

Lecture Note: Week 9

MATH 461: Probability Theory, Spring 2021
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Lecture 22. Joint Distribution Functions (Sec 6.1)

In this section, we consider a collection of random variables X_1, X_2, \dots, X_n defined on a sample space S . In particular, we are interested in modeling relationships between them. For example, collections of random variables are involved in the following:

- (i) X_1 is the price of a product today, X_2 the price tomorrow, and so on.
- (ii) X_1 is rainfall in IL, X_2 rainfall in IN, and so on.
- (iii) Statistical topics such as time series, multivariate analysis, multiple linear regression, factor models.
- (iv) Probability topics such as Markov chains, stochastic processes.

First, we focus on two random variable X and Y on a sample space S . The probability of X and Y can be realized by their joint cumulative distribution function.

Definition: Joint cumulative distribution functions

For random variables X and Y , the joint cumulative distribution function $F(a, b)$ on \mathbb{R}^2 is defined by

$$F(a, b) = \mathbb{P}(X \leq a, Y \leq b).$$

The distributions of each X and Y can be obtained from the joint distribution. Indeed, we have

$$\begin{aligned} F_X(a) &= \mathbb{P}(X \leq a) = \lim_{b \rightarrow \infty} \mathbb{P}(X \leq a, Y \leq b) = \lim_{b \rightarrow \infty} F(a, b), \\ F_Y(b) &= \mathbb{P}(Y \leq b) = \lim_{a \rightarrow \infty} \mathbb{P}(X \leq a, Y \leq b) = \lim_{a \rightarrow \infty} F(a, b). \end{aligned}$$

Example 1. Describe $\mathbb{P}(X > a, Y > b)$ and $\mathbb{P}(a_1 < X \leq a_2, b_1 < Y \leq b_2)$ for $a, a_1, a_2, b, b_1, b_2 \in \mathbb{R}$ in terms of distribution functions.

Definition: Joint probability mass function

For discrete random variables X and Y , the joint probability mass function is defined by

$$p(x, y) = \mathbb{P}(X = x, Y = y).$$

The probability mass functions for each X and Y are

$$\begin{aligned} p_X(x) &= \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y p(x, y), \\ p_Y(y) &= \mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y) = \sum_x p(x, y). \end{aligned}$$

Example 2. Two fair dice are rolled. Find the joint probability mass function of X and Y when X is the largest value obtained on any die and Y is the sum of the values.

Definition: Joint continuity

We say that X and Y are jointly continuous if there exists a nonnegative function $f(x, y)$ on \mathbb{R}^2 such that

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) \, dx dy$$

for every C in \mathbb{R}^2 . The function $f(x, y)$ is called the joint probability density function of X and Y .

The probability mass functions for each X and Y are

$$p_X(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y p(x, y),$$

$$p_Y(y) = \mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y) = \sum_x p(x, y).$$

Remark 3. Suppose X and Y are jointly continuous.

(i) For any sets $A, B \subseteq \mathbb{R}$,

$$\mathbb{P}(X \in A, Y \in B) = \int_B \int_A f(x, y) \, dx dy.$$

(ii) The cumulative distribution function $F(a, b)$ is

$$F(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^b \int_{-\infty}^a f(x, y) \, dx dy.$$

Differentiating with respect to a and b , we have

$$\frac{\partial^2}{\partial a \partial b} F(a, b) = f(a, b).$$

(iii) The density of X can be obtained from the joint density $f(x, y)$. For $A \subseteq \mathbb{R}$, one can see that

$$\mathbb{P}(X \in A) = \mathbb{P}(X \in A, Y \in \mathbb{R}) = \int_{\mathbb{R}} \int_A f(x, y) \, dx dy.$$

Let $A = (-\infty, x)$ and differentiate in x , then

$$f_X(x) = \int_{\mathbb{R}} f(x, y) \, dy.$$

Similarly, we have

$$f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx.$$

We call f_X and f_Y the marginal densities of X and Y .

Example 4. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\mathbb{P}(X < Y)$.

Example 5. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} c(y^2 - x^2)e^{-y}, & -y \leq x \leq y, 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

- (i) Find c .
- (ii) Find the marginal densities of X and Y .
- (iii) Compute $\mathbb{E}[X]$.
- (iv) Compute $\mathbb{P}(0 < X < 1, Y < 1)$.

More than two random Variables

For random variables X_1, X_2, \dots, X_n , the joint cumulative distribution function is defined by

$$F(a_1, a_2, \dots, a_n) = \mathbb{P}(X_1 \leq a_1, X_2 \leq a_2, \dots, X_n \leq a_n).$$

The random variables X_1, X_2, \dots, X_n are jointly continuous if there exists a nonnegative function $f(x_1, x_2, \dots, x_n)$ such that

$$\mathbb{P}((X_1, X_2, \dots, X_n) \in C) = \iiint \cdots \int_C f(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n$$

for all $C \subseteq \mathbb{R}^n$.

Lecture 23. Independent Random Variables (Sec 6.2)

Definition

Two random variables X and Y are independent if for any sets A and B

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

It is equivalent to the following:

- (i) $F(a, b) = F_X(a)F_Y(b)$ for all $a, b \in \mathbb{R}$;
- (ii) (Discrete case) $p(x, y) = p_X(x)p_Y(y)$ for all $x, y \in \mathbb{R}$;
- (iii) (Discrete case) $p(x, y) = h(x)g(y)$ for all $x, y \in \mathbb{R}$, for some h and g ;
- (iv) (Jointly continuous case) $f(x, y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbb{R}$;
- (v) (Jointly continuous case) $f(x, y) = h(x)g(y)$ for all $x, y \in \mathbb{R}$, for some h and g .

Otherwise, we say that X and Y are dependent.

Example 6. If the joint density function of X and Y is

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise,} \end{cases}$$

are the random variables independent?

Example 7. If the joint density function of X and Y is

$$f(x, y) = \begin{cases} 24xy, & 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0, & \text{otherwise,} \end{cases}$$

are the random variables independent?

Example 8. Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter λ . Each person who enters the post office is a male with probability p and a female with probability $1 - p$. Show that the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1 - p)$.

Example 9. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$. Find the probability that the distance between the two points is greater than $L/3$.

Example 10. Buffon's needle problem A table is ruled with equidistant parallel lines a distance D apart. A needle of length L , where $L < D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)? Let us determine the position of the needle by specifying

- (i) the distance X from the middle point of the needle to the nearest parallel line and
- (ii) the angle θ between the needle and the projected line of length X .

Definition

Random variables X_1, X_2, \dots, X_n are independent if, for any sets A_1, A_2, \dots, A_n

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \mathbb{P}(X_2 \in A_2) \cdots \mathbb{P}(X_n \in A_n).$$

If the random variables are jointly continuous, it is equivalent to

$$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2) \cdots f(x_n)$$

for all x_1, x_2, \dots, x_n .

Example 11. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over $(0, 1)$, compute the probability that the largest of the three is greater than the sum of the other two.

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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