Chapter 6: Functions of Random Variables

6.1 The distribution function of Y is $F_Y(y) = \int_0^y 2(1-t)dt = 2y - y^2$, $0 \le y \le 1$.

a. $F_{U_1}(u) = P(U_1 \le u) = P(2Y - 1 \le u) = P(Y \le \frac{u+1}{2}) = F_Y(\frac{u+1}{2}) = 2(\frac{u+1}{2}) - (\frac{u+1}{2})^2$. Thus,
$f_{U_1}(u) = F'_{U_1}(u) = \frac{1-u}{2}$, $-1 \le u \le 1$.

b. $F_{U_2}(u) = P(U_2 \le u) = P(1 - 2Y \le u) = P(Y \le \frac{1-u}{2}) = F_Y(\frac{1-u1}{2}) = 1 - 2(\frac{u+1}{2}) = (\frac{u+1}{2})^2$. Thus,
$f_{U_2}(u) = F'_{U_2}(u) = \frac{u+1}{2}$, $-1 \le u \le 1$.

c. $F_{U_3}(u) = P(U_3 \le u) = P(Y^2 \le u) = P(Y \le \sqrt{u}) = F_Y(\sqrt{u}) = 2\sqrt{u} - u$ Thus,
$f_{U_3}(u) = F'_{U_3}(u) = \frac{1}{\sqrt{u}} - 1$, $0 \le u \le 1$.

d. $E(U_1) = -1/3$, $E(U_2) = 1/3$, $E(U_3) = 1/6$.

e. $E(2Y - 1) = -1/3$, $E(1 - 2Y) = 1/3$, $E(Y^2) = 1/6$.

6.2 The distribution function of Y is $F_Y(y) = \int_{-1}^y (3/2)t^2 dt = (1/2)(y^3 - 1)$, $-1 \le y \le 1$.

a. $F_{U_1}(u) = P(U_1 \le u) = P(3Y \le u) = P(Y \le u/3) = F_Y(u/3) = \frac{1}{2}(u^3/18 - 1)$. Thus,
$f_{U_1}(u) = F'_{U_1}(u) = u^2/18$, $-3 \le u \le 3$.

b. $F_{U_2}(u) = P(U_2 \le u) = P(3 - Y \le u) = P(Y \ge 3 - u) = 1 - F_Y(3 - u) = \frac{1}{2}[1 - (3-u)^3]$.
Thus, $f_{U_2}(u) = F'_{U_2}(u) = \frac{3}{2}(3-u)^2$, $2 \le u \le 4$.

c. $F_{U_3}(u) = P(U_3 \le u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = u^{3/2}$.
Thus, $f_{U_3}(u) = F'_{U_3}(u) = \frac{3}{2}\sqrt{u}$, $0 \le u \le 1$.

6.3 The distribution function for Y is $F_Y(y) = \begin{cases} y^2/2 & 0 \le y \le 1 \\ y - 1/2 & 1 < y \le 1.5 \\ 1 & y > 1.5 \end{cases}$.

a. $F_U(u) = P(U \le u) = P(10Y - 4 \le u) = P(Y \le \frac{u+4}{10}) = F_Y(\frac{u+4}{10})$. So,

$F_U(u) = \begin{cases} \frac{(u+4)^2}{200} & -4 \le u \le 6 \\ \frac{u-1}{10} & 6 < u \le 11 \\ 1 & u > 11 \end{cases}$, and $f_U(u) = F'_U(u) = \begin{cases} \frac{u+4}{100} & -4 \le u \le 6 \\ \frac{1}{10} & 6 < u \le 11 \\ 0 & \text{elsewhere} \end{cases}$.

b. $E(U) = 5.583$.

c. $E(10Y - 4) = 10(23/24) - 4 = 5.583$.

6.4 The distribution function of Y is $F_Y(y) = 1 - e^{-y/4}$, $0 \le y$.

a. $F_U(u) = P(U \le u) = P(3Y + 1 \le u) = P(Y \le \frac{u-1}{3}) = F_Y(\frac{u-1}{3}) = 1 - e^{-(u-1)/12}$. Thus,
$f_U(u) = F'_U(u) = \frac{1}{12}e^{-(u-1)/12}$, $u \ge 1$.

b. $E(U) = 13$.

6.5 The distribution function of Y is $F_Y(y) = y/4$, $1 \le y \le 5$.

$F_U(u) = P(U \le u) = P(2Y^2 + 3 \le u) = P(Y \le \sqrt{\frac{u-3}{2}}) = F_Y(\sqrt{\frac{u-3}{2}}) = \frac{1}{4}\sqrt{\frac{u-3}{2}}$. Differentiating,

$f_U(u) = F_U'(u) = \frac{1}{16}\left(\frac{u-3}{2}\right)^{-1/2}$, $5 \le u \le 53$.

6.6 Refer to Ex. 5.10 ad 5.78. Define $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = P(Y_1 \le Y_2 + u)$.

a. For $u \le 0$, $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = 0$.

For $0 \le u < 1$, $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = \int_0^u \int_{2y_2}^{y_2+u} 1 dy_1 dy_2 = u^2/2$.

For $1 \le u \le 2$, $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = 1 - \int_0^{2-u} \int_{y_2+u}^{2} 1 dy_1 dy_2 = 1 - (2-u)^2/2$.

Thus, $f_U(u) = F_U'(u) = \begin{cases} u & 0 \le u < 1 \\ 2-u & 1 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$.

b. $E(U) = 1$.

6.7 Let $F_Z(z)$ and $f_Z(z)$ denote the standard normal distribution and density functions respectively.

a. $F_U(u) = P(U \le u) = P(Z^2 \le u) = P(-\sqrt{u} \le Z \le \sqrt{u}) = F_Z(\sqrt{u}) - F_Z(-\sqrt{u})$. The density function for U is then

$$f_U(u) = F_U'(u) = \frac{1}{2\sqrt{u}} f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_Z(-\sqrt{u}) = \frac{1}{\sqrt{u}} f_Z(\sqrt{u}),\ u \ge 0.$$

Evaluating, we find $f_U(u) = \frac{1}{\sqrt{\pi}\sqrt{2}} u^{-1/2} e^{-u/2}$ $u \ge 0$.

b. U has a gamma distribution with $\alpha = 1/2$ and $\beta = 2$ (recall that $\Gamma(1/2) = \sqrt{\pi}$).

c. This is the chi–square distribution with one degree of freedom.

6.8 Let $F_Y(y)$ and $f_Y(y)$ denote the beta distribution and density functions respectively.

a. $F_U(u) = P(U \le u) = P(1 - Y \le u) = P(Y \ge 1 - u) = 1 - F_Y(1-u)$. The density function for U is then $f_U(u) = F_U'(u) = f_Y(1-u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\beta-1}(1-u)^{\alpha-1}$, $0 \le u \le 1$.

b. $E(U) = 1 - E(Y) = \frac{\beta}{\alpha+\beta}$.

c. $V(U) = V(Y)$.

6.9 Note that this is the same density from Ex. 5.12: $f(y_1, y_2) = 2$, $0 \le y_1 \le 1$, $0 \le y_2 \le 1$, $0 \le y_1 + y_2 \le 1$.

a. $F_U(u) = P(U \le u) = P(Y_1 + Y_2 \le u) = P(Y_1 \le u - Y_2) = \int_0^u \int_0^{u-y_2} 2 dy_1 dy_2 = u^2$. Thus,

$f_U(u) = F_U'(u) = 2u$, $0 \le u \le 1$.

b. $E(U) = 2/3$.

c. (found in an earlier exercise in Chapter 5) $E(Y_1 + Y_2) = 2/3$.

6.10 Refer to Ex. 5.15 and Ex. 5.108.

a. $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = P(Y_1 \le u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$, so that

$f_U(u) = F'_U(u) = e^{-u}$, $u \ge 0$, so that U has an exponential distribution with $\beta = 1$.

b. From part a above, $E(U) = 1$.

6.11 It is given that $f_i(y_i) = e^{-y_i}$, $y_i \ge 0$ for $i = 1, 2$. Let $U = (Y_1 + Y_2)/2$.

a. $F_U(u) = P(U \le u) = P(\frac{Y_1+Y_2}{2} \le u) = P(Y_1 \le 2u - Y_2) = \int_0^{2u} \int_{y_2}^{2u-y_2} e^{-y_1-y_2} dy_1 dy_2 = 1 - e^{-2u} - 2ue^{-2u}$,

so that $f_U(u) = F'_U(u) = 4ue^{-2u}$, $u \ge 0$, a gamma density with $\alpha = 2$ and $\beta = 1/2$.

b. From part (a), $E(U) = 1$, $V(U) = 1/2$.

6.12 Let $F_Y(y)$ and $f_Y(y)$ denote the gamma distribution and density functions respectively.

a. $F_U(u) = P(U \le u) = P(cY \le u) = P(Y \le u/c)$. The density function for U is then $f_U(u) = F'_U(u) = \frac{1}{c} f_Y(u/c) = \frac{1}{\Gamma(\alpha)(c\beta)^\alpha} u^{\alpha-1} e^{-u/c\beta}$, $u \ge 0$. Note that this is another gamma distribution.

b. The shape parameter is the same (α), but the scale parameter is $c\beta$.

6.13 Refer to Ex. 5.8;

$$F_U(u) = P(U \le u) = P(Y_1 + Y_2 \le u) = P(Y_1 \le u - Y_2) = \int_0^u \int_0^{u-y_2} e^{-y_1-y_2} dy_1 dy_2 = 1 - e^{-u} - ue^{-u}.$$

Thus, $f_U(u) = F'_U(u) = ue^{-u}$, $u \ge 0$.

6.14 Since Y_1 and Y_2 are independent, so $f(y_1, y_2) = 18(y_1 - y_1^2)y_2^2$, for $0 \le y_1 \le 1$, $0 \le y_2 \le 1$. Let $U = Y_1 Y_2$. Then,

$$F_U(u) = P(U \le u) = P(Y_1 Y_2 \le u) = P(Y_1 \le u/Y_2) = P(Y_1 > u/Y_2) = 1 - \int_u^1 \int_{u/y_2}^1 18(y_1 - y_1^2)y_2^2 dy_1 dy_2$$

$$= 9u^2 - 8u^3 + 6u^3 \ln u.$$

$f_U(u) = F'_U(u) = 18u(1 - u + u\ln u)$, $0 \le u \le 1$.

6.15 Let U have a uniform distribution on (0, 1). The distribution function for U is $F_U(u) = P(U \le u) = u$, $0 \le u \le 1$. For a function G, we require $G(U) = Y$ where Y has distribution function $F_Y(y) = 1 - e^{-y^2}$, $y \ge 0$. Note that

$$F_Y(y) = P(Y \le y) = P(G(U) \le y) = P[U \le G^{-1}(y)] = F_U[G^{-1}(y)] = u.$$

So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$ so that $G(u) = [-\ln(1-u)]^{-1/2}$. Therefore, the random variable $Y = [-\ln(U - 1)]^{-1/2}$ has distribution function $F_Y(y)$.

6.16 Similar to Ex. 6.15. The distribution function for Y is $F_Y(y) = b\int_b^y t^{-2}dt = 1 - \frac{b}{y},\ y \geq b.$

$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P[U \leq G^{-1}(y)] = F_U[G^{-1}(y)] = u.$
So it must be true that $G^{-1}(y) = 1 - \frac{b}{y} = u$ so that $G(u) = \frac{b}{1-u}$. Therefore, the random variable $Y = b/(1 - U)$ has distribution function $F_Y(y)$.

6.17 **a.** Taking the derivative of $F(y)$, $f(y) = \frac{\alpha y^{\alpha-1}}{\theta^{\alpha}}$, $0 \leq y \leq \theta$.

b. Following Ex. 6.15 and 6.16, let $u = \left(\frac{y}{\theta}\right)^{\alpha}$ so that $y = \theta u^{1/\alpha}$. Thus, the random variable $Y = \theta U^{1/\alpha}$ has distribution function $F_Y(y)$.
c. From part (b), the transformation is $y = 4\sqrt{u}$. The values are 2.0785, 3.229, 1.5036, 1.5610, 2.403.

6.18 **a.** Taking the derivative of the distribution function yields $f(y) = \alpha\beta^{\alpha}y^{-\alpha-1}$, $y \geq \beta$.
b. Following Ex. 6.15, let $u = 1 - \left(\frac{\beta}{y}\right)^{\alpha}$ so that $y = \frac{\beta}{(1-u)^{1/\alpha}}$. Thus, $Y = \beta(1-U)^{-1/\alpha}$.
c. From part (b), $y = 3/\sqrt{1-u}$. The values are 3.0087, 3.3642, 6.2446, 3.4583, 4.7904.

6.19 The distribution function for X is:
$F_X(x) = P(X \leq x) = P(1/Y \leq x) = P(Y \geq 1/x) = 1 - F_Y(1/x)$
$= 1 - \left[1 - (\beta x)^{\alpha}\right] = (\beta x)^{\alpha}$, $0 < x < \beta^{-1}$, which is a power distribution with $\theta = \beta^{-1}$.

6.20 **a.** $F_W(w) = P(W \leq w) + P(Y^2 \leq w) = P(Y \leq \sqrt{w}) = F_Y(\sqrt{w}) = \sqrt{w},\ 0 \leq w \leq 1.$
b. $F_W(w) = P(W \leq w) + P(\sqrt{Y} \leq w) = P(Y \leq w^2) = F_Y(w^2) = w^2, 0 \leq w \leq 1.$

6.21 By definition, $P(X = i) = P[F(i - 1) < U \leq F(i)] = F(i) - F(i - 1)$, for $i = 1, 2, \ldots$, since for any $0 \leq a \leq 1$, $P(U \leq a) = a$ for any $0 \leq a \leq 1$. From Ex. 4.5, $P(Y = i) = F(i) - F(i - 1)$, for $i = 1, 2, \ldots$. Thus, X and Y have the same distribution.

6.22 Let U have a uniform distribution on the interval (0, 1). For a geometric distribution with parameter p and distribution function F, define the random variable X as:
$X = k$ if and only if $F(k - 1) < U \leq F(k)$, $k = 1, 2, \ldots$.
Or since $F(k) = 1 - q^k$, we have that:
$X = k$ if and only if $1 - q^{k-1} < U \leq 1 - q^k$, OR
$X = k$ if and only if $q^k, < 1-U \leq q^{k-1}$, OR
$X = k$ if and only if $k\ln q \leq \ln(1-U) \leq (k-1)\ln q$, OR
$X = k$ if and only if $k-1 < [\ln(1-U)]/\ln q \leq k$.

6.23 **a.** If $U = 2Y - 1$, then $Y = \frac{U+1}{2}$. Thus, $\frac{dy}{du} = \frac{1}{2}$ and $f_U(u) = \frac{1}{2}2(1 - \frac{u+1}{2}) = \frac{1-u}{2}, -1 \leq u \leq 1.$
b. If $U = 1 - 2Y$, then $Y = \frac{1-U}{2}$. Thus, $\frac{dy}{du} = \frac{1}{2}$ and $f_U(u) = \frac{1}{2}2(1 - \frac{1-u}{2}) = \frac{1+u}{2}, -1 \leq u \leq 1.$
c. If $U = Y^2$, then $Y = \sqrt{U}$. Thus, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{2\sqrt{u}}2(1 - \sqrt{u}) = \frac{1-\sqrt{u}}{\sqrt{u}}, 0 \leq u \leq 1.$

6.24 If $U = 3Y + 1$, then $Y = \frac{U-1}{3}$. Thus, $\frac{dy}{du} = \frac{1}{3}$. With $f_Y(y) = \frac{1}{4}e^{-y/4}$, we have that $f_U(u) = \frac{1}{3}\left[\frac{1}{4}e^{-(u-1)/12}\right] = \frac{1}{12}e^{-(u-1)/12}$, $1 \le u$.

6.25 Refer to Ex. 6.11. The variable of interest is $U = \frac{Y_1+Y_2}{2}$. Fix $Y_2 = y_2$. Then, $Y_1 = 2u - y_2$ and $\frac{dy_1}{du} = 2$. The joint density of U and Y_2 is $g(u, y_2) = 2e^{-2u}$, $u \ge 0$, $y_2 \ge 0$, and $y_2 < 2u$.

Thus, $f_U(u) = \int_0^{2u} 2e^{-2u}dy_2 = 4ue^{-2u}$ for $u \ge 0$.

6.26 **a.** Using the transformation approach, $Y = U^{1/m}$ so that $\frac{dy}{du} = \frac{1}{m}u^{-(m-1)/m}$ so that the density function for U is $f_U(u) = \frac{1}{\alpha}e^{-u/\alpha}$, $u \ge 0$. Note that this is the exponential distribution with mean α.

b. $E(Y^k) = E(U^{k/m}) = \int_0^{\infty} u^{k/m}\frac{1}{\alpha}e^{-u/\alpha}du = \Gamma\left(\frac{k}{m}+1\right)\alpha^{k/m}$, using the result from Ex. 4.111.

6.27 **a.** Let $W = \sqrt{Y}$. The random variable Y is exponential so $f_Y(y) = \frac{1}{\beta}e^{-y/\beta}$. Then, $Y = W^2$ and $\frac{dy}{dw} = 2w$. Then, $f_Y(y) = \frac{2}{\beta}we^{-w^2/\beta}$, $w \ge 0$, which is Weibull with $m = 2$.

b. It follows from Ex. 6.26 that $E(Y^{k/2}) = \Gamma\left(\frac{k}{2}+1\right)\beta^{k/2}$

6.28 If Y is uniform on the interval (0, 1), $f_U(u) = 1$. Then, $Y = e^{-U/2}$ and $\frac{dy}{du} = -\frac{1}{2}e^{-u/2}$. Then, $f_Y(y) = 1\,|-\frac{1}{2}e^{-u/2}| = \frac{1}{2}e^{-u/2}$, $u \ge 0$ which is exponential with mean 2.

6.29 **a.** With $W = \frac{mV^2}{2}$, $V = \sqrt{\frac{2W}{m}}$ and $|\frac{dv}{dw}| = \frac{1}{\sqrt{2mw}}$. Then,

$$f_W(w) = \frac{a(2w/m)}{\sqrt{2mw}}e^{-2bw/m} = \frac{a\sqrt{2}}{m^{3/2}}w^{1/2}e^{-w/kT}, \ w \ge 0.$$

The above expression is in the form of a gamma density, so the constant a must be chosen so that the density integrate to 1, or simply

$$\frac{a\sqrt{2}}{m^{3/2}} = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}}.$$

So, the density function for W is

$$f_W(w) = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}}w^{1/2}e^{-w/kT}.$$

b. For a gamma random variable, $E(W) = \frac{3}{2}kT$.

6.30 The density function for I is $f_I(i) = 1/2$, $9 \le i \le 11$. For $P = 2I^2$, $I = \sqrt{P/2}$ and $\frac{di}{dp} = (1/2)^{3/2}p^{-1/2}$. Then, $f_p(p) = \frac{1}{4\sqrt{2p}}$, $162 \le p \le 242$.

6.31 Similar to Ex. 6.25. Fix $Y_1 = y_1$. Then, $U = Y_2/y_1$, $Y_2 = y_1U$ and $|\frac{dy_2}{du}| = y_1$. The joint density of Y_1 and U is $f(y_1, u) = \frac{1}{8} y_1^2 e^{-y_1(1+u)/2}$, $y_1 \geq 0$, $u \geq 0$. So, the marginal density for U is $f_U(u) = \int_0^\infty \frac{1}{8} y_1^2 e^{-y_1(1+u)/2} dy_1 = \frac{2}{(1+u)^3}$, $u \geq 0$.

6.32 Now $f_Y(y) = 1/4$, $1 \leq y \leq 5$. If $U = 2Y^2 + 3$, then $Y = \left(\frac{U-3}{2}\right)^{1/2}$ and $|\frac{dy}{du}| = \frac{1}{4}\left(\frac{\sqrt{2}}{\sqrt{u-3}}\right)$. Thus, $f_U(u) = \frac{1}{8\sqrt{2(u-3)}}$, $5 \leq u \leq 53$.

6.33 If $U = 5 - (Y/2)$, $Y = 2(5 - U)$. Thus, $|\frac{dy}{du}| = 2$ and $f_U(u) = 4(80 - 31u + 3u^2)$, $4.5 \leq u \leq 5$.

6.34 **a.** If $U = Y^2$, $Y = \sqrt{U}$. Thus, $|\frac{dy}{du}| = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{\theta} e^{-u/\theta}$, $u \geq 0$. This is the exponential density with mean θ.

b. From part a, $E(Y) = E(U^{1/2}) = \frac{\sqrt{\pi\theta}}{2}$. Also, $E(Y^2) = E(U) = \theta$, so $V(Y) = \theta[1 - \frac{\pi}{4}]$.

6.35 By independence, $f(y_1, y_2) = 1$, $0 \leq y_1 \leq 0$, $0 \leq y_2 \leq 1$. Let $U = Y_1Y_2$. For a fixed value of Y_1 at y_1, then $y_2 = u/y_1$. So that $\frac{dy_2}{du} = \frac{1}{y_1}$. So, the joint density of Y_1 and U is

$$g(y_1, u) = 1/y_1, \ 0 \leq y_1 \leq 0, \ 0 \leq u \leq y_1.$$

Thus, $f_U(u) = \int_u^1 (1/y_1) dy_1 = -\ln(u)$, $0 \leq u \leq 1$.

6.36 By independence, $f(y_1, y_2) = \frac{4y_1y_2}{\theta^2} e^{-(y_1^2+y_2^2)}$, $y_1 > 0$, $y_2 > 0$. Let $U = Y_1^2 + Y_2^2$. For a fixed value of Y_1 at y_1, then $U = y_1^2 + Y_2^2$ so we can write $y_2 = \sqrt{u - y_1^2}$. Then, $\frac{dy_2}{du} = \frac{1}{2\sqrt{u-y_1^2}}$ so that the joint density of Y_1 and U is

$$g(y_1, u) = \frac{4y_1\sqrt{u-y_1^2}}{\theta^2} e^{-u/\theta} \frac{1}{2\sqrt{u-y_1^2}} = \frac{2}{\theta^2} y_1 e^{-u/\theta}, \text{ for } 0 < y_1 < \sqrt{u}.$$

Then, $f_U(u) = \int_0^{\sqrt{u}} \frac{2}{\theta^2} y_1 e^{-u/\theta} dy_1 = \frac{1}{\theta^2} u e^{-u/\theta}$. Thus, U has a gamma distribution with $\alpha = 2$.

6.37 The mass function for the Bernoulli distribution is $p(y) = p^y(1-p)^{1-y}$, $y = 0, 1$.

a. $m_{Y_1}(t) = E(e^{tY_1}) = \sum_{x=0}^{1} e^{ty} p(y) = 1 - p + pe^t$.

b. $m_W(t) = E(e^{tW}) = \prod_{i=1}^{n} m_{Y_i}(t) = [1 - p + pe^t]^n$

c. Since the mgf for W is in the form of a binomial mgf with n trials and success probability p, this is the distribution for W.

6.38 Let Y_1 and Y_2 have mgfs as given, and let $U = a_1Y_1 + a_2Y_2$. The mdf for U is

$$m_U(t) = E(e^{Ut}) = E(e^{(a_1Y_1+a_2Y_2)t}) = E(e^{(a_1t)Y_1})E(e^{(a_2t)Y_2}) = m_{Y_1}(a_1t)m_{Y_2}(a_2t).$$

6.39 The mgf for the exponential distribution with $\beta = 1$ is $m(t) = (1-t)^{-1}$, $t < 1$. Thus, with Y_1 and Y_2 each having this distribution and $U = (Y_1 + Y_2)/2$. Using the result from Ex. 6.38, let $a_1 = a_2 = 1/2$ so the mgf for U is $m_U(t) = m(t/2)m(t/2) = (1-t/2)^{-2}$. Note that this is the mgf for a gamma random variable with $\alpha = 2$, $\beta = 1/2$, so the density function for U is $f_U(u) = 4ue^{-2u}$, $u \ge 0$.

6.40 It has been shown that the distribution of both Y_1^2 and Y_2^2 is chi–square with $\nu = 1$. Thus, both have mgf $m(t) = (1-2t)^{-1/2}$, $t < 1/2$. With $U = Y_1^2 + Y_2^2$, use the result from Ex. 6.38 with $a_1 = a_2 = 1$ so that $m_U(t) = m(t)m(t) = (1-2t)^{-1}$. Note that this is the mgf for a exponential random variable with $\beta = 2$, so the density function for U is $f_U(u) = \frac{1}{2}e^{-u/2}$, $u \ge 0$ (this is also the chi–square distribution with $\nu = 2$.)

6.41 (Special case of Theorem 6.3) The mgf for the normal distribution with parameters μ and σ is $m(t) = e^{\mu t+\sigma^2t^2/2}$. Since the Y_i's are independent, the mgf for U is given by

$$m_U(t) = E(e^{Ut}) = \prod_{i=1}^{n} E(e^{a_itY_i}) = \prod_{i=1}^{n} m(a_it) = \exp\left[\mu t\sum_i a_i + (t^2\sigma^2/2)\sum_i a_i^2\right].$$

This is the mgf for a normal variable with mean $\mu\sum_i a_i$ and variance $\sigma^2\sum_i a_i^2$.

6.42 The probability of interest is $P(Y_2 > Y_1) = P(Y_2 - Y_1 > 0)$. By Theorem 6.3, the distribution of $Y_2 - Y_1$ is normal with $\mu = 4000 - 5000 = -1000$ and $\sigma^2 = 400^2 + 300^2 =$ 250,000. Thus, $P(Y_2 - Y_1 > 0) = P(Z > \frac{0-(-1000)}{\sqrt{250{,}000}}) = P(Z > 2) = .0228$.

6.43 **a.** From Ex. 6.41, $\overline{Y}$ has a normal distribution with mean μ and variance σ^2/n.

b. For the given values, $\overline{Y}$ has a normal distribution with variance $\sigma^2/n = 16/25$. Thus, the standard deviation is 4/5 so that

$$P(|\overline{Y} - \mu| \le 1) = P(-1 \le \overline{Y} - \mu \le 1) = P(-1.25 \le Z \le 1.25) = .7888.$$

c. Similar to the above, the probabilities are .8664, .9544, .9756. So, as the sample size increases, so does the probability that $P(|\overline{Y} - \mu| \le 1)$.

6.44 The total weight of the watermelons in the packing container is given by $U = \sum_{i=1}^{n} Y_i$, so by Theorem 6.3 U has a normal distribution with mean $15n$ and variance $4n$. We require that $.05 = P(U > 140) = P(Z > \frac{140-15n}{\sqrt{4n}})$. Thus, $\frac{140-15n}{\sqrt{4n}} = z_{.05} = 1.645$. Solving this nonlinear expression for n, we see that $n \approx 8.687$. Therefore, the maximum number of watermelons that should be put in the container is 8 (note that with this value n, we have $P(U > 140) = .0002$).

6.45 By Theorem 6.3 we have that $U = 100 + 7Y_1 + 3Y_2$ is a normal random variable with mean $\mu = 100 + 7(10) + 3(4) = 182$ and variance $\sigma^2 = 49(.5)^2 + 9(.2)^2 = 12.61$. We require a value c such that $P(U > c) = P(Z > \frac{c-182}{\sqrt{12.61}})$. So, $\frac{c-182}{\sqrt{12.61}} = 2.33$ and $c = \$190.27$.

6.46 The mgf for W is $m_W(t) = E(e^{Wt}) = E(e^{(2Y/\beta)t}) = m_Y(2t/\beta) = (1-2t)^{-n/2}$. This is the mgf for a chi–square variable with n degrees of freedom.

6.47 By Ex. 6.46, $U = 2Y/4.2$ has a chi–square distribution with $\nu = 7$. So, by Table III, $P(Y > 33.627) = P(U > 2(33.627)/4.2) = P(U > 16.0128) = .025$.

6.48 From Ex. 6.40, we know that $V = Y_1^2 + Y_2^2$ has a chi–square distribution with $\nu = 2$. The density function for V is $f_V(v) = \frac{1}{2}e^{-v/2}$, $v \geq 0$. The distribution function of $U = \sqrt{V}$ is $F_U(u) = P(U \leq u) = P(V \leq u^2) = F_V(u^2)$, so that $f_U(u) = F'_U(u) = ue^{-u^2/2}$, $u \geq 0$. A sharp observer would note that this is a Weibull density with shape parameter 2 and scale 2.

6.49 The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = [1 - p + pe^t]^{n_1}$, $m_{Y_2}(t) = [1 - p + pe^t]^{n_2}$. Since Y_1 and Y_2 are independent, the mgf for $Y_1 + Y_2$ is $m_{Y_1}(t) \times m_{Y_2}(t) = [1 - p + pe^t]^{n_1+n_2}$. This is the mgf of a binomial with $n_1 + n_2$ trials and success probability p.

6.50 The mgf for Y is $m_Y(t) = [1 - p + pe^t]^n$. Now, define $X = n - Y$. The mgf for X is

$$m_X(t) = E(e^{tX}) = E(e^{t(n-Y)}) = e^{tn}m_Y(-t) = [p + (1-p)e^t]^n.$$

This is an mgf for a binomial with n trials and "success" probability $(1 - p)$. Note that the random variable X = # of *failures* observed in the experiment.

6.51 From Ex. 6.50, the distribution of $n_2 - Y_2$ is binomial with n_2 trials and "success" probability $1 - .8 = .2$. Thus, by Ex. 6.49, the distribution of $Y_1 + (n_2 - Y_2)$ is binomial with $n_1 + n_2$ trials and success probability $p = .2$.

6.52 The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = e^{\lambda_1(e^t-1)}$, $m_{Y_2}(t) = e^{\lambda_2(e^t-1)}$.

a. Since Y_1 and Y_2 are independent, the mgf for $Y_1 + Y_2$ is $m_{Y_1}(t) \times m_{Y_2}(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$. This is the mgf of a Poisson with mean $\lambda_1 + \lambda_2$.

b. From Ex. 5.39, the distribution is binomial with m trials and $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

6.53 The mgf for a binomial variable Y_i with n_i trials and success probability p_i is given by $m_{Y_i}(t) = [1 - p_i + p_ie^t]^{n_i}$. Thus, the mgf for $U = \sum_{i=1}^n Y_i$ is $m_U(t) = \prod_i [1 - p_i + p_ie^t]^{n_i}$.

a. Let $p_i = p$ and $n_i = m$ for all i. Here, U is binomial with $m(n)$ trials and success probability p.

b. Let $p_i = p$. Here, U is binomial with $\sum_{i=1}^n n_i$ trials and success probability p.

c. (Similar to Ex. 5.40) The cond. distribution is hypergeometric w/ $r = n_i$, $N = \sum n_i$.

d. By definition,

$$P(Y_1+Y_2=k \mid \sum_{i=1}^{n} Y_i) = \frac{P(Y_1+Y_2=k, \sum Y_i = m)}{P(\sum Y_i = m)} = \frac{P(Y_1+Y_2=k, \sum_{i=3}^{n} Y_i = m-k)}{P(\sum Y_i = m)} = \frac{P(Y_1+Y_2=k)P(\sum_{i=3}^{n} Y_i = m-k)}{P(\sum Y_i = m)}$$

$$= \frac{\binom{n_1+n_2}{k}\binom{\sum_{i=3}^{n} n_i}{m-k}}{\binom{\sum_{i=1}^{n} n_i}{m}}$$, which is hypergeometric with $r = n_1 + n_2$.

e. No, the mgf for U does not simplify into a recognizable form.

6.54 **a.** The mgf for $U = \sum_{i=1}^{n} Y_i$ is $m_U(t) = e^{(e^t-1)\sum_i \lambda_i}$, which is recognized as the mgf for a Poisson w/ mean $\sum_i \lambda_i$.

b. This is similar to 6.52. The distribution is binomial with m trials and $p = \frac{\lambda_1}{\sum \lambda_i}$.

c. Following the same steps as in part d of Ex. 6.53, it is easily shown that the conditional distribution is binomial with m trials and success probability $\frac{\lambda_1+\lambda_2}{\sum \lambda_i}$.

6.55 Let $Y = Y_1 + Y_2$. Then, by Ex. 6.52, Y is Poisson with mean $7 + 7 = 14$. Thus, $P(Y \geq 20) = 1 - P(Y \leq 19) = .077$.

6.56 Let U = total service time for two cars. Similar to Ex. 6.13, U has a gamma distribution with $\alpha = 2$, $\beta = 1/2$. Then, $P(U > 1.5) = \int_{1.5}^{\infty} 4ue^{-2u}\,du = .1991$.

6.57 For each Y_i, the mgf is $m_{Y_i}(t) = (1-\beta t)^{-\alpha_i}$, $t < 1/\beta$. Since the Y_i are independent, the mgf for $U = \sum_{i=1}^{n} Y_i$ is $m_U(t) = \prod (1-\beta t)^{-\alpha_i} = (1-\beta t)^{-\sum_{i=1}^{n} \alpha_i}$.

This is the mgf for the gamma with shape parameter $\sum_{i=1}^{n} \alpha_i$ and scale parameter β.

6.58 **a.** The mgf for each W_i is $m(t) = \frac{pe^t}{(1-qe^t)}$. The mgf for Y is $[m(t)]^r = \left(\frac{pe^t}{1-qe^t}\right)^r$, which is the mgf for the negative binomial distribution.

b. Differentiating with respect to t, we have

$$m'(t)\Big|_{t=0} = r\left(\frac{pe^t}{1-qe^t}\right)^{r-1} \times \frac{pe^t}{(1-qe^t)^2}\Big|_{t=0} = \frac{r}{p} = E(Y).$$

Taking another derivative with respect to t yields

$$m''(t)\Big|_{t=0} = \frac{(1-qe^t)^{r+1} r^2 pe^t (pe^t)^{r-1} - r(pe^t)^r (r+1)(-qe^t)(1-qe^t)^r}{(1-qe^t)^{2(r+1)}}\Big|_{t=0} = \frac{pr^2 + r(r+1)q}{p^2} = E(Y^2).$$

Thus, $V(Y) = E(Y^2) - [E(Y)]^2 = rq/p^2$.

c. This is similar to Ex. 6.53. By definition,

$$P(W_1 = k \mid \Sigma W_i) = \frac{P(W_1 = k, \Sigma W_i = m)}{P(\Sigma W_i = m)} = \frac{P(W_1 = k, \sum_{i=2}^{n} W_i = m-k)}{P(\Sigma W_i = m)} = \frac{P(W_1 = k) P(\sum_{i=2}^{n} W_i = m-k)}{P(\Sigma W_i = m)} = \frac{\binom{m-k-1}{r-2}}{\binom{m-1}{r-1}}.$$

6.59 The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = (1-2t)^{-\nu_1/2}$, $m_{Y_2}(t) = (1-2t)^{-\nu_2/2}$. Thus the mgf for $U = Y_1 + Y_2 = m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) = (1-2t)^{-(\nu_1+\nu_2)/2}$, which is the mgf for a chi–square variable with $\nu_1 + \nu_2$ degrees of freedom.

6.60 Note that since Y_1 and Y_2 are independent, $m_W(t) = m_{Y_1}(t) \times m_{Y_2}(t)$. Therefore, it must be so that $m_W(t)/m_{Y_1}(t) = m_{Y_2}(t)$. Given the mgfs for W and Y_1, we can solve for $m_{Y_2}(t)$:

$$m_{Y_2}(t) = \frac{(1-2t)^{-\nu}}{(1-2t)^{-\nu_1}} = (1-2t)^{-(\nu-\nu_1)/2}.$$

This is the mgf for a chi–squared variable with $\nu - \nu_1$ degrees of freedom.

6.61 Similar to Ex. 6.60. Since Y_1 and Y_2 are independent, $m_W(t) = m_{Y_1}(t) \times m_{Y_2}(t)$. Therefore, it must be so that $m_W(t)/m_{Y_1}(t) = m_{Y_2}(t)$. Given the mgfs for W and Y_1,

$$m_{Y_2}(t) = \frac{e^{\lambda(e^t-1)}}{e^{\lambda_1(e^t-1)}} = e^{(\lambda-\lambda_1)(e^t-1)}.$$

This is the mgf for a Poisson variable with mean $\lambda - \lambda_1$.

6.62

$$\begin{aligned} E\{\exp[t_1(Y_1+Y_2) + t_2(Y_1 - Y_2)]\} &= E\{\exp[(t_1+t_2)Y_1 + (t_1+t_2)Y_2]\} = m_{Y_1}(t_1+t_2) m_{Y_2}(t_1+t_2) \\ &= \exp[\tfrac{\sigma^2}{2}(t_1+t_2)^2]\exp[\tfrac{\sigma^2}{2}(t_1-t_2)^2] = \exp[\tfrac{\sigma^2}{2}t_1^{\,2}]\exp[\tfrac{\sigma^2}{2}t_2]^2 \\ &= m_{U_1}(t_1) m_{U_1}(t_2). \end{aligned}$$

Since the joint mgf factors, U_1 and U_2 are independent.

6.63 **a.** The marginal distribution for U_1 is $f_{U_1}(u_1) = \int_0^\infty \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_2 = 1$, $0 < u_1 < 1$.

b. The marginal distribution for U_2 is $f_{U_2}(u_2) = \int_0^1 \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_1 = \frac{1}{\beta^2} u_2 e^{-u_2/\beta}$, $u_2 > 0$. This is a gamma density with $\alpha = 2$ and scale parameter β.

c. Since the joint distribution factors into the product of the two marginal densities, they are independent.

6.64 **a.** By independence, the joint distribution of Y_1 and Y_2 is the product of the two marginal densities:

$$f(y_1, y_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)\beta^{\alpha_1+\alpha_2}} y_1^{\alpha_1-1} y_2^{\alpha_2-1} e^{-(y_1+y_2)/\beta}, \; y_1 \ge 0, \; y_2 \ge 0.$$

With U and V as defined, we have that $y_1 = u_1 u_2$ and $y_2 = u_2(1-u_1)$. Thus, the Jacobian of transformation $J = u_2$ (see Example 6.14). Thus, the joint density of U_1 and U_2 is

$$f(u_1,u_2)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)\beta^{\alpha_1+\alpha_2}}(u_1u_2)^{\alpha_1-1}[u_2(1-u_1)]^{\alpha_2-1}e^{-u_2/\beta}u_2$$
$$=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)\beta^{\alpha_1+\alpha_2}}u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}u_2^{\alpha_1+\alpha_2-1}e^{-u_2/\beta}$$, with $0 < u_1 < 1$, and $u_2 > 0$.

b. $f_{U_1}(u_1)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)}u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}\int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}}v^{\alpha_1+\alpha_2-1}e^{-v/\beta}dv=\frac{\Gamma(\alpha_1+\alpha_a)}{\Gamma(\alpha_1)\Gamma(\alpha_a)}u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}$, with $0 < u_1 < 1$. This is the beta density as defined.

c. $f_{U_2}(u_2)=\frac{1}{\beta^{\alpha_1+\alpha_2}}u_2^{\alpha_1+\alpha_2-1}e^{-u_2/\beta}\int_0^1 \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)}u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}du_1 = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}u_2^{\alpha_1+\alpha_2-1}e^{-u_2/\beta}$, with $u_2 > 0$. This is the gamma density as defined.

d. Since the joint distribution factors into the product of the two marginal densities, they are independent.

6.65 **a.** By independence, the joint distribution of Z_1 and Z_2 is the product of the two marginal densities:
$$f(z_1,z_2)=\tfrac{1}{2\pi}e^{-(z_1^2+z_2^2)/2}.$$

With $U_1 = Z_1$ and $U_2 = Z_1 + Z_2$, we have that $z_1 = u_1$ and $z_2 = u_2 - u_1$. Thus, the Jacobian of transformation is
$$J=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}=1.$$
Thus, the joint density of U_1 and U_2 is
$$f(u_1,u_2)=\tfrac{1}{2\pi}e^{-[u_1^2+(u_2-u_1)^2]/2}=\tfrac{1}{2\pi}e^{-(2u_1^2-2u_1u_2+u_2^2)/2}.$$

b. $E(U_1)=E(Z_1)=0$, $E(U_2)=E(Z_1+Z_2)=0$, $V(U_1)=V(Z_1)=1$,
$V(U_2)=V(Z_1+Z_2)=V(Z_1)+V(Z_2)=2$, $Cov(U_1,U_2)=E(Z_1^2)=1$

c. Not independent since $\rho \neq 0$.

d. This is the bivariate normal distribution with $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = 1$, $\sigma_2^2 = 2$, and $\rho = \frac{1}{\sqrt{2}}$.

6.66 **a.** Similar to Ex. 6.65, we have that $y_1 = u_1 - u_2$ and $y_2 = u_2$. So, the Jacobian of transformation is
$$J=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1.$$
Thus, by definition the joint density is as given.

b. By definition of a marginal density, the marginal density for U_1 is as given.

c. If Y_1 and Y_2 are independent, their joint density factors into the product of the marginal densities, so we have the given form.

6.67 **a.** We have that $y_1 = u_1u_2$ and $y_2 = u_2$. So, the Jacobian of transformation is

$$J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = |u_2|.$$

Thus, by definition the joint density is as given.

b. By definition of a marginal density, the marginal density for U_1 is as given.

c. If Y_1 and Y_2 are independent, their joint density factors into the product of the marginal densities, so we have the given form.

6.68 **a.** Using the result from Ex. 6.67,

$$f(u_1,u_2) = 8(u_1u_2)u_2u_2 = 8u_1u_2^3,\ 0 \le u_1 \le 1,\ 0 \le u_2 \le 1.$$

b. The marginal density for U_1 is

$$f_{U_1}(u_1) = \int_0^1 8u_1u_2^3 du_2 = 2u_1,\ 0 \le u_1 \le 1.$$

The marginal density for U_1 is

$$f_{U_2}(u_2) = \int_0^1 8u_1u_2^3 du_1 = 4u_2^3,\ 0 \le u_2 \le 1.$$

The joint density factors into the product of the marginal densities, thus independence.

6.69 **a.** The joint density is $f(y_1,y_2) = \frac{1}{y_1^2y_2^2}$, $y_1 > 1$, $y_2 > 1$.

b. We have that $y_1 = u_1u_2$ and $y_2 = u_2(1 - u_1)$. The Jacobian of transformation is u_2. So,

$$f(u_1,u_2) = \frac{1}{u_1^2u_2^3(1-u_1)^2},$$

with limits as specified in the problem.

c. The limits may be simplified to: $1/u_1 < u_2$, $0 < u_1 < 1/2$, or $1/(1-u_1) < u_2$, $1/2 \le u_1 \le 1$.

d. If $0 < u_1 < 1/2$, then $f_{U_1}(u_1) = \int_{1/u_1}^{\infty} \frac{1}{u_1^2u_2^3(1-u_1)^2} du_2 = \frac{1}{2(1-u_1)^2}$.

If $1/2 \le u_1 \le 1$, then $f_{U_1}(u_1) = \int_{1/(1-u_1)}^{\infty} \frac{1}{u_1^2u_2^3(1-u_1)^2} du_2 = \frac{1}{2u_1^2}$.

e. Not independent since the joint density does not factor. Also note that the support is not rectangular.

6.70 **a.** Since Y_1 and Y_2 are independent, their joint density is $f(y_1, y_2) = 1$. The inverse transformations are $y_1 = \frac{u_1+u_2}{2}$ and $y_2 = \frac{u_1-u_2}{2}$. Thus the Jacobian is

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \tfrac{1}{2}\text{, so that}$$

$$f(u_1, u_2) = \tfrac{1}{2}\text{, with limits as specified in the problem.}$$

b. The support is in the shape of a square with corners located (0, 0), (1, 1), (2, 0), (1, –1).

c. If $0 < u_1 < 1$, then $f_{U_1}(u_1) = \int_{-u_1}^{u_1} \frac{1}{2} du_2 = u_1$.

If $1 \le u_1 < 2$, then $f_{U_1}(u_1) = \int_{u_1-2}^{2-u_1} \frac{1}{2} du_2 = 2 - u_1$.

d. If $-1 < u_2 < 0$, then $f_{U_2}(u_2) = \int_{-u_2}^{2+u_2} \frac{1}{2} du_2 = 1 + u_2$.

If $0 \le u_2 < 1$, then $f_{U_2}(u_2) = \int_{u_2}^{2-u_2} \frac{1}{2} du_2 = 1 - u_2$.

6.71 **a.** The joint density of Y_1 and Y_2 is $f(y_1, y_2) = \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta}$. The inverse transformations are $y_1 = \frac{u_1 u_2}{1+u_2}$ and $y_2 = \frac{u_1}{1+u_2}$ and the Jacobian is

$$J = \begin{vmatrix} \frac{u_2}{1+u_2} & \frac{u_1}{(1+u_2)^2} \\ \frac{1}{1+u_2} & \frac{-u_1}{(1+u_2)^2} \end{vmatrix} = \left| \frac{-u_1}{(1+u_2)^2} \right|$$

So, the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \tfrac{1}{\beta^2} e^{-u_1/\beta} \frac{u_1}{(1+u_2)^2},\ u_1 > 0,\ u_2 > 0.$$

b. Yes, U_1 and U_2 are independent since the joint density factors and the support is rectangular (Theorem 5.5).

6.72 Since the distribution function is $F(y) = y$ for $0 \le y \le 1$,

a. $g_{(1)}(u) = 2(1-u)$, $0 \le u \le 1$.

b. Since the above is a beta density with $\alpha = 1$ and $\beta = 2$, $E(U_1) = 1/3$, $V(U_1) = 1/18$.

6.73 Following Ex. 6.72,

a. $g_{(2)}(u) = 2u$, $0 \le u \le 1$.

b. Since the above is a beta density with $\alpha = 2$ and $\beta = 1$, $E(U_2) = 2/3$, $V(U_2) = 1/18$.

6.74 Since the distribution function is $F(y) = y/\theta$ for $0 \le y \le \theta$,

a. $G_{(n)}(y) = (y/\theta)^n$, $0 \le y \le \theta$.

b. $g_{(n)}(y) = G'_{(n)}(y) = ny^{n-1}/\theta^n$, $0 \le y \le \theta$.

c. It is easily shown that $E(Y_{(n)}) = \frac{n}{n+1}\theta$, $V(Y_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$.

6.75 Following Ex. 6.74, the required probability is $P(Y_{(n)} < 10) = (10/15)^5 = .1317$.

6.76 Following Ex. 6.74 with $f(y) = 1/\theta$ for $0 \le y \le \theta$,

a. By Theorem 6.5, $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}\left(\frac{y}{\theta}\right)^{k-1}\left(\frac{\theta-y}{\theta}\right)^{n-k}\frac{1}{\theta} = \frac{n!}{(k-1)!(n-k)!}\frac{y^{k-1}(\theta-y)^{n-k}}{\theta^n}$, $0 \le y \le \theta$.

b. $E(Y_{(k)}) = \frac{n!}{(k-1)!(n-k)!}\int_0^\theta \frac{y^k(\theta-y)^{n-k}}{\theta^n}dy = \frac{k}{n+1}\frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)}\int_0^\theta \left(\frac{y}{\theta}\right)^k\left(1-\frac{y}{\theta}\right)^{n-k}dy$. To evaluate this integral, apply the transformation $z = \frac{y}{\theta}$ and relate the resulting integral to that of a beta density with $\alpha = k + 1$ and $\beta = n - k + 1$. Thus, $E(Y_{(k)}) = \frac{k}{n+1}\theta$.

c. Using the same techniques in part b above, it can be shown that $E(Y_{(k)}^2) = \frac{k(k+1)}{(n+1)(n+2)}\theta^2$ so that $V(Y_{(k)}) = \frac{(n-k+1)k}{(n+1)^2(n+2)}\theta^2$.

d. $E(Y_{(k)} - Y_{(k-1)}) = E(Y_{(k)}) - E(Y_{(k-1)}) = \frac{k}{n+1}\theta - \frac{k-1}{n+1}\theta = \frac{1}{n+1}\theta$. Note that this is constant for all k, so that the expected order statistics are equally spaced.

6.77 **a.** Using Theorem 6.5, the joint density of $Y_{(j)}$ and $Y_{(k)}$ is given by

$$g_{(j)(k)}(y_j, y_k) = \frac{n!}{(j-1)!(k-1-j)!(n-k)!}\left(\frac{y_j}{\theta}\right)^{j-1}\left(\frac{y_k}{\theta}-\frac{y_j}{\theta}\right)^{k-1-j}\left(1-\frac{y_k}{\theta}\right)^{n-k}\left(\frac{1}{\theta}\right)^2, \quad 0 \le y_j \le y_k \le \theta.$$

b. $\text{Cov}(Y_{(j)}, Y_{(k)}) = E(Y_{(j)}Y_{(k)}) - E(Y_{(j)})E(Y_{(k)})$. The expectations $E(Y_{(j)})$ and $E(Y_{(k)})$ were derived in Ex. 6.76. To find $E(Y_{(j)}Y_{(k)})$, let $u = y_j/\theta$ and $v = y_k/\theta$ and write

$$E(Y_{(j)}Y_{(k)}) = c\theta\int_0^1\int_0^v u^j(v-u)^{k-1-j}v(1-v)^{n-k}\,du\,dv,$$

where $c = \frac{n!}{(j-1)!(k-1-j)!(n-k)!}$. Now, let $w = u/v$ so $u = wv$ and $du = vdw$. Then, the integral is

$$c\theta^2\left[\int_0^1 u^{k+1}(1-u)^{n-k}du\right]\left[\int_0^1 w^j(1-w)^{k-1-j}dw\right] = c\theta^2\left[B(k+2, n-k+1)\right]\left[B(j+1, k-j)\right].$$

Simplifying, this is $\frac{(k+1)j}{(n+1)(n+2)}\theta^2$. Thus, $\text{Cov}(Y_{(j)}, Y_{(k)}) = \frac{(k+1)j}{(n+1)(n+2)}\theta^2 - \frac{jk}{(n+1)^2}\theta^2 = \frac{n-k+1}{(n+1)^2(n+2)}\theta^2$.

c. $V(Y_{(k)} - Y_{(j)}) = V(Y_{(k)}) + V(Y_{(j)}) - 2\text{Cov}(Y_{(j)}, Y_{(k)})$

$$= \frac{(n-k+1)k}{(n+1)^2(n+2)}\theta^2 + \frac{(n-j+1)j}{(n+1)^2(n+2)}\theta^2 - \frac{2(n-k+1)}{(n+1)^2(n+2)}\theta^2 = \frac{(k-j)(n-k+k+1)}{(n+1)^2(n+2)}\theta^2.$$

6.78 From Ex. 6.76 with $\theta = 1$, $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}y^{k-1}(1-y)^{n-k} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}y^{k-1}(1-y)^{n-k}$. Since $0 \le y \le 1$, this is the beta density as described.

6.79 The joint density of $Y_{(1)}$ and $Y_{(n)}$ is given by (see Ex. 6.77 with $j = 1$, $k = n$),

$$g_{(1)(n)}(y_1, y_n) = n(n-1)\left(\frac{y_n}{\theta}-\frac{y_1}{\theta}\right)^n\left(\frac{1}{\theta}\right)^2 = n(n-1)\left(\frac{1}{\theta}\right)^n(y_n - y_1)^{n-2}, \quad 0 \le y_1 \le y_n \le \theta.$$

Applying the transformation $U = Y_{(1)}/Y_{(n)}$ and $V = Y_{(n)}$, we have that $y_1 = uv$, $y_n = v$ and the Jacobian of transformation is v. Thus,

$$f(u, v) = n(n-1)\left(\frac{1}{\theta}\right)^n(v - uv)^{n-2}v = n(n-1)\left(\frac{1}{\theta}\right)^n(1-u)^{n-2}v^{n-1}, \quad 0 \le u \le 1,\ 0 \le v \le \theta.$$

Since this joint density factors into separate functions of u and v and the support is rectangular, thus $Y_{(1)}/Y_{(n)}$ and $V = Y_{(n)}$ are independent.

6.80 The density and distribution function for Y are $f(y) = 6y(1-y)$ and $F(y) = 3y^2 - 2y^3$, respectively, for $0 \le y \le 1$.

a. $G_{(n)}(y) = \left(3y^2 - 2y^3\right)^n$, $0 \le y \le 1$.

b. $g_{(n)}(y) = G'_{(n)}(y) = n\left(3y^2 - 2y^3\right)^{n-1}(6y - 6y^2) = 6ny(1-y)\left(3y^2 - 2y^3\right)^{n-1}$, $0 \le y \le 1$.

c. Using the above density with $n = 2$, it is found that $E(Y_{(2)})$=.6286.

6.81 **a.** With $f(y) = \frac{1}{\beta}e^{-y/\beta}$ and $F(y) = 1 - e^{-y/\beta}$, $y \ge 0$:

$$g_{(1)}(y) = n\left[e^{-y/\beta}\right]^{n-1}\frac{1}{\beta}e^{-y/\beta} = \frac{n}{\beta}e^{-ny/\beta},\ y \ge 0.$$

This is the exponential density with mean β/n.

b. With $n = 5$, $\beta = 2$, $Y_{(1)}$ has and exponential distribution with mean .4. Thus

$$P(Y_{(1)} \le 3.6) = 1 - e^{-9} = .99988.$$

6.82 Note that the distribution function for the largest order statistic is

$$G_{(n)}(y) = \left[F(y)\right]^n = \left[1 - e^{-y/\beta}\right]^n,\ y \ge 0.$$

It is easily shown that the median m is given by $m = \phi_{.5} = \beta\ln 2$. Now,

$$P(Y_{(m)} > m) = 1 - P(Y_{(m)} \le m) = 1 - \left[F(\beta\ln 2)\right]^n = 1 - (.5)^n.$$

6.83 Since $F(m) = P(Y \le m) = .5$, $P(Y_{(m)} > m) = 1 - P(Y_{(n)} \le m) = 1 - G_{(n)}(m) = 1 - (.5)^n$. So, the answer holds regardless of the continuous distribution.

6.84 The distribution function for the Weibull is $F(y) = 1 - e^{-y^m/\alpha}$, $y > 0$. Thus, the distribution function for $Y_{(1)}$, the smallest order statistic, is given by

$$G_{(1)}(y) = 1 - \left[1 - F(y)\right]^n = 1 - \left[e^{-y^m/\alpha}\right]^n = 1 - e^{-ny^m/\alpha},\ y > 0.$$

This is the Weibull distribution function with shape parameter m and scale parameter α/n.

6.85 Using Theorem 6.5, the joint density of $Y_{(1)}$ and $Y_{(2)}$ is given by

$$g_{(1)(2)}(y_1, y_2) = 2,\ 0 \le y_1 \le y_2 \le 1.$$

$$\text{Thus, } P(2Y_{(1)} < Y_{(2)}) = \int_0^{1/2}\int_{2y_1}^{1} 2\,dy_2\,dy_1 = .5.$$

6.86 Using Theorem 6.5 with $f(y) = \frac{1}{\beta}e^{-y/\beta}$ and $F(y) = 1 - e^{-y/\beta}$, $y \ge 0$:

a. $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}\left(1 - e^{-y/\beta}\right)^{k-1}\left(e^{-y/\beta}\right)^{n-k}\frac{e^{-y/\beta}}{\beta} = \frac{n!}{(k-1)!(n-k)!}\left(1 - e^{-y/\beta}\right)^{k-1}\left(e^{-y/\beta}\right)^{n-k+1}\frac{1}{\beta}$, $y \ge 0$.

b. $g_{(j)(k)}(y_j, y_k) = \frac{n!}{(j-1)!(k-1-j)!(n-k)!}\left(1 - e^{-y_j/\beta}\right)^{j-1}\left(e^{-y_j/\beta} - e^{-y_k/\beta}\right)^{k-1-j}\left(e^{-y_k/\beta}\right)^{n-k+1}\frac{1}{\beta^2}e^{-y_j/\beta}$,
$0 \le y_j \le y_k < \infty$.

6.87 For this problem, we need the distribution of $Y_{(1)}$ (similar to Ex. 6.72). The distribution function of Y is

$$F(y) = P(Y \le y) = \int_4^y (1/2)e^{-(1/2)(t-4)}dy = 1 - e^{-(1/2)(y-4)}\ , y \ge 4.$$

a. $g_{(1)}(y) = 2\left[e^{-(1/2)(y-4)}\right]^1 \frac{1}{2}e^{-(1/2)(y-4)} = e^{-(y-4)}, y \ge 4.$

b. $E(Y_{(1)}) = 5$.

6.88 This is somewhat of a generalization of Ex. 6.87. The distribution function of Y is

$$F(y) = P(Y \le y) = \int_\theta^y e^{-(t-\theta)}dy = 1 - e^{-(y-\theta)}\ , y > \theta.$$

a. $g_{(1)}(y) = n\left[e^{-(y-\theta)}\right]^{n-1} e^{-(y-\theta)} = ne^{-n(y-4)}, y > \theta.$

b. $E(Y_{(1)}) = \frac{1}{n} + \theta$.

6.89 Theorem 6.5 gives the joint density of $Y_{(1)}$ and $Y_{(n)}$ is given by (also see Ex. 6.79)

$$g_{(1)(n)}(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2},\ 0 \le y_1 \le y_n \le 1.$$

Using the method of transformations, let $R = Y_{(n)} - Y_{(1)}$ and $S = Y_{(1)}$. The inverse transformations are $y_1 = s$ and $y_n = r + s$ and Jacobian of transformation is 1. Thus, the joint density of R and S is given by

$$f(r,s) = n(n-1)(r+s-s)^{n-2} = n(n-1)r^{n-2},\ 0 \le s \le 1 - r \le 1.$$

(Note that since $r = y_n - y_1$, $r \le 1 - y_1$ or equivalently $r \le 1 - s$ and then $s \le 1 - r$).
The marginal density of R is then

$$f_R(r) = \int_0^{1-r} n(n-1)r^{n-2}ds = n(n-1)r^{n-2}(1-r),\ 0 \le r \le 1.$$

FYI, this is a beta density with $\alpha = n - 1$ and $\beta = 2$.

6.90 Since the points on the interval $(0, t)$ at which the calls occur are uniformly distributed, we have that $F(w) = w/t$, $0 \le w \le t$.

a. The distribution of $W_{(4)}$ is $G_{(4)}(w) = [F(w)]^4 = w^4/t^4$, $0 \le w \le t$. Thus $P(W_{(4)} \le 1) = G_{(4)}(1) = 1/16$.

b. With $t = 2$, $E(W_{(4)}) = \int_0^2 4w^4/2^4 dw = \int_0^2 w^4/4dw = 1.6$.

6.91 With the exponential distribution with mean θ, we have $f(y) = \frac{1}{\theta}e^{-y/\theta}$, $F(y) = 1 - e^{-y/\theta}$, for $y \ge 0$.

a. Using Theorem 6.5, the joint distribution of order statistics $W_{(j)}$ and $W_{(j-1)}$ is given by

$$g_{(j-1)(j)}(w_{j-1}, w_j) = \frac{n!}{(j-2)!(n-j)!}\left(1 - e^{-w_{j-1}/\theta}\right)^{j-2}\left(e^{-w_j/\theta}\right)^{n-j}\frac{1}{\theta^2}\left(e^{-(w_{j-1}+w_j)/\theta}\right),\ 0 \le w_{j-1} \le w_j < \infty.$$

Define the random variables $S = W_{(j-1)}$, $T_j = W_{(j)} - W_{(j-1)}$. The inverse transformations are $w_{j-1} = s$ and $w_j = t_j + s$ and Jacobian of transformation is 1. Thus, the joint density of S and T_j is given by

$$f(s,t_j) = \frac{n!}{(j-2)!(n-j)!}\left(1-e^{-s/\theta}\right)^{j-2}\left(e^{-(t_j+s)/\theta}\right)^{n-j}\frac{1}{\theta^2}\left(e^{-(2s+t_j)/\theta}\right)$$
$$= \frac{n!}{(j-2)!(n-j)!}e^{-(n-j+1)t_j/\theta}\frac{1}{\theta^2}\left(1-e^{-s/\theta}\right)^{j-2}\left(e^{-(n-j+2)s/\theta}\right),\ s\geq 0,\ t_j\geq 0.$$

The marginal density of T_j is then

$$f_{T_j}(t_j) = \frac{n!}{(j-2)!(n-j)!}e^{-(n-j+1)t_j/\theta}\frac{1}{\theta^2}\int_0^\infty\left(1-e^{-s/\theta}\right)^{j-2}\left(e^{-(n-j+2)s/\theta}\right)ds.$$

Employ the change of variables $u = e^{-s/\theta}$ and the above integral becomes the integral of a scaled beta density. Evaluating this, the marginal density becomes

$$f_{T_j}(t_j) = \frac{n-j+1}{\theta}e^{-(n-j+1)t_j/\theta},\ t_j\geq 0.$$

This is the density of an exponential distribution with mean $\theta/(n-j+1)$.

b. Observe that

$$\sum_{j=1}^{r}(n-j+1)T_j = nW_1 + (n-1)(W_2-W_1) + (n-2)(W_3-W_2) + \ldots + (n-r+1)(W_r - W_{r-1})$$
$$= W_1 + W_2 + \ldots + W_{r-1} + (n-r+1)W_r = \sum\nolimits_{j=1}^{r}W_j + (n-r)W_r = U_r.$$

Hence, $E(U_r) = \sum\nolimits_{j=1}^{r}(n-r+1)E(T_j) = r\theta$.

6.92 By Theorem 6.3, U will have a normal distribution with mean $(1/2)(\mu - 3\mu) = -\mu$ and variance $(1/4)(\sigma^2 + 9\sigma^2) = 2.5\sigma^2$.

6.93 By independence, the joint distribution of I and R is $f(i,r) = 2r$, $0\leq i\leq 1$ and $0\leq r\leq 1$. To find the density for W, fix $R = r$. Then, $W = I^2r$ so $I = \sqrt{W/r}$ and $\left|\frac{di}{dw}\right| = \frac{1}{2r}\left(\frac{w}{r}\right)^{-1/2}$ for the range $0\leq w\leq r\leq 1$. Thus, $f(w,r) = \sqrt{r/w}$ and

$$f(w) = \int_w^1\sqrt{r/w}\,dr = \tfrac{2}{3}\left(\tfrac{1}{\sqrt{w}} - w\right),\ 0\leq w\leq 1.$$

6.94 Note that Y_1 and Y_2 have identical gamma distributions with $\alpha = 2$, $\beta = 2$. The mgf is

$$m(t) = (1-2t)^{-2},\ t < 1/2.$$

The mgf for $U = (Y_1 + Y_2)/2$ is

$$m_U(t) = E(e^{tU}) = E(e^{t(Y_1+Y_2)/2}) = m(t/2)m(t/2) = (1-t)^{-4}.$$

This is the mgf for a gamma distribution with $\alpha = 4$ and $\beta = 1$, so that is the distribution of U.

6.95 By independence, $f(y_1,y_2) = 1$, $0\leq y_1\leq 0$, $0\leq y_2\leq 1$.

a. Consider the joint distribution of $U_1 = Y_1/Y_2$ and $V = Y_2$. Fixing V at v, we can write $U_1 = Y_1/v$. Then, $Y_1 = vU_1$ and $\frac{dy_1}{du} = v$. The joint density of U_1 and V is $g(u,v) = v$. The ranges of u and v are as follows:

- if $y_1 \le y_2$, then $0 \le u \le 1$ and $0 \le v \le 1$
- if $y_1 > y_2$, then u has a minimum value of 1 and a maximum at $1/y_2 = 1/v$. Similarly, $0 \le v \le 1$

So, the marginal distribution of U_1 is given by

$$f_{U_1}(u) = \begin{cases} \int_0^1 v dv = \frac{1}{2} & 0 \le u \le 1 \\ \int_0^{1/u} v dv = \frac{1}{2u^2} & u > 1 \end{cases}.$$

b. Consider the joint distribution of $U_2 = -\ln(Y_1Y_2)$ and $V = Y_1$. Fixing V at v, we can write $U_2 = -\ln(vY_2)$. Then, $Y_2 = e^{-U_2}/v$ and $\frac{dy_2}{du} = -e^{-u}/v$. The joint density of U_2 and V is $g(u,v) = -e^{-u}/v$, with $-\ln v \le u < \infty$ and $0 \le v \le 1$. Or, written another way, $e^{-u} \le v \le 1$.

So, the marginal distribution of U_2 is given by

$$f_{U_2}(u) = \int_{e^{-u}}^{1} -e^{-u}/v dv = ue^{-u},\ 0 \le u.$$

c. Same as Ex. 6.35.

6.96 Note that $P(Y_1 > Y_2) = P(Y_1 - Y_2 > 0)$. By Theorem 6.3, $Y_1 - Y_2$ has a normal distribution with mean $5 - 4 = 1$ and variance $1 + 3 = 4$. Thus,

$$P(Y_1 - Y_2 > 0) = P(Z > -1/2) = .6915.$$

6.97 The probability mass functions for Y_1 and Y_2 are:

y_1	0	1	2	3	4
$p_1(y_1)$	.4096	.4096	.1536	.0256	.0016

y_2	0	1	2	3
$p_2(y_2)$	.125	.375	.375	.125

Note that $W = Y_1 + Y_2$ is a random variable with support (0, 1, 2, 3, 4, 5, 6, 7). Using the hint given in the problem, the mass function for W is given by

w	$p(w)$
0	$p_1(0)p_2(0) = .4096(.125) =$ **.0512**
1	$p_1(0)p_2(1) + p_1(1)p_2(0) = .4096(.375) + .4096(.125) =$ **.2048**
2	$p_1(0)p_2(2) + p_1(2)p_2(0) + p_1(1)p_2(1) = .4096(.375) + .1536(.125) + .4096(.375) =$ **.3264**
3	$p_1(0)p_2(3) + p_1(3)p_2(0) + p_1(1)p_2(2) + p_1(2)p_2(1) = .4096(.125) + .0256(.125) + .4096(.375) + .1536(.375) =$ **.2656**
4	$p_1(1)p_2(3) + p_1(3)p_2(1) + p_1(2)p_2(2) + p_1(4)p_2(0) = .4096(.125) + .0256(.375) + .1536(.375) + .0016(.125) =$ **.1186**
5	$p_1(2)p_2(3) + p_1(3)p_2(2) + p_1(4)p_2(1) = .1536(.125) + .0256(.375) + .0016(.375) =$ **.0294**
6	$p_1(4)p_2(2) + p_1(3)p_2(3) = .0016(.375) + .0256(.125) =$ **.0038**
7	$p_1(4)p_2(3) = .0016(.125) =$ **.0002**

Check: .0512 + .2048 + .3264 + .2656 + .1186 + .0294 + .0038 + .0002 = 1.

6.98 The joint distribution of Y_1 and Y_2 is $f(y_1, y_2) = e^{-(y_1+y_2)}$, $y_1 > 0$, $y_2 > 0$. Let $U_1 = \frac{Y_1}{Y_1+Y_2}$, $U_2 = Y_2$. The inverse transformations are $y_1 = u_1u_2/(1 - u_1)$ and $y_2 = u_2$ so the Jacobian of transformation is

$$J = \begin{vmatrix} \frac{u_2}{(1-u_1)^2} & \frac{u_1}{1-u_1} \\ 0 & 1 \end{vmatrix} = \frac{u_2}{(1-u_1)^2}.$$

Thus, the joint distribution of U_1 and U_2 is

$$f(u_1, u_2) = e^{-[u_1u_2/(1-u_1)+u_2]} \frac{u_2}{(1-u_1)^2} = e^{-[u_2/(1-u_1)} \frac{u_2}{(1-u_1)^2},\ 0 \le u_1 \le 1,\ u_2 > 0.$$

Therefore, the marginal distribution for U_1 is

$$f_{U_1}(u_1) = \int_0^\infty e^{-[u_2/(1-u_1)} \frac{u_2}{(1-u_1)^2} du_2 = 1,\ 0 \le u_1 \le 1.$$

Note that the integrand is a gamma density function with $\alpha = 1$, $\beta = 1 - u_1$.

6.99 This is a special case of Example 6.14 and Ex. 6.63.

6.100 Recall that by Ex. 6.81, $Y_{(1)}$ is exponential with mean $15/5 = 3$.

a. $P(Y_{(1)} > 9) = e^{-3}$.

b. $P(Y_{(1)} < 12) = 1 - e^{-4}$.

6.101 If we let $(A, B) = (-1, 1)$ and $T = 0$, the density function for X, the landing point is

$$f(x) = 1/2,\ -1 < x < 1.$$

We must find the distribution of $U = |X|$. Therefore,

$$F_U(u) = P(U \le u) = P(|X| \le u) = P(-u \le X \le u) = [u - (-u)]/2 = u.$$

So, $f_U(u) = F'_U(u) = 1$, $0 \le u \le 1$. Therefore, U has a uniform distribution on (0, 1).

6.102 Define Y_1 = point chosen for sentry 1 and Y_2 = point chosen for sentry 2. Both points are chosen along a one–mile stretch of highway, so assuming independent uniform distributions on (0, 1), the joint distribution for Y_1 and Y_2 is

$$f(y_1, y_2) = 1,\ 0 \le y_1 \le 1,\ 0 \le y_2 \le 1.$$

The probability of interest is $P(|Y_1 - Y_2| < \frac{1}{2})$. This is most easily solved using geometric considerations (similar to material in Chapter 5): $P(|Y_1 - Y_2| < \frac{1}{2}) = .75$ (this can easily be found by considering the complement of the event).

6.103 The joint distribution of Y_1 and Y_2 is $f(y_1, y_2) = \frac{1}{2\pi} e^{-(y_1^2+y_2^2)/2}$. Considering the transformations $U_1 = Y_1/Y_2$ and $U_2 = Y_2$. With $y_1 = u_1u_2$ and $y_2 = |u_2|$, the Jacobian of transformation is u_2 so that the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \frac{1}{2\pi}|u_2| e^{-[(u_1u_2)^2+u_2^2]/2} = \frac{1}{2\pi}|u_2| e^{-[u_2^2(1+u_1^2)]/2}.$$

The marginal density of U_1 is

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi}|u_2| e^{-[u_2^2(1+u_1^2)]/2} du_2 = \int_0^\infty \frac{1}{\pi} u_2 e^{-[u_2^2(1+u_1^2)]/2} du_2.$$

Using the change of variables $v = u_2^2$ so that $du_2 = \frac{1}{2\sqrt{v}} dv$ gives the integral

$$f_{U_1}(u_1) = \int_0^\infty \tfrac{1}{2\pi} e^{-[v(1+u_1^2)]/2} dv = \tfrac{1}{\pi(1+u_1^2)},\ \infty < u_1 < \infty.$$

The last expression above comes from noting the integrand is related an exponential density with mean $2/(1+u_1^2)$. The distribution of U_1 is called the Cauchy distribution.

6.104 **a.** The event $\{Y_1 = Y_2\}$ occurs if

$\{(Y_1 = 1, Y_2 = 1), (Y_1 = 2, Y_2 = 2), (Y_1 = 3, Y_2 = 3), \ldots\}$

So, since the probability mass function for the geometric is given by $p(y) = p(1-p)^{y-1}$, we can find the probability of this event by

$$P(Y_1 = Y_2) = p(1)^2 + p(2)^2 + p(3)^2 \ldots = p^2 + p^2(1-p)^2 + p^2(1-p)^4 + \ldots$$

$$= p^2 \sum_{j=0}^{\infty} (1-p)^{2j} = \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}.$$

b. Similar to part a, the event $\{Y_1 - Y_2 = 1\} = \{Y_1 = Y_2 + 1\}$ occurs if

$\{(Y_1 = 2, Y_2 = 1), (Y_1 = 3, Y_2 = 2), (Y_1 = 4, Y_2 = 3), \ldots\}$

Thus,

$$P(Y_1 - Y_2 = 1) = p(2)\,p(1) + p(3)\,p(2) + p(4)\,p(3) + \ldots$$

$$= p^2(1-p) + p^2(1-p)^3 + p^2(1-p)^5 + \ldots = \frac{p(1-p)}{2-p}.$$

c. Define $U = Y_1 - Y_2$. To find $p_U(u) = P(U = u)$, assume first that $u > 0$. Thus,

$$P(U = u) = P(Y_1 - Y_2 = u) = \sum_{y_2=1}^{\infty} P(Y_1 = u + y_2)P(Y_2 = y_2) = \sum_{y_2=1}^{\infty} p(1-p)^{u+y_2-1} p(1-p)^{y_2-1}$$

$$= p^2(1-p)^u \sum_{y_2=1}^{\infty} (1-p)^{2(y_2-1)} = p^2(1-p)^u \sum_{x=1}^{\infty} (1-p)^{2x} = \frac{p(1-p)^u}{2-p}.$$

If $u < 0$, proceed similarly with $y_2 = y_1 - u$ to obtain $P(U = u) = \frac{p(1-p)^{-u}}{2-u}$. These two results can be combined to yield $p_U(u) = P(U = u) = \frac{p(1-p)^{|u|}}{2-u}$, $u = 0, \pm1, \pm2, \ldots$.

6.105 The inverse transformation is $y = 1/u - 1$. Then,

$$f_U(u) = \tfrac{1}{B(\alpha,\beta)} \left(\tfrac{1-u}{u}\right)^{\alpha-1} u^{\alpha+\beta} \tfrac{1}{u^2} = \tfrac{1}{B(\alpha,\beta)} u^{\beta-1}(1-u)^{\alpha-1},\ 0 < u < 1.$$

This is the beta distribution with parameters β and α.

6.106 Recall that the distribution function for a continuous random variable is monotonic increasing and returns values on [0, 1]. Thus, the random variable $U = F(Y)$ has support on (0, 1) and has distribution function

$$F_U(u) = P(U \le u) = P(F(Y) \le u) = P(Y \le F^{-1}(u)) = F[F^{-1}(u)] = u,\ 0 \le u \le 1.$$

The density function is $f_U(u) = F_U'(u) = 1$, $0 \le u \le 1$, which is the density for the uniform distribution on (0, 1).

6.107 The density function for Y is $f(y) = \frac{1}{4}, -1 \le y \le 3$. For $U = Y^2$, the density function for U is given by

$$f_U(u) = \frac{1}{2\sqrt{u}}\left[f(\sqrt{u}) + f(-\sqrt{u})\right],$$

as with Example 6.4. If $-1 \le y \le 3$, then $0 \le u \le 9$. However, if $1 \le u \le 9$, $f(-\sqrt{u})$ is not positive. Therefore,

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4\sqrt{u}} & 0 \le u < 1 \\ \\ \frac{1}{2\sqrt{u}}\left(\frac{1}{4} + 0\right) = \frac{1}{8\sqrt{u}} & 1 \le u \le 9 \end{cases}.$$

6.108 The system will operate provided that C_1 *and* C_2 function *and* C_3 *or* C_4 function. That is, defining the system as S and using set notation, we have

$$S = (C_1 \cap C_2) \cap (C_3 \cup C_4) = (C_1 \cap C_2 \cap C_3) \cup (C_1 \cap C_2 \cap C_4).$$

At some y, the probability that a component is operational is given by $1 - F(y)$. Since the components are independent, we have

$$P(S) = P(C_1 \cap C_2 \cap C_3) + P(C_1 \cap C_2 \cap C_4) - P(C_1 \cap C_2 \cap C_3 \cap C_4).$$

Therefore, the reliability of the system is given by

$$[1 - F(y)]^3 + [1 - F(y)]^3 - [1 - F(y)]^4 = [1 - F(y)]^3[1 + F(y)].$$

6.109 Let C_3 be the production cost. Then U, the profit function (per gallon), is

$$U = \begin{cases} C_1 - C_3 & \frac{1}{3} < Y < \frac{2}{3} \\ C_2 - C_3 & \text{otherwise} \end{cases}.$$

So, U is a discrete random variable with probability mass function

$$P(U = C_1 - C_3) = \int_{1/3}^{2/3} 20y^3(1-y)dy = .4156.$$

$$P(U = C_2 - C_3) = 1 - ,4156 = .5844.$$

6.110 **a.** Let X = next gap time. Then, $P(X \le 60) = F_X(60) = 1 - e^{-6}$.

b. If the next four gap times are assumed to be independent, then $Y = X_1 + X_2 + X_3 + X_4$ has a gamma distribution with $\alpha = 4$ and $\beta = 10$. Thus,

$$f(y) = \frac{1}{\Gamma(4)10^4} y^3 e^{-y/10}, \; y \ge 0.$$

6.111 **a.** Let $U = \ln Y$. So, $\frac{du}{dy} = \frac{1}{y}$ and with $f_U(u)$ denoting the normal density function,

$$f_Y(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right], y > 0.$$

b. Note that $E(Y) = E(e^U) = m_U(1) = e^{\mu + \sigma^2/2}$, where $m_U(t)$ denotes the mgf for U. Also, $E(Y^2) = E(e^{2U}) = m_U(2) = e^{2\mu + 2\sigma^2}$ so $V(Y) = e^{2\mu+2\sigma^2} - \left(e^{\mu+\sigma^2/2}\right)^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

6.112 **a.** Let $U = \ln Y$. So, $\frac{du}{dy} = \frac{1}{y}$ and with $f_U(u)$ denoting the gamma density function,

$$f_Y(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y\Gamma(\alpha)\beta^\alpha}(\ln y)^{\alpha-1} e^{-(\ln y)/\beta} = \frac{1}{\Gamma(\alpha)\beta^\alpha}(\ln y)^{\alpha-1} y^{-(1+\beta)/\beta}, y > 1.$$

b. Similar to Ex. 6.111: $E(Y) = E(e^U) = m_U(1) = (1-\beta)^{-\alpha}$, $\beta < 1$, where $m_U(t)$ denotes the mgf for U.

c. $E(Y^2) = E(e^{2U}) = m_U(2) = (1-2\beta)^{-\alpha}$, $\beta < .5$, so that $V(Y) = (1-2\beta)^{-\alpha} - (1-\beta)^{-2\alpha}$.

6.113 **a.** The inverse transformations are $y_1 = u_1/u_2$ and $y_2 = u_2$ so that the Jacobian of transformation is $1/|u_2|$. Thus, the joint density of U_1 and U_2 is given by

$$f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(u_1/u_2, u_2)\frac{1}{|u_2|}.$$

b. The marginal density is found using standard techniques.

c. If Y_1 and Y_2 are independent, the joint density will factor into the product of the marginals, and this is applied to part b above.

6.114 The volume of the sphere is $V = \frac{4}{3}\pi R^3$, or $R = \left(\frac{3}{4\pi}V\right)^{1/3}$, so that $\frac{dr}{dv} = \frac{1}{3}\left(\frac{3}{4\pi}\right)^{1/3} v^{-2/3}$. Thus, $f_V(v) = \frac{2}{3}\left(\frac{3}{4\pi}\right)^{2/3} v^{-1/3}$, $0 \le v \le \frac{4}{3}\pi$.

6.115 **a.** Let R = distance from a randomly chosen point to the nearest particle. Therefore,

$P(R > r) = P(\text{no particles in the sphere of radius } r) = P(Y = 0 \text{ for volume } \frac{4}{3}\pi r^3)$.

Since Y = # of particles in a volume v has a Poisson distribution with mean λv, we have

$$P(R > r) = P(Y = 0) = e^{-(4/3)\pi r^3 \lambda}, \quad r > 0.$$

Therefore, the distribution function for R is $F(r) = 1 - P(R > r) = 1 - e^{-(4/3)\pi r^3\lambda}$ and the density function is

$$f(r) = F'(r) = 4\lambda\pi r^2 e^{-(4/3)\lambda\pi r^3}, \quad r > 0.$$

b. Let $U = R^3$. Then, $R = U^{1/3}$ and $\frac{dr}{du} = \frac{1}{3}u^{-2/3}$. Thus,

$$f_U(u) = \frac{4\lambda\pi}{3} e^{-(4\lambda\pi/3)u}, \quad u > 0.$$

This is the exponential density with mean $\frac{3}{4\lambda\pi}$.

6.116 **a.** The inverse transformations are $y_1 = u_1 + u_2$ and $y_2 = u_2$. The Jacobian of transformation is 1 so that the joint density of U_1 and U_2 is

$$f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(u_1+u_2, u_2).$$

b. The marginal density is found using standard techniques.

c. If Y_1 and Y_2 are independent, the joint density will factor into the product of the marginals, and this is applied to part b above.