Homework 2

Math 461: Probability Theory, Spring 2021
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Due date: Feb 12, 2021

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. (a) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n = k.$$

(b) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

(c) How many vectors $(x_1, x_2, \dots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

2. Consider the set S of numbers $\{1, 2, \dots, n\}$. One can see that the number of subsets of S size k is $\binom{n}{k}$. Count the same number in a different way depending on how many subsets of size k have i as their highest numbered member, to give a proof of the following identity known as Fermat's combinatorial identity: For all integers $n \geqslant k$

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

3. (a) In how many ways can n identical balls be distributed into r bins such that each bin contains at least two balls. Assume that $n \geqslant 2r$.
(b) Do the same problem as in (a), but now each bin contains at least three balls and $n \geqslant 3r$.

4. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i-th position, $1 \leqslant i \leqslant b+g$, is a girl?

5. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

6. A die is rolled until either 3 or 5 appears. Find the probability that a 5 occurs first. Simplify the answer.
Hint: Let E_n denote the event that a 5 occurs on the n-th roll and no 3 or 5 occurs on the first $n-1$ rolls. Find $\mathbb{P}(E_n)$ and express the above probability in terms of them.

7. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?
Hint: Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*).

8. For a group of 10 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.
   Hint: Let E_i be the event that there are no birthdays in the i-th season.

9. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly
   (a) all 5 problems?
   (b) at least 4 of the problems?

10. A closet contains 12 pairs of shoes. If 7 shoes are randomly selected without replacement, find the probability that there will be (a) at least one complete pair? (b) exactly 2 complete pairs? (c) exactly 2 complete pairs given that there is at least one complete pair.