

# Lecture Note: Week 11

MATH 461: Probability Theory, Spring 2021  
Daesung Kim

## Lecture 26. Joint Distribution of Functions of RVs (Sec 6.7)

Suppose  $X_1$  and  $X_2$  are jointly continuous random variables with joint probability density  $f_X(x_1, x_2)$ . Let  $g = (g_1, g_2)$  and  $Y_1 = g_1(X_1, X_2)$ ,  $Y_2 = g_2(X_1, X_2)$  with

$$|J_g(x_1, x_2)| = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \left| \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \right| \neq 0$$

where

$$J_g(x_1, x_2) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix}.$$

Suppose that there exists a map  $h = (h_1, h_2)$  such that the equations  $y_1 = g_1(x_1, x_2)$  and  $y_2 = g_2(x_1, x_2)$  can be uniquely solved for  $x_1$  and  $x_2$  so that  $x_1 = h_1(y_1, y_2)$  and  $x_2 = h_2(y_1, y_2)$ . Here,  $J_g$  is called the Jacobian of  $g$ , and  $h$  is the inverse of  $g$ . Note that if  $g$  is continuously differentiable (that is,  $g_1$  and  $g_2$  are differentiable and their derivatives are continuous), then  $J_h = J_g^{-1}$  (due to the inverse function theorem). Let  $Y_1 = g_1(X_1, X_2)$  and  $Y_2 = g_2(X_1, X_2)$ , then the change of variable formula provides

$$\begin{aligned} \iint_B f_Y(y_1, y_2) dy_1 dy_2 &= \mathbb{P}((Y_1, Y_2) \in B) \\ &= \mathbb{P}((X_1, X_2) \in h(B)) \\ &= \iint_{h(B)} f_X(x_1, x_2) dx_1 dx_2 \\ &= \iint_B f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_h(y_1, y_2)| dy_1 dy_2 \\ &= \iint_B f_X(x_1, x_2) |J_g(x_1, x_2)|^{-1} dy_1 dy_2. \end{aligned}$$

Thus, the joint density of  $Y_1$  and  $Y_2$  is

$$\begin{aligned} f_Y(y_1, y_2) &= f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_h(y_1, y_2)| \\ &= f_X(h_1(y_1, y_2), h_2(y_1, y_2)) |J_g(h_1(y_1, y_2), h_2(y_1, y_2))|^{-1}. \end{aligned}$$

**Example 1.** Let  $X_1$  and  $X_2$  be jointly continuous random variables with probability density function  $f(X_1, X_2)$ . Let  $Y_1 = X_1 + X_2$ ,  $Y_2 = X_1 - X_2$ . Find the joint density function of  $Y_1$  and  $Y_2$  in terms of  $f(X_1, X_2)$ .

**Example 2.** Let  $(X, Y)$  denote a random point in the plane, and assume that the rectangular coordinates  $X$  and  $Y$  are independent standard normal random variables. What is the joint distribution of  $R$ ,  $\Theta$ , the polar coordinate representation of  $(X, Y)$ .

## Lecture 27. Expectation of functions of RVs (Sec 7.2, 3)

Let  $g(x, y)$  be a function on  $\mathbb{R}^2$ . If  $X, Y$  are discrete with joint pmf  $p(x, y)$ , then

$$\mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y)p(x, y).$$

If  $X, Y$  are jointly continuous with joint density  $f(x, y)$ , then

$$\mathbb{E}[g(X, Y)] = \iint g(x, y)f(x, y) dx dy.$$

**Example 3.** Let  $X$  and  $Y$  be independent uniform random variables on  $(0, 1)$ . Find  $\mathbb{E}[|X - Y|^k]$ .

### Expectations of sum and product

(i)  $\mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$

(ii) If  $X$  and  $Y$  are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

**Example 4** (A random walk in the plane). Consider a particle initially located at the origin in the plane  $\mathbb{R}^2$ . Suppose that the new position after each step is one unit of distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over  $(0, 2\pi)$ . Compute the expected square of the distance from the origin after  $n$  steps.

### The number of events that occur

For events  $A_1, A_2, \dots, A_n$ , we consider the corresponding indicator random variables  $X_i = I_{A_i}$ , defined by

$$I_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $\mathbb{E}[I_{A_i}] = \mathbb{P}(A_i)$  and  $\text{Var}(I_{A_i}) = \mathbb{P}(A_i)(1 - \mathbb{P}(A_i))$ . The number of the events that occur can be written as the sum of  $X_i$ . Let  $X = \sum_i X_i$ . Then,

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \mathbb{P}(A_i), \\ \mathbb{E}[X^2] &= \sum_{i,j=1}^n \mathbb{E}[X_i X_j] = \sum_{i=1}^n \mathbb{E}[X_i^2] + 2 \sum_{i < j}^n \mathbb{E}[X_i X_j] = \sum_{i=1}^n \mathbb{P}(A_i) + 2 \sum_{i < j}^n \mathbb{P}(A_i \cap A_j), \\ \text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \sum_{i=1}^n \mathbb{P}(A_i)(1 - \mathbb{P}(A_i)) + 2 \sum_{i < j}^n (\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)), \\ &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j}^n (\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)). \end{aligned}$$

**Example 5.** Let  $X$  be a binomial random variable with parameters  $n$  and  $p$ . Compute  $\mathbb{E}[X]$ ,  $\mathbb{E}[X^2]$  and  $\text{Var}(X)$ .

**Example 6.** A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman.

## References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN  
*E-mail address:* `daesungk@illinois.edu`