PLEASE HAND IN

PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term Test Sample Solutions

CSC 236H1
Sections L0101/L2000
Duration — 50 minutes

Examination Aids: One 8.5”x11” sheet of paper, handwritten on one side.

Last Name: ____________________

First Name: ____________________

Student No: ____________________

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 4 questions on 6 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided.

Good Luck!

Question 1. [9 MARKS]

Prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Solution: Let $P(n)$ denote the assertion that $\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$.

Base Case: Let $k = 0$.

Then $\sum_{i=0}^{k}(2i+1)^2 = (2 \cdot 0 + 1)^2 = 1$.

Also, $\frac{(k+1)(2k+1)(2k+3)}{3} = \frac{(0+1)(0+1)(0+3)}{3} = 1$.

So $\sum_{i=0}^{k}(2i+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.

Induction Step: Let $k \in \mathbb{N}$. Suppose $P(k)$ is true, i.e., $\sum_{i=0}^{k}(2i+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$. **[IH]**

WTP: $P(k+1)$ holds, i.e., $\sum_{i=0}^{k+1}(2i+1)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$.

$$\begin{aligned}
\sum_{i=0}^{k+1}(2i+1)^2 &= \sum_{i=0}^{k}(2i+1)^2 + (2(k+1)+1)^2 \\
&= \sum_{i=0}^{k}(2i+1)^2 + (2k+3)^2 \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 && \text{by the IH} \\
&= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3} \\
&= \frac{(2k+3)[(k+1)(2k+1) + 3(2k+3)]}{3} && \text{factoring out } (2k+3) \\
&= \frac{(2k+1)(2k^2+k+2k+1+6k+9)}{3} \\
&= \frac{(2k+1)(2k^2+9k+10)}{3} \\
&= \frac{(2k+1)(2k+5)(k+2)}{3}.
\end{aligned}$$

Question 2. [13 MARKS]

Let $f_1, f_2, ...$ be a sequence of natural numbers defined as follows:

$$\begin{aligned} f_1 =& 1, \\ f_2 =& 1, \\ f_n =& f_{n-1} + f_{n-2}, \qquad n \geq 3. \end{aligned}$$

Let $a_0, a_1, a_2, ...$ be a sequence of natural numbers defined as follows:

$$\begin{aligned} a_0 =& 0, \\ a_1 =& 0, \\ a_n =& a_{n-1} + a_{n-2} + 2^{n-2}, \qquad n \geq 2. \end{aligned}$$

Prove that for all $n \in \mathbb{N}$, $a_n = 2^n - f_{n+2}$.

Solution: Let $P(n)$ denote the assertion that $a_n = 2^n - f_{n+2}$.

Base Case:
Let $k = 0$.
Then by definition, $a_k = 0$.
Also, $2^k - f_{k+2} = 1 - f_2 = 0$, since $f_2 = 1$.
So $a_k = 2^k - f_{k+2}$.

Let $k = 1$.
Then by definition, $a_k = 0$.
Also, $2^k - f_{n+2} = 2 - f_3 = 0$ since $f_3 = 2$.
So $a_k = 2^k - f_{k+2}$.

Induction Step: Let $k \in \mathbb{N}$, and $k \geq 2$. Suppose for all $0 \leq j < k$, $P(j)$ is true, i.e., $a_j = 2^j - f_{j+2}$. **[IH]**
WTP: $P(k)$ holds, i.e., $a_k = 2^k - f_{k+2}$.

$$\begin{aligned} a_k &= a_{k-1} + a_{k-2} + 2^{k-2} && \text{By the definition of } a_k \\ &= 2^{k-1} - f_{k+1} + 2^{k-2} - f_k + 2^{k-2} && \text{By the IH, and since } 0 \leq k-1, k-2 < k \\ &= 2^{k-2}(2+1+1) - (f_{k+1} + f_k) \\ &= 2^2 \cdot 2^{k-2} - f_{k+2} && \text{By the definition of } f_{k+2} \text{ as } k+2 \geq 3 \\ &= 2^k - f_{k+2}. \end{aligned}$$

Question 3. [12 MARKS]

Let m, n be integers, not both zero. Let L be a set of integers defined as follows:

- $m, n \in L$;
- if $j, k \in L$, then $k^2 - j^2 \in L$ and $j^2 + k^2 + 2j \cdot k \in L$.

Prove that every common divisor of m and n also divides every member of L.

Solution: Let d be a common divisor of m and n.
Let $P(r)$ denote the assertion that d divides r.
(Alternative: $P(r)$ denotes the assertion that every common divisor, d, of m and n divides r. Note that if you use the alternative definition for P, you will need to introduce d in the Base Case, Case 1, and Case 2).

Base Case:
Let $r = m$ or $r = n$.
Then by assumption, d divides r.
So, $P(r)$ holds.

Induction Step: Let j, k be arbitrary members of L. Suppose $P(j)$ and $P(k)$ holds, i.e., d divides both j and k. **[IH]**
WTP: $P(k^2 - j^2)$ and $P(j^2 + k^2 + 2j \cdot k)$.

Case 1: Let $r = k^2 - j^2$.
By the IH, d divides j and k. So it also divides j^2 and k^2, i.e., exist $t_1, t_2 \in \mathbb{Z}$ such that $j^2 = t_1 \cdot d$ and $k^2 = t_2 \cdot d$.
Then d also divides $k^2 - j^2$ since $k^2 - j^2 = (t_2 - t_1) \cdot d$ and $t_2 - t_1 \in \mathbb{Z}$.

Case 2: Let $r = j^2 + k^2 + 2j \cdot k$.
Then $r = (j + k)^2$.
By the IH, d divides j and k, i.e., exist $t_1, t_2 \in \mathbb{Z}$ such that $j = t_1 \cdot d$ and $k = t_2 \cdot d$.
Then d also divides $j + k$ (since $j + k = (t_1 + t_2) \cdot d$ and $t_1 + t_2 \in \mathbb{Z}$), as well as $(j + k)^2$.

Question 4. [6 MARKS]

Find the flaw with the following "proof" that for all natural numbers n, $5n = 0$.
Make sure to identify **all** errors and missing parts in the "proof", and provide enough explanations to justify your answer.
You will lose mark for identifying false errors.

Base Case: $5 \cdot 0 = 0$.

IS: Assume that $5t = 0$ for all natural numbers t with $t \leq k$.
Then $k + 1 = i + j$, where i and j are natural numbers less than $k + 1$. By the induction hypothesis,

$$5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0.$$

Solution: In the proof for the IS, in addition to stating that i and j are natural numbers less than $k + 1$, it must be stated that at least one of i or j is greater than or equal to 1, otherwise the other one would be equal to $k + 1$ and we wouldn't be able to apply he induction hypothesis (IH) for it. This means that in the base case we must prove the claim for $n = 1$. However, $5 \cdot 1$ is not equal to 0.

This page is left (nearly) blank to accommodate work that wouldn't fit elsewhere and/or scratch work.

# 1:	_______/ 9
# 2:	_______/13
# 3:	_______/12
# 4:	_______/ 6
TOTAL:	_______/40