Homework 11

Math 461: Probability Theory, Spring 2021
Daesung Kim

Due date: May 5, 2021

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. The joint density of X and Y is given by
$$f(x,y) = \begin{cases} e^{-x/y-y}/y & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\mathbb{E}(X^2|Y=y)$.

2. The joint density of X and Y is given by
$$f(x,y) = \begin{cases} e^{-y}/y & \text{if } 0 < x < y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\mathbb{E}(X^3|Y=y)$.

3. The number of people who enter an elevator on the ground floor is a Geometric random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

4. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2.5. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

5. The moment generating function for X is given by $M_X(t) = \exp(2e^t - 2)$ and that of Y by $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$. If X, Y are independent what are (a) $\mathbb{P}(X+Y=2)$; (b) $\mathbb{P}(XY=0)$ and (c) $\mathbb{E}(XY)$.

6. The joint density of X and Y is given by
$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2} & \text{if } 0 < y < \infty, -\infty < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$
(a) Compute the joint moment generating function of X and Y.
(b) Compute the individual moment generating functions.

7. From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.
(a) Give an upper bound for the probability that a student's test score will exceed 85. Suppose, in addition,

that the professor knows that the variance of a student's test score is equal to 25.
(b) What can be said about the probability that a student will score between 65 and 85?
(c) How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem.

8. Let $X_1, X_2, \ldots, X_{20}$ be independent Poisson random variables with mean 1.
(a) Use the Markov inequality to obtain a bound on

$$\mathbb{P}\left(\sum_{i=1}^{20} X_i > 25\right).$$

(b) Use the central limit theorem to approximate

$$\mathbb{P}\left(\sum_{i=1}^{20} X_i > 25\right).$$

9. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$, approximate the probability that the resultant sum differs from the exact sum by more than 3.

10. A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.