Homework 6

Math 461: Probability Theory, Spring 2021
Daesung Kim

Due date: Mar 19, 2021

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. An urn contains 4 white and 5 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

2. Let σ be a random ordering of the sequence $1, 2, \ldots, n$ (for example $2, 4, 5, 1, 3, \ldots$) where each of the $n!$ ordering has equal probability. Given an ordering we say that the i-th position is a *local maxima* if the value at the i-th position is bigger than the neighboring value/values. For example, if the ordering is $(3, 2, 4, 1, 5)$ the 1st, 3rd and 5th positions are local maxima and thee are 3 local maxima. Find the expected total number of local maxima in σ.

3. Let X be a negative binomial random variable with parameters r and p and let Y be a binomial random variable with parameters n and p. Show that
$$\mathbb{P}(X > n) = \mathbb{P}(Y < r).$$

4. Let X be a random variable with probability density function
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
(a) What is the value of c?
(b) What is the cumulative distribution function of X?

5. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by
$$f(x) = \begin{cases} Cxe^{-x/2} & x \geqslant 0 \\ 0 & x \leqslant 0. \end{cases}$$
What is the probability that the system functions for at least 5 months?

6. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by
$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leqslant 10. \end{cases}$$
(a) Find $\mathbb{P}(X > 20)$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

7. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

8. Compute $\mathbb{E}\, X$ if X has a density function given by

(a)
$$f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b)
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c)
$$f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & \text{otherwise.} \end{cases}$$

9. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.
(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?
(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

10. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
(a) What is the probability that you will have to wait longer than 10 minutes?
(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?