MAT224H1S - Linear Algebra II
Winter 2020

Notes on Vector Spaces:

Defintion: A *real vector space* is a set V together with two operations called *vector addition* and *scalar multiplication* such that the following *axioms* hold

1. (AC) Additive Closure. For all $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$
2. (SC) Scalar Closure. For all $\mathbf{x} \in V$, and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in V$
3. (AA) Additive Associativity. For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
4. (Z) Zero Vector (or additive identity). There exists a unique vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$
5. (AI) Additive Inverse. For each vector $\mathbf{x} \in V$, there exists a unique vector $-\mathbf{x} \in \mathbf{V}$ with the property that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
6. (SMA) Scalar Multiplication Associativity. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$
7. (DVA) Distributivity of Vector Addition. For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x}+c\mathbf{y}$
8. (DSA) Distributivity of Scalar Addition. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c+d)\mathbf{x} = c\mathbf{x}+d\mathbf{x}$
9. (O) One. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$

Notes:

- The elements in V can be anything but we call them *vectors*.
- When describing a vector space, we need to define (or understand) exactly what the operations of vector addition and scalar multiplication are.
- The scalars need not be real numbers, they can be any *field* F. In this case we call V a vector space over the field F. We will detail this idea in week 9.
- A vector space is composed of three things: a set V together with the two operations of vector addition and scalar multiplication. We use V to describe both the vector space and the set itself.

Standard Examples:

(i) $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ is a vector space with respect to the usual *entry-wise* vector addition and scalar multiplication. The zero vector $\mathbf{0} = (0, 0, \ldots, 0)$.

(ii) $M_{m\times n}(\mathbb{R})$, the set of all $m \times n$ matrices with real entries, is a vector space with respect to the usual *entry-wise* matrix addition and scalar multiplication. The zero vector $\mathbf{0}$ is the $m \times n$ matrix all of whose entries are zero.

(iii) $P_n(\mathbb{R})$, the set of polynomials of degree at most n with real coefficients, is a vector space with respect to the usual polynomial addition and scalar multiplication. The zero vector $\mathbf{0}$ is the constant polynomial $p(x) = 0$.

Exercise and Discussion: Let $P_n'(\mathbb{R})$ be the set of polynomials of degree *exactly* n with real coefficients with the usual polynomial addition and scalar multiplication. Is P_n' a vector space? Why or why not?

Exercise and Discussion : Consider the set $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ of ordered pairs of real numbers with the operations of vector addition and scalar multiplication defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1 + 1, x_2 + y_2 + 1) \\ c\mathbf{x} &= (cx_1 + c - 1, cx_2 + c - 1) \end{aligned}$$

for all $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in V and for all $c \in \mathbb{R}$.

(a) Determine the zero vector $\mathbf{0}$.

(b) For any $\mathbf{x} = (x_1, x_2) \in V$, determine its additive inverse $-\mathbf{x}$.

(c) Show that V is a vector space.

Exercise and Discussion : Use the definition of scalar multiplication in the above Exercise and Discussion to evaluate $0\mathbf{x}$ and $(-1)\mathbf{x}$. What do you notice?

Some properties of Vector Spaces:

Proposition: Let V be a vector space and $\mathbf{x} \in V$. Then $0\mathbf{x} = \mathbf{0}$.

Proposition: Let V be a vector space and $\mathbf{x} \in V$. Then $(-1)\mathbf{x} = -\mathbf{x}$.

Exercise and Discussion: You may have noticed that the axiom

(C) Commutativity. For all $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

is missing from our definition of a vector space. Show that commutativity does indeed hold using our defintion of a vector space.

Hint: What are two possible expressions for $-(\mathbf{x} + \mathbf{y})$?

Bottom Line: A vector space V consists of a bunch of things, called vectors, that you can add to each other and multiply by real numbers, and those operations behave exactly how you would expect.