Lecture Note: Week 10

MATH 461: Probability Theory, Spring 2021
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Lecture 24.   Sums of Independent Random Variables (Sec 6.3)

In this section, we consider the sum of two independent random variables X and Y. If X and Y are jointly continuous and independent, then the joint density is $f(x,y) = f_X(x)f_Y(y)$ where f_X and f_Y are the densities for X and Y respectively. Then, the cdf of $X + Y$ is

$$F_{X+Y}(t) = \mathbb{P}(X + Y \leq t) = \iint_{x+y \leq t} f_X(x)f_Y(y)\,dxdy = \int_{\mathbb{R}} F_X(t - y)f_Y(y)\,dy.$$

The cdf of $X + Y$ is called the convolution of F_X and F_Y. Taking derivative with respect to t, we get

$$f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t - y)f_Y(y)\,dy.$$

Example 1. If X and Y are independent uniform random variables on $(0, 1)$, find the density of $X + Y$.

Example 2. Let $X \sim U(0, 1)$ and $Y \sim \text{Exp}(1)$ be independent. Find the distribution of $Z = X + Y$.

Sum of independent random variables
Suppose X and Y are independent. Let $Z = X + Y$.

 (i) If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then $Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

 (ii) If $X \sim \Gamma(s, \lambda)$ and $Y \sim \Gamma(t, \lambda)$, then $Z \sim \Gamma(s + t, \lambda)$.

 (iii) If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$, then $Z \sim \text{Bin}(n + m, p)$.

 (iv) If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$, then $Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

 (v) If $X \sim \text{NegBin}(r, p)$ and $Y \sim \text{NegBin}(s, p)$, then $Z \sim \text{NegBin}(r + s, p)$.

Example 3. If $X \sim N(0, \frac{1}{2})$ and $Y \sim N(0, \frac{1}{2})$ are independent, then what is $f_{X+Y}(t)$?

Example 4. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that the total gross sales over the next 2 weeks exceeds \$5000?

Example 5. If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$, then what is $\mathbb{P}(X + Y = n)$?

Lecture 25.   Conditional Distribution (Sec 6.4-6)

Suppose X and Y are discrete with the joint pmf $p(x, y)$, that is $\mathbb{P}(X = x, Y = y) = p(x, y)$. Let y satisfy $p_Y(y) = \sum_x p(x, y) > 0$. The conditional pmf of X given $Y = y$ is defined by

$$p_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{p(x, y)}{p_Y(y)}.$$

Note that if X and Y are independent, then $p_{X|Y}(x|y) = p_X(x)$. The conditional cdf of X given $Y = y$ is

$$F_{X|Y}(t|y) = \mathbb{P}(X \leq t | Y = y) = \sum_{x \leq t} p_{X|Y}(x|y).$$

Example 6. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

Suppose X and Y are jointly continuous with joint density $f(x, y)$. For y with $f_Y(y) > 0$, the conditional density of X given $Y = y$ is defined by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

If X and Y are independent, then $f_{X|Y}(x|y) = f_X(x)$. Then, the conditional probability and the conditional cdf of X given $Y = y$ can be written as

$$\mathbb{P}(X \in A | Y = y) = \int_A f_{X|Y}(x|y)\, dx$$

$$F_{X|Y}(t|y) = \mathbb{P}(X \leq t | Y = y) = \int_{-\infty}^{t} f_{X|Y}(x|y)\, dx.$$

Example 7. Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}} e^{-y}}{y}, & 0 < x, y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{X|Y}(x|y)$ and $\mathbb{P}(X > 1 | Y = y)$.

Bivariate normal random variable

Jointly continuous random variables X and Y are bivariate normal if their density is given by

$$f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right)}$$

where $\sigma_X, \sigma_Y > 0$, $\rho \in (-1, 1)$, and $\mu_X, \mu_Y \in \mathbb{R}$. We denote by

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right).$$

Proposition 8. (i) $X \sim N(\mu_X, \sigma_X)$ and $Y \sim N(\mu_Y, \sigma_Y)$. In particular, $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$, $\text{Var}(X) = \sigma_X$, and $\text{Var}(Y) = \sigma_Y$.

(ii) *The random variable X given $Y = y$ is normal with mean $\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$ and variance $\sigma_X(1-\rho^2)$.*

Remark 9. The parameter ρ represents how X and Y correlated.

Joint distribution of maximum and minimum

Let $X_1, X_2, \cdots, X_n$ be independent random variables with the common cdf $F(t)$. Let $U = \max\{X_1, X_2, \cdots, X_n\}$ and $V = \min\{X_1, X_2, \cdots, X_n\}$.

Proposition 10. *The joint density of U and V is*

$$f_{U,V}(u,v) = n(n-1)(F(u) - F(v))^{n-2} f(u) f(v).$$

References

[SR]   Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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