Lecture Note: Week 10

MATH 461: Probability Theory, Spring 2021
Daesung Kim

Lecture 24. Sums of Independent Random Variables (Sec 6.3)

In this section, we consider the sum of two independent random variables X and Y. If X and Y are jointly continuous and independent, then the joint density is $f(x,y) = f_X(x)f_Y(y)$ where f_X and f_Y are the densities for X and Y respectively. Then, the cdf of $X+Y$ is

$$F_{X+Y}(t) = \mathbb{P}(X+Y \leq t) = \iint_{x+y\leq t} f_X(x)f_Y(y)\,dxdy = \int_{\mathbb{R}} F_X(t-y)f_Y(y)\,dy.$$

The cdf of $X+Y$ is called the convolution of F_X and F_Y. Taking derivative with respect to t, we get

$$f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t-y)f_Y(y)\,dy.$$

Example 1. If X and Y are independent uniform random variables on $(0,1)$, find the density of $X+Y$.

Sum of independent random variables
Suppose X and Y are independent. Let $Z = X+Y$.

(i) If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then $Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

(ii) If $X \sim \Gamma(s,\lambda)$ and $Y \sim \Gamma(t,\lambda)$, then $Z \sim \Gamma(s+t,\lambda)$.

(iii) If $X \sim \mathrm{Bin}(n,p)$ and $Y \sim \mathrm{Bin}(m,p)$, then $Z \sim \mathrm{Bin}(n+m,p)$.

(iv) If $X \sim \mathrm{Poisson}(\lambda_1)$ and $Y \sim \mathrm{Poisson}(\lambda_2)$, then $Z \sim \mathrm{Poisson}(\lambda_1+\lambda_2)$.

(v) If $X \sim \mathrm{NegBin}(r,p)$ and $Y \sim \mathrm{NegBin}(s,p)$, then $Z \sim \mathrm{NegBin}(r+s,p)$.

Example 2. If $X \sim N(0, \frac{1}{2})$ and $Y \sim N(0, \frac{1}{2})$ are independent, then what is $f_{X+Y}(t)$?

Example 3. If $X \sim \mathrm{Poisson}(\lambda_1)$ and $Y \sim \mathrm{Poisson}(\lambda_2)$, then what is $\mathbb{P}(X+Y=n)$?

Further examples

Example 4. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that the total gross sales over the next 2 weeks exceeds \$5000?

Example 5. Let $X \sim U(0,1)$ and $Y \sim \mathrm{Exp}(1)$ be independent. Find the distribution of $Z = X+Y$.

Lecture 25. Conditional Distribution (Sec 6.4-6)

Suppose X and Y are discrete with the joint pmf $p(x,y)$, that is $\mathbb{P}(X=x, Y=y) = p(x,y)$. Let y satisfy $p_Y(y) = \sum_x p(x,y) > 0$. The conditional pmf of X given $Y=y$ is defined by

$$p_{X|Y}(x|y) = \mathbb{P}(X=x|Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \frac{p(x,y)}{p_Y(y)}.$$

Note that if X and Y are independent, then $p_{X|Y}(x|y) = p_X(x)$. The conditional cdf of X given $Y=y$ is

$$F_{X|Y}(t|y) = \mathbb{P}(X \le t|Y=y) = \sum_{x \le t} p_{X|Y}(x|y).$$

Example 6. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X+Y=n$.

Suppose X and Y are jointly continuous with joint density $f(x,y)$. For y with $f_Y(y) > 0$, the conditional density of X given $Y=y$ is defined by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

If X and Y are independent, then $f_{X|Y}(x|y) = f_X(x)$. Then, the conditional probability and the conditional cdf of X given $Y=y$ can be written as

$$\mathbb{P}(X \in A|Y=y) = \int_A f_{X|Y}(x|y)\,dx$$
$$F_{X|Y}(t|y) = \mathbb{P}(X \le t|Y=y) = \int_{-\infty}^{t} f_{X|Y}(x|y)\,dx.$$

Example 7. Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}} e^{-y}}{y}, & 0 < x, y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{X|Y}(x|y)$ and $\mathbb{P}(X > 1|Y=y)$.

Bivariate normal random variable

Jointly continuous random variables X and Y are bivariate normal if their density is given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)}$$

where $\sigma_X, \sigma_Y > 0$, $\rho \in (-1,1)$, and $\mu_X, \mu_Y \in \mathbb{R}$. We denote by

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right).$$

Proposition 8. (i) *$X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. In particular, $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$, $\mathrm{Var}(X) = \sigma_X^2$, and $\mathrm{Var}(Y) = \sigma_Y^2$.*

(ii) *The random variable X given $Y=y$ is normal with mean $\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$ and variance $\sigma_X^2(1-\rho^2)$.*

Proof. Let $\overline{x} = \frac{x-\mu_X}{\sigma_X}$ and $\overline{y} = \frac{y-\mu_Y}{\sigma_Y}$, then

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\overline{x}^2 + \overline{y}^2 - 2\rho\overline{x}\overline{y}\right)} \\ &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(\overline{x}-\rho\overline{y})^2} e^{-\frac{1}{2}\overline{y}^2}. \end{aligned}$$

Since

$$\int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}(\overline{x}-\rho\overline{y})^2}\,dx = \int_{\mathbb{R}} e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\left(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)\right)^2}\,dx$$
$$= \int_{\mathbb{R}} e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}x^2}\,dx$$
$$= \sqrt{2\pi\sigma_X^2(1-\rho^2)},$$

we have

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{1}{2}\overline{y}^2} = \frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

and so $Y \sim N(\mu_Y, \sigma_Y^2)$. The same argument for X yields $X \sim N(\mu_X, \sigma_X^2)$. A direct computation leads to

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi\sigma_X^2(1-\rho^2)}}e^{-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\left(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)\right)^2}$$

as desired. $\square$

Remark 9. The parameter ρ represents how X and Y correlated.

Joint distribution of maximum and minimum

Let $X_1, X_2, \cdots, X_n$ be independent jointly continuous random variables with the common cdf $F(t)$. Let $U = \max\{X_1, X_2, \cdots, X_n\}$ and $V = \min\{X_1, X_2, \cdots, X_n\}$.

Proposition 10. *The joint density of U and V is*

$$f_{U,V}(u,v) = n(n-1)(F(u)-F(v))^{n-2}f(u)f(v).$$

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu