Greedy Strategy #1. (Intuition: keep the tank as full as possible.) Stop at *every* gas station.

Clearly not optimal: in an extreme case, if $k \geqslant d$, the entire trip could be completed without stopping at all, but this greedy strategy would cause us to stop n times.

Greedy Strategy #2. (Intuition: fill up only when necessary.)

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set d_{n+1} = d to handle the end of the trip
S = ∅   # set of stops
ℓ = 0   # location of last stop
for j = 1, 2, ..., n:
    if d_{j+1} − ℓ > k:
        # Stop at d_j iff we cannot make it to d_{j+1}.
        S = S ∪ {j}
        ℓ = d_j
return S
```

To prove correctness, let $S_0, S_1, \dots, S_n$ be the partial solutions generated by the algorithm at the end of each iteration. Say that an optimum solution S^* *extends* S_i iff $S_i \subseteq S^* \subseteq S_i \cup \{i+1, \dots, n\}$. We prove that $\forall i \in \{0, 1, \dots, n\}, S_i$ is promising.

Base Case: Every optimum solution extends $S_0 = \varnothing$.

Ind. Hyp.: Suppose for some $i \geqslant 0$ that S_i is promising (with optimum solution S_i^*).

Ind. Step: Either $S_{i+1} = S_i$ or $S_{i+1} = S_i \cup \{i+1\}$.

- **Case 1:** If $S_{i+1} = S_i$, then the greedy algorithm has decided to skip d_{i+1}. Either $i+1 \in S_i^*$ or $i+1 \notin S_i^*$.
  - **Subcase A:** If $i+1 \in S_i^*$, then let d_a be the last stop in S_i (for some $a \leqslant i$) and let d_b be the next stop after d_{i+1} in S_i^* (for some $b \geqslant i+1$). From the greedy algorithm, we know that $d_{i+2} - d_a \leqslant k$. From the fact that S_i^* is a solution, we know that $d_b - d_{i+1} \leqslant k$. Consider $S_{i+1}^* = S_i^* - \{i+1\} \cup \{i+2\}$. Then S_{i+1}^* is still a solution because $d_{i+2} - d_a \leqslant k$ and $d_b - d_{i+2} < d_b - d_{i+1} \leqslant k$. Also, S_{i+1}^* is optimum (it has the same size as S_i^*) and it extends S_{i+1}.
  - **Subcase B:** If $i+1 \notin S_i^*$, then S_i^* already extends S_{i+1}.
- **Case 2:** If $S_{i+1} = S_i \cup \{i+1\}$, then let d_j be the last stop in S_i (for some $j \leqslant i$). Because the greedy algorithm stops only when necessary, $d_{i+2} - d_j > k$. Since $S_i^* \subseteq S_i \cup \{i+1, \dots, n\}$, S_i^* contains no stop strictly between j and $i+1$. These two facts together imply $i+1 \in S_i^*$ —otherwise, S_i^* would have a gap longer than k after d_j. Hence, S_i^* already extends S_{i+1}.

Hence, S_i is promising for every i. In particular, S_n is promising: it can be extended to some optimum S_n^* using only stops in $\{d_{n+1}, \dots, d_n\} = \varnothing$. But this means $S_n = S_n^*$ is optimum.