Moment Generating Function

Let X be a random variable. Suppose the moment generating function $M_X(t)$ of X exists and is finite for $-\infty < t < \infty$. Then $M_X''(0) = \mathbb{E}[X^2]$.

- (a) False
- X (b) True

Let X be a random variable. Suppose the moment generating function $M_X(t)$ of X exists and is finite for $-\infty < t < \infty$. Then $M_X''(0) = \mathrm{Var}(X)$.

- X (a) False
- (b) True

In general, $M_X^{(n)}(0) = \mathbb{E}[X^n]$.

and $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = M_X''(0) - (M_X'(0))^2$

$\neq M_X''(0)$

Conditional Expectation

Let X, Y be two continuous random variables with joint density given by

$$f(x,y) = \frac{15}{y} e^{-5xy^3} \qquad \text{for } x > 0, y > 1$$

and 0 otherwise.

Compute

$$\mathbb{E}[X \mid Y = 1.7].$$

Answer =	number (2 significant figures) 0.040708

In general, Consider

$$f(x,y) = \begin{cases} \frac{A}{y} e^{-Bxy^k}, & x > 0, \ y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$f_Y(y) = \int_0^\infty \frac{A}{y} e^{-Bxy^k} dx$$

$$= \frac{A}{y} \left[-\frac{1}{By^k} e^{-Bxy^k} \right]_0^\infty = \frac{A}{B} \cdot \frac{1}{y^{k+1}}$$

$$f_{X|Y}(x|y) = By^k e^{-Bxy^k} \quad \text{for } x > 0.$$

i.e. $X \mid Y = y \sim \mathrm{Exp}(By^k)$

$$\mathbb{E}[X \mid Y = y] = \frac{1}{By^k}$$

$A = 15$, $B = 5$, $k = 3$. $\Rightarrow$ $\mathbb{E}[X \mid Y = 1.7] = \frac{1}{5 \cdot (1.7)^3}$

$= 0.040708$

Covariance

Let $X_1, X_2, \ldots$ be independent random variables with common mean 7 and common variance 10. Set

$$Y_n = 4X_n + 2X_{n+1}, \quad n \geq 1.$$

Find $\mathrm{Cov}(Y_2, Y_1)$.

Answer = number (2 significant figures) 60

Consider mean $= \mu = 7$, variance $= \sigma^2 = 10$,

and $Y_n = A X_n + B X_{n+1}$, $n \geq 1$

$$\mathrm{Cov}(Y_2, Y_i) = \mathrm{Cov}(A X_2 + B X_3, A X_i + B X_{i+1}) \qquad i = 1, 2, 3$$

$$= A^2 \mathrm{Cov}(X_2, X_i) + AB\, \mathrm{Cov}(X_2, X_{i+1}) + AB\, \mathrm{Cov}(X_3, X_i) + B^2 \mathrm{Cov}(X_3, X_{i+1})$$

$$\mathrm{Cov}(Y_2, Y_1) = AB\, \mathrm{Var}(X_2) = AB\sigma^2 = 80$$

$$\mathrm{Cov}(Y_2, Y_2) = A^2 \mathrm{Var}(X_2) + B^2 \mathrm{Var}(X_3) = 2\sigma^2 (A^2 + B^2)$$

$$\mathrm{Cov}(Y_2, Y_3) = AB\, \mathrm{Var}(X_3) = AB\sigma^2$$

Compute Expectation by Conditioning

Let X be random variable with $\mathbb{E}[X] = 19$ and $\mathbb{E}[X^2] = 363$.

Assume that X takes values bigger than 2 and the conditional distribution of Y given $X = x$ is a geometric random variable with success probability $1/x^2$.

Find $\mathbb{E}[Y]$.

Answer = number (2 significant figures) 363

Suppose $\mathbb{E}[X] = A = 19$, $\mathbb{E}[X^2] = 363 = B$

$X \geq C = 2$, $Y \mid X = x \sim \text{Geom}(D/x^k)$

$(D = 1, k = 2)$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$$

$$= \mathbb{E}\left[\frac{X^k}{D}\right]$$

$$= \frac{B}{D} = 363$$