NOTE TO STUDENTS: This file contains sample solutions to the term test together with the marking scheme and comments for each question. Please read the solutions and the marking schemes and comments carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. For this test in particular, note that the marking was a little more strict than usual. We will remark your test only if you clearly demonstrate that the marking scheme was not followed correctly.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

Question 1. [7 MARKS]

Part (a) [2 MARKS]

Write a *precise* definition of the SUBSETSUM decision problem. In particular, be specific about the exact *type* of each input element. If you do not remember exactly, make an assumption and state it clearly.

SAMPLE SOLUTIONS:

> **Input:** Finite set of positive integers $S \subset \mathbb{Z}^+$, positive integer "target" $t \in \mathbb{Z}^+$.
> **Output:** Does S contain some subset S' with sum exactly t $\left(\exists S' \subseteq S, \sum_{x \in S'} x = t\right)$?

MARKING SCHEME:

- **Input** [1 mark]: correct input description ("positive" was **not** required but made Q3 much simpler)
- **Output** [1 mark]: correct output description

MARKING COMMENTS:

- Too many students lost marks here, given that *you had an aid sheet*!

Now consider the following ZEROSUM decision problem.

Input: A finite set of integers $S \subset \mathbb{Z}$.

Output: Does S contain some subset S' whose sum is exactly zero $\left(\exists S' \subseteq S, \sum_{x \in S'} x = 0\right)$?

Part (b) [2 MARKS]

State the two properties of ZEROSUM that you must prove to conclude that ZEROSUM is *NP*-complete.

1. ZEROSUM $\in NP$
   (the "easy" property)

2. ZEROSUM is *NP*-hard
   (the "hard" property)

SAMPLE SOLUTIONS: (See above.)

MARKING SCHEME:

- **Easy** [1 mark]: correct "easy" property ($-1/2$ for **not** explicitly mentioning "*NP*")
- **Hard** [1 mark]: correct "hard" property ($-1/2$ for **not** explicitly mentioning "*NP*-hard")

MARKING COMMENTS:

- The marking scheme was a little harsh here: I was looking specifically for the *names* of the two properties. In the interest of time and consistency, I will stick with my original marking on this.

Part (c) [3 MARKS]

The "hard" property you stated in Part (b) is usually proven with a suitable polytime reduction. Fill in the blanks in the statement below to describe this process.

To prove that ZEROSUM is *NP*-hard (property),
we show that SUBSETSUM (decision problem) $\leqslant_p$ ZEROSUM (decision problem)
because we know that SUBSETSUM (decision problem) is *NP*-hard (property).

SAMPLE SOLUTIONS: (See above.)

MARKING SCHEME:

- **Property** [1 mark]: correct property filled in — the same in both places
- **Decision Problems** [1 mark]: correct decision problems filled in
- **Direction** [1 mark]: correct direction for the reduction

MARKING COMMENTS:

- Writing "*NP*-complete" in the last blank was fine, but not in the first one (one reduction alone is not enough to conclude *NP*-completeness).

Question 2. [8 MARKS]

Prove that ZEROSUM satisfies the "easy" property you stated in Question 1(b). Write a *detailed* answer and expain what you are doing at each step.

SAMPLE SOLUTIONS:

Algorithm:

VERIFYZS(S, C): # $C \subseteq S$
  return $\left(\sum_{x \in C} x = 0\right)$

Runtime: VERIFYZS performs a linear number of additions, each one taking polytime for total polynomial time.

Correctness: Clearly, VERIFYZS(S, C) = TRUE for some C iff $\exists S' \subseteq S, \sum_{x \in S'} x = 0$.

MARKING SCHEME:

- **Structure** [3 marks]: clear attempt to write a verifier and argue it runs in polytime and is correct
- **Verifier** [2 marks]: runs in polytime and is correct
- **Runtime** [1 mark]: good analysis or "clearly" polytime
- **Correctness** [2 marks]: good argument
- **error code A** [−1]: counting arithmetic operations as constant time (either explicitly or implicitly)
- **error code ∃** [−1]: not mentioning existence/non-existence of certificate when justifying correctness

MARKING COMMENTS:

- **common error** [−1 to −3]: Writing a "generate-and-verify" algorithm instead of just a verifier. Penalty depends on details: runtime, whether it was acknowledged as generate-and-verify, etc.

Question 3. [9 marks]

Prove the reduction you stated in Question 1(c). Write a *detailed* answer and expain what you are doing at each step.

Sample Solutions:

SubsetSum $\leqslant_p$ ZeroSum:

On input (S, t), output $S \cup \{-t\}$.

Clearly, $S \cup \{-t\}$ can be computed in polytime from (S, t).

Also, if $\exists S' \subseteq S, \sum_{x \in S'} x = t$, then $\sum_{x \in S' \cup \{-t\}} x = t - t = 0$ so $\exists S'' \subseteq S \cup \{-t\}, \sum_{x \in S''} x = 0$.

Finally, if $\exists S' \subseteq S \cup \{-t\}, \sum_{x \in S'} x = 0$, then $-t \in S'$ because all other elements in $S \cup \{-t\}$ are positive (so no subset of S alone can have sum equal to zero). Then, $\sum_{x \in S' - \{-t\}} x = t$ so $\exists S'' \subseteq S, \sum_{x \in S''} x = t$.

Marking Scheme:

- **Structure** [3 marks]: clear attempt to construct inputs for ZeroSum from inputs to SubsetSum, analyse runtime and argue correctness
- **Function** [2 marks]: correct reduction (polytime and preserves ansers)
- **Runtime** [1 mark]: good analysis or "clearly" polytime
- **Correctness** [3 marks]: good argument [1 mark for "yes implies yes"; 2 marks for "no implies no"]
- **error code D** [−4]: wrong direction for reduction
- **error code E** [−4]: reduction uses certificate (not polytime)
- **error code F** [−5]: reduction based on idea "let $t = 0$" (input cannot be restricted in this way)
- **error code W** [−4]: "two-way" reduction: two separate constructions, one in each direction
- **error code $\mathbb{Z}$** [−2]: SubsetSum input defined over $\mathbb{Z}$ instead of $\mathbb{Z}^+$ (reduction idea above fails)

Marking Comments:

- **common error**: Error code "F" was very common.
- Many students tried to give a polytime Turing reduction ("$\underset{p}{\rightarrow}$") instead of a polytime reduction ("$\leqslant_p$"). Make sure you understand the difference!