Algorithm

UPDATE(G, w, T, e_0, w_0):
  Run DFS on T to find a path P between the endpoints of e_0.
  Find an edge $e_1 \in P$ with the maximum weight over all edges in P.
  if $w_0 < w(e_1)$:
    return $T \cup \{e_0\} - \{e_1\}$
  else:
    return T

Running Time

- DFS runs in worst-case linear time; because T contains $n-1$ edges, this means $\Theta(n)$.
- Finding an edge of maximum weight in $P \cup \{e_0\}$ takes linear time $\Theta(n)$ in the worst-case.
- Total is $\Theta(n)$—better than recomputing T_0.

Correctness

First, a few observations.

- By definition of "spanning tree," T contains a path P between the endpoints of e_0. As discussed in class, this means that $T \cup \{e_0\} - \{e\}$ is another spanning tree for every edge $e \in P$.
- Because T is a MST in G, $w(e) \leqslant w(e_0)$ for all $e \in P$ (using the original value of $w(e_0)$)—otherwise, we could swap e_0 for some edge $e \in P$ to obtain a spanning tree with total weight strictly less than $w(T)$.
- Let $T_0 = \text{UPDATE}(G, w, T, e_0, w_0)$. Either $T_0 = T$, in which case $w(T_0) = w(T)$, or $T_0 = T \cup \{e_0\} - \{e_1\}$, in which case $w(T_0) < w(T)$ (because this happens only when $w_0 < w(e_1)$).

Now, let $T_0 = \text{UPDATE}(G, w, T, e_0, w_0)$ and, for a contradiction, suppose T_0 is not a MST in G_0, i.e., there is some spanning tree T' in G_0 with $w(T') < w(T_0)$. Either $e_0 \notin T'$ or $e_0 \in T'$.

- If $e_0 \notin T'$, then T' is a spanning tree in the original graph G with $w(T') < w(T_0) \leqslant w(T)$—a contradiction since T is a MST in G.
- If $e_0 \in T'$, then consider $T' - \{e_0\}$. This consists of two connected components A and B, each containing one endpoint of e_0.

  Claim: at least one edge $e' \in P$ has one endpoint in A and the other in B (otherwise, T' would contain every edge in P in addition to e_0, creating a cycle). Note that $w(e') \leqslant w(e_1)$ because e_1 is an edge of maximum weight on path P. Then,

  $$\begin{aligned} w(T') < w(T_0) &\Rightarrow w(T') - \{e_0\} < w(T_0) - \{e_0\} \\ &\Rightarrow w(T') - \{e_0\} \cup \{e'\} < w(T_0) - \{e_0\} \cup \{e_1\} = w(T) \end{aligned}$$

  i.e., $T' - \{e_0\} \cup \{e'\}$ is a spanning tree in the original graph G, with weight $w(T') < w(T)$—a contradiction since T is a MST in G.

In both cases, we reach a contradiction. Hence, T_0 is a MST in G_0.