Lecture Note: Week 3

MATH 461: Probability Theory, Spring 2021
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Lecture 7. Bayes's Formula (Sec 3.3)

Let S be a sample space and E, F events. Since E can be decomposed into two disjoint events EF and EF^c, we have

$$\mathbb{P}(E) = \mathbb{P}(EF) + \mathbb{P}(EF^c) = \mathbb{P}(E|F)\mathbb{P}(F) + \mathbb{P}(E|F^c)\mathbb{P}(F^c).$$

In general, we have the following.

Law of Total Probability
If $S = \bigcup_{i=1}^n F_i$ and F_i's are mutually disjoint (exclusive), then

$$\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(EF_i) = \sum_{i=1}^n \mathbb{P}(E|F_i)\mathbb{P}(F_i).$$

Example 1. There are 15 tennis balls in a box, of which 9 have not previously been used. Three of the balls are randomly chosen, played with, and then returned to the box. Later, another 3 balls are randomly chosen from the box. Find the probability that none of these balls has ever been used.

Example 2. An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

When computing conditional probability $\mathbb{P}(F|E)$, sometimes it is the case that computing $\mathbb{P}(E|F)$ is easier. In that case, we use the following identity

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(FE)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|F)\mathbb{P}(F)}{\mathbb{P}(E)}.$$

In general, we have

Bayes formula
If $S = \bigcup_{i=1}^n F_i$ and F_i's are mutually disjoint (exclusive), then

$$\mathbb{P}(F_i|E) = \frac{\mathbb{P}(E|F_i)\mathbb{P}(F_i)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|F_i)\mathbb{P}(F_i)}{\sum_{i=1}^n \mathbb{P}(E|F_i)\mathbb{P}(F_i)}.$$

Example 3. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested, i.e., if a healthy person is tested, then, with probability .01, the test result will imply that he or she has the disease. If .5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Example 4. A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let $1 - \beta i$, $i = 1, 2, 3$, denote the probability that the plane will be found upon a search of the ith region when the plane is, in fact, in that region. (The constants βi are called overlook probabilities, because they represent the probability of overlooking the plane; they are generally attributable to the geographical and environmental conditions of the regions.) What is the conditional probability that the plane is in the ith region given that a search of region 1 is unsuccessful?

Example 5. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are .7, .2, and .1, respectively.

(i) How certain is she that she will receive the new job offer?

(ii) Given that she does receive the offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

(iii) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

Lecture 8. Independent Events (Sec 3.4)

Consider tossing two coins. The sample space is $S = \{(H, H), (H, T), (T, H), (T, T)\}$ and assume that the probability of each outcome is uniform, $\frac{1}{4}$. Let E (and F) be the events that the first coin (respectively the second coin) is heads. It is natural to expect that the two events are independent, that is, one event does not affect on the other. In fact, the conditional probabilities $\mathbb{P}(E|F)$ and $\mathbb{P}(F|E)$ are both $\frac{1}{2}$, which are the same as $\mathbb{P}(E)$ and $\mathbb{P}(F)$ "unconditional" probabilities.

Independent Events
Events E and F are independent if $\mathbb{P}(EF) = \mathbb{P}(E)\mathbb{P}(F)$.

Note also that this is effectively a consequence of the model. If we believe or data show that there is independence, the model has to incorporate this. If you had to construct a mathematical model for events E and F, as described below, would you assume that they were independent events? Explain your reasoning.

(i) E is the event that a businesswoman has blue eyes, and F is the event that her secretary has blue eyes.

(ii) E is the event that a man is under 6 feet tall, and F is the event that he weighs over 200 pounds.

Example 6. Suppose two cards are drawn at random from a 52-card deck. Let $E = \{\text{first card is black}\}$, $F = \{\text{second card is black}\}$. Are E and F independent?

Example 7. Let E_1 denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Are E_1 and F independent? Suppose that we let E_2 be the event that the sum of the dice equals 7. Is E_2 independent of F?

Example 8. (i) Suppose E and F are independent. Are E and F^c independent?

(ii) Suppose E and F are disjoint. Are E and F independent?

Independent Events: More than two events
Events E, F and G are independent if

$$\mathbb{P}(EF) = \mathbb{P}(E)\mathbb{P}(F), \qquad \mathbb{P}(FG) = \mathbb{P}(F)\mathbb{P}(G),$$
$$\mathbb{P}(GE) = \mathbb{P}(G)\mathbb{P}(E), \qquad \mathbb{P}(EFG) = \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G).$$

In general, a sequence of events $E_1, E_2, \cdots, E_n$ are independent if, for every subsequence $E_{r_1}, E_{r_2}, \cdots, E_{r_k}$,

$$\mathbb{P}(E_{r_1} E_{r_2} \cdots E_{r_k}) = \mathbb{P}(E_{r_1})\mathbb{P}(E_{r_2}) \cdots \mathbb{P}(E_{r_k})$$

Example 9. An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that

(i) at least 1 success occurs in the first n trials;

(ii) exactly k successes occur in the first n trials;

(iii) all trials result in successes?

Lecture 9. Random Variables (Sec 4.1)

Suppose that our experiment consists of tossing 3 fair coins. Let X denote the number of heads that appear. For instance, if the outcome is (H,H,T), then the corresponding X is 2. That means, X is a function of outcomes in the sample space.

Definition: Random Variables
A random variable is a real-valued function defined on the sample space.

If X is 2, then possible outcomes are $(H,H,T),(H,T,H),(T,H,H)$. We use the notation $\{X = 2\} = \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability that $X = 2$ is then defined by

$$\mathbb{P}(X = 2) = \mathbb{P}(\{(H,H,T),(H,T,H),(T,H,H)\}) = \frac{3}{8}.$$

Example 10. Three balls are to be randomly selected without replacement from an urn containing 20 balls numbered 1 through 20. Let X be the largest ball selected. Indicate what values it takes and with what probabilities.

Example 11. Independent trials consisting of the flipping of a coin having probability p of coming up heads are continually performed until either a head occurs or a total of n flips is made. Indicate what values it takes and with what probabilities.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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