More on the Calculations of Yule-Walker equations

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Recall that PACF at lag k may be thought of as the k-th regression coefficient in the following regression

$$z_{k+1} = \phi_{k,1} z_k + \phi_{k,2} z_{k-1} + \ \ldots + \phi_{k,k} z_1 + e_t, \quad k = 1,2,3,\ldots,n.$$

And, as discussed in class, we can calculate the partial autocorrelation functions (PACP) using the Cramer's rule which requires calculating complicated determinants for large k. Alternatively, we can calculate PACF using a recursive method, Durbin-Levinson algorithm. Specifically, the Durbin-Levinson recursive may be summarized below.

$$\boldsymbol{\phi}_1 = \phi_{1,1}, \qquad \phi_{1,1} = \frac{\gamma(1)}{\gamma(0)} = \rho(1)$$

$$\boldsymbol{\phi}_k = \begin{pmatrix} \boldsymbol{\phi}_{k-1} - \phi_{k,k}\, \tilde{\boldsymbol{\phi}}_{k-1} \\ \phi_{k,k} \end{pmatrix}, \qquad \phi_{k,k} = \frac{\gamma(k) - \boldsymbol{\phi}'_{k-1}\, \tilde{\boldsymbol{\gamma}}_{k-1}}{\gamma(0) - \boldsymbol{\phi}'_{k-1}\, \boldsymbol{\gamma}_{k-1}},$$

where $\boldsymbol{\phi}_k = \left(\phi_{k,1}, \ldots, \phi_{k,k}\right)'$, $\tilde{\boldsymbol{\phi}}_k = \left(\phi_{k,k}, \ldots, \phi_{k,1}\right)'$, $\boldsymbol{\gamma}_k = (\gamma(1), \ldots, \gamma(k))'$, and $\boldsymbol{\gamma}_k = (\gamma(1), \ldots, \gamma(k))'$, $k = 1,2,\ldots,n$. Alternatively, we may calculate the Durbin-Levinson algorithm using

$$\hat{\phi}_{11} = \hat{\rho}_1, \qquad (1)$$

$$\hat{\phi}_{k+1,k+1} = \frac{\hat{\rho}_{k+1} - \sum_{j=1}^{k} \hat{\phi}_{kj}\, \hat{\rho}_{k+1-j}}{1 - \sum_{j=1}^{k} \hat{\phi}_{kj}\, \hat{\rho}_j}, \qquad (2)$$

$$\hat{\phi}_{k+1,j} = \hat{\phi}_{kj} - \hat{\phi}_{k+1,k+1}\ \hat{\phi}_{k,k+1-j}, \qquad j = 1,\ldots,k. \ (3)$$

Example (Wei, 2005, pp. 21-23)

Suppose that we observe the following sample autocorrelation functions[1]

$$\hat{\rho}_1 = -0.188, \qquad \hat{\rho}_2 = -0.201, \qquad \hat{\rho}_3 = 0.181, \ldots$$

Using Equations (1) to (3), we have

1. $k = 0$:

$$\hat{\phi}_{11} = \hat{\rho}_1 = -0.188$$

2. $k = 1$:

$$\phi_{2,2} = \frac{\hat{\rho}_{1+1} - \hat{\phi}_{1,1}\hat{\rho}_1}{1 - \hat{\phi}_{1,1}\hat{\rho}_1} = \frac{\hat{\rho}_2 - \hat{\rho}_1^2}{1 - \hat{\rho}_1^2} = -0.245$$

$$\hat{\phi}_{2,1} = \hat{\phi}_{1,1} - \hat{\phi}_{2,2}\,\hat{\phi}_{1,1} = -0.234$$

3. $k = 2$:

$$\hat{\phi}_{\underbrace{\underbrace{2}_{k}+1,2+1}_{33}} = \frac{\hat{\rho}_{2+1} - \sum_{j=1}^{2} \hat{\phi}_{2j}\hat{\rho}_{2+1-j}}{1 - \sum_{j=1}^{2} \hat{\phi}_{2j}\hat{\rho}_j} = \frac{\hat{\rho}_3 - \hat{\phi}_{21}\hat{\rho}_2 - \hat{\phi}_{22}\hat{\rho}_1}{1 - \hat{\phi}_{21}\hat{\rho}_1 - \hat{\phi}_{22}\hat{\rho}_2} = 0.097$$

$$\hat{\phi}_{3,\underbrace{1}_{j=1}} = \hat{\phi}_{2,1} - \hat{\phi}_{2+1,2+1}\hat{\phi}_{2,2+1-\underbrace{1}_{j}} = \hat{\phi}_{2,1} - \hat{\phi}_{3,3}\hat{\phi}_{22}$$

$$\hat{\phi}_{3,\underbrace{2}_{j=2}} = \hat{\phi}_{2,2} - \hat{\phi}_{3,3}\hat{\phi}_{2,2+1-\underbrace{2}_{j}} = \hat{\phi}_{2,2} - \hat{\phi}_{3,3}\hat{\phi}_{2,1}$$

4. $k = 3, 4, \ldots, n$

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[1] Like ACF, the sample autocorrelation functions are symmetric around the origin $k = 0$ since

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k}(z_t - \bar{z})(z_{t+k} - \bar{z})}{\sum_{t=1}^{n}(z_t - \bar{z})^2} = \frac{\sum_{t=k+1}^{n}(z_t - \bar{z})(z_{t-k} - \bar{z})}{\sum_{t=1}^{n}(z_t - \bar{z})^2} = \hat{\rho}_{-k}.$$