Probability of Winning

7 distinct numbers are randomly distributed to players numbered 1 through 7. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on.

Let X denote the number of times player 2 is a winner. Find P$(X = 5)$.

Answer = number (3 significant figures)

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Solution

Player 2 wins 5 times = Player 2's number is 6.
and
Player 7's number is 7.

Player 1, 3, 4, 5, 6 can have any number among $\{1, 2, \cdots, 5\}$

$\Rightarrow$ $5!$ number of ways

$\therefore$ $P(X=5) = \frac{5!}{7!} = \frac{1}{42}$.

Compute Mean

Let X be a random variable taking only the values $0, 1, 2, 3, 4$ with probabilities given by $P(X=0)=5c, P(X=1)=4c, P(X=2)=4c, P(X=3)=3c, P(X=4)=5c$ for some constant $c>0$. Find E(X).

Answer = number (3 significant figures)

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Solution

$$\sum_{i=0}^{4} P(X=i) = c\cdot(5+4+4+3+5)$$
$$= 21c$$
$$= 1 \qquad \text{i.e.}\quad c=\frac{1}{21}.$$

$$\mathbb{E}X = \frac{1}{21}\cdot(0\cdot 5+1\cdot 4+2\cdot 4+3\cdot 3+4\cdot 5)$$
$$= \frac{41}{21}$$

Expectation of a Polynomial from mean and variance

Let X be a random variable with E$(X) = 4$ and Var$(X) = 2$. Find E$((4X+5)^2)$.

Answer = number (3 significant figures)

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$$\mathrm{Var}(X) = 2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}X^2 - 16$$

i.e. $\mathbb{E}X^2 = 18$

$$\mathbb{E}(4X+5)^2 = \mathbb{E}\left[16X^2 + 40X + 25\right]$$

$$= 16 \cdot \mathbb{E}X^2 + 40\,\mathbb{E}X + 25$$

$$= 16 \cdot 18 + 40 \cdot 4 + 25$$

$$= 473$$

Expectation of a function

Let X be a Poisson distributed random variable with mean 1.75. Find $\mathbb{E}(5^X)$.

Answer = number (3 significant figures)

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Solution

$$\mathbb{E}[5^X] = \sum_{k=0}^{\infty} 5^k \cdot e^{-\lambda} \frac{\lambda^k}{k!} \quad (\lambda = 1.75)$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{(5\lambda)^k}{k!}$$

$$= e^{-4\lambda} \underbrace{\sum_{k=0}^{\infty} e^{-5\lambda} \frac{(5\lambda)^k}{k!}}_{=1}$$

$$= e^{-7}$$