Worth: 12% **Due:** By 8:59pm on Tuesday 31 March

Remember to write the *full name* and *student number* of *every group member* prominently on your submission.

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*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission **every** source of information you used to complete this homework (other than your own lecture and tutorial notes). For example, indicate clearly the **name** of every student from another group with whom you had discussions, the **title and sections** of every textbook you consulted (including the course textbook), the **source** of every web document you used (including documents from the course webpage), etc.*

*For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.*

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1. (a) Give a decision problem D_0 such that $D_0 \leqslant_p \overline{D_0}$.

   Define D_0 precisely, describe the reduction $D_0 \leqslant_p \overline{D_0}$ in detail, and write a detailed proof that your reduction is correct.

   (b) Does your decision problem D_0 from part (1a) belong to NP? Justify.

   (c) Suppose that D_1 is a decision problem such that $D_1 \leqslant_p \overline{D_1}$ and D_1 is NP-complete. What can you conclude? Make the strongest claim you can and prove it.

2. Consider the following (strangely familiar) GOLDDIGGER decision problem.

   Input: A positive integer h (the initial *drill hardness*), **a positive integer g (the *desired gold*)** and two $[m \times n]$ matrices H, G containing non-negative integers with $H[i,j] = 0 \Rightarrow G[i,j] = 0$. (Everything is the same as for question 2 on Assignment 1, **except** for the additional integer parameter g.)

   Output: **Does there exist** a drilling path $j_1, j_2, \ldots, j_\ell$ for some $\ell \leqslant m$ such that:
   - $1 \leqslant j_k \leqslant n$ for $k = 1, 2, \ldots, \ell$ (each coordinate on the path is valid);
   - $j_{k-1} - 1 \leqslant j_k \leqslant j_{k-1} + 1$ for $k = 2, \ldots, \ell$ (each block is underneath the one just above, either directly or diagonally);
   - $H[1, j_1] + H[2, j_2] + \cdots + H[\ell, j_\ell] \leqslant h$ (the total hardness of all the blocks on the path is no more than the initial drill hardness);
   - $G[1, j_1] + G[2, j_2] + \cdots + G[\ell, j_\ell] \geqslant g$ (the path collects at least the desired amount of gold)?

   Write a *detailed* proof that GOLDDIGGER is NP-complete.

3. Write a detailed argument that GOLDDIGGER is polytime self-reducible. (Let GOLDDIGGEROPT represent the optimization version of the problem from question 2 on Assignment 1.)

4. Consider the following PRICEISRIGHT optimization problem.

   Input: A positive integer bound B and a sequence of positive integers $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ with $B \geqslant x_1$.

   Output: A subsequence $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ with the maximum possible sum $x_{i_1} + \cdots + x_{i_k}$ such that $x_{i_1} + \cdots + x_{i_k} \leqslant B$.

(a) Write an algorithm that attempts to solve the PRICEISRIGHT problem. (HINT: There is a natural greedy strategy to try here: use it! The goal is **not** to come up with a fancy algorithm.)

(b) Prove that your algorithm has approximation ratio *at most* 2. *Explain what you are doing and why.* (HINT: Divide your proof into cases, depending on how numbers in the input sequence compare with certain fractions of the bound B.)

[NOTE: This question is at the "Problem Set" level of difficulty. In other words, its purpose is to make you review and understand certain definitions and concepts, and apply them to a simple problem. It should *not* require a great deal of insigh and the final solution should be quite short.]