Chapter 7: Sampling Distributions and the Central Limit Theorem

7.1 **a. – c.** Answers vary.
d. The histogram exhibits a mound shape. The sample mean should be close to $3.5 = \mu$
e. The standard deviation should be close to $\sigma/\sqrt{3} = 1.708/\sqrt{3} = .9860$.
f. Very similar pictures.

7.2 **a.** $P(\overline{Y} = 2) = P(W = 6) = p(4, 1, 1) + p(1, 4, 1) + p(1, 1, 4) + p(3, 2, 1) + p(3, 1, 2)$
$\qquad\qquad\qquad\qquad = p(2, 3, 1) + p(2, 1, 3) + p(1, 3, 2) + p(1, 2, 3) + p(2, 2, 2) = \frac{10}{216}$.
b. Answers vary, but the relative frequency should be fairly close.
c. The relative frequency should be even closer than what was observed in part b.

7.3 **a.** The histogram should be similar in shape, but this histogram has a smaller spread.
b. Answers vary.
c. The normal curve should approximate the histogram fairly well.

7.4 **a.** The histogram has a right–skewed shape. It appears to follow $p(y) = y/21$, $y = 1, \ldots, 6$.
b. From the Stat Report window, $\mu = 2.667$, $\sigma = 1.491$.
c. Answers vary.
d.   i. It has a right–skewed shape.   ii. The mean is larger, but the std. dev. is smaller.
e.   i. sample mean $= 2.667$, sample std. dev $= 1.491/\sqrt{12} = .4304$.
ii. The histogram is closely mound shaped.
iii. Very close indeed.

7.5 **a.** Answers vary.
b. Answers vary, but the means are probably not equal.
c. The sample mean values cluster around the population mean.
d. The theoretical standard deviation for the sample mean is $6.03/\sqrt{5} = 2.6967$.
e. The histogram has a mound shape.
f. Yes.

7.6 The larger the sample size, the smaller the spread of the histogram. The normal curves approximate the histograms equally well.

7.7 **a. – b.** Answers vary.
c. The mean should be close to the population variance
d. The sampling distribution is not mound–shaped for this case.
e. The theoretical density should fit well.
f. Yes, because the chi–square density is right–skewed.

7.8 **a.** $\sigma^2 = (6.03)^2 = 36.3609$.
b. The two histograms have similar shapes, but the histogram generated from the smaller sample size exhibits a greater spread. The means are similar (and close to the value found in part a). The theoretical density should fit well in both cases.
c. The histogram generated with n = 50 exhibits a mound shape. Here, the theoretical density is chi–square with $v = 50 - 1 = 49$ degrees of freedom (a large value).

7.9   **a.** $P(|\overline{Y} - \mu| \le .3) = P(-1.2 \le Z \le 1.2) = .7698$.

    b. $P(|\overline{Y} - \mu| \le .3) = P(-.3\sqrt{n} \le Z \le .3\sqrt{n}) = 1 - 2P(Z > .3\sqrt{n})$. For $n = 25, 36, 69$, and 64, the probabilities are (respectively) .8664, .9284, .9642, and .9836.

    c. The probabilities increase with n, which is intuitive since the variance of $\overline{Y}$ decreases with n.

    d. Yes, these results are consistent since the probability was less than .95 for values of n less than 43.

7.10   **a.** $P(|\overline{Y} - \mu| \le .3) = P(-.15\sqrt{n} \le Z \le .15\sqrt{n}) = 1 - 2P(Z > .15\sqrt{n})$. For $n = 9$, the probability is .3472 (a smaller value).

    b.   For $n = 25$: $P(|\overline{Y} - \mu| \le .3) = 1 - 2P(Z > .75) = .5468$

           For $n = 36$: $P(|\overline{Y} - \mu| \le .3) = 1 - 2P(Z > .9) = .6318$

           For $n = 49$: $P(|\overline{Y} - \mu| \le .3) = 1 - 2P(Z > 1.05) = .7062$

           For $n = 64$: $P(|\overline{Y} - \mu| \le .3) = 1 - 2P(Z > 1.2) = .7698$

    c. The probabilities increase with n.

    d. The probabilities are smaller with a larger standard deviation (more diffuse density).

7.11 $P(|\overline{Y} - \mu| \le 2) = P(-1.5 \le Z \le 1.5) = 1 - 2P(Z > 1.5) = 1 - 2(.0668) = .8664$.

7.12   From Ex. 7.11, we require $P(|\overline{Y} - \mu| \le 1) = P(-.25\sqrt{n} \le Z \le .25\sqrt{n}) = .90$. This will be solved by taking $.25\sqrt{n} = 1.645$, so $n = 43.296$. Hence, sample 44 trees.

7.13   Similar to Ex. 7.11: $P(|\overline{Y} - \mu| \le .5) = P(-2.5 \le Z \le 2.5) = .9876$.

7.14   Similar to Ex. 7.12: we require $P(|\overline{Y} - \mu| \le .5) = P(-\frac{.5}{\sqrt{.4}}\sqrt{n} \le Z \le \frac{.5}{\sqrt{.4}}\sqrt{n}) = .95$. Thus, $\frac{.5}{\sqrt{.4}}\sqrt{n} = 1.96$ so that $n = 6.15$. Hence, run 7 tests.

7.15   Using Theorems 6.3 and 7.1:

    a. $E(\overline{X} - \overline{Y}) = \mu_1 - \mu_2$.

    b. $V(\overline{X} - \overline{Y}) = \sigma_1^2/m + \sigma_2^2/n$.

    c. It is required that $P(|\overline{X} - \overline{Y} - (\mu_1 - \mu_2)| \le 1) = .95$. Using the result in part b for standardization with $n = m$, $\sigma_1^2 = 2$, and $\sigma_2^2 = 2.5$, we obtain $n = 17.29$. Thus, the two sample sizes should be at least 18.

7.16   Following the result in Ex. 7.15 and since the two population means are equal, we find
$$P(\overline{X}_A - \overline{Y}_B \ge 1) = P\left(\frac{\overline{X}_A - \overline{Y}_B}{\sqrt{\frac{4}{10} + \frac{8}{10}}} \ge \frac{1}{\sqrt{\frac{4}{10} + \frac{8}{10}}}\right) = P(Z \ge 2.89) = .0019.$$

7.17 $P\left(\sum_{i=1}^{6} Z_i^2 \le 6\right) = .57681$.

7.18 $P(S^2 \ge 3) = P(9S^2 \ge 27) = .0014$.

7.19   Given that $s^2 = .065$ and $n = 10$, suppose $\sigma^2 = .04$. The probability of observing a value
of s^2 that is as extreme or more so is given by
$$P(S^2 \geq .065) = P(9S^2/.04 \geq 9(.065)/.04) = P(9S^2/.04 \geq 14.925) = .10$$
Thus, it is fairly unlikely, so this casts some doubt that $\sigma^2 = .04$.

7.20   **a.** Using the fact that the chi–square distribution is a special case of the gamma
distribution, $E(U) = \nu$, $V(U) = 2\nu$.
b. Using Theorem 7.3 and the result from part a:
$$E(S^2) = \tfrac{\sigma^2}{n-1} E(\tfrac{n-1}{\sigma^2} S^2) = \tfrac{\sigma^2}{n-1}(n-1) = \sigma^2.$$
$$V(S^2) = \left(\tfrac{\sigma^2}{n-1}\right)^2 V(\tfrac{n-1}{\sigma^2} S^2) = \left(\tfrac{\sigma^2}{n-1}\right)^2 [2(n-1)] = 2\sigma^4/(n-1).$$

7.21   These values can be found by using percentiles from the chi–square distribution.
With $\sigma^2 = 1.4$ and $n = 20$, $\tfrac{19}{1.4} S^2$ has a chi–square distribution with 19 degrees of freedom.
a. $P(S^2 \leq b) = P(\tfrac{n-1}{\sigma^2} S^2 \leq \tfrac{n-1}{\sigma^2} b) = P(\tfrac{19}{1.4} S^2 \leq \tfrac{19}{1.4} b) = .975$. It must be true that
$\tfrac{19}{1.4} b = 32.8523$, the 97.5%-tile of this chi–square distribution, and so $b = 2.42$.

b.   Similarly, $P(S^2 \geq a) = P(\tfrac{n-1}{\sigma^2} S^2 \geq \tfrac{n-1}{\sigma^2} a) = .974$. Thus, $\tfrac{19}{1.4} a = 8.96055$, the 2.5%-tile
of this chi–square distribution, and so $a = .656$.

c. $P(a \leq S^2 \leq b) = .95$.

7.22   **a.** The corresponding gamma densities with parameters (α, β) are (5, 2), (20, 2), (40, 2),
respectively.
b. The chi–square densities become more symmetric with larger values of ν.
c. They are the same.
d. Not surprising, given the answer to part b.

7.23   **a.** The three probabilities are found to be .44049, .47026, and .47898, respectively.
b. As the degrees of freedom increase, so do the probabilities.
c. Since the density is becoming more symmetric, the probability is approaching .5.

7.24   **a.** .05097
b. .05097
c. $1 - 2(.05097) = .8806$.
d. The t–distribution with 5 degrees of freedom exhibits greater variability.

7.25   **a.** Using Table 5, $t_{.10} = 1.476$. Using the applet, $t_{.10} = 1.47588$.
b. The value $t_{.10}$ is the 90th percentile/quantile.
c. The values are 1.31042, 1.29582, 1.28865, respectively.
d. The t–distribution exhibits greater variability than the standard normal, so the
percentiles are more extreme than $z_{.10}$.
e. As the degrees of freedom increase, the t–distribution approaches the standard normal
distribution.

7.26   From Definition 7.2,

$P(g_1 \le \overline{Y} - \mu \le g_2) = P(\frac{\sqrt{n}g_1}{S} \le T \le \frac{\sqrt{n}g_2}{S}) = .90$. Thus, it must be true that $\frac{\sqrt{n}g_1}{S} = -t_{.05}$ and

$\frac{\sqrt{n}g_2}{S} = t_{.05}$. Thus, with $n = 9$ and $t_{.05} = 1.86$, $g_1 = -\frac{1.86}{3}S$, $g_2 = \frac{1.86}{3}S$.

7.27   By Definition 7.3, S_1^2 / S_2^2 has an F–distribution with 5 numerator and 9 denominator degrees of freedom. Then,

a. $P(S_1^2 / S_2^2 > 2) = .17271$.

b. $P(S_1^2 / S_2^2 < .5) = .23041$.

c. $P(S_1^2 / S_2^2 > 2) + P(S_1^2 / S_2^2 < .5) = .17271 + .23041 = .40312$.

7.28   **a.** Using Table 7, $F_{.025} = 6.23$.
    b. The value $F_{.025}$ is the 97.5%-tile/quantile.
    c. Using the applet, $F_{.975} = .10873$.
    d. Using the applet, $F_{.025} = 9.19731$.
    e. The relationship is $1/.10873 \approx 9.19731$.

7.29   By Definition 7.3, $Y = (W_1 / \nu_1) \div (W_2 / \nu_2)$ has an F distribution with ν_1 numerator and ν_2 denominator degrees of freedom. Therefore, $U = 1/Y = (W_2 / \nu_2) \div (W_1 / \nu_1)$ has an F distribution with ν_2 numerator and ν_1 denominator degrees of freedom.

7.30   **a.** $E(Z) = 0$, $E(Z^2) = V(Z) + [E(Z)]^2 = 1$.
    b. This is very similar to Ex. 5.86, part a. Using that result, it is clear that
        i. $E(T) = 0$
        ii. $V(T) = E(T^2) = \nu E(Z^2/Y) = \nu/(\nu-2)$, $\nu > 2$.

7.31   **a.** The values for $F_{.01}$ are 5.99, 4.89, 4.02, 3.65, 3.48, and 3.32, respectively.
    b. The values for $F_{.01}$ are decreasing as the denominator degrees of freedom increase.
    c. From Table 6, $\chi_{.01}^2 = 13.2767$.
    d. $13.2767/3.32 \approx 4$. This follows from the fact that the F ratio as given in Definition 7.3 converges to W_1/ν_1 as ν_2 increases without bound.

7.32   **a.** Using the applet, $t_{.05} = 2.01505$.
    b. $P(T^2 > t_{.05}^2) = P(T > t_{.05}) + P(T < -t_{.05}) = .10$.
    c. Using the applet, $F_{.10} = 4.06042$.
    d. $F_{.10} = 4.06042 = (2.01505)^2 = t_{.05}^2$.
    e. Let $F = T^2$. Then, $.10 = P(F > F_{.10}) = P(T^2 > F_{.10}) = P(T < -\sqrt{F_{.10}}) + P(T > \sqrt{F_{.10}})$.
    This must be equal to the expression given in part b.

7.33   Define $T = Z / \sqrt{W / \nu}$ as in Definition 7.2. Then, $T^2 = Z^2 /(W / \nu)$. Since Z^2 has a chi–square distribution with 1 degree of freedom, and Z and W are independent, T^2 has an F distribution with 1 numerator and ν denominator degrees of freedom.

7.34   This exercise is very similar to Ex. 5.86, part b. Using that result, is can be shown that

    a. $E(F) = \frac{v_2}{v_1} E(W_1)E(W_2^{-1}) = \frac{v_2}{v_1} \times \left(\frac{v_1}{v_2-2}\right) = v_2/(v_2 - 2)$, $v_2 > 2$.

    b. $V(F) = E(F^2) - [E(F)]^2 = \left(\frac{v_2}{v_1}\right)^2 E(W_1^2)E(W_2^{-2}) - \left(\frac{v_2}{v_2-2}\right)^2$

$$= \left(\tfrac{v_2}{v_1}\right)^2 v_1(v_1+2)\tfrac{1}{(v_2-2)(v_2-4)} - \left(\tfrac{v_2}{v_2-2}\right)^2$$

$$= \left[2v_2^2(v_1 + v_2 - 2)\right] / \left[v_1(v_2-2)^2(v_2-4)\right], \; v_2 > 4.$$

7.35   Using the result from Ex. 7.34,

    a. $E(F) = 70/(70–2) = 1.029$.

    b. $V(F) = [2(70)^2(118)]/[50(68)^2(66)] = .076$

    c.   Note that the value 3 is $(3 - 1.029)/\sqrt{.076} = 7.15$ standard deviations above this mean. This represents and unlikely value.

7.36   We are given that $\sigma_1^2 = 2\sigma_2^2$. Thus, $\sigma_1^2/\sigma_2^2 = 2$ and $S_1^2/(2S_2^2)$ has an F distribution with $10 - 1 = 9$ numerator and $8 - 1 = 7$ denominator degrees of freedom.

    a.   We have $P(S_1^2/S_2^2 \le b) = P(S_1^2/(2S_2^2) \le b/2) = .95$. It must be that $b/2 = F_{.05} = 3.68$, so $b = 7.36$.

    b.   Similarly, $a/2 = F_{.95}$, but we must use the relation $a/2 = 1/F_{.05}$, where $F_{.05}$ is the 95th percentile of the F distribution with 7 numerator and 9 denominator degrees of freedom (see Ex. 7.29). Thus, with $F_{.05} = 3.29 = .304$, $a/2 = 2/3.29 = .608$.

    c. $P(.608 \le S_1^2/S_2^2 \le 7.36) = .90$.

7.37   **a.** By Theorem 7.2, χ^2 with 5 degrees of freedom.

    b. By Theorem 7.3, χ^2 with 4 degrees of freedom (recall that $\sigma^2 = 1$).

    c. Since Y_6^2 is distributed as χ^2 with 1 degrees of freedom, and $\sum_{i=1}^{5}(Y_i - \bar{Y})^2$ and Y_6^2 are independent, the distribution of $W + U$ is χ^2 with $4 + 1 = 5$ degrees of freedom.

7.38   **a.** By Definition 7.2, t–distribution with 5 degrees of freedom.

    b. By Definition 7.2, t–distribution with 4 degrees of freedom.

    c. $\bar{Y}$ follows a normal distribution with $\mu = 0$, $\sigma^2 = 1/5$. So, $\sqrt{5}\bar{Y}$ is standard normal and $\left(\sqrt{5}\bar{Y}\right)^2$ is chi–square with 1 degree of freedom. Therefore, $5\bar{Y}^2 + Y_6^2$ has a chi–square distribution with 2 degrees of freedom (the two random variables are independent). Now, the quotient

$$2(5\bar{Y}^2 + Y_6^2)/U = [(5\bar{Y}^2 + Y_6^2)/2] \div [U/4]$$

has an F-distribution with 2 numerator and 4 denominator degrees of freedom.

Note: we have assumed that $\bar{Y}$ and U are independent (as in Theorem 7.3).

7.39   **a.** Note that for $i = 1, 2, \ldots, k$, the $\overline{X}_i$ have independent a normal distributions with mean μ_i and variance σ/n_i. Since $\hat{\theta}$, a linear combination of independent normal random variables, by Theorem 6.3 $\hat{\theta}$ has a normal distribution with mean given by

$$E(\hat{\theta}) = E(c_1\overline{X}_1 + \ldots + c_k\overline{X}_k) = \sum\nolimits_{i=1}^{k} c_i\mu_i$$

and variance

$$V(\hat{\theta}) = V(c_1\overline{X}_1 + \ldots + c_k\overline{X}_k) = \sigma^2 \sum\nolimits_{i=1}^{k} c_i^2 / n_i^2 \;.$$

b. For $i = 1, 2, \ldots, k$, $(n_i - 1)S_i^2 / \sigma^2$ follows a chi–square distribution with $n_i - 1$ degrees of freedom. In addition, since the S_i^2 are independent,

$$\frac{SSE}{\sigma^2} = \sum\nolimits_{i=1}^{k} (n_i - 1)S_i^2 / \sigma^2$$

is a sum of independent chi–square variables. Thus, the above quantity is also distributed as chi–square with degrees of freedom $\sum\nolimits_{i=1}^{k} (n_i - 1) = \sum\nolimits_{i=1}^{k} n_i - k.$

c. From part a, we have that $\dfrac{\hat{\theta} - \theta}{\sigma\sqrt{\sum\nolimits_{i=1}^{k} c_i^2 / n_i^2}}$

has a standard normal distribution. Therefore, by Definition 7.2, a random variable constructed as

$$\frac{\hat{\theta} - \theta}{\sigma\sqrt{\sum\nolimits_{i=1}^{k} c_i^2 / n_i^2}} \Bigg/ \sqrt{\frac{\sum\nolimits_{i=1}^{k} (n_i - 1)S_i^2 / \sigma^2}{\sum\nolimits_{i=1}^{k} n_i - k}} = \frac{\hat{\theta} - \theta}{\sqrt{\mathrm{MSE}\sum\nolimits_{i=1}^{k} c_i^2 / n_i^2}}$$

has the t–distribution with $\sum\nolimits_{i=1}^{k} n_i - k$ degrees of freedom. Here, we are assuming that $\hat{\theta}$ and SSE are independent (similar to $\overline{Y}$ and S^2 as in Theorem 7.3).

7.40   **a.** Both histograms are centered about the mean M = 16.50, but the variation is larger for sample means of size 1.
b. For sample means of size 1, the histogram closely resembles the population. For sample means of size 3, the histogram resembles the shape of the population but the variability is smaller.
c. Yes, the means are very close and the standard deviations are related by a scale of $\sqrt{3}$.
d. The normal densities approximate the histograms fairly well.
e. The normal density has the best approximation for the sample size of 25.

7.41   **a.** For sample means of size 1, the histogram closely resembles the population. For sample means of size 3, the histogram resembles that of a multi–modal population. The means and standard deviations follow the result of Ex. 7.40 (c), but the normal densities are not appropriate for either case. The normal density is better with $n = 10$, but it is best with $n = 25$.
b. For the "U–shaped population," the probability is greatest in the two extremes in the distribution.

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7.42   Let $\overline{Y}$ denote the sample mean strength of 100 random selected pieces of glass. Thus, the quantity $(\overline{Y} - 14.5)/.2$ has an approximate standard normal distribution.

  a. $P(\overline{Y} > 14) \approx P(Z > 2.5) = .0062$.
  b.   We have that $P(-1.96 < Z < 1.96) = .95$. So, denoting the required interval as (a, b) such that $P(a < \overline{Y} < b) = .95$, we have that $-1.96 = (a - 14)/.2$ and $1.96 = (b - 14)/.2$. Thus, $a = 13.608$, $b = 14.392$.

7.43   Let $\overline{Y}$ denote the mean height and $\sigma = 2.5$ inches. By the Central Limit Theorem,
$$P(|\overline{Y} - \mu| \le .5) = P(-.5 \le \overline{Y} - \mu \le .5) \approx P(\tfrac{-.5(10)}{2.5} \le Z \le \tfrac{-5(10)}{2.5}) = P(-2 \le Z \le 2) = .9544.$$

7.44   Following Ex. 7.43, we now require
$$P(|\overline{Y} - \mu| \le .4) = P(-.4 \le \overline{Y} - \mu \le .4) \approx P(\tfrac{-.5\sqrt{n}}{2.5} \le Z \le \tfrac{.5\sqrt{n}}{2.5}) = .95.$$
Thus, it must be true that $\frac{.5\sqrt{n}}{2.5} = 1.96$, or $n = 150.0625$. So, 151 men should be sampled.

7.45   Let $\overline{Y}$ denote the mean wage calculated from a sample of 64 workers. Then,
$$P(\overline{Y} \le 6.90) \approx P(Z \le \tfrac{\sqrt{8}(6.90-7.00)}{.5}) = P(Z \le -1.60) = .0548.$$

7.46   With $n = 40$ and $\sigma \approx (\text{range})/4 = (8 - 5)/4 = .75$, the approximation is
$$P(|\overline{Y} - \mu| \le .2) \approx P(|Z| \le \tfrac{\sqrt{40}(.2)}{.75}) = P(-1.69 \le Z \le 1.69) = .9090.$$

7.47   (Similar to Ex. 7.44). Following Ex. 7.47, we require
$$P(|\overline{Y} - \mu| \le .1) \approx P(|Z| \le \tfrac{\sqrt{n}(.1)}{.75}) = .90.$$
Thus, we have that $\frac{\sqrt{n}(.1)}{.75} = 1.645$, so $n = 152.21$. Therefore, 153 core samples should be taken.

7.48   **a.** Although the population is not normally distributed, with n $= 35$ the sampling distribution of $\overline{Y}$ will be approximately normal. The probability of interest is
$$P(|\overline{Y} - \mu| \le 1) = P(-1 \le \overline{Y} - \mu \le 1).$$
In order to evaluate this probability, the population standard deviation σ is needed. Since it is unknown, we will estimate its value by using the sample standard deviation $s = 12$ so that the estimated standard deviation of $\overline{Y}$ is $12/\sqrt{35} = 2.028$. Thus,
$$P(|\overline{Y} - \mu| \le 1) = P(-1 \le \overline{Y} - \mu \le 1) \approx P(-\tfrac{1}{2.028} \le Z \le \tfrac{1}{2.028}) = P(-.49 \le Z \le .49) = .3758.$$

  b. No, the measurements are still only *estimates*.

7.49   With $\mu = 1.4$ hours, $\sigma = .7$ hour, let $\overline{Y} = $ mean service time for $n = 50$ cars. Then,
$$P(\overline{Y} > 1.6) \approx P(Z > \tfrac{\sqrt{50}(16-14)}{.7}) = P(Z > 2.02) = .0217.$$

7.50   We have $P(|\overline{Y} - \mu| < 1) = P(|Z| < \tfrac{1}{\sigma/\sqrt{n}}) = P(-1 < Z < 1) = .6826.$

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7.51   We require $P(|\overline{Y} - \mu| < 1) = P(|Z| < \frac{1}{\sigma/\sqrt{n}}) = P(-\frac{1}{10/\sqrt{n}} < Z < \frac{1}{10/\sqrt{n}}) = .99$. Thus it must be

true that $\frac{1}{10/\sqrt{n}} = z_{.005} = 2.576$. So, $n = 663.57$, or 664 measurements should be taken.

7.52   Let $\overline{Y}$ denote the average resistance for the 25 resistors. With $\mu = 200$ and $\sigma = 10$ ohms,
   a. $P(199 \le \overline{Y} \le 202) \approx P(-.5 \le Z \le 1) = .5328$.
   b.   Let $X = $ *total resistance* of the 25 resistors. Then,
$$P(X \le 5100) = P(\overline{Y} \le 204) \approx P(Z \le 2) = .9772.$$

7.53   **a.** With these given values for μ and σ, note that the value 0 has a z–score of $(0 - 12)/9 = 1.33$. This is not considered extreme, and yet this is the smallest possible value for CO concentration in air. So, a normal distribution is not possible for these measurements.
   b. $\overline{Y}$ is approximately normal: $P(\overline{Y} > 14) \approx P(Z > \frac{\sqrt{100}(14-12)}{9}) = P(Z > 2.22) = .0132$.

7.54 $P(\overline{Y} < 1.3) \approx P(Z < \frac{\sqrt{25}(1.3-1.4)}{.05}) = P(Z < -10) \approx 0$, so it is very unlikely.

7.55   **a.**   i.   We assume that we have a random sample
            ii.   Note that the standard deviation for the sample mean is $.8/\sqrt{30} = .146$. The endpoints of the interval (1, 5) are substantially beyond 3 standard deviations from the mean. Thus, the probability is approximately 1.

   b. Let Y_i denote the downtime for day i, $i = 1, 2, \ldots, 30$. Then,
$$P(\sum_{i=1}^{30} Y_i < 115) = P(\overline{Y} < 3.833) \approx P(Z < \frac{\sqrt{30}(3.833-4)}{.8}) = P(Z < -1.14) = .1271.$$

7.56   Let Y_i denote the volume for sample i, $i = 1, 2, \ldots, 30$. We require
$$P(\sum_{i=1}^{50} Y_i > 200) = P(\overline{Y} - \mu < \frac{200}{50} - \mu) \approx P(Z < \frac{\sqrt{50}(4-\mu)}{2}) = .95.$$
   Thus, $\frac{\sqrt{50}(4-\mu)}{2} = -z_{.05} = -1.645$, and then $\mu = 4.47$.

7.57   Let Y_i denote the lifetime of the ith lamp, $i = 1, 2, \ldots, 25$, and the mean and standard deviation are given as 50 and 4, respectively. The random variable of interest is $\sum_{i=1}^{25} Y_i$, which is the lifetime of the lamp system. So,
$$P(\sum_{i=1}^{25} Y_i \ge 1300) = P(\overline{Y} \ge 52) \approx P(Z \ge \frac{\sqrt{25}(52-50)}{4}) = P(Z \ge 2.5) = .0062.$$

7.58   For $W_i = X_i - Y_i$, we have that $E(W_i) = E(X_i) - E(Y_i) = \mu_1 - \mu_2$ and $V(W_i) = V(X_i) - V(Y_i) = \sigma_1^2 + \sigma_2^2$ since X_i and Y_i are independent. Thus, $\overline{W} = \frac{1}{n}\sum_{i=1}^{n} W_i = \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i) = \overline{X} - \overline{Y}$
   so $E(\overline{W}) = \mu_1 - \mu_2$, and $V(\overline{W}) = (\sigma_1^2 + \sigma_2^2)/n$. Thus, since the W_i are independent,
$$U_n = \frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}} = \frac{\overline{W} - E(\overline{W})}{\sqrt{V(\overline{W})}}$$

   satisfies the conditions of Theorem 7.4 and has a limiting standard normal distribution.

7.59 Using the result of Ex. 7.58, we have that $n = 50$, $\sigma_1 = \sigma_2 = 2$ and $\mu_1 = \mu_2$. Let $\overline{X}$ denote the mean time for operator A and let $\overline{Y}$ denote the mean time for operator B (both measured in seconds) Then, operator A will get the job if $\overline{X} - \overline{Y} < -1$. This probability is

$$P(\overline{X} - \overline{Y} < -1) \approx P\left(Z < \tfrac{-1}{\sqrt{(4+4)/50}}\right) = P(Z < -2.5) = .0062.$$

7.60 Extending the result from Ex. 7.58, let $\overline{X}$ denote the mean measurement for soil A and $\overline{Y}$ the mean measurement for soil B. Then, we require

$$P\left[\left|\overline{X} - \overline{Y} - (\mu_1 - \mu_2)\right| \le .05\right] \approx P\left[|Z| \le \tfrac{.05}{\sqrt{\frac{.01}{50} + \frac{.02}{100}}}\right] = P\left[|Z| \le 2.5\right] = .9876.$$

7.61 It is necessary to have

$$P\left[\left|\overline{X} - \overline{Y} - (\mu_1 - \mu_2)\right| \le .04\right] \approx P\left[|Z| \le \tfrac{.05}{\sqrt{\frac{.01}{n} + \frac{.02}{n}}}\right] = P\left[|Z| \le \tfrac{.05\sqrt{n}}{\sqrt{.01+.02}}\right] = .90.$$

Thus, $\tfrac{.05\sqrt{n}}{\sqrt{.01+.02}} = z_{.05} = 1.645$, so $n = 50.74$. Each sample size must be at least $n = 51$.

7.62 Let Y_i represent the time required to process the ith person's order, $i = 1, 2, \ldots, 100$. We have that $\mu = 2.5$ minutes and $\sigma = 2$ minutes. So, since 4 hours = 240 minutes,

$$P(\textstyle\sum_{i=1}^{100} Y_i > 240) = P(\overline{Y} > 2.4) \approx P(Z > \tfrac{\sqrt{100}(2.4-2.5)}{2}) = P(Z > -.5) = .6915.$$

7.63 Following Ex. 7.62, consider the relationship $P(\sum_{i=1}^{n} Y_i < 120) = .1$ as a function of n:

Then, $P(\sum_{i=1}^{n} Y_i < 120) = P(\overline{Y} < 120/n) \approx P(Z < \tfrac{\sqrt{n}(120/n-2.5)}{2}) = .1$. So, we have that

$$\tfrac{\sqrt{n}(120/n-2.5)}{2} = -z_{.10} = -1.282.$$

Solving this nonlinear relationship (for example, this can be expressed as a quadratic relation in $\sqrt{n}$), we find that $n = 55.65$ so we should take a sample of 56 customers.

7.64 **a.** two.
b. exact: .27353, normal approximation: .27014
c. this is the continuity correction

7.65 **a.** exact: .91854, normal approximation: .86396.
b. the mass function does not resemble a mound–shaped distribution (n is not large here).

7.66 Since $P(|Y - E(Y)| \le 1) = P(E(Y) - 1 \le Y \le E(Y) + 1) = P(np - 1 \le Y \le np + 1)$, if $n = 20$ and $p = .1$, $P(1 \le Y \le 3) = .74547$. Normal Approximation: .73645.

7.67 **a.** $n = 5$ (exact: ..99968, approximate: .95319), $n = 10$ (exact: ..99363, approximate: .97312), $n = 15$ (exact: .98194, approximate: .97613), $n = 20$ (exact: .96786, approximate: .96886).
b. The binomial histograms appear more mound shaped with increasing values of n. The exact and approximate probabilities are closer for larger n values.
c. rule of thumb: $n > 9(.8/.2) = 36$, which is conservative since $n = 20$ is quite good.

7.68   **a.** The probability of interest is $P(Y \geq 29)$, where Y has a binomial distribution with $n = 50$ and $p = .48$. Exact: .10135, approximate: .10137.
b. The two probabilities are close. With $n = 50$ and $p = .48$, the binomial histogram is mound shaped.

7.69   **a.** Probably not, since current residents would have learned their lesson.
b. (Answers vary). With $b = 32$, we have exact: ..03268, approximate: .03289.

7.70   **a.** $p + 3\sqrt{pq/n} < 1 \Leftrightarrow 3\sqrt{pq/n} < q \Leftrightarrow 9pq/n < q^2 \Leftrightarrow 9p/q < n$.
b. $p - 3\sqrt{pq/n} < 1 \Leftrightarrow 3\sqrt{pq/n} < p \Leftrightarrow 9pq/n < p^2 \Leftrightarrow 9q/p < n$.
c. Parts **a** and **b** imply that $n > 9\max\left(\frac{p}{q}, \frac{q}{p}\right)$, and it is trivial to show that
$\max\left(\frac{p}{q}, \frac{q}{p}\right) = \frac{\max(p,q)}{\min(p,q)}$ (consider the three cases where $p = q$, $p > q$, $p < q$.

7.71   **a.** $n > 9$.
b. $n > 14$, $n > 14$, $n > 36$, $n > 36$, $n > 891$, $n > 8991$.

7.72   Using the normal approximation, $P(Y \geq 15) \approx P(Z \geq \frac{14.5-10}{\sqrt{100(.1)(.9)}}) = P(Z \geq 1.5) = .0668$.

7.73   Let $Y = \#$ the show up for a flight. Then, Y is binomial with $n = 160$ and $p = .95$. The probability of interest is $P(Y \leq 155)$, which gives the probability that the airline will be able to accommodate all passengers. Using the normal approximation, this is
$$P(Y \leq 155) \approx P(Z \leq \frac{155.5-160(.95)}{\sqrt{160(.95)(.05)}}) = P(Z \leq 1.27) = .8980.$$

7.74   **a.** Note that calculating the exact probability is easier: with $n = 1500$, $p = 1/410$,
$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - (409/410)^{1500} = .9504.$$
b. Here, $n = 1500$, $p = 1/64$. So,
$$P(Y > 30) \approx P(Z > \frac{30.5-23.4375}{\sqrt{23.0713}}) = P(Z > 1.47) = .0708.$$
c. The value $y = 30$ is $(30 - 23.4375)/\sqrt{23.0713} = 1.37$ standard deviations above the mean. This does not represent an unlikely value.

7.75   Let $Y = \#$ the favor the bond issue. Then, the probability of interest is
$$P\left(\left|\tfrac{Y}{n} - p\right| \leq .06\right) = P\left(-.06 \leq \tfrac{Y}{n} - p \leq .06\right) \approx P\left(\frac{-.06}{\sqrt{\frac{.2(.8)}{64}}} \leq Z \leq \frac{.06}{\sqrt{\frac{.2(.8)}{64}}}\right) = P(-1.2 \leq Z \leq 1.2) = .7698.$$

7.76   **a.** We know that $V(Y/n) = p(1-p)/n$. Consider n fixed and let $g(p) = p(1-p)/n$. This function is maximized at $p = 1/2$ (verify using standard calculus techniques).

b. It is necessary to have $P\left(\left|\tfrac{Y}{n} - p\right| \leq .1\right) = .95$, or approximately $P\left(|Z| \leq \frac{.1}{\sqrt{pq/n}}\right) = .95$.
Thus, it must be true that $\frac{.1}{\sqrt{pq/n}} = 1.96$. Since p is unknown, replace it with the value $1/2$ found in part a (this represents the "worse case scenario") and solve for n. In so doing, it is found that $n = 96.04$, so that 97 items should be sampled.

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7.77   (Similar to Ex. 7.76). Here, we must solve $\frac{.15}{\sqrt{pq/n}} = z_{.01} = 2.33$. Using $p = 1/2$, we find
           that $n = 60.32$, so 61 customers should be sampled.

7.78   Following Ex. 7.77: if $p = .9$, then
           $$P\left(\left|\tfrac{Y}{n} - p\right| \le .15\right) \approx P\left(|Z| \le \tfrac{.15}{\sqrt{.9(.1)/50}}\right) = P(|Z| \le 3.54) \approx 1.$$

7.79   **a.** Using the normal approximation:
           $$P(Y \ge 2) = P(Y \ge 1.5) = P(Z \ge \tfrac{1.5-2.5}{\sqrt{25(.1)(.9)}}) = P(Z \ge -.67) = .7486.$$
           b. Using the exact binomial probability:
           $$P(Y \ge 2) = 1 - P(Y \le 1) = 1 - .271 = .729.$$

7.80   Let $Y = \#$ in the sample that are younger than 31 years of age. Since 31 is the median
           age, Y will have a binomial distribution with $n = 100$ and $p = 1/2$ (here, we are being
           rather lax about the specific age of 31 in the population). Then,
           $$P(Y \ge 60) = P(Y \ge 59.5) \approx P(Z \ge \tfrac{59.5-50}{\sqrt{100(.5)(.5)}}) = P(Z \ge 1.9) = .0287.$$

7.81   Let $Y = \#$ of non–conforming items in our lot. Thus, with $n = 50$:
           a. With $p = .1$, P(lot is accepted) $= P(Y \le 5) = P(Y \le 5.5) = P(Z \le \tfrac{5.5-50(.1)}{\sqrt{50(.1)(.9)}}) =$
           $P(Z \le .24) = .5948.$
           b. With $p = .2$ and $.3$, the probabilities are $.0559$ and $.0017$ respectively.

7.82   Let $Y = \#$ of disks with missing pulses. Then, Y is binomial with $n = 100$ and $p = .2$.
           Thus, $P(Y \ge 15) = P(Y \ge 14.5) \approx P(Z \ge \tfrac{14.5-100(.2)}{\sqrt{100(.2)(.8)}}) = P(Z \ge -1.38) = .9162.$

7.83   **a.** Let $Y = \#$ that turn right. Then, Y is binomial with $n = 50$ and $p = 1/3$. Using the
           applet, $P(Y \le 15) = .36897.$
           b. Let $Y = \#$ that turn (left or right). Then, Y is binomial with $n = 50$ and $p = 2/3$. Using
           the applet, $P(Y \ge (2/3)50) = P(Y \ge 33.333) = P(Y \ge 34) = .48679.$

7.84   **a.** $E\left(\tfrac{Y_1}{n_1} - \tfrac{Y_2}{n_2}\right) = \tfrac{E(Y_1)}{n_1} - \tfrac{E(Y_2)}{n_2} = \tfrac{n_1 p_1}{n_1} - \tfrac{n_2 p_2}{n_2} = p_1 - p_2.$
           b. $V\left(\tfrac{Y_1}{n_1} - \tfrac{Y_2}{n_2}\right) = \tfrac{V(Y_1)}{n_1^2} + \tfrac{V(Y_2)}{n_2^2} = \tfrac{n_1 p_1 q_1}{n_1^2} + \tfrac{n_2 p_2 q_2}{n_2^2} = \tfrac{p_1 q_1}{n_1} + \tfrac{p_2 q_2}{n_2}.$

7.85   It is given that $p_1 = .1$ and $p_2 = .2$. Using the result of Ex. 7.58, we obtain
           $$P\left(\left|\tfrac{Y_1}{n_1} - \tfrac{Y_2}{n_2} - (p_1 - p_2)\right| \le .1\right) \approx P\left(|Z| \le \tfrac{.1}{\sqrt{\tfrac{.1(.9)}{50} + \tfrac{.2(.8)}{50}}}\right) = P(|Z| \le 1.4) = .8414.$$

7.86   Let $Y = \#$ of travel vouchers that are improperly documented. Then, Y has a binomial
           distribution with $n = 100$, $p = .20$. Then, the probability of observing more than 30 is
           $$P(Y > 30) = P(Y > 30.5) \approx P(Z > \tfrac{30.5-100(.2)}{\sqrt{100(.2)(.8)}}) = P(Z > 2.63) = .0043.$$
           We conclude that the claim is probably incorrect since this probability is very small.

7.87 Let X = waiting time over a 2–day period. Then, X is exponential with $\beta = 10$ minutes.
Let Y = # of customers whose waiting times is greater than 10 minutes. Then, Y is
binomial with $n = 100$ and p is given by

$$p = \int_{10}^{\infty} \tfrac{1}{10} e^{-y/10} dy = e^{-1} = .3679.$$

Thus, $P(Y \geq 50) \approx P(Z \geq \frac{50-100(.3697)}{\sqrt{1000(.3679)(.6321)}}) = P(Z \geq 2.636) = .0041.$

7.88 Since the efficiency measurements follow a normal distribution with mean $\mu = 9.5$
lumens and $\sigma = .5$ lumens, then

$$\overline{Y} = \text{mean efficiency of eight bulbs}$$

follows a normal distribution with mean 9.5 lumens and standard deviation $.5/\sqrt{8}$.
Thus, $P(\overline{Y} > 10) = P(Z > \frac{10-9.5}{.5/\sqrt{8}}) = P(Z > 2.83) = .0023.$

7.89 Following Ex. 7.88, it is necessary that $P(\overline{Y} > 10) = P(Z > \frac{10-\mu}{.5/\sqrt{8}}) = .80$, where μ denotes

the mean efficiency. Thus, $\frac{10-\mu}{.5/\sqrt{8}} = z_{.2} = -.84$ so $\mu = 10.15$.

7.90 Denote Y = # of successful transplants. Then, Y has a binomial distribution with $n = 100$
and $p = .65$. Then, using the normal approximation to the binomial,

$$P(Y > 70) \approx P(Z > \frac{70-100(.65)}{\sqrt{100(.65)(.35)}}) = P(Z > 1.15) = .1251.$$

7.91 Since X, Y, and W are normally distributed, so are $\overline{X}$, $\overline{Y}$, and $\overline{W}$. In addition, by
Theorem 6.3 U follows a normal distribution such that

$$\mu_U = E(U) = .4\mu_1 + .2\mu_2 + .4\mu_3$$
$$\sigma_U^2 = V(U) = .16\left(\tfrac{\sigma_1^2}{n_1}\right) + .04\left(\tfrac{\sigma_2^2}{n_2}\right) + .16\left(\tfrac{\sigma_3^2}{n_3}\right).$$

7.92 The desired probability is

$$P\left[\left|\overline{X} - \overline{Y}\right| > .6\right] = P\left[|Z| \leq \frac{.06}{\sqrt{[(6.4)^2 + (6.4)^2]/64}}\right] = P\left[|Z| \leq .50\right] = .6170.$$

7.93 Using the mgf approach, the mgf for the exponential distribution with mean θ is
$$m_Y(t) = (1 - \theta t)^{-1}, \; t < 1/\theta.$$
The mgf for $U = 2Y/\theta$ is
$$m_U(t) = E(e^{tU}) = E(e^{(t2/\theta)Y}) = m_Y(2t/\theta) = (1 - 2t)^{-1}, \; t < 1/2.$$
This is the mgf for the chi–square distribution with 2 degrees of freedom.

7.94 Using the result from Ex. 7.93, the quantity $2Y_i/20$ is chi–square with 2 degrees of
freedom. Further, since the Y_i are independent, $U = \sum_{i=1}^{5} 2Y_i/20$ is chi–square with 10

degrees of freedom. Thus, $P\left(\sum_{i=1}^{5} Y_i > c\right) = P(U > \tfrac{c}{10}) = .05$. So, it must be true that

$\frac{c}{10} = \chi_{.05}^2 = 18.307$, or $c = 183.07$.

7.95 **a.** Since $\mu = 0$ and by Definition 2, $T = \dfrac{\overline{Y}}{S/\sqrt{10}}$ has a t–distribution with 9 degrees of

freedom. Also, $T^2 = \dfrac{\overline{Y}^2}{S^2/10} = \dfrac{10\overline{Y}^2}{S^2}$ has an F–distribution with 1 numerator and 9
denominator degrees of freedom (see Ex. 7.33).

b. By Definition 3, $T^{-2} = \dfrac{S^2}{10\overline{Y}^2}$ has an F–distribution with 9 numerator and 1
denominator degrees of freedom (see Ex. 7.29).

c. With 9 numerator and 1 denominator degrees of freedom, $F_{.05} = 240.5$. Thus,
$$.95 = P\left(\frac{S^2}{10\overline{Y}^2} < 240.5\right) = P\left(\frac{S^2}{\overline{Y}^2} < 2405\right) = P\left(-49.04 < \frac{S}{\overline{Y}} < 49.04\right),$$
so $c = 49.04$.

7.96 Note that Y has a beta distribution with $\alpha = 3$ and $\beta = 1$. So, $\mu = 3/4$ and $\sigma^2 = 3/80$. By
the Central Limit Theorem, $P(\overline{Y} > .7) \approx P(Z > \frac{.7-.75}{\sqrt{.0375/40}}) = P(Z > -1.63) = .9484$.

7.97 **a.** Since the X_i are independent and identically distributed chi–square random variables
with 1 degree of freedom, if $Y = \sum_{i=1}^{n} X_i$, then $E(Y) = n$ and $V(Y) = 2n$. Thus, the
conditions of the Central Limit Theorem are satisfied and
$$Z = \frac{Y-n}{\sqrt{2n}} = \frac{\overline{X}-1}{\sqrt{2/n}} .$$

b. Since each Y_i is normal with mean 6 and variance .2, we have that
$$U = \sum_{i=1}^{50} \frac{(Y_i - 6)^2}{.2}$$
is chi–square with 50 degrees of freedom. For $i = 1, 2, \ldots, 50$, let C_i be the cost for a
single rod, Then, $C_i = 4(Y_i - 6)^2$ and the total cost is $T = \sum_{i=1}^{50} C_i = .8U$. By Ex. 7.97,
$$P(T > 48) = P(.8U > 48) = P(U > 60) \approx P\left(Z > \frac{60-50}{\sqrt{100}}\right) = P(Z > 1) = .1587.$$

7.98 **a.** Note that since Z has a standard normal distribution, the random variable Z/c also has a
normal distribution with mean 0 and variance $1/c^2 = v/w$. Thus, we can write the
conditional density of T given $W = w$ as
$$f(t \mid w) = \tfrac{1}{\sqrt{2\pi}} \sqrt{\tfrac{w}{v}} e^{-wt^2/(2v)} , \quad -\infty < t < \infty .$$
b. Since W has a chi–square distribution with v degrees of freedom,
$$f(t,w) = f(t \mid w)f(w) = \tfrac{1}{\sqrt{2\pi}} \sqrt{\tfrac{w}{v}} e^{-wt^2/(2v)} \left(\tfrac{1}{\Gamma(v/2)2^{v/2}} w^{v/2-1} e^{-w/2}\right).$$

c. Integrating over w, we obtain

$$f(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \sqrt{\frac{w}{v}} e^{-wt^2/(2v)} \left(\frac{1}{\Gamma(v/2)2^{v/2}} w^{v/2-1} e^{-w/2} \right) dw = \int_0^\infty \frac{1}{\sqrt{\pi v}} \frac{1}{\Gamma(v/2)2^{(v+1)/2}} \exp\left[-\frac{w}{2}\left(1 + \frac{t^2}{v}\right) \right] w^{[(v+1)/2]-1} dw.$$

Writing another way this is,

$$f(t) = \frac{\left(1 + t^2/v\right)^{-(v+1)/2}}{\sqrt{\pi v}} \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)} \int_0^\infty \frac{1}{\Gamma[(v+1)/2]} \frac{1}{2^{(v+1)/2}} \frac{1}{\left(1 + t^2/v\right)^{-(v+1)/2}} \exp\left[-\frac{w}{2}\left(1 + \frac{t^2}{v}\right) \right] w^{[(v+1)/2]-1} dw.$$

The integrand is that of a gamma density with shape parameter $(v+1)/2$ and scale parameter $2/\left[1 + t^2/v\right]$, so it must integrate to one. Thus, the given form for $f(t)$ is correct.

7.99   **a.** Similar to Ex. 7.98. For fixed $W_2 = w_2$, $F = W_1/c$, where $c = w_2 v_1/v_2$. To find this conditional density of F, note that the mgf for W_1 is

$$m_{W_1}(t) = (1 - 2t)^{-v_1/2}.$$

The mgf for $F = W_1/c$ is

$$m_F(t) = m_{W_1}(t/c) = (1 - 2t/c)^{-v_1/2}.$$

Since this mgf is in the form of a gamma mgf, the conditional density of F, conditioned that $W_2 = w_2$, is gamma with shape parameter v_1 and scale parameter $2v_2/(w_2 v_1)$.

b. Since W_2 has a chi–square distribution with v_2 degrees of freedom, the joint density is

$$g(f, w_2) = g(f \mid w_2) f(w_2) = \frac{f^{(v_1/2)-1} e^{-f w_2 v_1/(2v_2)} w_2^{(v_2/2)-1} e^{-w_2/2}}{\Gamma\left(\frac{v_1}{2}\right)\left(\frac{2v_2}{w_2 v_1}\right)^{v_1/2} \Gamma\left(\frac{v_2}{2}\right) 2^{v_2/2}}$$

$$= \frac{f^{(v_1/2)-1} w_2^{[(v_1+v_2)/2]-1} e^{-(w_2/2)[f v_1/v_2 + 1]}}{\Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{v_1}\right)^{v_1/2} \Gamma\left(\frac{v_2}{2}\right) 2^{(v_1+v_2)/2}}.$$

c. Integrating over w_2, we obtain,

$$g(f) = \frac{f^{(v_1/2)-1}}{\Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{v_1}\right)^{v_1/2} \Gamma\left(\frac{v_2}{2}\right) 2^{(v_1+v_2)/2}} \int_0^\infty w_2^{[(v_1+v_2)/2]-1} e^{-(w_2/2)[f v_1/v_2 + 1]} dw_2.$$

The integrand can be related to a gamma density with shape parameter $(v_1 + v_2)/2$ and scale parameter $\frac{1}{2}\left(1 + f v_1/v_2\right)^{-1}$ in order to evaluate the integral. Thus:

$$g(f) = \frac{\Gamma\left(\frac{v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \frac{f^{(v_1/2)-1}\left(1 + f v_1/v_2\right)^{-(v_1+v_2)/2}}{\left(\frac{v_2}{v_1}\right)^{v_1/2} 2^{(v_1+v_2)/2}}, f \geq 0.$$

7.100   The mgf for X is $m_X(t) = \exp\left(\lambda e^t - 1\right)$.

   a.   The mgf for $Y = (X - \lambda)/\sqrt{\lambda}$ is given by

$$m_Y(t) = E(e^{tY}) = e^{-t\sqrt{\lambda}} m_X(t/\sqrt{\lambda}) = \exp\left(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda\right).$$

   b.   Using the expansion as given, we have

$$m_Y(t) = \exp\left[-t\sqrt{\lambda} + \lambda\left(\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^{3/2}} + \cdots \right) \right] = \exp\left(\frac{t^2}{2} + \frac{t^3}{6\lambda^{1/2}} + \cdots \right).$$

As $\lambda \to \infty$, all terms after the first in the series will go to zero so that the limiting form of the mgf is $m_Y(t) = \exp\left(\frac{t^2}{2}\right)$

c. Since the limiting mgf is the mgf of the standard normal distribution, by Theorem 7.5 the result is proven.

7.101 Using the result in Ex. 7.100,
$$P(X \leq 110) \approx P(Z \leq \tfrac{110-100}{\sqrt{100}}) = P(Z \leq 1) = .8413.$$

7.102 Again use the result in Ex. 7.101,
$$P(Y \geq 45) \approx P(Z \geq \tfrac{45-36}{\sqrt{36}}) = P(Z \geq 1.5) = .0668.$$

7.103 Following the result in Ex. 7.101, and that X and Y are independent, the quantity
$$\frac{X - Y - (\lambda_1 - \lambda_2)}{\sqrt{\lambda_1 + \lambda_2}}$$
has a limiting standard normal distribution (see Ex. 7.58 as applied to the Poisson). Therefore, the approximation is
$$P(X - Y > 10) \approx P(Z > 1) = .1587.$$

7.104 The mgf for Y_n is given by
$$m_{Y_n}(t) = \left[1 - p + pe^t\right]^n.$$
Let $p = \lambda/n$ and this becomes
$$m_{Y_n}(t) = \left[1 - \tfrac{\lambda}{n} + \tfrac{\lambda}{n}e^t\right]^n = \left[1 + \tfrac{1}{n}(\lambda e^t - 1)\right]^n.$$
As $n \to \infty$, this is $\exp(\lambda e^t - 1)$, the mgf for the Poisson with mean λ.

7.105 Let $Y = \#$ of people that suffer an adverse reaction. Then, Y is binomial with $n = 1000$ and $p = .001$. Using the result in Ex. 7.104, we let $\lambda = 1000(.001) = 1$ and evaluate

$$P(Y \geq 2) = 1 - P(Y \leq 1) \approx 1 - .736 = .264,$$

using the Poisson table in Appendix 3.