Final Take-Home Exam

Math 461: Probability Theory, Spring 2021
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Due date: May 12, 2021

1. Let X be a uniform random variable on $(-1,1)$ and $Y=|X|$.
   (a) Show that Y is also a uniform random variable.
   (b) Determine whether X and Y are independent.
   (c) Compute $\mathrm{Cov}(X,Y)$.

Solution:

(a) If $a \geqslant 1$, then $F_Y(a)=\mathbb{P}(Y\leqslant a)=1$. If $a\leqslant 0$, then $F_Y(a)=\mathbb{P}(Y\leqslant a)=0$. For $a\in(0,1)$,

$$F_Y(a)=\mathbb{P}(Y\leqslant a)=\mathbb{P}(-a\leqslant X\leqslant a)=a.$$

By differentiating F_Y with respect to a, we get

$$f_Y(a)=\begin{cases}1, & a\in(0,1),\\ 0, & \text{otherwise.}\end{cases}$$

(b) Since

$$\mathbb{P}(X\leqslant \frac{1}{2}, Y\leqslant\frac{1}{2})=\mathbb{P}(-\frac{1}{2}\leqslant X\leqslant\frac{1}{2})=\frac{1}{2}$$
$$\mathbb{P}(X\leqslant\frac{1}{2})\mathbb{P}(Y\leqslant\frac{1}{2})=\mathbb{P}(X\leqslant\frac{1}{2})\mathbb{P}(-\frac{1}{2}\leqslant X\leqslant\frac{1}{2})=\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8},$$

X and Y are dependent.

(c) Since $\mathbb{E}[XY]=\mathbb{E}[X|X|]=0$, $\mathbb{E}[X]=0$, and $\mathbb{E}[Y]=\frac{1}{2}$, we have

$$\mathrm{Cov}(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\,\mathbb{E}[Y]=0.$$

2. Let X,Y be discrete random variables defined by, for some $p_i,q_i>0$ for $i=1,2,\cdots,n$

$$p_i=\mathbb{P}(X=i),\quad q_i=\mathbb{P}(Y=i),\quad \sum_{i=1}^n p_i=1,\quad \sum_{i=1}^n q_i=1.$$

   (a) The entropy of X is defined by

$$H(X)=-\sum_{i=1}^n p_i\log p_i.$$

Show that if $q_1=q_2=\cdots=q_n=\frac{1}{n}$, then $H(X)\leqslant H(Y)$. In other words, the entropy of X is maximized when X is uniform. (Hint: Consider two random variables Z,W defined by

$$\mathbb{P}(Z=\frac{1}{p_i})=p_i,\quad \mathbb{P}(W=\frac{1}{q_i})=q_i,$$

for $i=1,2,\cdots,n$. Then, use Jensen's inequality for $f(t)=\log t$.)

(b) The Kullback–Leibler divergence between X and Y is defined by

$$D(X||Y) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

Show that $D(X||Y) \geqslant 0$. (Hint: Consider a random variable Z defined by

$$\mathbb{P}(Z = \frac{p_i}{q_i}) = q_i,$$

for $i = 1, 2, \cdots, n$. Then, use Jensen's inequality for $f(t) = t \log t$.)

Solution:

(a) Let Z, W be random variables defined by

$$\mathbb{P}(Z = \frac{1}{p_i}) = p_i, \quad \mathbb{P}(W = \frac{1}{q_i}) = q_i,$$

for $i = 1, 2, \cdots, n$. Then, $H(X) = \mathbb{E}[\log Z]$ and $H(Y) = \mathbb{E}[\log W]$. If $q_1 = q_2 = \cdots = q_n = \frac{1}{n}$, then $H(Y) = \log n$. Since $t \mapsto \log t$ is concave and $\mathbb{E}[Z] = n$, it follows from Jensen's inequality that

$$H(X) = \mathbb{E}[\log Z] \leqslant \log(\mathbb{E}[Z]) = \log n = H(Y).$$

(b) Let Z be a random variable defined by

$$\mathbb{P}(Z = \frac{p_i}{q_i}) = q_i,$$

for $i = 1, 2, \cdots, n$. Since $\mathbb{E}[Z] = 1$ and $t \mapsto t \log t$ is convex, we have

$$D(X||Y) = \sum_{i=1}^{n} \frac{p_i}{q_i} \log\left(\frac{p_i}{q_i}\right) q_i. = \mathbb{E}[Z \log Z] \geqslant \mathbb{E}[Z] \log(\mathbb{E}[Z]) = 0.$$

3. Let X and Y be random variables with $0 < \mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$.

(a) Show that $\mathbb{E}[XY]^2 \leqslant \mathbb{E}[X^2]\,\mathbb{E}[Y^2]$. (Hint: Consider $\mathbb{E}[(X - tY)^2] \geqslant 0$ and choose an appropriate value t to get the inequality.)

(b) Assume that $X \geqslant 0$. Show that $\mathbb{P}(X > 0) \geqslant \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$. (Hint: Consider $Y = I_{\{X>0\}}$.)

(c) Let $A_1, A_2, \cdots, A_n$ be events with

$$m = \sum_{i=1}^{n} \mathbb{P}(A_i), \qquad v = \sum_{i<j} \mathbb{P}(A_i \cap A_j).$$

Show that

$$\frac{m^2}{m + 2v} \leqslant \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leqslant m.$$

(Hint: Consider $X_i = I_{A_i}$, $X = \sum_{i=1}^{n} X_i$, and $Y = I_{\{X>0\}}$.)

Solution:

(a) For $t = (\mathbb{E}[X^2]/\,\mathbb{E}[Y^2])^{\frac{1}{2}}$, it follows from $\mathbb{E}[(X - tY)^2] \geqslant 0$ that

$$\mathbb{E}[XY] \leqslant \frac{1}{2t}\,\mathbb{E}[X^2] + \frac{t}{2}\,\mathbb{E}[Y^2] = \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}.$$

(b) Let $Y = I_{\{X>0\}}$, then

$$\mathbb{E}[X]^2 = \mathbb{E}[XY]^2 \leqslant \mathbb{E}[X^2]\,\mathbb{E}[Y^2] = \mathbb{E}[X^2]\mathbb{P}(X > 0).$$

(c) Let $X_i = I_{A_i}$ and $X = \sum_{i=1}^{n} X_i$. Let $Y = I_{\{X>0\}}$, then $Y \leqslant X$ and so

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{P}(X > 0) = \mathbb{E}[Y] \leqslant \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) = m.$$

Applying (b), we also have

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{P}(X > 0) \geqslant \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} = \frac{m^2}{m + 2v}.$$