Homework 10 Solution

Math 461: Probability Theory, Spring 2021
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Due date: Apr 19, 2021

1. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over $(0,1)$, compute the probability that the largest of the three is greater than the sum of the other two.

Solution: By symmetry,

$$\begin{aligned}
&\mathbb{P}\left(\text{the largest of the three is greater than the sum of the other two}\right)\\
&= \mathbb{P}\left(X_1 \geqslant X_2 + X_3\right) + \mathbb{P}\left(X_2 \geqslant X_3 + X_1\right) + \mathbb{P}\left(X_3 \geqslant X_1 + X_2\right)\\
&= 3\mathbb{P}\left(X_1 \geqslant X_2 + X_3\right)\\
&= 3\int_0^1 \int_0^z \int_0^{z-y} dxdydz\\
&= \frac{1}{2}.
\end{aligned}$$

2. Let X_1, X_2, X_3, X_4 be independent uniform random variables on $(0,1)$ and $X_{(i)}$ be the i-th smallest random variable between X_1, X_2, X_3, X_4, for $i = 1,2,3,4$. (That is, $X_{(1)} \leqslant X_{(2)} \leqslant X_{(3)} \leqslant X_{(4)}$.) Define $X = X_{(2)}$ and $Y = 1 - X_{(3)}$.

   (a) Find the joint distribution of X and Y.

   (b) Find the conditional density of Y given $X = x$ for $x \in (0,1)$.

Solution:

(a) For $a, b \in (0,1)$ with $a < 1 - b$,

$$\begin{aligned}
F_{X,Y}(a,b) &= \mathbb{P}\left(X \leqslant a, Y \leqslant b\right)\\
&= \mathbb{P}\left(X_{(2)} \leqslant a, X_{(3)} \geqslant 1 - b\right)\\
&= \binom{4}{2}\mathbb{P}\left(X_1, X_2 \leqslant a \leqslant 1 - b \leqslant X_3, X_4\right)\\
&= 6a^2b^2.
\end{aligned}$$

By differentiating with respect to a and b, we have

$$f_{X,Y}(a,b) = \begin{cases} 24ab, & 0 < a + b < 1, a, b \in (0,1),\\ 0, & \text{otherwise.}\end{cases}$$

(b) The marginal density of X for $x \in (0,1)$ is

$$f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_0^{1-x} 24xy\,dy = 12x(1-x)^2.$$

So, the conditional density of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{(1-x)^2}, & 0 < y < 1-x, \\ 0, & \text{otherwise.} \end{cases}$$

3. Let X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2}, \ x \geqslant 1, y \geqslant 1.$$

(a) Compute the joint density function of $U = XY, V = X/Y$.
(b) What are the marginal densities?

Solution: (a) $u = xy, v = x/y \Rightarrow x = \sqrt{uv}, y = \sqrt{u/v}$. Note that $x \geqslant 1, y \geqslant 1$ implies $v \geqslant 1/u, u \geqslant v, u \geqslant 1, v > 0$. Thus

$$J = \det \begin{bmatrix} y & x \\ 1/y & -x/y^2 \end{bmatrix} = -2x/y$$

and

$$f_{U,V}(u,v) = f(x,y) \cdot \frac{y}{2x} = \frac{1}{2u^2v}, \qquad u \geqslant v \geqslant 1/u, u \geqslant 1, v > 0.$$

(b) The marginal density of U is

$$f_U(u) = \int_0^\infty f_{U,V}(u,v)dv = \int_{1/u}^{u} \frac{1}{2u^2v}dv = \frac{\log u}{u^2}, u \geqslant 1.$$

The marginal density of V is

$$f_V(v) = \int_0^\infty f_{U,V}(u,v)du = \int_{\max\{v,1/v\}}^\infty \frac{1}{2u^2v}du = \begin{cases} \frac{1}{2v^2} & \text{if } v \geqslant 1 \\ \frac{1}{2} & \text{if } 0 < v < 1. \end{cases}$$

Here we use that $f_{U,V}(u,v)$ is positive only when $u \geqslant \max\{1, v, 1/v\}$.

4. If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}[(X-Y)^2]$.

Solution: We have $\mathbb{E}[X-Y] = 0$ and thus $\mathbb{E}[(X-Y)^2] = \operatorname{Var}(X-Y) = \operatorname{Var}(X) + \operatorname{Var}(-Y) = 2\sigma^2$.

5. If $\mathbb{E}[X] = 1$ and $\operatorname{Var}(X) = 4$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\operatorname{Var}(4+2X)$.

Solution: (a) $\mathbb{E}[(2+X)^2] = \mathbb{E}[4+4X+X^2] = 4 + 4 \cdot \mathbb{E}[X] + (\mathbb{E}[X]^2 + \operatorname{Var}(X)) = 13$ and (b) $\operatorname{Var}(4+2X) = 4\operatorname{Var}(X) = 16$.

6. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 2e^{-2x}/x & \text{if } 0 < x < \infty, 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\operatorname{Cov}(X,Y)$.

Solution: We have

$$\mathbb{E}[X] = \int_0^\infty \int_0^x x \cdot 2e^{-2x}/x dy dx = \int_0^\infty x \cdot 2e^{-2x} dx = 1/2$$
$$\mathbb{E}[Y] = \int_0^\infty \int_0^x y \cdot 2e^{-2x}/x dy dx = \int_0^\infty x \cdot e^{-2x} dx = 1/4$$
$$\mathbb{E}[XY] = \int_0^\infty \int_0^x xy \cdot 2e^{-2x}/x dy dx = \int_0^\infty x^2 \cdot e^{-2x} dx = 1/4.$$

Thus $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 1/4 - 1/2 \cdot 1/4 = 1/8$.

7. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find
(a) the expected number of urns that are empty;
(b) the probability that none of the urns is empty.

Solution: Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$$\mathbb{E}[X_j] = \mathbb{P}(\text{ ball } i \text{ is not in urn } j, i \geqslant j) = \prod_{i=j}^{n} (1 - 1/i) = \frac{j-1}{n}.$$

Hence,

$$(a) \qquad \mathbb{E}[\text{Number of empty bins}] = \sum_{j=1}^{n} \frac{j-1}{n} = \frac{n-1}{2}.$$

$$(b) \qquad \mathbb{P}(\text{None are empty}) = \mathbb{P}(\text{ball } j \text{ is in urn } j, \text{ for all } j) = \prod_{j=1}^{n} \frac{1}{j} = \frac{1}{n!}.$$

8. Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers. Find the expected number of changeovers.

Solution: Let X_i equal 1 if a changeover occurs on the i-th flip and 0 otherwise. Then

$$\mathbb{E}[X_i] = \mathbb{P}((i-1) \text{ is H}, i \text{ is T}) + \mathbb{P}((i-1) \text{ is T}, i \text{ is H}) = 2p(1-p), i \geqslant 2.$$

Thus, expected number of changeovers is $(n-1) \cdot 2p(1-p)$.

9. A group of 20 people consisting of 10 married couples is randomly arranged into 10 pairs of 2 each. Compute the mean and variance of the number of married couples that are paired together.

Solution: Let $X_i = \mathbf{1}_{\{\text{pair } i \text{ consists of a married couple}\}}$. Thus $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = 1/19, \mathrm{Var}(X_i) = \frac{1}{19}\left(1 - \frac{1}{19}\right)$, $\mathrm{Cov}(X_i, X_j) = \mathbb{P}(X_i = 1, X_j = 1) - \mathbb{P}(X_i = 1)\,\mathbb{P}(X_j = 1) = \frac{1}{19 \cdot 17} - \left(\frac{1}{19}\right)^2$ for $i \neq j$. Hence

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_{10}) = 10 \cdot \frac{1}{19}\left(1 - \frac{1}{19}\right) + 10 \cdot 9 \cdot \left[\frac{1}{19 \cdot 17} - \left(\frac{1}{19}\right)^2\right] = \frac{180 \cdot 18}{19^2 \cdot 17}.$$

10. Let $X_1, X_2, \ldots$ be independent random variables with common mean μ and common variance σ^2. Set $Y_n = X_n + X_{n+1} + X_{n+2}$, $n \geqslant 1$. For $j \geqslant 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$.

Solution:

$$
\begin{aligned}
\operatorname{Cov}(Y_n, Y_n) &= \operatorname{Var}(Y_n) = 3\sigma^2 \\
\operatorname{Cov}(Y_n, Y_{n+1}) &= \operatorname{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \operatorname{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) = 2\sigma^2 \\
\operatorname{Cov}(Y_n, Y_{n+2}) &= \operatorname{Cov}(X_{n+2}, X_{n+2}) = \sigma^2 \\
\operatorname{Cov}(Y_n, Y_{n+j}) &= 0 \text{ when } j \geqslant 3.
\end{aligned}
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