $\alpha = 0$, then $f(y_1,y_2)=e^{-y_1-y_2}=f_1(y_1)f_2(y_2)$ and so Y_1 and Y_2 are independent. Now, suppose Y_1 and Y_2 are independent. Then, $E(Y_1Y_2) = E(Y_1)E(Y_2) = 1$. So,

$$E(Y_1Y_2)=\int_0^{\infty}\int_0^{\infty}y_1y_2[(1-\alpha(1-2e^{-y_1})(1-2e^{-y_2})]e^{-y_1-y_2}dy_1dy_2$$

$$=\int_0^{\infty}\int_0^{\infty}y_1y_2e^{-y_1-y_2}dy_1dy_2-\alpha\left[\int_0^{\infty}y_1(1-2e^{-y_1})e^{-y_1}dy_1\right]\times\left[\int_0^{\infty}y_2(1-2e^{-y_2})e^{-y_2}dy_2\right]$$

$=1-\alpha\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)=1-\alpha/4$. This equals 1 only if $\alpha = 0$.

5.66 **a.** Since $F_2(\infty)=1$, $F(y_1,\infty)=F_1(y_1)\cdot 1\cdot[1-\alpha\{1-F_1(y_1)\}\{1-1\}]=F_1(y_1)$.

b. Similarly, it is $F_2(y_2)$ from $F(y_1,y_2)$

c. If $\alpha = 0$, $F(y_1,y_2)=F_1(y_1)F_2(y_2)$, so by Definition 5.8 they are independent.

d. If $\alpha \neq 0$, $F(y_1,y_2)\neq F_1(y_1)F_2(y_2)$, so by Definition 5.8 they are not independent.

5.67 $P(a < Y_1 \le b, c < Y_2 \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$
$= F_1(b)F_2(d) - F_1(b)F_2(c) - F_1(a)F_2(d) + F_1(a)F_2(c)$
$= F_1(b)[F_2(d) - F_2(c)] - F_1(a)[F_2(d) - F_2(c)]$
$= [F_1(b) - F_1(a)] \times [F_2(d) - F_2(c)]$
$= P(a < Y_1 \le b) \times P(c < Y_2 \le d)$.

5.68 With $f(y_1, y_2) = f_1(y_1)f_2(y_2) = 1$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$,

$$P(Y_2 \le Y_1 \le Y_2 + 1/4) = \int_0^{1/4}\int_0^{y_1} 1dy_2dy_1 + \int_{1/4}^{1}\int_{y_1-1/4}^{y_1} 1dy_2dy_1 = 7/32.$$

5.69 **a.** $f(y_1, y_2) = f_1(y_1)f_2(y_2) = (1/9)e^{-(y_1+y_2)/3}$, $y_1 > 0$, $y_2 > 0$.

b. $P(Y_1 + Y_2 \le 1) = \int_0^1\int_0^{1-y_2} (1/9)e^{-(y_1+y_2)/3}dy_1dy_2 = 1 - \frac{4}{3}e^{-1/3} = .0446$.

5.70 Given that $p_1(y_1) = \binom{2}{y_1}(.2)^{y_1}(.8)^{2-y_1}$, $y_1 = 0, 1, 2$, and $p_2(y_2) = (.3)^{y_2}(.7)^{1-y_1}$, $y_2 = 0, 1$:

a. $p(y_1, y_2) = p_1(y_1)p_2(y_2) = \binom{2}{y_1}(.2)^{y_1}(.8)^{2-y_1}(.3)^{y_2}(.7)^{1-y_1}$, $y_1 = 0, 1, 2$ and $y_2 = 0, 1$.

b. The probability of interest is $P(Y_1 + Y_2 \le 1) = p(0,0) + p(1,0) + p(0,1) = .864$.

5.71 Assume uniform distributions for the call times over the 1–hour period. Then,

a. $P(Y_1 \le 1/2, Y_2 \le 1/2) = P(Y_1 \le 1/2 P(Y_2 \le 1/2) = (1/2)(1/2) = 1/4$.

b. Note that 5 minutes = 1/12 hour. To find $P(|Y_1 - Y_2| \le 1/12)$, we must break the region into three parts in the integration:

$$P(|Y_1 - Y_2| \le 1/12) = \int_0^{1/12}\int_0^{y_1+1/12} 1dy_2dy_1 + \int_{1/12}^{11/12}\int_{y_1-1/12}^{y_1+1/12} 1dy_2dy_1 + \int_{11/12}^{1}\int_{y_1-1/12}^{1} 1dy_2dy_1 = 23/144.$$

5.72 **a.** $E(Y_1) = 2(1/3) = 2/3$.
b. $V(Y_1) = 2(1/3)(2/3) = 4/9$
c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.

5.73 Use the mean of the hypergeometric: $E(Y_1) = 3(4)/9 = 4/3$.

5.74 The marginal distributions for Y_1 and Y_2 are uniform on the interval (0, 1). And it was found in Ex. 5.50 that Y_1 and Y_2 are independent. So:
a. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.
b. $E(Y_1Y_2) = E(Y_1)E(Y_2) = (1/2)(1/2) = 1/4$.
c. $E(Y_1^2 + Y_2^2) = E(Y_1^2) + E(Y_2^2) = (1/12 + 1/4) + (1/12 + 1/4) = 2/3$

d. $V(Y_1Y_2) = V(Y_1)V(Y_2) = (1/12)(1/12) = 1/144$.

5.75 The marginal distributions for Y_1 and Y_2 are exponential with $\beta = 1$. And it was found in Ex. 5.51 that Y_1 and Y_2 are independent. So:

a. $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 2$, $V(Y_1 + Y_2) = V(Y_1) + V(Y_2) = 2$.

b. $P(Y_1 - Y_2 > 3) = P(Y_1 > 3 + Y_2) = \int_0^\infty \int_{3+y_2}^\infty e^{-y_1-y_2}dy_1dy_2 = (1/2)e^{-3} = .0249$.

c. $P(Y_1 - Y_2 < -3) = P(Y_1 > Y_2 - 3) = \int_0^\infty \int_{3+y_1}^\infty e^{-y_1-y_2}dy_2dy_1 = (1/2)e^{-3} = .0249$.

d. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$, $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = 2$.

e. They are equal.

5.76 From Ex. 5.52, we found that Y_1 and Y_2 are independent. So,

a. $E(Y_1) = \int_0^1 2y_1^2dy_1 = 2/3$.

b. $E(Y_1^2) = \int_0^1 2y_1^3dy_1 = 2/4$, so $V(Y_1) = 2/4 - 4/9 = 1/18$.

c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.

5.77 Following Ex. 5.27, the marginal densities can be used:

a. $E(Y_1) = \int_0^1 3y_1(1-y_1)^2dy_1 = 1/4$, $E(Y_2) = \int_0^1 6y_2(1-y_2)dy_2 = 1/2$.

b. $E(Y_1^2) = \int_0^1 3y_1^2(1-y_1)^2dy_1 = 1/10$, $V(Y_1) = 1/10 - (1/4)^2 = 3/80$,

$E(Y_2^2) = \int_0^1 6y_2^2(1-y_2)dy_2 = 3/10$, $V(Y_2) = 3/10 - (1/2)^2 = 1/20$.

c. $E(Y_1 - 3Y_2) = E(Y_1) - 3\cdot E(Y_2) = 1/4 - 3/2 = -5/4$.

5.78 **a.** The marginal distribution for Y_1 is $f_1(y_1) = y_1/2$, $0 \le y_1 \le 2$. $E(Y_1) = 4/3$, $V(Y_1) = 2/9$.
b. Similarly, $f_2(y_2) = 2(1 - y_2)$, $0 \le y_2 \le 1$. So, $E(Y_2) = 1/3$, $V(Y_1) = 1/18$.
c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 4/3 - 1/3 = 1$.
d. $V(Y_1 - Y_2) = E[(Y_1 - Y_2)^2] - [E(Y_1 - Y_2)]^2 = E(Y_1^2) - 2E(Y_1Y_2) + E(Y_2^2) - 1$.

Since $E(Y_1Y_2) = \int_0^1 \int_{2y_2}^2 y_1y_2dy_1dy_2 = 1/2$, we have that

$$V(Y_1 - Y_2) = [2/9 + (4/3)^2] - 1 + [1/18 + (1/3)^2] - 1 = 1/6.$$

Using Tchebysheff's theorem, two standard deviations about the mean is (.19, 1.81).

5.79 Referring to Ex. 5.16, integrating the joint density over the two regions of integration:

$$E(Y_1Y_2) = \int_{-1}^{0}\int_{0}^{1+y_1} y_1y_2dy_2dy_1 + \int_{0}^{1}\int_{0}^{1-y_1} y_1y_2dy_2dy_1 = 0$$

5.80 From Ex. 5.36, $f_1(y_1) = y_1 + \frac{1}{2}$, $0 \le y_1 \le 1$, and $f_2(y_2) = y_2 + \frac{1}{2}$, $0 \le y_2 \le 1$. Thus, $E(Y_1) = 7/12$ and $E(Y_2) = 7/12$. So, $E(30Y_1 + 25Y_2) = 30(7/12) + 25(7/12) = 32.08$.

5.81 Since Y_1 and Y_2 are independent, $E(Y_2/Y_1) = E(Y_2)E(1/Y_1)$. Thus, using the marginal densities found in Ex. 5.61,

$$E(Y_2/Y_1) = E(Y_2)E(1/Y_1) = \tfrac{1}{2}\int_{0}^{\infty} y_2e^{-y_2/2}dy_2\left[\tfrac{1}{4}\int_{0}^{\infty} e^{-y_1/2}dy_1\right] = 2(\tfrac{1}{2}) = 1.$$

5.82 The marginal densities were found in Ex. 5.34. So,

$$E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 1/2 - \int_{0}^{1} -y_2\ln(y_2)dy_2 = 1/2 - 1/4 = 1/4.$$

5.83 From Ex. 3.88 and 5.42, $E(Y) = 2 - 1 = 1$.

5.84 All answers use results proven for the geometric distribution and independence:
a. $E(Y_1) = E(Y_2) = 1/p$, $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.
b. $E(Y_1^2) = E(Y_2^2) = (1 - p)/p^2 + (1/p)^2 = (2 - p)/p^2$. $E(Y_1Y_2) = E(Y_1)E(Y_2) = 1/p^2$.
c. $E[(Y_1 - Y_2)^2] = E(Y_1^2) - 2E(Y_1Y_2) + E(Y_2^2) = 2(1 - p)/p^2$.
$V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = 2(1 - p)/p^2$.
d. Use Tchebysheff's theorem with $k = 3$.

5.85 **a.** $E(Y_1) = E(Y_2) = 1$ (both marginal distributions are exponential with mean 1)
b. $V(Y_1) = V(Y_2) = 1$
c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0$.
d. $E(Y_1Y_2) = 1 - \alpha/4$, so $\text{Cov}(Y_1, Y_2) = -\alpha/4$.
e. $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2) = 1 + \alpha/2$. Using Tchebysheff's theorem with $k = 2$, the interval is $(-2\sqrt{2+\alpha/2}, -2\sqrt{2+\alpha/2})$.

5.86 Using the hint and Theorem 5.9:
a. $E(W) = E(Z)E(Y_1^{-1/2}) = 0E(Y_1^{-1/2}) = 0$. Also, $V(W) = E(W^2) - [E(W)]^2 = E(W^2)$.
Now, $E(W^2) = E(Z^2)E(Y_1^{-1}) = 1\cdot E(Y_1^{-1}) = E(Y_1^{-1}) = \frac{1}{\nu_1-2}$, $\nu_1 > 2$ (using Ex. 4.82).

b. $E(U) = E(Y_1)E(Y_2^{-1}) = \frac{\nu_1}{\nu_2 - 2}$, $\nu_2 > 2$, $V(U) = E(U^2) - [E(U)]^2 = E(Y_1^2)E(Y_2^{-2}) - \left(\frac{\nu_1}{\nu_2-2}\right)^2$

$= \nu_1(\nu_1 + 2)\frac{1}{(\nu_2-2)(\nu_2-4)} - \left(\frac{\nu_1}{\nu_2-2}\right)^2 = \frac{2\nu_1(\nu\cdot + \nu_2 - 2)}{(\nu_2-2)^2(\nu_2-4)}$, $\nu_2 > 4$.

5.87 **a.** $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = \nu_1 + \nu_2$.
b. By independence, $V(Y_1 + Y_2) = V(Y_1) + V(Y_2) = 2\nu_1 + 2\nu_2$.

5.88 It is clear that $E(Y) = E(Y_1) + E(Y_2) + \ldots + E(Y_6)$. Using the result that Yi follows a geometric distribution with success probability $(7 - i)/6$, we have

$$E(Y) = \sum_{i=1}^{6} \frac{6}{7-i} = 1 + 6/5 + 6/4 + 6/3 + 6/2 + 6 = 14.7.$$

5.89 $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \sum_{y_1}\sum_{y_2} y_1 y_2 p(y_1, y_2) - [2(1/3)]^2 = 2/9 - 4/9 = -2/9$.

As the value of Y_1 increases, the value of Y_2 tends to decrease.

5.90 From Ex. 5.3 and 5.21, $E(Y_1) = 4/3$ and $E(Y_2) = 1$. Thus,

$$E(Y_1Y_2) = 1(1)\tfrac{24}{84} + 2(1)\tfrac{12}{84} + 1(2)\tfrac{18}{84} = 1$$

So, $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 1 - (4/3)(1) = -1/3$.

5.91 From Ex. 5.76, $E(Y_1) = E(Y_2) = 2/3$. $E(Y_1Y_2) = \int_0^1\int_0^1 4y_1^2y_2^2\,dy_1dy_2 = 4/9$. So,

$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 4/9 - 4/9 = 0$ as expected since Y_1 and Y_2 are independent.

5.92 From Ex. 5.77, $E(Y_1) = 1/4$ and $E(Y_2) = 1/2$. $E(Y_1Y_2) = \int_0^1\int_0^{y_2} 6y_1y_2(1 - y_2)\,dy_1dy_2 = 3/20$.

So, $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 3/20 - 1/8 = 1/40$ as expected since Y_1 and Y_2 are dependent.

5.93 Note that the marginal distributions for Y_1 and Y_2 are

y_1	-1	0	1
$p_1(y_1)$	1/3	1/3	1/3

y_2	0	1
$p_2(y_2)$	2/3	1/3

So, Y_1 and Y_2 not independent since $p(-1, 0) \neq p_1(-1)p_2(0)$. However, $E(Y_1) = 0$ and $E(Y_1Y_2) = (-1)(0)1/3 + (0)(1)(1/3) + (1)(0)(1/3) = 0$, so $\text{Cov}(Y_1, Y_2) = 0$.

5.94 **a.** $\text{Cov}(U_1, U_2) = E[(Y_1 + Y_2)(Y_1 - Y_2)] - E(Y_1 + Y_2)E(Y_1 - Y_2)$

$= E(Y_1^2) - E(Y_2^2) - [E(Y_1)]^2 - [E(Y_2)]^2$

$= (\sigma_1^2 + \mu_1^2) - (\sigma_2^2 + \mu_2^2) - (\mu_1^2 - \mu_2^2) = \sigma_1^2 - \sigma_2^2$.

b. Since $V(U_1) = V(U_2) = \sigma_1^2 + \sigma_2^2$ (Y_1 and Y_2 are uncorrelated), $\rho = \dfrac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$.

c. If $\sigma_1^2 = \sigma_2^2$, U_1 and U_2 are uncorrelated.

5.95 **a.** From Ex. 5.55 and 5.79, $E(Y_1Y_2) = 0$ and $E(Y_1) = 0$. So,
$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0 - 0E(Y_2) = 0.$$
b. Y_1 and Y_2 are dependent.
c. Since $\text{Cov}(Y_1, Y_2) = 0$, $\rho = 0$.
d. If $\text{Cov}(Y_1, Y_2) = 0$, Y_1 and Y_2 are not necessarily independent.

5.96 **a.** $\text{Cov}(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = E[(Y_2 - \mu_2)(Y_1 - \mu_1)] = \text{Cov}(Y_2, Y_1)$.
b. $\text{Cov}(Y_1, Y_1) = E[(Y_1 - \mu_1)(Y_1 - \mu_1)] = E[(Y_1 - \mu_1)^2] = V(Y_1)$.

5.97 **a.** From Ex. 5.96, $\text{Cov}(Y_1, Y_1) = V(Y_1) = 2$.
b. If $\text{Cov}(Y_1, Y_2) = 7$, $\rho = 7/4 > 1$, impossible.
c. With $\rho = 1$, $\text{Cov}(Y_1, Y_2) = 1(4) = 4$ (a perfect positive linear association).
d. With $\rho = -1$, $\text{Cov}(Y_1, Y_2) = -1(4) = -4$ (a perfect negative linear association).

5.98 Since $\rho^2 \le 1$, we have that $-1 \le \rho \le 1$ or $-1 \le \dfrac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)}\sqrt{V(Y_2)}} \le 1$.

5.99 Since $E(c) = c$, $\text{Cov}(c, Y) = E[(c - c)(Y - \mu)] = 0$.

5.100 **a.** $E(Y_1) = E(Z) = 0$, $E(Y_2) = E(Z^2) = 1$.
b. $E(Y_1Y_2) = E(Z^3) = 0$ (odd moments are 0).
c. $\text{Cov}(Y_1, Y_1) = E(Z^3) - E(Z)E(Z^2) = 0$.
d. $P(Y_2 > 1 \mid Y_1 > 1) = P(Z^2 > 1 \mid Z > 1) = 1 \ne P(Z^2 > 1)$. Thus, Y_1 and Y_2 are dependent.

5.101 **a.** $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 1 - \alpha/4 - (1)(1) = -\alpha/4$.
b. This is clear from part a.
c. We showed previously that Y_1 and Y_2 are independent only if $\alpha = 0$. If $\rho = 0$, if must be true that $\alpha = 0$.

5.102 The quantity $3Y_1 + 5Y_2$ = dollar amount spend per week. Thus:
$$E(3Y_1 + 5Y_2) = 3(40) + 5(65) = 445.$$
$$E(3Y_1 + 5Y_2) = 9V(Y_1) + 25V(Y_2) = 9(4) + 25(8) = 236.$$

5.103 $E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E(Y_2) - 6E(Y_3) = 3(2) + 4(-1) - 6(-4) = -22$,
$V(3Y_1 + 4Y_2 - 6Y_3) = 9V(Y_1) + 16V(Y_2) + 36E(Y_3) + 24\text{Cov}(Y_1, Y_2) - 36\text{Cov}(Y_1, Y_3) - 48\text{Cov}(Y_2, Y_3) = 9(4) + 16(6) + 36(8) + 24(1) - 36(-1) - 48(0) = 480$.

5.104 **a.** Let $X = Y_1 + Y_2$. Then, the probability distribution for X is

x	1	2	3
$p(x)$	7/84	42/84	35/84

Thus, $E(X) = 7/3$ and $V(X) = .3889$.

b. $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 4/3 + 1 = 7/3$. We have that $V(Y_1) = 10/18$, $V(Y_2) = 42/84$, and $\text{Cov}(Y_1, Y_1) = -1/3$, so

$$V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_2, Y_3) = 10/18 + 42/84 - 2/3 = 7/18 = .3889.$$

5.105 Since Y_1 and Y_2 are independent, $V(Y_1 + Y_2) = V(Y_1) + V(Y_1) = 1/18 + 1/18 = 1/9$.

5.106 $V(Y_1 - 3Y_2) = V(Y_1) + 9V(Y_2) - 6\text{Cov}(Y_1, Y_2) = 3/80 + 9(1/20) - 6(1/40) = 27/80 = .3375$.

5.107 Since $E(Y_1) = E(Y_2) = 1/3$, $V(Y_1) = V(Y_2) = 1/18$ and $E(Y_1Y_2) = \int_0^1 \int_0^{1-y_2} 2y_1y_2 dy_1 dy_2 = 1/12$, we have that $\text{Cov}(Y_1, Y_1) = 1/12 - 1/9 = -1/36$. Therefore,

$$E(Y_1 + Y_2) = 1/3 + 1/3 = 2/3 \text{ and } V(Y_1 + Y_2) = 1/18 + 1/18 + 2(-1/36) = 1/18.$$

5.108 From Ex. 5.33, Y_1 has a gamma distribution with $\alpha = 2$ and $\beta = 1$, and Y_2 has an exponential distribution with $\beta = 1$. Thus, $E(Y_1 + Y_2) = 2(1) + 1 = 3$. Also, since

$$E(Y_1Y_2) = \int_0^\infty \int_0^{y_1} y_1 y_2 e^{-y_1} dy_2 dy_1 = 3, \text{ Cov}(Y_1, Y_1) = 3 - 2(1) = 1,$$

$$V(Y_1 - Y_2) = 2(1)^2 + 1^2 - 2(1) = 1.$$

Since a value of 4 minutes is four three standard deviations above the mean of 1 minute, this is not likely.

5.109 We have $E(Y_1) = E(Y_2) = 7/12$. Intermediate calculations give $V(Y_1) = V(Y_2) = 11/144$. Thus, $E(Y_1Y_2) = \int_0^1 \int_0^1 y_1y_2(y_1 + y_2)dy_1dy_2 = 1/3$, $\text{Cov}(Y_1, Y_1) = 1/3 - (7/12)^2 = -1/144$.

From Ex. 5.80, $E(30Y_1 + 25Y_2) = 32.08$, so

$$V(30Y_1 + 25Y_2) = 900V(Y_1) + 625V(Y_2) + 2(30)(25)\text{ Cov}(Y_1, Y_1) = 106.08.$$

The standard deviation of $30Y_1 + 25Y_2$ is $\sqrt{106.08} = 10.30$. Using Tchebysheff's theorem with $k = 2$, the interval is $(11.48, 52.68)$.

5.110 **a.** $V(1 + 2Y_1) = 4V(Y_1)$, $V(3 + 4Y_2) = 16V(Y_2)$, and $\text{Cov}(1 + 2Y_1, 3 + 4Y_2) = 8\text{Cov}(Y_1, Y_2)$.

So, $\dfrac{8\text{Cov}(Y_1, Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = \rho = .2$.

b. $V(1 + 2Y_1) = 4V(Y_1)$, $V(3 - 4Y_2) = 16V(Y_2)$, and $\text{Cov}(1 + 2Y_1, 3 - 4Y_2) = -8\text{Cov}(Y_1, Y_2)$.

So, $\dfrac{-8\text{Cov}(Y_1, Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = -\rho = -.2$.

c. $V(1 - 2Y_1) = 4V(Y_1)$, $V(3 - 4Y_2) = 16V(Y_2)$, and $\text{Cov}(1 - 2Y_1, 3 - 4Y_2) = 8\text{Cov}(Y_1, Y_2)$.
So, $\dfrac{8\text{Cov}(Y_1,Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = \rho = .2$.

5.111 The net daily gain is given by the random variable $G = X - Y$. Thus, given the distributions for X and Y in the problem,

$$E(G) = E(X) - E(Y) = 50 - (4)(2) = 48$$
$$V(G) = V(G) + V(G) = 3^2 + 4(2^2) = 25.$$

The value \$70 is $(70 - 48)/5 = 5.6$ standard deviations above the mean, an unlikely value.

5.112 In Ex. 5.61, it was showed that Y_1 and Y_2 are independent. In addition, Y_1 has a gamma distribution with $\alpha = 2$ and $\beta = 2$, and Y_2 has an exponential distribution with $\beta = 2$. So, with $C = 50 + 2Y_1 + 4Y_2$, it is clear that

$$E(C) = 50 + 2E(Y_1) + 4E(Y_2) = 50 + (2)(4) + (4)(2) = 66$$
$$V(C) = 4V(Y_1) + 16V(Y_2) = 4(2)(4) + 16(4) = 96.$$

5.113 **a.** $V(a + bY_1) = b^2V(Y_1)$, $V(c + dY_2) = d^2V(Y_2)$, and $\text{Cov}(a + bY_1, c + dY_2) = bd\text{Cov}(Y_1, Y_2)$.
So, $\rho_{W_1,W_2} = \dfrac{bd\text{Cov}(Y_1,Y_2)}{\sqrt{b^2V(Y_1)}\sqrt{d^2V(Y_2)}} = \dfrac{bd}{|bd|}\rho_{Y_1,Y_2}$. Provided that the constants b and d are nonzero, $\dfrac{bd}{|bd|}$ is either 1 or –1. Thus, $|\rho_{W_1,W_2}| = |\rho_{Y_1,Y_2}|$.

b. Yes, the answers agree.

5.114 Observe that Y_1 has a gamma distribution with $\alpha = 4$ and $\beta = 1$ and Y_2 has an exponential distribution with $\beta = 2$. Thus, with $U = Y_1 - Y_2$,

a. $E(U) = 4(1) - 2 = 2$
b. $V(U) = 4(1^2) + 2^2 = 8$
c. The value 0 has a z–score of $(0 - 2)/\sqrt{8} = -.707$, or it is $-.707$ standard deviations *below* the mean. This is not extreme so it is likely the profit drops below 0.

5.115 Following Ex. 5.88:
a. Note that for non–negative integers a and b and $i \neq j$,

$$P(Y_i = a, Y_j = b) = P(Y_j = b \mid Y_i = a)P(Y_i = a)$$

But, $P(Y_j = b \mid Y_i = a) = P(Y_j = b)$ since the trials (i.e. die tosses) are independent — the experiments that generate Y_i and Y_j represent independent experiments via the memoryless property. So, Y_i and Y_j are independent and thus $\text{Cov}(Y_i. Y_j) = 0$.

b. $V(Y) = V(Y_1) + \ldots + V(Y_6) = 0 + \frac{1/6}{(5/6)^2} + \frac{2/6}{(4/6)^2} + \frac{3/6}{(3/6)^2} + \frac{4/6}{(2/6)^2} + \frac{5/6}{(1/6)^2} = 38.99$.

c. From Ex. 5.88, E(Y) = 14.7. Using Tchebysheff's theorem with $k = 2$, the interval is $14.7 \pm 2\sqrt{38.99}$ or (0 , 27.188)

5.116 $V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1, Y_2)$, $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2)$. When Y_1 and Y_2 are independent, $\text{Cov}(Y_1, Y_2) = 0$ so the quantities are the same.

5.117 Refer to Example 5.29 in the text. The situation here is analogous to drawing n balls from an urn containing N balls, r_1 of which are red, r_2 of which are black, and $N - r_1 - r_2$ are neither red nor black. Using the argument given there, we can deduce that:

$$E(Y_1) = np_1 \qquad V(Y_1) = np_1(1 - p_1)\left(\tfrac{N-n}{N-1}\right) \qquad \text{where } p_1 = r_1/N$$
$$E(Y_2) = np_2 \qquad V(Y_2) = np_2(1 - p_2)\left(\tfrac{N-n}{N-1}\right) \qquad \text{where } p_2 = r_2/N$$

Now, define new random variables for $i = 1, 2, \ldots, n$:

$$U_i = \begin{cases} 1 & \text{if alligator } i \text{ is a mature female} \\ 0 & \text{otherwise} \end{cases} \qquad V_i = \begin{cases} 1 & \text{if alligator } i \text{ is a mature male} \\ 0 & \text{otherwise} \end{cases}$$

Then, $Y_1 = \sum_{i=1}^{n} U_i$ and $Y_2 = \sum_{i=1}^{n} V_i$. Now, we must find $\text{Cov}(Y_1, Y_2)$. Note that:

$$E(Y_1Y_2) = E\left(\sum_{i=1}^{n} U_i, \sum_{i=1}^{n} V_i\right) = \sum_{i=1}^{n} E(U_iV_i) + \sum_{i \neq j} E(U_iV_j).$$

Now, since for all i, $E(U_i, V_i) = P(U_i = 1, V_i = 1) = 0$ (an alligator can't be both female and male), we have that $E(U_i, V_i) = 0$ for all i. Now, for $i \neq j$,

$$E(U_i, V_j) = P(U_i = 1, V_i = 1) = P(U_i = 1)P(V_i = 1|U_i = 1) = \tfrac{r_1}{N}\left(\tfrac{r_2}{N-1}\right) = \tfrac{N}{N-1} p_1 p_2 .$$

Since there are $n(n - 1)$ terms in $\sum_{i \neq j} E(U_iV_j)$, we have that $E(Y_1Y_2) = n(n - 1)\frac{N}{N-1} p_1 p_2$.

Thus, $$\text{Cov}(Y_1, Y_2) = n(n - 1)\tfrac{N}{N-1} p_1 p_2 - (np_1)(np_2) = -\tfrac{n(N-n)}{N-1} p_1 p_2 .$$

So, $$E\left[\tfrac{Y_1}{n} - \tfrac{Y_2}{n}\right] = \tfrac{1}{n}(np_1 - np_2) = p_1 - p_2 ,$$
$$V\left[\tfrac{Y_1}{n} - \tfrac{Y_2}{n}\right] = \tfrac{1}{n^2}\left[V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2)\right] = \tfrac{N-n}{n(N-1)}\left(p_1 + p_2 - (p_1 - p_2)^2\right)$$

5.118 Let $Y = X_1 + X_2$, the total sustained load on the footing.

a. Since X_1 and X_2 have gamma distributions and are independent, we have that

$E(Y) = 50(2) + 20(2) = 140$
$V(Y) = 50(2^2) + 20(2^2) = 280$.

b. Consider Tchebysheff's theorem with $k = 4$: the corresponding interval is

$$140 + 4\sqrt{280} \text{ or } (73.07, 206.93).$$

So, we can say that the sustained load will exceed 206.93 kips with probability less than 1/16.

5.119 **a.** Using the multinomial distribution with $p_1 = p_2 = p_3 = 1/3$,

$$P(Y_1 = 3, Y_2 = 1, Y_3 = 2) = \tfrac{6!}{3!1!2!}\left(\tfrac{1}{3}\right)^6 = .0823.$$

b. $E(Y_1) = n/3$, $V(Y_1) = n(1/3)(2/3) = 2n/9$.
c. $\text{Cov}(Y_2, Y_3) = -n(1/3)(1/3) = -n/9$.
d. $E(Y_2 - Y_3) = n/3 - n/3 = 0$, $V(Y_2 - Y_3) = V(Y_2) + \text{V}(Y_3) - 2\text{Cov}(Y_2, Y_3) = 2n/3$.

5.120 $E(C) = E(Y_1) + 3E(Y_2) = np_1 + 3np_2$.
$V(C) = V(Y_1) + 9V(Y_2) + 6\text{Cov}(Y_1, Y_2) = np_1q_1 + 9np_2q_2 - 6np_1p_2$.

5.121 If N is large, the multinomial distribution is appropriate:

a. $P(Y_1 = 2, Y_2 = 1) = \tfrac{5!}{2!1!2!}(.3)^2(.1)^1(.6)^2 = .0972$.

b. $E\left[\tfrac{Y_1}{n} - \tfrac{Y_2}{n}\right] = = p_1 - p_2 = .3 - .1 = .2$

$$V\left[\tfrac{Y_1}{n} - \tfrac{Y_2}{n}\right] = \tfrac{1}{n^2}\left[V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2)\right] = \tfrac{p_1q_1}{n} + \tfrac{p_2q_2}{n} + 2\tfrac{p_1p_2}{n} = .072.$$

5.122 Let Y_1 = # of mice weighing between 80 and 100 grams, and let Y_2 = # weighing over 100 grams. Thus, with X having a normal distribution with $\mu = 100$ g. and $\sigma = 20$ g.,

$$p_1 = P(80 \le X \le 100) = P(-1 \le Z \le 0) = .3413$$
$$p_2 = P(X > 100) = P(Z > 0) = .5$$

a. $P(Y_1 = 2, Y_2 = 1) = \tfrac{4!}{2!1!1!}(.3413)^2(.5)^1(.1587)^1 = .1109$.

b. $P(Y_2 = 4) = \tfrac{4!}{0!4!0!}(.5)^4 = .0625$.

5.123 Let Y_1 = # of family home fires, Y_2 = # of apartment fires, and Y_3 = # of fires in other types. Thus, (Y_1, Y_2, Y_3) is multinomial with $n = 4$, $p_1 = .73$, $p_2 = .2$ and $p_3 = .07$. Thus,

$$P(Y_1 = 2, Y_2 = 1, Y_3 = 1) = 6(.73)^2(.2)(.07) = .08953.$$

5.124 Define C = total cost = $20{,}000Y_1 + 10{,}000Y_2 + 2000Y_3$

a. $E(C) = 20{,}000E(Y_1) + 10{,}000E(Y_2) + 2000E(Y_3)$
$= 20{,}000(2.92) + 10{,}000(.8) + 2000(.28) = 66{,}960$.

b. $V(C) = (20{,}000)^2V(Y_1) + (10{,}000)^2V(Y_2) + (2000)^2V(Y_3)$ + covariance terms
$= (20{,}000)^2(4)(.73)(.27) + (10{,}000)^2(4)(.8)(.2) + (2000)^2(4)(.07)(.93)$
$+ 2[20{,}000(10{,}000)(-4)(.73)(.2) + 20{,}000(2000)(-4)(.73)(.07) +$
$10{,}000(2000)(-4)(.2)(.07)] = 380{,}401{,}600 - 252{,}192{,}000 = 128{,}209{,}600$.

5.125 Let Y_1 = # of planes with no wine cracks, Y_2 = # of planes with detectable wing cracks, and Y_3 = # of planes with critical wing cracks. Therefore, (Y_1, Y_2, Y_3) is multinomial with $n = 5$, $p_1 = .7$, $p_2 = .25$ and $p_3 = .05$.

a. $P(Y_1 = 2, Y_2 = 2, Y_3 = 1) = 30(.7)^2(.25)^2(.05) = .046$.

b. The distribution of Y_3 is binomial with $n = 5$, $p_3 = .05$, so

$$P(Y_3 \geq 1) = 1 - P(Y_3 = 0) = 1 - (.95)^5 = .2262.$$

5.126 Using formulas for means, variances, and covariances for the multinomial:

$E(Y_1) = 10(.1) = 1$ $\qquad V(Y_1) = 10(.1)(.9) = .9$
$E(Y_2) = 10(.05) = .5$ $\qquad V(Y_2) = 10(.05)(.95) = .475$
$\text{Cov}(Y_1, Y_2) = -10(.1)(.05) = -.05$

So,

$E(Y_1 + 3Y_2) = 1 + 3(.5) = 2.5$
$V(Y_1 + 3Y_2) = .9 + 9(.475) + 6(-.05) = 4.875$.

5.127 Y is binomial with $n = 10$, $p = .10 + .05 = .15$.

a. $P(Y = 2) = \binom{10}{2}(.15)^2(.85)^8 = .2759$.

b. $P(Y \geq 1) = 1 - P(Y = 0) = 1 - (.85)^{10} = .8031$.

5.128 The marginal distribution for Y_1 is found by

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)dy_2 .$$

Making the change of variables $u = (y_1 - \mu_1)/\sigma_1$ and $v = (y_2 - \mu_2)/\sigma_2$ yields

$$f_1(y_1) = \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(u^2 + v^2 - 2\rho uv)\right]dv .$$

To evaluate this, note that $u^2 + v^2 - 2\rho uv = (v - \rho u)^2 + u^2(1-\rho^2)$ so that

$$f_1(y_1) = \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} e^{-u^2/2} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(v - \rho u)^2\right]dv ,$$

So, the integral is that of a normal density with mean ρu and variance $1 - \rho^2$. Therefore,

$$f_1(y_1) = \frac{1}{2\pi\sigma_1} e^{-(y_1-\mu_1)^2/2\sigma_1^2} , -\infty < y_1 < \infty,$$

which is a normal density with mean μ_1 and standard deviation σ_1. A similar procedure will show that the marginal distribution of Y_2 is normal with mean μ_2 and standard deviation σ_2.

5.129 The result follows from Ex. 5.128 and defining $f(y_1 \mid y_2) = f(y_1, y_2)/f_2(y_2)$, which yields a density function of a normal distribution with mean $\mu_1 + \rho(\sigma_1/\sigma_2)(y_2 - \mu_2)$ and variance $\sigma_1^2(1-\rho^2)$.

5.130 **a.** $\text{Cov}(U_1, U_2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \text{Cov}(Y_i, Y_j) = \sum_{i=1}^{n} a_i b_j V(Y_i) = \sigma^2 \sum_{i=1}^{n} a_i b_j$, since the Y_i's are independent. If $\text{Cov}(U_1, U_2) = 0$, it must be true that $\sum_{i=1}^{n} a_i b_j = 0$ since $\sigma^2 > 0$. But, it is trivial to see if $\sum_{i=1}^{n} a_i b_j = 0$, $\text{Cov}(U_1, U_2) = 0$. So, U_1 and U_2 are orthogonal.

b. Given in the problem, (U_1, U_2) has a bivariate normal distribution. Note that $E(U_1) = \mu\sum_{i=1}^{n} a_i$, $E(U_2) = \mu\sum_{i=1}^{n} b_i$, $V(U_1) = \sigma^2\sum_{i=1}^{n} a_i^2$, and $V(U_2) = \sigma^2\sum_{i=1}^{n} b_i^2$. If they are orthogonal, $\text{Cov}(U_1, U_2) = 0$ and then $\rho_{U_1, U_2} = 0$. So, they are also independent.

5.131 **a.** The joint distribution of Y_1 and Y_2 is simply the product of the marginals $f_1(y_1)$ and $f_2(y_2)$ since they are independent. It is trivial to show that this product of density has the form of the bivariate normal density with $\rho = 0$.

b. Following the result of Ex. 5.130, let $a_1 = a_2 = b_1 = 1$ and $b_2 = -1$. Thus, $\sum_{i=1}^{n} a_i b_j = 0$ so U_1 and U_2 are independent.

5.132 Following Ex. 5.130 and 5.131, U_1 is normal with mean $\mu_1 + \mu_2$ and variance $2\sigma^2$ and U_2 is normal with mean $\mu_1 - \mu_2$ and variance $2\sigma^2$.

5.133 From Ex. 5.27, $f(y_1 \mid y_2) = 1/y_2$, $0 \le y_1 \le y_2$ and $f_2(y_2) = 6y_2(1 - y_2)$, $0 \le y_2 \le 1$.

a. To find $E(Y_1 \mid Y_2 = y_2)$, note that the conditional distribution of Y_1 given Y_2 is uniform on the interval $(0, y_2)$. So, $E(Y_1 \mid Y_2 = y_2) = y_2/2$.

b. To find $E(E(Y_1 \mid Y_2))$, note that the marginal distribution is beta with $\alpha = 2$ and $\beta = 2$. So, from part a, $E(E(Y_1 \mid Y_2)) = E(Y_2/2) = 1/4$. This is the same answer as in Ex. 5.77.

5.134 The z–score is $(6 - 1.25)/\sqrt{1.5625} = 3.8$, so the value 6 is 3.8 standard deviations above the mean. This is not likely.

5.135 Refer to Ex. 5.41:

a. Since Y is binomial, $E(Y|p) = 3p$. Now p has a uniform distribution on (0, 1), thus

$$E(Y) = E[E(Y|p)] = E(3p) = 3(1/2) = 3/2.$$

b. Following part a, $V(Y|p) = 3p(1 - p)$. Therefore,

$$V(p) = E[3p(1 - p)] + V(3p) = 3E(p - p^2) + 9V(p)$$
$$= 3E(p) - 3[V(p) + (E(p))^2] + 9V(p) = 1.25$$

5.136 **a.** For a given value of λ, Y has a Poisson distribution. Thus, $E(Y \mid \lambda) = \lambda$. Since the marginal distribution of λ is exponential with mean 1, $E(Y) = E[E(Y \mid \lambda)] = E(\lambda) = 1$.

b. From part a, $E(Y \mid \lambda) = \lambda$ and so $V(Y \mid \lambda) = \lambda$. So, $V(Y) = E[V(Y \mid \lambda)] + E[V(Y \mid \lambda)] = 2$
c. The value 9 is $(9 - 1)/\sqrt{2} = 5.657$ standard deviations above the mean (unlikely score).

5.137 Refer to Ex. 5.38: $E(Y_2 \mid Y_1 = y_1) = y_1/2$. For $y_1 = 3/4$, $E(Y_2 \mid Y_1 = 3/4) = 3/8$.

5.138 If Y = # of bacteria per cubic centimeter,
a. $E(Y) = E(Y) = E[E(Y \mid \lambda)] = E(\lambda) = \alpha\beta$.
b. $V(Y) = E[V(Y \mid \lambda)] + V[E(Y \mid \lambda)] = \alpha\beta + \alpha\beta^2 = \alpha\beta(1+\beta)$. Thus, $\sigma = \sqrt{\alpha\beta(1+\beta)}$.

5.139 **a.** $E(T \mid N = n) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = n\alpha\beta$.

b. $E(T) = E[E(T \mid N)] = E(N\alpha\beta) = \lambda\alpha\beta$. Note that this is $E(N)E(Y)$.

5.140 Note that $V(Y_1) = E[V(Y_1 \mid Y_2)] + V[E(Y_1 \mid Y_2)]$, so $E[V(Y_1 \mid Y_2)] = V(Y_1) - V[E(Y_1 \mid Y_2)]$. Thus, $E[V(Y_1 \mid Y_2)] \le V(Y_1)$.

5.141 $E(Y_2) = E(E(Y_2 \mid Y_1)) = E(Y_1/2) = \lambda/2$
$V(Y_2) = E[V(Y_2 \mid Y_1)] + V[E(Y_2 \mid Y_1)] = E[Y_1^2/12] + V[Y_1/2] = (2\lambda^2)/12 + (\lambda^2)/2 = 2\lambda^2/3$.

5.142 **a.** $E(Y) = E[E(Y|p)] = E(np) = nE(p) = \dfrac{n\alpha}{\alpha+\beta}$.

b. $V(Y) = E[V(Y \mid p)] + V[E(Y \mid p)] = E[np(1-p)] + V(np) = nE(p - p^2) + n^2V(p)$. Now:

$$nE(p - p^2) = \frac{n\alpha}{\alpha+\beta} - \frac{n\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$n^2V(p) = \frac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

So, $V(Y) = \dfrac{n\alpha}{\alpha+\beta} - \dfrac{n\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} + \dfrac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \dfrac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

5.143 Consider the random variable y_1Y_2 for the fixed value of Y_1. It is clear that y_1Y_2 has a normal distribution with mean 0 and variance y_1^2 and the mgf for this random variable is

$$m(t) = E(e^{ty_1Y_2}) = e^{t^2y_1^2/2}.$$

$$\text{Thus, } m_U(t) = E(e^{tU}) = E(e^{tY_1Y_2}) = E[E(e^{tY_1Y_2} \mid Y_1)] = E(e^{tY_1^2/2}) = \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{(-y_1^2/2)(1-t^2)} dy_1.$$

Note that this integral is essentially that of a normal density with mean 0 and variance $\frac{1}{1-t^2}$, so the necessary constant that makes the integral equal to 0 is the reciprocal of the standard deviation. Thus, $m_U(t) = (1-t^2)^{-1/2}$. Direct calculations give $m'_U(0) = 0$ and

$m''_U(0) = 1$. To compare, note that $E(U) = E(Y_1Y_2) = E(Y_1)E(Y_2) = 0$ and $V(U) = E(U^2) = E(Y_1^2Y_2^2) = E(Y_1^2)E(Y_2^2) = (1)(1) = 1$.

5.144 $E[g(Y_1)h(Y_2)] = \sum_{y_1}\sum_{y_2} g(y_1)h(y_2)p(y_1,y_2) = \sum_{y_1}\sum_{y_2} g(y_1)h(y_2)p_1(y_1)p_2(y_2) = \sum_{y_1} g(y_1)p_1(y_1)\sum_{y_2} h(y_2)p_2(y_2) = E[g(Y_1)]\times E[h(Y_2)]$.

5.145 The probability of interest is $P(Y_1 + Y_2 < 30)$, where Y_1 is uniform on the interval (0, 15) and Y_2 is uniform on the interval (20, 30). Thus, we have

$$P(Y_1 + Y_2 < 30) = \int_{20}^{30}\int_{0}^{30-y2}\left(\frac{1}{15}\right)\left(\frac{1}{10}\right)dy_1dy_2 = 1/3.$$

5.146 Let (Y_1, Y_2) represent the coordinates of the landing point of the bomb. Since the radius is one mile, we have that $0 \le y_1^2 + y_2^2 \le 1$. Now,

P(target is destroyed) = P(bomb destroys everything within 1/2 of landing point)

This is given by $P(Y_1^2 + Y_2^2 \le (\frac{1}{2})^2)$. Since (Y_1, Y_2) are uniformly distributed over the unit circle, the probability in question is simply the area of a circle with radius 1/2 divided by the area of the unit circle, or simply 1/4.

5.147 Let Y_1 = arrival time for 1st friend, $0 \le y_1 \le 1$, Y_2 = arrival time for 2nd friend, $0 \le y_2 \le 1$. Thus $f(y_1, y_2) = 1$. If friend 2 arrives 1/6 hour (10 minutes) before or after friend 1, they will meet. We can represent this event as $|Y_1 - Y_2| < 1/3$. To find the probability of this event, we must find:

$$P(|Y_1 - Y_2| < 1/3) = \int_0^{1/6}\int_0^{y_1+1/6} 1dy_2dy_1 + \int_{1/6}^{5/6}\int_{y_1-1/6}^{y_1+1/6} 1dy_2dy_1 + \int_{5/6}^{1}\int_{y_1-1/6}^{1} 1dy_2dy_1 = 11/36.$$

5.148 **a.** $p(y_1, y_2) = \dfrac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}$, $y_1 = 0, 1, 2, 3$, $y_2 = 0, 1, 2, 3$, $y_1 + y_2 \le 3$.

b. Y_1 is hypergeometric w/ $r = 4$, $N = 9$, $n = 3$; Y_2 is hypergeometric w/ $r = 3$, $N = 9$, $n = 3$

c. $P(Y_1 = 1 \mid Y_2 \ge 1) = [p(1, 1) + p(1, 2)]/[1 - p_2(0)] = 9/16$

5.149 **a.** $f_1(y_1) = \int_0^{y_1} 3y_1dy_2 = 3y_1^2$, $0 \le y_1 \le 1$, $f_1(y_1) = \int_{y_2}^{1} 3y_1dy_1 = \frac{3}{2}(1 - y_2^2)$, $0 \le y_2 \le 1$.

b. $P(Y_1 \le 3/4 \mid Y_2 \le 1/2) = 23/44$.

c. $f(y_1 \mid y_2) = 2y_1/(1 - y_2^2)$, $y_2 \le y_1 \le 1$.

d. $P(Y_1 \le 3/4 \mid Y_2 = 1/2) = 2/3$.

5.150 **a.** Note that $f(y_2 \mid y_1) = f(y_1, y_2)/f(y_1) = 1/y_1$, $0 \le y_2 \le y_1$. This is the same conditional density as seen in Ex. 5.38 and Ex. 5.137. So, $E(Y_2 \mid Y_1 = y_1) = y_1/2$.

b. $E(Y_2) = E[E(Y_2 \mid Y_1)] = E(Y_1/2) = \int_0^1 \frac{y_1}{2} 3y_1^2 dy_1 = 3/8$.

c. $E(Y_2) = \int_0^1 y_2 \frac{3}{2}(1 - y_2^2) dy_2 = 3/8$.

5.151 **a.** The joint density is the product of the marginals: $f(y_1, y_2) = \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta}$, $y_1 \ge \infty$, $y_2 \ge \infty$

b. $P(Y_1 + Y_2 \le a) = \int_0^a \int_0^{a-y_2} \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta} dy_1 dy_2 = 1 - [1 + a/\beta]e^{-a/\beta}$.

5.152 The joint density of (Y_1, Y_2) is $f(y_1, y_2) = 18(y_1 - y_1^2)y_2^2$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$. Thus,

$P(Y_1Y_2 \le .5) = P(Y_1 \le .5/Y_2) = 1 - P(Y_1 > .5/Y_2) = 1 - \int_{.5}^1 \int_{.5/y_2}^1 18(y_1 - y_1^2) y_2^2 dy_1 dy_2$. Using straightforward integration, this is equal to $(5 - 3\ln 2)/4 = .73014$.

5.153 This is similar to Ex. 5.139:

a. Let N = # of eggs laid by the insect and Y = # of eggs that hatch. Given $N = n$, Y has a binomial distribution with n trials and success probability p. Thus, $E(Y \mid N = n) = np$. Since N follows as Poisson with parameter λ, $E(Y) = E[E(Y \mid N)] = E(Np) = \lambda p$.

b. $V(Y) = E[V(Y \mid N)] + V[E(Y \mid N)] = E[Np(1 - p)] + V[Np] = \lambda p$.

5.154 The conditional distribution of Y given p is binomial with parameter p, and note that the marginal distribution of p is beta with $\alpha = 3$ and $\beta = 2$.

a. Note that $f(y) = \int_0^1 f(y, p) = \int_0^1 f(y \mid p) f(p) dp = 12\binom{n}{y} \int_0^1 p^{y+2}(1-p)^{n-y+1} dp$. This integral can be evaluated by relating it to a beta density w/ $\alpha = y + 3$, $\beta = n + y + 2$. Thus,

$$f(y) = 12\binom{n}{y} \frac{\Gamma(n - y + 2)\Gamma(y + 3)}{\Gamma(n + 5)}, \; y = 0, 1, 2, \ldots, n.$$

b. For $n = 2$, $E(Y \mid p) = 2p$. Thus, $E(Y) = E[E(Y|p)] = E(2p) = 2E(p) = 2(3/5) = 6/5$.

5.155 **a.** It is easy to show that

$$\begin{aligned}
\text{Cov}(W_1, W_2) &= \text{Cov}(Y_1 + Y_2, Y_1 + Y_3) \\
&= \text{Cov}(Y_1, Y_1) + \text{Cov}(Y_1, Y_3) + \text{Cov}(Y_2, Y_1) + \text{Cov}(Y_2, Y_3) \\
&= \text{Cov}(Y_1, Y_1) = \text{V}(Y_1) = 2\nu_1.
\end{aligned}$$

b. It follows from part a above (i.e. the variance is positive).

5.156 **a.** Since $E(Z) = E(W) = 0$, $\text{Cov}(Z, W) = E(ZW) = E(Z^2 Y^{-1/2}) = E(Z^2)E(Y^{-1/2}) = E(Y^{-1/2})$. This expectation can be found by using the result Ex. 4.112 with $a = -1/2$. So,

$$\text{Cov}(Z, W) = E(Y^{-1/2}) = \frac{\Gamma(\frac{\nu}{2} - \frac{1}{2})}{\sqrt{2}\Gamma(\frac{\nu}{2})}, \text{ provided } \nu > 1.$$

b. Similar to part a, $\text{Cov}(Y, W) = E(YW) = E(\sqrt{Y}\, W) = E(\sqrt{Y})E(W) = 0$.

c. This is clear from parts a and b above.

5.157 $p(y) = \int_0^\infty p(y\,|\,\lambda)f(\lambda)d\lambda = \int_0^\infty \frac{\lambda^{y+\alpha-1}e^{-\lambda[(\beta+1)/\beta]}}{\Gamma(y+1)\Gamma(\alpha)\beta^\alpha}d\lambda = \frac{\Gamma(y+\alpha)\left(\frac{\beta}{\beta+1}\right)^{y+\alpha}}{\Gamma(y+1)\Gamma(\alpha)\beta^\alpha}$, $y = 0, 1, 2, \ldots$. Since it was assumed that α was an integer, this can be written as

$$p(y) = \binom{y+\alpha-1}{y}\left(\frac{\beta}{\beta+1}\right)^y\left(\frac{1}{\beta+1}\right)^\alpha, \; y = 0, 1, 2, \ldots .$$

5.158 Note that for each X_i, $E(X_i) = p$ and $V(X_i) = pq$. Then, $E(Y) = \Sigma E(X_i) = np$ and $V(Y) = npq$. The second result follows from the fact that the X_i are independent so therefore all covariance expressions are 0.

5.159 For each W_i, $E(W_i) = 1/p$ and $V(W_i) = q/p^2$. Then, $E(Y) = \Sigma E(X_i) = r/p$ and $V(Y) = rq/p^2$. The second result follows from the fact that the W_i are independent so therefore all covariance expressions are 0.

5.160 The marginal probabilities can be written directly:

$P(X_1 = 1) = P(\text{select ball 1 or 2}) = .5$ $\qquad$ $P(X_1 = 0) = .5$
$P(X_2 = 1) = P(\text{select ball 1 or 3}) = .5$ $\qquad$ $P(X_2 = 0) = .5$
$P(X_3 = 1) = P(\text{select ball 1 or 4}) = .5$ $\qquad$ $P(X_3 = 0) = .5$

Now, for $i \neq j$, X_i and X_j are clearly pairwise independent since, for example,

$P(X_1 = 1, X_2 = 1) = P(\text{select ball 1}) = .25 = P(X_1 = 1)P(X_2 = 1)$
$P(X_1 = 0, X_2 = 1) = P(\text{select ball 3}) = .25 = P(X_1 = 0)P(X_2 = 1)$

However, X_1, X_2, and X_3 are <u>not</u> mutually independent since

$P(X_1 = 1, X_2 = 1, X_3 = 1) = P(\text{select ball 1}) = .25 \neq P(X_1 = 1)P(X_2 = 1)P(X_1 = 3)$.

5.161 $E(\bar{Y} - \bar{X}) = E(\bar{Y}) - E(\bar{X}) = \frac{1}{n}\sum E(Y_i) - \frac{1}{m}\sum E(X_i) = \mu_1 - \mu_2$
$V(\bar{Y} - \bar{X}) = V(\bar{Y}) + V(\bar{X}) = \frac{1}{n^2}\sum V(Y_i) + \frac{1}{m^2}\sum V(X_i) = \sigma_1^2/n + \sigma_2^2/m$

5.162 Using the result from Ex. 5.65, choose two different values for α with $-1 \le \alpha \le 1$.

5.163 **a.** The distribution functions with the exponential distribution are:

$F_1(y_1) = 1 - e^{-y_1}$, $y_1 \ge 0$; $\quad F_2(y_2) = 1 - e^{-y_2}$, $y_2 \ge 0$.

Then, the joint distribution function is

$$F(y_1, y_2) = [1 - e^{-y_1}][1 - e^{-y_2}][1 - \alpha(e^{-y_1})(e^{-y_2})].$$

Finally, show that $\frac{\partial^2}{\partial y_1 \partial y_2} F(y_1, y_2)$ gives the joint density function seen in Ex. 5.162.

b. The distribution functions with the uniform distribution on (0, 1) are:

$F_1(y_1) = y_1$, $0 \le y_1 \le 1$; $\quad F_2(y_2) = y_2$, $0 \le y_2 \le 1$.

Then, the joint distribution function is

$$F(y_1, y_2) = y_1 y_2 [1 - \alpha(1 - y_1)(1 - y_2)].$$

c. $\frac{\partial^2}{\partial y_1 \partial y_2} F(y_1, y_2) = f(y_1, y_2) = 1 - \alpha[(1 - 2y_1)(1 - 2y_2)]$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$.

d. Choose two different values for α with $-1 \le \alpha \le 1$.

5.164 **a.** If $t_1 = t_2 = t_3 = t$, then $m(t, t, t) = E\left(e^{t(X_1 + X_2 + X_3)}\right)$. This, by definition, is the mgf for the random variable $X_1 + X_2 + X_3$.

b. Similarly with $t_1 = t_2 = t$ and $t_3 = 0$, $m(t, t, 0) = E\left(e^{t(X_1 + X_2)}\right)$.

c. We prove the continuous case here (the discrete case is similar). Let (X_1, X_2, X_3) be continuous random variables with joint density function $f(x_1, x_2, x_3)$. Then,

$$m(t_1, t_2, t_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x_1} e^{t_2 x_2} e^{t_3 x_3} f(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 .$$

Then,

$$\frac{\partial^{k_1 + k_2 + k_3}}{\partial t_1^{k_1} \partial t_2^{k_2} \partial t_3^{k_3}} m(t_1, t_2, t_3)\Big|_{t_1 = t_2 = t_3 = 0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{k_1} x_2^{k_2} x_3^{k_3} f(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 .$$

This is easily recognized as $E\left(X_1^{k_1} X_2^{k_2} X_3^{k_3}\right)$.

5.165 **a.** $m(t_1, t_2, t_3) = \sum_{x_1}\sum_{x_2}\sum_{x_3} \frac{n!}{x_1!x_2!x_3!} e^{t_1x_1+t_2x_2+t_3x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$

$$= \sum_{x_1}\sum_{x_2}\sum_{x_3} \frac{n!}{x_1!x_2!x_3!}(p_1e^{t_1})^{x_1}(p_2e^{t_2})^{x_2}(p_3e^{t_3})^{x_3} = (p_1e^{t_1} + p_2e^{t_2} + p_3e^{t_3})^n.$$ The final form follows from the multinomial theorem.

b. The mgf for X_1 can be found by evaluating $m(t, 0, 0)$. Note that $q = p_2 + p_3 = 1 - p_1$.

c. Since $\text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$ and $E(X_1) = np_1$ and $E(X_2) = np_2$ since X_1 and X_2 have marginal binomial distributions. To find $E(X_1X_2)$, note that

$$\frac{\partial^2}{\partial t_1 \partial t_2} m(t_1, t_2, 0)\Big|_{t_1=t_2=0} = n(n-1)p_1p_2.$$

Thus, $\text{Cov}(X_1, X_2) = n(n-1)p_1p_2 - (np_1)(np_2) = -np_1p_2$.

5.166 The joint probability mass function of (Y_1, Y_2, Y_3) is given by

$$p(y_1, y_2, y_3) = \frac{\binom{N_1}{y_1}\binom{N_2}{y_2}\binom{N_3}{y_3}}{\binom{N}{n}} = \frac{\binom{Np_1}{y_1}\binom{Np_2}{y_2}\binom{Np_3}{y_3}}{\binom{N}{n}},$$

where $y_1 + y_2 + y_3 = n$. The marginal distribution of Y_1 is hypergeometric with $r = Np_1$, so $E(Y_1) = np_1$, $V(Y_1) = np_1(1-p_1)\left(\frac{N-n}{N-1}\right)$. Similarly, $E(Y_2) = np_2$, $V(Y_2) = np_2(1-p_2)\left(\frac{N-n}{N-1}\right)$. It can be shown that (using mathematical expectation and straightforward albeit messy algebra) $E(Y_1Y_2) = n(n-1)p_1p_2\frac{N}{N-1}$. Using this, it is seen that

$$\text{Cov}(Y_1, Y_2) = n(n-1)p_1p_2\frac{N}{N-1} - (np_1)(np_2) = -np_1p2\left(\frac{N-n}{N-1}\right).$$

(Note the similar expressions in Ex. 5.165.) Finally, it can be found that

$$\rho = -\sqrt{\frac{p_1p_2}{(1-p_1)(1-p_2)}}.$$

5.167 **a.** For this exercise, the quadratic form of interest is

$$At^2 + Bt + C = E(Y_1^2)t^2 + [-2E(Y_1Y_2)]t + [E(Y_2^2)]^2.$$

Since $E[(tY_1 - Y_2)^2] \geq 0$ (it is the integral of a non–negative quantity), so we must have that $At^2 + Bt + C \geq 0$. In order to satisfy this inequality, the two roots of this quadratic must either be imaginary or equal. In terms of the discriminant, we have that

$$B^2 - 4AC \leq 0, \text{ or}$$

$$[-2E(Y_1Y_2)]^2 - 4E(Y_1^2)E(Y_2^2) \leq 0.$$

Thus, $[E(Y_1Y_2)]^2 \leq E(Y_1^2)E(Y_2^2)$.

b. Let $\mu_1 = E(Y_1)$, $\mu_2 = E(Y_2)$, and define $Z_1 = Y_1 - \mu_1$, $Z_2 = Y_2 - \mu_2$. Then,

$$\rho^2 = \frac{[E(Y_1 - \mu_1)(Y_2 - \mu_2)]^2}{[E(Y_1 - \mu_1)^2]E[(Y_2 - \mu_2)^2]} = \frac{[E(Z_1 Z_2)]^2}{E(Z_1{}^2)E(Z_2{}^2)} \le 1$$

by the result in part **a.**