

Duration: **50 minutes**
Aids Allowed: **one *single-sided handwritten* 8.5"×11" aid sheet**

Student Number: _____

Family Name(s): _____

Given Name(s): _____

Lecture Section: ☐ L0101 (A. Farzan) ☐ L0201 (F. Pitt)

*Do **not** turn this page until you have received the signal to start.*
*In the meantime, please read the instructions below **carefully**.*

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use one of the “blank” pages for rough work. If you need more space for one of your solutions, use a “blank” page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any theorem or result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

MARKING GUIDE

1: _____/ 6

2: _____/ 8

3: _____/12

BONUS

MARKS: _____/ 3

TOTAL: _____/26

Good Luck!

*Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.*

Question 1. [6 MARKS]

Consider the sequence of “Fibonacci Numbers” defined on the right:

$$F_0 = 0,$$

Use simple induction to prove that for all integers $n \geq 0$,

$$F_1 = 1,$$

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

$$F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2.$$

*Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.*

Question 2. [8 MARKS]

For $x, y \in \{\text{True}, \text{False}\}$, let “ $x \oplus y$ ” denote the *exclusive-or* of x and y , which is defined to be True iff exactly one of x and y is True. (The full truth table for \oplus is shown on the right.)

Use complete induction to prove that for all integers $n \geq 0$, every propositional formula F that contains n occurrences of the \oplus connective (and no other connective) is True iff F contains an *odd* number of True variables.

(For example, $(x_2 \oplus x_5) \oplus ((x_1 \oplus x_4) \oplus x_3)$ is False when x_1, x_3 are True and x_2, x_4, x_5 are False, but the same formula is True when x_1, x_3, x_4 are True and x_2, x_5 are False. Note that a propositional formula with *no* connective is simply equal to some propositional variable.)

$$\text{True} \oplus \text{True} = \text{False}$$

$$\text{True} \oplus \text{False} = \text{True}$$

$$\text{False} \oplus \text{True} = \text{True}$$

$$\text{False} \oplus \text{False} = \text{False}$$

*Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere.
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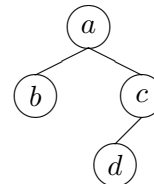
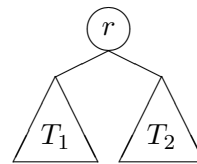
Question 3. [12 MARKS]

Consider the following recursive definition of *binary trees*.

- The *empty tree* (that contains **no** node and **no** edge) is a binary tree.
- If T_1 and T_2 are binary trees and r is a single node, then the tree obtained by making T_1 the left subtree of r and T_2 the right subtree of r is also a binary tree. (This is illustrated on the right.)
- Nothing else is a binary tree.

A *path* is a sequence of nodes with edges between them. For example, “ $b—a—c—d$ ” is a path between b and d in the binary tree pictured on the right. A sequence with just one node (like “ c ”) is a valid path, and so is the empty sequence “” (vacuously).

Use structural induction to prove that for all binary trees T , there is a unique path between any two nodes of T .



*Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.*

Bonus. [3 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this bonus only *after* you have completed the rest of the test.

Use the Principle of Well Ordering to prove that for all positive integers n ,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Attempts to prove this by induction will receive **no** credit.)

On this page, please write nothing except your name.

Family Name(s): _____

Given Name(s): _____

Total Marks = 26