Homework 7 Solution

Math 461: Probability Theory, Spring 2021
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Due date: Mar 26, 2021

1. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute (in terms of the Standard normal CDF $\Phi(\cdot)$)
(a) $\mathbb{P}(X > 5)$; (b) $\mathbb{P}(4 < X < 16)$; (c) $\mathbb{P}(X < 8)$; (d) $\mathbb{P}(X < 20)$; (e) $\mathbb{P}(X > 16)$.

Solution:

(a) $\mathbb{P}(X > 5) = \mathbb{P}\left(\frac{X-10}{6} > \frac{5-10}{6}\right) = 1 - -\frac{5}{6} = \frac{5}{6} = 0.7977$

(b) $\mathbb{P}(4 < X < 16) = \mathbb{P}\left(-1 < \frac{X-10}{6} < 1\right) = 1 - -1 = 21 - 1 = 0.6827$

(c) $\mathbb{P}(X < 8) = \mathbb{P}\left(\frac{X-10}{6} < -\frac{1}{3}\right) = -\frac{1}{3} = 1 - \frac{1}{3} = 0.3695$

(d) $\mathbb{P}(X < 20) = \mathbb{P}\left(\frac{X-10}{6} < -\frac{10}{6}\right) = \frac{5}{3} = 0.9522$

(e) $\mathbb{P}(X > 16) = \mathbb{P}\left(\frac{X-10}{6} > 1\right) = 1 - 1 = 0.1587$

2. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet, 2 inches tall? What percentage of men in the 6-footer club are over 6 feet, 5 inches?

Solution: Let X be a normal random variable with $\mu = 71$ and $\sigma^2 = 6.25$. Then $\mathbb{P}(X > 74) = \mathbb{P}\left(\frac{X-71}{2.5} > \frac{3}{2.5}\right) = 1 - \frac{6}{5} = 0.1151$.

Moreover, $\mathbb{P}(X > 77 \mid X \geqslant 72) = \frac{\mathbb{P}\left(\frac{X-71}{2.5} > \frac{6}{2.5}\right)}{\mathbb{P}\left(\frac{X-71}{2.5} \geqslant \frac{1}{2.5}\right)} = \frac{1-\frac{12}{5}}{1-\frac{2}{5}} = 0.024$.

3. The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = .900$ and $\sigma = .003$. The specification limits were given as $.900 \pm .005$.
(a) What percentage of forgings will be defective?
(b) What is the maximum allowable value of σ that will permit no more than 6 in 1000 defectives when the widths are normally distributed with $\mu = .9000$ and σ?

Solution: Let X be normal with $\mu = 0.9$ and $\sigma = 0.003$.

(a) $\mathbb{P}(|X - 0.9| > 0.005) = \mathbb{P}\left(\frac{|X-0.9|}{0.003} > \frac{5}{3}\right) = 2 - 2\frac{5}{3} = 0.095$.

(b) We want $\mathbb{P}\left(\frac{|X-0.9|}{\sigma} > 0.005\right) = 2 - 2\frac{0.005}{\sigma} \leqslant 0.006$, hence $\frac{0.005}{\sigma} \geqslant 0.997$, so that $\frac{0.005}{\sigma} \geqslant 3$, hence $\sigma = 0.0017$.

4. One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear strictly less than 150 times.

Solution: Let X be the number of times the number six appears.

$$\mathbb{P}(149.5 < X < 200.5) = \mathbb{P}\left(\frac{149.5 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{X - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{200.5 - \frac{5000}{36}}{\sqrt{\frac{5000}{36}}}\right)$$
$$\approx 2.87 + 1.46 - 1 = 0.9258.$$

Given that number 6 appears exactly 200 times out of 1000 rolls, Y = number of times 5 appears in the rest 800 rolls follows $\text{Bin}(800, 1/5)$. Thus

$$\mathbb{P}(Y < 149.5) = \mathbb{P}\left(\frac{Y - \frac{800}{5}}{\sqrt{\frac{3200}{25}}} < \frac{149.5 - \frac{800}{5}}{\sqrt{\frac{3200}{25}}}\right) \approx 1 - 0.92 = 0.1762.$$

5. Twelve percent of the population is left handed. Approximate the probability that there are at least 20 left-handers in a school of 200 students. State your assumptions.

Solution: Let X be the number of left handers. Then X is binomial with $p = 0.12$ and $n = 200$. Then

$$\mathbb{P}(X \geqslant 20) = \mathbb{P}(X > 19.5) = \mathbb{P}\left(\frac{X - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} > \frac{19.5 - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}}\right)$$
$$= 1 - -0.9792 = 0.9792 = 0.8363.$$

6. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is
   (a) the probability that a repair time exceeds 2 hours?
   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

Solution: Let X be exponential with parameter $\lambda = \frac{1}{2}$.

(a) $\mathbb{P}(X > 2) = 1 - F(2) = e^{-1}$

(b) $\mathbb{P}(X > 10 | X > 9) = \mathbb{P}(X > 1) = 1 - F(1) = e^{-\frac{1}{2}}$ because X is memoryless.

7. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with mean 20. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed, but rather is (in thousands of miles) uniformly distributed over $(0, 40)$.

Solution: Let X be an exponential random variable with parameter $\lambda = \frac{1}{20}$. Since X is memoryless, we have $\mathbb{P}(X > 30 \mid X > 10) = \mathbb{P}(X > 20) = e^{-1}$.

Let Y be a uniform random variable on $[0, 40]$. Then $\mathbb{P}(Y > 30 \mid Y > 10) = \frac{\mathbb{P}(Y>30)}{\mathbb{P}(Y>10)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.

8. If X is uniformly distributed over $(-1, 1)$, find (a) $\mathbb{P}(|X| > 1/2)$ and (b) the density function of the random variable $|X|$.

Solution: Let X be uniformly distributed over $(-1, 1)$.

(a) $\mathbb{P}\left(|X| > \frac{1}{2}\right) = \mathbb{P}\left(X > \frac{1}{2}\right) + \mathbb{P}\left(X < -\frac{1}{2}\right) = \frac{1}{2}$

(b) Let $Y = |X|$. If $y \in (0, 1)$, then $F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(-y \leqslant Y \leqslant y) = y$, so that

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

9. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

Solution: Let X be exponential with $\lambda = 1$, and let $Y = \log X$. Then $F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(\log X \leqslant y) = \mathbb{P}(X \leqslant e^y) = 1 - e^{-e^y}$, so that

$$f_Y(y) = e^y e^{-e^y} \text{ for } y \in \mathbb{R}.$$

10. If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

Solution: Let X be uniform on $(0, 1)$, and $Y = e^X$. Then, for $1 < y < e$, $F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}\left(e^X \leqslant y\right) = \mathbb{P}(X \leqslant \log Y) = \log Y$, so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$