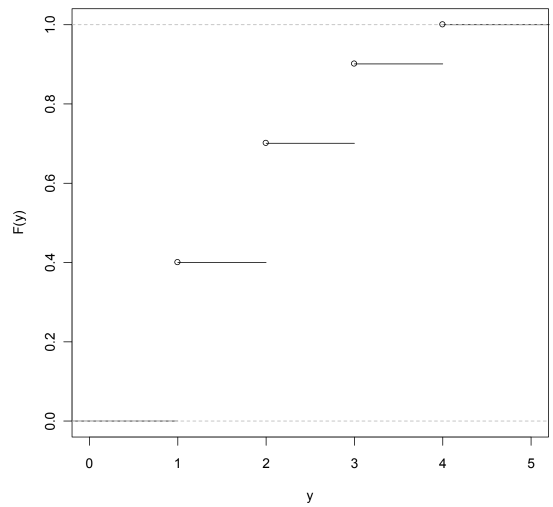
Chapter 4: Continuous Variables and Their Probability Distributions

4.1 **a.** $F(y) = P(Y \le y) = \begin{cases} 0 & y < 1 \\ .4 & 1 \le y < 2 \\ .7 & 2 \le y < 3 \\ .9 & 3 \le y < 4 \\ 1 & y \ge 4 \end{cases}$



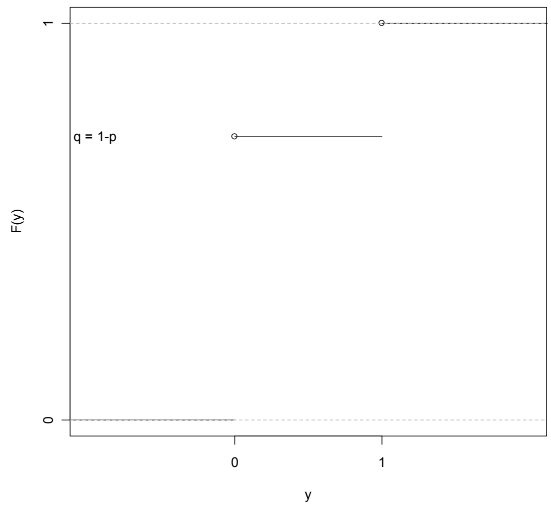
b. The graph is above.

4.2 **a.** $p(1) = .2$, $p(2) = (1/4)4/5 = .2$, $p(3) = (1/3)(3/4)(4/5) = 2.$, $p(4) = .2$, $p(5) = .2$.

b. $F(y) = P(Y \le y) = \begin{cases} 0 & y < 1 \\ .2 & 1 \le y < 2 \\ .4 & 2 \le y < 3 \\ .6 & 3 \le y < 4 \\ .8 & 4 \le y < 5 \\ 1 & y \ge 5 \end{cases}$

c. $P(Y < 3) = F(2) = .4$, $P(Y \le 3) = .6$, $P(Y = 3) = p(3) = .2$

d. No, since Y is a discrete random variable.



4.3 **a.** The graph is above.

b. It is easily shown that all three properties hold.

4.4 A binomial variable with $n = 1$ has the Bernoulli distribution.

4.5 For $y = 2, 3, \ldots$, $F(y) - F(y - 1) = P(Y \le y) - P(Y \le y - 1) = P(Y = y) = p(y)$. Also, $F(1) = P(Y \le 1) = P(Y = 1) = p(1)$.

4.6 **a.** $F(i) = P(Y \le i) = 1 - P(Y > i) = 1 - P(1^{\text{st}}\ i$ trials are failures$) = 1 - q^i$.

b. It is easily shown that all three properties hold.

4.7 **a.** $P(2 \le Y < 5) = P(Y \le 4) - P(Y \le 1) = .967 - .376 = 0.591$
$P(2 < Y < 5) = P(Y \le 4) - P(Y \le 2) = .967 - .678 = .289$.
Y is a discrete variable, so they are not equal.

b. $P(2 \le Y \le 5) = P(Y \le 5) - P(Y \le 1) = .994 - .376 = 0.618$
$P(2 < Y \le 5) = P(Y \le 5) - P(Y \le 2) = .994 - .678 = 0.316$.
Y is a discrete variable, so they are not equal.

c. Y is not a continuous random variable, so the earlier result do not hold.

4.8 **a.** The constant $k = 6$ is required so the density function integrates to 1.

b. $P(.4 \le Y \le 1) = .648$.

c. Same as part b. above.

d. $P(Y \le .4 \mid Y \le .8) = P(Y \le .4)/P(Y \le .8) = .352/.896 = 0.393$.

e. Same as part **d.** above.

4.9 **a.** Y is a discrete random variable because $F(y)$ is not a continuous function. Also, the set of possible values of Y represents a countable set.

b. These values are 2, 2.5, 4, 5.5, 6, and 7.

c. $p(2) = 1/8$, $p(2.5) = 3/16 - 1/8 = 1/16$, $p(4) = 1/2 - 3/16 = 5/16$, $p(5.5) = 5/8 - 1/2 = 1/8$, $p(6) = 11/16 - 5/8 = 1/16$, $p(7) = 1 - 11/16 = 5/16$.

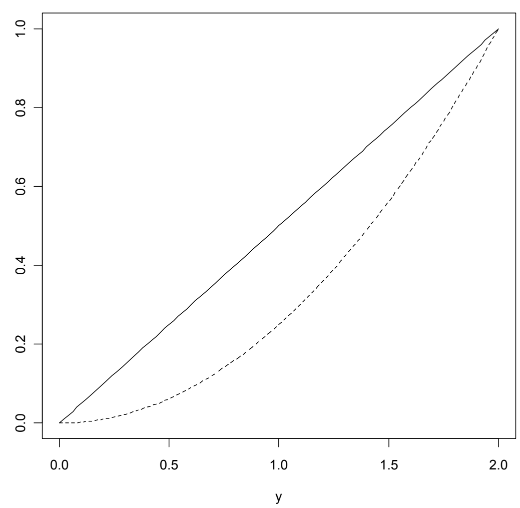
d. $P(Y \le \phi_{.5}) = F(\phi_{.5}) = .5$, so $\phi_{.5} = 4$.

4.10 **a.** $F(\phi_{.95}) = \int_0^{\phi_{.95}} 6y(1-y)dy = .95$, so $\phi_{.95} = 0.865$.

b. Since Y is a continuous random variable, $y_0 = \phi_{.95} = 0.865$.

4.11 **a.** $\int_0^2 cydy = \left[cy^2/2\right]_0^2 = 2c = 1$, so $c = 1/2$.

b. $F(y) = \int_{-\infty}^{y} f(t)dt = \int_0^y \frac{t}{2}dt = \frac{y^2}{4}$, $0 \le y \le 4$.



c. Solid line: $f(y)$; dashed line: $F(y)$

d. $P(1 \le Y \le 2) = F(2) - F(1) = 1 - .25 = .75$.

e. Note that $P(1 \le Y \le 2) = 1 - P(0 \le Y < 1)$. The region $(0 \le y < 1)$ forms a triangle (in the density graph above) with a base of 1 and a height of .5. So, $P(0 \le Y < 1) = \frac{1}{2}(1)(.5) = .25$ and $P(1 \le Y \le 2) = 1 - .25 = .75$.

4.12 **a.** $F(-\infty) = 0$, $F(\infty) = 1$, and $F(y_1) - F(y_2) = e^{-y_2^2} - e^{-y_1^2} > 0$ provided $y_1 > y_2$.

b. $F(\phi_{.3}) = 1 - e^{-\phi_{.3}^2} = .3$, so $\phi_{.3} = \sqrt{-\ln(.7)} = 0.5972$.

c. $f(y) = F'(y) = 2ye^{-y^2}$ for $y \geq 0$ and 0 elsewhere.

d. $P(Y \geq 200) = 1 - P(Y < 200) = 1 - P(Y \leq 200) = 1 - F(2) = e^{-4}$.

e. $P(Y > 100 \mid Y \leq 200) = P(100 < Y \leq 200)/P(Y \leq 200) = [F(2) - F(1)]/F(2) = \frac{e^{-1} - e^{-4}}{1 - e^{-4}}$.

4.13 **a.** For $0 \leq y \leq 1$, $F(y) = \int_0^y t\,dt = y^2/2$. For $1 < y \leq 1.5$, $F(y) = \int_0^1 t\,dt + \int_1^y dt = 1/2 + y - 1$ $= y - 1/2$. Hence,

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \leq y \leq 1 \\ y - 1/2 & 1 < y \leq 1.5 \\ 1 & y > 1.5 \end{cases}$$

b. $P(0 \leq Y \leq .5) = F(.5) = 1/8$.

c. $P(.5 \leq Y \leq 1.2) = F(1.2) - F(.5) = 1.2 - 1/2 - 1/8 = .575$.

4.14 **a.** A triangular distribution.

b. For $0 < y < 1$, $F(y) = \int_0^y t\,dt = y^2/2$. For $1 \leq y < 2$, $F(y) = \int_0^1 t\,dt + \int_1^y (2-t)\,dt = 2y - \frac{y^2}{2} - 1$.

c. $P(.8 \leq Y \leq 1.2) = F(1.2) - F(.8) = .36$.

d. $P(Y > 1.5 \mid Y > 1) = P(Y > 1.5)/P(Y > 1) = .125/.5 = .25$.

4.15 **a.** For $b \geq 0, f(y) \geq 0$. Also, $\int_{-\infty}^{\infty} f(y) = \int_b^{\infty} b/y^2 = -b/y\Big]_b^{\infty} = 1$.

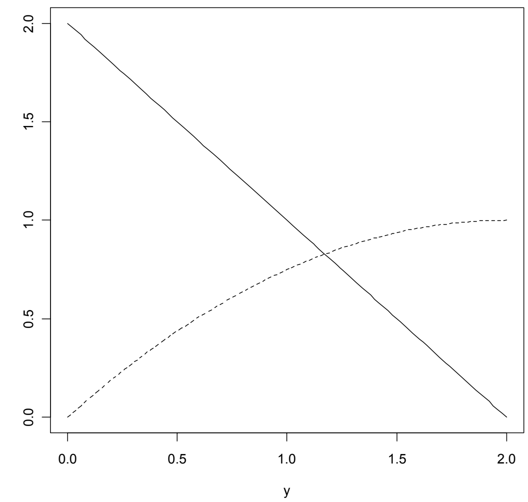
b. $F(y) = 1 - b/y$, for $y \geq b$, 0 elsewhere.

c. $P(Y > b + c) = 1 - F(b + c) = b/(b + c)$.

d. Applying part c., $P(Y > b + d \mid Y > b + c) = (b + c)/(b + d)$.

4.16 **a.** $\int_0^2 c(2-y)dy = c\left[2y - y^2/2\right]_0^2 = 2c = 1$, so $c = 1/2$.

b. $F(y) = y - y^2/4$, for $0 \le y \le 2$.

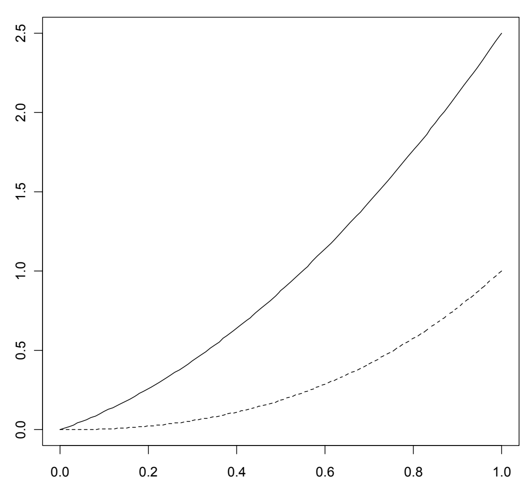


c. Solid line: $f(y)$; dashed line: $F(y)$

d. $P(1 \le Y \le 2) = F(2) - F(1) = 1/4$.

4.17 **a.** $\int_0^1 (cy^2 + y)dy = \left[cy^3/3 + y^2/2\right]_0^2 = 1$, $c = 3/2$.

b. $F(y) = y^3/2 + y^2/2$ for $0 \le y \le 1$.



c. Solid line: $f(y)$; dashed line: $F(y)$

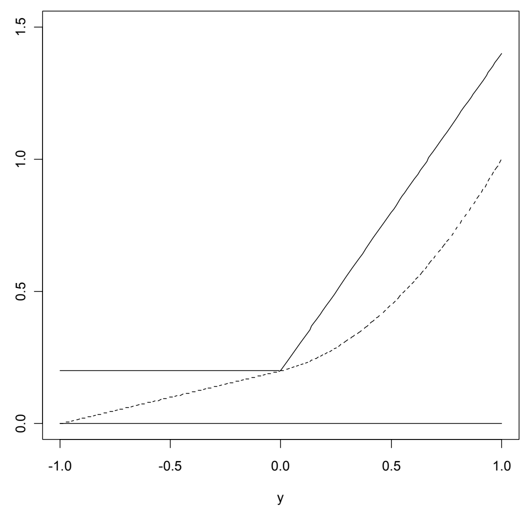
d. $F(-1) = 0$, $F(0) = 0$, $F(1) = 1$.

e. $P(Y < .5) = F(.5) = 3/16$.

f. $P(Y \geq .5 \mid Y \geq .25) = P(Y \geq .5)/P(Y \geq .25) = 104/123$.

4.18 **a.** $\int_{-1}^{0} .2dy + \int_{0}^{1} (.2 + cy)dy = .4 + c/2 = 1$, so $c = 1.2$.

b. $F(y) = \begin{cases} 0 & y \leq -1 \\ .2(1+y) & -1 < y \leq 0 \\ .2(1+y+3y^2) & 0 < y \leq 1 \\ 1 & y > 1 \end{cases}$



c. Solid line: $f(y)$; dashed line: $F(y)$

d. $F(-1) = 0$, $F(0) = .2$, $F(1) = 1$

e. $P(Y > .5 \mid Y > .1) = P(Y > .5)/P(Y > .1) = .55/.774 = .71$.

4.19 **a.** Differentiating $F(y)$ with respect to y, we have

$$f(y) = \begin{cases} 0 & y \leq 0 \\ .125 & 0 < y < 2 \\ .125y & 2 \leq y < 4 \\ 0 & y \geq 4 \end{cases}$$

b. $F(3) - F(1) = 7/16$

c. $1 - F(1.5) = 13/16$

d. $7/16/(9/16) = 7/9$.

4.20 From Ex. 4.16:

$$E(Y)=\int_0^2 .5y(2-y)dy = \frac{y^2}{2}+\frac{y^3}{6}\Big]_0^2 = 2/3\,,\; E(Y^2)=\int_0^2 .5y^2(2-y)dy = \frac{y^3}{3}+\frac{y^4}{8}\Big]_0^2 = 2/3\,.$$

So, $V(Y) = 2/3 - (2/3)^2 = 2/9$.

4.21 From Ex. 4.17:

$$E(Y)=\int_0^1 1.5y^3+y^2)dy = \frac{3y^4}{8}+\frac{y^3}{3}\Big]_0^1 = 17/24 = .708\,.$$

$$E(Y^2)=\int_0^1 1.5y^4+y^3)dy = \frac{3y^5}{10}+\frac{y^4}{4}\Big]_0^1 = 3/10+1/4 = .55\,.$$

So, $V(Y) = .55 - (.708)2 = .0487$.

4.22 From Ex. 4.18:

$$E(Y)=\int_{-1}^0 .2ydy+\int_0^1(.2y+1.2y^2)dy = .4\,,\; E(Y^2)=\int_{-1}^0 .2y^2dy+\int_0^1(.2y^2+1.2y^3)dy = 1.3/3\,.$$

So, $V(Y) = 1.3/3 - (.4)2 = .2733$.

4.23 **1.** $E(c)=\int_{-\infty}^{\infty} cf(y)dy = c\int_{-\infty}^{\infty} f(y)dy = c(1) = c\,.$

2. $E[cg(Y)]=\int_{-\infty}^{\infty} cg(y)f(y)dy = c\int_{-\infty}^{\infty} g(y)f(y)dy = cE[g(Y)].$

3. $E[g_1(Y)+g_2(Y)+\cdots g_k(Y)] = \int_{-\infty}^{\infty}[g_1(y)+g_2(y)+\cdots g_k(y)]f(y)dy$

$$= \int_{-\infty}^{\infty} g_1(y)f(y)dy+\int_{-\infty}^{\infty} g_2(y)f(y)dy+\cdots\int_{-\infty}^{\infty} g_k(y)f(y)dy$$

$$= E[g_1(Y)]+E[g_2(Y)]+\cdots E[g_k(Y)].$$

4.24 $V(Y)=E\{[Y-E(Y)]^2\}=E\{Y^2-2YE(Y)+[E(Y)]^2\}=E(Y^2)-2[E(Y)]^2+[E(Y)]^2$
$=E(Y^2)-[E(Y)]^2 = \sigma^2.$

4.25 Ex. 4.19:

$$E(Y)=\int_0^2 .125ydy+\int_2^4 .125y^2dy = 31/12\,,\; E(Y^2)=\int_0^2 .125y^2dy+\int_2^4 .125y^3dy = 47/6.$$

So, $V(Y) = 47/6 - (31/12)^2 = 1.16$.

4.26 **a.** $E(aY+b)=\int_{-\infty}^{\infty}(ay+b)f(y)dy = \int_{-\infty}^{\infty} ayf(y)dy+\int_{-\infty}^{\infty} bf(y)dy = aE(Y)+b = a\mu+b\,.$

b. $V(aY+b)=E\{[aY+b-E(aY+b)]^2\}=E\{[aY+b-a\mu-b]^2\}=E\{a^2[Y-\mu]^2\}$
$= a^2V(Y) = a^2\sigma^2.$

4.27 First note that from Ex. 4.21, $E(Y) = .708$ and $V(Y) = .0487$. Then,
$E(W) = E(5 - .5Y) = 5 - .5E(Y) = 5 - .5(.708) = \$4.65$.
$V(W) = V(5 - .5Y) = .25V(Y) = .25(.0487) = .012$.

4.28 **a.** By using the methods learned in this chapter, $c = 105$.

b. $E(Y) = 105\int_0^1 y^3(1-y)^4\,dy = 3/8$.

4.29 $E(Y) = .5\int_{59}^{61} y\,dy = .5\frac{y^2}{2}\Big]_{59}^{61} = 60$, $E(Y^2) = .5\int_{59}^{61} y^2\,dy = .5\frac{y^3}{3}\Big]_{59}^{61} = 3600\frac{1}{3}$. Thus,
$V(Y) = 3600\frac{1}{3} - (60)^2 = \frac{1}{3}$.

4.30 **a.** $E(Y) = \int_0^1 2y^2\,dy = 2/3$, $E(Y^2) = \int_0^1 2y^3\,dy = 1/2$. Thus, $V(Y) = 1/2 - (2/3)^2 = 1/18$.

b. With $X = 200Y - 60$, $E(X) = 200(2/3) - 60 = 220/3$, $V(X) = 20000/9$.

c. Using Tchebysheff's theorem, a two standard deviation interval about the mean is given by $220/3 \pm 2\sqrt{20000/9}$ or (–20.948, 167.614).

4.31 **a.** $E(Y) = \frac{3}{8}\int_5^7 y(7-y)^2\,dy = \frac{3}{8}\left[\frac{49}{2}y^2 - \frac{14}{3}y^3 + \frac{y^4}{4}\right]_5^7 = 5.5$

$E(Y^2) = \frac{3}{8}\int_5^7 y^2(7-y)^2\,dy = \frac{3}{8}\left[\frac{49}{3}y^3 - \frac{14}{4}y^4 + \frac{y^5}{5}\right]_5^7 = 30.4$, so $V(Y) = .15$.

b. Using Tchebysheff's theorem, a two standard deviation interval about the mean is given by $5.5 \pm 2\sqrt{.15}$ or (4.725, 6.275). Since $Y \geq 5$, the interval is (5, 6.275).

c. $P(Y < 5.5) = \frac{3}{8}\int_5^{5.5} (7-y)^2\,dy = .5775$, or about 58% of the time (quite common).

4.32 **a.** $E(Y) = \frac{3}{64}\int_0^4 y^3(4-y)\,dy = \frac{3}{64}\left[y^4 - \frac{y^5}{5}\right]_0^4 = 2.4$. V(Y) = .64.

b. $E(200Y) = 200(2.4) = \$480$, $V(200Y) = 200^2(.64) = 25{,}600$.

c. $P(200Y > 600) = P(Y > 3) = \frac{3}{64}\int_3^4 y^2(4-y)\,dy = .2616$, or about 26% of the time the cost will exceed \$600 (fairly common).

4.33 $E(Y)=\int_2^6 y\left(\tfrac{3}{32}\right)(y-2)(6-y)dy = 4.$

4.34 $E(Y)=\int_0^\infty yf(y)dy=\int_0^\infty\left(\int_0^y dt\right)f(y)dy=\int_0^\infty\left(\int_y^\infty f(y)dy\right)dt=\int_0^\infty P(Y>y)dy=\int_0^\infty[1-F(y)]dy.$

4.35 Let $\mu = E(Y)$. Then, $E[(Y-a)^2]=E[(Y-\mu+\mu-a)^2]$

$$= E[(Y-\mu)^2]-2E[(Y-\mu)(\mu-a)]+(\mu-a)^2$$
$$= \sigma^2+(\mu-a)^2.$$

The above quantity is minimized when $\mu = a$.

4.36 This is also valid for discrete random variables — the properties of expected values used in the proof hold for both continuous and discrete random variables.

4.37 $E(Y)=\int_{-\infty}^{\infty} yf(y)dy=\int_{-\infty}^{0} yf(y)dy+\int_0^\infty yf(y)dy$. In the first integral, let $w = -y$, Then,

$$E(Y)=-\int_0^\infty wf(-w)dy+\int_0^\infty yf(y)dy=-\int_0^\infty wf(w)dy+\int_0^\infty yf(y)dy=0.$$

4.38 **a.** $F(y)=\begin{cases}0 & y<0\\ \int_0^y 1dy=y & 0\le y\le 1\\ 1 & y>1\end{cases}$

b. $P(a \le Y \le a + b) = F(a + b) - F(a) = a + b - a = b.$

4.39 The distance Y is uniformly distributed on the interval A to B, If she is closer to A, she has landed in the interval $(A, \frac{A+B}{2})$. This is one half the total interval length, so the probability is 1/2. If her distance to A is more than three times her distance to B, she has landed in the interval $(\frac{3B+A}{4}, B)$. This is one quarter the total interval length, so the probability is .25.

4.40 The probability of landing past the midpoint is 1/2 according to the uniform distribution. Let X = # parachutists that land past the midpoint of (A, B). Therefore, X is binomial with $n = 3$ and $p = 1/2$. $P(X = 1) = 3(1/2)^3 = .375$.

4.41 First find $E(Y^2)=\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} y^2dy=\frac{1}{\theta_2-\theta_1}\left[\frac{y^3}{3}\right]_{\theta_1}^{\theta_2}=\frac{\theta_2^3-\theta_1^3}{3(\theta_2-\theta_1)}=\frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3}$. Thus,

$$V(Y) = \frac{\theta_1^2 + \theta_1\theta_2 + \theta_2^2}{3} - \left(\frac{\theta_2 + \theta_1}{2}\right)^2 = \frac{(\theta_2 - \theta_1)^2}{12}.$$

4.42 The distribution function is $F(y) = \frac{y - \theta_1}{\theta_2 - \theta_1}$, for $\theta_1 \le y \le \theta_2$. For $F(\phi_{.5}) = .5$, then $\phi_{.5} = \theta_1 + .5(\theta_2 - \theta_1) = .5(\theta_2 + \theta_1)$. This is also the mean if the distribution.

4.43 Let $A = \pi R^2$, where R has a uniform distribution on the interval (0, 1). Then,

$$E(A) = \pi E(R^2) = \pi\int_0^1 r^2 dr = \frac{\pi}{3}$$

$$V(A) = \pi^2 V(R^2) = \pi^2[E(R^4) - \left(\frac{1}{3}\right)^2] = \pi^2\left[\int_0^1 r^4 dr - \left(\frac{1}{3}\right)^2\right] = \pi^2\left[\frac{1}{5} - \left(\frac{1}{3}\right)^2\right] = \frac{4\pi^2}{45}.$$

4.44 **a.** Y has a uniform distribution (constant density function), so $k = 1/4$.

b. $$F(y) = \begin{cases} 0 & y < -2 \\ \int_{-2}^{y} \frac{1}{4}dy = \frac{y+2}{4} & -2 \le y \le 2 \\ 1 & y > 2 \end{cases}$$

4.45 Let Y = low bid (in thousands of dollars) on the next intrastate shipping contract. Then, Y is uniform on the interval (20, 25).
a. $P(Y < 22) = 2/5 = .4$
b. $P(Y > 24) = 1/5 = .2$.

4.46 Mean of the uniform: $(25 + 20)/2 = 22.5$.

4.47 The density for Y = delivery time is $f(y) = \frac{1}{4}$, $1 \le y \le 5$. Also, $E(Y) = 3$, $V(Y) = 4/3$.
a. $P(Y > 2) = 3/4$.
b. $E(C) = E(c_0 + c_1Y^2) = c_0 + c_1E(Y^2) = c_0 + c_1[V(Y) + (E(Y))^2] = c_0 + c_1[4/3 + 9]$

4.48 Let Y = time when the phone call comes in. Then, Y has a uniform distribution on the interval (0, 5). The probability is $P(0 < Y < 1) + P(3 < Y < 4) = .4$.

4.49 If Y has a uniform distribution on the interval (0, 1), then $P(Y > 1/4) = 3/4$.

4.50 Let Y = location of the selected point. Then, Y has a uniform distribution on the interval (0, 500).
a. $P(475 \le Y \le 500) = 1/20$
b. $P(0 \le Y \le 25) = 1/20$
c. $P(0 < Y < 250) = 1/2$.

4.51 Let Y = cycle time. Thus, Y has a uniform distribution on the interval (50, 70). Then,

$P(Y > 65 \mid Y > 55) = P(Y > 65)/P(Y > 55) = .25/(.75) = 1/3$.

4.52 Mean and variance of a uniform distribution: $\mu = 60$, $\sigma^2 = (70–50)^2/12 = 100/3$.

4.53 Let Y = time when the defective circuit board was produced. Then, Y has an approximate uniform distribution on the interval (0, 8).

a. $P(0 < Y < 1) = 1/8$.
b. $P(7 < Y < 8) = 1/8$
c. $P(4 < Y < 5 \mid Y > 4) = P(4 < Y < 5)/P(Y > 4) = (1/8)/(1/2) = 1/4$.

4.54 Let Y = amount of measurement error. Then, Y is uniform on the interval (–.05, .05).

a. $P(–.01 < Y < .01) = .2$
b. $E(Y) = 0$, $V(Y) = (.05 + .05)^2/12 = .00083$.

4.55 Let Y = amount of measurement error. Then, Y is uniform on the interval (–.02, .05).

a. $P(–.01 < Y < .01) = 2/7$
b. $E(Y) = (–.02 + .05)/2 = .015$, $V(Y) = (.05 + .02)^2/12 = .00041$.

4.56 From Example 4.7, the arrival time Y has a uniform distribution on the interval (0, 30). Then, $P(25 < Y < 30 \mid Y > 10) = 1/6/(2/3) = 1/4$.

4.57 The volume of a sphere is given by $(4/3)\pi r^3 = (1/6)\pi d^3$, where r is the radius and d is the diameter. Let D = diameter such that D is uniform distribution on the interval (.01, .05).

Thus, $E(\frac{\pi}{6}D^3) = \frac{\pi}{6}\int_{.01}^{.05} d^3 \frac{1}{4} dd = .0000065\pi$. By similar logic used in Ex. 4.43, it can be found that $V(\frac{\pi}{6}D^3) = .0003525\pi^2$.

4.58 **a.** $P(0 \le Z \le 1.2) = .5 – .1151 = .3849$
b. $P(–.9 \le Z \le 0) = .5 – .1841 – .3159$.
c. $P(.3 \le Z \le 1.56) = .3821 – .0594 = .3227$.
d. $P(–.2 \le Z \le .2) = 1 – 2(.4207) = .1586$.
e. $P(–1.56 \le Z \le –.2) = .4207 – .0594 = .3613$
f. $P(0 \le Z \le 1.2) = .38493$. The desired probability is for a standard normal.

4.59 **a.** $z_0 = 0$.
b. $z_0 = 1.10$
c. $z_0 = 1.645$
d. $z_0 = 2.576$

4.60 The parameter σ must be positive, otherwise the density function could obtain a negative value (a violation).

4.61 Since the density function is symmetric about the parameter μ, $P(Y < \mu) = P(Y > \mu) = .5$. Thus, μ is the median of the distribution, regardless of the value of σ.

4.62 **a.** $P(Z^2 < 1) = P(–1 < Z < 1) = .6826$.

b. $P(Z^2 < 3.84146) = P(-1.96 < Z < 1.96) = .95$.

4.63 **a.** Note that the value 17 is $(17 - 16)/1 = 1$ standard deviation above the mean.
So, $P(Z > 1) = .1587$.
b. The same answer is obtained.

4.64 **a.** Note that the value 450 is $(450 - 400)/20 = 2.5$ standard deviations above the mean.
So, $P(Z > 2.5) = .0062$.
b. The probability is .00618.
c. The top scale is for the standard normal and the bottom scale is for a normal distribution with mean 400 and standard deviation 20.

4.65 For the standard normal, $P(Z > z_0) = .1$ if $z_0 = 1.28$. So, $y_0 = 400 + 1.28(20) = \$425.60$.

4.66 Let Y = bearing diameter, so Y is normal with $\mu = 3.0005$ and $\sigma = .0010$. Thus,
Fraction of scrap $= P(Y > 3.002) + P(Y < 2.998) = P(Z > 1.5) + P(Z < -2.5) = .0730$.

4.67 In order to minimize the scrap fraction, we need the maximum amount in the specifications interval. Since the normal distribution is symmetric, the mean diameter should be set to be the midpoint of the interval, or $\mu = 3.000$ in.

4.68 The GPA 3.0 is $(3.0 - 2.4)/.8 = .75$ standard deviations above the mean. So, $P(Z > .75) = .2266$.

4.69 The z–score for 1.9 is $(1.9 - 2.4)/.8 = -.625$. Thus, $P(Z < -.625) = .2660$.

4.70 From Ex. 4.68, the proportion of students with a GPA greater than 3.0 is .2266. Let X = # in the sample with a GPA greater than 3.0. Thus, X is binomial with $n = 3$ and $p = .2266$. Then, $P(X = 3) = (.2266)^3 = .0116$.

4.71 Let Y = the measured resistance of a randomly selected wire.
a. $P(.12 \le Y \le .14) = P(\frac{.12-.13}{.005} \le Z \le \frac{.14-.13}{.005}) = P(-2 \le Z \le 2) = .9544$.
b. Let X = # of wires that do not meet specifications. Then, X is binomial with $n = 4$ and $p = .9544$. Thus, $P(X = 4) = (.9544)^4 = .8297$.

4.72 Let Y = interest rate forecast, so Y has a normal distribution with $\mu = .07$ and $\sigma = .026$.
a. $P(Y > .11) = P(Z > \frac{.11-.07}{.026}) = P(Z > 1.54) = .0618$.
b. $P(Y < .09) = P(Z > \frac{.09-.07}{.026}) = P(Z > .77) = .7794$.

4.73 Let Y = width of a bolt of fabric, so Y has a normal distribution with $\mu = 950$ mm and $\sigma = 10$ mm.
a. $P(947 \le Y \le 958) = P(\frac{947-950}{10} \le Z \le \frac{958-950}{10}) = P(-.3 \le Z \le .8) = .406$

b. It is necessary that $P(Y \le c) = .8531$. Note that for the standard normal, we find that $P(Z \le z_0) = .8531$ when $z_0 = 1.05$. So, $c = 950 + (1.05)(10) = 960.5$ mm.

4.74 Let Y = examination score, so Y has a normal distribution with $\mu = 78$ and $\sigma^2 = 36$.

a. $P(Y > 72) = P(Z > -1) = .8413$.

b. We seek c such that $P(Y > c) = .1$. For the standard normal, $P(Z > z_0) = .1$ when $z_0 = 1.28$. So $c = 78 + (1.28)(6) = 85.68$.

c. We seek c such that $P(Y > c) = .281$. For the standard normal, $P(Z > z_0) = .281$ when $z_0 = .58$. So, $c = 78 + (.58)(6) = 81.48$.

d. For the standard normal, $P(Z < -.67) = .25$. So, the score that cuts off the lowest 25% is given by $(-.67)(6) + 78 = 73.98$.

e. Similar answers are obtained.

f. $P(Y > 84 \mid Y > 72) = P(Y > 84)/P(Y > 72) = P(Z > 1)/P(Z > -1) = .1587/.8413 = .1886$.

4.75 Let Y = volume filled, so that Y is normal with mean μ and $\sigma = .3$ oz. They require that $P(Y > 8) = .01$. For the standard normal, $P(Z > z_0) = .01$ when $z_0 = 2.33$. Therefore, it must hold that $2.33 = (8 - \mu)/.3$, so $\mu = 7.301$.

4.76 It follows that $.95 = P(|Y - \mu| < 1) = P(|Z| < 1/\sigma)$, so that $1/\sigma = 1.96$ or $\sigma = 1/1.96 = .5102$.

4.77 **a.** Let Y = SAT math score. Then, $P(Y < 550) = P(Z < .7) = 0.758$.

b. If we choose the same percentile, $18 + 6(.7) = 22.2$ would be comparable on the ACT math test.

4.78 Easiest way: maximize the function $\ln f(y) = -\ln(\sigma\sqrt{2\pi}) - \frac{(y-\mu)^2}{2\sigma^2}$ to obtain the maximum at $y = \mu$ and observe that $f(\mu) = 1/(\sigma\sqrt{2\pi})$.

4.79 The second derivative of $f(y)$ is found to be $f''(y) = \left(\frac{1}{\sigma^3\sqrt{2\pi}}\right)e^{-(y-\mu)^2/2\sigma^2}\left[1 - \frac{(\mu-y)^2}{\sigma^2}\right]$. Setting this equal to 0, we must have that $\left[1 - \frac{(\mu-y)^2}{\sigma^2}\right] = 0$ (the other quantities are strictly positive). The two solutions are $y = \mu + \sigma$ and $\mu - \sigma$.

4.80 Observe that $A = L*W = |Y|\times 3|Y| = 3Y^2$. Thus, $E(A) = 3E(Y^2) = 3(\sigma^2 + \mu^2)$.

4.81 **a.** $\Gamma(1) = \int_0^\infty e^{-y}dy = -e^{-y}\Big]_0^\infty = 1$.

b. $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy = \left[-y^{\alpha-1}e^{-y}\right]_0^\infty + \int_0^\infty (\alpha-1)y^{\alpha-2}e^{-y}dy = (\alpha-1)\Gamma(\alpha-1)$.

4.82 From above we have $\Gamma(1) = 1$, so that $\Gamma(2) = 1\Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2(1)$, and generally $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$ $\Gamma(4) = 3! = 6$ and $\Gamma(7) = 6! = 720$.

4.83 Applet Exercise — the results should agree.

4.84 **a.** The larger the value of α, the more symmetric the density curve.
b. The location of the distribution centers are increasing with α.
c. The means of the distributions are increasing with α.

4.85 **a.** These are all exponential densities.
b. Yes, they are all skewed densities (decaying exponential).
c. The spread is increasing with β.

4.86 **a.** $P(Y < 3.5) = .37412$
b. $P(W < 1.75) = P(Y/2 < 1.75) = P(Y < 3.5) = .37412$.
c. They are identical.

4.87 **a.** For the gamma distribution, $\phi_{.05} = .70369$.
b. For the χ^2 distribution, $\phi_{.05} = .35185$.
c. The .05–quantile for the χ^2 distribution is exactly one–half that of the .05–quantile for the gamma distribution. It is due to the relationship stated in Ex. 4.86.

4.88 Let Y have an exponential distribution with $\beta = 2.4$.

a. $P(Y > 3) = \int_3^\infty \frac{1}{2.4} e^{-y/2.4} dy = e^{-3/2.4} = .2865$.

b. $P(2 \le Y \le 3) = \int_2^3 \frac{1}{2.4} e^{-y/2.4} dy = .1481$.

4.89 Let Y = water demand in the early afternoon. Then, Y is exponential with $\beta = 100$ cfs.

a. $P(Y > 200) = \int_{200}^\infty \frac{1}{100} e^{-y/100} dy = e^{-2} = .1353$.

b. We require the 99th percentile of the distribution of Y:

$$P(Y > \phi_{.99}) = \int_{\phi_{.99}}^\infty \frac{1}{100} e^{-y/100} dy = e^{-\phi_{.99}/100} = .01. \text{ So, } \phi_{.99} = -100\ln(.01) = 460.52 \text{ cfs.}$$

4.90 Let Y = magnitude of the earthquake which is exponential with $\beta = 2.4$. Let X = # of earthquakes that exceed 5.0 on the Richter scale. Therefore, X is binomial with $n = 10$ and $p = P(Y > 5) = \int_5^\infty \frac{1}{2.4} e^{-y/2.4} dy = e^{-5/2.4} = .1245$. Finally, the probability of interest is

$$P(X \ge 1) = 1 - P(X = 0) = 1 - (.8755)^{10} = 1 - .2646 = .7354.$$

4.91 **a.** Note that $\int_2^\infty \frac{1}{\beta} e^{-y/\beta} dy = e^{-2/\beta} = .0821$, so β = .8.

b. $P(Y \le 1.7) = 1 - e^{-1.7/2.4} = .5075$.

4.92 The random variable Y has an exponential distribution with β = 10. The cost C is related to Y by the formula $C = 100 + 40Y + 3Y^2$. Thus,

$E(C) = E(100 + 40Y + 3Y^2) = 100 + 40(10) + 3E(Y^2) = 100 + 400 + 3(100 + 10^2) = 1100$.

To find $V(C)$, note that $V(C) = E(C^2) - [E(C)]^2$. Therefore,

$E(C^2) = E[(100 + 40Y + 3Y^2)^2] = 10{,}000 + 2200E(Y^2) + 9E(Y^4) + 8000E(Y) + 240E(Y^3)$.

$E(Y) = 10 \qquad E(Y^2) = 200$

$$E(Y^3) = \int_0^\infty y^3 \tfrac{1}{100} e^{-y/100} dy = \Gamma(4)100^3 = 6000.$$

$$E(Y^4) = \int_0^\infty y^4 \tfrac{1}{100} e^{-y/100} dy = \Gamma(5)100^4 = 240{,}000.$$

Thus, $E(C^2) = 10{,}000 + 2200(200) + 9(240{,}000) + 8000(10) + 240(6000) = 4{,}130{,}000$.

So, $V(C) = 4{,}130{,}000 - (1100)^2 = 2{,}920{,}000$.

4.93 Let Y = time between fatal airplane accidents. So, Y is exponential with β = 44 days.

a. $P(Y \le 31) = \int_0^{31} \frac{1}{44} e^{-y/44} dy = 1 - e^{-31/44} = .5057$.

b. $V(Y) = 44^2 = 1936$.

4.94 Let Y = CO concentration in air samples. So, Y is exponential with β = 3.6 ppm.

a. $P(Y > 9) = \int_9^\infty \frac{1}{3.6} e^{-y/3.6} dy = e^{-9/3.6} = .0821$

b. $P(Y > 9) = \int_9^\infty \frac{1}{2.5} e^{-y/3.6} dy = e^{-9/2.5} = .0273$

4.95 **a.** For any k = 1, 2, 3, …

$$P(X = k) = P(k - 1 \le Y < k) = P(Y < k) - P(Y \le k - 1) = 1 - e^{-k/\beta} - (1 - e^{-(k-1)/\beta})$$
$$= e^{-(k-1)/\beta} - e^{-k/\beta}.$$

b. $P(X = k) = e^{-(k-1)/\beta} - e^{-k/\beta} = e^{-(k-1)/\beta} - e^{-(k-1)/\beta}(e^{1/\beta}) = e^{-(k-1)/\beta}(1 - e^{1/\beta}) = [e^{-1/\beta}]^{k-1}(1 - e^{1/\beta})$.

Thus, X has a geometric distribution with $p = 1 - e^{1/\beta}$.

4.96 **a.** The density function $f(y)$ is in the form of a gamma density with $\alpha = 4$ and $\beta = 2$. Thus, $k = \frac{1}{\Gamma(4)2^4} = \frac{1}{96}$.

b. Y has a χ^2 distribution with $\nu = 2(4) = 8$ degrees of freedom.

c. $E(Y) = 4(2) = 8$, $V(Y) = 4(2^2) = 16$.

d. Note that $\sigma = \sqrt{16} = 4$. Thus, $P(|Y - 8| < 2(4)) = P(0 < Y < 16) = .95762$.

4.97 $P(Y > 4) = \int_4^\infty \frac{1}{4}e^{-y/4}dy = e^{-1} = .3679$.

4.98 We require the 95th percentile of the distribution of Y:

$$P(Y > \phi_{.95}) = \int_{\phi_{.95}}^\infty \frac{1}{4}e^{-y/4}dy = e^{-\phi_{.95}/4} = .05. \text{ So, } \phi_{.95} = -4\ln(.05) = 11.98.$$

4.99 **a.** $P(Y > 1) = \sum_{y=0}^{1} \frac{e^{-1}}{y!} = e^{-1} + e^{-1} = .7358$.

b. The same answer is found.

4.100 **a.** $P(X_1 = 0) = e^{-\lambda_1}$ and $P(X_2 = 0) = e^{-\lambda_2}$. Since $\lambda_2 > \lambda_1$, $e^{-\lambda_2} < e^{-\lambda_1}$.

b. The result follows from Ex. 4.100.

c. Since distribution function is a nondecreasing function, it follows from part b that

$$P(X_1 \le k) = P(Y > \lambda_1) > P(Y > \lambda_2) = P(X_2 \le k)$$

d. We say that X_2 is "stochastically greater" than X_1.

4.101 Let Y have a gamma distribution with $\alpha = .8$, $\beta = 2.4$.

a. $E(Y) = (.8)(2.4) = 1.92$
b. $P(Y > 3) = .21036$
c. The probability found in Ex. 4.88 (a) is larger. There is greater variability with the exponential distribution.
d. $P(2 \le Y \le 3) = P(Y > 2) - P(Y > 3) = .33979 - .21036 = .12943$.

4.102 Let Y have a gamma distribution with $\alpha = 1.5$, $\beta = 3$.
a. $P(Y > 4) = .44592$.
b. We require the 95th percentile: $\phi_{.95} = 11.72209$.

4.103 Let R denote the radius of a crater. Therefore, R is exponential w/ $\beta = 10$ and the area is $A = \pi R^2$. Thus,

$$E(A) = E(\pi R^2) = \pi E(R^2) = \pi(100 + 100) = 200\pi.$$

$$V(A) = E(A^2) - [E(A)]^2 = \pi^2[E(R^4) - 200^2] = \pi^2[240{,}000 - 200^2] = 200{,}000\pi^2,$$

where $E(R^4) = \int_0^\infty \frac{1}{10} r^4 e^{-r/10} dr = 10^4\Gamma(5) = 240.000.$

4.104 Y has an exponential distribution with $\beta = 100$. Then, $P(Y > 200) = e^{-200/100} = e^{-2}$. Let the random variable X = # of componential that operate in the equipment for more than 200 hours. Then, X has a binomial distribution and

$$P(\text{equipment operates}) = P(X \geq 2) = P(X = 2) + P(X = 3) = 3(e^{-2})^2(1-e^{-2}) + (e^{-2})^3 = .05.$$

4.105 Let the random variable Y = four–week summer rainfall totals
a. $E(Y) = 1.6(2) = 3.2$, $V(Y) = 1.6(2^2) = 6.4$
b. $P(Y > 4) = .28955$.

4.106 Let Y = response time. If $\mu = 4$ and $\sigma^2 = 8$, then it is clear that $\alpha = 2$ and $\beta = 2$.
a. $f(y) = \frac{1}{4} y e^{-y/2}$, $y > 0$.
b. $P(Y < 5) = 1 - .2873 = .7127$.

4.107 **a.** Using Tchebysheff's theorem, two standard deviations about the mean is given by $4 \pm 2\sqrt{8} = 4 \pm 5.657$ or $(-1.657, 9.657)$, or simply $(0, 9.657)$ since Y must be positive.
b. $P(Y < 9.657) = 1 - .04662 = 0.95338$.

4.108 Let Y = annual income. Then, Y has a gamma distribution with $\alpha = 20$ and $\beta = 1000$.
a. $E(Y) = 20(1000) = 20{,}000$, $V(Y) = 20(1000)^2 = 20{,}000{,}000$.
b. The standard deviation $\sigma = \sqrt{20{,}000{,}000} = 4472.14$. The value 30,000 is $\frac{30{,}000-20{,}000}{4472.14}$ $= 2.236$ standard deviations above the mean. This represents a fairly extreme value.
c. $P(Y > 30{,}000) = .02187$

4.109 Let Y have a gamma distribution with $\alpha = 3$ and $\beta = 2$. Then, the loss $L = 30Y + 2Y^2$. Then,

$$E(L) = E(30Y + 2Y^2) = 30E(Y) + 2E(Y^2) = 30(6) + 2(12 + 6^2) = 276,$$
$$V(L) = E(L^2) - [E(L)]^2 = E(900Y^2 + 120Y^3 + 4Y^4) - 276^2.$$

$$E(Y^3) = \int_0^\infty \tfrac{y^5}{16} e^{-y/2} = 480 \qquad E(Y^4) = \int_0^\infty \tfrac{y^6}{16} e^{-y/2} = 5760$$

Thus, $V(L) = 900(48) + 120(480) + 4(5760) - 276^2 = 47{,}664$.

4.110 Y has a gamma distribution with $\alpha = 3$ and $\beta = .5$. Thus, $E(Y) = 1.5$ and $V(Y) = .75$.

4.111 **a.** $E(Y^a) = \int_0^\infty y^a \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{a+\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{\Gamma(a+\alpha)\beta^{a+\alpha}}{1} = \frac{\beta^a \Gamma(a+\alpha)}{\Gamma(\alpha)}$.

b. For the gamma function $\Gamma(t)$, we require $t > 0$.

c. $E(Y^1) = \frac{\beta^1 \Gamma(1+\alpha)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta.$

d. $E(\sqrt{Y}) = E(Y^{.5}) = \frac{\beta^{.5}\Gamma(.5+\alpha)}{\Gamma(\alpha)}$, $\alpha > 0$.

e. $E(1/Y) = E(Y^{-1}) = \frac{\beta^{-1}\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{1}{\beta(\alpha-1)}$, $\alpha > 1$.

$E(1/\sqrt{Y}) = E(Y^{-.5}) = \frac{\beta^{-.5}\Gamma(-.5+\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-.5)}{\sqrt{\beta}\Gamma(\alpha)}$, $\alpha > .5$.

$E(1/Y^2) = E(Y^{-2}) = \frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{\beta^{-2}\Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} \frac{1}{\beta^2(\alpha-1)(\alpha-2)}$, $\alpha > 2$.

4.112 The chi–square distribution with ν degrees of freedom is the same as a gamma distribution with $\alpha = \nu/2$ and $\beta = 2$.

a. From Ex. 4.111, $E(Y^a) = \frac{2^a \Gamma(a+\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})}$.

b. As in Ex. 4.111 with $\alpha + a > 0$ and $\alpha = \nu/2$, it must hold that $\nu > -2a$

c. $E(\sqrt{Y}) = E(Y^{.5}) = \frac{\sqrt{2}\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}$, $\nu > 0$.

d. $E(1/Y) = E(Y^{-1}) = \frac{2^{-1}\Gamma(-1+\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} = \frac{1}{\nu-2}$, $\nu > 2$.

$E(1/\sqrt{Y}) = E(Y^{-.5}) = \frac{\Gamma(\frac{\nu-1}{2})}{\sqrt{2}\Gamma(\frac{\nu}{2})}$, $\nu > 1$.

$E(1/Y^2) = E(Y^{-2}) = \frac{1}{2^2(\frac{\nu}{2}-1)(\frac{\nu}{2}-2)} = \frac{1}{(\nu-2)(\nu-4)}$, $\alpha > 4$.

4.113 Applet exercise.

4.114 **a.** This is the (standard) uniform distribution.
b. The beta density with $\alpha = 1$, $\beta = 1$ is symmetric.
c. The beta density with $\alpha = 1$, $\beta = 2$ is skewed right.
d. The beta density with $\alpha = 2$, $\beta = 1$ is skewed left.
e. Yes.

4.115 **a.** The means of all three distributions are .5.
b. They are all symmetric.
c. The spread decreases with larger (and equal) values of α and β.
d. The standard deviations are .2236, .1900, and .1147 respectively. The standard deviations are decreasing which agrees with the density plots.
e. They are always symmetric when $\alpha = \beta$.

4.116 **a.** All of the densities are skewed right.
b. The density obtains a more symmetric appearance.
c. They are always skewed right when $\alpha < \beta$ and $\alpha > 1$ and $\beta > 1$.

4.117 **a.** All of the densities are skewed left.
b. The density obtains a more symmetric appearance.
c. They are always skewed right when $\alpha > \beta$ and $\alpha > 1$ and $\beta > 1$.

4.118 **a.** All of the densities are skewed right (similar to an exponential shape).
b. The spread decreases as the value of β gets closer to 12.
c. The distribution with $\alpha = .3$ and $\beta = 4$ has the highest probability.
d. The shapes are all similar.

4.119 **a.** All of the densities are skewed left (a mirror image of those from Ex. 4.118).
b. The spread decreases as the value of α gets closer to 12.
c. The distribution with $\alpha = 4$ and $\beta = .3$ has the highest probability.
d. The shapes are all similar.

4.120 Yes, the mapping explains the mirror image.

4.121 **a.** These distributions exhibit a "U" shape.
b. The area beneath the curve is greater closer to "1" than "0".

4.122 **a.** $P(Y > .1) = .13418$
b. $P(Y < .1) = 1 - .13418 = .86582$.
c. Values smaller than .1 have greater probability.
d. $P(Y < .1) = 1 - .45176 = .54824$
e. $P(Y > .9) = .21951$.
f. $P(0.1 < Y < 0.9) = 1 - .54824 - .21951 = .23225$.
g. Values of $Y < .1$ have the greatest probability.

4.123 **a.** The random variable Y follows the beta distribution with $\alpha = 4$ and $\beta = 3$, so the constant $k = \frac{\Gamma(4+3)}{\Gamma(4)\Gamma(3)} = \frac{6!}{3!2!} = 60$.
b. We require the 95th percentile of this distribution, so it is found that $\phi_{.95} = 0.84684$.

4.124 **a.** $P(Y > .4) = \int_{.4}^{1}(12y^2 - 12y^3)dy = \left[4y^3 - 3y^4\right]_{.4}^{1} = .8208$.
b. $P(Y > .4) = .82080$.

4.125 From Ex. 4.124 and using the formulas for the mean and variance of beta random variables, $E(Y) = 3/5$ and $V(Y) = 1/25$.

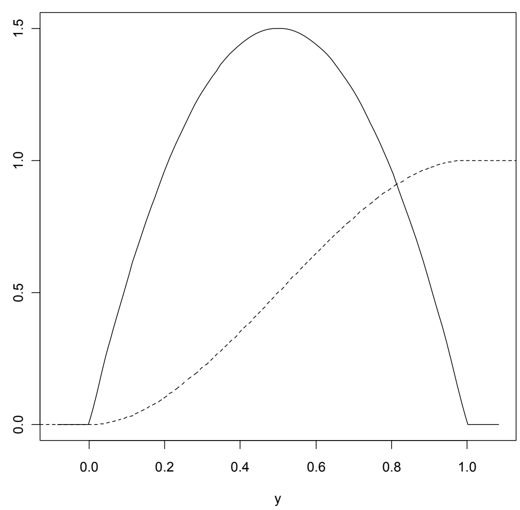
4.126 The random variable Y = weekly repair cost (in hundreds of dollars) has a beta distribution with $\alpha = 1$ and $\beta = 3$. We require the 90th percentile of this distribution:

$$P(Y > \phi_{.9}) = \int_{\phi_{.9}}^{1} 3(1-y)^2\,dy = (1-\phi_{.9})^3 = .1.$$

Therefore, $\phi_{.9} = 1 - (.1)^{1/3} = .5358$. So, the budgeted cost should be \$53.58.

4.127 For $\alpha = \beta = 1$, $f(y) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} y^{1-1}(1-y)^{1-1} = 1,\ 0 \le y \le 1$, which is the uniform distribution.

4.128 **a.** $F(y) = \int_0^y (6t - 6t^2)\,dt = 3y^2 - 2y^3$, $0 \le y \le 1$. $F(y) = 0$ for $y < 0$ and $F(y) = 1$ for $y > 1$.



b. Solid line: $f(y)$; dashed line: $F(y)$

c. $P(.5 \le Y \le .8) = F(.8) - F(.5) = 1.92 - 1.092 - .75 + .25 = .396$.

4.129 $E(C) = 10 + 20E(Y) + 4E(Y^2) = 10 + 20\left(\frac{1}{3}\right) + 4\left(\frac{2}{9*4} + \frac{1}{9}\right) = \frac{52}{3}$

$V(C) = E(C^2) - [E(C)]^2 = E[(10 + 20Y + 4Y^2)^2] - \left(\frac{52}{3}\right)^2$

$E[(10 + 20Y + 4Y^2)^2] = 100 + 400E(Y) + 480E(Y^2) + 160E(Y^3) + 16E(Y^4)$

Using mathematical expectation, $E(Y^3) = \frac{1}{10}$ and $E(Y^4) = \frac{1}{15}$. So,

$V(C) = E(C^2) - [E(C)]^2 = (100 + 400/3 + 480/6 + 160/10 + 16/15) - (52/3)^2 = 29.96.$

4.130 To find the variance $\sigma^2 = E(Y^2) - \mu^2$:

$$E(Y^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha+1}(1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\sigma^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

4.131 This is the same beta distribution used in Ex. 4.129.

a. $P(Y < .5) = \int_0^{.5} 2(1-y)dy = 2y - y^2\Big]_0^{.5} = .75$

b. $E(Y) = 1/3$, $V(Y) = 1/18$, so $\sigma = 1/\sqrt{18} = .2357$.

4.132 Let Y = proportion of weight contributed by the fine powders
a. $E(Y) = .5$, $V(Y) = 9/(36*7) = 1/28$
b. $E(Y) = .5$, $V(Y) = 4/(16*5) = 1/20$
c. $E(Y) = .5$, $V(Y) = 1/(4*3) = 1/12$
d. Case (a) will yield the most homogenous blend since the variance is the smallest.

4.133 The random variable Y has a beta distribution with $\alpha = 3$, $\beta = 5$.
a. The constant $c = \frac{\Gamma(3+5)}{\Gamma(3)\Gamma(5)} = \frac{7!}{2!4!} = 105$.
b. $E(Y) = 3/8$.
c. $V(Y) = 15/(64*9) = 5/192$, so $\sigma = .1614$.
d. $P(Y > .375 + 2(.1614)) = P(Y > .6978) = .02972$.

4.134 **a.** If $\alpha = 4$ and $\beta = 7$, then we must find

$P(Y \le .7) = F(.7) = \sum_{i=4}^{10} \binom{10}{i} (.7)^i (.3)^{10-i} = P(4 \le X \le 10)$, for the random variable X distributed as binomial with $n = 10$ and $p = .7$. Using Table I in Appendix III, this is .989.

b. Similarly, $F(.6) = P(12 \le X \le 25)$, for the random variable X distributed as binomial with $n = 25$ and $p = .6$. Using Table I in Appendix III, this is .922.

c. Similar answers are found.

4.135 **a.** $P(Y_1 = 0) = (1 - p_1)^n > P(Y_2 = 0) = (1 - p_2)^n$, since $p_1 < p_2$.

b. $P(Y_1 \le k) = 1 - P(Y_1 \ge k+1) = 1 - \sum_{i=k+1}^{n} \binom{n}{i} p_1^i (1-p_1)^{n-i} = 1 - \int_0^{p_1} \frac{t^k (1-t)^{n-k-1}}{B(k+1, n-k)}$

$= 1 - P(X \le p_1) = P(X > p_1)$, where is X beta with parameters $k+1$, $n-k$.

c. From part b, we see the integrands for $P(Y_1 \le k)$ and $P(Y_2 \le k)$ are identical but since $p_1 < p_2$, the regions of integration are different. So, Y_2 is "stochastically greater" than Y_1.

4.136 **a.** Observing that the exponential distribution is a special case of the gamma distribution, we can use the gamma moment–generating function with $\alpha = 1$ and $\beta = \theta$:

$$m(t) = \frac{1}{1-\theta t},\ t < 1/\theta.$$

b. The first two moments are found by $m'(t) = \frac{\theta}{(1-\theta t)^2}$, $E(Y) = m'(0) = \theta$.

$m''(t) = \frac{2\theta}{(1-\theta t)^3}$, $E(Y^2) = m''(0) = 2\theta^2$. So, $V(Y) = 2\theta^2 - \theta^2 = \theta^2$.

4.137 The mgf for U is $m_U(t) = E(e^{tU}) = E(e^{t(aY+b)}) = E(e^{bt}e^{(at)Y}) = e^{bt}m(at)$. Thus,

$m'_U(t) = be^{bt}m(at) + ae^{bt}m'(at)$. So, $m'_U(0) = b + am'(0) = b + a\mu = E(U)$.

$m''_U(t) = b^2e^{bt}m(at) + abe^{bt}m'(at) + abe^{bt}m'(at) + a^2e^{bt}m''(at)$, so
$m''_U(0) = b^2 + 2ab\mu + a^2E(Y^2) = E(U^2)$.

Therefore, $V(U) = b^2 + 2ab\mu + a^2E(Y^2) - (b + a\mu)^2 = a^2[E(Y^2) - \mu^2] = a^2\sigma^2$.

4.138 **a.** For $U = Y - \mu$, the mgf $m_U(t)$ is given in Example 4.16. To find the mgf for $Y = U + \mu$, use the result in Ex. 4.137 with $a = 1$, $b = -\mu$:

$$m_Y(t) = e^{-\mu t}m_U(t) = e^{\mu t + \sigma^2 t^2/2}$$

b. $m'_Y(t) = (\mu + t\sigma^2)e^{\mu t + \sigma^2 t^2/2}$, so $m'_Y(0) = \mu$
$m''_Y(t) = (\mu + t\sigma^2)^2 e^{\mu t + \sigma^2 t^2/2} + \sigma^2 e^{\mu t + \sigma^2 t^2/2}$, so $m''_Y(0) = \mu^2 + \sigma^2$. Finally, $V(Y) = \sigma^2$.

4.139 Using Ex. 4.137 with $a = -3$ and $b = 4$, it is trivial to see that the mgf for X is

$$m_X(t) = e^{4t}m(-3t) = e^{(4-3\mu)t + 9\sigma^2 t^2/2}.$$

By the uniqueness of mgfs, X is normal with mean $4 - 3\mu$ and variance $9\sigma^2$.

4.140 **a.** Gamma with $\alpha = 2$, $\beta = 4$

b. Exponential with $\beta = 3.2$
c. Normal with $\mu = -5$, $\sigma^2 = 12$

4.141 $m(t) = E(e^{tY}) = \int_{\theta_1}^{\theta_2} \frac{e^{ty}}{\theta_2 - \theta_1} dy = \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$.

4.142 **a.** $m_Y(t) = \frac{e^t - 1}{t}$

b. From the cited exercises, $m_W(t) = \frac{e^{at} - 1}{at}$. From the uniqueness property of mgfs, W is uniform on the interval (0, a).

c. The mgf for X is $m_X(t) = \frac{e^{-at} - 1}{-at}$, which implies that X is uniform on the interval ($-a$, 0).

d. The mgf for V is $m_V(t) = e^{bt} \frac{e^{at} - 1}{at} = \frac{e^{(b+a)t} - e^{bt}}{at}$, which implies that V is uniform on the interval (b, $b + a$).

4.143 The mgf for the gamma distribution is $m(t) = (1 - \beta t)^{-\alpha}$. Thus,

$$m'(t) = \alpha\beta(1 - \beta t)^{-\alpha - 1}, \text{ so } m'(0) = \alpha\beta = E(Y)$$

$$m''(t) = (\alpha + 1)\alpha\beta^2(1 - \beta t)^{-\alpha - 2}, \text{ so } m''(0) = (\alpha + 1)\alpha\beta^2 = E(Y^2). \text{ So,}$$
$$V(Y) = (\alpha + 1)\alpha\beta^2 - (\alpha\beta)^2 = \alpha\beta^2.$$

4.144 **a.** The density shown is a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Thus, $k = 1/\sqrt{2\pi}$.
b. From Ex. 4.138, the mgf is $m(t) = e^{t^2/2}$.
c. $E(Y) = 0$ and $V(Y) = 1$.

4.145 **a.** $E(e^{3T/2}) = \int_{-\infty}^{0} e^{3y/2} e^{y} dy = \frac{2}{5} e^{5y/2} \Big]_{-\infty}^{0} = \frac{2}{5}$.

b. $m(t) = E(e^{tY}) = \int_{-\infty}^{0} e^{ty} e^{y} dy = \frac{1}{t+1}, \; t > -1$.

c. By using the methods with mgfs, $E(Y) = -1$, $E(Y^2) = 2$, so $V(Y) = 2 - (-1)^2 = 1$.

4.146 We require $P(|Y - \mu| \leq k\sigma) \geq .90 = 1 - 1/k^2$. Solving for k, we see that $k = 3.1622$. Thus, the necessary interval is $|Y - 25{,}000| \leq (3.1622)(4000)$ or $12{,}351 \leq 37{,}649$.

4.147 We require $P(|Y - \mu| \leq .1) \geq .75 = 1 - 1/k^2$. Thus, $k = 2$. Using Tchebysheff's inequality, $1 = k\sigma$ and so $\sigma = 1/2$.

4.148 In Exercise 4.16, $\mu = 2/3$ and $\sigma = \sqrt{2/9} = .4714$. Thus,

$$P(|Y - \mu| \leq 2\sigma) = P(|Y - 2/3| \leq .9428) = P(-.2761 \leq Y \leq 1.609) = F(1.609) = .962.$$

Note that the negative portion of the interval in the probability statement is irrelevant since Y is non–negative. According to Tchsebysheff's inequality, the probability is at least 75%. The empirical rule states that the probability is approximately 95%. The above probability is closest to the empirical rule, even though the density function is far from mound shaped.

4.149 For the uniform distribution on (θ_1, θ_2), $\mu = \frac{\theta_1+\theta_2}{2}$ and $\sigma^2 = \frac{(\theta_2-\theta_1)^2}{12}$. Thus,

$$2\sigma = \frac{(\theta_2-\theta_1)}{\sqrt{3}}.$$

The probability of interest is

$$P(|Y-\mu| \le 2\sigma) = P(\mu - 2\sigma \le Y \le \mu + 2\sigma) = P(\tfrac{\theta_1+\theta_2}{2} - \tfrac{(\theta_2-\theta_1)}{\sqrt{3}} \le Y \le \tfrac{\theta_1+\theta_2}{2} + \tfrac{(\theta_2-\theta_1)}{\sqrt{3}})$$

It is not difficult to show that the range in the last probability statement is greater than the actual interval that Y is restricted to, so

$$P(\tfrac{\theta_1+\theta_2}{2} - \tfrac{(\theta_2-\theta_1)}{\sqrt{3}} \le Y \le \tfrac{\theta_1+\theta_2}{2} + \tfrac{(\theta_2-\theta_1)}{\sqrt{3}}) = P(\theta_1 \le Y \le \theta_2) = 1.$$

Note that Tchebysheff's theorem is satisfied, but the probability is greater than what is given by the empirical rule. The uniform is not a mound–shaped distribution.

4.150 For the exponential distribution, $\mu = \beta$ and $\sigma^2 = \beta^2$. Thus, $2\sigma = 2\beta$. The probability of interest is

$$P(|Y-\mu| \le 2\sigma) = P(\mu - 2\sigma \le Y \le \mu + 2\sigma) = P(-\beta \le Y \le 3\beta) = P(0 \le Y \le 3\beta)$$

This is simply $F(3\beta) = 1 - e^{-3\beta} = .9502$. The empirical rule and Tchebysheff's theorem are both valid.

4.151 From Exercise 4.92, $E(C) = 1000$ and $V(C) = 2{,}920{,}000$ so that the standard deviation is $\sqrt{2{,}920{,}000} = 1708.80$. The value 2000 is only $(2000 - 1100)/1708.8 = .53$ standard deviations above the mean. Thus, we would expect C to exceed 2000 fair often.

4.152 We require $P(|L-\mu| \le k\sigma) \ge .89 = 1 - 1/k^2$. Solving for k, we have $k = 3.015$. From Ex. 4.109, $\mu = 276$ and $\sigma = 218.32$. The interval is

$$|L-276| \le 3.015(218.32) \text{ or } (-382.23, 934.23)$$

Since L must be positive, the interval is (0, 934.23)

4.153 From Ex. 4.129, it is shown that $E(C) = \frac{52}{3}$ and $V(C) = 29.96$, so, the standard deviation is $\sqrt{29.96} = 5.474$. Thus, using Tchebysheff's theorem with $k = 2$, the interval is

$$|Y - \tfrac{52}{3}| \le 2(5.474) \text{ or } (6.38, 28.28)$$

4.154 **a.** $\mu = 7$, $\sigma^2 = 2(7) = 14$.
b. Note that $\sigma = \sqrt{14} = 3.742$. The value 23 is $(23 - 7)/3.742 = 4.276$ standard deviations above the mean, which is unlikely.
c. With $\alpha = 3.5$ and $\beta = 2$, $P(Y > 23) = .00170$.

4.155 The random variable Y is uniform over the interval (1, 4). Thus, $f(y) = \frac{1}{3}$ for $1 \le y \le 4$ and $f(y) = 0$ elsewhere. The random variable C = cost of the delay is given as

$$C = g(Y) = \begin{cases} 100 & 1 \le y \le 2 \\ 100 + 20(Y-2) & 2 < y \le 4 \end{cases}$$

Thus, $E(C) = E[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y) dy = \int_{1}^{2} \frac{100}{3} dy + \int_{2}^{4} [100 + 20(y-2)]\frac{1}{3} dy = \$113.33$.

4.156 Note that Y is a discrete random variable with probability $.2 + .1 = .3$ and it is continuous with probability $1 - .3 = .7$. Hence, by using Definition 4.15, we can write Y as a mixture of two random variables X_1 and X_2. The random variable X_1 is discrete and can assume two values with probabilities $P(X_1 = 3) = .2/.3 = 2/3$ and $P(X_1 = 6) = .1/.3 = 1/3$. Thus, $E(X_1) = 3(2/3) + 6(1/3) = 4$. The random variable X_2 is continuous and follows a gamma distribution (as given in the problem) so that $E(X_2) = 2(2) = 4$. Therefore,

$$E(Y) = .3(4) + .7(4) = 4.$$

4.157 **a.** The distribution function for X is $F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{1}{100} e^{-t/100} dx = 1 - e^{-x/100} & 0 \le x < 200 \\ 1 & x \ge 200 \end{cases}$.

b. $E(X) = \int_0^{200} x \frac{1}{100} e^{-x/100} dx + .1353(200) = 86.47$, where $.1353 = P(Y > 200)$.

4.158 The distribution for V is gamma with $\alpha = 4$ and $\beta = 500$. Since there is one discrete point at 0 with probability .02, using Definition 4.15 we have that $c_1 = .02$ and $c_2 = .98$. Denoting the kinetic energy as $K = \frac{m}{2} V^2$ we can solve for the expected value:

$$E(K) = (.98)\tfrac{m}{2} E(V^2) = (.98)\tfrac{m}{2}\{V(V) + [E(V)]^2\} = (.98)\tfrac{m}{2}\{4(500)^2 + 2000^2\} = 2{,}450{,}000m.$$

4.159 **a.** The distribution function has jumps at two points: $y = 0$ (of size .1) and $y = .5$ (of size .15). So, the discrete component of $F(y)$ is given by

$$F_1(y) = \begin{cases} 0 & y < 0 \\ \frac{.1}{.1 + .15} = .4 & 0 \le y < 5 \\ 1 & y \ge .5 \end{cases}$$

The continuous component of $F(y)$ can then by determined:

$$F_2(y) = \begin{cases} 0 & y < 0 \\ 4y^2/3 & 0 \le y < .5 \\ (4y-1)/3 & .5 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

b. Note that $c_1 = .1 + .15 = .25$. So, $F(y) = 0.25F_1(y) + 0.75F_2(y)$.

c. First, observe that $f_2(y) = F_2'(y) = \begin{cases} 8y/3 & 0 \le y < .5 \\ 4/3 & y \ge .5 \end{cases}$. Thus,

$E(Y) = .25(.6)(.5) + \int_0^{.5} 8y^2/3dy + \int_{.5}^{1} 4y/3dy = .533$. Similarly, $E(Y^2) = .3604$ so that $V(Y) = .076$.

4.160 a. $F(y) = \int_{-1}^{y} \frac{2}{\pi(1+y^2)} dy = \frac{2}{\pi}\tan^{-1}(y) + \frac{1}{2}, -1 \le y \le 1, F(y) = 0 \text{ if } y < 0, F(y) = 1 \text{ if } y > 1$.

b. Find $E(Y)$ directly using mathematical expectation, or observe that $f(y)$ is symmetric about 0 so using the result from Ex. 4.27, $E(Y) = 0$.

4.161 Here, $\mu = 70$ and $\sigma = 12$ with the normal distribution. We require $\phi_{.9}$, the 90th percentile of the distribution of test times. Since for the standard normal distribution, $P(Z < z_0) = .9$ for $z_0 = 1.28$, thus

$$\phi_{.9} = 70 + 12(1.28) = 85.36.$$

4.162 Here, $\mu = 500$ and $\sigma = 50$ with the normal distribution. We require $\phi_{.01}$, the 1st percentile of the distribution of light bulb lives. For the standard normal distribution, $P(Z < z_0) = .01$ for $z_0 = -2.33$, thus

$$\phi_{.01} = 500 + 50(-2.33) = 383.5$$

4.163 Referring to Ex. 4.66, let X = # of defective bearings. Thus, X is binomial with $n = 5$ and $p = P(\text{defective}) = .073$. Thus,

$$P(X > 1) = 1 - P(X = 0) = 1 - (.927)^5 = .3155.$$

4.164 Let Y = lifetime of a drill bit. Then, Y has a normal distribution with $\mu = 75$ hours and $\sigma = 12$ hours.

a. $P(Y < 60) = P(Z < -1.25) = .1056$

b. $P(Y \ge 60) = 1 - P(Y < 60) = 1 - .1056 = .8944$.

c. $P(Y > 90) = P(Z > 1.25) = .1056$

4.165 The density function for Y is in the form of a gamma density with $\alpha = 2$ and $\beta = .5$.

a. $c = \frac{1}{\Gamma(2)(.5)^2} = 4$.

b. $E(Y) = 2(.5) = 1$, $V(Y) = 2(.5)^2 = .5$.

c. $m(t) = \frac{1}{(1-.5t)^2}$, $t < 2$.

4.166 In Example 4.16, the mgf is $m(t) = e^{t^2\sigma^2/2}$. The infinite series expansion of this is

$$m(t) = 1 + \left(\tfrac{t^2\sigma^2}{2}\right) + \left(\tfrac{t^2\sigma^2}{2}\right)^2 \tfrac{1}{2!} + \left(\tfrac{t^2\sigma^2}{2}\right)^3 \tfrac{1}{3!} + \cdots = 1 + \tfrac{t^2\sigma^2}{2} + \tfrac{t^4\sigma^4}{8} + \tfrac{t^6\sigma^6}{48} + \cdots$$

Then, μ_1 = coefficient of t, so $\mu_1 = 0$
μ_2 = coefficient of $t^2/2!$, so $\mu_2 = \sigma^2$
μ_3 = coefficient of $t^3/3!$, so $\mu_3 = 0$
μ_4 = coefficient of $t^4/4!$, so $\mu_4 = 3\sigma^4$

4.167 For the beta distribution,

$$E(Y^k) = \int_0^1 y^k \tfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy = \tfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{k+\alpha-1}(1-y)^{\beta-1}\,dy = \tfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \tfrac{\Gamma(k+\alpha)\Gamma(\beta)}{\Gamma(k+\alpha+\beta)}.$$

Thus, $E(Y^k) = \frac{\Gamma(\alpha+\beta)\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+\alpha+\beta)}$.

4.168 Let T = length of time until the first arrival. Thus, the distribution function for T is given by

$$F(t) = P(T \le t) = 1 - P(T > t) = 1 - P[\text{no arrivals in } (0, t)] = 1 - P[N = 0 \text{ in } (0, t)]$$

The probability $P[N = 0 \text{ in } (0, t)]$ is given by $\frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$. Thus, $F(t) = 1 - e^{-\lambda t}$ and

$$f(t) = \lambda e^{-\lambda t},\ t > 0.$$

This is the exponential distribution with $\beta = 1/\lambda$.

4.169 Let Y = time between the arrival of two call, measured in hours. To find $P(Y > .25)$, note that $\lambda t = 10$ and $t = 1$. So, the density function for Y is given by $f(y) = 10e^{-10y}$, $y > 0$. Thus,

$$P(Y > .25) = e^{-10(.25)} = e^{-2.5} = .082.$$

4.170 a. Similar to Ex. 4.168, the second arrival will occur after time t if either one arrival has occurred in (0, t) or no arrivals have occurred in $(0, t)$. Thus:

$P(U > t) = P[\text{one arrival in } (0, t)] + P[\text{no arrivals in } (0, t)] = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} + \frac{(\lambda t)^1 e^{-\lambda t}}{1!}$. So,

$$F(t) = 1 - P(U > t) = 1 - \tfrac{(\lambda t)^0 e^{-\lambda t}}{0!} + \tfrac{(\lambda t)^1 e^{-\lambda t}}{1!} = 1 - (\lambda t + 1)e^{-\lambda t}.$$

The density function is given by $f(t) = F'(t) = \lambda^2 t e^{-\lambda t}$, $t > 0$. This is a gamma density with $\alpha = 2$ and $\beta = 1/\lambda$.

b. Similar to part a, but let X = time until the k^{th} arrival. Thus, $P(X > t) = \sum_{n=0}^{k-1} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$. So,

$$F(t) = 1 - \sum_{n=0}^{k-1} \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

The density function is given by

$$f(t) = F'(t) = -\left[-\lambda e^{-\lambda t}\sum_{n=0}^{k-1}\frac{(\lambda t)^n}{n!} + e^{-\lambda t}\sum_{n=1}^{k-1}\frac{\lambda^n t^{n-1}}{(n-1)!}\right] = \lambda e^{-\lambda t}\left[\sum_{n=0}^{k-1}\frac{(\lambda t)^n}{n!} - \sum_{n=1}^{k-1}\frac{(\lambda t)^{n-1}}{(n-1)!}\right].$$ Or,

$f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$, $t > 0$. This is a gamma density with $\alpha = k$ and $\beta = 1/\lambda$.

4.171 From Ex. 4.169, W = waiting time follow an exponential distribution with $\beta = 1/2$.

a. $E(W) = 1/2$, $V(W) = 1/4$.

b. P(at least one more customer waiting) = 1 – P(no customers waiting in three minutes) = $1 - e^{-6}$.

4.172 Twenty seconds is 1/5 a minute. The random variable Y = time between calls follows an exponential distribution with $\beta = .25$. Thus:

$$P(Y > 1/5) = \int_{1/5}^{\infty} 4e^{-4y}\,dy = e^{-4/5}.$$

4.173 Let R = distance to the nearest neighbor. Then,

$$P(R > r) = P(\text{no plants in a circle of radius } r)$$

Since the number of plants in a area of one unit has a Poisson distribution with mean λ, the number of plants in a area of πr^2 units has a Poisson distribution with mean $\lambda\pi r^2$. Thus,

$$F(r) = P(R \le r) = 1 - P(R > r) = 1 - e^{-\lambda\pi r^2}.$$

So, $f(r) = F'(t) = 2\lambda\pi r e^{-\lambda\pi r^2}$, $r > 0$.

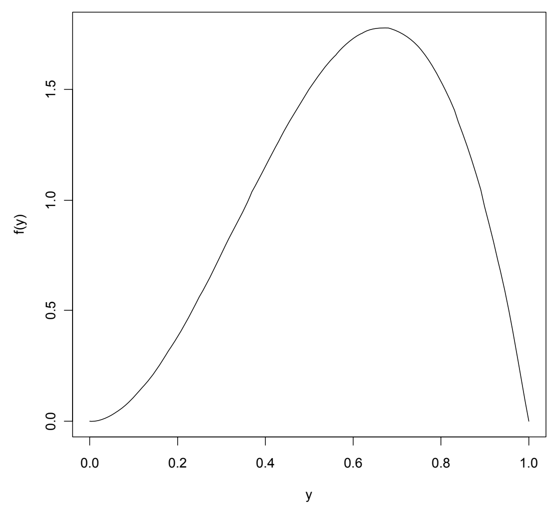
4.174 Let Y = interview time (in hours). The second applicant will have to wait only if the time to interview the first applicant exceeds 15 minutes, or .25 hour. So,

$$P(Y > .25) = \int_{.25}^{\infty} 2e^{-2y}\,dy = e^{-.5} = .61.$$

4.175 From Ex. 4.11, the median value will satisfy $F(y) = y^2/2 = .5$. Thus, the median is given by $\sqrt{2} = 1.414$.

4.176 The distribution function for the exponential distribution with mean β is $F(y) = 1 - e^{-y/\beta}$. Thus, we require the value *y* such that $F(y) = 1 - e^{-y/\beta} = .5$. Solving for *y*, this is βln(2).

4.177 **a.** 2.07944 = 3ln(2)
b. 3.35669 < 4, the mean of the gamma distribution.
c. 46.70909 < 50, the mean of the gamma distribution.
d. In all cases the median is *less* than the mean, indicating right skewed distributions.



4.178 The graph of this beta density is above.

a. Using the relationship with binomial probabilities,
$P(.1 \le Y \le .2) = 4(.2)^3(.8) + (.2)^4 - 4(.1)^3(.9) - (.1)^4 = .0235$.
b. $P(.1 \le Y \le .2) = .9963 - .9728 = .0235$, which is the same answer as above.
c. $\phi_{.05} = .24860$, $\phi_{.95} = .90239$.
d. $P(\phi_{.05} \le Y \le \phi_{.95}) = .9$.

4.179 Let *X* represent the grocer's profit. In general, her profit (in cents) on a order of 100*k* pounds of food will be $X = 1000Y - 600k$ as long as $Y < k$. But, when $Y \ge k$ the grocer's profit will be $X = 1000k - 600k = 400k$. Define the random variable Y' as

$$Y' = \begin{cases} Y & 0 \le Y < k \\ k & Y \ge k \end{cases}$$

Then, we can write $g(Y') = X = 1000Y' + 600k$. The random variable Y' has a mixed distribution with one discrete point at *k*. Therefore,

$$c_1 = P(Y' = k) = P(Y \ge k) = \int_k^1 3y^2 dy = 1 - k^3, \text{ so that } c_2 = k^3.$$

Thus, $F_2(y) = \begin{cases} 0 & 0 \le y < k \\ 1 & y \ge k \end{cases}$ and $F_1(y) = P(Y_2 \le y \mid 0 \le Y' < k) = \frac{\int_0^y 3t^2 dt}{k^3} = \frac{y^3}{k^3}$, $0 \le y < k$.

Thus, from Definition 4.15,

$$E(X) = E[g(Y')] = c_1 E[g(Y_1)] + c_2 E[g(Y_2)] = (1-k^3)400k + k^3 \int_0^k (1000y - 600k)\tfrac{3y^2}{k^3} dy,$$

or $E(X) = 400k - 250k^2$. This is maximized at $k = (.4)^{1/3} = .7368$. (2^{nd} derivative is –.)

4.180 **a.** Using the result of Ex. 4.99, $P(Y \le 4) = 1 - \sum_{y=0}^{2} \frac{4^y e^{-4}}{y!} = .7619$.

b. A similar result is found.

4.181 The mgf for Z is $m_Z(t) = E(e^{Zt}) = E(e^{(\frac{Y-\mu}{\sigma})t}) = e^{-\frac{\mu}{\sigma}t} m_Y(t/\sigma) = e^{t^2/2}$, which is a mgf for a normal distribution with $\mu = 0$ and $\sigma = 1$.

4.182 **a.** $P(Y \le 4) = P(X \le \ln 4) = P[Z \le (\ln 4 - 4)/1] = P(Z \le -2.61) = .0045$.
b. $P(Y > 8) = P(X > \ln 8) = P[Z > (\ln 8 - 4)/1] = P(Z > -1.92) = .9726$.

4.183 **a.** $E(Y) = e^{3+16/2} = e^{11}$ (598.74 g), $V(Y) = e^{22}(e^{16} - 1)$.

b. With $k = 2$, the interval is given by $E(Y) \pm 2\sqrt{V(Y)}$ or 598.74 ± 3,569,038.7. Since the weights must be positive, the final interval is (0, 3,570,236.1)
c. $P(Y < 598.74) = P(\ln Y < 6.3948) = P[Z < (6.3948 - 3)/4] = P(Z < .8487) = .9726$.

4.184 The mgf for Y is $m_Y(t) = E(e^{tY}) = \frac{1}{2}\int_{\infty}^{0} e^{ty}e^{y} dy + \frac{1}{2}\int_{0}^{\infty} e^{ty}e^{-y} dy = \frac{1}{2}\int_{\infty}^{0} e^{(t+1)y} dy + \frac{1}{2}\int_{0}^{\infty} e^{-y(1-t)} dy$.

This simplifies to $m_Y(t) = \frac{1}{1-t^2}$. Using this, $E(Y) = m'(t)\big|_{t=0} = \frac{2t}{(1-t^2)}\big|_{t=0} = 0$.

4.185 **a.** $\int_{-\infty}^{\infty} f(y)dy = a\int_{-\infty}^{\infty} f_1(y)dy + (1-a)\int_{-\infty}^{\infty} f_2(y)dy = a + (1-a) = 1$.

b. i. $E(Y) = \int_{-\infty}^{\infty} yf(y)dy = a\int_{-\infty}^{\infty} yf_1(y)dy + (1-a)\int_{-\infty}^{\infty} yf_2(y)dy = a\mu_1 + (1-a)\mu_2$

ii. $E(Y_2) = a\int_{-\infty}^{\infty} y^2 f_1(y)dy + (1-a)\int_{-\infty}^{\infty} y^2 f_2(y)dy = a(\mu_1^2 + \sigma_1^2) + (1-a)(\mu_2^2 + \sigma_2^2)$. So,

$V(Y) = E(Y^2) - [E(Y)]^2 = a(\mu_1^2 + \sigma_1^2) + (1-a)(\mu_2^2 + \sigma_2^2) - [a\mu_1 + (1-a)\mu_2]^2$, which simplifies to $a\sigma_1^2 + (1-a)\sigma_2^2 + a(1-a)[\mu_1 - \mu_2]^2$

4.186 For $m = 2$, $E(Y) = \int_0^\infty y \frac{2y}{\alpha} e^{-y^2/\alpha} dy$. Let $u = y^2/\alpha$. Then, $dy = \frac{\sqrt{\alpha}}{2\sqrt{u}} du$. Then,

$$E(Y) = \int_0^\infty \frac{2y^2}{\alpha} e^{-y^2/\alpha} dy = \sqrt{\alpha} \int_0^\infty u^{1/2} e^{-u} du = \sqrt{\alpha}\Gamma(3/2) = \frac{\sqrt{\alpha}\Gamma(1/2)}{2}.$$ Using similar methods,

it can be shown that $E(Y^2) = \alpha$ so that $V(Y) = \alpha - \left[\frac{\sqrt{\alpha}\Gamma(1/2)}{2}\right]^2 = \alpha\left[1 - \frac{\pi}{4}\right]$, since it will

be shown in Ex. 4.196 that $\Gamma(1/2) = \sqrt{\pi}$.

4.187 The density for Y = the life length of a resistor (in thousands of hours) is

$$f(y) = \frac{2ye^{-y^2/10}}{10}, \quad y > 0.$$

a. $P(Y > 5) = \int_5^\infty \frac{2ye^{-y^2/10}}{10} dy = -e^{-y^2/10}\Big]_5^\infty = e^{-2.5} = .082.$

b. Let X = # of resistors that burn out prior to 5000 hours. Thus, X is a binomial random variable with $n = 3$ and $p = .082$. Then, $P(X = 1) = 3(1 - .082)(.082)^2 = .0186$.

4.188 **a.** This is the exponential distribution with $\beta = \alpha$.
b. Using the substitution $u = y^m/\alpha$ in the integrals below, we find:

$$E(Y) = \int_0^\infty \tfrac{m}{\alpha} y^m e^{-y^m/\alpha} dy = \alpha^{1/m} \int_0^\infty u^{1/m} e^{-u} du = \alpha^{1/m}\Gamma(1 + 1/m)$$

$$E(Y^2) = \int_0^\infty \tfrac{m}{\alpha} y^{m+1} e^{-y^m/\alpha} dy = \alpha^{2/m} \int_0^\infty u^{2/m} e^{-u} du = \alpha^{1/m}\Gamma(1 + 2/m).$$

Thus,
$V(Y) = \alpha^{2/m}[\Gamma(1 + 2/m) + \Gamma^2(1 + 1/m)]$.

4.189 Since this density is symmetric about 0, so using the result from Ex. 4.27, $E(Y) = 0$. Also, it is clear that $V(Y) = E(Y^2)$. Thus,

$$E(Y^2) = \int_{-1}^1 \frac{1}{B(1/2,(n-2)/2)} y^2 (1 - y^2)^{(n-4)/2} dy = \frac{B(3/2,(n-2)/2)}{B(1/2,(n-2)/2)} = \frac{1}{n-1} = V(Y).$$ This

equality follows after making the substitution $u = y^2$.

4.190 **a.** For the exponential distribution, $f(t) = \frac{1}{\beta} e^{-t/\beta}$ and $F(t) = 1 - e^{-t/\beta}$. Thus, $r(t) = 1/\beta$.

b. For the Weibull, $f(y) = \frac{my^{m-1}}{\alpha} e^{-y^m/\alpha}$ and $F(y) = 1 - e^{-y^m/\alpha}$. Thus, $r(t) = \frac{my^{m-1}}{\alpha}$, which is an increasing function of t when $m > 1$.

4.191 **a.** $G(y) = P(Y \le y \mid Y \ge c) = \frac{P(c \le Y \le y)}{P(Y \ge c)} = \frac{F(y) - F(c)}{1 - F(c)}$.

b. Refer to the properties of distribution functions; namely, show $G(-\infty) = 0$, $G(\infty) = 1$, and for constants a and b such that $a \le b$, $G(a) \le G(b)$.

c. It is given that $F(y) = 1 - e^{-y^2/3}$. Thus, by using the result in part b above,

$$P(Y \le 4 \mid Y \ge 2) = \frac{1 - e^{-4^2/3} - (1 - e^{-2^2/2}}{e^{-2^2/2}} = 1 - e^{-4}.$$

4.192 **a.** $E(V) = 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2} \int_0^\infty v^3 e^{-v^2(m/2KT)} dv$. To evaluate this integral, let $u = v^2\left(\frac{m}{2KT}\right)$ so that $dv = \sqrt{\frac{2KT}{m}} \frac{1}{2\sqrt{u}} du$ to obtain $E(V) = 2\sqrt{\frac{2KT}{m\pi}} \int_0^\infty u e^{-u} du = 2\sqrt{\frac{2KT}{m\pi}} \Gamma(2) = 2\sqrt{\frac{2KT}{m\pi}}$.

b. $E(\frac{1}{2}mV^2) = \frac{1}{2}mE(V^2) = 2\pi m\left(\frac{m}{2\pi KT}\right)^{3/2} \int_0^\infty v^4 e^{-v^2(m/2KT)} dv$. Here, let $u = v^2\left(\frac{m}{2KT}\right)$ so that $dv = \sqrt{\frac{2KT}{m}} \frac{1}{2\sqrt{u}} du$ to obtain $E(\frac{1}{2}mV^2) = \frac{2KT}{\sqrt{\pi}} \Gamma(\frac{5}{2}) = \frac{3}{2}KT$ (here, we again used the result from Ex. 4.196 where it is shown that $\Gamma(1/2) = \sqrt{\pi}$).

4.193 For $f(y) = \frac{1}{100} e^{-y/100}$, we have that $F(y) = 1 - e^{-y/100}$. Thus,

$$E(Y \mid Y \ge 50) = \frac{1}{e^{-1/2}} \int_{50}^\infty \frac{y e^{-y/100}}{100} dy = 150.$$

Note that this value is 50 + 100, where 100 is the (unconditional) mean of Y. This illustrates the memoryless property of the exponential distribution.

4.194 $\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(1/2)uy^2} dy\right]\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(1/2)ux^2} dx\right] = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(1/2)u(x^2+y^2)} dx dy$. By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$. Thus, the desired integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty e^{-(1/2)ur^2} r dr d\theta = \frac{1}{u}.$$

Note that the result proves that the standard normal density integrates to 1 with $u = 1$.

4.195 **a.** First note that $W = (Z^2 + 3Z)^2 = Z^4 + 6Z^3 + 9Z^2$. The odd moments of the standard normal are equal to 0, and $E(Z^2) = V(Z) + [E(Z)]^2 = 1 + 0 = 1$. Also, using the result in Ex. 4.199, $E(Z^4) = 3$ so that $E(W) = 3 + 9(1) = 12$.

b. Applying Ex. 4.198 and the result in part a:

$$P(W \le w) \ge 1 - \frac{E(W)}{w} = .9,$$

so that $w = 120$.

4.196 $\Gamma(1/2) = \int_0^\infty y^{-1/2} e^{-y} dy = \int_0^\infty \sqrt{2} e^{-(1/2)x^2} dx = \sqrt{2}\sqrt{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} dx = 2\sqrt{\pi}\left[\frac{1}{2}\right] = \sqrt{\pi}$ (relating the last integral to that $P(Z > 0)$, where Z is a standard normal random variable).

4.197 a. Let $y = \sin^2\theta$, so that $dy = 2\sin\theta\cos\theta d\theta$. Thus,

$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = 2\int_0^{\pi/2} \sin^{2\alpha-2}\theta(1-\sin^2\theta)^{\beta-1} d\theta = 2\int_0^{\pi/2} \sin^{2\alpha-1}\theta\cos^{2\beta-1} d\theta$, using the trig identity $1 - \sin^2\theta = \cos^2\theta$.

b. Following the text, $\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty y^{\alpha-1}e^{-y} dy \int_0^\infty z^{\beta-1}e^{-z} dz = \int_0^\infty\int_0^\infty y^{\alpha-1}z^{\beta-1}e^{-y-z} dydz$. Now, use the transformation $y = r^2\cos^2\theta,\ x = r^2\sin^2\theta$ so that $dydz = 4r^3\cos\theta\sin\theta$. Following this and using the result in part a, we find

$$\Gamma(\alpha)\Gamma(\beta) = B(\alpha,\beta)\int_0^\infty r^{2(\alpha+\beta-1)}e^{-r^2} 2rdr.$$

A final transformation with $x = r^2$ gives $\Gamma(\alpha)\Gamma(\beta) = B(\alpha,\beta)\Gamma(\alpha+\beta)$, proving the result.

4.198 Note that

$$E[|g(Y)|] = \int_{-\infty}^{\infty} |g(y)| f(y)dy \ge \int_{|g(y)|>k}^{\infty} |g(y)| f(y)dy > \int_{|g(y)|>k}^{\infty} kf(y)dy = kP(|g(Y)| > k),$$

Since $|g(y)| > k$ for this integral. Therefore,

$$P(|g(Y)| \le k) \ge 1 - E(|g(Y)|)/k.$$

4.199 **a.** Define $g(y) = y^{2i-1}e^{-y^2/2}$ for positive integer values of i. Observe that $g(-y) = -g(y)$ so that $g(y)$ is an odd function. The expected value $E(Z^{2i-1})$ can be written as $E(Z^{2i-1}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} g(y)dy$ which is thus equal to 0.

b. Now, define $h(y) = y^{2i}e^{-y^2/2}$ for positive integer values of i. Observe that $h(-y) = h(y)$ so that $h(y)$ is an even function. The expected value $E(Z^{2i})$ can be written as $E(Z^{2i}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} h(y)dy = \int_0^\infty \frac{2}{\sqrt{2\pi}} h(y)dy$. Therefore, the integral becomes

$$E(Z^{2i}) = \int_0^\infty \frac{2}{\sqrt{2\pi}} y^{2i}e^{-y^2/2}dy = \frac{1}{\sqrt{\pi}}\int_0^\infty 2^i w^{i-1/2}e^{-w}dw = \frac{1}{\sqrt{\pi}} 2^i\Gamma(i + 1/2).$$

In the last integral, we applied the transformation $w = z^2/2$.

c. $$E(Z^2) = \tfrac{1}{\sqrt{\pi}} 2^1 \Gamma(1+1/2) = \tfrac{1}{\sqrt{\pi}} 2^1 (1/2)\sqrt{\pi} = 1$$
$$E(Z^4) = \tfrac{1}{\sqrt{\pi}} 2^2 \Gamma(2+1/2) = \tfrac{1}{\sqrt{\pi}} 2^2 (3/2)(1/2)\sqrt{\pi} = 3$$
$$E(Z^6) = \tfrac{1}{\sqrt{\pi}} 2^3 \Gamma(3+1/2) = \tfrac{1}{\sqrt{\pi}} 2^3 (5/2)(3/2)(1/2)\sqrt{\pi} = 15$$
$$E(Z^8) = \tfrac{1}{\sqrt{\pi}} 2^4 \Gamma(4+1/2) = \tfrac{1}{\sqrt{\pi}} 2^4 (7/2)(5/2)(3/2)(1/2)\sqrt{\pi} = 105.$$

d. The result follows from:
$$\prod_{j=i}^{i}(2j-1) = \prod_{j=i}^{i} 2(j-1/2) = 2^i \prod_{j=i}^{i}(j-1/2) = 2^i \Gamma(i+1/2)\left(\tfrac{1}{\sqrt{\pi}}\right) = E(Z^{2i}).$$

4.200 **a.** $E(Y^a) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{a+\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(a+\alpha)\Gamma(\beta)}{\Gamma(a+\alpha+\beta)} = \frac{\Gamma(\alpha+\beta)\Gamma(a+\alpha)}{\Gamma(\alpha)\Gamma(a+\alpha+\beta)}$.

b. The value $\alpha + a$ must be positive in the beta density.

c. With $a = 1$, $E(Y^1) = \frac{\Gamma(\alpha+\beta)\Gamma(1+\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$.

d. With $a = 1/2$, $E(Y^{1/2}) = \frac{\Gamma(\alpha+\beta)\Gamma(1/2+\alpha)}{\Gamma(\alpha)\Gamma(1/2+\alpha+\beta)}$.

e. With $a = -1$, $E(Y^{-1}) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\alpha+\beta-1)} = \frac{\alpha+\beta-1}{\alpha-1}$, $\alpha > 1$

With $a = -1/2$, $E(Y^{-1/2}) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha-1/2)}{\Gamma(\alpha)\Gamma(\alpha+\beta-1/2)}$, $\alpha > 1/2$

With $a = -2$, $E(Y^{-2}) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha-2)}{\Gamma(\alpha)\Gamma(\alpha+\beta-2)} = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)}$, $\alpha > 2$.