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University of Toronto
Department of Mathematics

Term Test 1
MAT224H1F
Linear Algebra II

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Instructions.

1. There are **53** possible marks to be earned in this exam. The examination booklet contains a total of 9 pages. It is your responsibility to ensure that *no pages are missing from your examination.* DO NOT DETACH ANY PAGES FROM YOUR EXAMINATION.
2. DO NOT WRITE ON THE QR CODE AT THE TOP RIGHT-HAND CORNER OF EACH PAGE OF YOUR EXAMINATION.
3. For the full answer questions, WRITE YOUR SOLUTIONS ON THE FRONT OF THE QUESTION PAGES THEMSELVES. THE BACK OF EVERY PAGE WILL **NOT** BE SCANNED NOR SEEN BY THE GRADERS.
4. Ensure that your solutions are LEGIBLE.
5. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
6. Have your student card ready for inspection.
7. You may use the two blank pages at the end for rough work. The last two pages of the examination WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
8. **Show all of your work and justify your answers** but do not include extraneous information.

1. (a) Let $V = \{\mathbf{x}, \mathbf{y}\}$ be a set with exactly two vectors. Define vector addition and scalar multiplication in V by the following rules:

Vector addition: $\mathbf{x}+\mathbf{x}=\mathbf{x}$, $\mathbf{y}+\mathbf{y}=\mathbf{x}$, $\mathbf{x}+\mathbf{y}=\mathbf{y}$, and $\mathbf{y}+\mathbf{x}=\mathbf{y}$.
Scalar multiplication: $c\mathbf{x}=\mathbf{x}$, and $c\mathbf{y}=\mathbf{y}$ for all $c \in \mathbb{R}$.

Show that V is **not** a vector space by citing one axiom in the definition of a vector space that fails to hold. You must both state the axiom clearly and show it does not hold. [4 marks]

Axiom: For all $c, d \in \mathbb{R}$ and all $\mathbf{x} \in V$

$$(c+d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$$

Notice that $(1+1)\mathbf{y} = 2\mathbf{y} = \mathbf{y}$ but $\mathbf{y}+\mathbf{y}=\mathbf{x}$.
Therefore, we have $(1+1)\mathbf{y} \neq \mathbf{y}+\mathbf{y}$, contradicting the above axiom.

1. (b) Define what it means for a subset W of a vector space V to be a *subspace* of V. [2 marks]

W is a vector subspace of V if W is a vector space itself under the same operations of vector sum and scalar multiplication from V.

Alternative Solution: W is a subspace of V if $0 \in W$ and for all $\mathbf{x}, \mathbf{y} \in W$ and for all $c \in \mathbb{R}$, we have $c\mathbf{x}+\mathbf{y} \in W$.

2. A vector $\mathbf{x} \in \mathbb{R}^n$ is *symmetric* if $x_k = x_{n-k+1}$ for $k = 1, 2, \ldots, n$. It is *anti-symmetric* if $x_k = -x_{n-k+1}$ for $k = 1, 2, \ldots, n$. Let

$$U = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is symmetric}\}$$
$$W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is anti-symmetric}\}$$

(a) Both U and W are subspaces of $\mathbb{R}^n$ but pick only one (your choice) and show it is a subspace. [5 marks]

Proving U is a subspace:

First note that $\mathbf{x} = \mathbf{0} \in U$ since $x_k = x_{n-k+1} = 0$ for all $k = 1, 2, \ldots, n$. Now pick any $c \in \mathbb{R}$, and vectors $\mathbf{x}, \mathbf{y} \in U$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Note that since $\mathbf{x}, \mathbf{y} \in U$, we have that $x_k = x_{n-k+1}$ and $y_k = y_{n-k+1}$ for $k = 1, 2, \ldots, n$.

We claim that $c\mathbf{x} + \mathbf{y} = (cx_1 + y_1, \ldots, cx_n + y_n) \in U$. Note that $(c\mathbf{x} + \mathbf{y})_k = cx_k + y_k = cx_{n-k+1} + y_{n-k+1} = (c\mathbf{x} + \mathbf{y})_{n-k+1} \Rightarrow c\mathbf{x} + \mathbf{y} \in U$.

Proving W is a subspace:

First note that $\mathbf{x} = \mathbf{0} \in W$ since $x_k = -x_{n-k+1} = 0$ for all $k = 1, 2, \ldots, n$. Now pick any $c \in \mathbb{R}$, and vectors $\mathbf{x}, \mathbf{y} \in W$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Note that since $\mathbf{x}, \mathbf{y} \in W$, we have that $x_k = -x_{n-k+1}$ and $y_k = -y_{n-k+1}$ for $k = 1, 2, \ldots, n$.

We claim that $c\mathbf{x} + \mathbf{y} = (cx_1 + y_1, \ldots, cx_n + y_n) \in W$. Note that $(c\mathbf{x} + \mathbf{y})_k = cx_k + y_k = -cx_{n-k+1} - y_{n-k+1} = -(c\mathbf{x} + \mathbf{y})_{n-k+1} \Rightarrow c\mathbf{x} + \mathbf{y} \in W$.

(b) Is $\mathbb{R}^n = U \oplus W$? Explain your answer. [5 marks]

This is true. First, note that $U \cap W = \{\mathbf{0}\}$. Let $\mathbf{x} \in U \cap W$. Then for $k = 1, 2, \ldots, n$, $x_k = x_{n-k+1} = -x_{n-k+1} \Rightarrow x_k = 0$ for all $k = 1, 2, \ldots, n$. Thus, $\mathbf{x} = \mathbf{0}$.

Now we claim that $\mathbb{R}^n = U + W$. Pick any $\mathbf{x} \in \mathbb{R}$. Let $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}$ be such that $y_k = x_{n-k+1}$ for $k = 1, 2, \ldots n$. Notice that

$$\mathbf{x} = \frac{\mathbf{x} + \mathbf{y}}{2} + \frac{\mathbf{x} - \mathbf{y}}{2} = \mathbf{u} + \mathbf{w}$$

where $\mathbf{u} = \frac{\mathbf{x}+\mathbf{y}}{2}$ and $\mathbf{w} = \frac{\mathbf{x}-\mathbf{y}}{2}$. Notice that $u_k = u_{n-k+1}$ and $w_k = w_{n-k+1}$ for $k = 1, 2, \ldots, n$. Thus, $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

3. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in vector space V.

(a) Define what it means for the list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ to be linearly dependent. [2 marks]

The list $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linear dependent if there exists $a_1, a_2, \ldots a_k \in \mathbb{R}$ where not all of the a_i are zero and

$$a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_k = \mathbf{0}$$

(b) Define $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. [2 marks]

$\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is the set of all all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.

Alternatively, may write $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k | a_1, \ldots, a_k \in \mathbb{R}\}$

(c) Either prove the following statement is true or find a counterexample to show it is false: Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be a linearly independent list of vectors in a vector space V. If the pair $\mathbf{u}, \mathbf{v} \in V$ is linearly independent and both $\mathbf{u}, \mathbf{v} \notin \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, then the list $\mathbf{u}, \mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly independent. [4 marks]

This statement is false. Suppose our vector space is $V = \mathbb{R}^2$, and our list of vectors contains only 1 vector, $\mathbf{x}_1 = (1, 0)$. Let $\mathbf{u} = (0, 1)$ and $\mathbf{v} = (1, 1)$. Note that $\mathbf{u}$ and $\mathbf{v}$ are both linear independent and not in $\text{span}\{\mathbf{x}_1\}$. However, the list $\{\mathbf{u}, \mathbf{v}, \mathbf{x}_1\}$ is linear independent since $\mathbf{u} + \mathbf{x}_1 - \mathbf{v} = \mathbf{0}$

4. (a) Let V be a finite dimensional vector space. Define the *dimension* of V. [2 marks]

The dimension of V is the number of vectors in any basis of V.

4. (b) Let V be an n-dimensional vector space. Let W_1 and W_2 be unequal subspaces of V, each with dimension $(n-1)$. Prove that $V = W_1 + W_2$, and that $\dim(W_1 \cap W_2) = n - 2$. [5 marks]

Let $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n-1}}\}$ be any basis for W_1. Since W_2 is unequal to W_1 and has the same dimension as W_1, there exists some $\mathbf{y} \in W_2$ where $\mathbf{y} \notin W_1$. Since $\mathbf{y} \notin W_1 = \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$, it follows that $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{y}\}$ is a linear independent set of n vectors, and hence a basis for V. Thus, for any $\mathbf{v} \in V$, there exists scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$\mathbf{v} = c_1\mathbf{x}_1 + \ldots + c_{n-1}\mathbf{x}_{n-1} + c_n\mathbf{y}$$

Since $c_1\mathbf{x}_1 + \ldots + c_{n-1}\mathbf{x}_{n-1} \in W_1$ and $c_n\mathbf{y} \in W_2$, it follows that $\mathbf{v} \in W_1 + W_2$ and thus $V = W_1 + W_2$.

Now recall the following formula:

$$\begin{aligned}
\dim(W_1 + W_2) &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \\
\dim(W_1 \cap W_2) &= \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \\
\dim(W_1 \cap W_2) &= (n-1) + (n-1) - n \\
\dim(W_1 \cap W_2) &= n - 2
\end{aligned}$$

5. (a) Define what it means for a function $T : V \to W$ to be a *linear transformation.* [2 marks]

A function $T : V \to W$ is a linear transformation if:
(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
(ii) $T(a\mathbf{v}) = aT(\mathbf{v})$ for all $a \in \mathbb{R}$ and $\mathbf{v} \in V$.

5. (b) Suppose $\mathbf{a} \in \mathbb{R}^n$ be a fixed vector. Show $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T\mathbf{x} = \mathbf{a}^{\mathrm{T}}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ is a linear transformation. [4 marks]

Suppose $\mathbf{a} = (a_1, \dots, a_n)$. Pick any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. We show that $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

$$\begin{aligned}
T(c\mathbf{u} + d\mathbf{v}) &= \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} cu_1 + dv_1 \\ \vdots \\ cu_n + dv_n \end{pmatrix} \\
&= a_1(cu_1 + dv_1) + \cdots + a_n(cu_n + dv_n) \\
&= c\big(a_1u_1 + \cdots + a_nu_n\big) + d\big(a_1v_1 + \cdots + a_nv_n\big) \\
&= c\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + d\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= cT(\mathbf{u}) + dT(\mathbf{v})
\end{aligned}$$

5. (c) Suppose you are given that the function $T : P_2(\mathbb{R}) \to \mathbb{R}$ satisfies

$$T(1+x) = 1,\ T(-1+x+x^2) = -1, \text{and } T(1-x-x^2) = -1$$

Choose one of the following (i) T must be linear, (ii) T might be linear, or (iii) T cannot be linear, and justify your choice. [4 marks]

The correct option is (iii), T cannot be linear. Note that $T(-1+x+x^2) = T\big(-1\cdot(-1+x+x^2)\big) = T(1-x-x^2) = -1 \Rightarrow -T(-1+x+x^2) \neq T\big(-(-1+x+x^2)\big)$. Thus, T does not satisfy property (ii) from part (a).

6. Determine if each statement below is True or False and indicate your answer by circling one of the options. No explanation is necessary. Each correct answer is worth 2 marks and each incorrect answer is worth 0 marks.

(i) Let V and W be vector spaces. For $\mathbf{x}, \mathbf{y} \in V$, the *line segment* joining $\mathbf{x}$ and $\mathbf{y}$ in V is $L = \{\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \mid 0 \leq t \leq 1\}$. If $T : V \to W$ is a linear transformation, then $T(L)$ is a line segment in W.

(**True**) (**False**)

False. This is not necessarily true. For instance, if T is the zero transformation, then $T(L) = \{0\}$ which is just a point.

(ii) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is a list of linearly independent vectors in a vector space V, then $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} \cap \operatorname{span}\{\mathbf{x}_3, \mathbf{x}_4\} = \{\mathbf{0}\}$.

(**True**) (**False**)

True. Suppose otherwise and let $\mathbf{v} \neq \mathbf{0}$ be such that $\mathbf{v} \in \operatorname{span}\{x_1, x_2\} \cap \operatorname{span}\{x_3, x_4\}$. Then there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ where not all the c_i are zero and $\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_3\mathbf{x}_3 + c_4\mathbf{x}_4$. But then $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 - c_3\mathbf{x}_3 - c_4\mathbf{x}_4 = 0$, which is a contradiction since $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a linearly independent set.

(iii) If $\mathbf{x}_1, \mathbf{x}_2$ are vectors in a vector space V, then $\operatorname{span}\{\mathbf{x}_1\} + \operatorname{span}\{\mathbf{x}_2\} = \operatorname{span}\{\mathbf{x}_1 + \mathbf{x}_2\}$.

(**True**) (**False**)

False. Let $V = \mathbb{R}^2$ and $\mathbf{x}_1 = (1, 0)$ and $\mathbf{x}_2 = (0, 1)$. Then $\operatorname{span}\{\mathbf{x}_1\} + \operatorname{span}\{\mathbf{x}_2\} = \mathbb{R}^2$ but $\operatorname{span}\{\mathbf{x}_1 + \mathbf{x}_2\} = \operatorname{span}\{(1, 1)\} \neq \mathbb{R}^2$.

(iv) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for a vector space V and $S_1, S_2, \ldots, S_n$ are subsets of V such that $\mathbf{x}_i \in S_i$ for each $i = 1, 2, \ldots, n$, then $V = \operatorname{span} S_1 + \operatorname{span} S_2 + \cdots + \operatorname{span} S_n$.

(**True**) (**False**)

True. Note that $\operatorname{span}\{\mathbf{x}_i\} \subset \operatorname{span} S_i$ and $\operatorname{span}\{\mathbf{x}_1\} + \cdots + \operatorname{span}\{\mathbf{x}_n\} = V$

(v) If W_1, W_2, W_3 are subspaces of a vector space V such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$ then $W_2 = W_3$.

(**True**) (**False**)

False. Let $V = \mathbb{R}^2$, $W_1 = \operatorname{span}\{(1, 0)\}$, $W_2 = \operatorname{span}\{(0, 1)\}$ and $W_3 = \operatorname{span}\{(1, 1)\}$. Notice that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$ but $W_2 \neq W_3$

(vi) Let k be a positive integer. If W_1, W_2 are subspaces of a finite dimensional vector space V such that $\dim W_1 + \dim W_2 \geq \dim V + k$, then $W_1 \cap W_2$ contains at least k linearly independent vectors.

(**True**) (**False**)

True. Recall $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \geq k$

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