Lecture Note: Week 2

MATH 461: Probability Theory, Spring 2021
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Lecture 4. Axioms of Probability II (Sec 2.4)

Probability Space
Let S be a sample space. For each event E, the probability $\mathbb{P}(E)$ is an assignment so that the following axioms are satisfied:

(i) For all events, $0 \leq \mathbb{P}(E) \leq 1$.

(ii) $\mathbb{P}(S) = 1$.

(iii) For any sequence of mutually exclusive events $E_1, E_2, \cdots$, (meaning tthat $E_iE_j = \varnothing$ for $i \neq j$),

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

The pair of a sample space and probability $(S, \mathbb{P})$ satisfying the three axioms is called a probabilty space.

Example 1. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = \mathbb{P}(\{5\}) = \mathbb{P}(\{6\}) = \frac{1}{6}$. For an event E in S, we define the probability $\mathbb{P}(E)$ by

$$\mathbb{P}(E) = \sum_{i \in E} \mathbb{P}(\{i\}).$$

Then the three axioms are satisfied.

Proposition: Properties of Probability

(i) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

(ii) If $E \subset F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$.

(iii) For any two events E and F, $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(EF)$.

Proof. (i) Let $E_1 = E$, $E_2 = E^c$, and $E_i = \varnothing$ for all $i > 2$. The axiom (iii) yields that $\mathbb{P}(S) = 1 = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$. Since $\mathbb{P}(\varnothing) = 0$ (why?), we have $1 = \mathbb{P}(E) + \mathbb{P}(E^c)$.

(ii) Let $E_1 = E$, $E_2 = F \cap E^c$, and $E_i = \varnothing$ for all $i > 2$. The axiom (iii) yields that $\mathbb{P}(F) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \mathbb{P}(E) + \mathbb{P}(F \cap E^c)$. Since $\mathbb{P}(F \cap E^c) \geq 0$ by the axiom (i), we conclude that $\mathbb{P}(E) \leq \mathbb{P}(F)$.

(iii) Using the axiom (iii), one has

$$\begin{aligned}
\mathbb{P}(E \cup F) &= \mathbb{P}(EF^c) + \mathbb{P}(EF) + \mathbb{P}(FE^c) \\
\mathbb{P}(E) &= \mathbb{P}(EF^c) + \mathbb{P}(EF) \\
\mathbb{P}(F) &= \mathbb{P}(EF) + \mathbb{P}(FE^c),
\end{aligned}$$

which leads to

$$\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cup F) = \mathbb{P}(EF).$$

□

Example 2. J is taking two books along on her holiday vacation. With probability .5, she will like the first book; with probability .4, she will like the second book; and with probability .3, she will like both books. What is the probability that she likes neither book?

What if we have more than two events? Let E, F, G be events and consider $\mathbb{P}(E \cup F \cup G)$. Using the proposition (iii) for $(E \cup F)$ and G and for E and F, one has

$$\begin{aligned}\mathbb{P}(E \cup F \cup G) &= \mathbb{P}(E \cup F) + \mathbb{P}(G) + \mathbb{P}((E \cup F)G) \\ &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(EF) + \mathbb{P}(G) + \mathbb{P}(EG \cup FG).\end{aligned}$$

Using the same proposition for EG and FG, we have

$$\mathbb{P}(EG \cup FG) = \mathbb{P}(EG) + \mathbb{P}(FG) - \mathbb{P}(EFG).$$

Therefore, we get

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G) - \mathbb{P}(EF) - \mathbb{P}(FG) - \mathbb{P}(GE) + \mathbb{P}(EFG).$$

In general, we have the following.

Lemma: Inclusion-Exclusion Principle
For events $E_1, E_2, \cdots, E_n$,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup \cdots E_n) = (&\mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots + \mathbb{P}(E_n)) \\ &- (\mathbb{P}(E_1E_2) + \mathbb{P}(E_1E_3) + \cdots + \mathbb{P}(E_iE_j) + \cdots) + \cdots \\ &+ (-1)^{r+1} \sum \mathbb{P}(E_{i_1}E_{i_2} \cdots E_{i_r}) + \cdots \\ &+ (-1)^{n+1}\mathbb{P}(E_1E_2 \cdots E_n).\end{aligned}$$

Lecture 5. Sample spaces having equally likely (Sec 2.5)

Let S be a sample space with finitely many outcomes. For convenience, let $S = \{1, 2, 3, \cdots, N\}$. In many cases, it is natural to assume that all outcomes in S are equally likey to occur. In other words, we assume

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{N\}).$$

By the axioms (ii) and (iii), we have

$$1 = \mathbb{P}(S) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \cdots + \mathbb{P}(\{N\}).$$

Therefore, $\mathbb{P}(\{i\}) = \frac{1}{N}$ for each $i = 1, 2, \cdots, N$. Define the probability of an event E by

$$\mathbb{P}(E) = \sum_{i \in E} \mathbb{P}(\{i\}) = \frac{\text{Number of Outcomes in } E}{\text{Number of Outcomes in } S}.$$

Then one can see that $(S, \mathbb{P})$ is a probability space.

Example 3. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

(i) of the same color?

(ii) of different colors?

Suppose we draw a ball, note its color, and replace it into the urn. If we draw 3 balls in this way, what is the probability that each of the balls are of the same color? or of different colors?

Example 4. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

Example 5. A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs? What is the probability that there are 4 offensive–defensive roommate pairs?

Recall: Inclusion-Exclusion Principle
For events $E_1, E_2, \cdots, E_n$,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup \cdots E_n) = (&\mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots + \mathbb{P}(E_n)) \\ &- (\mathbb{P}(E_1E_2) + \mathbb{P}(E_1E_3) + \cdots + \mathbb{P}(E_iE_j) + \cdots) + \cdots \\ &+ (-1)^{r+1} \sum \mathbb{P}(E_{i_1}E_{i_2} \cdots E_{i_r}) + \cdots \\ &+ (-1)^{n+1}\mathbb{P}(E_1E_2 \cdots E_n).\end{aligned}$$

Example 6. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Example 7. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be (a) no complete pair? (b) exactly 1 complete pair?

Lecture 6. Conditional Probabilities (Sec 3.2)

Let S be a sample space and E, F events. We consider the probability that E occurs given that F has given. This probability is called the conditional probability of E given F and denoted by $\mathbb{P}(E|F)$.

Suppose $S = \{(i,j) : 1 \le i,j \le 6\}$, $E = \{(3,j) : 1 \le j \le 6\}$, and $F = \{(i,j) : i + j = 8\}$. Assume that each outcome has the same probability. Since $F = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ and only $(3,5)$ belongs to E, it is natural to think

$$\mathbb{P}(E|F) = \frac{1}{5} = \frac{\text{Number of outcomes in } E \text{ and } F}{\text{Number of outcomes in } F}.$$

In general, we can define the conditional probability as follows.

Definition: Conditional Probability
If $\mathbb{P}(F) > 0$, then

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)}.$$

Example 8. A coin is flipped twice. Assuming that all four points in the sample space

$$S = (H,H),(H,T),(T,H),(T,T)$$

are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads? (b) at least one flip lands on heads?

Example 9. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces and let A be the event that at least one ace is chosen. Find $\mathbb{P}(B|A)$.

Computing Probabilities via Conditioning
If $\mathbb{P}(F) > 0$, then

$$\mathbb{P}(EF) = \mathbb{P}(F)\frac{\mathbb{P}(EF)}{\mathbb{P}(F)} = \mathbb{P}(E|F)\mathbb{P}(F).$$

In general,

$$\mathbb{P}(E_1E_2\cdots E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)\mathbb{P}(E_3|E_2E_1)\cdots\mathbb{P}(E_n|E_1E_2\cdots E_{n-1}).$$

Example 10. Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Example 11. An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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