Homework 2 Solution

Math 461: Probability Theory, Spring 2021
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Due date: Feb 12, 2021

1. (a) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n = k.$$

(b) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

(c) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

Solution: (a) This is equivalent to choosing k positions out of n positions where the value will be 1 and 0 for the rest. Thus the answer is $\binom{n}{k}$.
(b) The answer is

$$\sum_{i=0}^{k} \binom{n}{i}$$

as the sum can be any number from k to n.
(c) This can be done in two ways:

i. Number of vectors $(x_1, x_2, \ldots, x_n)$ where each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n = i,$$

is $\binom{n+i-1}{i}$. So the answer is

$$\sum_{i=0}^{k} \binom{n+i-1}{i}.$$

ii. This is same as number of vectors $(x_1, x_2, \ldots, x_n, x_{n+1})$ where each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_{n+1} = k,$$

which is

$$\binom{n+k}{k}$$

2. Consider the set S of numbers $\{1, 2, \ldots, n\}$. One can see that the number of subsets of S size k is $\binom{n}{k}$. Count the same number in a different way depending on how many subsets of size k have i as their highest numbered member, to give a proof of the following identity known as Fermat's combinatorial identity: For all integers $n \geqslant k$

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

Solution: Suppose we want to choose a group of k many numbers from the set of numbers 1 through n. Clearly the number of choices is $\binom{n}{k}$. Now we can count the number in a different way. The largest number, say i, in the group of selected numbers can be anything from k to n. Given the largest number i, the number of ways to choose the remaining numbers is $\binom{i-1}{k-1}$. Thus the total number of choices is

$$\sum_{i=k}^{n} \binom{i-1}{k-1}.$$

Thus we have,

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

3. (a) In how many ways can n identical balls be distributed into r bins such that each bin contains at least two balls. Assume that $n \geqslant 2r$.
(b) Do the same problem as in (a), but now each bin contains at least three balls and $n \geqslant 3r$.

Solution: (a) This is equivalent to in how many ways we can write $x_1 + x_2 + \cdots + x_r = n$ where $x_i \geqslant 2$ for all i.

Given a sequence $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_r = n$ and $x_i \geqslant 1$ for all i, write $y_i = x_i - 2$ for all i. Then we have $y_i \geqslant 0$ for all i and

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

Note that we can get back x_i's from y_i's by adding 2 to each of them. Thus the answer is same as in how many ways we can write

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

where $y_i \geqslant 0$ for all i. We did this in class and the answer is $\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}$.

(b) $\binom{n-3r+r-1}{r-1} = \binom{n-2r-1}{r-1}$.

4. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i-th position, $1 \leqslant i \leqslant b+g$, is a girl?

Solution: We can compute all permutations of the $b+g$ people that have a girl in the i-th spot as follows. We have g choices for the specific girl we place in the i-th spot. Once this girl is selected we have $b+g-1$ other people to place in the $b+g-1$ slots around this i-th spot. This can be done in $(b+g-1)!$ ways. So the total number of ways to place a girl at position i is $g \cdot (b+g-1)!$. Thus the probability of finding a girl in the i-th spot is given by $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$.

5. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

Solution: Number of ways to choose two cards (ordered) from the deck of 52 cards is $52 \cdot 51$. Among them number of ways in which one of them is an ace (there are 4 aces with one of 4 suits) and the other is either a ten, a jack, a queen, or a king (there are total 16 cards) is $4 \cdot 16 \cdot 2$, where the last 2 is for ordering the cards. Thus the answer is

$$\frac{4 \cdot 16 \cdot 2}{52 \cdot 51} = 0.048.$$

6. A die is rolled until either 3 or 5 appears. Find the probability that a 5 occurs first. Simplify the answer.
Hint: Let E_n denote the event that a 5 occurs on the n-th roll and no 3 or 5 occurs on the first $n-1$ rolls. Find $\mathbb{P}(E_n)$ and express the above probability in terms of them.

Solution: Following the hint we let E_n be the event that a 5 occurs on the n-th roll and no three or five occurs on the $(n-1)$ rolls up to that point. Then

$$\mathbb{P}(E_n) = \frac{4^{n-1} \cdot 1}{6^n},$$

as for n rolls total number of possible outcomes are 6^n and out of them E_n has 4^{n-1} many outcomes. Since we want the probability that a five comes first, this can happen at roll number one ($n = 1$), at roll number two ($n = 2$) or any subsequent roll. Moreover, the events E_n are mutually exclusive. Thus the probability that a five comes first is given by

$$\mathbb{P}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}(E_n) = \sum_{n=1}^{\infty} \frac{4^{n-1}}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{4 \cdot 3^n} = \frac{1}{4} \cdot \frac{2/3}{1-2/3} = \frac{1}{2} = 0.5.$$

7. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?
Hint: Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*).

Solution: We want the probability that a given hand of bridge is void in *at least* one suit which means the hand could be void in more than one suit out of four suits. Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*). Then the probability we want is $\mathbb{P}(\cup_{i=1}^{4} E_i)$ which we can calculate by using the inclusion-exclusion identity and the symmetry of the suits, given in this case by

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} \mathbb{P}(E_i) - \sum_{i=1}^{3} \sum_{j>i} \mathbb{P}(E_i E_j) + \sum_{i=1}^{3} \sum_{j>i} \sum_{k>j} \mathbb{P}(E_i E_j E_k) \\ &= 4\,\mathbb{P}(E_1) - 6\,\mathbb{P}(E_1 E_2) + 4\,\mathbb{P}(E_1 E_2 E_3).\end{aligned}$$

Note there is no terms $\mathbb{P}(E_i E_j E_k E_l)$ (which is zero) since we must be dealt some cards. We have $\mathbb{P}(E_1) = \frac{\binom{39}{13}}{\binom{52}{13}}$. Similarly

$$\mathbb{P}(E_1 E_2) = \frac{\binom{26}{13}}{\binom{52}{13}} \text{ and } \mathbb{P}(E_1 E_2 E_3) = \frac{\binom{13}{13}}{\binom{52}{13}} = \frac{1}{\binom{52}{13}}.$$

Thus we get

$$\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{1}{\binom{52}{13}} \left(4\binom{39}{13} - 6\binom{26}{13} + 4 \right) = 0.051.$$

8. For a group of 10 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.
Hint: Let E_i be the event that there are no birthdays in the i-th season.

Solution: Let E_i be the event that there are no birthdays in the i-th season. The probability that all seasons occur at least once is $1 - \mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4)$. Note that $E_1E_2E_3E_4 = \emptyset$;. Using the inclusion-exclusion principle and the symmetry of the seasons,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} \mathbb{P}(E_i) - \sum_{i=1}^{3}\sum_{j>i} \mathbb{P}(E_iE_j) + \sum_{i=1}^{3}\sum_{j>i}\sum_{k>j} \mathbb{P}(E_iE_jE_k) \\ &= 4\,\mathbb{P}(E_1) - 6\,\mathbb{P}(E_1E_2) + 4\,\mathbb{P}(E_1E_2E_3).\end{aligned}$$

We have $\mathbb{P}(E_1) = (3/4)^{10}$. Similarly

$$\mathbb{P}(E_1E_2) = \frac{1}{2^{10}} \text{ and } \mathbb{P}(E_1E_2E_3) = \frac{1}{4^{10}}.$$

Therefore, $\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = 4(\frac{3}{4})^{10} - \frac{6}{2^{10}} + \frac{4}{4^{10}}$. So the probability that all four seasons occur at least once is $1 - \left(4(\frac{3}{4})^{10} - \frac{6}{2^{10}} + \frac{4}{4^{10}}\right) = 0.781$.

9. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly
(a) all 5 problems?
(b) at least 4 of the problems?

Solution: (a) There are $\binom{10}{5}$ selections for the final exam. The number of selections that allow the student to solve all problems is $\binom{7}{5}$, so that the desired probability is $\frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333$.
(b) There are $\binom{7}{4} \cdot \binom{3}{1}$ selections that'll let the student solve exactly four problems, so that the probability of solving at least four problems is $\frac{\binom{7}{5}+\binom{7}{4}\cdot\binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$.

10. A closet contains 12 pairs of shoes. If 7 shoes are randomly selected without replacement, find the probability that there will be (a) at least one complete pair? (b) exactly 2 complete pairs? (c) exactly 2 complete pairs given that there is at least one complete pair.

Solution: Let A be the event that there are no complete pairs and B be the event that there is exactly one complete pair. Then
(a) $\mathbb{P}(A) = 1 - \frac{2^7\binom{12}{7}}{\binom{24}{7}}$.
(b) $\mathbb{P}(B) = \frac{2^3\cdot\binom{12}{2}\cdot\binom{10}{3}}{\binom{24}{7}}$.
(c) $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(BA)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{1-\mathbb{P}(A)} = \frac{2^3\cdot\binom{12}{2}\cdot\binom{10}{3}}{\binom{24}{7}-2^7\binom{12}{7}}$.