Solutions to Final Review

1. Consider the following initial value problem for $y(x)$:

$$y' = \frac{\sqrt{y-a}}{(x-b)^2}, \qquad y(x_0) = y_0.$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

A. $x_0 = \beta \neq b$ **and** $y_0 = \alpha > a$

B. $x_0 = b$ and $y_0 = \alpha > a$

C. $x_0 = \beta \neq b$ and $y_0 = a$

D. $x_0 = b$ and $y_0 = a$

E. $x_0 = \beta \neq b$ and $y_0 = a$

F. None of these

Solution: The equation is of the form

$$y' = f(x, y)$$

with $f(x,y) = \frac{\sqrt{y-a}}{(x-b)^2}$. The initial value $x_0 = \beta \neq b$ and $y_0 = \alpha > a$ is the only one for which $f(x,y) = \frac{\sqrt{y-a}}{(x-b)^2}$ and $\frac{\partial f}{\partial y} = \frac{1}{(x-b)^2\sqrt{y-a}}$ are continuous.

2. Consider the following initial value problem for $y(x)$:

$$y' = \frac{x}{(y-a)(x-b)^2}, \qquad y(x_0) = y_0$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

A. $x_0 = \beta \neq b$ **and** $y_0 = \alpha \neq a$

B. $x_0 = b$ and $y_0 = \alpha \neq a$

C. $x_0 = \beta \neq b$ and $y_0 = a$

D. $x_0 = b$ and $y_0 = a$

E. $x_0 = \beta \neq b$ and $y_0 = a$

F. None of these

Solution: The equation is of the form

$$y' = f(x, y)$$

with $f(x,y) = \frac{x}{(y-a)(x-b)^2}$. The initial value $x_0 = \beta \neq b$ and $y_0 = \alpha \neq a$ is the only one for which $f(x,y) = \frac{x}{(y-a)(x-b)^2}$ and $\frac{\partial f}{\partial y} = -\frac{x}{(y-a)^2(x-b)^2}$ are continuous.

3. Select all sets of solutions that are linearly independent on the whole real axis

   A. $e^x,\ e^{2x},\ ae^x + be^{2x},$

   B. $\sin(2x),\ \cos(x)\sin(x),\ x,$

   C. $x,\ x^2,\ ax^2 + bx,$

   D. $x + 1,\ x - 1,\ 1,$

   E. $e^x,\ e^{2x},\ e^{3x},$

   F. $\sin(2x),\ \cos(2x),\ 1,$

   G. $x + 1,\ x - 1,\ x^2,$

   H. $1,\ ae^x, +b\,e^{2x}$

   I. $e^x + 1,\ e^{-x} - 1, x$

4. Assume that a linear homogeneous ODE for $y(x)$ has one of the following characteristic equation

   (i) $r^3 - 4r^2 + 4r$

   (ii) $r^3 - 2r^2$

   (iii) $r^3 + 6r^2 + 9r$

   (iv) $r^3 + 3r^2$

   (v) $^4 + r^3 - 6r^2$

   In each case pare item (i)-(v) above with a solution below

   A. $y = c_1 + c_2e^{2x} + c_3xe^{2x}$

   B. $y = c_1 + c_2x + c_3e^{2x}$

   C. $c_1 + c_2e^{-3x} + c_3xe^{-3x}$

   D. $c_1 + c_2x + c_3e^{-3x}$

   E. $c_1 + c_2x + c_3e^{2x} + c_4e^{-3x}$

   F. $c_1 + c_2x + c_3x^2$

   G. $c_1e^{2x} + c_2e^{-3x}$

   H. $c_1e^{2x} + c_2xe^{2x} + c_3e^{-3x}$

   I. $c_1e^{2x} + c_2e^{-3x} + c_3xe^{-3x}$

   Solution:

5. Consider the following initial value problem for $y(x)$:

$$(x^2 - (a+b)\,x\ + ab)\,y' + \frac{\gamma\,x}{x^2 - 2c\,x + c^2}\,y = \gamma_2\,\frac{x^2 + 2c\,x + c^2}{x^2}$$

with $y(x_0) = \gamma_2$. Assuming that $a < 0 < b < c$, determine the interval in which we are guaranteed one and only one solution:

$$x \in \Big(\;, \; \Big).$$

Solution: Dividing by $(x^2 - (a+b)\,x + ab) = (x-a)(x-b)$ and using $x^2 - 2c\,x + c^2 = (x-c)^2$

$$y' + \frac{\gamma\, x}{(x-a)(x-b)(x-c)^2}\, y = \gamma_2\, \frac{x^2 + 2c\,x + c^2}{(x-a)(x-b)x^2}.$$

For the uniqueness and existence theorem we need $\frac{\gamma\, x}{(x-a)(x-b)(x-c)^2}$ and $\frac{x^2+2c\,x+c^2}{(x-a)(x-b)x^2}$ to be continuous so the interval must be

(i) $(-\infty, a)$ if $x_0 < a$

(ii) $(a, 0)$ if $x_0 \in (a, 0)$

(iii) $(0, b)$ if $x_0 \in (0, b)$

(iv) (b, c) if $x_0 \in (b, c)$

(v) $(c, +\infty)$ if $x_0 > a$

6. Determine the integrating factor for the following ODE for $y(x)$:

$$x^2\, y' + A\, x\, y = Bx^C, \qquad x > 0$$

Solution: After dividing by x^2 the equation becomes

$$y' + \frac{A}{x}\, y = Bx^{C-2}.$$

The integrating factor is then

$$\rho = \exp\left(\int \frac{A}{x} dx \right) = e^{A \ln x} = x^A.$$

7. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = \gamma(P^2 - (a+b)P + ab)(P^2 - 2c\,P + c^2)$$

with $c < 0 < a < b$. Identify the correct equilibrium solutions and their stability

A. if $\gamma > 0$ $P = c$, semistable; $P = a$, stable; $P = b$, unstable

B. if $\gamma < 0$ $P = c$, semistable; $P = a$, unstable; $P = b$, stable

C. $P = c$, stable; $P = a$, unstable; $P = b$, stable

D. $P = c$, stable; $P = a$, stable; $P = b$, unstable

E. $P = 0$, semistable; $P = a$, stable; $P = b$, unstable

F. $P = 0$, semistable; $P = a$, unstable; $P = b$, stable

G. $P = c$, stable; $P = 0$, unstable; $P = b$, stable

H. $P = c$, stable; $P = 0$, stable; $P = b$, unstable

I. $P = c$, stable; $P = a$, unstable; $P = 0$, stable

J. $P = c$, unstable; $P = a$, stable; $P = 0$, unstable

K. None of these

Solution: The equation is of the form

$$\frac{dP}{dt} = f(P)$$

with $f(P) = \gamma(P^2 - (a+b)P + ab)(P^2 - 2c\,P + c^2) = \gamma(P-a)(P-b)(P-c)^2$ whose roots are $P = c, 0, a, b$. If $\gamma > 0$, then $f(P) = \gamma(P-a)(P-b)(P-c)^2$ is positive on $(-\infty, a) \cup (b, +\infty)$ and negative in (a, b) and therefore solutions $P(t)$ are increasing before a and after b and decreasing in between (a, b) whence $P = a$, stable; $P = b$, unstable. Since $c < 0 < a$ then $f(P)$ is positive near $P = c$ and therefore $P = c$, is semistable; When $\gamma < 0$ the above analysis is flipped.

8. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = (P^4 + (b-a)P^3 - ab\,P^2)$$

with $b > a > 0$. Determine the value of the stable equilibrium solution $P =$. Determine the value of the unstable equilibrium solution $P =$.

Solution: The equation is of the form

$$\frac{dP}{dt} = f(P)$$

with $f(P) = \gamma(P^4 + (b-a)P^3 - abP^2) = \gamma P^2(P-a)(P+b)$. When $\gamma > 0$, $f(P) = \gamma P^2(P-a)(P+b)$ is positive on $(-\infty, -b) \cup (a, +\infty)$ and negative in $(-b, a)$. So if $\gamma > 0$, $P = -b$ is stabel and $P = a$ is unstable. When $\gamma < 0$ the above analysis is flipped.

9. Consider the following oscillator equation for $x(t)$ and the given initial conditions:

$$x'' + \omega_0^2\, x = F_0\, f(\omega t),\ x(0) = 0;\ x'(0) = 0$$

with $\omega \neq \omega_0$ and where either $f = \cos$ or $f = \sin$. What is the long term behavior of the solution?

A. The solution will oscillate forever

B. The solution is 0 at all times

C. The solution will oscillates with amplitude growing to infinity

D. The solution will decay to 0

E. The solution will oscillates with amplitude decaying to 0

F. There is no solution

G. The solution will reach a finite asymptote

H. The solution will be lost in a forest

I. None of these

Solution: Since $\omega \neq \omega_0$ the general solution is $x(t) = R\,\cos(\omega_0 t - \delta) + F_0 f(\omega t)$ and since $f(\omega\, t)$ is either $\cos(\omega t)$ or $\sin(\omega t)$, the claim follows.

10. Consider the following ODE's for $y(x)$

(i) $y'' - y' - 6\,y = -4\,e^{x} + 3\,e^{-2x}$

(ii) $y'' + 3\,y' - 4\,y = 2\,e^{-2x} - e^{-4x}$

(iii) $y'' + y' - 6\,y = e^{2x} + 4\,e^{-2x}$

(iv) $y'' - 2\,y' - 8\,y = 2\,e^{4x} - 4\,e^{2x}$,

(v) $y'' + 2\,y' - 3\,y = 2\,e^{x} + e^{-4x}$

If you were to use the method of variation of parameters, what would be the correct particular solution to use?

A. $u_1(x)\,e^{3x} + u_2(x)\,e^{-2x}$ **for item 1**

B. $u_1(x)\,e^{x} + u_2(x)\,e^{-4x}$ **for item 2**

C. $u_1(x)\,e^{2x} + u_2(x)\,e^{-3x}$ **for item 3**

D. $u_1(x)\,e^{4x} + u_2(x)\,e^{-2x}$ **for item 4**

E. $u_1(x)\,e^{x} + u2(x)\,e^{-3x}$ **for item 5**

F. $A\,e^{x} + B\,e^{-2x}$

G. $A\,e^{-2x} + B\,e^{-4x}$

H. $A\,e^{2x} + B\,e^{-2x}$

I. $A\,e^{4x} + B\,e^{2x}$

J. $A\,e^{x} + B\,e^{-4x}$

K. $A\,e^{x} + B\,x\,e^{-2x}$

L. $A\,e^{-2x} + B\,x\,e^{-4x}$

M. $A\,x\,e^{2x} + B\,e^{-2x}$

N. $A\,x\,e^{4x} + B\,e^{2x}$

O. $A\,x\,e^{x} + B\,e^{-4x}$

Solution: Using the method of variation of parameters $y_p = u_1 y_1 + u_2 y_2$ where y_1 and y_2 are solutions to the the homogeneous equation. The characteristic polynomials of the equations given are

(i) $r^2 - r - 6 = (r-3)(r+2)$

(ii) $r^2 + 3r - 4 = (r+4)(r-1)$

(iii) $r^2 + r - 6 = (r-3)(r+2)$

(iv) $r^2 - 2r - 8 = (r-4)(r+2)$

(v) $r^2 + 2r - 3 = (r+2)(r-1)$

11. Consider the following eigenvalue problem for $y(x)$

$$\begin{aligned} y'' + \lambda\, y &= 0, \\ y(0) - y'(0) &= 0, \\ y(0) &= 0, \qquad y'(0) = 0, \\ y(L) &= 0, \qquad y(L) + y'(L) = 0, y(L) + y'(L) = 0. \end{aligned}$$

Which are the correct eigenvalues and corresponding eigenfunctions?

A. First choice of boundary conditions $\lambda_n = \alpha_n^2, \qquad y_n(x) = \alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)$ **where** $\tan(L\,\alpha_n) = -\alpha_n$

B. Second choice of boundary conditions$\lambda_n = \alpha_n^2, \qquad y_n(x) = \sin(\alpha_n x)$, **where** $tan(L\,\alpha_n) = -\alpha_n$

C. Third choice of boundary conditions $\lambda_n = \alpha_n^2, \qquad y_n(x) = \cos(\alpha_n x)$, **where** , $\tan(L\,\alpha_n) = 1/\alpha_n$

D. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

E. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

F. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}; \qquad y_n(x) = \cos\left(\frac{(2n-1)\pi x}{L}\right)$

G. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}$; $\qquad y_n(x) = \sin\left(\frac{(2n-1)\pi x}{L}\right)$

H. None of these

12. Calculate the coefficients of the Cosine Fourier Series expansion of $f(x) = a\,x$ for $0 < x < L$.

Solution:

$$a_0 = \frac{2}{L}\int_0^L a\,x dx = aL$$
$$a_n = \frac{2}{L}\int_0^L a\,x\,\cos\left(\frac{n\pi x}{L}\right)\,dx = 2aL\frac{(\cos(n\pi)-1)}{((n\pi)^2)}$$

13. Using separation of variables solve the following diffusion equation problem for $u(x,t)$ where $0 < x < L$ and $t > 0$

$$\begin{cases} u_t = \kappa\,u_{xx} & \text{for } 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \;\; u(L,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = a\,x. \end{cases}$$

Assume that the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n\,T_n(t)\,X_n(x).$$

Calculate c_n, $T_n(t)$, and $X_n(x)$.

Solution:

$$\begin{aligned} X_n(x) &= \sin\left(\frac{n\pi x}{L}\right), \\ T_n(t) &= e^{-\frac{n^2\pi^2\kappa^2 t}{L}}, \\ c_n &= \frac{2}{L}\int_0^L a\,x\,\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\frac{L}{2}\left(-ax\cos\left(\frac{n\pi x}{L}\right)|_0^L + \int_0^L a\,\cos\left(\frac{n\pi x}{L}\right)dx\right) \\ &= -2aL\frac{\cos(n\pi)}{(n\pi)} = (-1)^{n+1}\frac{aL}{n\pi}. \end{aligned}$$

14. Consider the Laplace equation problem in the rectangle $0 < x < a$ and $0 < y < b$:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(0, y) = bc_1, \qquad u(a, y) = bc_2, \\ &u(x, 0) = bc_3, \qquad u(x, b) = bc_4. \end{aligned}$$

Where bc_i means that the i-th condition is a function $f(x)$ and everything else is 0. If you were to solve this problem by separation of variables by writing $u(x, y) = X(x)Y(y)$, what would be the solutions for X_n and Y_n?

A. Correct in case bc_1 $X_n = -\tanh\left(\frac{an\pi}{b}\right)\cosh\left(\frac{n\pi x}{b}\right) + \sinh\left(\frac{n\pi x}{b}\right)$; $\quad Y_n = \sin\left(\frac{n\pi y}{b}\right)$;

B. Correct in case bc_2 $X_n = \sinh\left(\frac{n\pi x}{b}\right)$; $\quad Y_n = \sin\left(\frac{n\pi y}{b}\right)$

C. Correct in case bc_3 $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = -\tanh\left(\frac{bn\pi}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)$

D. Correct in case bc_4 $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = \sinh\left(\frac{n\pi y}{a}\right)$

E. None of these

F. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = \sin\left(\frac{n\pi y}{b}\right)$

G. $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = \cos\left(\frac{n\pi y}{b}\right)$

H. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = \cos\left(\frac{n\pi y}{b}\right)$

I. $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $\quad Y_n = \sin\left(\frac{n\pi y}{b}\right)$

J. There is no solution