Homework 6 Solution

Math 461: Probability Theory, Spring 2021
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Due date: Mar 19, 2021

1. An urn contains 4 white and 5 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

> **Solution**: Let E be the event that a single drawing results in two white and two black balls. Then $\mathbb{P}(E) = \frac{\binom{4}{2}\binom{5}{2}}{\binom{9}{4}} = \frac{10}{21}$. Let X be the number of selections until E occurs. Thus X is a Geometric random variable with parameter $p = 10/21$. Then
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> $$\mathbb{P}(X = n) = \frac{11^{n-1} \cdot 10}{21^n}.$$

2. Let σ be a random ordering of the sequence $1, 2, \ldots, n$ (for example $2, 4, 5, 1, 3, \ldots$) where each of the $n!$ ordering has equal probability. Given an ordering we say that the i-th position is a *local maxima* if the value at the i-th position is bigger than the neighboring value/values. For example, if the ordering is $(3, 2, 4, 1, 5)$ the 1st, 3rd and 5th positions are local maxima and thee are 3 local maxima. Find the expected total number of local maxima in σ.

> **Solution**: Let X_i be the indicator random variable whether the i-th location in σ is a local maxima for $i = 1, 2, \ldots, n$. Total number of local maxima is
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> $$X = X_1 + X_2 + \cdots + X_n.$$
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> Clearly
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> $$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_n = 1) = \frac{1}{2}.$$
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> However,
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> $$\mathbb{P}(X_2 = 1) = \frac{1}{3}$$
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> as any one of the 1st, 2nd or 3rd location can have the largest value. Thus
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> $$\mathbb{E}\, X = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{3} + \frac{1}{2} = 1 + \frac{n-2}{3} = \frac{n+1}{3}.$$

3. Let X be a negative binomial random variable with parameters r and p and let Y be a binomial random variable with parameters n and p. Show that

$$\mathbb{P}(X > n) = \mathbb{P}(Y < r).$$

Solution:

(a) **Probabilistic proof:** Consider a sequence of independent Bernoulli trials with success probability p. Then, Y is the number of success in the first n trials and X is the number of trials until r successes are obtained. If $X > n$, then it means that the number of success in the first n trials is less than r, that is, $Y < r$. Thus, we have $\{X > n\} = \{Y < r\}$.

(b) **Combinatoric proof:** Since

$$\mathbb{P}(X > n) = 1 - \mathbb{P}(X \leqslant n) = 1 - \sum_{k=r}^{n} \binom{k-1}{r-1} p^r (1-p)^{k-r} = 1 - \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \mathbb{P}(Y < r),$$

it suffices to show that

$$\sum_{k=r}^{n} \binom{k-1}{r-1} p^r (1-p)^{k-r} = \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

for all $r \leqslant n$. We fix $r > 0$ and prove the claim by induction on $n = r, r+1, \cdots$. If $n = r$, then

$$\sum_{k=r}^{r} \binom{k-1}{r-1} p^r (1-p)^{k-r} = p^r = \sum_{k=r}^{r} \binom{r}{k} p^k (1-p)^{r-k}.$$

Suppose (1) holds for $n \geqslant r$. Then,

$$\begin{aligned}
\sum_{k=n+1}^{r} \binom{k-1}{r-1} p^r (1-p)^{k-r} &= \binom{n}{r-1} p^r (1-p)^{n+1-r} + \sum_{k=n+1}^{r} \binom{k-1}{r-1} p^r (1-p)^{k-r} \\
&= \binom{n}{r-1} p^r (1-p)^{n+1-r} + \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k}.
\end{aligned}$$

Using Pascal's identity ($\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$),

$$\begin{aligned}
\sum_{k=r}^{n+1} \binom{n+1}{k} p^k (1-p)^{n+1-k} &= \sum_{k=r}^{n+1} \binom{n}{k-1} p^k (1-p)^{n+1-k} + \sum_{k=r}^{n+1} \binom{n}{k} p^k (1-p)^{n+1-k} \\
&= \binom{n}{r-1} p^r (1-p)^{n+1-r} + \sum_{k=r}^{n} \binom{n}{k} p^{k+1} (1-p)^{n-k} + \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n+1-k} \\
&= \binom{n}{r-1} p^r (1-p)^{n+1-r} + \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k}.
\end{aligned}$$

Thus, (1) holds for $n+1$.

4. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of c?
(b) What is the cumulative distribution function of X?

Solution: (a) We have $1 = \int_{-1}^{1} c(1-x^2)dx = cx\left(1 - \frac{x^2}{3}\right)|_{-1}^{1} = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^{x} f(y)dy = \frac{3}{4}y\left(1-\frac{y^2}{3}\right)|_{-1}^{x} = \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right)$ if $-1 \leqslant x \leqslant 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right) & -1 \leqslant x < 1, \\ 1 & x \geqslant 1. \end{cases}$$

5. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x \geqslant 0 \\ 0 & x \leqslant 0. \end{cases}$$

What is the probability that the system functions for at least 5 months?

Solution: Determine C: $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4$, so that $C = \frac{1}{4}$. Now, we have $\mathbb{P}(X \geqslant 5) = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}\Big|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$.

6. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leqslant 10. \end{cases}$$

(a) Find $\mathbb{P}(X > 20)$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

Solution:

(a) $\mathbb{P}(X > 20) = \int_{20}^\infty \frac{10}{x^2}dx = -\frac{10}{x}\Big|_{20}^\infty = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1-\frac{10}{x} & x \geqslant 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

7. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

> **Solution**: We want to find C such that $F(C) \geqslant 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5|_0^C = 1-(1-C)^5$. We want $1-(1-C)^5 \geqslant 0.99$, i.e., $(1-C)^5 \leqslant 0.01$, hence $C \geqslant 1-(0.01)^{0.2}$.

8. Compute $\mathbb{E}\, X$ if X has a density function given by

(a)
$$f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b)
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c)
$$f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & \text{otherwise.} \end{cases}$$

> **Solution**:
>
> (a) $\mathbb{E}\, X = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{4}\int_0^\infty x^2 e^{-\frac{x}{2}} dx = \frac{1}{4}\left(-2x^2 - 8x - 16\right) e^{-\frac{x}{2}}|_0^\infty = 4.$
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> (b) $\mathbb{E}\, X = \int_{-1}^{1} c(1-x^2) x dx = 0$ by symmetry.
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> (c) $\mathbb{E}\, X = \int_5^\infty x \frac{5}{x^2} dx = \int_5^\infty \frac{5}{x} dx = \infty$.

9. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.
(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?
(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

> **Solution**: (a) Let X be uniform on $[0, 60]$ where X is the time in minutes after 7am when the passenger arrives at the station. Then
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> $$\begin{aligned} \mathbb{P}(\text{passenger goes to } A) &= \mathbb{P}(5 \leqslant X < 15) + \mathbb{P}(20 \leqslant X < 30) + \mathbb{P}(35 \leqslant X < 45) + \mathbb{P}(50 \leqslant X < 60) \\ &= \frac{2}{3}. \end{aligned}$$
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> (b) Same as above.

10. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
(a) What is the probability that you will have to wait longer than 10 minutes?
(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

> **Solution**: (a) $\mathbb{P}(X > 10) = \frac{2}{3}$
> (b) $\mathbb{P}(X > 25 \mid X > 15) = \frac{\mathbb{P}(X>25)}{\mathbb{P}(X>15)} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.