CSC 373 H1 | Quiz # 12 | 29 November 2012
Aids Allowed: **none** | *Worth*: 1.5% | *Duration*: **10 minutes**

Warning! This quiz involves a *lot* of reading, but only a *little* bit of writing.
Suppose you want to transfer a number of songs to CDs, using as few CDs as possible (the final order of the songs on the CDs does not matter). Formally, we define the problem as follows.

Input: A list of positive integer song *sizes* $[s_1, s_2, \ldots, s_n]$, and a positive integer *capacity* C.

Output: A partition of $[1, 2, \ldots, n]$ into sublists $L_1, L_2, \ldots, L_k$ (*i.e.*, each i belongs to exactly one L_j) such that k is **as small as possible** and no CD uses more than capacity C ($\forall j, \sum_{i \in L_j} s_i \leqslant C$).

This problem is *NP*-hard, but consider the following "First-Fit" approximation algorithm.

```
k ← 0
for i = 1, 2, ..., n:   # for each song
    # Put song i on the first CD L_j on which it fits.
    for j = 1, 2, ..., k:
        if s_i + ℓ_j ⩽ C:   # ℓ_j is the total size of the songs already on CD L_j
            L_j.append(i)
            ℓ_j ← ℓ_j + s_i
            continue with the next value of i
    # We get here only if s_i does not fit on any existing L_j; in this case, create a new CD.
    k ← k + 1
    L_k = [i]
    ℓ_k = s_i
```

1. For any input $s_1, s_2, \ldots, s_n$, let k be the number of CDs generated by this algorithm, and k^* be the minimum number of CDs required to store all the songs. Give a precise definition of what it means for the algorithm to have approximation ratio r.

2. Use the following facts to show that the algorithm has an approximation ratio of at most 2.

- Every solution uses *at least* $\sum_{i=1}^{n} s_i / C$ CDs.
- The solution produced by the algorithm leaves *at most* one CD less than half full. In other words, the total size of the songs on any *two* CDs is greater than the capacity C.

Hint: Think about $L_1 + L_2 + \cdots + L_k$ and how it relates to the total size of all the songs $\sum_{i=1}^{n} s_i$.