Solutions to Practice for Final exam

1. Consider the following initial value problem for $y(x)$:

$$y' = \frac{\sqrt{y-a}}{(x-b)^2}, \qquad y(x_0) = y_0.$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

    A. $x_0 = \beta \neq b$ and $y_0 = \alpha > a$

    B. $x_0 = b$ and $y_0 = \alpha > a$

    C. $x_0 = \beta \neq b$ and $y_0 = a$

    D. $x_0 = b$ and $y_0 = a$

    E. $x_0 = \beta \neq b$ and $y_0 = a$

    F. None of these

> **Solution:** The equation is of the form
>
> $$y' = f(x, y)$$
>
> with $f(x,y) = \frac{\sqrt{y-a}}{(x-b)^2}$. The initial value $x_0 = \beta \neq b$ and $y_0 = \alpha > a$ is the only one for which $f(x,y) = \frac{\sqrt{y-a}}{(x-b)^2}$ and $\frac{\partial f}{\partial y} = \frac{1}{(x-b)^2\sqrt{y-a}}$ are continuous.

2. Consider the following initial value problem for $y(x)$:

$$y' = \frac{x}{(y-a)(x-b)^2}, \qquad y(x_0) = y_0$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

    A. $x_0 = \beta \neq b$ and $y_0 = \alpha \neq a$

    B. $x_0 = b$ and $y_0 = \alpha \neq a$

    C. $x_0 = \beta \neq b$ and $y_0 = a$

    D. $x_0 = b$ and $y_0 = a$

    E. $x_0 = \beta \neq b$ and $y_0 = a$

    F. None of these

> **Solution:** The equation is of the form
>
> $$y' = f(x, y)$$
>
> with $f(x,y) = \frac{x}{(y-a)(x-b)^2}$. The initial value $x_0 = \beta \neq b$ and $y_0 = \alpha \neq a$ is the only one for which $f(x,y) = \frac{x}{(y-a)(x-b)^2}$ and $\frac{\partial f}{\partial y} = -\frac{x}{(y-a)^2(x-b)^2}$ are continuous.

3. Select all sets of solutions that are linearly independent on the whole real axis

   A. e^x, e^{2x}, $ae^x + be^{2x}$,

   B. $\sin(2x)$, $\cos(x)\sin(x)$, x,

   C. x, x^2, $ax^2 + bx$,

   D. $x + 1$, $x - 1$, 1,

   E. e^x, e^{2x}, e^{3x},

   F. $\sin(2x)$, $\cos(2x)$, 1,

   G. $x + 1$, $x - 1$, x^2,

   H. 1, ae^x, $+be^{2x}$

   I. $e^x + 1$, $e^{-x} - 1$, x

4. Assume that a linear homogeneous ODE for $y(x)$ has one of the following characteristic equation

   (i) $r^3 - 4r^2 + 4r$

   (ii) $r^3 - 2r^2$

   (iii) $r^3 + 6r^2 + 9r$

   (iv) $r^3 + 3r^2$

   (v) $r^4 + r^3 - 6r^2$

   In each case pare item (i)-(v) above with a solution below

   A. $c_1 + c_2 e^{2x} + c_3 x e^{2x}$

   B. $c_1 + c_2 x + c_3 e^{2x}$

   C. $c_1 + c_2 e^{-3x} + c_3 x e^{-3x}$

   D. $c_1 + c_2 x + c_3 e^{-3x}$

   E. $c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-3x}$

   F. $c_1 + c_2 x + c_3 x^2$

   G. $c_1 e^{2x} + c_2 e^{-3x}$

   H. $c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x}$

   I. $c_1 e^{2x} + c_2 e^{-3x} + c_3 x e^{-3x}$

Solution:

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5. Consider the following initial value problem for $y(x)$:

$$(x^2 - (a + b)x + ab)\, y' + \frac{\gamma x}{x^2 - 2cx + c^2}\, y = \gamma_2 \frac{x^2 + 2cx + c^2}{x^2}$$

with $y(x_0) = \gamma_2$. Assuming that $a < 0 < b < c$, determine the interval in which we are guaranteed one and only one solution:

$$x \in \left(\ , \ \right).$$

Solution: Dividing by $(x^2 - (a+b)\,x + ab) = (x-a)(x-b)$ and using $x^2 - 2c\,x + c^2 = (x-c)^2$

$$y' + \frac{\gamma\,x}{(x-a)(x-b)(x-c)^2}\,y = \gamma_2\,\frac{x^2 + 2c\,x + c^2}{(x-a)(x-b)x^2}.$$

For the uniqueness and existence theorem we need $\frac{\gamma\,x}{(x-a)(x-b)(x-c)^2}$ and $\frac{x^2+2c\,x+c^2}{(x-a)(x-b)x^2}$ to be continuous so the interval must be

   (i) $(-\infty, a)$ if $x_0 < a$

   (ii) $(a, 0)$ if $x_0 \in (a, 0)$

   (iii) $(0, b)$ if $x_0 \in (0, b)$

   (iv) (b, c) if $x_0 \in (b, c)$

   (v) $(c, +\infty)$ if $x_0 > a$

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6. Determine the integrating factor for the following ODE for $y(x)$:

$$x^2\,y' + A\,x\,y = B x^C, \qquad x > 0$$

Solution: After dividing by x^2 the equation becomes

$$y' + \frac{A}{x}\,y = Bx^{C-2}.$$

The integrating factor is then

$$\rho = \exp\left(\int \frac{A}{x}dx\right) = e^{A\ln x} = x^A.$$

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7. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = \gamma(P^2 - (a+b)P + ab)(P^2 - 2c\,P + c^2)$$

with $c < 0 < a < b$. Identify the correct equilibrium solutions and their stability

A. if $\gamma > 0$ $P = c$, **semistable;** $P = a$, **stable;** $P = b$, **unstable**

B. if $\gamma < 0$ $P = c$, **semistable;** $P = a$, **unstable;** $P = b$, **stable**

C. $P = c$, stable; $P = a$, unstable; $P = b$, stable

D. $P = c$, stable; $P = a$, stable; $P = b$, unstable

E. $P = 0$, semistable; $P = a$, stable; $P = b$, unstable

F. $P = 0$, semistable; $P = a$, unstable; $P = b$, stable

G. $P = c$, stable; $P = 0$, unstable; $P = b$, stable

H. $P = c$, stable; $P = 0$, stable; $P = b$, unstable

I. $P = c$, stable; $P = a$, unstable; $P = 0$, stable

J. $P = c$, unstable; $P = a$, stable; $P = 0$, unstable

K. None of these

Solution: The equation is of the form

$$\frac{dP}{dt} = f(P)$$

with $f(P) = \gamma(P^2 - (a+b)P + ab)(P^2 - 2cP + c^2) = \gamma(P-a)(P-b)(P-c)^2$ whose roots are $P = c, 0, a, b$. If $\gamma > 0$, then $f(P) = \gamma(P-a)(P-b)(P-c)^2$ is positive on $(-\infty, a) \cup (b, +\infty)$ and negative in (a, b) and therefore solutions $P(t)$ are increasing before a and after b and decreasing in between (a, b) whence $P = a$, stable; $P = b$, unstable. Since $c < 0 < a$ then $f(P)$ is positive near $P = c$ and therefore $P = c$, is semistable; When $\gamma < 0$ the above analysis is flipped.

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8. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = (P^4 + (b-a)P^3 - ab\,P^2)$$

with $b > a > 0$. Determine the value of the stable equilibrium solution $P =$. Determine the value of the unstable equilibrium solution $P =$.

Solution: The equation is of the form

$$\frac{dP}{dt} = f(P)$$

with $f(P) = \gamma(P^4 + (b-a)P^3 - abP^2) = \gamma P^2(P-a)(P+b)$. When $\gamma > 0$, $f(P) = \gamma P^2(P-a)(P+b)$ is positive on $(-\infty, -b) \cup (a, +\infty)$ and negative in $(-b, a)$. So if $\gamma > 0$, $P = -b$ is stabel and $P = a$ is unstable. When $\gamma < 0$ the above analysis is flipped.

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9. Consider the following oscillator equation for $x(t)$ and the given initial conditions:

$$x'' + \omega_0^2 x = F_0 f(\omega t), \ x(0) = 0; \ x'(0) = 0$$

with $\omega \neq \omega_0$ and where either $f = \cos$ or $f = \sin$. What is the long term behavior of the solution?

A. The solution will oscillate forever

B. The solution is 0 at all times

C. The solution will oscillates with amplitude growing to infinity

D. The solution will decay to 0

E. The solution will oscillates with amplitude decaying to 0

F. There is no solution

G. The solution will reach a finite asymptote

H. The solution will be lost in a forest

I. None of these

Solution: Since $\omega \neq \omega_0$ the general solution is $x(t) = R \cos(\omega_0 t - \delta) + F_0 f(\omega t)$ and since $f(\omega t)$ is either $\cos(\omega t)$ or $\sin(\omega t)$, the claim follows.

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10. Consider the following ODE's for $y(x)$

(i) $y'' - y' - 6y = -4e^x + 3e^{-2x}$

(ii) $y'' + 3y' - 4y = 2e^{-2x} - e^{-4x}$

(iii) $y'' + y' - 6y = e^{2x} + 4e^{-2x}$

(iv) $y'' - 2y' - 8y = 2e^{4x} - 4e^{2x}$,

(v) $y'' + 2y' - 3y = 2e^x + e^{-4x}$

If you were to use the method of variation of parameters, what would be the correct particular solution to use?

A. $u_1(x)e^{3x} + u_2(x)e^{-2x}$ **for item 1**

B. $u_1(x)e^x + u_2(x)e^{-4x}$ **for item 2**

C. $u_1(x)e^{2x} + u_2(x)e^{-3x}$ **for item 3**

D. $u_1(x)e^{4x} + u_2(x)e^{-2x}$ **for item 4**

E. $u_1(x)e^x + u2(x)e^{-3x}$ **for item 5**

F. $Ae^x + Be^{-2x}$

G. $Ae^{-2x} + Be^{-4x}$

H. $Ae^{2x} + Be^{-2x}$

   I. $A e^{4x} + B e^{2x}$

   J. $A e^{x} + B e^{-4x}$

   K. $A e^{x} + B x e^{-2x}$

   L. $A e^{-2x} + B x e^{-4x}$

   M. $A x e^{2x} + B e^{-2x}$

   N. $A x e^{4x} + B e^{2x}$

   O. $A x e^{x} + B e^{-4x}$

Solution: Using the method of variation of parameters $y_p = u_1 y_1 + u_2 y_2$ where y_1 and y_2 are solutions to the the homogeneous equation. The characteristic polynomials of the equations given are

(i) $r^2 - r - 6 = (r - 3)(r + 2)$

(ii) $r^2 + 3r - 4 = (r + 4)(r - 1)$

(iii) $r^2 + r - 6 = (r - 3)(r + 2)$

(iv) $r^2 - 2r - 8 = (r - 4)(r + 2)$

(v) $r^2 + 2r - 3 = (r + 2)(r - 1)$

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11. Consider the following eigenvalue problem for $y(x)$

$$y'' + \lambda y = 0,$$

with

$$
\begin{aligned}
y(0) - y'(0) &= 0, & y(0) &= 0, & \text{(A)} \\
y'(0) &= 0, & y(L) &= 0, & \text{(B)} \\
y(L) + y'(L) &= 0, & y(L) + y'(L) &= 0. & \text{(C)}
\end{aligned}
$$

Which are the correct eigenvalues and corresponding eigenfunctions?

   A. First choice of boundary conditions $\lambda_n = \alpha_n^2$, $y_n(x) = \alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)$ **where** $\tan(L\alpha_n) = -\alpha_n$

   B. Second choice of boundary conditions $\lambda_n = \alpha_n^2$, $y_n(x) = \sin(\alpha_n x)$, **where** $\tan(L\alpha_n) = -\alpha_n$

   C. Third choice of boundary conditions $\lambda_n = \alpha_n^2$, $y_n(x) = \cos(\alpha_n x)$, **where** , $\tan(L\alpha_n) = 1/\alpha_n$

   D. $\lambda_n = \frac{n^2 \pi^2}{L^2}$; $y_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

   E. $\lambda_n = \frac{n^2 \pi^2}{L^2}$; $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

F. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}$; $\qquad y_n(x) = \cos\left(\frac{(2n-1)\pi x}{L}\right)$

G. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}$; $\qquad y_n(x) = \sin\left(\frac{(2n-1)\pi x}{L}\right)$

H. None of these

12. Calculate the coefficients of the Cosine Fourier Series expansion of $f(x) = a\,x$ for $0 < x < L$.

> **Solution:**
>
> $$a_0 = \frac{2}{L}\int_0^L a\,x\,dx = aL$$
>
> $$a_n = \frac{2}{L}\int_0^L a\,x\,\cos\left(\frac{n\pi x}{L}\right)\,dx = 2aL\frac{(\cos(n\pi)-1)}{((n\pi)^2)}$$

13. Using separation of variables solve the following diffusion equation problem for $u(x,t)$ where $0 < x < L$ and $t > 0$

$$\begin{cases} u_t = \kappa\,u_{xx} & \text{for } 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = a\,x. \end{cases}$$

Assume that the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n\,T_n(t)\,X_n(x).$$

Calculate c_n, $T_n(t)$, and $X_n(x)$.

> **Solution:**
>
> $$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$
>
> $$T_n(t) = e^{-\frac{n^2\pi^2\kappa^2 t}{L}},$$
>
> $$\begin{aligned} c_n &= \frac{2}{L}\int_0^L a\,x\,\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\frac{L}{2}\left(-ax\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \int_0^L a\cos\left(\frac{n\pi x}{L}\right)\,dx\right) \\ &= -2aL\frac{\cos(n\pi)}{(n\pi)} = (-1)^{n+1}\frac{aL}{n\pi}. \end{aligned}$$

14. Consider the Laplace equation problem in the rectangle $0 < x < a$ and $0 < y < b$:

$$u_{xx} + u_{yy} = 0,$$
$$u(0, y) = bc_1, \qquad u(a, y) = bc_2,$$
$$u(x, 0) = bc_3, \qquad u(x, b) = bc_4.$$

Where bc_i means that the i-th condition is a function $f(x)$ and everything else is 0. If you were to solve this problem by separation of variables by writing $u(x, y) = X(x)Y(y)$, what would be the solutions for X_n and Y_n?

A. Correct in case bc_1 $X_n = -\tanh\left(\frac{an\pi}{b}\right)\cosh\left(\frac{n\pi x}{b}\right) + \sinh\left(\frac{n\pi x}{b}\right);$ $Y_n = \sin\left(\frac{n\pi y}{b}\right);$

B. Correct in case bc_2 $X_n = \sinh\left(\frac{n\pi x}{b}\right);$ $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

C. Correct in case bc_3 $X_n = \sin\left(\frac{n\pi x}{a}\right);$ $Y_n = -\tanh\left(\frac{bn\pi}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)$

D. Correct in case bc_4 $X_n = \sin\left(\frac{n\pi x}{a}\right);$ $Y_n = \sinh\left(\frac{n\pi y}{a}\right)$

E. None of these

F. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right);$ $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

G. $X_n = \sin\left(\frac{n\pi x}{a}\right);$ $Y_n = \cos\left(\frac{n\pi y}{b}\right)$

H. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right);$ $Y_n = \cos\left(\frac{n\pi y}{b}\right)$

I. $X_n = \sin\left(\frac{n\pi x}{a}\right);$ $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

J. There is no solution