Worth: 2% **Due:** By 8:59pm on Tuesday 17 March

Remember to write your *full name* and *student number* prominently on your submission.

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*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission **every** source of information you used to complete this homework (other than your own lecture and tutorial notes). For example, indicate clearly the **name** of every student with whom you had discussions, the **title and sections** of every textbook you consulted (including the course textbook), the **source** of every web document you used (including documents from the course webpage), etc.*

*For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.*

Recall that a path in a graph is *simple* iff it contains no repeated vertex or edge. The definition of simple cycle is similar (except, of course, that the first and last vertex are the same). Consider the following decision problems.

- HamiltonianPath (HP):

  Input: Graph $G = (V, E)$, directed or undirected.

  Output: Does G contain some simple path that includes every vertex of V?

- HamiltonianCycle (HC):

  Input: Graph $G = (V, E)$, directed or undirected.

  Output: Does G contain some simple cycle that includes every vertex of V?

The textbook proves that HC is NP-complete (subsection 34.5.3).

Below you will find different "proofs" that HP is NP-**hard**. State clearly whether or not each "proof" is correct and, if not, explain **every** error committed in the "proof."

1. HP is NP-hard because HC $\leqslant_p$ HP:

   On input $G = (V, E)$, output $G = (V, E)$.

   Clearly, this can be computed in polytime!

   Also, G contains a Hamiltonian cycle iff G contains a Hamiltonian path.

2. HP is NP-hard because HP $\leqslant_p$ HC:

   On input $G = (V, E)$, output $G' = \Big(V \cup \{v_0\}, E \cup \{(v_0, u) : u \in V\} \cup \{(u, v_0) : u \in V\}\Big)$.

   Clearly, G' can be computed in polytime from G: we only add one new vertex v_0 together with edges that connect v_0 to and from every vertex of G.

   Also, if G contains some Hamiltonian path $(v_1, v_2), \ldots, (v_{n-1}, v_n)$, then (v_0, v_1), (v_1, v_2), $\ldots$, (v_{n-1}, v_n), (v_n, v_0) is a Hamiltonian cycle in G'.

   Finally, if G' contains some Hamiltonian cycle (v_0, v_1), (v_1, v_2), $\ldots$, (v_{n-1}, v_n), (v_n, v_0), then (v_1, v_2), $\ldots$, (v_{n-1}, v_n) is a Hamiltonian path in G.

3. HP is NP-hard because HC $\leqslant_p$ HP:

   On input $G = (V, E)$, where $C = (v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ is a Hamiltonian cycle in G, output $G' = \big(V, E - \{(v_n, v_1)\}\big)$.

   Clearly, G' can be computed in polytime from G.

   Also, C is a Hamiltonian cycle in G iff $C - (v_n, v_1)$ is a Hamiltonian path in G'.

4. HP is NP-hard because HC $\leqslant_p$ HP:

   On input $G = (V, E)$, let $v_1 \in V$ be any vertex of G and $T_1 = \{u \in V : (u, v_1) \in E\}$ be the subset of vertices that have an edge to v_1.
   Output $G' = \big(V \cup \{s_1, v_1', t_1\}, E \cup \{(s_1, v_1), (v_1', t_1)\} \cup \{(u, v_1') : u \in T_1\}\big)$.

   Clearly, G' can be computed from G in polytime: T_1 can be found in time $\mathcal{O}(n)$ and we only add three new vertices and a linear number of additional edges that can easily be found in linear time.

   Also, if G contains some Hamiltonian cycle $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$, then G' contains the Hamiltonian path $(s_1, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1'), (v_1', t_1)$.

   Finally, if G' contains some Hamiltonian path, then the path must start with the edge (s_1, v_1) and end with the edge (v_1', t_1) because s_1 and t_1 both have only a single edge connecting them to the rest of G'. This means that the Hamiltonian path has the following form: $(s_1, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1'), (v_1', t_1)$. Because G' contains edges (u, v_1') exactly for vertices u such that G contains an edge (u, v_1), this implies that G contains the Hamiltonian cycle $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$.

5. HP is NP-hard because HC $\leqslant_p$ HP:

   On input $G = (V, E)$, let $v_1 \in V$ be any vertex of G and $T_1 = \{u \in V : (u, v_1) \in E\}$ be the subset of vertices that have an edge to v_1.
   Output $G' = \big(V \cup \{v_1'\}, E \cup \{(u, v_1') : u \in T_1\}\big)$.

   Clearly, G' can be computed from G in polytime: T_1 can be found in time $\mathcal{O}(n)$ and we only add one new vertex and a linear number of additional edges that can easily be found in linear time.

   Also, if G contains some Hamiltonian cycle $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$, then G' contains the Hamiltonian path $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1')$.

   Finally, if G' contains some Hamiltonian path $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1')$, then G contains the Hamiltonian cycle $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ because G' contains edges (u, v_1') exactly for vertices u such that G contains an edge (u, v_1).