Homework 3 Solution

Math 461: Probability Theory, Spring 2021
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1. An urn contains 4 red and 8 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

Solution:

$$\begin{aligned}
\mathbb{P}(A \text{ wins in one move}) &= \frac{4}{12} = \frac{1}{3} \\
\mathbb{P}(A \text{ wins in three moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{4}{10} = \frac{56}{330} \\
\mathbb{P}(A \text{ wins in five moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{7}{99} \\
\mathbb{P}(A \text{ wins in seven moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} = \frac{2}{99} \\
\mathbb{P}(A \text{ wins in nine moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4} = \frac{2}{990} \\
\mathbb{P}(A \text{ wins}) &= \frac{1}{3} + \frac{56}{330} + \frac{7}{99} + \frac{2}{99} + \frac{2}{990} = \frac{59}{99} = 0.596.
\end{aligned}$$

2. Two fair dice are rolled. What is the conditional probability that none lands on 6 given that the dice land on different numbers?

Solution: Let E be the event that no dice lands on six, and let F be the event that the dice land of different numbers. Then $\mathbb{P}(EF) = \frac{5}{6} \cdot \frac{4}{6} = \frac{5}{9}$ and $\mathbb{P}(F) = \frac{6}{6} \cdot \frac{5}{6} = \frac{5}{6}$. Hence,

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)} = \frac{5/9}{5/6} = \frac{2}{3}.$$

3. Consider an urn containing 15 balls, of which 8 are red, 5 are green and 2 are blue. A sample of size 4 is to be drawn with replacement (without replacement). What is the conditional probability (**in each case**) that the first and third balls drawn will be red given that the sample drawn contains exactly 2 red balls?

Solution: Let A be the event that the sample drawn contains exactly 2 red balls. Let B be the event that the first and third ball drawn are red.

WITH REPLACEMENT: $\mathbb{P}(A) = \binom{4}{2}\frac{8^2 \cdot 7^2}{15^4}$ and $\mathbb{P}(AB) = \frac{8^2 \cdot 7^2}{15^4}$, hence $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \frac{1}{6}$.

WITHOUT REPLACEMENT: $\mathbb{P}(A) = \binom{4}{2}\frac{8 \cdot 7 \cdot 7 \cdot 6}{15 \cdot 14 \cdot 13 \cdot 12}$ and $\mathbb{P}(AB) = \frac{8 \cdot 7 \cdot 7 \cdot 6}{15 \cdot 14 \cdot 13 \cdot 12}$, hence $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \frac{1}{6}$.

4. A closet contains 12 pairs of shoes. If 7 shoes are randomly selected without replacement, find the probability that there will be (a) at least one complete pair? (b) exactly 2 complete pairs? (c) exactly 2 complete pairs given that there is at least one complete pair.

Solution: Let A be the event that there are no complete pairs and B be the event that there is exactly one complete pair. Then
(a) $\mathbb{P}(A) = 1 - \frac{2^7\binom{12}{7}}{\binom{24}{7}}$.
(b) $\mathbb{P}(B) = \frac{2^3\cdot\binom{12}{2}\cdot\binom{10}{3}}{\binom{24}{7}}$.
(c) $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(BA)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{1-\mathbb{P}(A)} = \frac{2^3\cdot\binom{12}{2}\cdot\binom{10}{3}}{\binom{24}{7}-2^7\binom{12}{7}}$.

5. Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Solution: Let E_i be the event that the ball drawn from the i-th urn is white, for $i = 1, 2, 3$ (for urn A, B, C). Let F be the event that exactly two white balls were drawn. Then

$$\mathbb{P}(E_1 \mid F) = \frac{\mathbb{P}(E_1F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E_1E_2E_3^c) + \mathbb{P}(E_1E_2^cE_3)}{\mathbb{P}(E_1E_2E_3^c) + \mathbb{P}(E_1E_2^cE_3) + \mathbb{P}(E_1^cE_2E_3)} = \frac{\frac{2\cdot8\cdot3}{6\cdot12\cdot4} + \frac{2\cdot4\cdot1}{6\cdot12\cdot4}}{\frac{2\cdot8\cdot3}{6\cdot12\cdot4} + \frac{2\cdot4\cdot1}{6\cdot12\cdot4} + \frac{4\cdot8\cdot1}{6\cdot12\cdot4}} = \frac{7}{11}.$$

6. Urn I contains 2 white and 4 red balls, whereas urn II contains 1 white and 1 red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is
(a) the probability that the ball selected from urn II is white?
(b) the conditional probability that the transferred ball was white given that a white ball is selected from urn II?

Solution: (a) Let A be the event that the ball selected from urn I is white and let B be the event that the ball selected from urn II is white. The we have

$$\mathbb{P}(A) = 2/(2+4) = 1/3, \ \mathbb{P}(A^c) = 4/(2+4) = 2/3.$$

If the selected ball from urn I is white that after the transfer there are 2 white balls and 1 ed ball in urn II, thus $\mathbb{P}(B \mid A) = 2/(2+1) = 2/3$. Similarly we have $\mathbb{P}(B \mid A^c) = 1/3$. Thus we have by LoTP

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c) = (2/3)(1/3) + (1/3)(2/3) = 4/9.$$

(b) Using Bayes' rule we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c)} = \frac{(2/3)(1/3)}{(2/3)(1/3) + (1/3)(2/3)} = 1/2.$$

7. Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random.
(a) What is the probability that the marble is black?
(b) What is the probability that the first box was the one selected given that the marble is white?

Solution: (a) Let C be the event that the chosen box is box 1 containing 1 black and 1 white marble. Let B be the event that the chosen marble is black. (a) We want $\mathbb{P}(B)$. Using LoTP we have

$$\mathbb{P}(B) = \mathbb{P}(B \mid C)\,\mathbb{P}(C) + \mathbb{P}(B \mid C^c)\,\mathbb{P}(C^c) = \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} = \frac{7}{12} \approx 0.583.$$

(b) We want $\mathbb{P}(C \mid B^c)$. Using Bayes' rule we have

$$\mathbb{P}(C \mid B^c) = \frac{\mathbb{P}(B^c \mid C)\,\mathbb{P}(C)}{\mathbb{P}(B^c)} = \frac{1/2 \cdot 1/2}{1 - \mathbb{P}(B)} = \frac{1/4}{1 - 7/12} = \frac{3}{5} = 0.6.$$

8. Die A has 4 red and 2 white faces, whereas die B has 2 red and 4 white faces. A fair coin is flipped once. If it lands on heads, the game continues with die A; if it lands on tails, then die B is to be used.
(a) Show that the probability of red at any throw is 1/2.
(b) If the first two throws result in red, what is the probability of red at the third throw?
(c) If red turns up at the first two throws, what is the probability that it is die A that is being used?

Solution: Let H be the event the coin lands heads up and we select the die A. From this H^c is then the event that the coin lands tails up and we select die B. Let R_n be the event a red face is showing on the n-th roll of the die. (a) Compute $\mathbb{P}(R_n)$ by conditioning on the result of the coin flip. By Law of Total probability we have

$$\mathbb{P}(R_n) = \mathbb{P}(R_n \mid H)\,\mathbb{P}(H) + \mathbb{P}(R_n \mid H^c)\,\mathbb{P}(H^c) = \frac{4}{6} \cdot \frac{1}{2} + \frac{2}{6} \cdot \frac{1}{2} = \frac{1}{2} = 0.5.$$

(b) We use the definition of conditional probability and LoTP to get

$$\mathbb{P}(R_3 \mid R_1 R_2) = \frac{\mathbb{P}(R_1 R_2 R_3)}{\mathbb{P}(R_1 R_2)} = \frac{\mathbb{P}(R_1 R_2 R_3 \mid H)\,\mathbb{P}(H) + \mathbb{P}(R_1 R_2 R_3 \mid H^c)\,\mathbb{P}(H^c)}{\mathbb{P}(R_1 R_2 \mid H)\,\mathbb{P}(H) + \mathbb{P}(R_1 R_2 \mid H^c)\,\mathbb{P}(H^c)}.$$

Given the choice of the die we have

$$\mathbb{P}(R_1 R_2 R_3 \mid H) = \frac{4^3}{6^3} \text{ and } \mathbb{P}(R_1 R_2 R_3 \mid H^c) = \frac{2^3}{6^3}.$$

Similarly we have

$$\mathbb{P}(R_1 R_2 \mid H) = \frac{4^2}{6^2}, \text{ and } \mathbb{P}(R_1 R_2 \mid H^c) = \frac{2^2}{6^2}.$$

Combining we have

$$\mathbb{P}(R_3 \mid R_1 R_2) = \frac{\left(\frac{4}{6}\right)^3 \cdot \frac{1}{2} + \left(\frac{2}{6}\right)^3 \cdot \frac{1}{2}}{\left(\frac{4}{6}\right)^2 \cdot \frac{1}{2} + \left(\frac{2}{6}\right)^2 \cdot \frac{1}{2}} = \frac{3}{5} = 0.6.$$

(c) Use Bayes' Rule to get

$$\mathbb{P}(H \mid R_1 R_2) = \frac{\mathbb{P}(R_1 R_2 \mid H)\,\mathbb{P}(H)}{\mathbb{P}(R_1 R_2 \mid H)\,\mathbb{P}(H) + \mathbb{P}(R_1 R_2 \mid H^c)\,\mathbb{P}(H^c)} = \frac{4}{5} = 0.8.$$

9. Suppose that you continually collect coupons and that there are m different types. Suppose also that each time a new coupon is obtained, it is a type i coupon with probability $p_i, i = 1, 2, \ldots, m$. Suppose that you have just collected your n-th coupon. What is the probability that it is a new type?
Hint: Condition on the type of the n-th coupon.

Solution: $\mathbb{P}(\text{new type}) = \sum_{i=1}^m p_i \cdot \mathbb{P}(\text{new} \mid \text{the } n\text{-th coupon is of type } i) = \sum_{i=1}^m p_i (1 - p_i)^{n-1}$.

10. Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first four outcomes are
(a) HHHH?
(b) THHH?
(c) What is the probability that the pattern THHH occurs before the pattern HHHH?
Hint for part (c): How can the pattern HHHH occur first?

Solution: (a) $\mathbb{P}(\text{HHHH}) = p^4$.
(b) $\mathbb{P}(\text{THHH}) = p^3(1-p)$.
(c) The only way in which the pattern HHHH can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so THHH will appear first). Hence, $\mathbb{P}(\text{THHH occurs before HHHH}) = 1 - p^4$.