Slide 12, chapter 4

$Y \sim U(\theta_1, \theta_2)$; $\Rightarrow f(y) = \dfrac{1}{\theta_2 - \theta_1}$ $\theta_1 < y < \theta_2$

a) $E(Y) = \dfrac{\theta_2 + \theta_1}{2}$;

$$E(Y) = \int_{-\infty}^{+\infty} y\, f(y)\, dy = \int_{\theta_1}^{\theta_2} y \cdot \frac{1}{\theta_2 - \theta_1}\, dy = \frac{1}{\theta_2 - \theta_1} \cdot \frac{y^2}{2}\Big|_{\theta_1}^{\theta_2}$$

$$= \frac{\theta_2^2 - \theta_1^2}{2(\theta_2 - \theta_1)} = \frac{\cancel{(\theta_2 - \theta_1)}(\theta_2 + \theta_1)}{2\cancel{(\theta_2 - \theta_1)}} = \frac{\theta_2 + \theta_1}{2}$$

$$E(Y^2) = \int_{\theta_1}^{\theta_2} y^2 \cdot \frac{1}{\theta_2 - \theta_1}\, dy = \frac{1}{\theta_2 - \theta_1} \cdot \frac{y^3}{3}\Big|_{\theta_1}^{\theta_2}$$

$$= \frac{1}{\theta_2 - \theta_1} \cdot \frac{\theta_2^3 - \theta_1^3}{3} = \frac{(\theta_2 - \theta_1)(\theta_2^2 + \theta_1\theta_2 + \theta_1^2)}{(\theta_2 - \theta_1) \cdot 3}$$

$$= \frac{(\theta_2^2 + \theta_1\theta_2 + \theta_1^2)}{3}$$

$$Var(Y) = E(Y^2) - (EY)^2 = \frac{\theta_2^2 + \theta_1\theta_2 + \theta_1^2}{3} - \frac{(\theta_1 + \theta_2)^2}{4}$$

$$= \frac{\theta_2^2 + \theta_1\theta_2 + \theta_1^2}{3} - \frac{\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2}{4} = \frac{\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2}{12} = \frac{(\theta_2 - \theta_1)^2}{12}$$

Ex. #4.50 ; Slide 14 ; chapter 4

$Y =$ Time when the call comes in

$$Y \sim \text{Unif}(0,5) \Rightarrow f(y) = \begin{cases} \frac{1}{5} & 0 \leq y \leq 1 \\ 0 & \text{o.w} \end{cases}$$

$$P(\text{center is up when call comes in}) = P(0 < Y < 1) + P(3 < Y < 4)$$

$$= \int_0^1 \frac{1}{5}\,dy + \int_3^4 \frac{1}{5}\,dy = \frac{1}{5}\left(y\Big|_0^1 + y\Big|_3^4 \right)$$

$$= \frac{1}{5}\left((1-0)+(4-3)\right) = \frac{1}{5}(1+1) = \frac{2}{5}$$

Ex. #4.88 ; Slide 23 , chapter 4

$Y \sim Exp(2.4) \qquad f(y) = \frac{1}{\beta} e^{-y/\beta}$

a) $P(Y > 3) = \int_3^{\infty} \frac{1}{2.4} e^{-y/2.4} \, dy = -\frac{2.4}{2.4} \left(e^{-y/2.4} \right) \Big|_3^{\infty} = e^{-3/2.4} = 0.2865$

b) $P(2 \leqslant Y \leqslant 3) = \int_2^3 \frac{1}{2.4} e^{-y/2.4} \, dy = -e^{-y/2.4} \Big|_2^3 = e^{-2/2.4} - e^{-3/2.4}$

$= .1481$

Let $Y \sim \text{Gamma}(\alpha, \beta)$; slide 33, chapter 4

mgf of Y ?

$$m(t) = E(e^{tY}) = \int_0^\infty e^{ty} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} \exp\left(-y\left(\frac{1}{\beta}-t\right)\right) dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} \exp\left(\frac{-y}{\beta/(1-\beta t)}\right) dy$$

$$\Rightarrow \frac{\beta}{1-\beta t} = \beta_0 > 0 \Rightarrow 1-\beta t > 0 \Rightarrow t < \frac{1}{\beta}$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1} e^{-y/\beta_0}}{\beta_0^\alpha} dy \cdot \beta_0^\alpha$$

$$= \frac{1}{\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{\alpha-1} e^{-y/\beta_0}}{\beta_0^\alpha \Gamma(\alpha)} dy}_{=1} \cdot \beta_0^\alpha = \frac{\beta_0^\alpha}{\beta^\alpha} = \frac{\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\beta^\alpha} = \left(\frac{1}{1-\beta t}\right)^\alpha$$

$$\Rightarrow m(t) = (1-\beta t)^{-\alpha} \qquad t < 1/\beta$$

$$E(Y) = m^{(1)}(0) = -\alpha (1-\beta t)^{-\alpha-1} \cdot (-\beta) \Big|_{t=0} = \alpha\beta$$

$$E(y^2) = m^{(2)}(t)\Big|_{t=0} = (-\alpha)(-\alpha-1)(1-\beta t)^{-\alpha-2}\beta^2\Big|_{t=0}$$

$$= \alpha(\alpha+1)\beta^2$$

$$Var(y) = E(y^2) - (Ey)^2 = \left(\alpha(\alpha+1)\beta^2\right) - \alpha^2\beta^2$$

$$= \left(\alpha^2\beta^2 + \alpha\beta^2\right) - \alpha^2\beta^2 = \alpha\beta^2$$

Ex. # 4.137 , Slide 35 ; Chapter 4

a. Y with $m(t)$; $U = aY + b$

$$m_U(t) = E\left(e^{tu}\right) = E\left(e^{t(aY+b)}\right) = E\left(e^{taY} \cdot e^{tb}\right)$$

$$= e^{tb} E\left(e^{taY}\right) = e^{tb} M_Y(at)$$

$$E(u) = m_u^{(1)}(0) = be^{tb} \cdot M_Y(at) + e^{tb} M_Y^{(1)}(at)\, a$$

$$= b\, M_Y(0) + a \underbrace{M_Y^{(1)}(0)}_{E(Y)} = b + a\mu = E(u)$$

$$m_u^{(2)}(t) = b^2 e^{tb} m_y(at) + abe^{bt} m_y^{(1)}(at) + ba\, e^{tb} m_y^{(1)}(at) + a^2 m_y^{(2)}(at)\, e^{bt}$$

$$m_u^{(2)}(0) = b^2 + 2ab\mu + a^2 E(Y^2) = E(u^2)$$

$$\left(\text{Since } m_y(0) = 1 \;;\; m_y^{(1)}(0) = \mu \;;\; m_y^{(2)}(0) = E(Y^2)\right)$$

Therefore $V(U) = E(U^2) - (EU)^2 = b^2 + 2ab\mu + a^2 E(Y^2) - (b + a\mu)^2$

$$= a^2\left(E(Y^2) - \mu^2\right) = a^2\sigma^2$$

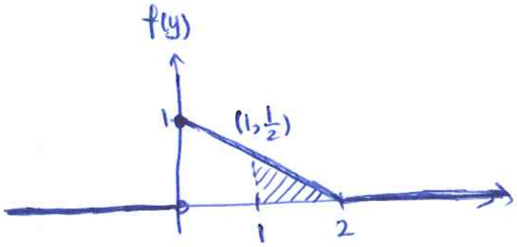
Ex. # 4.16 ; Slide 38 ; chapter 4

$$f(y) = \begin{cases} c(2-y) & 0 \leq y < 2 \\ 0 & \text{o.w} \end{cases}$$

a) $\displaystyle\int_{-\infty}^{+\infty} f(y)\,dy = 1 \Rightarrow \int_0^2 c(2-y)\,dy = 1$

$$\Rightarrow 1 = \int_0^2 c(2-y)\,dy = c\left(2y - \frac{y^2}{2}\right)\Big|_0^2 = c\left[(4-2)-(0)\right] = 2c \Rightarrow c = \frac{1}{2}$$

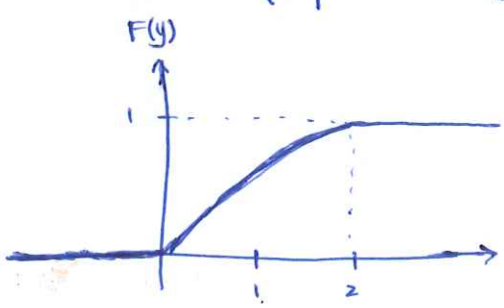
$$\Rightarrow f(y) = \begin{cases} \frac{1}{2}(2-y) & 0 < y < 2 \\ 0 & \text{o.w} \end{cases}$$



$$F(y) = \int_0^y \frac{1}{2}(2-t)\,dt = \frac{1}{2}\left(2t - \frac{t^2}{2}\right)\Big|_0^y = \frac{1}{2}\left(2y - \frac{y^2}{2}\right) = y - \frac{y^2}{4}, \quad 0 \leq y \leq 2$$

$$\Rightarrow F(y) = \begin{cases} 0 & y < 0 \\ y - \frac{y^2}{4} & 0 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$

b)



c) $P(1 \leqslant Y \leqslant 2) = F(2) - F(1) = \left(2 - \frac{4}{4}\right) - \left(1 - \frac{1}{4}\right)$

$= 1 - \left(\frac{3}{4}\right) = \frac{1}{4}$

Ex.#4.96 ; slide: 42 , chapter 4

a) $Y \sim \text{Gamma}(\alpha=4, \beta=2)$

$$1=\int_0^\infty f(y)\,dy=\int_0^\infty K\,y^3 e^{-y/2}\,dy = K\int_0^\infty y^{3} e^{-y/2} = K * \beta^{\alpha}\,\Gamma(\alpha) = K\,2^4\,\Gamma(4)$$

$$\Rightarrow K=\frac{1}{16\cdot 3!}=\frac{1}{96}$$

b) $\beta=2,\ \alpha=\frac{v}{2}=4 \Rightarrow v=8 \Rightarrow Y\sim\chi^2_8$

c) $E(y)=\alpha\beta = \overset{4*2}{\cancel{2*4}} = 8$;

$Var(y)=\alpha\beta^2 = 4*(2)^2=16$

$sd(y)=\sqrt{16}=4$