Lecture Note: Week 11

MATH 461: Probability Theory, Spring 2021
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Lecture 26. Joint Distribution of Functions of RVs (Sec 6.7)

Suppose X_1 and X_2 are jointly continuous random variables with joint probability density $f_X(x_1, x_2)$. Let $g = (g_1, g_2)$ and $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ with

$$|J_g(x_1, x_2)| = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \left| \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \right| \neq 0$$

where

$$J_g(x_1, x_2) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix}.$$

Suppose that there exists a map $h = (h_1, h_2)$ such that the equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 so that $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$. Here, J_g is called the Jacobian of g, and h is the inverse of g. Note that g is continuously differentiable (that is, g_1 and g_2 are differentiable and their derivatives are continuous), then $J_h = J_g^{-1}$ (due to the inverse function theorem). Let $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$, then the change of variable formula provides

$$\begin{aligned} \iint_B f_Y(y_1, y_2)\, dy_1 dy_2 &= \mathbb{P}((Y_1, Y_2) \in B) \\ &= \mathbb{P}((X_1, X_2) \in h(B)) \\ &= \iint_{h(B)} f_X(x_1, x_2)\, dx_1 dx_2 \\ &= \iint_B f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_h(y_1, y_2)|\, dy_1 dy_2 \\ &= \iint_B f_X(x_1, x_2) |J_g(x_1, x_2)|^{-1}\, dy_1 dy_2. \end{aligned}$$

Thus, the joint density of Y_1 and Y_2 is

$$\begin{aligned} f_Y(y_1, y_2) &= f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_h(y_1, y_2)| \\ &= f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_g(h_1(y_1, y_2), h_2(y_1, y_2))|^{-1}. \end{aligned}$$

Example 1. Let X_1 and X_2 be jointly continuous random variables with probability density function $f(X_1, X_2)$. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of $f(X_1, X_2)$.

Example 2. Let (X, Y) denote a random point in the plane, and assume that the rectangular coordinates X and Y are independent standard normal random variables. What is the joint distribution of R, Θ, the polar coordinate representation of (X, Y).

Lecture 27. Sums of random variables (Sec 7.2, 3)

Let $g(x, y)$ be a function on $\mathbb{R}^2$. If X, Y are discrete with joint pmf $p(x, y)$, then

$$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y)p(x,y).$$

If X, Y are jointly continuous with joint density $f(x, y)$, then

$$\mathbb{E}[g(X,Y)] = \iint g(x,y)f(x,y)\,dxdy.$$

Example 3. Let X and Y be independent uniform random variables on $(0, 1)$. Find $\mathbb{E}[|X - Y|^k]$.

As a consequence, we have the following:

Expectations of sum and product

(i) $\mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$

(ii) If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

Example 4. Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters are at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size f flies overhead.

Example 5 (A random walk in the plane)**.** Consider a particle initially located at a given point in the plane, and suppose that it undergoes a sequence of steps of fixed length, but in a completely random direction. Specifically, suppose that the new position after each step is one unit of distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over $(0, 2\pi)$. Compute the expected square of the distance from the origin after n steps.

The number of events that occur

For events $A_1, A_2, \cdots, A_n$, we consider the corresponding indicator random variables $X_i = I_{A_i}$, defined by

$$I_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, the number of the events that occur can be written as the sum of X_i. Let $X = \sum_i X_i$. Then, the expectation and the variance of X are

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \mathbb{P}(A_i), \\
\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \sum_{i,j=1}^n \mathbb{E}[X_i X_j] - (\mathbb{E}[X])^2 \\
&= 2 \sum_{i,j=1, i<j}^n \mathbb{P}(A_i A_j) + \sum_{i=1}^n \mathbb{P}(A_i) - (\sum_{i=1}^n \mathbb{P}(A_i))^2.
\end{aligned}$$

Example 6. Let X be a binomial random variable with parameters n and p. Compute $\mathbb{E}[X]$, $E[X^2]$ and $\mathrm{Var}(X)$.

Example 7. A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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