Lecture Note: Week 4

MATH 461: Probability Theory, Spring 2021
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Lecture 10. Discrete Random Variables (Sec 4.2, 10)

Let X be a random variable. A discrete random variable is a random variable that takes at most a countable number of possible values. For a discrete random variable, we define the probability mass function $p(a)$ by $p(a) = \mathbb{P}(X = a)$. If X takes the values $x_1, x_2, \cdots$, then

$$\begin{aligned} &p(x_i) \geq 0, \text{ for } i = 1, 2, \cdots, \\ &p(x) = 0, \text{ otherwise.} \end{aligned}$$

If X is a discrete random variable taking values x_i with probability $p(x_i)$, its expected value (or mean) is defined as

$$\mathbb{E}[X] = \sum_i x_i p(x_i).$$

Example 1. Let $\lambda > 0$. The probability mass function of a random variable X is given by

$$p(k) = c\frac{\lambda^k}{k!}.$$

for some c.

(i) Find c in terms of λ.

(ii) Find $\mathbb{P}(X = 0)$.

(iii) Find $\mathbb{P}(X > 2)$.

Distribution function
The (cumulative) distributtion function (d.f.) is

$$F(x) = \mathbb{P}(X \leq x)$$

for $-\infty < x < \infty$. We have the following properties:

(i) F is nondecreasing.

(ii) $\lim_{b\to\infty} F(b) = 1$.

(iii) $\lim_{b\to-\infty} F(b) = 0$.

(iv) F is right continuous.

Example 2. If the distribution function of X is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \leq x < 1, \\ \frac{3}{5}, & 1 \leq x < 2, \\ \frac{4}{5}, & 2 \leq x < 3, \\ \frac{9}{10}, & 3 \leq x < 3.5, \\ 1, & x \geq 3.5, \end{cases}$$

calculate the probability mass function of X.

Lecture 11. Expectation (Sec 4.3, 4, 6)

Definition
If X is a discrete random variable taking values x_i with probability $p(x_i)$, its expected value (or mean or expectation) is defined as

$$\mathbb{E}[X] = \sum_i x_i p(x_i).$$

One can think that the expected value of X is a weighted average of the possible values that X taks on.

Example 3. Find $\mathbb{E}[X]$, where X is the outcome when we roll a fair die.

The expectation can be understood as a long run average of values of X in n repeated experiments. That is,

$$\begin{aligned} \mathbb{E}[X] &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} X^{(i)} \\ &= \lim_{n\to\infty} x_k \frac{\text{Number of } x_k \text{ in } n \text{ repeated experiments}}{n} \\ &= \sum_k x_k p(x_k). \end{aligned}$$

Example 4. We say that I is an indicator variable for the event A if

$$I = \begin{cases} 1, & \text{if } A \text{ occurs}, \\ 0, & \text{if } A^c \text{ occurs}. \end{cases}$$

Find $\mathbb{E}[I]$.

If X is a discrete random variable and g is a function, then $g(X)$ is also a discrete random variable. Suppose X takes values $1, 0$ and 1 with probabilities 0.2, 0.5 and 0.3. Let $Y = X^2$. Then, Y takes values either 0 or 1 with probabilities 0.5, 0.5. Thus, the expected value is $\mathbb{E}[Y] = 0.5$.

Expectation of a function of RV
If X is a discrete random variable taking values x_i with probability $p(x_i)$, and g is a function, then

$$\mathbb{E}[g(X)] = \sum_i g(x_i)p(x_i).$$

In particular, for $g(x) = x^n$ and a positive integer n, we call $\mathbb{E}[g(X)] = \mathbb{E}[X^n]$ the n-th moment of X.

Proof. Let $Y = g(X)$ and $p_X(a), p_Y(a)$ be the probability mass functions of X and Y respectively. Then,

$$p_Y(y_j) = \sum_{i:g(x_i)=y_j} p_X(x_i).$$

$$\begin{aligned}\mathbb{E}[g(X)] = \mathbb{E}[Y] = \sum_j y_j p_Y(y_j) &= \sum_j y_j \sum_{i:g(x_i)=y_j} p_X(x_i) \\ &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p_X(x_i) \\ &= \sum_i g(x_i)p_X(x_i).\end{aligned}$$

□

Linearity of Expectation
Let X be a discrete random variable with probability mass function $p(a)$. If a and b are real numbers, then

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

Proof. Using $g(x) = ax + b$ and the fact that $\sum_i p(x_i) = 1$, we have

$$\mathbb{E}[aX + b] = \sum_i (ax_i + b)p(x_i) = a\sum_i x_i p(x_i) + b\sum_i p(x_i) = a\mathbb{E}[X] + b.$$

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References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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