Solve all the following questions. Show all your work. Please write your answers in the Space provided.

1. If A and B are two events such that $P(A) = 0.2, P(B) = 0.3$ and $P(A U B) = 0.4$. Find the following probabilities.

(a) $P(A \cap B)$. **[2 points]**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.4 = 0.2 + 0.3 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 0.5 - 0.4 = 0.1$$

(b) $P(\overline{A}|B)$. **[2 points]**

$$P(\bar{A} | B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= \frac{0.3 - 0.1}{0.3} = \frac{0.2}{0.3} = \frac{2}{3}$$

(c) Are the events are A and B are mutually exclusive? Explain. **[1 point]**

Since $P(A \cap B) = 0.4 \neq 0$

A and B are not mutually exclusive.

2. If A and B are independent events, show that $\overline{A}$ and $\overline{B}$ are independent. **[2 points]**

We need to show that $P(\bar{A} \cap \bar{B}) = P(\bar{A}) * P(\bar{B})$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B) = 1 - P(A) - P(B)\left(1 - P(A)\right)$$

$$= \left(1 - P(A)\right)\left(1 - P(B)\right) = P(\bar{A}) * P(\bar{B})$$

3. A multiple-choice examination has 5 question, each with five possible answers, only one of which is correct. Suppose that one of the students who takes the examination answers each of the questions with an independent random guess. What is the probability that he answers at least two questions correctly? **[2 points]**

Let Y = no. of correctly answer questions.

$$Y \sim Bin\left(n=5,\ p=\tfrac{1}{5}=0.2\right)$$

$$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$$

$$= 1 - \left\{\binom{5}{0}(0.2)^0(0.8)^5\right\} - \left\{\binom{5}{1}(0.2)^1(0.8)^4\right\}$$

$$= 1 - 0.32768 - 0.4096 = 0.2627$$

4. A box contains 3 different types of phone chips. The probability that a type 1 chip will give over 1 year of the use is 0.7, with the corresponding probabilities for type 2 and type 3 chips being 0.4 and 0.3, respectively. Suppose that 20 percent of the chips in the box are type 1, 30 percent are type 2 and 50 percent are type 3. Suppose that a chosen chip has lasted more than 1 year, what is the probability that it comes from type 2? **[2 points]**

Let A = a chip will test more than 1 year of use.

I: type 1 , II: type 2 ; III : type 3

$P(I) = 0.2$

$P(II) = 0.3$

$P(III) = 0.5$

$P(A|I) = 0.7$

$P(A|II) = 0.4$

$P(A|III) = 0.3$

$$P(II|A) = \frac{P(II \cap A)}{P(A)} = \frac{P(A|II)\,P(II)}{P(A|I)P(I) + P(A|II)P(II) + P(A|III)P(III)}$$

$$= \frac{0.4 * 0.5}{(0.7 * 0.2) + (0.4 * 0.3) + (0.3 * 0.5)}$$

$$= 0.2927$$

5. A group of three undergraduate and five graduate students are available to fill certain student government posts. If four students are to be randomly selected from this group, find the probability that exactly two undergraduates will be among the four chosen. **[2 points]**

$$\frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{3 * 10}{70} = \frac{3}{7} = 0.4286$$

6. The number of telephone calls arriving at an exchange during any one hour period is a Poisson random variable with mean 10. Find the probability that during a fifteen-minutes period there will be 4 calls.
[2 points]

Y = no. of calls arriving on an exchange during any one hour

unit

$Y \sim \text{Poisson}(\lambda_1 = 10)$

$P(X=4)$; where X: no. of call in fifteen minutes

new unit

$X \sim \text{Poisson}\left(\lambda_2 = 10 * \frac{1}{4} = 2.5\right)$

$$P(X=4) = \frac{e^{-\lambda_2} \lambda_2^{x}}{x!} = \frac{e^{-2.5} (2.5)^4}{4!} = 0.1335$$

7. Every day, a lecture may be cancelled due to bad weather with probability 0.05. Class cancellation on days are independent. Compare the probability that the tenth class this semester is the third class that gets cancelled. **[2 points]**

Let Y = the no. of classes until three classes are cancelled

$Y \sim \text{Negbin}(r=3, p=0.05)$

$$P(Y=3) = \binom{10-1}{3-1} (0.05)^3 (0.95)^7$$

$0.95 \equiv 1-0.05$

$$= \binom{9}{2} (0.05)^3 (0.95)^7 = 0.0031$$

8. Suppose that a study of a certain computer system reveals that the response time, in seconds, has a exponential distribution with a mean of 3 seconds. What is the probability that response time exceeds 5 seconds? **[2 points]**

Y = response time. $\quad Y \sim \exp(\beta=3)$

$$P(Y>5) = \int_5^\infty \frac{1}{\beta} e^{-y/\beta}\, dy = \int_5^\infty \frac{1}{3} e^{-y/3} = \frac{1}{3} \cdot \frac{e^{-y/3}}{-1/3} = -e^{-y/3} \Big|_5^\infty$$

$$= -\left(0 - e^{-5/3}\right) = e^{-5/3} \approx 0.1889$$

9. The moment generating function of a random variable Y is **[2 points]**

$$m(t) = E(e^{tY}) = \frac{1}{t+1}, \quad t > -1,$$

Find $Var(Y)$.

$$m(t) = (t+1)^{-1} \Rightarrow m^{(1)}(t) = -1\,(t+1)^{-2}$$

$$m^{(2)}(t) = -1 * -2\,(t+1)^{-3}$$

$$E(Y) = m^{(1)}(t)\Big|_{t=0} = -1$$

$$E(Y^2) = m^{(2)}(t)\Big|_{t=0} = 2$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 2 - (-1)^2 = 1$$

10. A fair (unbiased) die is thrown. Let Y be the number shown up.

(a) Find $E(Y)$. **[2 points]**

y	1	2	3	4	5	6
$p(y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(Y) = \left(1 * \frac{1}{6}\right) + \left(2 * \frac{1}{6}\right) + \cdots + \left(6 * \frac{1}{6}\right) = 3.5$$

(b) Find $Var(Y)$. **[2 points]**

$$E(Y^2) = \left(1^2 * \frac{1}{6}\right) + \left(2^2 * \frac{1}{6}\right) + \cdots + \left(6^2 * \frac{1}{6}\right) = \frac{91}{6} = 15.1667$$

$$Var(Y) = E(Y^2) - (EY)^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{35}{12} = 2.9167$$

(c) Apply Chebysheff's inequality to find an upper bound to $P(|Y - E(Y)| \geq 2.5)$. **[2 points]**

$$P\left(|Y - EY| \geq 2.5\right) \leq \frac{1}{K^2} = \frac{35}{12 * (2.5)^2} = 0.4667$$

$$2.5 = K\sigma \Rightarrow K = \frac{2.5}{\sigma} = \frac{2.5}{\sqrt{\frac{35}{12}}}$$

(d) Find the moment generating function of Y. **[2 points]**

$$m_Y(t) = E\left(e^{tY}\right) = \sum_y e^{ty} P(y)$$

$$= \frac{1}{6} e^{t} + \frac{1}{6} e^{2t} + \cdots + \frac{1}{6} e^{6t}$$

11. Let Y be continuous random variable with probability density function

$$f(y) = \begin{cases} ky^2 + y & 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

where k is a constant.

(a) Find the constant k. **[2 points]**

$$\int_0^1 (Ky^2+y)\,dy = 1 \Rightarrow \frac{Ky^3}{3} + \frac{y^2}{2}\Big|_0^1 = 1$$

$$\Rightarrow \frac{K}{3} + \frac{1}{2} = 1$$

$$\Rightarrow \frac{K}{3} = \frac{1}{2} \Rightarrow K = 3/2$$

(b) Find $E(Y^2)$. **[2 points]**

$$E(Y^2) = \int_0^1 y^2\left(\frac{3}{2}y^2 + y\right) dy$$

$$= \int_0^1 \left(\frac{3}{2}y^4 + y^3\right) dy$$

$$= 3/2 \cdot \frac{y^5}{5} + \frac{y^4}{4}\Big|_0^1 = \frac{3}{10} + \frac{1}{4} = \frac{12+10}{40} = \frac{22}{40} = 0.5$$

(c) Find the cumulative distribution function of Y. **[2 points]**

$$F(y) = \begin{cases} 0 & \text{if } y<0 \\ \int_0^y \left(\frac{3}{2}t^2+t\right)dt & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases} = \begin{cases} 0 & \text{if } y<0 \\ \frac{1}{2}y^3 + \frac{y^2}{2} & \text{if } 0 \leq y \leq 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$