Homework 5

Math 461: Probability Theory, Spring 2021
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Due date: Mar 5, 2021

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.
(a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?
(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the one week period, what strategy should you employ?

2. To determine whether they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to place the people into groups of 10. The blood samples of the 10 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 10 people, whereas if the test is positive, each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume that the probability that a person has the disease is .1 for all people, independently of each other, and compute the expected number of tests necessary for each group. (Note that we are assuming that the pooled test will be positive if at least one person in the pool has the disease.)

3. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win $-$\$1.00. (That is, you lose \$1.00.) Calculate
(a) the expected value of the amount you win;
(b) the variance of the amount you win.

4. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\mathrm{Var}(4+3X)$.

5. Let N be a nennegative integer-valued random variable. For nonnegative values a_j, $j \geqslant 1$, show that

$$\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)\,\mathbb{P}(N = j) = \sum_{i=1}^{\infty} a_i\,\mathbb{P}(N \geqslant i).$$

Then show that

$$\mathbb{E}[N] = \sum_{i=1}^{\infty}\mathbb{P}(N \geqslant i),$$

$$\mathbb{E}[N(N+1)] = 2\sum_{i=1}^{\infty} i\,\mathbb{P}(N \geqslant i).$$

6. Let $p > 0$ be fixed and X be a random variable with pmf

$$p_X(k) = c\binom{n}{k}p^k \text{ for } k = 0, 1, \ldots, n.$$

Find the value of c. Identify the distribution of X and find its mean and variance.

7. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.
(a) Find the probability that 3 or more accidents occur today.
(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

8. The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that, in 1000 hands of poker, you will be dealt at least 2 full houses.

9. Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability $p_i, \sum_{i=1}^{5} p_i = 1$.
(a) Find the expected number of boxes that do not have any balls.
(b) Find the expected number of boxes that have exactly 1 ball.

10. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0, \\ b/4 & 0 \leqslant b < 1, \\ (b+1)/4 & 1 \leqslant b < 2, \\ 11/12 & 2 \leqslant b < 3, \\ 1 & 3 \leqslant b. \end{cases}$$

(a) Find $\mathbb{P}(X = i), i = 1, 2, 3$.
(b) Find $\mathbb{P}(1/2 < X < 3/2)$.