Lecture Note: Week 8

MATH 461: Probability Theory, Spring 2021
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Lecture 19. Normal Random Variables (Sec 5.4-7)

Definition: Normal random variables
X is a normal random variable with parameters μ and σ^2 if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $x \in \mathbb{R}$ and denoted by $X \sim N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma = 1$, then we call X the standard normal random variable.

In real life, there are a lot of cases where its randomness can be understood by a normal distribution. Later in Section 8, we will see that normal distributions arise in an important result known as Central Limit Theorem. Note that the constant $\frac{1}{\sqrt{2\pi\sigma^2}}$ is chosen so that $\int f(x)\,dx = 1$.

Proposition 1. *Let $X \sim N(\mu, \sigma^2)$.*

(i) *For any $a, b \in \mathbb{R}$, $aX + b \sim N(a\mu + b, a^2\sigma^2)$. In particular, $Z = (X-\mu)/\sigma \sim N(0,1)$ is standard normal.*

(ii) $\mathbb{E}[X] = \mu$ *and* $\mathrm{Var}(X) = \sigma^2$.

The cumulative distribution function of $N(0,1)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt.$$

Note that the cdf cannot be computed explicitly. Note also that $\Phi(x) = 1 - \Phi(-x)$.

Example 2. If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find

(i) $\mathbb{P}(2 < X < 5)$,

(ii) $\mathbb{P}(X > 0)$,

(iii) $\mathbb{P}(|X - 3| > 6)$.

Normal approximations to binomial
Let $S_n \sim \mathrm{Bin}(n, p)$ be the number of successes in n independent Bernoulli trials. Then, we have seen that $\mathbb{E}[S_n] = np$, $\mathrm{Var}(S_n) = np(1-p)$. For large n,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - np}{\sqrt{np(1-p)}} \approx N(0,1).$$

The approximation is good for $np(1-p) \geq 10$. Compared to Poisson approximation, the success probability p needs not to be small.

Example 3. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Example 4. The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.

Lecture 20. Exponential Random Variables (Sec 5.4-7)

Definition: Exponential random variable
A random variable X is exponential with parameter $\lambda > 0$ if its density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We denote by $X \sim \exp(\lambda)$.

Proposition 5. *Let $X \sim \exp(\lambda)$ for $\lambda > 0$.*

(i) *The cumulative distribution function $F(x) = 1 - e^{-\lambda x}$.*

(ii) $\mathbb{E}[X] = \frac{1}{\lambda}$.

(iii) $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

Proposition 6 (Memoryless property)**.** *Let $s, t > 0$ and $X \sim \exp(\lambda)$ for $\lambda > 0$, then*

$$\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s).$$

Example 7. Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery? What can be said when the distribution is not exponential?

Lecture 21. Other Continuous RVs (Sec 5.4-7)

Gamma random variables
A random variable X is a Gamma random variable with parameter $\lambda > 0$ and $\alpha > 0$ if its density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha - 1} \, dy$ is the gamma function. We denote by $X \sim \Gamma(\alpha, \lambda)$.

Remark 8. Note that $\Gamma(1, \lambda) \sim \exp(\lambda)$.

Remark 9. By integration by parts, one can show that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$. In particular, $\Gamma(1) = 1$ and $\Gamma(n) = (n-1)!$. Thus, the gamma function is a generalization of the factorial.

Proposition 10. *Let* $X \sim \Gamma(\alpha, \lambda)$*, then* $\mathbb{E}[X] = \frac{\alpha}{\lambda}$ *and* $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.

Remark 11. Suppose that the number of events occur in the time interval $[0, t]$ is a Poisson random variable Poisson(λt) for some $\lambda > 0$. (See [SR, p.144] for this approximation.) If T_n is the time at which the n-th event occurs, then T_n is a gamma random variable with n and λ.

Weibull random variables
A random variable X is a Weibull random variable with parameter $\nu, \alpha > 0$ if its cdf is given by

$$F(x) = \begin{cases} 0, & x < \nu \\ 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, & x > \nu. \end{cases}$$

Weibull is often used to model lifetime of some object, device, individual, etc.

Beta random variables
A random variable X is a beta random variable with parameter $a, b > 0$ if its density is given by

$$f(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$ is the beta function.

The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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