CSC373 Winter 2015 Problem Set # 9

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March 23, 2015

First, define the decision problem and the search problem of SubsetSum.

SubsetSum decision problem:
Input: A set $S \subseteq \mathbf{Z}^+$ and $t \in \mathbf{Z}^+$
Output: Is there some subset of S whose sum is exactly t?

SubsetSum search problem:
Input: A set $S \subseteq \mathbf{Z}^+$ and $t \in \mathbf{Z}^+$
Output: A subset of S whose sum is exactly t, or NIL if there is no such subset.

We show that SubsetSum problem is self-reducible by showing

$$\text{SubsetSumSearch} \underset{p}{\longrightarrow} \text{SubsetSumDecision}.$$

Proof: Suppose algorithm $\text{Ssd}(S, t)$ solves SubsetSumDecision in polynomial time $T(n)$ where $n = |S|$.
Consider the following algorithm $\text{Sss}(S, t)$ that solves SubsetSumSearch in polynomial time.

$\text{Sss}(S, t)$
if not $\text{Ssd}(S, t)$
    return FALSE
for each $x \in S$
    if $\text{Ssd}(S - \{x\}, t)$
        $S = S - \{x\}$
return S

Runtime analysis
In the worst case, the for-loop iterates n times. It takes $O(n)$ time to form a subset of S in each iteration. It takes $O(T(n))$ time to call Ssd in each iteration. Hence, the algorithm runs in $nO(n + T(n)) = O(n^2 + nT(n))$. If Ssd runs in polynomial time, then Sss runs in polynomial time.

Justification of correctness
Let S_i denote the value of S after each iteration. In particular, S_0 is equal to the initial set S.
Claim: $C \subseteq S_i$ is a loop invariant for some $C \subseteq S$ and $\Sigma_{x \in C} x = t$.
Base case: $S_0 \subseteq S$ is correct clearly since $S_0 = S$.
Induction step: Assume the claim is true for the i^{th} iteration. Then Consider the $(i+1)^{th}$ iteration. If $\text{Ssd}(S_i - \{x\}, t)$ is TRUE, then $x \notin C$ so $C \subseteq S_{i+1} = S_i - \{x\}$. If $\text{Ssd}(S_i - \{x\}, t)$ is FALSE,

$C \subseteq S_{i+1} = S_i$.
Hence, the claim is true.
Finally, since the loop iterates over all the elements in S, all the elements that cannot form a sum t are removed from S. Therefore, SSS is correct.

By the above argument, SUBSETSUM is polytime self-reducible. ■