Homework 4 Solution

Math 461: Probability Theory, Spring 2021
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Due date: Feb 26, 2021

1. A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1-p$. The changes on different days are assumed to be independent.
(a) What is the probability that after 2 days the stock will be at its original price?
(b) What is the probability that after 3 days the stock's price will have increased by 1 unit?
(c) Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

Solution: (a) $\mathbb{P}(\text{original price after two days}) = \binom{2}{1}p(1-p) = 2p(1-p)$.
(b) $\mathbb{P}(\text{increase by one after three days}) = \binom{3}{2}p^2(1-p) = 3p^2(1-p)$.
(c) $\mathbb{P}(\text{increase on first day} \mid \text{increase by one after three days}) = \frac{p\cdot 2p(1-p)}{3p^2(1-p)} = \frac{2}{3}$.

2. A and B play a series of games. Each game is independently won by A with probability p and by B with probability $1-p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series.
(a) Find the probability that a total of 4 games are played.
(b) Find the probability that A is the winner of the series.

Solution:

(a) $\mathbb{P}(\text{exactly four games are played}) = \mathbb{P}(ABAA) + \mathbb{P}(BAAA) + \mathbb{P}(ABBB) + \mathbb{P}(BABB) = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)\left(p^2+(1-p)^2\right) = 2p(1-p)(1-2p+2p^2)$.

(b) Let E be the event that A wins the match. Conditioning on the first two games of the match, we get

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \mid A, A)p^2 + \mathbb{P}(E \mid A, B)p(1-p) + \mathbb{P}(E \mid B, A)(1-p)p + \mathbb{P}(E \mid B, B)(1-p)^2 \\ &= p^2 + 2\,\mathbb{P}(E)p(1-p)\end{aligned}$$

because $\mathbb{P}(E \mid A, B) = \mathbb{P}(E \mid B, A) = \mathbb{P}(E)$. Hence, $\mathbb{P}(E) = \frac{p^2}{1-2p+2p^2}$.

3. An urn contains 12 balls, of which 4 are white. Three player—A, B, and C—successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the probability of winning for each player if
(a) each ball is replaced after it is drawn;
(b) the balls that are withdrawn are not replaced.

Solution:

(a)
$$\begin{aligned}
\mathbb{P}(A \text{ win}) &= \sum_{i=0}^{\infty} \mathbb{P}(\text{the first white appears on draw number } 3i+1) \\
&= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i} \frac{1}{3} = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^{i} = \frac{1}{3} \frac{1}{1-\frac{8}{27}} = \frac{9}{19} \approx 0.474. \\
\mathbb{P}(B \text{ win}) &= \sum_{i=0}^{\infty} \mathbb{P}(\text{the first white appears on draw number } 3i+2) \\
&= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+1} \frac{1}{3} = \frac{2}{9} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^{i} = \frac{2}{9} \frac{1}{1-\frac{8}{27}} = \frac{6}{19} \approx 0.316. \\
\mathbb{P}(C \text{ win}) &= \sum_{i=0}^{\infty} \mathbb{P}(\text{the first white appears on draw number } 3i+3) \\
&= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+2} \frac{1}{3} = \frac{4}{27} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^{i} = \frac{4}{27} \frac{1}{1-\frac{8}{27}} = \frac{4}{19} \approx 0.211.
\end{aligned}$$

(b)
$$\begin{aligned}
\mathbb{P}(A \text{ win}) &= \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} = \frac{7}{15} \approx 0.467 \\
\mathbb{P}(B \text{ win}) &= \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5} = \frac{53}{165} \approx 0.321 \\
\mathbb{P}(C \text{ win}) &= \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5} = \frac{7}{33} \approx 0.212.
\end{aligned}$$

4. Find an example that three events A, B, C are pairwise independent (that is, A and B are independent, B and C are independent, and C and A are independent) but not independent.

Solution: Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Then, $A = \{HH, HT\}$, $B = \{HH, TH\}$, and $C = \{HH, TT\}$. So,

$$\begin{aligned}
\mathbb{P}(AB) &= \frac{1}{4} = \mathbb{P}(A)\,\mathbb{P}(B), \\
\mathbb{P}(BC) &= \frac{1}{4} = \mathbb{P}(B)\,\mathbb{P}(C), \\
\mathbb{P}(CA) &= \frac{1}{4} = \mathbb{P}(C)\,\mathbb{P}(A), \\
\mathbb{P}(ABC) &= \frac{1}{4} \neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).
\end{aligned}$$

5. Two balls are chosen randomly from an urn containing 8 white and 4 black. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

Solution: Possible values of X: $4, 1, -2$ with probabilities:

$$\mathbb{P}(X=4) = \frac{\binom{4}{2}}{\binom{12}{2}} = \frac{1}{11}, \qquad \mathbb{P}(X=1) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{12}{2}} = \frac{16}{33}, \qquad \mathbb{P}(X=-2) = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{14}{33}.$$

6. Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X be rank of the best scoring woman (ranks are w.r.t. decreasing values of score, so that rank 1 has the best score, rank 2 has the second best score and so on). Find the pmf of X.

Solution:

$$\begin{aligned}
&\mathbb{P}(X=1)=\frac{\binom{5}{1}9!}{10!}=\frac{1}{2}, && \mathbb{P}(X=2)=\frac{\binom{5}{1}\binom{5}{1}8!}{10!}=\frac{5}{18}\\
&\mathbb{P}(X=3)=\frac{\binom{5}{2}2!\binom{5}{1}7!}{10!}=\frac{5}{36}, && \mathbb{P}(X=4)=\frac{\binom{5}{3}3!\binom{5}{1}6!}{10!}=\frac{5}{84}\\
&\mathbb{P}(X=5)=\frac{\binom{5}{4}4!\binom{5}{1}5!}{10!}=\frac{5}{252}, && \mathbb{P}(X=6)=\frac{\binom{5}{5}5!\binom{5}{1}4!}{10!}=\frac{1}{252}\\
&\mathbb{P}(X=7)=\mathbb{P}(X=8)=\mathbb{P}(X=9)=\mathbb{P}(X=10)=0.
\end{aligned}$$

7. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .4, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the possible values and probabilities of X, the total dollar value of all sales.

Solution: Let X be the total dollar value of all sales. Then X can take the values $0, 500, 1000, 1500, 2000$.

Let U be the dollar value from the first sale and V be the dollar value from the second sale. So that U takes values $0, 500, 1000$ with probabilities $0.6, 0.2, 0.2$, respectively and V takes values $0, 500, 1000$ with probabilities $0.4, 0.3, 0.3$, respectively. Note that the two sales are independent. Thus we have

$$\begin{aligned}
\mathbb{P}(X=0)&=\mathbb{P}(U=0,V=0)=0.6\cdot 0.4=0.24\\
\mathbb{P}(X=500)&=\mathbb{P}(U=0,V=500)+\mathbb{P}(U=500,V=0)=0.6\cdot 0.3+0.2\cdot 0.4=0.26\\
\mathbb{P}(X=1000)&=\mathbb{P}(U=500,V=500)+\mathbb{P}(U=0,V=1000)+\mathbb{P}(U=1000,V=0)\\
&=0.2\cdot 0.3+0.6\cdot 0.3+0.2\cdot 0.4=0.32\\
\mathbb{P}(X=1500)&=\mathbb{P}(U=500,V=1000)+\mathbb{P}(U=1000,V=500)=0.2\cdot 0.3+0.2\cdot 0.3=0.12\\
\mathbb{P}(X=2000)&=\mathbb{P}(U=1000,V=1000)=0.2\cdot 0.3=0.06.
\end{aligned}$$

8. Five distinct numbers are randomly distributed to five players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $\mathbb{P}(X=i), i=0,1,2,3,4$.

Solution:

$$\begin{aligned}
\mathbb{P}(X=0)&=\frac{0!}{2!}=\frac{1}{2}\\
\mathbb{P}(X=1)&=\frac{1!}{3!}=\frac{1}{6}\\
\mathbb{P}(X=2)&=\frac{2!}{4!}=\frac{1}{12}\\
\mathbb{P}(X=3)&=\frac{3!}{5!}=\frac{1}{20}\\
\mathbb{P}(X=4)&=\frac{4!}{5!}=\frac{1}{5}
\end{aligned}$$

9. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.
(a) Which of $\mathbb{E}(X)$ or $\mathbb{E}(Y)$ do you think is larger? Why?
(b) Find the pmf of X and Y.
(c) Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

Solution: (a) $\mathbb{E}(X)$ is larger than $\mathbb{E}(Y)$ because the random selection of students favors larger busloads.
(b) X, Y take values $40, 33, 25, 50$. $\mathbb{P}(X = i) = i/(40 + 33 + 25 + 50)$ and $\mathbb{P}(Y = i) = 1/4$ for $i = 40, 33, 25, 50$.
(c) $\mathbb{E}(X) = \frac{40 \cdot 40 + 33 \cdot 33 + 25 \cdot 25 + 50 \cdot 50}{40 + 33 + 25 + 50} = \frac{5814}{148} = 39.3$, $\mathbb{E}(Y) = \frac{148}{4} = 37$.

10. On a multiple-choice exam with 3 possible answers (that is, one answer among 3 choices) for each of the 5 questions, what is the probability that a student will get 4 or more correct answers just by guessing?

Solution: Let X be the number of correct answers. Then

$$\mathbb{P}(X \geqslant 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) = \binom{5}{4} \frac{1}{3^4} \cdot \frac{2}{3} + \frac{1}{3^5} = \frac{11}{243}.$$