Homework 10

Math 461: Probability Theory, Spring 2021
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Due date: Apr 19, 2021

Instruction

1. Each problem is worth 10 points and only five randomly chosen problems will be graded.
2. Convert a photocopy of your solutions to **one single pdf file** and upload it on Moodle.
3. Please indicate whom you worked with, it will not affect your grade in any way.

1. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over $(0,1)$, compute the probability that the largest of the three is greater than the sum of the other two.

2. Let X_1, X_2, X_3, X_4 be independent uniform random variables on $(0,1)$ and $X_{(i)}$ be the i-th smallest random variable between X_1, X_2, X_3, X_4, for $i = 1,2,3,4$. (That is, $X_{(1)} \leqslant X_{(2)} \leqslant X_{(3)} \leqslant X_{(4)}$.) Define $X = X_{(2)}$ and $Y = 1 - X_{(3)}$.
   (a) Find the joint distribution of X and Y.
   (b) Find the conditional density of Y given $X = x$ for $x \in (0,1)$.

3. Let X and Y have joint density function
$$f(x,y) = \frac{1}{x^2y^2}, \; x \geqslant 1, y \geqslant 1.$$
   (a) Compute the joint density function of $U = XY, V = X/Y$.
   (b) What are the marginal densities?

4. If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}[(X-Y)^2]$.

5. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 4$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\mathrm{Var}(4+2X)$.

6. The random variables X and Y have a joint density function given by
$$f(x,y) = \begin{cases} 2e^{-2x}/x & \text{if } 0 < x < \infty, 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$
   Compute $\mathrm{Cov}(X,Y)$.

7. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \dots, i$. Find
   (a) the expected number of urns that are empty;
   (b) the probability that none of the urns is empty.

8. Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers. Find the expected number of changeovers.

9. A group of 20 people consisting of 10 married couples is randomly arranged into 10 pairs of 2 each. Compute the mean and variance of the number of married couples that are paired together.

10. Let $X_1, X_2, \dots$ be independent random variables with common mean μ and common variance σ^2. Set $Y_n = X_n + X_{n+1} + X_{n+2}$, $n \geqslant 1$. For $j \geqslant 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$.