STA255: Statistical Theory

Chapter 6: Functions of Random Variables

Summer 2017

Introduction

- In statistical methods, we are generally not interested in one random variables but rather functions of one or more random variables.
- **For example**: the average of random variables.
- **Objective:** to derive the probability distribution of a function of one or more random variables.
- We assume that populations are large in comparison to the sample size.
- Thus, the random variables obtained through a random sample are in fact independent of one another.

Introduction

- In the discrete case, the joint probability function for $Y_1, Y_2, \ldots, Y_n$, all sampled from the same population, is given by

$$p(y_1, y_2, \ldots, y_n) = p_1(y_1)p_2(y_2)\ldots p_n(y_n).$$

- In the continuous case, the joint density function is

$$f(y_1, y_2, \ldots, y_n) = f_1(y_1)f_2(y_2)\ldots f_n(y_n).$$

- The statement $Y_1, Y_2, \ldots, Y_n$ is a random sample from a population with density $f(y)$ will mean that the random variables are independent with common density function $f(y)$.

Introduction

- **We will study three methods:**
  - The Method of Distribution Functions: Univariate Case.
  - The Method of Transformations : Univariate Case
  - The Method of Moment-Generating Functions: Univariate and Multivariate Cases

The Method of Distribution Functions: Univariate Case

The Method

- *Suppose that Y is a random variable with density function $f(y)$.*
- *Let U be a function of Y.*
- *Find $F_U(u) = P(U \leq u)$ by integrating $f(y)$ over $\{U \leq u\}$.*
- *Then $f_U(u) = \frac{dF_U(u)}{du}$.*

Example

Let $Y \sim U[-1, 1]$. Find the probability density function for $U = Y^2$.

Solution:

The Method

- *Suppose that Y is a random variable with probability distribution $f(y)$.*
- *Let U be a function of Y. That is, $U = h(Y)$*
- *We want to find the pdf of U*
- **Case 1:** *$h(y)$ is increasing function (hence, $h^{-1}(u)$ is also an increasing function)*
    - *The cdf of U:*

$$\begin{aligned} F_U(u) &= P(U \le u) \\ &= P(h(Y) \le u) \\ &= P(Y \le h^{-1}(u)) \\ &= F_Y(h^{-1}(u)). \end{aligned}$$

    - *Thus, the pdf of U is:*

$$f_U(u) = F_Y^{'}(h^{-1}(u)) \frac{dh^{-1}}{du}$$

The Method of Transformations

The Method

- **Case 2:** $h(y)$ *is decreasing function (hence,* $h^{-1}(u)$ *is also a decreasing function)*
  - *The cdf of* U*:*

$$
\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(h(Y) \leq u) \\
&= P(Y \geq h^{-1}(u)) \\
&= 1 - F_Y(h^{-1}(u)).
\end{aligned}
$$

  - *Thus, the pdf of* U *is:*

$$
f_U(u) = -F^{'}(h^{-1}(u))\frac{dh^{-1}}{du}
$$

The Method of Transformations

The Method

- *Thus, in both cases,*
$$f_U(u) = F^{'}(h^{-1}(u))|\frac{dh^{-1}}{du}|.$$
- **Note:** *To apply the method of transformations, $h(y)$ must be either increasing or decreasing for all y such that $f_Y(y) > 0$.*
- **Note:** *The set of points $\{y : f_Y > 0\}$ is called the support of the density $f_Y(y)$.*
- **Summary:**
    - *Let $U = h(Y) : h(y)$ is either increasing of decreasing function of y for all y such that $f_Y(y) > 0$.*
    - *Compute $\frac{dh^{-1}}{du} = \frac{dh^{-1}(u)}{du}$.*
    - $f_U(u) = f_Y(h^{-1}(u))|\frac{dh^{-1}}{du}| = f_Y(h^{-1}(u))|\frac{dy}{du}| = f_Y(y)|_{y=h^{-1}(u)}|\frac{dy}{du}|.$

Example

Let Y has the pdf

$$f(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $U = -4Y + 3$. Find the pdf of U by the transformation method.

Solution:

Example: # 6.26

Let Y have a uniform $(0, 1)$ distribution. Show that $U = -ln(Y)$ has an
2
exponential distribution with mean 2.

Solution:

Moment-Generating Functions

Definition (Moment Generating Function)

The moment generating function (mgf) of a random variable Y, denoted by $m_Y(t)$, is defined to be

$$m_Y(t) = E(e^{tY}) = \begin{cases} \int_{-\infty}^{\infty} e^{ty} f(y)dy & \text{if } Y \text{ is continuous} \\ \sum_y e^{ty} p(y) & \text{if } Y \text{ is discrete.} \end{cases}$$

When the interest is to find the distribution of **a linear combination of independent random variables**, then using the moment-generating functions approach is preferred over the methods discussed earlier.

The Method of Moment-Generating Function

Uniqueness Theorem

Let X and Y be two random variables with moment-generating functions $m_X(t)$ and $m_Y(t)$, respectively. If $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Main Theorem

If $Y_1, \ldots, Y_n$ are independent random variables with moment-generating functions $m_{Y_1}(t), \ldots, m_{Y_n}(t)$, respectively. If $U = Y_1 + \ldots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \cdots m_{Y_n}(t).$$

- **Summary:**
  - Let U be a function of $Y_1, \cdots, Y_n$.
  - Find $m_U(t)$, the mgf of U.
  - Compare $m_U(t)$ with a well-known mgf and use the uniqueness

Example: #6.50

Let $Y \sim Bin(n, p)$. Show that $n - Y \sim Bin(n, 1-p)$.

Solution:

Example: Standard Normal and Chi-squared Distributions

Theorem

$Y \sim N(\mu, \sigma^2)$, then $Z^2 = \left(\frac{Y-\mu}{\sigma}\right)^2 \sim \chi^2(1)$. That is, $[N(0,1)]^2 = \chi^2(1)$.

Proof:

Example

Let Y_1 and Y_2 be two independent random variables having Poisson distributions with parameters μ_1 and μ_2, respectively. Find the distribution of the random variable $U = Y1 + Y2$.

Solution:

Linear Combinations of Normal Distributions

Theorem

Let $Y_1 \ldots, Y_n$ *are independent random variables having normal distributions with means* $\mu_1, \ldots, \mu_n$ *and variances* $\sigma_1^2, \ldots, \sigma_n^2$*, respectively. If*

$$U = a_1 Y_1 + \ldots + a_n Y_n,$$

then U *has a normal distribution with mean*

$$\mu_U = a_1 \mu_1 + \ldots + a_n \mu_n$$

and variance

$$\sigma_U^2 = a_1^2 \sigma_1^2 + \ldots + a_n^2 \sigma_n^2.$$

Proof

Special Cases

1. $Y_1, \ldots, Y_n \overset{iid}{\sim} N(0,1)$, then

$$U = Y_1 + \ldots + Y_n \sim N(0, n).$$

2. If $Y_i \sim (\mu_i, \sigma_i^2)$ (and independent), then

$$U = Y_1 + \ldots + Y_n \sim N(\mu_1, \ldots, \mu_n, \sigma_1^2, \ldots, \sigma_n^2)$$

3. $Y_1, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$,then

$$U = Y_1 + \ldots + Y_n \sim N(n\mu, n\sigma^2)$$

and

$$\bar{Y} = \bar{Y}_n = \frac{Y_1 + \ldots + Y_n}{n} = \frac{U}{n} \sim N(\mu, \sigma^2/n)$$

[set $a_i = 1/n$ in the preceding theorem]

Linear Combinations of Chi-squared Distributions

Theorem

If $Y_1, \ldots, Y_n$ are independent random variables that have, respectively, chi-squared distributions with $v_1, \ldots, v_n$ degrees of freedom, then the random variable

$$U = Y_1 + \ldots + Y_n$$

has a chi-squared distribution with $v = v_1 + \ldots + v_n$ degrees of freedom.

Proof:

Linear Combinations of Chi-squared Distributions

Corollary

If $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$, *then*

$$U = \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma} \right)^2 \sim \chi^2(n).$$

Corollary

If $Y_i \overset{iid}{\sim} N(\mu_i, \sigma_i^2), i = 1, \ldots, n$ *(and independent) then*

$$U = \sum_{i=1}^{n} \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi^2(n).$$

Exercises

(1) Let Y_1 and Y_2 be two independent discrete random variables such that

$$p_1(y_1) = \frac{1}{3}, y_1 = -1, 0, 1$$

and

$$p_2(y_2) = \frac{1}{2}, y_2 = 2, 4.$$

Let $U = Y_1 + Y_2$.

(a) Using the probability mass functions of Y_1 and Y_2, find the probability mass function of U.

(b) Find the moment generating function of U.

(c) Using part (b), find the probability mass function of U. Does your answer agree with (a)?

Exercises

(2) Let Y_1, Y_2 be independent and identically distributed exponential random variables with mean λ.

(a) Find the probability mass function of $U = max(Y_1, Y_2)$.

(b) Find the probability mass function of $V = min(Y_1, Y_2)$.