Lecture Note: Week 13

MATH 461: Probability Theory, Spring 2021
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Lecture 30. Moment Generating Functions (Sec 7.7)

Definition
A moment generating function (in short, mgf) of a random variable X is defined as

$$\begin{aligned} M_X(t) = M(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_x e^{tx} p(x) \qquad \text{(discrete)} \\ &= \int e^{tx} f(x)\, dx \qquad \text{(continuous).} \end{aligned}$$

Example 1. Let $X \sim \text{Bin}(n, p)$. Find the mgf of X.

Example 2. Let $X \sim \text{Poisson}(\lambda)$. Find the mgf of X.

Example 3. Let $X \sim N(0, 1)$. Find the mgf of X.

Suppose $M_X(t)$ is well-defined (in a sense that $M_X(t) < \infty$ for t around the origin). Then, one can see that $M'(0) = \mathbb{E}[X]$ and $M''(0) = \mathbb{E}[X^2]$. In general, we have $M^{(n)}(0) = \mathbb{E}[X^n]$ for all $n \geq 1$. This is why $M(t)$ is called a moment generating function.

Proposition 4. (i) *If $M_X(t)$ is well-defined near 0 (that is, $M_X(t)$ exists and is finite for $x \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$), then $M_X(t)$ uniquely determines the distribution of X.*

(ii) *Random variables X and Y are independent if and only if*

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

Example 5. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. What is the distribution of $X+Y$?

Example 6. Let $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$ be independent. What is the distribution of $X + Y$?

Example 7. Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ be independent. What is the joint distribution of $X + Y$ and $X - Y$?

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

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