Q1

(a)

let (e_1) be a basis of $\mathbb{R}$

$\text{im}T \subseteq \mathbb{R}$, then (e_1) is a basis of $\text{im}T$

$\Leftrightarrow \dim(\text{im}T) = 1$

Since $\dim(\ker T) + \dim(\text{im}T) = \dim V$ and $\dim V = n$

then $\dim(\ker T) = n-1$

(b) let $x \in V$, not in $\ker T$ s.t. $Tx \neq 0$.

Show $V = \ker T \oplus \text{span}\{x\}$

Let $(x_1, x_2, \dots x_n)$ be a basis for V

i) Show $V \subseteq \ker T \oplus \text{span}\{x\}$

let $y \in V$, $(x_1, x_2, \dots x_{n-1})$ be a basis of $\ker T$ wlog,

case 1 $y \in \ker T$

case 2 $y \notin \ker T \Leftrightarrow y \in \text{span}\{x_n\}$

since $x \notin \ker T \Leftrightarrow x \in \text{span}\{x_n\}$,

then $y \in \text{span}\{x\}$

we also know $\ker T \cap \operatorname{span}\{x\} = \{0\}$ (Since $x \notin \ker T$, T is Linear)

then $y \in \ker T \oplus \operatorname{span}\{x\}$

$\Leftrightarrow V \in \ker T \oplus \operatorname{span}\{x\}$

ii) Show $\ker T \oplus \operatorname{span}\{x\} \subseteq V$

We know $\ker T \subseteq V$ and $\operatorname{span}\{x\} \subseteq V$

let $z \in \ker T \oplus \operatorname{span}\{x\}$

Then $z = c_1 q_1 + c_2 q_2$ $\quad c_1, c_2 \in \mathbb{R}$, $q_1 \in \ker T$, $q_2 \in \operatorname{span}\{x\}$

$\Rightarrow z \in V$ (V is vector space)

Thus $V = \ker T \oplus \operatorname{span}\{x\}$ $\square$