Homework 9 Solution

Math 461: Probability Theory, Spring 2021
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1. An ambulance travels back and forth at a constant speed along a road of length L. At a certain moment of time, an accident occurs at a point uniformly distributed on the road. (That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.) Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the probability density function and mean of the distance of the ambulance from the accident.

Solution: Let X, Y be uniform random variables on $(0, L)$. Let $Z = |Y - X|$. We want to find $\mathbb{E}\, Z$. First, find $F_Z(a)$, for $a \geqslant 0$. We have $F_Z(a) = \mathbb{P}\left(Z \leqslant a\right) = \mathbb{P}\left(|Y - X| \leqslant a\right) = \mathbb{P}\left(-a \leqslant Y - X \leqslant a\right) = 1 - \frac{(L-a)^2}{L^2} = \frac{2aL - a^2}{L^2}$ using geometric considerations. Hence, $f_Z(x) = \frac{2L-2x}{L^2}$ if $0 \leqslant a \leqslant L$. Hence,

$$\mathbb{E}\, Z = \int_0^L x \cdot \frac{2L - 2x}{L^2} dx = \frac{2}{L^2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right)\Big|_0^L = \frac{L}{3}.$$

2. Let X be uniform on $(0, 1)$, and let Y be exponential with $\lambda = 1$ and let X, Y be independent.
(a) Find the pdf of $U = X + Y$.
(b) Also find the pdf of $V = X/Y$.

Solution: Let X be uniform on $(0, 1)$, and let Y be exponential with $\lambda = 1$.

(a) If $U = X + Y$, then

$$f_U(a) = \int_{-\infty}^{\infty} f_Y(a - x) f_X(x) dx = \int_0^1 f_Y(a - x) dx$$
$$= \begin{cases} 0 & a < 0 \\ \int_0^a e^{x-a} dx & 0 < a < 1 \\ \int_0^1 e^{x-a} dx & 1 < a \end{cases} = \begin{cases} 0 & a < 0 \\ 1 - e^{-a} & 0 < a < 1 \\ e^{-a}(e - 1) & 1 < a. \end{cases}$$

(b) If $V = X/Y$, we first find the cumulative distribution function of V. Note that $F_Z(a) = 0$ if $a \leqslant 0$. Assume that $a > 0$.

$$F_V(a) = \mathbb{P}\left(V \leqslant a\right) = \mathbb{P}\left(X \leqslant aY\right) = \int_0^1 \int_{\frac{x}{a}}^{\infty} f_Y(y) dy dx = a\left(1 - e^{-\frac{1}{a}}\right).$$

Now, we have

$$f_V(a) = \frac{d}{da} F_V(a) = \begin{cases} 0 & a \leqslant 0 \\ 1 - e^{-\frac{1}{a}}\left(1 + \frac{1}{a}\right) & a > 0. \end{cases}$$

3. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that

(a) the total gross sales over the next 2 weeks exceeds \$5000;
(b) weekly sales exceed \$2000 in at least 2 of the next 3 weeks?

Solution: Let X_1, X_2 be independent normal random variables with $\mu = 2200$ and $\sigma^2 = 230^2$, representing the gross sales over this week and next week, respectively. Then $X = X_1 + X_2$ is normal with mean 4400 and variance $2 \cdot 230^2 = 105800$.

(a) $\mathbb{P}(X > 5000) = \mathbb{P}\left(\frac{X-4400}{\sqrt{105800}} > \frac{600}{\sqrt{105800}}\right) = 1 - 1.84 = 1 - 0.9671 = 0.0329.$

(b) Let $p = \mathbb{P}(X_1 > 2000) = \mathbb{P}\left(\frac{X_1-2200}{230} > \frac{-200}{230}\right) = 1 - -\frac{20}{23} = 0.87 = 0.8078.$
Let N be the number of weeks (out of three) in which the sales exceed \$2000. Then N is binomial with parameters $(p, 3)$, so that $\mathbb{P}(N \geqslant 2) = p^3 + 3p^2(1-p) = 0.9034.$

4. According to the U.S. National Center for Health Statistics, 25.2 percent of males and 23.6 percent of females never eat breakfast. Suppose that random samples of 200 men and 200 women are chosen. Approximate the probability that
(a) at least 110 of these 400 people never eat breakfast;
(b) the number of the women who never eat breakfast is at least as large as the number of the men who never eat breakfast.

Solution: Let X be the number of women who never eat breakfast, and let Y be the number of men who never eat breakfast. Let $Z = X + Y$. By normal approximation to binomial, X is approximated by a normal random variable with mean $200 \cdot 0.236 = 47.2$ and variance $47.2 \cdot 0.764 = 36.061$, and Y is normal with mean $200 \cdot 0.252 = 50.4$ and variance $50.4 \cdot 0.748 = 37.699$.

Let $Z_1 = X+Y$ and $Z_2 = X-Y$. Then Z_1 is normal with mean 97.6 and variance $36.061+37.699 = 73.76$, and Z_2 is normal with mean -3.2 and variance 73.76.

(a) $\mathbb{P}(Z_1 \geqslant 110) = \mathbb{P}(Z_1 > 109.5) = \mathbb{P}\left(\frac{Z_1-97.6}{\sqrt{73.76}} > \frac{11.9}{\sqrt{73.76}}\right) = 1 - 1.39 = 1 - 0.9177 = 0.0823.$

(b) $\mathbb{P}(X \geqslant Y) = \mathbb{P}(X - Y \geqslant 0) = \mathbb{P}(Z_2 \geqslant 0) = \mathbb{P}(Z_2 > -0.5) = \mathbb{P}\left(\frac{Z_2+3.2}{\sqrt{73.76}} > \frac{2.7}{\sqrt{73.76}}\right) = 1 - 0.31 = 0.3783.$

5. The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be
(a) more than 2 such accidents in the next month?
(b) more than 4 such accidents in the next 2 months?
(c) more than 5 such accidents in the next 3 months?
Explain your reasoning!

Solution: Let X_1 be the number of accidents in the next month, X_2 the number of accidents in the month after that, and X_3 the number of accidents in the third month. It makes sense to think of X_1, X_2, and X_3 as independent Poisson random variables with parameter $\lambda = 2.2$.

Let $X = X_1$, $Y = X_1 + X_2$, and $Z = X_1 + X_2 + X_3$. Then X, Y, and Z are Poisson with parameter $2.2, 4.4$, and 6.6, respectively.

(a) $\mathbb{P}(X > 2) = 1 - e^{-2.2}\left(1 + 2.2 + \frac{2.2^2}{2}\right) = 0.3773.$

(b) $\mathbb{P}(Y > 4) = 1 - e^{-4.4}\left(1 + 4.4 + \frac{4.4^2}{2} + \frac{4.4^3}{3!} + \frac{4.4^4}{4!}\right) = 0.4488.$

(c) $\mathbb{P}(Z > 5) = 1 - e^{-6.6}\left(1 + 6.6 + \frac{6.6^2}{2} + \frac{6.6^3}{3!} + \frac{6.6^4}{4!} + \frac{6.6^5}{5!}\right) = 0.6453.$

6. Choose a number X at random from the set of numbers $\{1,2,3,4,5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1,\ldots,X\}$. Call this second number Y.
(a) Find the joint mass function of X and Y.
(b) Find the conditional mass function of X given that $Y=i$. Do it for $i=1,2,3,4,5$.
(c) Are X and Y independent? Why?

Solution:

(a) $\mathbb{P}(X=i,Y=j)=\frac{1}{5i}$ for $i=1,\ldots,5$ and $j=1,\ldots,i$, 0 otherwise.

$\mathbb{P}(X=i,Y=j)$	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$	$\mathbb{P}(X=i)$
$X=1$	$\frac{1}{5}$	0	0	0	0	$\frac{1}{5}$
$X=2$	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	$\frac{1}{5}$
$X=3$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	0	0	$\frac{1}{5}$
$X=4$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{5}$
$X=5$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{5}$
$\mathbb{P}(Y=j)$	$\frac{137}{300}$	$\frac{77}{300}$	$\frac{47}{300}$	$\frac{9}{100}$	$\frac{1}{25}$	1

(b) $\mathbb{P}(X=i\mid Y=j)=\frac{1}{5i}/\sum_{k=i}^{5}\frac{1}{5k}$ for $i=1,\ldots,5$ and $j=1,\ldots,i$.

$\mathbb{P}(X=i\mid Y=j)$	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$
$X=1$	$\frac{60}{137}$	0	0	0	0
$X=2$	$\frac{30}{137}$	$\frac{30}{77}$	0	0	0
$X=3$	$\frac{20}{137}$	$\frac{20}{77}$	$\frac{20}{47}$	0	0
$X=4$	$\frac{15}{137}$	$\frac{15}{77}$	$\frac{15}{47}$	$\frac{5}{9}$	0
$X=5$	$\frac{12}{137}$	$\frac{12}{77}$	$\frac{12}{47}$	$\frac{4}{9}$	1

(c) No. As X is always bigger than or equal to Y.

7. The joint probability mass function of X and Y is given by

$p(i,j)$	$j=1$	$j=2$
$i=1$	$\frac{1}{8}$	$\frac{1}{4}$
$i=2$	$\frac{1}{8}$	$\frac{1}{2}$

(a) Compute the conditional mass function of X given $Y=j, j=1,2$.
(b) Are X and Y independent?
(c) Compute $\mathbb{P}(XY\leqslant 3),\mathbb{P}(X+Y>2),\mathbb{P}(X/Y>1)$.

Solution:

(a)

$\mathbb{P}(X=i\mid Y=j)$	$j=1$	$j=2$
$i=1$	$\frac{1}{2}$	$\frac{1}{3}$
$i=2$	$\frac{1}{2}$	$\frac{2}{3}$

(b) No.

(c) $\mathbb{P}(XY\leqslant 3)=1-p(2,2)=\frac{1}{2}$, $\mathbb{P}(X+Y>2)=1-p(1,1)=\frac{7}{8}$, $\mathbb{P}(X/Y>1)=p(2,1)=\frac{1}{8}$.

8. The joint density function of X and Y is given by

$$f(x,y)=2xe^{-x(y+2)},\ x>0,y>0.$$

(a) Find the conditional density of X, given $Y=y$, and that of Y, given $X=x$.
(b) Find the density function of $Z=XY$.

Solution: Let X and Y be jointly continuous with density function $f(x,y) = 2xe^{-x(y+2)}$ for $x > 0, y > 0$. Note that

$$f_X(x) = \int_0^\infty f(x,y)dy = 2e^{-2x} \text{ for } x > 0,$$

and

$$f_Y(y) = \int_0^\infty f(x,y)dx = \frac{2}{(y+2)^2} \text{ for } y > 0.$$

(a) $f_{X|Y}(x|y) = (y+2)^2 xe^{-x(y+2)}$ for $x > 0, y > 0$, 0 otherwise, and $f_{Y|X}(y|x) = xe^{-xy}$ for $x > 0, y > 0$.

(b) Let $Z = XY$. Find $F_Z(a) = \mathbb{P}(XY < a) = \int_0^\infty \int_0^{\frac{a}{x}} 2xe^{-x(y+2)}dydx = 1 - e^{-a}$ for $a > 0$. Hence, $f_Z(a) = \frac{d}{da}F_Z(a) = e^{-a}$ for $a > 0$, 0 otherwise.

9. The joint density function of X and Y is given by

$$f(x,y) = c(x^2 - y^2)e^{-x}, 0 \leqslant x < \infty, -x \leqslant y \leqslant x.$$

Find the conditional distribution of Y, given $X = x$.

Solution: Let X and Y be jointly continuous with density function

$$f(x,y) = c(x^2 - y^2)e^{-x}$$

for $0 \leqslant x < \infty, -x \leqslant y \leqslant x$. For $x > 0$, we have

$$f_X(x) = \int_{-x}^{x} c(x^2 - y^2)e^{-x}dy = \frac{4c}{3}x^3e^{-x}.$$

Hence, $f_{Y|X}(y|x) = \frac{3}{4}\frac{x^2-y^2}{x^3}$ for $-x < y < x$, 0 otherwise. We conclude that

$$F_{Y|X}(y|x) = \begin{cases} 0 & y \leqslant -x \\ \frac{3}{4}\int_{-x}^{y} \frac{x^2-y^2}{x^3}dy = \frac{1}{4}\left(\frac{y(3x^2-y^2)}{x^3} + 2\right) & -x < y < x \\ 1 & x \leqslant y. \end{cases}$$

10. If $X_1, X_2, \ldots, X_6$ are independent and identically distributed exponential random variables with the parameter λ, compute (a) $\mathbb{P}(\min(X_1, X_2, \ldots, X_6) \leqslant a)$ and (b) $\mathbb{P}(\max(X_1, X_2, \ldots, X_6) \leqslant a)$.

Solution: Let $X_1, \ldots, X_6$ be independent exponential random variables with parameter λ.

(a)
$$\begin{aligned}\mathbb{P}(\min(X_1,\ldots,X_6) \leqslant a) = 1 - \mathbb{P}(\min(X_1,\ldots,X_6) > a) &= 1 - \mathbb{P}(X_1 > a, \ldots, X_6 > a) \\ &= 1 - \mathbb{P}(X_1 > a)\cdots\mathbb{P}(X_6 > a) \\ &= \begin{cases} 1 - (e^{-\lambda a})^6 & a > 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(b)
$$\begin{aligned}\mathbb{P}(\max(X_1,\ldots,X_6) \leqslant a) = \mathbb{P}(X_1 \leqslant a, \ldots, X_6 \leqslant a) &= \mathbb{P}(X_1 \leqslant a)\cdots\mathbb{P}(X_6 \leqslant a) \\ &= \begin{cases} (1 - e^{-\lambda a})^6 & a > 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$