Question 4

(a). The procedure PriceIsRightGreedy employs a greedy strategy as an attempt to solve the problem. Since the input is sorted in non-increasing order, the next element of the sequence is greedily chosen as the first unseen element in the sequence X that "fits" (so that the collective sum is at most B).

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PriceIsRightGreedy(B, X = x_1, x_2, ..., x_n):
S = ∅ // subsequence of elements in X
sumSoFar = 0
for i = 1 to n:
    if sumSoFar + x_i ≤ B:
        S = S ∪ {x_i}
        sumSoFar = sumSoFar + x_i
return S
```

(b). ***Solution.*** We now show that PriceIsRightGreedy has approximation ratio *at most* 2. Specifically, let the sum of S (that is, $\sum_{x \in S} x$) as returned by the procedure above be denoted by $|S|$. Then, we want to show that,

$$\frac{OPT}{|S|} \leq 2,$$

where OPT is the maximal possible subsequence sum satisfying the constraint of the problem. To see this, we consider two exhaustive cases.

Case 1: $x_1 \geq \frac{B}{2}$. Thus, here we know that $\frac{B}{2} \leq x_1 \leq B$, and so it is clear that the algorithm adds x_1 to S. Thus, $|S| \geq \frac{B}{2}$. Of course, we know that $OPT \leq B$, and so,

$$\begin{aligned} \frac{OPT}{|S|} &\leq \frac{B}{|S|} \\ &\leq \frac{B}{\frac{B}{2}} \\ &= 2, \end{aligned}$$

as desired.

Case 2: $x_1 < \frac{B}{2}$. Since the input sequence is non-increasing, we know that every subsequent element is also less than $\frac{B}{2}$. We either have that $|S| \geq \frac{B}{2}$ or $|S| < \frac{B}{2}$. If $|S| \geq \frac{B}{2}$, then we are done by the argument in Case 1. So, suppose that $|S| < \frac{B}{2}$. Each element x_i is less than $\frac{B}{2}$, and thus if it is left out of S, $|S| + x_i < B$ — which shows that every x_i must be chosen by the algorithm. Thus, the optimal algorithm must also choose every element in the input — in other words, OPT is the sum of all elements in the input, and so $OPT = |S|$. Thus, $\frac{OPT}{|S|} = 1 \leq 2$, and again we have the desired result.

Together, these cases encompass all possible scenarios, and thus we are done.

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